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# Generalized diffusion problems in a conical domain, part I

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## Abstract

The purpose of this article (composed of two parts) is the study of the generalized dispersal operator of a reaction-diffusion equation in  $L^p$ -spaces set in the finite conical domain  $S_{\omega,\rho}$  of angle  $\omega > 0$  and radius  $\rho > 0$  in  $\mathbb{R}^2$ .

This first part is devoted to the behaviour of the solution near the top of the cone which is completely described in the weighted Sobolev space  $W_{3-\frac{1}{p}}^{4,p}(S_{\omega,\rho})$ , see Theorem 2.2.

**Key Words and Phrases:** Fourth order boundary value problem, conical domain, weighted Sobolev spaces.

**2020 Mathematics Subject Classification:** 35B65, 35J40, 35J75, 35K35, 46E35.

## 1 Introduction

This work required the use of many calculations and non-trivial checks linked, among other, to the theory of sums of linear operators. This is why we were forced to split this work into two more or less independent parts.

In this first part, we consider the generalized following reaction-diffusion equation

$$\begin{cases} \frac{\partial u}{\partial t} = -\Delta^2 u + k\Delta u + f^*(u) & \text{in } \mathbb{R}_+ \times \Omega, \\ u(0) = u_0 \text{ given} \\ \text{Boundary conditions for } u \text{ on } \partial\Omega, \end{cases} \quad (1)$$

where  $k$  is a positive number,  $f^*$  is a non-linear reaction function and  $\Omega$  is a bounded conical domain. This work is a natural continuation of the one studied in [5]. The originality of this work lies in the fact that the open set  $\Omega$  is conical whereas in [5], it was cylindrical.

The study of the spatial operator, that is the linear combination of the laplacian and the bilaplacian, in (1) required the analysis of a sum of unbounded linear operators in a Banach space, carried out in part II, see [6]. Note that, by similar techniques, we can prove that the dispersal operator in (1) generates an analytic semigroup, see for instance [7].

Such problems, set in conical domains, model many concrete situations related to pollution for instance.

More precisely, in this work, we consider the following 2 dimensional sector

$$\Omega = S_\omega = \{(x, y) = (r \cos \theta, r \sin \theta) : r > 0 \text{ and } 0 < \theta < \omega\},$$

with its lateral edges

$$\begin{cases} \Gamma_0 = (0, +\infty) \times \{0\} \\ \Gamma_\omega = \{(r \cos \omega, r \sin \omega) : r > 0\}; \end{cases}$$

here  $\omega \in (0, 2\pi]$ .

It is known that the study of problem (1) in  $L^p$ -spaces needs the analyze of the following linear stationary problem

$$\begin{cases} \Delta^2 u - k\Delta u = f & \text{in } S_\omega \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_0 \cup \Gamma_\omega. \end{cases} \quad (2)$$

This work is inspired by the one done in [3] and [4], where the author has considered, in a hilbertian framework, the following boundary problem

$$\begin{cases} \Delta^2 u = 0 & \text{in } S_\omega \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_0 \cup \Gamma_\omega. \end{cases}$$

The author has proved that the solution of this problem writes as a "superposition" of particular solutions with separate variables of the form  $\chi_{1,j}(r)\chi_{2,j}(\theta)$ , for  $j \in \mathbb{N}$ . The basic tools used are based on the compact operators belonging to the so-called Carleman class and the Fredholm determinants.

Let us recall some known results concerning the biharmonic equation in a conical domain or in a Lipschitz domain. In [9], the authors gave many estimates concerning the solution of the Dirichlet problem in  $L^p$  for the biharmonic equation in Lipschitz domain. In [1], many results are given for general higher-order elliptic equations in non smooth-domains. In [10], the author has studied the following problem

$$\begin{cases} \Delta^2 u = f & \text{in } S_{\omega,1} \\ u = \Delta u = 0 & \text{on } \partial S_{\omega,1}, \end{cases}$$

where  $f \in L^2(S_{\omega,1})$ . He has proved the two following results

1. If  $\omega < \pi$ , the variational solution writes, in the neighbourhood of  $O$ , as

$$u_\omega = u_{1,\omega} + u_{2,\omega} + u_{3,\omega},$$

with  $u_{1,\omega} \in H^{1+\frac{\pi}{\omega}-\varepsilon}$ ,  $u_{2,\omega} \in H^{2+\frac{\pi}{\omega}-\varepsilon}$  and  $u_{3,\omega} \in H^4$ , for a small  $\varepsilon > 0$ .

2. If  $\omega = \pi$ , in the neighbourhood of  $O$ , the variational solution is verifies

$$u_\pi \in H^4.$$

This article is organized as follows. In Section 2, we state our problem in polar coordinates and our main result in Theorem 2.2. In Section 3 we reformulate our problem as a sum of linear operators. Then, Section 4 is devoted to its proof.

