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Abstract

After different variables and functions changes, the generalized dispersal problem, recalled in (1) below and considered in part I, see [14], leads us to invert a sum of linear operators in a suitable Banach space, see (2) below.

The essential result of this second part lies in the complete study of this sum using the two well-known strategies: the one of Da Prato-Grisvard [4] and the one of Dore-Venni [6].
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1 Introduction and main result

This work is a natural continuation of [14], where we have studied, for $k > 0$, the following problem

$$\begin{cases} \Delta^2 u - k\Delta u = f & \text{in } S_{\omega,\rho} \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \Gamma_0 \cup \Gamma_{\omega,\rho}, \end{cases} \tag{1}$$

where, for given $\rho > 0$ and $\omega \in (0, 2\pi]$:

$$\begin{cases} S_{\omega,\rho} &= \{(x,y) = (r\cos\theta, r\sin\theta) : 0 < r < \rho \text{ and } 0 < \theta < \omega\}, \\ \Gamma_0 &= (0,+\infty) \times \{0\} \\ \Gamma_{\omega,\rho} &= \{(r\cos\omega, r\sin\omega) : 0 < r < \rho\}. \end{cases}$$

Note that a similar problem set in a cylindrical domain has been already studied by [15], but in the present paper, since the domain is conical, the study is completely different.

After using the polar coordinates $v(r,\theta) = u(r\cos\theta, r\sin\theta)$, the function

$$\phi(t)(\theta) := \phi(t,\theta) = \frac{v(\rho e^{-t}, \theta)}{\rho e^{-t}}, \quad t > 0,$$

and the abstract vector-valued function

$$V(t) = \begin{pmatrix} e^{\nu t}\phi(t) \\ e^{\nu t}\phi''(t) \end{pmatrix}, \quad \text{with} \quad \nu = 3 - \frac{2}{p} \in (1,3),$$

the above study led us, to solve the following abstract equation

$$(\mathcal{L}_1 + \mathcal{L}_2)\, V + k\rho^2 (\mathcal{P}_1 + \mathcal{P}_2)\, V = \mathcal{F}, \tag{2}$$

where $\mathcal{F} \in L^p(0,+\infty; W_0^{2,p}(0,\omega) \times L^p(0,\omega))$, with $p \in (1,+\infty)$. Operators $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{P}_1$ and $\mathcal{P}_2$, introduced in [14], are recalled here for the reader convenience:

$$\begin{cases} D(\mathcal{L}_1) &= \left\{V \in W^{2,p}(0,+\infty;X) : V(0) = V(+\infty) = 0\right\} \\ [\mathcal{L}_1(V)](t) &= (\partial_t - \nu I)^2 V(t) = V''(t) - 2\nu V'(t) + \nu^2 V(t), \end{cases}$$

with $\nu \in \mathbb{R}$ fixed,

$$\left\{ \begin{array}{lcl} D(\mathcal{L}_2) & = & \{V \in L^p(0,+\infty;X) : \text{for } a.e.\ t \in (0,+\infty),\ V(t) \in D(\mathcal{A})\} \\ [\mathcal{L}_2(V)](t) & = & -\mathcal{A}V(t), \end{array} \right.$$

with

$$\left\{ \begin{array}{lcl} D(\mathcal{A}) & = & \left[W^{4,p}(0,\omega) \cap W_0^{2,p}(0,\omega)\right] \times W_0^{2,p}(0,\omega) \subset X \\ \mathcal{A}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} & = & \begin{pmatrix} \psi_2 \\ -\left(\dfrac{\partial^2}{\partial\theta^2}+1\right)^2 \psi_1 - 2\left(\dfrac{\partial^2}{\partial\theta^2}-1\right)\psi_2 \end{pmatrix}, \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in D(\mathcal{A}), \end{array} \right.$$

$$\left\{ \begin{array}{lcl} D(\mathcal{P}_1) & = & \{V \in L^p(0,+\infty;X) : \text{for } a.e.\ t \in (0,+\infty),\ V(t) \in D(\mathcal{A}_0)\} \\ [\mathcal{P}_1(V)](t) & = & -e^{-2t}\mathcal{A}_0 V(t), \end{array} \right.$$

with

$$\left\{ \begin{array}{lcl} D(\mathcal{A}_0) & = & W_0^{2,p}(0,\omega) \times L^p(0,\omega) = X \\ \mathcal{A}_0\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} & = & \begin{pmatrix} 0 \\ \left(\dfrac{\partial^2}{\partial\theta^2}+1\right)\psi_1 + \psi_2 \end{pmatrix}, \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in D(\mathcal{A}_0), \end{array} \right.$$

and

$$\left\{ \begin{array}{lcl} D(\mathcal{P}_2) & = & W^{1,p}(0,+\infty;X) \\ [\mathcal{P}_2(V)](t) & = & -e^{-2t}\left(\mathcal{B}_2 V\right)(t), \end{array} \right.$$

with

$$\mathcal{B}_2 = \begin{pmatrix} 0 & 0 \\ -2(\partial_t - \nu I) & 0 \end{pmatrix}.$$

In the present paper, we will focus ourselves on the resolution of equation (2). Among others, we need to use the fact that the roots of the following equation

$$(\sinh(z)+z)(\sinh(z)-z) = 0,$$

in $\mathbb{C}_+ := \{w \in \mathbb{C} : \mathrm{Re}(w) > 0\}$, constitute a family of complex numbers $(z_j)_{j\geqslant 1}$ such that

$$\tau := \min_{j\geqslant 1} |\mathrm{Im}(z_j)| > 0 \quad \text{and} \quad |z_j| \longrightarrow +\infty.$$

These roots are computed in [7] with $\tau \simeq 4.21239$.

Our main result is the following.

Theorem 1.1. Let $\mathcal{F} \in L^p(0,+\infty;X)$ and assume that

$$\omega\nu < \tau. \tag{3}$$

Then, there exists $\rho_0 > 0$ such that for all $\rho \in (0,\rho_0]$, the abstract equation

$$(\mathcal{L}_1+\mathcal{L}_2)V + k\rho^2(\mathcal{P}_1+\mathcal{P}_2)V = \mathcal{F},$$

has a unique classical solution $V \in L^p(0,+\infty;X)$, that is

$$V \in W^{2,p}(0,+\infty;X) \cap L^p(0,+\infty;D(\mathcal{A})).$$

In particular, $\mathcal{L}_1+\mathcal{L}_2$ is a closed operator and $V \in D(\mathcal{L}_1+\mathcal{L}_2)$.

This second part is organized as follows. Section 2 is devoted to some recalls. In Section 3, we analyze the spectral properties of operators $\mathcal{L}_1$ and $\mathcal{L}_2$ in view to study the invertibility of the closedness of the sum $\overline{\mathcal{L}_1+\mathcal{L}_2}$ in Section 4. In Section 5, by considering that operator $k\rho^2(\mathcal{P}_1+\mathcal{P}_2)$ is a perturbation, we deduce the existence and the uniqueness of a strong solution of equation (2). Finally, Section 6 is devoted to the proof of our main result given in Theorem 1.1.

2 Definitions and prerequisites

2.1 The class of Bounded Imaginary Powers of operators

Definition 2.1. A Banach space X is a UMD space if and only if for all $p \in (1, +\infty)$, the Hilbert transform is bounded from $L^p(\mathbb{R}, X)$ into itself (see [2] and [3]).

Definition 2.2. Let $\alpha \in (0, \pi)$. Sect(α) denotes the space of closed linear operators T_1 which satisfying

$$\begin{aligned} &i) \quad \sigma(T_1) \subset \overline{S_\alpha}, \\ &ii) \quad \forall\ \alpha' \in (\alpha, \pi), \quad \sup\left\{\|\lambda(\lambda I - T_1)^{-1}\|_{\mathcal{L}(X)} :\ \lambda \in \mathbb{C} \setminus \overline{S_{\alpha'}}\right\} < +\infty, \end{aligned}$$

where

$$S_\alpha\ := \begin{cases} \{z \in \mathbb{C} : z \neq 0 \ \text{ and } \ |\arg(z)| < \alpha\} & \text{if} \quad \alpha \in (0, \pi] \\ (0, +\infty) & \text{if} \quad \alpha = 0, \end{cases} \tag{4}$$

see [9], p. 19. Such an operator T_1 is called sectorial operator of angle α.

Remark 2.3. From [11], p. 342, we know that any injective sectorial operator T_1 admits imaginary powers T_1^{is} for all $s \in \mathbb{R}$; but in general, T_1^{is} is not bounded.

Definition 2.4. Let $\theta \in [0, \pi)$. We denote by BIP(X, θ), the class of sectorial injective operators T_2 such that

$$\begin{aligned} &i) \quad \overline{D(T_2)} = \overline{R(T_2)} = X, \\ &ii) \quad \forall\ s \in \mathbb{R}, \quad T_2^{is} \in \mathcal{L}(X), \\ &iii) \quad \exists\ C \geqslant 1,\ \forall\ s \in \mathbb{R}, \quad ||T_2^{is}||_{\mathcal{L}(X)} \leqslant Ce^{|s|\theta}, \end{aligned}$$

see [19], p. 430.

2.2 Recall on the sum of linear operators

Let us fix a pair of two closed linear densely defined operators $\mathcal{M}_1$ and $\mathcal{M}_2$ in a general Banach space $\mathcal{E}$. We note their domains by $D(\mathcal{M}_1)$ and $D(\mathcal{M}_2)$ respectively. Then we can define their sum by

$$\begin{cases} \mathcal{M}_1 w + \mathcal{M}_2 w \\ w \in D(\mathcal{M}_1) \cap D(\mathcal{M}_2). \end{cases}$$

We assume the following hypotheses

(H_1) There exist $\theta_{\mathcal{M}_1} \in [0, \pi)$, $\theta_{\mathcal{M}_2} \in [0, \pi)$, $C > 0$ and $R > 0$ such that

$$\begin{cases} \rho(\mathcal{M}_1) \supset \Sigma_{1,R} = \{z \in \mathbb{C} \setminus \{0\} : |z| \geqslant R \text{ and } |\arg(z)| < \pi - \theta_{\mathcal{M}_1}\} \\ \forall\, z \in \Sigma_{1,R}, \quad \left\|(\mathcal{M}_1 - zI)^{-1}\right\| \leqslant \dfrac{C}{|z|}, \end{cases}$$

and

$$\begin{cases} \rho(\mathcal{M}_2) \supset \Sigma_{2,R} = \{z \in \mathbb{C} \setminus \{0\} : |z| \geqslant R \text{ and } |\arg(z)| < \pi - \theta_{\mathcal{M}_2}\} \\ \forall\, z \in \Sigma_{2,R}, \quad \left\|(\mathcal{M}_2 - zI)^{-1}\right\| \leqslant \dfrac{C}{|z|}, \end{cases}$$

with

$$\theta_{\mathcal{M}_1} + \theta_{\mathcal{M}_2} < \pi.$$

(H_2) $\sigma(\mathcal{M}_1) \cap \sigma(-\mathcal{M}_2) = \emptyset$.

(H_3) The resolvents of $\mathcal{M}_1$ and $\mathcal{M}_2$ commute, that is

$$(\mathcal{M}_1 - \lambda I)^{-1} (\mathcal{M}_2 - \mu I)^{-1} = (\mathcal{M}_2 - \mu I)^{-1} (\mathcal{M}_1 - \lambda I)^{-1},$$

for all $\lambda \in \rho(\mathcal{M}_1)$ and all $\mu \in \rho(\mathcal{M}_2)$.

Remark 2.5. Note that from (H_2), we have $\rho(\mathcal{M}_1) \cup \rho(-\mathcal{M}_2) = \mathbb{C}$ and in particular $\mathcal{M}_1$ or $\mathcal{M}_2$ is boundedly invertible.

Theorem 2.6 ([4], [8])**.** Assume that (H_1), (H_2) and (H_3) hold. Then, operator $\mathcal{M}_1 + \mathcal{M}_2$ is closable. Its closure $\overline{\mathcal{M}_1 + \mathcal{M}_2}$ is boundedly invertible and

$$\left(\overline{\mathcal{M}_1 + \mathcal{M}_2}\right)^{-1} = \frac{-1}{2i\pi} \int_\Gamma (\mathcal{M}_1 - zI)^{-1} (\mathcal{M}_2 + zI)^{-1} dz; \tag{5}$$

where Γ is a path which separates $\sigma(\mathcal{M}_1)$ and $\sigma(-\mathcal{M}_2)$ and joins $\infty e^{-i\theta_0}$ to $\infty e^{i\theta_0}$ with θ_0 such that

$$\theta_{\mathcal{M}_1} < \theta_0 < \pi - \theta_{\mathcal{M}_2}.$$

This Theorem is proved in [4] (Theorem 3.7, p. 324), when $R = 0$ and has been extended to the case $R \geqslant 0$ in [8] (Theorem 2.1, p. 7). In this last case, the curve Γ does not need to be connected.

Corollary 2.7. Assume (H_1), (H_2) and (H_3) hold. Let $i = 1, 2$ and $\mathcal{E}_i$ a Banach space with $D(\mathcal{M}_i) \hookrightarrow \mathcal{E}_i \hookrightarrow \mathcal{E}$ such that there exist $C > 0$, $\delta \in (0, 1)$ satisfying

$$\begin{cases} \|w\|_{\mathcal{E}_i} \leqslant C \left(\|w\|_{\mathcal{E}} + \|w\|_{\mathcal{E}}^{1-\delta} \|\mathcal{M}_i w\|_{\mathcal{E}}^{\delta} \right) \\ \text{for every } w \in D(\mathcal{M}_i), \end{cases} \tag{6}$$

then $D\left(\overline{\mathcal{M}_1 + \mathcal{M}_2}\right) \subset \mathcal{E}_i$.

Proof. It is enough to prove that the integral in (5) converges in $\mathcal{E}_1$. For all $\xi \in \mathcal{E}$, we have

$$\left\| \int_\Gamma (\mathcal{M}_1 - zI)^{-1} (\mathcal{M}_2 + zI)^{-1} \xi dz \right\|_{\mathcal{E}_1} \leqslant \int_\Gamma \left\| (\mathcal{M}_1 - zI)^{-1} (\mathcal{M}_2 + zI)^{-1} \xi \right\|_{\mathcal{E}_1} |dz|,$$

then, applying (6), we obtain

$$\begin{aligned} &\left\| (\mathcal{M}_1 - zI)^{-1} (\mathcal{M}_2 + zI)^{-1} \xi \right\|_{\mathcal{E}_1} \\ &\leqslant C \left\| (\mathcal{M}_1 - zI)^{-1} (\mathcal{M}_2 + zI)^{-1} \xi \right\|_{\mathcal{E}} \\ &+ C \left\| (\mathcal{M}_1 - zI)^{-1} (\mathcal{M}_2 + zI)^{-1} \xi \right\|_{\mathcal{E}}^{1-\delta} \left\| \mathcal{M}_1 (\mathcal{M}_1 - zI)^{-1} (\mathcal{M}_2 + zI)^{-1} \xi \right\|_{\mathcal{E}}^{\delta}; \end{aligned}$$

Now, for all $z \in \Gamma$, we have

$$\begin{aligned} &\left\| (\mathcal{M}_1 - zI)^{-1} (\mathcal{M}_2 + zI)^{-1} \xi \right\|_{\mathcal{E}}^{1-\delta} \left\| \mathcal{M}_1 (\mathcal{M}_1 - zI)^{-1} (\mathcal{M}_2 + zI)^{-1} \xi \right\|_{\mathcal{E}}^{\delta} \\ &\leqslant \frac{(C_1(\theta_1) C_2(\theta_2))^{1-\delta}}{|z|^{2(1-\delta)}} \frac{C_1(\theta_1)^{\delta} C_2(\theta_2)^{\delta}}{|z|^{\delta}} \|\xi\|_{\mathcal{E}} = \frac{C_1(\theta_1) C_2(\theta_2)}{|z|^{1+(1-\delta)}} \|\xi\|_{\mathcal{E}}, \end{aligned}$$

from which we deduce the convergence of the integral in (5). The same result holds true replacing $\mathcal{M}_1$ by $\mathcal{M}_2$. □

3 Spectral study of operators

In all the sequel, in view to apply the above results, we will consider the following particular Banach space

$$\mathcal{E} = L^p(0, +\infty; X),$$

equipped with its natural norm.