## 2 Statement of the main result

We introduce the following polar variables function

$$v(r, \theta) = u(r \cos \theta, r \sin \theta) = u(x, y).$$

It is known that the laplacian and the bilaplacian, in polar coordinates, respectively write

$$\Delta u = \frac{1}{r^2} \left[ \left( r \frac{\partial}{\partial r} \right)^2 + \frac{\partial^2}{\partial \theta^2} \right] v = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} := \Lambda_1 v, \quad (3)$$

and

$$\begin{aligned}\Delta^2 u &= \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)^2 v := \Lambda_2 v \\ &= \frac{\partial^4 v}{\partial r^4} + \frac{2}{r^2} \frac{\partial^4 v}{\partial r^2 \partial \theta^2} + \frac{1}{r^4} \frac{\partial^4 v}{\partial \theta^4} + \frac{2}{r} \frac{\partial^3 v}{\partial r^3} - \frac{2}{r^3} \frac{\partial^3 v}{\partial r \partial \theta^2} - \frac{1}{r^2} \frac{\partial^2 v}{\partial r^2} + \frac{4}{r^4} \frac{\partial^2 v}{\partial \theta^2} + \frac{1}{r^3} \frac{\partial v}{\partial r}.\end{aligned}\quad (4)$$

**Remark 2.1.** We can generalize this work to the dimension  $n$  :

$$\Delta u = \frac{\partial^2 v}{\partial r^2} + \frac{n-1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \Delta' v,$$

where  $\Delta'$  is the Laplace-Beltrami operator.

We set

$$f(x, y) = f(r \cos \theta, r \sin \theta) = g(r, \theta).$$

Let  $\gamma \in \mathbb{R}$ . In the sequel, we will use the following weighted space Sobolev spaces on  $S_\omega$  by:

$$W_\gamma^{m,p}(S_\omega) = \left\{ v \in \mathcal{D}'(S_\omega), \forall (i, j) \in \mathbb{N}^2 : 0 \leq i + j \leq m, r^{\gamma-m+i+j} \frac{\partial^{i+j} v}{\partial r^i \partial \theta^j} \in L^p(S_\omega) \right\}. \quad (5)$$

Note that  $f \in L^p(\Omega)$  means that  $g \in L_{\frac{1}{p}}^p(S_\omega) := W_{\frac{1}{p}}^{0,p}(S_\omega)$ . In fact, we have

$$\int_{\Omega} |f(x, y)|^p dx dy = \int_{S_\omega} |g(r, \theta)|^p r dr d\theta = \int_{S_\omega} |r^{\frac{1}{p}} g(r, \theta)|^p dr d\theta.$$

We will focus ourselves, on the case when  $S_\omega$  is replaced by the finite sector:

$$S_{\omega,\rho} = \{(x, y) = (r \cos \theta, r \sin \theta) : 0 < r < \rho \text{ and } 0 < \theta < \omega\},$$

and its two lateral edges:

$$\begin{cases} \Gamma_0 &= (0, \rho) \times \{0\} \\ \Gamma_{\omega,\rho} &= \{(r \cos \omega, r \sin \omega) : 0 < r < \rho\}, \end{cases}$$

where  $\rho > 0$  is given small enough.

Then, in polar coordinates, problem (2) writes

$$\begin{cases} \Lambda_2 v - k \Lambda_1 v = g & \text{in } S_{\omega,\rho} \\ v(r, 0) = v(r, \omega) = \frac{\partial v}{\partial \theta}(r, 0) = \frac{\partial v}{\partial \theta}(r, \omega) = 0 \\ v(\rho, \theta) \text{ given,} \end{cases} \quad (6)$$

where  $\Lambda_1$  and  $\Lambda_2$  are given by (3) and (4).

In this article, we will focus ourselves on the resolution of problem (6) to obtain the behavior of the solution  $v$  to problem (6) in  $L^p$ -weighted spaces, in the neighborhood of the top of the cone. To this  $v$  corresponds a solution  $u_0$  to problem (2) by applying the inverse changes of variables and functions. We point out, that there is no reason for this solution  $u_0$  to coincide with the variational solution  $u_{var}$  belonging to the Hilbert space  $H^2(S_{\omega,\rho}) \cap H_0^1(S_{\omega,\rho})$ .

To solve problem (6), we will use results given in [6] and to this end, we need to consider

$$\tau = \min_{j \geq 1} |\text{Im}(z_j)| > 0,$$

where  $(z_j)_{j \geq 1}$  are the solutions of the following transcendent equation

$$(\sinh(z) + z)(\sinh(z) - z) = 0, \quad \text{with } \operatorname{Re}(z) > 0.$$

Actually, according to [2], we have

$$\tau \simeq 4.21239.$$

We assume that

$$\omega \left( 3 - \frac{2}{p} \right) < \tau. \quad (7)$$

Recall that  $3 - \frac{2}{p}$  is exactly the Sobolev exponent of the space  $W^{3,p}$  in two variables.

Let us remark that we have two cases:

1. if  $0 < \omega \leq \frac{\tau}{3} \simeq 0.45\pi$ , then (7) is satisfied for all  $p \in (1, +\infty)$ .
2. if  $\frac{\tau}{3} < \omega < \tau \simeq 1.34\pi$ , then (7) is satisfied for  $1 < p < \frac{2\omega}{3\omega - \tau}$ .

Our main result is the following:

**Theorem 2.2.** There exists  $\rho_0 > 0$  such that for all  $\rho \in (0, \rho_0]$ ,  $g \in L^{\frac{p}{1-p}}(S_{\omega, \rho})$  with  $p$  satisfying assumption (7), problem (6) has a unique strong solution

$$v \in W_{3-\frac{1}{p}}^{4,p}(S_{\omega, \rho}).$$

More precisely, for  $i = 0, 1, 2$  and  $j = 0, 1, 2, 3, 4$  such that  $0 \leq i + j \leq 4$ , we have

$$\frac{\partial^{i+j} v}{\partial r^i \partial \theta^j} \in L^p(S_{\omega, \rho}),$$

and

$$\frac{\partial^3 v}{\partial r^3}, \frac{\partial^4 v}{\partial r^3 \partial \theta} \in L_{2-\frac{1}{p}}^p(S_{\omega, \rho}) \quad \text{and} \quad \frac{\partial^4 v}{\partial r^4} \in L_{3-\frac{1}{p}}^p(S_{\omega, \rho}).$$

### 3 Reformulations of problem (6)

#### 3.1 Some preliminary calculus

Let us introduce the auxiliary function  $\frac{v}{r}$ . Then

$$\begin{aligned} \left( r \frac{\partial}{\partial r} \right)^2 \left( \frac{v}{r} \right) &= \left( r \frac{\partial}{\partial r} \right) \left( r \frac{\partial}{\partial r} \right) \left( \frac{v}{r} \right) \\ &= \frac{v}{r} - \frac{\partial v}{\partial r} + r \frac{\partial^2 v}{\partial r^2} \\ &= \frac{v}{r} - 2 \frac{\partial}{\partial r} \left( r \cdot \frac{v}{r} \right) + r \left( \frac{1}{r} \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial r^2} \right) \\ &= -\frac{v}{r} - 2r \frac{\partial}{\partial r} \left( \frac{v}{r} \right) + r \left( \frac{1}{r} \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial r^2} \right), \end{aligned}$$

so

$$\Delta u = \left( \frac{1}{r} \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial r^2} \right) + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} = \frac{1}{r} \left[ \left( r \frac{\partial}{\partial r} \right)^2 \left( \frac{v}{r} \right) + \frac{v}{r} + 2r \frac{\partial}{\partial r} \left( \frac{v}{r} \right) \right] + \frac{1}{r} \frac{\partial^2}{\partial \theta^2} \left( \frac{v}{r} \right).$$

Moreover, we have

$$\begin{aligned}
\Pi_1 &:= \left(r \frac{\partial}{\partial r}\right)^2 \left[ \left(r \frac{\partial}{\partial r}\right)^2 \left[\frac{v}{r}\right] \right] \\
&= \left(r \frac{\partial}{\partial r}\right)^2 \left[ \left(r \frac{\partial}{\partial r}\right) \left(-\frac{v}{r} + \frac{\partial v}{\partial r}\right) \right] \\
&= \left(r \frac{\partial}{\partial r}\right)^2 \left[ \frac{v}{r} - \frac{\partial v}{\partial r} + r \frac{\partial^2 v}{\partial r^2} \right] \\
&= \left(r \frac{\partial}{\partial r}\right) \left[ -\frac{v}{r} + \frac{\partial v}{\partial r} + r^2 \frac{\partial^3 v}{\partial r^3} \right] \\
&= \frac{v}{r} - \frac{\partial v}{\partial r} + r \frac{\partial^2 v}{\partial r^2} + 2r^2 \frac{\partial^3 v}{\partial r^3} + r^3 \frac{\partial^4 v}{\partial r^4},
\end{aligned}$$

and

$$\begin{aligned}
\Pi_2 &:= 2 \left(\frac{\partial^2}{\partial \theta^2} - 1\right) \left(r \frac{\partial}{\partial r}\right)^2 \left(\frac{v}{r}\right) + \left(\frac{\partial^2}{\partial \theta^2} + 1\right)^2 \left(\frac{v}{r}\right) \\
&= 2 \left(\frac{\partial^2}{\partial \theta^2} - 1\right) \left(\frac{v}{r} - \frac{\partial v}{\partial r} + r \frac{\partial^2 v}{\partial r^2}\right) + \frac{1}{r} \left(\frac{\partial^2}{\partial \theta^2} + 1\right)^2 v \\
&= 2 \frac{\partial v}{\partial r} - 2r \frac{\partial^2 v}{\partial r^2} + \frac{4}{r} \frac{\partial^2 v}{\partial \theta^2} - 2 \frac{\partial^3 v}{\partial \theta^2 \partial r} + 2r \frac{\partial^4 v}{\partial \theta^2 \partial r^2} + \frac{1}{r} \frac{\partial^4 v}{\partial \theta^4} - \frac{v}{r}.
\end{aligned}$$