3.1 Study of operator $\mathcal{L}_1$

We study the spectral equation

$$\mathcal{L}_1 V - \lambda V = R,$$

where $V \in D(\mathcal{L}_1)$, $R \in \mathcal{E}$ and $\lambda \in \mathbb{C}$ (which will be precised below), that is

$$\begin{cases} V''(t) - 2\nu V'(t) + (\nu^2 - \lambda)V(t) = R(t), \quad t > 0 \\ V(0) = 0, \quad V(+\infty) = 0. \end{cases} \tag{7}$$

We set

$$\Sigma_\nu = \{z \in \mathbb{C} \setminus \mathbb{R}_- : \mathrm{Re}(\sqrt{z}) > \nu\}.$$

Now, let us precise this set. For all $z = x + iy \in \mathbb{C} \setminus \mathbb{R}_-$, we have

$$\mathrm{Re}(\sqrt{z}) > \nu \Longleftrightarrow \sqrt{\frac{|z| + \mathrm{Re}(z)}{2}} > \nu \Longleftrightarrow \sqrt{x^2 + y^2} > 2\nu^2 - x.$$

- First case : if $x > \nu^2$, we have $\sqrt{x^2 + y^2} + x \geqslant 2x > 2\nu^2$, then $\mathrm{Re}(\sqrt{z}) > \nu$.
- Second case : if $x \leqslant \nu^2$, then $y^2 + 4\nu^2 x - 4\nu^4 > 0$. Thus, we deduce that Σ_ν is strictly outside the parabola of equation

  $$y^2 + 4\nu^2 x - 4\nu^4 = 0,$$

  turned towards the negative real axis and passing through the points $(\nu^2, 0)$, $(0, 2\nu^2)$ and $(0, -2\nu^2)$.

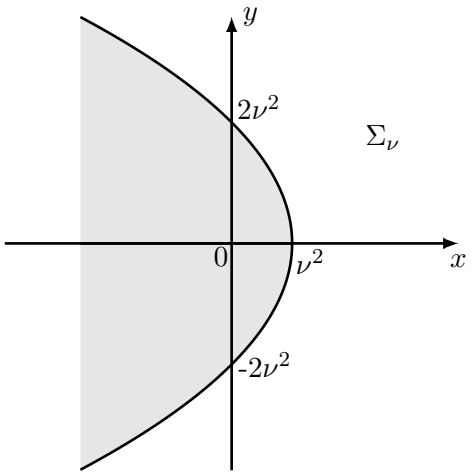

Figure 1: This figure represents Σ_ν.

Now, let $\varepsilon_{\mathcal{L}_1}$ be a small fixed positive number and consider the following set

$$\Sigma_{\mathcal{L}_1} := \left\{ \lambda \in \Sigma_\nu, \quad |\arg(\lambda)| \leqslant \pi - 2\varepsilon_{\mathcal{L}_1} \quad \text{and} \quad |\lambda| \geqslant \frac{4\nu^2}{\sin^2(\varepsilon_{\mathcal{L}_1})} \right\}. \tag{8}$$

We then obtain the following proposition.

Proposition 3.1. The linear operator $\mathcal{L}_1$ is closed and densely defined in $W^{2,p}(0,+\infty;X)$. Moreover, there exists a constant $M_{\mathcal{L}_1} > 0$ such that for all $\lambda \in \Sigma_{\mathcal{L}_1}$, operator $\mathcal{L}_1 - \lambda I$ is invertible with bounded inverse and

$$\left\|(\mathcal{L}_1 - \lambda I)^{-1}\right\|_{\mathcal{L}(\mathcal{E})} \leqslant \frac{M_{\mathcal{L}_1}}{|\lambda|}.$$

Therefore, assumption (H_1) in Section 2.2 is verified for $\mathcal{L}_1$ with

$$\theta_{\mathcal{L}_1} = 2\varepsilon_{\mathcal{L}_1}. \tag{9}$$

Proof. Let $\lambda \in \Sigma_\nu$. From [5], Theorem 2, p. 712, there exists a unique solution $V \in W^{2,p}(0,+\infty;X)$ of problem (7), given by

$$\begin{aligned} V(t) &= \frac{e^{t(\nu-\sqrt{\lambda})}}{2\sqrt{\lambda}} \int_0^{+\infty} e^{-s(\nu+\sqrt{\lambda})} R(s)\, ds \\ &\quad - \frac{1}{2\sqrt{\lambda}} \left(\int_0^t e^{(t-s)(\nu-\sqrt{\lambda})} R(s)\, ds + \int_t^{+\infty} e^{-(s-t)(\nu+\sqrt{\lambda})} R(s)\, ds \right), \end{aligned} \tag{10}$$

see formula (15) in [5] where $L_1 := -\nu I - \sqrt{\lambda} I$ and $L_2 := \nu I - \sqrt{\lambda} I$. It follows that $\Sigma_\nu \subset \rho(\mathcal{L}_1)$. This proves that $\mathcal{L}_1$ is closed. The boundary conditions are verified by using Lemma 8, p. 718 in [5].

Moreover, for all $\lambda \in \Sigma_\nu$, since $\mathrm{Re}(\sqrt{\lambda}) > \nu$, from (10), we obtain

$$\begin{aligned} \|V\|_{\mathcal{E}} &\leqslant \frac{1}{2\sqrt{|\lambda|}} \left(\int_0^{+\infty} e^{-tp(\mathrm{Re}(\sqrt{\lambda})-\nu)}\, dt \right)^{1/p} \int_0^{+\infty} e^{-s(\nu+\mathrm{Re}(\sqrt{\lambda}))} \|R(s)\|_X\, ds \\ &\quad + \sup_{t\in\mathbb{R}_+} \left(\int_0^t \left| e^{(t-s)(\nu-\sqrt{\lambda})} \right| ds + \int_t^{+\infty} \left| e^{-(s-t)(\nu+\sqrt{\lambda})} \right| ds \right) \frac{\|R\|_{\mathcal{E}}}{2\sqrt{|\lambda|}}, \end{aligned}$$

hence, noting q the conjugate exponent of p, we have

$$\begin{aligned} \|V\|_{\mathcal{E}} &\leqslant \left(\frac{1}{p(\mathrm{Re}(\sqrt{\lambda})-\nu)} \right)^{1/p} \left(\int_0^{+\infty} e^{-sq(\nu+\mathrm{Re}(\sqrt{\lambda}))}\, ds \right)^{1/q} \frac{\|R\|_{\mathcal{E}}}{2\sqrt{|\lambda|}} \\ &\quad + \sup_{t\in\mathbb{R}_+} \left(\frac{1 - e^{-t(\mathrm{Re}(\sqrt{\lambda})-\nu)}}{\mathrm{Re}(\sqrt{\lambda})-\nu} + \frac{1}{\mathrm{Re}(\sqrt{\lambda})+\nu} \right) \frac{\|R\|_{\mathcal{E}}}{2\sqrt{|\lambda|}} \\ &\leqslant \frac{1}{p^{1/p}(\mathrm{Re}(\sqrt{\lambda})-\nu)^{1/p}} \frac{1}{q^{1/q}(\mathrm{Re}(\sqrt{\lambda})+\nu)^{1/q}} \frac{\|R\|_{\mathcal{E}}}{2\sqrt{|\lambda|}} \\ &\quad + \frac{2}{\mathrm{Re}(\sqrt{\lambda})-\nu} \frac{\|R\|_{\mathcal{E}}}{2\sqrt{|\lambda|}} \\ &\leqslant \frac{2}{\mathrm{Re}(\sqrt{\lambda})-\nu} \frac{\|R\|_{\mathcal{E}}}{\sqrt{|\lambda|}}. \end{aligned}$$

Let $\lambda = |\lambda| e^{i \arg(\lambda)} \in \Sigma_{\mathcal{L}_1}$, then $|\arg(\lambda)| \leqslant \pi - 2\varepsilon_{\mathcal{L}_1}$. Thus

$$\begin{aligned} \mathrm{Re}(\sqrt{\lambda}) - \nu &\geqslant \sqrt{|\lambda|} \cos\left(\frac{\arg(\lambda)}{2}\right) - \nu \geqslant \sqrt{|\lambda|} \cos\left(\frac{\pi}{2} - \varepsilon_{\mathcal{L}_1}\right) - \nu \\ &\geqslant \sqrt{|\lambda|} \sin(\varepsilon_{\mathcal{L}_1}) - \frac{\sqrt{|\lambda|}}{2} \sin(\varepsilon_{\mathcal{L}_1}) \geqslant \frac{\sqrt{|\lambda|}}{2} \sin(\varepsilon_{\mathcal{L}_1}). \end{aligned}$$

Therefore, setting $M_{\mathcal{L}_1} = \dfrac{4}{\sin(\varepsilon_{\mathcal{L}_1})} > 0$ such that

$$\|V\|_{\mathcal{E}} \leqslant \frac{M_{\mathcal{L}_1}}{|\lambda|} \|R\|_{\mathcal{E}}. \tag{11}$$

□

3.2 Study of operator $\mathcal{L}_2$

In this section, in view to determine the spectral properties of $\mathcal{L}_2$, since $\mathcal{L}_2 = -\mathcal{A}$, we will study the spectral properties of $\mathcal{A}$.

We focus ourselves, for $\lambda \leqslant 0$, on the following spectral equation

$$\mathcal{A}\Psi - \lambda\Psi = F, \tag{12}$$

which writes

$$\begin{pmatrix} 0 & 1 \\ -\left(\dfrac{\partial^2}{\partial\theta^2}+1\right)^2 & -2\left(\dfrac{\partial^2}{\partial\theta^2}-1\right) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} - \lambda \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix},$$

with $F_1 \in W_0^{2,p}(0,\omega)$ and $F_2 \in L^p(0,\omega)$.

We have to find the unique couple $(\psi_1,\psi_2) \in \left(W^{4,p}(0,\omega) \cap W_0^{2,p}(0,\omega)\right) \times W_0^{2,p}(0,\omega)$, which satisfies the following system

$$\begin{cases} \psi_2 - \lambda\psi_1 & = & F_1 \\ -\dfrac{\partial^4}{\partial\theta^4}\psi_1 - 2\dfrac{\partial^2}{\partial\theta^2}\psi_1 - \psi_1 - 2\dfrac{\partial^2}{\partial\theta^2}\psi_2 + 2\psi_2 - \lambda\psi_2 & = & F_2. \end{cases}$$

Thus, we first have to solve

$$\begin{cases} -\dfrac{\partial^4}{\partial\theta^4}\psi_1 - 2\dfrac{\partial^2}{\partial\theta^2}\psi_1 - \psi_1 - 2\dfrac{\partial^2}{\partial\theta^2}(\lambda\psi_1 + F_1) + (2-\lambda)(\lambda\psi_1 + F_1) = F_2 \\ \psi_1 \in W^{4,p}(0,\omega) \cap W_0^{2,p}(0,\omega), \end{cases}$$

that is

$$\begin{cases} -\psi_1^{(4)} - 2(1+\lambda)\psi_1'' - (\lambda-1)^2\psi_1 = F_2 + 2F_1'' + (\lambda-2)F_1 \\ \psi_1(0) = \psi_1(\omega) = \psi_1'(0) = \psi_1'(\omega) = 0. \end{cases}$$

Set $G_\lambda = -F_2 - 2(F_1'' - F_1) - \lambda F_1$, it follows that the previous system writes

$$\begin{cases} \psi_1^{(4)} + 2(\lambda+1)\psi_1'' + (\lambda-1)^2\psi_1 = G_\lambda \\ \psi_1(0) = \psi_1(\omega) = \psi_1'(0) = \psi_1'(\omega) = 0. \end{cases} \tag{13}$$

Then, the characteristic equation

$$\chi^4 + 2\,(1+\lambda)\,\chi^2 + (\lambda-1)^2 = 0,$$

admits, for $\lambda < 0$, the following four distinct solutions

$$\begin{cases} \alpha_1 = \sqrt{-\lambda} + i, \quad \alpha_3 = -\alpha_1 \\ \alpha_2 = \sqrt{-\lambda} - i, \quad \alpha_4 = -\alpha_2, \end{cases} \tag{14}$$

and for $\lambda = 0$, two double solutions that are i and $-i$.

We have to distinguish the two cases : $\lambda = 0$ and $\lambda < 0$.

3.2.1 Case $\lambda = 0$: Invertibility of $\mathcal{A}$

Proposition 3.2. $\mathcal{A}$ is invertible with bounded inverse *i.e.* $0 \in \rho(\mathcal{A})$ and there exists $\varepsilon_0 > 0$, such that $\overline{B(0,\varepsilon_0)} \subset \rho(\mathcal{A})$.

Proof. Here $\lambda = 0$. We have to solve (12). This is equivalent to solve (13) which writes as

$$\begin{cases} \psi_1^{(4)} + 2\psi_1'' + \psi_1 = G_0 \\ \psi_1(0) = \psi_1(\omega) = \psi_1'(0) = \psi_1'(\omega) = 0. \end{cases} \tag{15}$$

From [22], Theorem 2.8, statement 2., there exists a unique classical solution of problem (15). Thus, using the closed graph theorem, there exists $C_1 > 0$ such that

$$\|\psi_1''\|_{L^p(0,\omega)} \leqslant C_1 \|G_1\|_{L^p(0,\omega)} \leqslant C_1 \left(\|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{W^{2,p}(0,\omega)} \right) \leqslant 2C_1 \|F\|_X,$$

and from the Poincaré inequality, there exists $C_\omega > 0$ such that

$$\|\psi_1\|_{W_0^{2,p}(0,\omega)} \leqslant C_\omega \|\psi_1''\|_{L^p(0,\omega)} \leqslant 2C_1 C_\omega \|F\|_X.$$

Finally, since $\psi_2 = F_1$, then we have

$$\|\psi_2\|_{L^p(0,\omega)} = \|F_1\|_{L^p(0,\omega)} \leqslant \|F_2\|_{L^p(0,\omega)} + \|F_1\|_{W^{2,p}(0,\omega)} = \|F\|_X.$$

Therefore, there exists a unique solution Ψ of $\mathcal{A}\Psi = F$ with $\|\Psi\|_X \leqslant (1 + 2C_1C_\omega)\|F\|_X$, from which we deduce that there exists $\varepsilon_0 > 0$, such that $\overline{B(0,\varepsilon_0)} \subset \rho(\mathcal{A})$. □

3.2.2 Case $\lambda < 0$: Spectral study of $\mathcal{A}$

In order to prove Proposition 3.9, we first have to state the following technical results.

Lemma 3.3. Let $\alpha \in \mathbb{C} \setminus \{0\}$, $a, b \in \mathbb{R}$ with $a < b$ and $f \in W_0^{2,p}(a,b)$. For all $x \in [a,b]$, we set

$$K(x) = \int_a^x e^{-(x-s)\alpha} f(s)\, ds + \int_x^b e^{-(s-x)\alpha} f(s)\, ds.$$

Then, we have

$$K(x) = \frac{2}{\alpha} f(x) + \frac{1}{\alpha^2} \int_a^x e^{-(x-s)\alpha} f''(s)\, ds + \frac{1}{\alpha^2} \int_x^b e^{-(s-x)\alpha} f''(s)\, ds.$$

Proof. The result is easily obtained by two integrations by parts. □

Proposition 3.4. For any $\lambda < 0$ problem (13) has a unique solution given by

$$\begin{aligned} \psi_1(\theta) \;:=\; & e^{-\theta\alpha_2}(\beta_1 + \beta_2 + \beta_3 + \beta_4) + e^{-(\omega-\theta)\alpha_2}(\beta_3 + \beta_4 - \beta_1 - \beta_2) + S(\theta) \\ & + \left(e^{-\theta\alpha_1} - e^{-\theta\alpha_2}\right)(\beta_2 + \beta_4) + \left(e^{-(\omega-\theta)\alpha_1} - e^{-(\omega-\theta)\alpha_2}\right)(\beta_4 - \beta_2), \end{aligned} \tag{16}$$

where

$$\begin{cases} \beta_1 &= \dfrac{1}{4i} U_-^{-1} \dfrac{1 - e^{-\omega\alpha_1}}{1 - e^{-\omega\alpha_2}} \left(J(0) - J(\omega)\right) \\ \beta_2 &= -\dfrac{1}{4i} U_-^{-1} \left(J(0) - J(\omega)\right) \\ \beta_3 &= -\dfrac{1}{4i} U_+^{-1} \dfrac{1 + e^{-\omega\alpha_1}}{1 + e^{-\omega\alpha_2}} \left(J(0) + J(\omega)\right) \\ \beta_4 &= \dfrac{1}{4i} U_+^{-1} \left(J(0) + J(\omega)\right), \end{cases} \tag{17}$$

with

$$\begin{cases} U_- &:= 1 - e^{-2\omega\sqrt{-\lambda}} - 2\omega\sqrt{-\lambda}\, e^{-\omega\sqrt{-\lambda}} \\ U_+ &:= 1 - e^{-2\omega\sqrt{-\lambda}} + 2\omega\sqrt{-\lambda}\, e^{-\omega\sqrt{-\lambda}}, \end{cases} \tag{18}$$

and S is a particular solution of the equation of problem (13) which is given, for all $\theta \in [0,\omega]$, by

$$\begin{aligned} S(\theta) &:= \frac{e^{-\theta\alpha_2}}{2\alpha_2\left(1-e^{-2\omega\alpha_2}\right)}\left(J(0)-e^{-\omega\alpha_2}J(\omega)\right)-\frac{\lambda}{\alpha_1^2\alpha_2^2}F_1(\theta) \\ &+\frac{e^{-(\omega-\theta)\alpha_2}}{2\alpha_2\left(1-e^{-2\omega\alpha_2}\right)}\left(J(\omega)-e^{-\omega\alpha_2}J(0)\right)-\frac{1}{2\alpha_2}J(\theta), \end{aligned} \tag{19}$$

with

$$J(\theta) := \int_0^\theta e^{-(\theta-s)\alpha_2}v(s)\,ds+\int_\theta^\omega e^{-(s-\theta)\alpha_2}v(s)\,ds, \tag{20}$$

where

$$\begin{aligned} v(\theta) &:= \frac{e^{-\theta\alpha_1}}{2\alpha_1\left(1-e^{-2\omega\alpha_1}\right)}\left(I(0)-e^{-\omega\alpha_1}I(\omega)\right)+\frac{\lambda}{\alpha_1^2\alpha_2^2}F_1''(\theta) \\ &+\frac{e^{-(\omega-\theta)\alpha_1}}{2\alpha_1\left(1-e^{-2\omega\alpha_1}\right)}\left(I(\omega)-e^{-\omega\alpha_1}I(0)\right)-\frac{1}{2\alpha_1}I(\theta), \end{aligned} \tag{21}$$

and

$$\begin{aligned} I(\theta) &= \int_0^\theta e^{-(\theta-s)\alpha_1}\left(-F_2-2(F_1''-F_1)+\frac{\lambda}{\alpha_1^2}F_1''\right)(s)\,ds \\ &+\int_\theta^\omega e^{-(s-\theta)\alpha_1}\left(-F_2-2(F_1''-F_1)+\frac{\lambda}{\alpha_1^2}F_1''\right)(s)\,ds. \end{aligned} \tag{22}$$