Therefore, we obtain that

$$r^3 \Delta^2 u = \Pi_1 + \Pi_2.$$

We set

$$w = \frac{v}{r}.$$

Then, in  $S_{\omega, \rho}$ , problem (6) writes

$$\begin{cases} \frac{1}{r^3} \left[ \left(r \frac{\partial}{\partial r}\right)^2 \left[ \left(r \frac{\partial}{\partial r}\right)^2 w \right] + 2 \left(\frac{\partial^2}{\partial \theta^2} - 1\right) \left(r \frac{\partial}{\partial r}\right)^2 w + \left(\frac{\partial^2}{\partial \theta^2} + 1\right)^2 w \right] \\ - \frac{k}{r} \left[ \left(\left(r \frac{\partial}{\partial r}\right)^2 w + 2 \left(r \frac{\partial}{\partial r}\right) w + w\right) + \frac{\partial^2 w}{\partial \theta^2} \right] = g \\ w(r, 0) = w(r, \omega) = \frac{\partial w}{\partial \theta}(r, 0) = \frac{\partial w}{\partial \theta}(r, \omega) = 0 \\ w(\rho, \theta) \text{ given.} \end{cases} \quad (8)$$

### 3.2 Vector formulation of problem (8)

Now, let us consider the vector variable  $\Psi(r, \theta)$ :

$$\Psi = \begin{pmatrix} w \\ \left(r \frac{\partial}{\partial r}\right)^2 w \end{pmatrix},$$

and the following matrix

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ -\left(\frac{\partial^2}{\partial \theta^2} + 1\right)^2 & -2\left(\frac{\partial^2}{\partial \theta^2} - 1\right) \end{pmatrix}.$$

We have

$$\begin{aligned}
& \left(r \frac{\partial}{\partial r}\right)^2 \Psi - \mathcal{A}\Psi \\
&= \begin{pmatrix} \left(r \frac{\partial}{\partial r}\right)^2 w \\ \left(r \frac{\partial}{\partial r}\right)^2 \left(r \frac{\partial}{\partial r}\right)^2 w \end{pmatrix} + \begin{pmatrix} -\left(r \frac{\partial}{\partial r}\right)^2 w \\ \left(\frac{\partial^2}{\partial \theta^2} + 1\right)^2 w + 2\left(\frac{\partial^2}{\partial \theta^2} - 1\right)\left(r \frac{\partial}{\partial r}\right)^2 w \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \left(r \frac{\partial}{\partial r}\right)^2 \left(r \frac{\partial}{\partial r}\right)^2 w + 2\left(\frac{\partial^2}{\partial \theta^2} - 1\right)\left(r \frac{\partial}{\partial r}\right)^2 w + \left(\frac{\partial^2}{\partial \theta^2} + 1\right)^2 w \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \Pi_1 + \Pi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ r^3 \Delta^2 u \end{pmatrix}.
\end{aligned}$$

We set

$$\mathcal{A}_0 = \begin{pmatrix} 0 & 0 \\ \frac{\partial^2}{\partial \theta^2} + 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathcal{B}_0 = \begin{pmatrix} 0 & 0 \\ 2\left(r \frac{\partial}{\partial r}\right) & 0 \end{pmatrix}.$$

We will precise the domain of all these operators in section 3.4. It is clear that the action of these operators are independent. Then

$$(\mathcal{A}_0 + \mathcal{B}_0) \Psi = \begin{pmatrix} 0 & 0 \\ \frac{\partial^2}{\partial \theta^2} + 1 & 1 \end{pmatrix} \begin{pmatrix} w \\ \left(r \frac{\partial}{\partial r}\right)^2 w \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 2\left(r \frac{\partial}{\partial r}\right) & 0 \end{pmatrix} \begin{pmatrix} w \\ \left(r \frac{\partial}{\partial r}\right)^2 w \end{pmatrix}$$

hence

$$\begin{aligned}
(\mathcal{A}_0 + \mathcal{B}_0) \Psi &= \begin{pmatrix} 0 \\ \frac{\partial^2}{\partial \theta^2} w + w + \left(r \frac{\partial}{\partial r}\right)^2 w \end{pmatrix} + \begin{pmatrix} 0 \\ 2\left(r \frac{\partial}{\partial r}\right) w \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 2r \frac{\partial}{\partial r} w + \frac{\partial^2}{\partial \theta^2} w + w + \left(r \frac{\partial}{\partial r}\right)^2 w \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ r \Delta u \end{pmatrix} = r \begin{pmatrix} 0 \\ \Delta u \end{pmatrix}.
\end{aligned}$$

The generalized diffusion equation writes as

$$\frac{1}{r^3} \left[ \left(r \frac{\partial}{\partial r}\right)^2 \Psi - \mathcal{A}\Psi \right] - \frac{k}{r} (\mathcal{A}_0 + \mathcal{B}_0) \Psi = \begin{pmatrix} 0 \\ g \end{pmatrix}.$$

Finally, we obtain the following complete equation

$$\left[ \left(r \frac{\partial}{\partial r}\right)^2 \Psi - \mathcal{A}\Psi \right] - kr^2 \mathcal{A}_0 \Psi - kr^2 \mathcal{B}_0 \Psi = \begin{pmatrix} 0 \\ r^3 g \end{pmatrix}. \quad (9)$$

Note that linear operators  $\mathcal{A}$  and  $\mathcal{A}_0$  act with respect to variable  $\theta$  whereas operator  $\mathcal{B}_0$  acts with respect to variable  $r \frac{\partial}{\partial r}$ .