Proof. In order to apply results obtained in [12] and [13], we set $L_- = -\alpha_1 I$, $M = -\alpha_2 I$, $r_- = \alpha_1^2-\alpha_2^2$, $a=0$ and $b=\omega$. Then, problem (13) reads as

$$\begin{cases} \psi_1^{(4)}-(L_-^2+M^2)\psi_1''+L_-^2M^2\psi_1 = G_\lambda \\ \psi_1(0)=\psi_1(\omega)=\psi_1'(0)=\psi_1'(\omega)=0. \end{cases}$$

From [13], there exists a unique solution whose representation formula is explicitly given in [12] by (14)-(15)-(16). This representation formula shows that ψ_1 writes as in (16), with $\beta_1,\beta_2,\beta_3,\beta_4$ given by (17) and

$$\begin{aligned} S(\theta) &= \frac{1}{2\alpha_2}e^{-\theta\alpha_2}Z\int_0^\omega e^{-s\alpha_2}v_0(s)\,ds+\frac{1}{2\alpha_2}e^{-(\omega-\theta)\alpha_2}Z\int_0^\omega e^{-(\omega-s)\alpha_2}v_0(s)\,ds \\ &-\frac{1}{2\alpha_2}\int_0^\theta e^{-(\theta-s)\alpha_2}v_0(s)\,ds-\frac{1}{2\alpha_2}\int_\theta^\omega e^{-(s-\theta)\alpha_2}v_0(s)\,ds \\ &-\frac{1}{2\alpha_2}e^{-\theta\alpha_2}Ze^{-\omega\alpha_2}\int_0^\omega e^{-(\omega-s)\alpha_2}v_0(s)\,ds \\ &-\frac{1}{2\alpha_2}e^{-(\omega-\theta)\alpha_2}Ze^{-\omega\alpha_2}\int_0^\omega e^{-s\alpha_2}v_0(s)\,ds, \quad \theta\in[0,\omega], \end{aligned} \tag{23}$$

where

$$\begin{aligned} v_0(\theta) &:= \frac{1}{2\alpha_1}e^{-\theta\alpha_1}W\int_0^\omega e^{-s\alpha_1}G_\lambda(s)\,ds+\frac{1}{2\alpha_1}e^{-(\omega-\theta)\alpha_1}W\int_0^\omega e^{-(\omega-s)\alpha_1}G_\lambda(s)\,ds \\ &-\frac{1}{2\alpha_1}e^{-\theta\alpha_1}We^{-\omega\alpha_1}\int_0^\omega e^{-(\omega-s)\alpha_1}G_\lambda(s)\,ds \\ &-\frac{1}{2\alpha_1}e^{-(\omega-\theta)\alpha_1}We^{-\omega\alpha_1}\int_0^\omega e^{-s\alpha_1}G_\lambda(s)\,ds \\ &-\frac{1}{2\alpha_1}I_1(\theta), \quad \theta\in[0,\omega], \end{aligned} \tag{24}$$

with $Z := \left(1-e^{-2\omega\alpha_2}\right)^{-1}$, $W := \left(1-e^{-2\omega\alpha_1}\right)^{-1}$ and

$$I_1(\theta) = \int_0^\theta e^{-(\theta-s)\alpha_1}G_\lambda(s)\,ds+\int_\theta^\omega e^{-(s-\theta)\alpha_1}G_\lambda(s)\,ds. \tag{25}$$

Then, since $G_\lambda = -F_2 - 2(F_1'' - F_1) - \lambda F_1$, we have

$$\begin{aligned} I_1(\theta) &= \int_0^\theta e^{-(\theta-s)\alpha_1} \left(-F_2 - 2(F_1'' - F_1)\right)(s)\, ds \\ &\quad + \int_\theta^\omega e^{-(s-\theta)\alpha_1} \left(-F_2 - 2(F_1'' - F_1)\right)(s)\, ds \\ &\quad - \lambda \int_0^\theta e^{-(\theta-s)\alpha_1} F_1(s)\, ds - \lambda \int_\theta^\omega e^{-(s-\theta)\alpha_1} F_1(s)\, ds. \end{aligned}$$

Then, from Lemma 3.3, since $F_1 \in W_0^{2,p}(0,\omega)$, it follows

$$\begin{aligned} I_1(\theta) &= \int_0^\theta e^{-(\theta-s)\alpha_1} \left(-F_2 - 2(F_1'' - F_1)\right)(s)\, ds \\ &\quad + \int_\theta^\omega e^{-(s-\theta)\alpha_1} \left(-F_2 - 2(F_1'' - F_1)\right)(s)\, ds \\ &\quad - \frac{2\lambda}{\alpha_1} F_1(\theta) + \frac{\lambda}{\alpha_1^2} \left(\int_0^\theta e^{-(\theta-s)\alpha_1} F_1''(s)\, ds + \int_\theta^\omega e^{-(s-\theta)\alpha_1} F_1''(s)\, ds \right) \\ &= -\frac{2\lambda}{\alpha_1} F_1(\theta) + \int_0^\theta e^{-(\theta-s)\alpha_1} \left(-F_2 - 2(F_1'' - F_1) + \frac{\lambda}{\alpha_1^2} F_1'' \right)(s)\, ds \\ &\quad + \int_\theta^\omega e^{-(s-\theta)\alpha_1} \left(-F_2 - 2(F_1'' - F_1) + \frac{\lambda}{\alpha_1^2} F_1'' \right)(s)\, ds. \end{aligned}$$

Hence I, given by (22), satisfies

$$I(\theta) = I_1(\theta) + \frac{2\lambda}{\alpha_1} F_1(\theta). \tag{26}$$

Note that, from (25) and (26), we have

$$\int_0^\omega e^{-s\alpha_1} G_\lambda(s)\, ds = I_1(0) = I(0) \quad \text{and} \quad \int_0^\omega e^{-(\omega-s)\alpha_1} G_\lambda(s)\, ds = I_1(\omega) = I(\omega).$$

Therefore, from (24), for all $\theta \in [0,\omega]$, we deduce that

$$\begin{aligned} v_0(\theta) &= \frac{1}{2\alpha_1} e^{-\theta\alpha_1} W \left(I(0) - e^{-\omega\alpha_1} I(\omega)\right) \\ &\quad + \frac{1}{2\alpha_1} e^{-(\omega-\theta)\alpha_1} W \left(I(\omega) - e^{-\omega\alpha_1} I(0)\right) - \frac{1}{2\alpha_1} I_1(\theta) \\ &= \frac{1}{2\alpha_1} e^{-\theta\alpha_1} W \left(I(0) - e^{-\omega\alpha_1} I(\omega)\right) + \frac{\lambda}{\alpha_1^2} F_1(\theta) \\ &\quad + \frac{1}{2\alpha_1} e^{-(\omega-\theta)\alpha_1} W \left(I(\omega) - e^{-\omega\alpha_1} I(0)\right) - \frac{1}{2\alpha_1} I(\theta). \end{aligned}$$

Moreover, setting

$$v_1 = v_0 - \frac{\lambda}{\alpha_1^2} F_1. \tag{27}$$

Thus, for all $\theta \in [0,\omega]$, noting

$$J_1(\theta) = \int_0^\theta e^{-(\theta-s)\alpha_2} v_0(s)\, ds + \int_\theta^\omega e^{-(s-\theta)\alpha_2} v_0(s)\, ds, \tag{28}$$

from (27) and Lemma 3.3, since $F_1 \in W_0^{2,p}(0,\omega)$, we obtain

$$\begin{aligned} J_1(\theta) &= \int_0^\theta e^{-(\theta-s)\alpha_2} v_1(s)\, ds + \int_\theta^\omega e^{-(s-\theta)\alpha_2} v_1(s)\, ds \\ &\quad + \frac{\lambda}{\alpha_1^2}\left(\int_0^\theta e^{-(\theta-s)\alpha_2} F_1(s)\, ds + \int_\theta^\omega e^{-(s-\theta)\alpha_2} F_1(s)\, ds \right), \end{aligned}$$

hence

$$\begin{aligned} J_1(\theta) &= \int_0^\theta e^{-(\theta-s)\alpha_2} v_1(s)\, ds + \int_\theta^\omega e^{-(s-\theta)\alpha_2} v_1(s)\, ds + \frac{2\lambda}{\alpha_1^2\alpha_2} F_1(\theta) \\ &\quad + \frac{\lambda}{\alpha_1^2\alpha_2^2}\left(\int_0^\theta e^{-(\theta-s)\alpha_2} F_1''(s)\, ds + \int_\theta^\omega e^{-(s-\theta)\alpha_2} F_1''(s)\, ds \right) \\ &= \int_0^\theta e^{-(\theta-s)\alpha_2} \left(v_1(s) + \frac{\lambda}{\alpha_1^2\alpha_2^2} F_1''(s) \right) ds \\ &\quad + \int_\theta^\omega e^{-(s-\theta)\alpha_2} \left(v_1(s) + \frac{\lambda}{\alpha_1^2\alpha_2^2} F_1''(s) \right) ds + \frac{2\lambda}{\alpha_1^2\alpha_2} F_1(\theta). \end{aligned}$$

From (27), for all $\theta \in [0,\omega]$, we deduce that v given by (21) and J given by (20), satisfy

$$v(\theta) = v_1(\theta) + \frac{\lambda}{\alpha_1^2\alpha_2^2} F_1''(\theta),$$

and

$$J(\theta) = J_1(\theta) - \frac{2\lambda}{\alpha_1^2\alpha_2} F_1(\theta). \tag{29}$$

Note that, from (28) and (29), one has

$$\int_0^\omega e^{-s\alpha_2} v_0(s)\, ds = J_1(0) = J(0) \quad \text{and} \quad \int_0^\omega e^{-(\omega-s)\alpha_2} v_0(s)\, ds = J_1(\omega) = J(\omega).$$

Finally, from (20), (23) and (29), for all $\theta \in [0,\omega]$, we deduce that

$$\begin{aligned} S(\theta) &= \frac{1}{2\alpha_2} e^{-\theta\alpha_2} Z \left(J(0) - e^{-\omega\alpha_2} J(\omega) \right) \\ &\quad + \frac{1}{2\alpha_2} e^{-(\omega-\theta)\alpha_2} Z \left(J(\omega) - e^{-\omega\alpha_2} J(0) \right) - \frac{1}{2\alpha_2} J_1(\theta), \end{aligned}$$

which leads to (19).

The constants β_i, $i = 1,2,3,4$, are given by (15) and (16) in [12]. U_-, V_- and $F_-'(a) \pm F_-'(\gamma)$ in [12] are replaced here by U_-, U_+ and $S'(0) \pm S'(\omega)$. So, in order to compute constants β_i, we now make explicit U_-, U_+ and $S'(0) \pm S'(\omega)$.

$$\begin{aligned} U_- &= 1 - e^{-\omega(\alpha_1+\alpha_2)} - (\alpha_1^2 - \alpha_2^2)^{-1}(\alpha_1+\alpha_2)^2 \left(e^{-\omega\alpha_2} - e^{-\omega\alpha_1} \right) \\ &= 1 - e^{-2\omega\sqrt{-\lambda}} + i\sqrt{-\lambda}\left(e^{-\omega(\sqrt{-\lambda}-i)} - e^{-\omega(\sqrt{-\lambda}+i)} \right) \\ &= 1 - e^{-2\omega\sqrt{-\lambda}} + 2i\sqrt{-\lambda}\, e^{-\omega\sqrt{-\lambda}} \left(\frac{e^{\omega i} - e^{-\omega i}}{2} \right) \\ &= 1 - e^{-2\omega\sqrt{-\lambda}} - 2\sqrt{-\lambda}\, e^{-\omega\sqrt{-\lambda}} \sin(\omega), \end{aligned}$$

and

$$\begin{aligned} U_+ &= 1 - e^{-\omega(\alpha_1+\alpha_2)} + (\alpha_1^2 - \alpha_2^2)^{-1}(\alpha_1+\alpha_2)^2 \left(e^{-\omega\alpha_2} - e^{-\omega\alpha_1} \right) \\ &= 1 - e^{-2\omega\sqrt{-\lambda}} - i\sqrt{-\lambda}\left(e^{-\omega(\sqrt{-\lambda}-i)} - e^{-\omega(\sqrt{-\lambda}+i)} \right) \\ &= 1 - e^{-2\omega\sqrt{-\lambda}} + 2\sqrt{-\lambda}\, e^{-\omega\sqrt{-\lambda}} \sin(\omega). \end{aligned}$$

From (19), it follows that

$$\begin{aligned} S'(\theta) &= -\frac{1}{2}\,e^{-\theta\alpha_2} Z\left(J(0)-e^{-\omega\alpha_2}J(\omega)\right) - \frac{\lambda}{\alpha_1^2\alpha_2^2}\,F_1'(\theta) \\ &\quad +\frac{1}{2}\,e^{-(\omega-\theta)\alpha_2} Z\left(J(\omega)-e^{-\omega\alpha_2}J(0)\right) - \frac{1}{2\alpha_2}\,J'(\theta) \\ &= -\frac{1}{2}\,e^{-\theta\alpha_2} Z\left(J(0)-e^{-\omega\alpha_2}J(\omega)\right) + \frac{1}{2}\,e^{-(\omega-\theta)\alpha_2} Z\left(J(\omega)-e^{-\omega\alpha_2}J(0)\right) \\ &\quad +\frac{1}{2}\left(\int_0^\theta e^{-(\theta-s)\alpha_2}v(s)\,ds - \int_\theta^\omega e^{-(s-\theta)\alpha_2}v(s)\,ds\right) - \frac{\lambda}{\alpha_1^2\alpha_2^2}\,F_1'(\theta), \end{aligned}$$

from which we deduce that

$$S'(0)+S'(\omega) = -\frac{J(0)-J(\omega)}{(1-e^{-\omega\alpha_2})} \quad \text{and} \quad S'(0)-S'(\omega) = -\frac{J(0)+J(\omega)}{(1+e^{-\omega\alpha_2})}.$$

This prove that constants β_i, $i=1,2,3,4$, are given by (17). □

Remark 3.5. Since $0<\sin(\omega)<\omega$, for all $\omega>0$, then we have

$$U_- = 1-e^{-2\omega\sqrt{-\lambda}} - 2\sqrt{-\lambda}\,e^{-\omega\sqrt{-\lambda}}\sin(\omega) \geqslant 1-e^{-2\omega\sqrt{-\lambda}} - 2\omega\sqrt{-\lambda}\,e^{-\omega\sqrt{-\lambda}},$$

and

$$U_+ = 1-e^{-2\omega\sqrt{-\lambda}} + 2\sqrt{-\lambda}\,e^{-\omega\sqrt{-\lambda}}\sin(\omega) \geqslant 1-e^{-2\omega\sqrt{-\lambda}} - 2\omega\sqrt{-\lambda}\,e^{-\omega\sqrt{-\lambda}}.$$

Let $x>0$. Setting

$$f(x) = 1-e^{-2x}-2xe^{-x},$$

we have

$$f'(x) = 2e^{-2x}-2e^{-x}+2xe^{-x} = 2e^{-x}\left(e^{-x}+x-1\right) > 0.$$

It follows that $f(x)>f(0)=0$. Finally, for all $\omega, \sqrt{-\lambda}>0$, we deduce that

$$U_- \geqslant 1-e^{-2\omega\sqrt{-\lambda}} - 2\omega\sqrt{-\lambda}e^{-\omega\sqrt{-\lambda}} = f(\omega\sqrt{-\lambda}) > 0, \tag{30}$$

and

$$U_+ \geqslant 1-e^{-2\omega\sqrt{-\lambda}} - 2\omega\sqrt{-\lambda}e^{-\omega\sqrt{-\lambda}} = f(\omega\sqrt{-\lambda}) > 0. \tag{31}$$

Lemma 3.6. Let $F_1 \in W_0^{2,p}(0,\omega)$ and $F_2 \in L^p(0,\omega)$. Consider ε_0 defined in Proposition 3.2, then for all $\lambda \leqslant -\varepsilon_0$, J, v and I, given by (20), (21) and (22), satisfy the following estimates

1. $\|I\|_{L^p(0,\omega)} \leqslant \dfrac{2}{\sqrt{-\lambda}}\left(\|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)}\right)$.

2. $|I(0)|+|I(\omega)| \leqslant \dfrac{2}{\sqrt{-\lambda}^{1-1/p}}\left(\|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)}\right)$.

3. $\|v\|_{L^p(0,\omega)} \leqslant \dfrac{M_1}{-\lambda}\left(\|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)}\right)$,
   where $M_1 = 2+\frac{2}{1-e^{-2\omega\sqrt{\varepsilon_0}}}$.

4. $\|J\|_{L^p(0,\omega)} \leqslant \dfrac{2}{\sqrt{-\lambda}}\|v\|_{L^p(0,\omega)}$.

5. $|J(0)|+|J(\omega)| \leqslant \dfrac{2}{\sqrt{-\lambda}^{1-1/p}}\|v\|_{L^p(0,\omega)}$.

Proof.

1. From (14) and (22), we obtain

$$\begin{aligned}
\|I\|_{L^p(0,\omega)} &\leqslant \sup_{\theta\in[0,\omega]}\left(\int_0^\theta e^{-(\theta-s)\sqrt{-\lambda}}\,ds+\int_\theta^\omega e^{-(s-\theta)\sqrt{-\lambda}}\,ds\right)\|2F_1-F_2\|_{L^p(0,\omega)}\\
&\quad+\sup_{\theta\in[0,\omega]}\left(\int_0^\theta e^{-(\theta-s)\sqrt{-\lambda}}\,ds+\int_\theta^\omega e^{-(s-\theta)\sqrt{-\lambda}}\,ds\right)\left\|\left(\frac{\lambda}{\alpha_1^2}-2\right)F_1''\right\|_{L^p(0,\omega)}\\
&\leqslant \sup_{\theta\in[0,\omega]}\left(\frac{1-e^{-\theta\sqrt{-\lambda}}}{\sqrt{-\lambda}}+\frac{1-e^{-(\omega-\theta)\sqrt{-\lambda}}}{\sqrt{-\lambda}}\right)\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}\right)\\
&\quad+3\sup_{\theta\in[0,\omega]}\left(\frac{1-e^{-\theta\sqrt{-\lambda}}}{\sqrt{-\lambda}}+\frac{1-e^{-(\omega-\theta)\sqrt{-\lambda}}}{\sqrt{-\lambda}}\right)\|F_1''\|_{L^p(0,\omega)}\\
&\leqslant \frac{2}{\sqrt{-\lambda}}\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right).
\end{aligned}$$

2. Due to (14) and (22), from the Hölder inequality, it follows

$$\begin{aligned}
|I(0)|+|I(\omega)| &\leqslant \int_0^\omega e^{-s\sqrt{-\lambda}}\left|-F_2(s)-2(F_1''(s)-F_1(s))+\frac{\lambda}{\alpha_1^2}F_1''(s)\right|ds\\
&\quad+\int_0^\omega e^{-(\omega-s)\sqrt{-\lambda}}\left|-F_2(s)-2(F_1''(s)-F_1(s))+\frac{\lambda}{\alpha_1^2}F_1''(s)\right|ds\\
&\leqslant \left(\int_0^\omega e^{-q(\omega-s)\sqrt{-\lambda}}\,ds\right)^{1/q}\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right)\\
&\quad+\left(\int_0^\omega e^{-sq\sqrt{-\lambda}}\,ds\right)^{1/q}\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right)\\
&\leqslant \frac{2\left(1-e^{-\omega q\sqrt{-\lambda}}\right)^{1/q}}{q^{1/q}\sqrt{-\lambda}^{1/q}}\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right)\\
&\leqslant \frac{2}{\sqrt{-\lambda}^{1-1/p}}\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right).
\end{aligned}$$

3. From (14) and (21), since $|\alpha_1|=|\alpha_2|=\sqrt{1-\lambda}>\sqrt{-\lambda}$, we have

$$\begin{aligned}
\|v\|_{L^p(0,\omega)} &\leqslant \frac{|I(0)|+|I(\omega)|}{2\sqrt{1-\lambda}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}\right)}\left(\int_0^\omega e^{-p\theta\sqrt{-\lambda}}\,d\theta\right)^{1/p}+\frac{-\lambda}{(1-\lambda)^2}\|F_1''\|_{L^p(0,\omega)}\\
&\quad+\frac{|I(0)|+|I(\omega)|}{2\sqrt{1-\lambda}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}\right)}\left(\int_0^\omega e^{-p(\omega-\theta)\sqrt{-\lambda}}\,d\theta\right)^{1/p}+\frac{1}{2\sqrt{1-\lambda}}\|I\|_{L^p(0,\omega)}\\
&\leqslant \frac{(|I(0)|+|I(\omega)|)}{\sqrt{1-\lambda}\,\sqrt{-\lambda}^{1/p}p^{1/p}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}\right)}+\frac{1}{2\sqrt{1-\lambda}}\|I\|_{L^p(0,\omega)}\\
&\quad+\frac{1}{1-\lambda}\|F_1''\|_{L^p(0,\omega)},
\end{aligned}$$

hence

$$\begin{aligned}\|v\|_{L^p(0,\omega)} &\leqslant \frac{2\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right)}{\sqrt{1-\lambda}\,\sqrt{-\lambda}^{1/p+1/q}\,q^{1/q}\,p^{1/p}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}\right)}\\ &\quad+\frac{\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}}{\sqrt{1-\lambda}\,\sqrt{-\lambda}}+\frac{1}{1-\lambda}\,\|F_1''\|_{L^p(0,\omega)}\\ &\leqslant \frac{M_1}{\sqrt{1-\lambda}\,\sqrt{-\lambda}}\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right).\end{aligned}$$

4. From (14) and (20), we have

$$\begin{aligned}\|J\|_{L^p(0,\omega)} &\leqslant \sup_{\theta\in[0,\omega]}\left(\int_0^\theta e^{-(\theta-s)\sqrt{-\lambda}}\,ds+\int_\theta^\omega e^{-(s-\theta)\sqrt{-\lambda}}\,ds\right)\|v\|_{L^p(0,\omega)}\\ &\leqslant \sup_{\theta\in[0,\omega]}\left(\frac{1-e^{-\theta\sqrt{-\lambda}}}{\sqrt{-\lambda}}+\frac{1-e^{-(\omega-\theta)\sqrt{-\lambda}}}{\sqrt{-\lambda}}\right)\|v\|_{L^p(0,\omega)}\\ &\leqslant \frac{2}{\sqrt{-\lambda}}\,\|v\|_{L^p(0,\omega)}.\end{aligned}$$

5. Due to (14) and (20), from the Hölder inequality, we deduce that

$$\begin{aligned}|J(0)|+|J(\omega)| &\leqslant \int_0^\omega e^{-s\sqrt{-\lambda}}|v(s)|\,ds+\int_0^\omega e^{-(\omega-s)\sqrt{-\lambda}}|v(s)|\,ds\\ &\leqslant \left(\left(\int_0^\omega e^{-sq\sqrt{-\lambda}}\,ds\right)^{1/q}+\left(\int_0^\omega e^{-(\omega-s)q\sqrt{-\lambda}}\,ds\right)^{1/q}\right)\|v\|_{L^p(0,\omega)}\\ &\leqslant \frac{2\left(1-e^{-\omega q\sqrt{-\lambda}}\right)^{1/q}}{q^{1/q}\sqrt{-\lambda}^{1/q}}\,\|v\|_{L^p(0,\omega)}\\ &\leqslant \frac{2}{\sqrt{-\lambda}^{1-1/p}}\,\|v\|_{L^p(0,\omega)}.\end{aligned}$$

□

Lemma 3.7. Let $\lambda<0$. Then, we have

$$\left(\int_0^\omega\left|e^{-\theta\alpha_1}-e^{-\theta\alpha_2}\right|^p\,d\theta\right)^{1/p}\leqslant\frac{4}{\sqrt{-\lambda}^{1+1/p}},$$

and

$$\left(\int_0^\omega\left|e^{-(\omega-\theta)\alpha_1}-e^{-(\omega-\theta)\alpha_2}\right|^p\,d\theta\right)^{1/p}\leqslant\frac{4}{\sqrt{-\lambda}^{1+1/p}}.$$

Proof. For $x\geqslant 0$, we have $e^{-\frac{px}{2}}x^p<1$, so

$$\int_0^{+\infty}e^{-px}x^p\,dx=\int_0^{+\infty}e^{-\frac{px}{2}}e^{-\frac{px}{2}}x^p\,dx\leqslant\int_0^{+\infty}e^{-\frac{px}{2}}\,dx=\frac{2}{p}.$$

Then, from (14), we have

$$\int_0^\omega\left|e^{-\theta\alpha_1}-e^{-\theta\alpha_2}\right|^p\,d\theta=\int_0^\omega\left|e^{-\theta\sqrt{-\lambda}}\left(e^{-\theta i}-e^{\theta i}\right)\right|^p\,d\theta=2^p\int_0^\omega e^{-p\theta\sqrt{-\lambda}}|\sin(\theta)|^p\,d\theta,$$

hence, setting $x = \theta\sqrt{-\lambda}$, it follows that

$$\begin{aligned}\int_0^\omega \left|e^{-\theta\alpha_1} - e^{-\theta\alpha_2}\right|^p \, d\theta &= 2^p \int_0^{\omega\sqrt{-\lambda}} e^{-px} \left|\sin\left(\frac{x}{\sqrt{-\lambda}}\right)\right|^p \frac{dx}{\sqrt{-\lambda}} \\ &\leqslant \frac{2^p}{\sqrt{-\lambda}} \int_0^{\omega\sqrt{-\lambda}} e^{-px} \left(\frac{x}{\sqrt{-\lambda}}\right)^p \, dx \\ &\leqslant \frac{2^p}{\sqrt{-\lambda}^{p+1}} \int_0^{+\infty} e^{-px} x^p \, dx \\ &\leqslant \frac{2^{p+1}}{p\sqrt{-\lambda}^{p+1}} \leqslant \frac{2^{2p}}{\sqrt{-\lambda}^{p+1}}.\end{aligned}$$

The second estimate is obtained by change of variable, taking $\omega - \theta$ instead of θ. □

Lemma 3.8. Let $F_1 \in W_0^{2,p}(0,\omega)$ and $F_2 \in L^p(0,\omega)$. Consider ε_0 defined in Proposition 3.2, then for all $\lambda \leqslant -\varepsilon_0$, β_1, β_2, β_3 and β_4, defined by (17), satisfy

$$\max\left(|\beta_1+\beta_2|, |\beta_3+\beta_4|\right) \leqslant \frac{M_1\left(\|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)}\right)}{\omega(-\lambda)\sqrt{-\lambda}^{2-1/p}\left(1 - e^{-2\omega\sqrt{\varepsilon_0}} - 2\omega\sqrt{\varepsilon_0}\, e^{-\omega\sqrt{\varepsilon_0}}\right)\left(1 - e^{-\omega\sqrt{\varepsilon_0}}\right)},$$

and

$$\max\left(|\beta_2|, |\beta_4|\right) \leqslant \frac{M_1\left(\|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)}\right)}{2(-\lambda)\sqrt{-\lambda}^{1-1/p}\left(1 - e^{-2\omega\sqrt{\varepsilon_0}} - 2\omega\sqrt{\varepsilon_0}\, e^{-\omega\sqrt{\varepsilon_0}}\right)},$$

where $M_1 = 2 + \frac{2}{1-e^{-2\omega\sqrt{\varepsilon_0}}}$.

Proof. Recall that β_i, $i = 1,2,3,4$, depends on U_-^{-1} and U_+^{-1}. From (30) and (31), it follows

$$U_- \geqslant f(\omega\sqrt{-\lambda}) \geqslant f(\omega\sqrt{\varepsilon_0}) = 1 - e^{-2\omega\sqrt{\varepsilon_0}} - 2\omega\sqrt{\varepsilon_0}\, e^{-\omega\sqrt{\varepsilon_0}} > 0,$$

and

$$U_+ \geqslant f(\omega\sqrt{-\lambda}) \geqslant f(\omega\sqrt{\varepsilon_0}) = 1 - e^{-2\omega\sqrt{\varepsilon_0}} - 2\omega\sqrt{\varepsilon_0}\, e^{-\omega\sqrt{\varepsilon_0}} > 0.$$

Thus, we deduce

$$U_-^{-1} \leqslant \frac{1}{1 - e^{-2\omega\sqrt{\varepsilon_0}} - 2\omega\sqrt{\varepsilon_0}\, e^{-\omega\sqrt{\varepsilon_0}}} \quad \text{and} \quad U_+^{-1} \leqslant \frac{1}{1 - e^{-2\omega\sqrt{\varepsilon_0}} - 2\omega\sqrt{\varepsilon_0}\, e^{-\omega\sqrt{\varepsilon_0}}}.$$

Therefore, from (17) and Lemma 3.6, we have

$$\begin{aligned}|\beta_1+\beta_2| &\leqslant \frac{|J(0)| + |J(\omega)|}{4\left(1 - e^{-2\omega\sqrt{\varepsilon_0}} - 2\omega\sqrt{\varepsilon_0}\, e^{-\omega\sqrt{\varepsilon_0}}\right)} \left|\frac{1 - e^{-\omega\alpha_1}}{1 - e^{-\omega\alpha_2}} - 1\right| \\ &\leqslant \frac{\|v\|_{L^p(0,\omega)}}{2\sqrt{-\lambda}^{1-1/p}\left(1 - e^{-2\omega\sqrt{\varepsilon_0}} - 2\omega\sqrt{\varepsilon_0}\, e^{-\omega\sqrt{\varepsilon_0}}\right)} \left|\frac{e^{-\omega\alpha_2} - e^{-\omega\alpha_1}}{1 - e^{-\omega\alpha_2}}\right| \\ &\leqslant \frac{M_1\left(\|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)}\right)}{2(-\lambda)\sqrt{-\lambda}^{1-1/p}\left(1 - e^{-2\omega\sqrt{\varepsilon_0}} - 2\omega\sqrt{\varepsilon_0}\, e^{-\omega\sqrt{\varepsilon_0}}\right)} \frac{2e^{-\omega\sqrt{-\lambda}}}{1 - e^{-\omega\sqrt{\varepsilon_0}}} \\ &\leqslant \frac{M_1\left(\|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)}\right)}{\omega(-\lambda)\sqrt{-\lambda}^{2-1/p}\left(1 - e^{-2\omega\sqrt{\varepsilon_0}} - 2\omega\sqrt{\varepsilon_0}\, e^{-\omega\sqrt{\varepsilon_0}}\right)\left(1 - e^{-\omega\sqrt{\varepsilon_0}}\right)}\end{aligned}$$

and similarly

$$\begin{aligned}|\beta_3+\beta_4| &\leqslant \frac{|J(0)|+|J(\omega)|}{4\left(1-e^{-2\omega\sqrt{\varepsilon_0}}-2\omega\sqrt{\varepsilon_0}\,e^{-\omega\sqrt{\varepsilon_0}}\right)}\left|1-\frac{1+e^{-\omega\alpha_1}}{1+e^{-\omega\alpha_2}}\right| \\ &\leqslant \frac{M_1\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right)}{\omega(-\lambda)\sqrt{-\lambda}^{2-1/p}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}-2\omega\sqrt{\varepsilon_0}\,e^{-\omega\sqrt{\varepsilon_0}}\right)\left(1-e^{-\omega\sqrt{\varepsilon_0}}\right)}.\end{aligned}$$

In the same way, we obtain

$$|\beta_2|\leqslant \frac{M_1\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right)}{2(-\lambda)\sqrt{-\lambda}^{1-1/p}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}-2\omega\sqrt{\varepsilon_0}\,e^{-\omega\sqrt{\varepsilon_0}}\right)},$$

and

$$|\beta_4|\leqslant \frac{M_1\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right)}{2(-\lambda)\sqrt{-\lambda}^{1-1/p}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}-2\omega\sqrt{\varepsilon_0}\,e^{-\omega\sqrt{\varepsilon_0}}\right)}.$$

□

The following proposition will allow us to use the well-defined operator $-\sqrt{\mathcal{A}}$ which, generates a uniformly bounded analytic semigroup $\left(e^{-s\sqrt{\mathcal{A}}}\right)_{s\geqslant 0}$.