### 3.3 New formulation in a finite cone

We apply the following variables and functions change

$$r = \rho e^{-t}, \quad \phi(t, \theta) = w(\rho e^{-t}, \theta) \quad \text{and} \quad g(\rho e^{-t}, \theta) = G(t, \theta)$$

then, it is easy to verify that

$$\left( r \frac{\partial}{\partial r} \right) w = -\frac{\partial \phi}{\partial t} \quad \text{and} \quad \left( r \frac{\partial}{\partial r} \right)^2 w = \left[ r \frac{\partial}{\partial r} + r^2 \frac{\partial^2}{\partial r^2} \right] w = \frac{\partial^2 \phi}{\partial t^2}.$$

We set  $\Phi(t, \theta) = \Psi(\rho e^{-t}, \theta)$ ; then

$$\Phi = \begin{pmatrix} \phi \\ \frac{\partial^2 \phi}{\partial t^2} \end{pmatrix}.$$

Therefore, equation (9) is now set on the strip  $\Sigma = (0, +\infty) \times (0, \omega)$  and writes

$$\begin{aligned} & \left[ \left( r \frac{\partial}{\partial r} \right)^2 \Psi - \mathcal{A}\Psi \right] - kr^2 \mathcal{A}_0 \Psi - kr^2 \mathcal{B}_0 [\Psi] \\ &= \left[ \frac{\partial^2 \Phi}{\partial t^2} - \mathcal{A}\Phi \right] - k\rho^2 e^{-2t} \mathcal{A}_0 \Phi + k\rho^2 e^{-2t} \mathcal{B}_1 [\Phi] = \begin{pmatrix} 0 \\ \rho^3 e^{-3t} G \end{pmatrix}, \end{aligned}$$

where

$$\mathcal{B}_1 = \begin{pmatrix} 0 & 0 \\ -2 \frac{\partial}{\partial t} & 0 \end{pmatrix}.$$

The boundary conditions, in problem (8), on  $w$  become

$$\phi(r, 0) = \phi(r, \omega) = \frac{\partial \phi}{\partial \theta}(r, 0) = \frac{\partial \phi}{\partial \theta}(r, \omega) = 0.$$

As usual, we will use the vector valued notation:

$$\Phi(t)(\theta) := \Phi(t, \theta) = \begin{pmatrix} \phi(t, \theta) \\ \frac{\partial^2 \phi}{\partial t^2}(t, \theta) \end{pmatrix} = \begin{pmatrix} \phi(t, \cdot) \\ \frac{\partial^2 \phi}{\partial t^2}(t, \cdot) \end{pmatrix}(\theta) := \begin{pmatrix} \phi(t) \\ \frac{\partial^2 \phi}{\partial t^2}(t) \end{pmatrix}(\theta).$$

Hence, we deduce the following abstract vector valued equation

$$[\Phi''(t) - \mathcal{A}\Phi(t)] - k\rho^2 e^{-2t} \mathcal{A}_0 \Phi(t) + k\rho^2 e^{-2t} [\mathcal{B}_1 \Phi](t) = \begin{pmatrix} 0 \\ \rho^3 e^{-3t} G(t) \end{pmatrix},$$

where

$$[\mathcal{B}_1 \Phi](t) = \begin{pmatrix} 0 & 0 \\ -2 \frac{\partial}{\partial t} & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \frac{\partial^2 \phi}{\partial t^2} \end{pmatrix}(t) = \begin{pmatrix} 0 \\ -2 \frac{\partial \phi}{\partial t}(t) \end{pmatrix}.$$

Note that  $\mathcal{A}$  and  $\mathcal{A}_0$  act on  $\Phi(t)$  with respect to  $\theta$ , while  $\mathcal{B}_1$  acts on  $\Phi$  with respect to  $t$ .

To determine completely  $\Psi$  in a suitable space and also  $\Phi$ , it is necessary to give us a boundary condition on  $\Psi$  in  $\rho$ . It means that  $\Phi$  is given at  $t = 0$  :

$$\Phi(0) = \Phi_0.$$

Now, we have to solve on  $(0, +\infty)$  the following problem

$$\begin{cases} \Phi''(t) - \mathcal{A}\Phi(t) - k\rho^2 e^{-2t} \mathcal{A}_0 \Phi(t) + k\rho^2 e^{-2t} [\mathcal{B}_1 \Phi](t) = \begin{pmatrix} 0 \\ \rho^3 e^{-3t} G(t) \end{pmatrix} \\ \Phi(0) = \Phi_0. \end{cases} \quad (10)$$



**Remark 3.1.** Note that the boundary condition at  $t = +\infty$ , will be included in the vector valued space containing the solution  $\Phi$ .