Proposition 3.9. $\mathcal{A}$ is closed and densely defined in X. Moreover, there exists a constant $M>0$ such that for all $\lambda\leqslant 0$, operator $\mathcal{A}-\lambda I$ is invertible with bounded inverse and

$$\left\|(\mathcal{A}-\lambda I)^{-1}\right\|_{\mathcal{L}(X)}\leqslant\frac{M}{1+|\lambda|}.$$

Proof. It is clear that $\mathcal{D}(0,\omega)\times\mathcal{D}(0,\omega)\subset D(\mathcal{A})\subset X=W_0^{2,p}(0,\omega)\times L^p(0,\omega)$, where $\mathcal{D}(0,\omega)$ is the set of C^∞-functions with compact support in $(0,\omega)$. Since $\mathcal{D}(0,\omega)$ is dense in each spaces $W_0^{2,p}(0,\omega)$ and $L^p(0,\omega)$ for their respective norms, then $D(\mathcal{A})$ is dense.

Let $F_1\in W_0^{2,p}(0,\omega)$ and $F_2\in L^p(0,\omega)$. From Proposition 3.2, $0\in\rho(\mathcal{A})$. From Proposition 3.4, for all $\lambda<0$, there exist a unique couple

$$(\psi_1,\psi_2)\in\left(W^{4,p}(0,\omega)\cap W_0^{2,p}(0,\omega)\right)\times W_0^{2,p}(0,\omega)$$

which satisfies

$$\left\{\begin{array}{lcl}\psi_2 & = & \lambda\psi_1+F_1\\ \psi_1^{(4)}+2\lambda\psi_1''+\lambda^2\psi_1 & = & G_\lambda,\end{array}\right. \tag{32}$$

where $G_\lambda=-F_2-2(F_1''-F_1)-\lambda F_1$. Recall that $\Psi=(\mathcal{A}-\lambda I)^{-1}F$ reads as (32), then $\mathbb{R}_-\subset\rho(\mathcal{A})$, thus $\mathcal{A}$ is closed.

Moreover, ψ_1 is given by (16)-(17)-(19) and ψ_2 is given by

$$\begin{aligned}\psi_2(\theta) &:= \lambda e^{-\theta\alpha_2}(\beta_1+\beta_2+\beta_3+\beta_4)+\lambda e^{-(\omega-\theta)\alpha_2}(\beta_3+\beta_4-\beta_1-\beta_2)\\ &\quad+\lambda\left(e^{-\theta\alpha_1}-e^{-\theta\alpha_2}\right)(\beta_2+\beta_4)+\lambda\left(e^{-(\omega-\theta)\alpha_1}-e^{-(\omega-\theta)\alpha_2}\right)(\beta_4-\beta_2)\\ &\quad+\lambda S(\theta)+F_1(\theta),\end{aligned} \tag{33}$$

where β_i, $i=1,2,3,4$ are given by (17)-(18). From (19), $\lambda S(\theta)+F_1(\theta)$ is given by

$$\begin{aligned}\lambda S(\theta)+F_1(\theta) &= \frac{\lambda}{2\alpha_2\left(1-e^{-2\omega\alpha_2}\right)}e^{-\theta\alpha_2}\left(J(0)-e^{-\omega\alpha_2}J(\omega)\right)\\ &\quad+\frac{\lambda}{2\alpha_2\left(1-e^{-2\omega\alpha_2}\right)}e^{-(\omega-\theta)\alpha_2}\left(J(\omega)-e^{-\omega\alpha_2}J(0)\right)\\ &\quad-\frac{\lambda^2}{\alpha_1^2\alpha_2^2}F_1(\theta)+F_1(\theta)-\frac{\lambda}{2\alpha_2}J(\theta),\end{aligned} \tag{34}$$

with $J(\theta)$ given by (20). Our aim is to prove that, for all $\lambda \leqslant 0$, there exists $M > 0$, such that

$$\|(\mathcal{A} - \lambda I)^{-1} F\|_{\mathcal{L}(X)} \leqslant \frac{M}{1 + |\lambda|} \|F\|_X,$$

where

$$\|F\|_X = \left\| \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \right\|_X = \|F_1\|_{W_0^{2,p}(0,\omega)} + \|F_2\|_{L^p(0,\omega)}. \tag{35}$$

To this end, we consider that $\lambda \in (-\infty, -\varepsilon_0)$, where ε_0 is defined in Proposition 3.2. We first study ψ_1''. From (19), for *a.e.* $\theta \in [0, \omega]$, we have

$$\begin{aligned} S''(\theta) &= \frac{\alpha_2\, e^{-\theta\alpha_2}}{2\,(1 - e^{-2\omega\alpha_2})} \left(J(0) - e^{-\omega\alpha_2} J(\omega)\right) - \frac{\lambda}{\alpha_1^2 \alpha_2^2} F_1''(\theta) \\ &\quad + \frac{\alpha_2\, e^{-(\omega-\theta)\alpha_2}}{2\,(1 - e^{-2\omega\alpha_2})} \left(J(\omega) - e^{-\omega\alpha_2} J(0)\right) - \frac{1}{2\alpha_2} J''(\theta), \end{aligned}$$

and from (20), we obtain $J''(\theta) = \alpha_2^2 J(\theta) - 2\alpha_2 v(\theta)$, hence

$$\begin{aligned} S''(\theta) &= \frac{\alpha_2\, e^{-\theta\alpha_2}}{2\,(1 - e^{-2\omega\alpha_2})} \left(J(0) - e^{-\omega\alpha_2} J(\omega)\right) - \frac{\lambda}{\alpha_1^2 \alpha_2^2} F_1''(\theta) \\ &\quad + \frac{\alpha_2\, e^{-(\omega-\theta)\alpha_2}}{2\,(1 - e^{-2\omega\alpha_2})} \left(J(\omega) - e^{-\omega\alpha_2} J(0)\right) - \frac{\alpha_2}{2} J(\theta) + v(\theta), \end{aligned}$$

Then, since $\alpha_1 = \sqrt{-\lambda} + i$ and $\alpha_2 = \sqrt{-\lambda} - i$, we have $|e^{-\omega\alpha_1}| = |e^{-\omega\alpha_2}| = e^{-\omega\sqrt{-\lambda}} \leqslant 1$ with $-\lambda \geqslant \varepsilon_0$, thus

$$\begin{aligned} \|S''\|_{L^p(0,\omega)} &\leqslant \frac{\sqrt{1-\lambda}\,(|J(0)| + |J(\omega)|)}{2\left(1 - e^{-2\omega\sqrt{\varepsilon_0}}\right)} \left(\int_0^\omega e^{-p\theta\sqrt{-\lambda}}\, d\theta\right)^{1/p} \\ &\quad + \frac{\sqrt{1-\lambda}\,(|J(0)| + |J(\omega)|)}{2\left(1 - e^{-2\omega\sqrt{\varepsilon_0}}\right)} \left(\int_0^\omega e^{-p(\omega-\theta)\sqrt{-\lambda}}\, d\theta\right)^{1/p} \\ &\quad + \frac{-\lambda}{(1-\lambda)^2} \|F_1''\|_{L^p(0,\omega)} + \frac{\sqrt{1-\lambda}}{2} \|J\|_{L^p(0,\omega)} + \|v\|_{L^p(0,\omega)} \\ &\leqslant \frac{\sqrt{1-\lambda}\,(|J(0)| + |J(\omega)|)}{\sqrt{-\lambda}^{1/p}\left(1 - e^{-2\omega\sqrt{\varepsilon_0}}\right)} + \frac{1}{1-\lambda} \|F_1''\|_{L^p(0,\omega)} \\ &\quad + \frac{\sqrt{1-\lambda}}{2} \|J\|_{L^p(0,\omega)} + \|v\|_{L^p(0,\omega)}, \end{aligned}$$

From Lemma 3.6, we have

$$\begin{aligned} \|S''\|_{L^p(0,\omega)} &\leqslant \frac{2\sqrt{1-\lambda}}{\sqrt{-\lambda}^{1-1/p+1/p}\left(1 - e^{-2\omega\sqrt{\varepsilon_0}}\right)} \|v\|_{L^p(0,\omega)} \\ &\quad + \frac{\sqrt{1-\lambda}}{\sqrt{-\lambda}} \|v\|_{L^p(0,\omega)} + \|v\|_{L^p(0,\omega)} + \frac{1}{1-\lambda} \|F_1''\|_{L^p(0,\omega)} \\ &\leqslant \frac{2M_1}{-\lambda\left(1 - e^{-2\omega\sqrt{\varepsilon_0}}\right)} \left(\|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)}\right) \\ &\quad + \frac{2M_1}{-\lambda} \left(\|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)}\right) + \frac{1}{1-\lambda} \|F_1''\|_{L^p(0,\omega)}. \end{aligned}$$

Finally, we obtain

$$\|S''\|_{L^p(0,\omega)} \leqslant \frac{M_2}{-\lambda}\left(\|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)}\right), \tag{36}$$

where $M_2 = \frac{2M_1}{1-e^{-2\omega\sqrt{\varepsilon_0}}} + 2M_1 + 1$.

Now, we consider $\psi_1'' - S''$ which reads as

$$\begin{aligned}\psi_1''(\theta) - S''(\theta) &= \alpha_2^2\, e^{-\theta\alpha_2}(\beta_1+\beta_2+\beta_3+\beta_4) + \alpha_2^2\, e^{-(\omega-\theta)\alpha_2}(\beta_3+\beta_4-\beta_1-\beta_2)\\ &\quad +\alpha_2^2\left(e^{-\theta\alpha_1} - e^{-\theta\alpha_2}\right)(\beta_2+\beta_4)\\ &\quad +\alpha_2^2\left(e^{-(\omega-\theta)\alpha_1} - e^{-(\omega-\theta)\alpha_2}\right)(\beta_4-\beta_2),\end{aligned}$$

where β_i, $i = 1,2,3,4$ are given by (17) and (18). Then, we have

$$\begin{aligned}\|\psi_1'' - S''\|_{L^p(0,\omega)} &\leqslant (1-\lambda)(|\beta_1+\beta_2| + |\beta_3+\beta_4|)\left(\int_0^\omega e^{-p\theta\sqrt{-\lambda}}\, d\theta\right)^{1/p}\\ &\quad +(1-\lambda)(|\beta_1+\beta_2| + |\beta_3+\beta_4|)\left(\int_0^\omega e^{-p(\omega-\theta)\sqrt{-\lambda}}\, d\theta\right)^{1/p}\\ &\quad +(1-\lambda)(|\beta_2| + |\beta_4|)\left(\int_0^\omega \left|e^{-\theta\alpha_1} - e^{-\theta\alpha_2}\right|^p\, d\theta\right)^{1/p}\\ &\quad +(1-\lambda)(|\beta_2| + |\beta_4|)\left(\int_0^\omega \left|e^{-(\omega-\theta)\alpha_1} - e^{-(\omega-\theta)\alpha_2}\right|^p\, d\theta\right)^{1/p}.\end{aligned}$$

Since $-\lambda \geqslant \varepsilon_0 > 0$, we have

$$\frac{1-\lambda}{-\lambda} = 1 + \frac{1}{-\lambda} \leqslant 1 + \frac{1}{\varepsilon_0},$$

and thus, from Lemma 3.7 and Lemma 3.8, we obtain

$$\begin{aligned}\|\psi_1'' - S''\|_{L^p(0,\omega)} &\leqslant \frac{2(1-\lambda)\left(|\beta_1+\beta_2| + |\beta_3+\beta_4|\right)}{\sqrt{-\lambda}^{1/p}} + \frac{8(1-\lambda)(|\beta_2| + |\beta_4|)}{\sqrt{-\lambda}^{1+1/p}}\\ &\leqslant \frac{M_3\left(\|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)}\right)}{-\lambda},\end{aligned}$$

where $M_3 = \frac{4M_1\left(1+\frac{1}{\omega}\right)\left(1+\frac{1}{\varepsilon_0}\right)}{\left(1-e^{-2\omega\sqrt{\varepsilon_0}}-2\omega\sqrt{\varepsilon_0}\, e^{-\omega\sqrt{\varepsilon_0}}\right)\left(1-e^{-\omega\sqrt{\varepsilon_0}}\right)}$.

From (36), it follows that

$$\begin{aligned}\|\psi_1''\|_{L^p(0,\omega)} &\leqslant \|\psi_1'' - S''\|_{L^p(0,\omega)} + \|S''\|_{L^p(0,\omega)}\\ &\leqslant \frac{M_3+M_2}{-\lambda}\left(\|F_2\|_{L^p(0,\omega)} + 2\|F_1\|_{L^p(0,\omega)} + 3\|F_1''\|_{L^p(0,\omega)}\right)\\ &\leqslant \frac{3(M_2+M_3)}{-\lambda}\|F\|_X.\end{aligned}$$

From the Poincaré inequality, there exists $C_\omega > 0$ such that

$$\|\psi_1\|_{W_0^{2,p}(0,\omega)} \leqslant C_\omega\|\psi_1''\|_{L^p(0,\omega)} \leqslant \frac{3C_\omega(M_2+M_3)}{-\lambda}\|F\|_X. \tag{37}$$

Now, we focus ourselves on $\|\psi_2\|_{L^p(0,\omega)}$. As previously, by (33), using Lemma 3.7 and Lemma 3.8, we obtain

$$\begin{aligned}
\|\psi_2\|_{L^p(0,\omega)} &\leqslant |\lambda|(|\beta_1+\beta_2|+|\beta_3+\beta_4|)\left(\int_0^\omega e^{-p\theta\sqrt{-\lambda}}\,d\theta\right)^{1/p}\\
&\quad+|\lambda|(|\beta_1+\beta_2|+|\beta_3+\beta_4|)\left(\int_0^\omega e^{-p(\omega-\theta)\sqrt{-\lambda}}\,d\theta\right)^{1/p}\\
&\quad+|\lambda|(|\beta_2|+|\beta_4|)\left(\int_0^\omega \left|e^{-\theta\alpha_1}-e^{-\theta\alpha_2}\right|^p\,d\theta\right)^{1/p}\\
&\quad+|\lambda|(|\beta_4|+|\beta_2|)\left(\int_0^\omega \left|e^{-(\omega-\theta)\alpha_1}-e^{-(\omega-\theta)\alpha_2}\right|^p\,d\theta\right)^{1/p}\\
&\quad+\|\lambda S+F_1\|_{L^p(0,\omega)}\\
&\leqslant \frac{M_2+M_3}{-\lambda}\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right)\\
&\quad+\|\lambda S+F_1\|_{L^p(0,\omega)}.
\end{aligned}$$

Moreover, from (34) and Lemma 3.6, we deduce

$$\begin{aligned}
\|\lambda S+F_1\|_{L^p(0,\omega)} &\leqslant \frac{-\lambda\left(|J(0)|+|J(\omega)|\right)}{2\sqrt{1-\lambda}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}\right)}\left(\int_0^\omega e^{-p\theta\sqrt{-\lambda}}\,d\theta\right)^{1/p}\\
&\quad+\frac{-\lambda\left(|J(0)|+|J(\omega)|\right)}{2\sqrt{1-\lambda}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}\right)}\left(\int_0^\omega e^{-p(\omega-\theta)\sqrt{-\lambda}}\,d\theta\right)^{1/p}\\
&\quad+\left|1-\frac{\lambda^2}{\alpha_1^2\alpha_2^2}\right|\|F_1\|_{L^p(0,\omega)}+\frac{-\lambda}{2\sqrt{1-\lambda}}\|J\|_{L^p(0,\omega)}\\
&\leqslant \frac{-\lambda\left(|J(0)|+|J(\omega)|\right)}{\sqrt{-\lambda}^{1+1/p}\left(1-e^{-2\omega\sqrt{\varepsilon_0}}\right)}+\left|1-\frac{\lambda^2}{(1-\lambda)^2}\right|\|F_1\|_{L^p(0,\omega)}+\|v\|_{L^p(0,\omega)}\\
&\leqslant \frac{-2\lambda}{\sqrt{-\lambda}^2\left(1-e^{-2\omega\sqrt{\varepsilon_0}}\right)}\|v\|_{L^p(0,\omega)}+\frac{1-2\lambda}{(1-\lambda)^2}\|F_1\|_{L^p(0,\omega)}+\|v\|_{L^p(0,\omega)}\\
&\leqslant \left(\frac{2}{1-e^{-2\omega\sqrt{\varepsilon_0}}}+1\right)\|v\|_{L^p(0,\omega)}+\left(\frac{1}{\lambda^2}+\frac{1}{-\lambda}\right)\|F_1\|_{L^p(0,\omega)}\\
&\leqslant \left(\frac{2}{1-e^{-2\omega\sqrt{\varepsilon_0}}}+1\right)\|v\|_{L^p(0,\omega)}+\frac{\frac{1}{\varepsilon_0}+1}{-\lambda}\|F_1\|_{L^p(0,\omega)}.
\end{aligned}$$

Then, from Lemma 3.6, we obtain

$$\|\lambda S+F_1\|_{L^p(0,\omega)}\leqslant\frac{M_4}{-\lambda}\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right),$$

where $M_4=\left(\frac{2}{1-e^{-2\omega\sqrt{\varepsilon_0}}}+1\right)M_1+\frac{1}{\varepsilon_0}+1$.