### 3.4 Sums of linear operators

In this section, we are going to write problem (10) as a sum of linear operators, firstly in the following Banach space

$$X = W_0^{2,p}(0, \omega) \times L^p(0, \omega),$$

see (12) below and secondly, in  $L^p(0, +\infty; X)$ , see (13) below.

Here  $X$  is equipped, for instance, with the following norm

$$\left\| \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_X = \|\psi_1\|_{W_0^{2,p}(0, \omega)} + \|\psi_2\|_{L^p(0, \omega)},$$

where

$$W_0^{2,p}(0, \omega) = \left\{ \varphi \in W^{2,p}(0, \omega) : \varphi(0) = \varphi(\omega) = \varphi'(0) = \varphi'(\omega) = 0 \right\}.$$

Then, we define the linear operator  $\mathcal{A}$  by

$$\begin{cases} D(\mathcal{A}) &= [W^{4,p}(0, \omega) \cap W_0^{2,p}(0, \omega)] \times W_0^{2,p}(0, \omega) \subset X \\ \mathcal{A} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} &= \begin{pmatrix} \psi_2 \\ -\left(\frac{\partial^2}{\partial \theta^2} + 1\right)^2 \psi_1 - 2\left(\frac{\partial^2}{\partial \theta^2} - 1\right) \psi_2 \end{pmatrix}, \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in D(\mathcal{A}). \end{cases}$$

In the same way, we define operator  $\mathcal{A}_0$  by

$$\begin{cases} D(\mathcal{A}_0) &= W_0^{2,p}(0, \omega) \times L^p(0, \omega) = X \\ \mathcal{A}_0 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ \left(\frac{\partial^2}{\partial \theta^2} + 1\right) \psi_1 + \psi_2 \end{pmatrix}, \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in D(\mathcal{A}_0). \end{cases}$$

It is clear that  $D(\mathcal{A}) \subset D(\mathcal{A}_0)$ . Note that operator  $\mathcal{A}_0$  is continuous from  $X$  into itself since

$$\begin{aligned} \left\| \mathcal{A}_0 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_X &= \left\| \begin{pmatrix} 0 \\ \left(\frac{\partial^2}{\partial \theta^2} + 1\right) \psi_1 + \psi_2 \end{pmatrix} \right\|_X = \left\| \left(\frac{\partial^2}{\partial \theta^2} + 1\right) \psi_1 + \psi_2 \right\|_{L^p(0, \omega)} \\ &\leq \|\psi_1\|_{W_0^{2,p}(0, \omega)} + \|\psi_2\|_{L^p(0, \omega)} = \left\| \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_X. \end{aligned}$$

Equation (10) is set in the Banach space  $X$ .

Recall that the second member in problem (2) satisfies

$$f \in L^p(S_{\omega, \rho}), \quad \text{for } p \in (1, +\infty).$$

Set

$$t \mapsto e^{-3t}G(t)(\cdot) = e^{-3t}G(t, \cdot) = H(t, \cdot) = H(t)(\cdot).$$

Therefore, we have

$$\begin{aligned}
\int_{S_{\omega,\rho}} |f(x,y)|^p dx dy &= \int_{S_{\omega,\rho}} |g(r,\theta)|^p r dr d\theta = \rho^2 \int_{\Sigma} |G(t,\theta)|^p e^{-2t} dt d\theta \\
&= \rho^2 \int_{\Sigma} \left| e^{(3-\frac{2}{p})t} H(t,\theta) \right|^p dt d\theta \\
&= \rho^2 \int_0^{+\infty} \left| e^{(3-\frac{2}{p})t} \right|^p \left[ \left( \int_0^\omega |H(t)(\theta)|^p d\theta \right)^{1/p} \right]^p dt \\
&= \rho^2 \int_0^{+\infty} \left[ e^{(3-\frac{2}{p})t} \|H(t)\|_{L^p(0,\omega)} \right]^p dt.
\end{aligned}$$

It follows that the second member  $H$  is in the weighted Sobolev space

$$\left\{ H : t \mapsto e^{(3-\frac{2}{p})t} H \in L^p(\Sigma) \right\} = L^p_\nu(0, +\infty; L^p(0, \omega)).$$

where

$$\nu = 3 - \frac{2}{p} > 1,$$

is exactly the Sobolev exponent of the space  $W^{3,p}(\Sigma)$  in dimension 2.

Then, since it would not be easy to work in weighted Sobolev spaces, we will use the following new vector valued function :

$$V(t) = e^{\nu t} \Phi(t) = \begin{pmatrix} e^{\nu t} \phi(t) \\ e^{\nu t} \phi''(t) \end{pmatrix} = \begin{pmatrix} V_1(t) \\ V_2(t) \end{pmatrix}. \quad (11)$$

Since we have

$$\Phi(t) = \begin{pmatrix} e^{-\nu t} V_1(t) \\ e^{-\nu t} V_2(t) \end{pmatrix},$$

we deduce that

$$\Phi'(t) = -\nu e^{-\nu t} V(t) + e^{-\nu t} V'(t) = e^{-\nu t} (\partial_t - \nu I) V(t),$$

and

$$\Phi''(t) = \nu^2 e^{-\nu t} V(t) - 2\nu e^{-\nu t} V'(t) + e^{-\nu t} V''(t) = e^{-\nu t} (\partial_t - \nu I)^2 V(t).$$