Thus, it follows that

$$\begin{aligned}
\|\psi_2\|_{L^p(0,\omega)} &\leqslant \frac{M_2+M_3+M_4}{-\lambda}\left(\|F_2\|_{L^p(0,\omega)}+2\|F_1\|_{L^p(0,\omega)}+3\|F_1''\|_{L^p(0,\omega)}\right)\\
&\leqslant \frac{3\left(M_2+M_3+M_4\right)}{-\lambda}\|F\|_X.
\end{aligned}$$

Finally, from (37), we have

$$\|(\mathcal{A}-\lambda I)^{-1}F\|_X = \|\psi_1\|_{W_0^{2,p}(0,\omega)} + \|\psi_2\|_{L^p(0,\omega)} \leqslant \frac{M}{|\lambda|}\|F\|_X,$$

where $M = 3\left((C_\omega+1)(M_2+M_3)+M_4\right)$, which gives the result since $0 \in \rho(\mathcal{A})$ from Proposition 3.2. □

Since $-\mathcal{A}$ is the realization of $\mathcal{L}_2$, we deduce the following corollary.

Corollary 3.10. There exist $\varepsilon_{\mathcal{L}_2} \in (0,\pi)$ small enough and $M_{\mathcal{L}_2} > 0$ such that

$$\forall\, z \in \Sigma_{\mathcal{L}_2} := \overline{B(0,\varepsilon_0)} \cup \{z \in \mathbb{C}\setminus\{0\} : |\arg(z)| \leqslant \varepsilon_{\mathcal{L}_2}\},$$

we have

$$\left\|(\mathcal{L}_2 - zI)^{-1}\right\|_{\mathcal{L}(X)} \leqslant \frac{M_{\mathcal{L}_2}}{1+|z|}.$$

Therefore, assumption (H_1) in Section 2.2 is verified for $\mathcal{L}_2$ with

$$\theta_{\mathcal{L}_2} = \pi - \varepsilon_{\mathcal{L}_2}. \tag{38}$$

Remark 3.11. $\mathcal{A}$ is anti-compact; since $\sigma(-\mathcal{L}_2) = \sigma(\mathcal{A})$ then $\sigma(-\mathcal{L}_2)$ is uniquely composed by isolated eigenvalues $(\lambda_j)_{j\geqslant 1}$ such that $|\lambda_j| \to +\infty$, see [10], Theorem 6.29, p. 187. More precisely, the calculus of the resolvent operator $(\mathcal{A}-\lambda I)^{-1}$ requires that, for all $\lambda \in \mathbb{C}\setminus\mathbb{R}_+$, U_- and U_+ defined by (18) do not vanish. Since $U_-U_+ = 0$ is equivalent to

$$\left(\sinh(\omega\sqrt{-\lambda}) - \omega\sqrt{-\lambda}\right)\left(\sinh(\omega\sqrt{-\lambda}) + \omega\sqrt{-\lambda}\right) = 0,$$

then, using $(z_j)_{j\geqslant 1}$ defined in Section 1, we deduce that

$$\forall\, j \geqslant 1, \quad \lambda_j = -\frac{z_j^2}{\omega^2} \in \mathbb{C}\setminus\mathbb{R}_+.$$

Now, we prove that operator $\mathcal{A}$ has Bounded Imaginary Powers, see Definition 2.4.

Proposition 3.12. $\mathcal{A} \in \mathrm{BIP}\,(X,\theta_{\mathcal{A}})$, for any $\theta_{\mathcal{A}} \in (0,\pi)$.

Proof. This proof follow the same steps than those in the proofs of Proposition 3.1 in [16] or Proposition 4.1 in [17]. From Proposition 3.9, we deduce that $\mathcal{A}$ is a sectorial operator and from Proposition 3.2, we have $0 \in \rho(\mathcal{A})$. Moreover, since $X = W_0^{2,p}(0,\omega)\times L^p(0,\omega)$ is a reflexive space, from [9], Proposition 2.1.1, h), statement i) of Definition 2.4 holds with $T_2 = \mathcal{A}$.

For all $\lambda > 0$, we have

$$\mathcal{A}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} + \lambda\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}F_1\\ F_2\end{pmatrix},$$

which writes

$$\begin{cases} \psi_2 = F_1 - \lambda\psi_1 \\ \psi_1^{(4)} + 2(1-\lambda)\psi_1'' - (\lambda+1)^2\psi_1 = G_\lambda \\ \psi_1(0) = \psi_1(\omega) = \psi_1'(0) = \psi_1'(\omega) = 0, \end{cases} \tag{39}$$

where

$$G_\lambda = -F_2 - 2(F_1'' - F_1) + \lambda F_1.$$

From Proposition 3.4, the explicit expression of the solution to problem (39) is

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = (\mathcal{A}+\lambda I)^{-1}\begin{pmatrix}F_1\\ F_2\end{pmatrix}.$$

Let $\varepsilon > 0$ and $r \in \mathbb{R}$. We have

$$\left[(\mathcal{A}+I)^{-\varepsilon+ir}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}\right](\theta) = \frac{1}{\Gamma_{\varepsilon,r}}\int_0^{+\infty}\lambda^{-\varepsilon+ir}\left[(\mathcal{A}+I+\lambda I)^{-1}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}\right](\theta)d\lambda,$$

where $\Gamma_{\varepsilon,r} = \Gamma(1-\varepsilon+ir)\Gamma(\varepsilon-ir)$, see for instance [23], (6), p. 100.

Now, we only focus on the first component ψ_1 since calculations are similar for ψ_2. Moreover, from [21], we only have to study the convolution term, which is the most singular term. In our case, this term is given by

$$\begin{aligned} I_S(\theta) &= \frac{1}{4\alpha_1\alpha_2}\int_0^{\omega} e^{-|\theta-s|\alpha_2}\int_0^{\omega} e^{-|s-t|\alpha_1}G_{\lambda+1}(t)\ dtds \\ &= \frac{1}{4\alpha_1\alpha_2}\int_0^{\omega} e^{-|\theta-s|\alpha_2}\int_0^{\omega} e^{-|s-t|\alpha_1}\left(-F_2-2(F_1''-F_1)+(\lambda+1)F_1\right)(t)\ dtds. \end{aligned}$$

Note that, in order to simplify calculations, we have used the convolutions terms given in (23) and (24). Moreover, since $F_1(0) = F_1''(0) = F_2(0) = F_1(\omega) = F_1''(\omega) = F_2(\omega) = 0$, we wet

$$\widetilde{G_0}(x) = \begin{cases} -F_2(x) - 2F_1''(x) + 2F_1(x), & \text{if } x \in [0,\omega] \\ 0, & \text{else,} \end{cases} \quad \text{with} \quad \widetilde{F_1}(x) = \begin{cases} F_1(x), & \text{if } x \in [0,\omega] \\ 0, & \text{else,} \end{cases}$$

and

$$E_\alpha(x) = e^{-|x|\alpha}.$$

It follows that

$$\begin{aligned} I_{\varepsilon,r}(\theta) &= \frac{1}{\Gamma_{\varepsilon,r}}\int_0^{+\infty}\lambda^{-\varepsilon+ir}I_S(\theta)\ d\lambda \\ &= \frac{1}{\Gamma_{\varepsilon,r}}\int_0^{+\infty}\frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2}\left(E_{\alpha_2}\star\left(E_{\alpha_1}\star\left(\widetilde{G_0}+(\lambda+1)\widetilde{F_1}\right)\right)\right)(\theta)\ d\lambda \\ &= \frac{1}{\Gamma_{\varepsilon,r}}\int_0^{+\infty}\frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2}\mathcal{F}^{-1}\left(\mathcal{F}\left(E_{\alpha_2}\star\left(E_{\alpha_1}\star\left(\widetilde{G_0}+(\lambda+1)\widetilde{F_1}\right)\right)\right)(\xi)\right)(\theta)\ d\lambda \\ &= \mathcal{F}^{-1}\left(\frac{1}{\Gamma_{\varepsilon,r}}\int_0^{+\infty}\frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2}\mathcal{F}\left(E_{\alpha_2}\right)(\xi)\mathcal{F}\left(E_{\alpha_1}\right)(\xi)\mathcal{F}\left(\widetilde{G_0}+(\lambda+1)\widetilde{F_1}\right)(\xi)\ d\lambda\right)(\theta) \\ &= \mathcal{F}^{-1}\left(\frac{1}{\Gamma_{\varepsilon,r}}\int_0^{+\infty}\frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2}\mathcal{F}\left(E_{\alpha_2}\right)(\xi)\mathcal{F}\left(E_{\alpha_1}\right)(\xi)\ d\lambda\,\mathcal{F}\left(\widetilde{G_0}\right)(\xi)\right)(\theta) \\ &\quad +\mathcal{F}^{-1}\left(\frac{1}{\Gamma_{\varepsilon,r}}\int_0^{+\infty}\frac{\lambda^{-\varepsilon+ir}(\lambda+1)}{4\alpha_1\alpha_2}\mathcal{F}\left(E_{\alpha_2}\right)(\xi)\mathcal{F}\left(E_{\alpha_1}\right)(\xi)\ d\lambda\,\mathcal{F}\left(\widetilde{F_1}\right)(\xi)\right)(\theta). \end{aligned}$$

We recall that

$$\mathcal{F}\left(E_\alpha\right)(\xi) = \frac{2\alpha}{\alpha^2+4\pi^2\xi^2},$$

and here $\alpha_1 = \sqrt{\lambda+1}+i$ and $\alpha_2 = \sqrt{\lambda+1}-i$. Hence

$$\begin{aligned} \frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2}\mathcal{F}\left(E_{\alpha_2}\right)(\xi)\mathcal{F}\left(E_{\alpha_1}\right)(\xi) &= \frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2}\frac{4\alpha_1\alpha_2}{(\alpha_1^2+4\pi^2\xi^2)(\alpha_2^2+4\pi^2\xi^2)} \\ &= \frac{\lambda^{-\varepsilon+ir}}{\alpha_1^2\alpha_2^2+4\pi^2\xi^2(\alpha_1^2+\alpha_2^2)+16\pi^4\xi^4} \\ &= \frac{\lambda^{-\varepsilon+ir}}{\lambda^2+4(1+2\pi^2\xi^2)\lambda+4(1+4\pi^4\xi^4)} \\ &= \frac{\lambda^{-\varepsilon+ir}}{(\lambda+\lambda_1)(\lambda+\lambda_2)}, \end{aligned}$$

where

$$\begin{cases} \lambda_1 = 2 + 4\pi\xi + 4\pi^2\xi^2 = 4\pi^2 \left(\xi - \dfrac{(1+i)}{2\pi}\right)\left(\xi - \dfrac{(1-i)}{2\pi}\right) \\ \lambda_2 = 2 - 4\pi\xi + 4\pi^2\xi^2 = 4\pi^2 \left(\xi + \dfrac{(1+i)}{2\pi}\right)\left(\xi + \dfrac{(1-i)}{2\pi}\right). \end{cases}$$

Thus, since we have

$$\frac{1}{(\lambda+\lambda_1)(\lambda+\lambda_2)} = \frac{1}{\lambda_1-\lambda_2}\left(-\frac{1}{\lambda+\lambda_1} + \frac{1}{\lambda+\lambda_2}\right),$$

it follows that

$$\frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2}\mathcal{F}\left(E_{\alpha_2}\right)(\xi)\mathcal{F}\left(E_{\alpha_1}\right)(\xi) = \frac{1}{8\pi\xi}\left(-\frac{\lambda^{-\varepsilon+ir}}{\lambda+\lambda_1} + \frac{\lambda^{-\varepsilon+ir}}{\lambda+\lambda_2}\right).$$

Then, setting

$$\sigma_1 = \frac{\lambda}{\lambda_1} \quad \text{and} \quad \sigma_2 = \frac{\lambda}{\lambda_2},$$

we obtain

$$\begin{aligned} \int_0^{+\infty} \frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2}\mathcal{F}\left(E_{\alpha_2}\right)(\xi)\mathcal{F}\left(E_{\alpha_1}\right)(\xi)\, d\lambda &= \frac{1}{8\pi\xi}\left(-\int_0^{+\infty}\frac{\lambda^{-\varepsilon+ir}}{\lambda+\lambda_1}\, d\lambda + \int_0^{+\infty}\frac{\lambda^{-\varepsilon+ir}}{\lambda+\lambda_2}\, d\lambda\right) \\ &= -\frac{\lambda_1^{-\varepsilon+ir}}{8\pi\xi}\int_0^{+\infty}\frac{\sigma_1^{-\varepsilon+ir}}{\sigma_1+1}\, d\sigma_1 \\ &\quad + \frac{\lambda_2^{-\varepsilon+ir}}{8\pi\xi}\int_0^{+\infty}\frac{\sigma_2^{-\varepsilon+ir}}{\sigma_2+1}\, d\sigma_2. \end{aligned}$$

Moreover, for all $z \in \mathbb{C}\setminus\mathbb{N}^-$, where $\mathbb{N}^-$ is the set of negative integer, we have

$$\int_0^{+\infty}\frac{\sigma^{-z}}{\sigma+1}\, d\sigma = \Gamma(z)\Gamma(1-z), \tag{40}$$

thus, it follows that

$$\int_0^{+\infty}\frac{\sigma^{-\varepsilon+ir}}{\sigma+1}\, d\sigma = \Gamma(\varepsilon-ir)\Gamma(1-\varepsilon+ir) = \Gamma_{\varepsilon,r}, \tag{41}$$

hence

$$\begin{aligned} \frac{1}{\Gamma_{\varepsilon,r}}\int_0^{+\infty}\frac{\lambda^{-\varepsilon+ir}}{4\alpha_1\alpha_2}\mathcal{F}\left(E_{\alpha_2}\right)(\xi)\mathcal{F}\left(E_{\alpha_1}\right)(\xi)\, d\lambda &= \frac{1}{8\pi\xi\,\Gamma_{\varepsilon,r}}\left(\lambda_2^{-\varepsilon+ir} - \lambda_1^{-\varepsilon+ir}\right)\Gamma_{\varepsilon,r} \\ &= \frac{1}{8\pi\xi}\left(\lambda_2^{-\varepsilon+ir} - \lambda_1^{-\varepsilon+ir}\right). \end{aligned}$$

In the same way, we have

$$\begin{aligned} \frac{\lambda^{-\varepsilon+ir}(\lambda+1)}{4\alpha_1\alpha_2}\mathcal{F}\left(E_{\alpha_2}\right)(\xi)\mathcal{F}\left(E_{\alpha_1}\right)(\xi) &= \frac{1}{\lambda_1-\lambda_2}\left(-\frac{(\lambda+1)\lambda^{-\varepsilon+ir}}{\lambda+\lambda_1} + \frac{(\lambda+1)\lambda^{-\varepsilon+ir}}{\lambda+\lambda_2}\right) \\ &= \frac{1}{8\pi\xi}\left(-\frac{\lambda^{1-\varepsilon+ir}}{\lambda+\lambda_1} + \frac{\lambda^{1-\varepsilon+ir}}{\lambda+\lambda_2}\right) \\ &\quad + \frac{1}{8\pi\xi}\left(-\frac{\lambda^{-\varepsilon+ir}}{\lambda+\lambda_1} + \frac{\lambda^{-\varepsilon+ir}}{\lambda+\lambda_2}\right). \end{aligned}$$

Then, setting

$$\sigma_1 = \frac{\lambda}{\lambda_1} \quad \text{and} \quad \sigma_2 = \frac{\lambda}{\lambda_2},$$

we obtain

$$\begin{aligned}
\Upsilon &= \int_0^{+\infty} \frac{\lambda^{-\varepsilon+ir}(\lambda+1)}{4\alpha_1\alpha_2} \mathcal{F}\left(E_{\alpha_2}\right)(\xi)\mathcal{F}\left(E_{\alpha_1}\right)(\xi)\, d\lambda \\
&= -\frac{1}{8\pi\xi}\int_0^{+\infty} \frac{\lambda^{-\varepsilon+ir}}{\lambda+\lambda_1}\, d\lambda + \frac{1}{8\pi\xi}\int_0^{+\infty} \frac{\lambda^{-\varepsilon+ir}}{\lambda+\lambda_2}\, d\lambda \\
&\quad -\frac{1}{8\pi\xi}\int_0^{+\infty} \frac{\lambda^{1-\varepsilon+ir}}{\lambda+\lambda_1}\, d\lambda + \frac{1}{8\pi\xi}\int_0^{+\infty} \frac{\lambda^{1-\varepsilon+ir}}{\lambda+\lambda_2}\, d\lambda \\
&= -\frac{\lambda_1^{-\varepsilon+ir}}{8\pi\xi}\int_0^{+\infty} \frac{\sigma_1^{-\varepsilon+ir}}{\sigma_1+1}\, d\sigma_1 + \frac{\lambda_2^{-\varepsilon+ir}}{8\pi\xi}\int_0^{+\infty} \frac{\sigma_2^{-\varepsilon+ir}}{\sigma_2+1}\, d\sigma_2 \\
&\quad -\frac{\lambda_1^{1-\varepsilon+ir}}{8\pi\xi}\int_0^{+\infty} \frac{\sigma_1^{1-\varepsilon+ir}}{\sigma_1+1}\, d\sigma_1 + \frac{\lambda_2^{1-\varepsilon+ir}}{8\pi\xi}\int_0^{+\infty} \frac{\sigma_2^{1-\varepsilon+ir}}{\sigma_2+1}\, d\sigma_2.
\end{aligned}$$

Moreover, from (40) and (41), we deduce that

$$\begin{aligned}
\Upsilon &= \left(\frac{\lambda_2^{-\varepsilon+ir}-\lambda_1^{-\varepsilon+ir}}{8\pi\xi}\right)\Gamma_{\varepsilon,r} + \left(\frac{\lambda_2^{-\varepsilon+ir}-\lambda_1^{-\varepsilon+ir}}{8\pi\xi}\right)\Gamma(\varepsilon-ir-1)\Gamma(1-(\varepsilon-ir-1)) \\
&= \left(\frac{\lambda_2^{-\varepsilon+ir}-\lambda_1^{-\varepsilon+ir}}{2\pi\xi}\right)\left(\Gamma_{\varepsilon,r}+\Gamma(\varepsilon-ir-1)\Gamma(1-(\varepsilon-ir-1))\right).
\end{aligned}$$

For all $z \in \mathbb{C}\setminus\mathbb{Z}$, we have

$$\Gamma(z-1)\Gamma(1-(z-1)) = \frac{\pi}{\sin(\pi(z-1))} = -\frac{\pi}{\sin(\pi z)} = -\Gamma(z)\Gamma(1-z),$$

Setting $z = \varepsilon - ir$, with $\varepsilon \in (0,1)$, it follows that

$$\Gamma(\varepsilon-ir-1)\Gamma(1-(\varepsilon-ir-1) = -\Gamma(\varepsilon-ir)\Gamma(1-\varepsilon+ir) = -\Gamma_{\varepsilon,r},$$

hence $\Upsilon = 0$. Finally, we obtain that

$$I_{\varepsilon,r}(\theta) = \mathcal{F}^{-1}\left(m_\varepsilon(\xi)\mathcal{F}\left(\widetilde{G_0}\right)(\xi)\right)(\theta),$$

where

$$m_\varepsilon(\xi) = \frac{\lambda_2^{-\varepsilon+ir}-\lambda_1^{-\varepsilon+ir}}{8\pi\xi}.$$

Setting

$$m(\xi) := \lim_{\varepsilon\to 0} m_\varepsilon(\xi) = \frac{\lambda_2^{ir}-\lambda_1^{ir}}{8\pi\xi},$$

due to the Lebesgue's dominated convergence Theorem, it follows that

$$I_{0,r}(\theta) := \lim_{\varepsilon\to 0} I_{\varepsilon,r}(\theta) = \mathcal{F}^{-1}\left(m(\xi)\mathcal{F}\left(\widetilde{G_0}\right)(\xi)\right)(\theta).$$

Moreover, for all $x_1, x_2 \in \mathbb{R}$, we have

$$\left|e^{ix_1}-e^{ix_2}\right| \leqslant |x_1-x_2|,$$

then, for all $\xi \in \mathbb{R}\setminus\{0\}$, we deduce that

$$|m(\xi)| = \frac{|\lambda_2^{ir}-\lambda_1^{ir}|}{8\pi|\xi|} = \frac{\left|e^{ir\ln(\lambda_2)}-e^{ir\ln(\lambda_1)}\right|}{8\pi|\xi|} \leqslant \frac{|r|\,|\ln(\lambda_2)-\ln(\lambda_1)|}{8\pi|\xi|}.$$

Thus

$$\sup_{\xi\in\mathbb{R}}|m(\xi)| = \lim_{\xi\to 0}|m(\xi)| = \left|\lim_{\xi\to 0}\frac{\lambda_2^{ir}-\lambda_1^{ir}}{8\pi\xi}\right|,$$

and

$$\begin{aligned}\lim_{\xi\to 0}\frac{\lambda_2^{ir}-\lambda_1^{ir}}{8\pi\xi} &= \lim_{\xi\to 0}2^{ir}\left(\frac{1+ir\left(-2\pi\xi+2\pi^2\xi^2\right)+2ir(ir-1)\pi^2\xi^2+o(\xi^2)}{8\pi\xi}\right)\\ &\quad -\lim_{\xi\to 0}2^{ir}\left(\frac{1+ir\left(2\pi\xi+2\pi^2\xi^2\right)+2ir(ir-1)\pi^2\xi^2+o(\xi^2)}{8\pi\xi}\right)\\ &= \lim_{\xi\to 0}2^{ir}\left(\frac{-4ir\pi\xi+o(\xi^2)}{8\pi\xi}\right)\\ &= -2^{ir-1}ir.\end{aligned}$$

Then

$$\sup_{\xi\in\mathbb{R}}|m(\xi)| = \frac{|r|}{2}.$$

Since we have

$$\begin{aligned}\xi\, m'(\xi) &= \frac{2\pi\xi^2\left(ir\lambda_2^{ir-1}(-4\pi+8\pi^2\xi)-ir\lambda_1^{ir-1}(4\pi+8\pi^2\xi)\right)-2\pi\xi\left(\lambda_2^{ir}-\lambda_1^{ir}\right)}{64\pi^2\xi^2}\\ &= \frac{ir}{32}\left(\lambda_2^{ir-1}(-1+2\pi\xi)-\lambda_1^{ir-1}(1+2\pi\xi)\right)-\frac{\lambda_2^{ir}-\lambda_1^{ir}}{32\pi\xi},\end{aligned}$$

we obtain in the same way that

$$\sup_{\xi\in\mathbb{R}}|\xi\, m'(\xi)| = \lim_{\xi\to 0}|\xi\, m'(\xi)| = \left|\lim_{\xi\to 0}\xi\, m'(\xi)\right|,$$

with

$$\lim_{\xi\to 0}\xi\, m'(\xi) = \frac{ir}{32}\lim_{\xi\to 0}\left(\lambda_2^{ir-1}(-1+2\pi\xi)-\lambda_1^{ir-1}(1+2\pi\xi)\right)-\frac{\lambda_2^{ir}-\lambda_1^{ir}}{32\pi\xi},$$

where

$$\lim_{\xi\to 0}\lambda_2^{ir-1} = \lim_{\xi\to 0}2^{ir-1}\left(1+(ir-1)\left(-2\pi\xi+2\pi^2\xi^2\right)+4ir(ir-1)\pi^2\xi^2+o(\xi^2)\right) = 2^{ir-1},$$

and

$$\lim_{\xi\to 0}\lambda_1^{ir-1} = \lim_{\xi\to 0}2^{ir-1}\left(1+(ir-1)\left(2\pi\xi+2\pi^2\xi^2\right)+4ir(ir-1)\pi^2\xi^2+o(\xi^2)\right) = 2^{ir-1}.$$

Thus, we obtain

$$\begin{aligned}\lim_{\xi\to 0}\xi\, m'(\xi) &= \frac{1}{4}\lim_{\xi\to 0}\frac{ir}{8}\left(-\left(\lambda_2^{ir-1}+\lambda_1^{ir-1}\right)+2\pi\xi\left(\lambda_2^{ir-1}-\lambda_1^{ir-1}\right)\right)-\frac{\lambda_2^{ir}-\lambda_1^{ir}}{8\pi\xi}\\ &= \frac{1}{4}\left(-\frac{2^{ir}ir}{8}+2^{ir-1}ir\right)\\ &= 3\times 2^{ir-5}ir.\end{aligned}$$

Then

$$\sup_{\xi\in\mathbb{R}}|\xi\, m'(\xi)| = \left|3\times 2^{ir-5}ir\right| = \frac{3}{32}|r|.$$

Therefore, we deduce that

$$\sup_{\xi\in\mathbb{R}}|m(\xi)|+\sup_{\xi\in\mathbb{R}}|\xi\, m'(\xi)|=\frac{|r|}{2}+\frac{3}{32}|r|=\frac{19}{32}|r|,$$

From the Mihlin Theorem, see [18], for all $\gamma>0$, there exists $C_{\gamma,p}>0$, such that, for all $r\in\mathbb{R}$, we have

$$\|I_{0,r}(.)\|_{\mathcal{L}(X)}=\left\|F^{-1}\left(m(\xi)F(\widetilde{G_0})(\xi)\right)(.)\right\|_{\mathcal{L}(X)}\leqslant C_{\gamma,p}e^{\gamma|r|}.$$

We can treat the others terms using [21] and obtain similar results. Finally, for all $\gamma>0$, there exists a constant $C_{\gamma,p}>0$ such that for all $r\in\mathbb{R}$, we obtain

$$\left\|(\mathcal{A}+I)^{ir}\right\|_{\mathcal{L}(X)}\leqslant C_{\gamma,p}\,e^{\gamma|r|}.$$

Therefore, taking $\theta_{\mathcal{A}}=\gamma>0$, we obtain that $\mathcal{A}+I\in\mathrm{BIP}\,(X,\theta_{\mathcal{A}})$. Finally, using Theorem 2.3, p. 69 in [1], we deduce that $\mathcal{A}=\mathcal{A}+I-I\in\mathrm{BIP}\,(X,\theta_{\mathcal{A}})$. □

4 Study of the sum $\mathcal{L}_1+\mathcal{L}_2$

4.1 Invertibility of the closure of the sum

In this section, we will apply the results described in Section 2.2. We take $\mathcal{L}_1=\mathcal{M}_1$ and $\mathcal{L}_2=\mathcal{M}_2$.

Theorem 4.1. Assume that (3) holds. Then $\mathcal{L}_1+\mathcal{L}_2$ is closable and its closure $\overline{\mathcal{L}_1+\mathcal{L}_2}$ is invertible.

Proof. Assumption (H_1) is satisfied from Proposition 3.1 and Corollary 3.10, with

$$\theta_{\mathcal{M}_1}+\theta_{\mathcal{M}_2}=\varepsilon_{\mathcal{L}_1}+\pi-\varepsilon_{\mathcal{L}_2},$$

where it suffices to take $\varepsilon_{\mathcal{L}_2}>\varepsilon_{\mathcal{L}_1}$ in order to obtain $\theta_{\mathcal{M}_1}+\theta_{\mathcal{M}_2}<\pi$.

For assumption (H_2), due to Proposition 3.2, it follows that $0\notin\sigma(\mathcal{L}_1)\cap\sigma(-\mathcal{L}_2)$. Moreover, from Proposition 3.1, we have

$$\sigma(\mathcal{L}_1)=\{\lambda\in\mathbb{C}:|\arg(\lambda)|<\pi\quad\text{and}\quad\mathrm{Re}(\sqrt{\lambda})\leqslant\nu\},$$

and from Remark 3.11, it follows that

$$\sigma(-\mathcal{L}_2)=\{\lambda\in\mathbb{C}\setminus\mathbb{R}_+:\sinh(\omega\sqrt{-\lambda})=\pm\omega\sqrt{-\lambda}\}=\left\{-\frac{z_j^2}{\omega^2}\in\mathbb{C}\setminus\mathbb{R}_+:j\in\mathbb{N}\setminus\{0\}\right\}.$$

Then, since

$$\mathrm{Re}\left(\sqrt{-\frac{z_j^2}{\omega^2}}\right)=\frac{1}{\omega}\left|\mathrm{Im}(z_j)\right|,$$

the condition $\sigma(\mathcal{L}_1)\cap\sigma(-\mathcal{L}_2)=\emptyset$ is fulfilled if (3) holds.

The commutativity assumption (H_3) is clearly verified since the actions of operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are independent.

Now, applying Theorem 2.6, we obtain the result. □

Remark 4.2. We can conjecture that, for the critical case $\nu=\tau$, the sum $\mathcal{L}_1+\mathcal{L}_2$ is not closable.

4.2 Convexity inequalities

In view to apply Corollary 2.7, we are going to verify inequality (6) in two situations.

Proposition 4.3. Let

$$\mathcal{E}_1 = W^{1,p}(0,+\infty;X) \subset \mathcal{E} = L^p(0,+\infty;X),$$

and

$$\mathcal{E}_2 = L^p\left(0,+\infty;\left[W^{3,p}(0,\omega)\cap W_0^{2,p}(0,\omega)\right]\times L^p(0,\omega)\right) \subset \mathcal{E}.$$

Then, we have

$$D\left(\overline{\mathcal{L}_1+\mathcal{L}_2}\right) \subset \mathcal{E}_1 \cap \mathcal{E}_2. \tag{42}$$

Proof. Let $V \in D(\mathcal{L}_1)$. We must prove that there exists $\delta \in (0,1)$ such that

$$\|V\|_{\mathcal{E}_1} \leqslant C\left[\|V\|_{\mathcal{E}} + \|V\|_{\mathcal{E}}^{1-\delta}\,\|\mathcal{L}_1(V)\|_{\mathcal{E}}^{\delta}\right].$$

For all $V \in W^{2,p}(0,+\infty;X)$, from [10], inequality (1.15), p. 192, we have the convexity inequality

$$\|V'\|_{\mathcal{E}} \leqslant 2\sqrt{2}\|V\|_{\mathcal{E}}^{1/2}\|V''\|_{\mathcal{E}}^{1/2}.$$

Thus, we deduce

$$\|V\|_{\mathcal{E}_1} = \|V\|_{\mathcal{E}} + \|V'\|_{\mathcal{E}} \leqslant \|V\|_{\mathcal{E}} + 2\sqrt{2}\|V\|_{\mathcal{E}}^{1/2}\|V''\|_{\mathcal{E}}^{1/2}.$$

Since $\mathcal{L}_1$ is not invertible, we are going to estimate $\|V''\|_{\mathcal{E}}$ by $\|\mathcal{L}_1(V)-\lambda_0 V\|_{\mathcal{E}}$, where $\lambda_0 \in \rho(\mathcal{L}_1)$. We have

$$V'' - 2\nu V' + (\nu^2-\lambda_0)V = \mathcal{L}_1(V) - \lambda_0 V.$$

Then, there exists a constant $C>0$ such that

$$\|V''\|_{\mathcal{E}} + \|V'\|_{\mathcal{E}} + \|V\|_{\mathcal{E}} \leqslant C\|\mathcal{L}_1(V)-\lambda_0 V\|_{\mathcal{E}},$$

hence

$$\|V''\|_{\mathcal{E}} \leqslant C\|\mathcal{L}_1(V)-\lambda_0 V\|_{\mathcal{E}} \leqslant C\|\mathcal{L}_1(V)\|_{\mathcal{E}} + |\lambda_0|C\|V\|_{\mathcal{E}}.$$