Moreover, we obtain

$$\begin{aligned}
[\mathcal{B}_1 \Phi](t) &= \begin{pmatrix} 0 & 0 \\ -2\partial_t & 0 \end{pmatrix} \begin{pmatrix} e^{-\nu t} V_1(t) \\ e^{-\nu t} V_2(t) \end{pmatrix} \\
&= e^{-\nu t} \begin{pmatrix} 0 & 0 \\ -2(\partial_t - \nu I) & 0 \end{pmatrix} \begin{pmatrix} V_1(t) \\ V_2(t) \end{pmatrix} \\
&= e^{-\nu t} [\mathcal{B}_2 V](t)
\end{aligned}$$

where

$$\mathcal{B}_2 = \begin{pmatrix} 0 & 0 \\ -2(\partial_t - \nu I) & 0 \end{pmatrix}.$$

Hence, problem (10) becomes

$$\begin{cases} e^{-\nu t} (\partial_t - \nu I)^2 V(t) - e^{-\nu t} \mathcal{A} V(t) - k\rho^2 e^{-\nu t} e^{-2t} \mathcal{A}_0 V(t) \\ -k\rho^2 e^{-\nu t} e^{-2t} [(\mathcal{B}_2 V)](t) = \begin{pmatrix} 0 \\ \rho^3 H(t) \end{pmatrix} \\ V(0) = \Phi_0, \end{cases}$$

then

$$\begin{cases} (\partial_t - \nu I)^2 V(t) - \mathcal{A}V(t) - k\rho^2 e^{-2t} \mathcal{A}_0 V(t) - k\rho^2 e^{-2t} [(\mathcal{B}_2 V)](t) = \begin{pmatrix} 0 \\ \rho^3 e^{\nu t} H(t) \end{pmatrix} \\ V(0) = \Phi_0. \end{cases}$$

Note that Remark 3.1 holds true for  $V$ .

We will be interested, in particular, in the following homogeneous problem

$$\begin{cases} (\partial_t - \nu I)^2 V(t) - \mathcal{A}V(t) - k\rho^2 e^{-2t} \mathcal{A}_0 V(t) - k\rho^2 e^{-2t} [(\mathcal{B}_2 V)](t) = \begin{pmatrix} 0 \\ \rho^3 e^{\nu t} H(t) \end{pmatrix} \\ V(0) = 0. \end{cases} \quad (12)$$

We have

$$t \mapsto \rho^3 e^{\nu t} H(t) \in L^p(\Sigma) = L^p(0, +\infty; L^p(0, \omega)),$$

and

$$t \mapsto \begin{pmatrix} 0 \\ \rho^3 e^{\nu t} H(t) \end{pmatrix} \in L^p(0, +\infty; W_0^{2,p}(0, \omega) \times L^p(0, \omega)) = L^p(0, +\infty; X).$$

Finally, let us introduce the following abstract linear operators:

$$\begin{cases} D(\mathcal{L}_1) & = \{V \in W^{2,p}(0, +\infty; X) : V(0) = V(+\infty) = 0\} \\ [\mathcal{L}_1(V)](t) & = (\partial_t - \nu I)^2 V(t) = V''(t) - 2\nu V'(t) + \nu^2 V(t), \end{cases}$$

with  $\nu = 3 - \frac{2}{p} \in (1, 3)$ ,

$$\begin{cases} D(\mathcal{L}_2) & = \{V \in L^p(0, +\infty; X) : \text{for a.e. } t \in (0, +\infty), V(t) \in D(\mathcal{A})\} \\ [\mathcal{L}_2(V)](t) & = -\mathcal{A}V(t), \end{cases}$$

$$\begin{cases} D(\mathcal{P}_1) & = \{V \in L^p(0, +\infty; X) : \text{for a.e. } t \in (0, +\infty), V(t) \in D(\mathcal{A}_0)\} \\ [\mathcal{P}_1(V)](t) & = -e^{-2t} \mathcal{A}_0 V(t), \end{cases}$$

and

$$\begin{cases} D(\mathcal{P}_2) & = W^{1,p}(0, +\infty; X) \\ [\mathcal{P}_2(V)](t) & = -e^{-2t} (\mathcal{B}_2 V)(t). \end{cases}$$

Then, problem (12) can be written as the following abstract equation

$$(\mathcal{L}_1 + \mathcal{L}_2) V + k\rho^2 (\mathcal{P}_1 + \mathcal{P}_2) V = \mathcal{F}, \quad (13)$$

set in  $L^p(0, +\infty; X)$ , with  $p \in (1, +\infty)$ , where, for almost every  $t \in (0, +\infty)$

$$\mathcal{F}(t) = \begin{pmatrix} 0 \\ \rho^3 e^{\sigma t} H(t) \end{pmatrix}.$$