Thus, we deduce

$$\begin{aligned}
\|V\|_{\mathcal{E}_1} = \|V\|_{\mathcal{E}} + \|V'\|_{\mathcal{E}} &\leqslant \|V\|_{\mathcal{E}} + 2\sqrt{2}\,\|V\|_{\mathcal{E}}^{1/2}\|V''\|_{\mathcal{E}}^{1/2} \\
&\leqslant \|V\|_{\mathcal{E}} + 2\sqrt{2C}\,\|V\|_{\mathcal{E}}^{1/2}\left(\|\mathcal{L}_1(V)\|_{\mathcal{E}} + |\lambda_0|\|V\|_{\mathcal{E}}\right)^{1/2} \\
&\leqslant \|V\|_{\mathcal{E}} + 2\sqrt{2C}\,\|V\|_{\mathcal{E}}^{1/2}\left(\|\mathcal{L}_1(V)\|_{\mathcal{E}}^{1/2} + |\lambda_0|^{1/2}\|V\|_{\mathcal{E}}^{1/2}\right) \\
&\leqslant \left(1+2\sqrt{2C}\,|\lambda_0|^{1/2}\right)\|V\|_{\mathcal{E}} + 2\sqrt{2C}\,\|V\|_{\mathcal{E}}^{1/2}\|\mathcal{L}_1(V)\|_{\mathcal{E}}^{1/2}.
\end{aligned}$$

Therefore, inequality (6) is satisfied for $\delta = 1/2$ and $\mathcal{M}_1 = \mathcal{L}_1$. Using Corollary 2.7, we obtain

$$D\left(\overline{\mathcal{L}_1+\mathcal{L}_2}\right) \subset \mathcal{E}_1.$$

Now, we must show that, for all $V \in D(\mathcal{L}_2)$, we have

$$\|V\|_{\mathcal{E}_2} \leqslant C\left[\|V\|_{\mathcal{E}} + \|V\|_{\mathcal{E}}^{1/2}\|\mathcal{L}_2(V)\|_{\mathcal{E}}^{1/2}\right].$$

To this end, it suffices to do it for $\mathcal{A}$. Set

$$\mathcal{G}_1 = \left[W^{3,p}(0,\omega)\cap W_0^{2,p}(0,\omega)\right]\times L^p(0,\omega) \subset X.$$

We must prove that

$$\forall \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in D(\mathcal{A}), \quad \left\| \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_{\mathcal{G}_1} \leqslant C \left[\left\| \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_X + \left\| \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_X^{1/2} \left\| \mathcal{A} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_X^{1/2} \right].$$

Here, we have

$$\begin{aligned} \left\| \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_{\mathcal{G}_1} &= \|\psi_1\|_{W^{3,p}(0,\omega)} + \|\psi_2\|_{L^p(0,\omega)} \\ &= \|\psi_1\|_{L^p(0,\omega)} + \|\psi_1'\|_{L^p(0,\omega)} + \|\psi_1''\|_{L^p(0,\omega)} + \|\psi_1'''\|_{L^p(0,\omega)} + \|\psi_2\|_{L^p(0,\omega)}. \end{aligned}$$

Set $\varphi = \psi_1''$. Then, for all $\eta > 0$, from [10], inequality (1.12), p. 192, taking $n = \eta + 1$ and $b - a = \omega$, we obtain

$$\|\varphi'\|_{L^p(0,\omega)} \leqslant \frac{\omega}{\eta}\|\varphi''\|_{L^p(0,\omega)} + \frac{2}{\omega}\left(\eta + 3 + \frac{2}{\eta}\right)\|\varphi\|_{L^p(0,\omega)}.$$

It is not difficult to see that the second member is minimal when

$$\eta = \frac{\sqrt{2}}{2} \frac{\left(\|\varphi''\|_{L^p(0,\omega)} + 4\|\varphi\|_{L^p(0,\omega)}\right)^{1/2}}{\|\varphi\|_{L^p(0,\omega)}^{1/2}}.$$

Therefore, we deduce that

$$\begin{aligned} \|\varphi'\|_{L^p(0,\omega)} &\leqslant \frac{\omega\sqrt{2}\,\|\varphi\|_{L^p(0,\omega)}^{1/2}\|\varphi''\|_{L^p(0,\omega)}}{\left(\|\varphi''\|_{L^p(0,\omega)} + 4\|\varphi\|_{L^p(0,\omega)}\right)^{1/2}} + \frac{4}{\omega}\frac{\sqrt{2}\,\|\varphi\|_{L^p(0,\omega)}^{1/2}\|\varphi\|_{L^p(0,\omega)}}{\left(\|\varphi''\|_{L^p(0,\omega)} + 4\|\varphi\|_{L^p(0,\omega)}\right)^{1/2}} \\ &\quad + \frac{\sqrt{2}}{\omega}\frac{\left(\|\varphi''\|_{L^p(0,\omega)} + 4\|\varphi\|_{L^p(0,\omega)}\right)^{1/2}\|\varphi\|_{L^p(0,\omega)}}{\|\varphi\|_{L^p(0,\omega)}^{1/2}} + \frac{6}{\omega}\|\varphi\|_{L^p(0,\omega)} \\ &\leqslant \frac{\sqrt{2}}{\omega}\left(\|\varphi''\|_{L^p(0,\omega)} + 4\|\varphi\|_{L^p(0,\omega)}\right)^{1/2}\|\varphi\|_{L^p(0,\omega)}^{1/2} + \frac{6}{\omega}\|\varphi\|_{L^p(0,\omega)} \\ &\quad + \left(\frac{4}{\omega}\|\varphi\|_{L^p(0,\omega)} + \omega\|\varphi''\|_{L^p(0,\omega)}\right)\frac{\sqrt{2}\,\|\varphi\|_{L^p(0,\omega)}^{1/2}}{\left(\|\varphi''\|_{L^p(0,\omega)} + 4\|\varphi\|_{L^p(0,\omega)}\right)^{1/2}} \\ &\leqslant C_\omega\left(\|\varphi\|_{L^p(0,\omega)} + \|\varphi\|_{L^p(0,\omega)}^{1/2}\|\varphi''\|_{L^p(0,\omega)}^{1/2}\right). \end{aligned}$$

Then, we have

$$\|\psi_1'''\|_{L^p(0,\omega)} \leqslant C_\omega\left(\|\psi_1''\|_{L^p(0,\omega)} + \|\psi_1''\|_{L^p(0,\omega)}^{1/2}\|\psi_1^{(4)}\|_{L^p(0,\omega)}^{1/2}\right).$$

Hence

$$\begin{aligned} \left\| \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_{\mathcal{G}_1} &\leqslant \|\psi_1\|_{L^p(0,\omega)} + \|\psi_1'\|_{L^p(0,\omega)} + \|\psi_1''\|_{L^p(0,\omega)} \\ &\quad + C_\omega\left(\|\psi_1''\|_{L^p(0,\omega)} + \|\psi_1''\|_{L^p(0,\omega)}^{1/2}\|\psi_1^{(4)}\|_{L^p(0,\omega)}^{1/2}\right) + \|\psi_2\|_{L^p(0,\omega)} \\ &\leqslant (1 + C_\omega)\,\|\psi_1\|_{W_0^{2,p}(0,\omega)} + C_\omega\|\psi_1\|_{W_0^{2,p}(0,\omega)}^{1/2}\|\psi_1^{(4)}\|_{L^p(0,\omega)}^{1/2} + \|\psi_2\|_{L^p(0,\omega)} \end{aligned}$$

Now, in virtue of the invertibility of $\mathcal{A}$, see Proposition 3.2, we have proved that there exists a constant C'_ω depending only on ω such that

$$\|\psi_1^{(4)}\|_{L^p(0,\omega)} \leqslant C'_\omega \left\| \mathcal{A} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_X .$$

Moreover, it follows

$$\left\| \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_{\mathcal{G}_1} \leqslant (1 + C_\omega) \left\| \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_X + C_\omega C'_\omega \left\| \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_X^{1/2} \left\| \mathcal{A} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_X^{1/2} .$$

Therefore, inequality (6) is satisfied for $\delta = 1/2$ and $\mathcal{M}_2 = \mathcal{L}_2$. Using Corollary 2.7, we obtain

$$D\left(\overline{\mathcal{L}_1 + \mathcal{L}_2}\right) \subset \mathcal{E}_2,$$

which gives the expected result. □

5 Back to the abstract problem

Now, we are in position to solve the following equation

$$\left(\overline{\mathcal{L}_1 + \mathcal{L}_2}\right) V + k\rho^2 \left(\mathcal{P}_1 + \mathcal{P}_2\right) V = \mathcal{F}. \tag{43}$$

Theorem 5.1. Let $\mathcal{F} \in L^p(0, +\infty; X)$ and assume that (3) holds. Then, there exists $\rho_0 > 0$ such that for all $\rho \in (0, \rho_0]$, equation (43) has a unique strong solution $V \in L^p(0, +\infty; X)$, that is

$$\begin{cases} \exists\ (V_n)_{n\geqslant 0} \in D(\mathcal{L}_1) \cap D(\mathcal{L}_2) : \\ \\ V_n \underset{n\to+\infty}{\longrightarrow} V \text{ in } L^p(0, +\infty; X) \text{ and} \\ \\ \left(\mathcal{L}_1 + \mathcal{L}_2\right) V_n + k\rho^2 \left(\mathcal{P}_1 + \mathcal{P}_2\right) V_n \underset{n\to+\infty}{\longrightarrow} \mathcal{F} \text{ in } L^p(0, +\infty; X), \end{cases} \tag{44}$$

satisfying

$$V \in W^{1,p}(0, +\infty; X) \cap L^p\left(0, +\infty; \left[W^{3,p}(0,\omega) \cap W_0^{2,p}(0,\omega)\right] \times L^p(0,\omega)\right). \tag{45}$$

Proof. Due to Theorem 4.1, if (3) holds, then $\overline{\mathcal{L}_1 + \mathcal{L}_2}$ is invertible. Thus, it follows that

$$\left[I + k\rho^2 \left(\mathcal{P}_1 + \mathcal{P}_2\right) \left(\overline{\mathcal{L}_1 + \mathcal{L}_2}\right)^{-1}\right] \left(\overline{\mathcal{L}_1 + \mathcal{L}_2}\right) V = \mathcal{F}.$$

From (42), we deduce that $V \in D(\overline{\mathcal{L}_1 + \mathcal{L}_2}) \subset \mathcal{E}_1 \cap \mathcal{E}_2$, that is (45) which involves that

$$(\mathcal{P}_1 + \mathcal{P}_2)(\overline{\mathcal{L}_1 + \mathcal{L}_2})^{-1} \in \mathcal{L}(X).$$

Then, there exists $\rho_0 > 0$ small enough such that, for all $\rho \in (0, \rho_0]$, we have

$$V = \left(\overline{\mathcal{L}_1 + \mathcal{L}_2}\right)^{-1} \left[I + k\rho^2 \left(\mathcal{P}_1 + \mathcal{P}_2\right) \left(\overline{\mathcal{L}_1 + \mathcal{L}_2}\right)^{-1}\right]^{-1} \mathcal{F}, \tag{46}$$

which means that V is the unique strong solution of (43). □

6 Proof of Theorem 1.1

From Theorem 5.1, there exists $\rho_0 > 0$ such that for all $\rho \in (0, \rho_0]$, equation (43) has a unique strong solution $V \in L^p(0, +\infty; X)$ satisfying (45). Then, due to (44), there exists $(V_n)_{n\in\mathbb{N}} \in D(\mathcal{L}_1 + \mathcal{L}_2)$ such that $V_n \underset{n\to+\infty}{\longrightarrow} V$ and

$$\lim_{n\to+\infty} (\mathcal{L}_1 + \mathcal{L}_2) V_n + k\rho^2 (\mathcal{P}_1 + \mathcal{P}_2) V_n = \mathcal{F}.$$

Since $(V_n)_{n\in\mathbb{N}} \in D(\mathcal{L}_1 + \mathcal{L}_2)$, then the previous equality writes

$$\begin{cases} \lim_{n\to+\infty} (V_n''(t) - \mathcal{A}V_n(t) - \mathcal{F}_n(t)) = 0 \\ \lim_{n\to+\infty} V_n(0) = 0, \quad \lim_{n\to+\infty} V_n(+\infty) = 0, \end{cases} \tag{47}$$

where

$$\mathcal{F}_n(t) = k\rho^2 e^{-2t}\mathcal{A}_0 V_n(t) + k\rho^2 e^{-2t} [(\mathcal{B}_2 V_n)](t) + 2\nu V_n'(t) - \nu^2 V_n(t) + \mathcal{F}(t).$$

Since $V_n \underset{n\to+\infty}{\longrightarrow} V$ and V satisfies (45), we deduce that

$$\lim_{n\to+\infty} V_n(0) = V(0) = 0 \quad \text{with} \quad \lim_{n\to+\infty} V_n(+\infty) = V(+\infty) = 0,$$

and

$$\lim_{n\to+\infty} \mathcal{F}_n(t) = \mathcal{F}_\infty(t) \in L^p(0, +\infty; X),$$

where

$$\mathcal{F}_\infty(t) = k\rho^2 e^{-2t}\mathcal{A}_0 V(t) + k\rho^2 e^{-2t} [(\mathcal{B}_2 V)](t) + 2\nu V'(t) - \nu^2 V(t) + \mathcal{F}(t).$$

Thus, problem (47) writes

$$\begin{cases} \lim_{n\to+\infty} (V_n''(t) - \mathcal{A}V_n(t)) = \mathcal{F}_\infty(t) \\ V(0) = 0, \quad V(+\infty) = 0, \end{cases}$$

Moreover, from Proposition 3.12, $\mathcal{A} \in \mathrm{BIP}(X, \theta_{\mathcal{A}})$, with $\theta_{\mathcal{A}} \in (0, \pi)$ and due to [9], Proposition 3.2.1, e), p. 71, it follows that $\sqrt{\mathcal{A}} \in \mathrm{BIP}(X, \theta_{\mathcal{A}}/2)$ with $\theta_{\mathcal{A}}/2 \in (0, \pi/2)$. Therefore, due to [5], Theorem 2, p. 712, with $L_1 = L_2 = -\sqrt{\mathcal{A}}$, there exists a unique classical solution to the following problem

$$\begin{cases} \mathcal{V}''(t) - \mathcal{A}\mathcal{V}(t) = \mathcal{F}_\infty(t) \\ \mathcal{V}(0) = 0, \quad \mathcal{V}(+\infty) = 0, \end{cases}$$

that is

$$\mathcal{V} \in W^{2,p}(0, +\infty; X) \cap L^p(0, +\infty; D(\mathcal{A})).$$

Thus, it follows that

$$\lim_{n\to+\infty} (V_n''(t) - \mathcal{A}V_n(t)) = \mathcal{V}''(t) - \mathcal{A}\mathcal{V},$$

hence

$$\lim_{n\to+\infty} \left[(V_n(t) - \mathcal{V}(t))'' - \mathcal{A}(V_n(t) - \mathcal{V}(t))\right] = 0.$$

Now, set

$$\begin{cases} D(\delta_2) &= \{\varphi \in W^{2,p}(0, +\infty; X) : \varphi(0) = \varphi(+\infty) = 0\} \\ \delta_2\varphi &= \varphi'', \quad \varphi \in D(\delta_2). \end{cases}$$

Then, we can write

$$0 = \lim_{n\to+\infty} \left[(V_n(t) - \mathcal{V}(t))'' - \mathcal{A}(V_n(t) - \mathcal{V}(t))\right] = \lim_{n\to+\infty} -(-\delta_2 + \mathcal{A})(V_n(t) - \mathcal{V}(t)). \tag{48}$$

From [20], Theorem C, p. 166-167, it follows that $-\delta_2 \in \mathrm{BIP}\,(X, \theta_{\delta_2})$, for every $\theta_{\delta_2} \in (0, \pi)$ and due to Proposition 3.12, $\mathcal{A} \in \mathrm{BIP}\,(X, \theta_{\mathcal{A}})$, for all $\theta_{\mathcal{A}} \in (0, \pi)$. Thus, since $-\delta_2$ and $\mathcal{A}$ are resolvent commuting with $\theta_{\delta_2} + \theta_{\mathcal{A}} < \pi$, from [19], Theorem 5, p. 443, we obtain that

$$-\delta_2 + \mathcal{A} \in \mathrm{BIP}\,(X, \theta), \quad \theta = \max(\theta_{\delta_2}, \theta_{\mathcal{A}}).$$

Moreover, due to Proposition 3.2, we have $0 \in \rho(\mathcal{A})$, then we deduce from [19], remark at the end of p. 445, that $0 \in \rho(\delta_2 + \mathcal{A})$. Therefore, due to (48), we obtain that

$$\lim_{n \to +\infty} V_n(t) - \mathcal{V}(t) = 0,$$

hence, since $V_n \underset{n \to +\infty}{\longrightarrow} V$, by uniqueness of the limit, we deduce that

$$V = \mathcal{V} \in W^{2,p}(0, +\infty; X) \cap L^p(0, +\infty; D(\mathcal{A})).$$

This prove that $\mathcal{L}_1 + \mathcal{L}_2$ is closed and that $V \in D(\mathcal{L}_1 + \mathcal{L}_2)$.

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