## 4 Proof of Theorem 2.2

### 4.1 Resolution of equation (13)

Equation (13) will be completely studied in the second part of this work by using the sum theory of linear operators, where the main result described by Theorem 1.1 in [6] states that

there exists  $\rho_0 > 0$  such that for all  $\rho \in (0, \rho_0]$ , there exists a unique solution  $V \in D(\mathcal{L}_1 + \mathcal{L}_2)$  to equation (13) that is

$$V \in W^{2,p}(0, +\infty; X) \cap L^p(0, +\infty; D(\mathcal{A})).$$

Thus, we know that there exists a continuous extension from  $W^{2,p}(0, +\infty; X)$  into  $W^{2,p}(\mathbb{R}; X)$  and also from  $L^p(0, +\infty; D(\mathcal{A}))$  into  $L^p(\mathbb{R}; D(\mathcal{A}))$ ; it suffices, for instance to use the well-known Babich techniques.

Set  $\tilde{V}$ , this extension of  $V$  which writes

$$\tilde{V} = \begin{pmatrix} \tilde{V}_1 \\ \tilde{V}_2 \end{pmatrix}.$$

Then,

$$\tilde{V}_1 \in W^{2,p}(\mathbb{R}; W_0^{2,p}(0, \omega)) \cap L^p(\mathbb{R}; W^{4,p}(0, \omega) \cap W_0^{2,p}(0, \omega)),$$

and

$$\tilde{V}_2 \in W^{2,p}(\mathbb{R}; L^p(0, \omega)) \cap L^p(\mathbb{R}; W_0^{2,p}(0, \omega)).$$

We deduce, for instance, that

$$\tilde{V}_1, \tilde{V}_2 \in W^{2,p}(\mathbb{R} \times (0, \omega)),$$

by using the Mihlin's theorem (see [8]) and

$$\tilde{V}_1 \in W^{2,p}(\mathbb{R}; W_0^{2,p}(0, \omega)) \cap L^p(\mathbb{R}; W^{4,p}(0, \omega)).$$

Therefore, we deduce that  $V_1$  and  $V_2$  have the same regularities on  $(0, \rho) \times (0, \omega)$ , with  $\rho \in (0, \rho_0]$ .

## 4.2 Regularity of $V(t, \theta) = V(t)(\theta)$

Recall that, from (11), we have

$$V(t) = e^{\nu t} \Phi(t) \quad \text{and} \quad \Phi(t) = e^{-\nu t} V(t).$$

Moreover, since  $r = \rho e^{-t}$ , we have

$$V_1(t, \theta) = \frac{v(\rho e^{-t}, \theta)}{(\rho e^{-t})^{\nu+1}},$$

where  $\nu = 3 - \frac{2}{p} \in (1, 3)$  and

$$V_2(t, \theta) = \left( r \frac{\partial}{\partial r} \right)^2 \left( \frac{v}{r} \right) (\rho e^{-t}, \theta).$$

### 4.2.1 Regularity of $V_1$

Here, we explicit the fact that

$$V_1 \in W^{2,p}((0, +\infty) \times (0, \omega)). \tag{14}$$

We have

$$\int_0^{+\infty} \int_0^\omega |V_1(t, \theta)|^p d\theta dt = \int_0^{+\infty} \int_0^\omega \left| \frac{v(\rho e^{-t}, \theta)}{\rho^{\nu+1} e^{-t(\nu+1)}} \right|^p d\theta dt < +\infty.$$

Setting  $r = \rho e^{-t}$ , we obtain

$$\begin{aligned} \int_0^{+\infty} \int_0^\omega \left| \frac{v(\rho e^{-t}, \theta)}{(\rho e^{-t})^{\nu+1}} \right|^p d\theta dt &= \int_0^\rho \int_0^\omega \left| \frac{v(r, \theta)}{r^{\nu+1+\frac{1}{p}}} \right|^p d\theta dr \\ &= \int_0^\rho \int_0^\omega r^{-4p+1} |v(r, \theta)|^p d\theta dr. \end{aligned}$$

Then, we have

$$v \in L_{\gamma_0}^p(S_{\omega, \rho}), \quad (15)$$

where  $\gamma_0 = -4 + \frac{1}{p}$ . Moreover

$$\begin{aligned} \frac{\partial V_1}{\partial t}(t, \theta) &= \frac{1}{\rho^{\nu+1}} \frac{\partial}{\partial t} \left( v(\rho e^{-t}, \theta) e^{(\nu+1)t} \right) \\ &= \frac{1}{\rho^{\nu+1}} \left( (\nu+1) e^{(\nu+1)t} v(\rho e^{-t}, \theta) - \rho e^{\nu t} \frac{\partial v}{\partial r}(\rho e^{-t}, \theta) \right), \end{aligned}$$

hence

$$\frac{1}{(\rho e^{-t})^\nu} \frac{\partial v}{\partial r}(\rho e^{-t}, \theta) = \frac{(\nu+1)}{(\rho e^{-t})^{\nu+1}} v(\rho e^{-t}, \theta) - \frac{\partial V_1(t, \theta)}{\partial t}.$$

Thus, in virtue of (14) and (15), it follows that

$$(t, \theta) \mapsto \frac{1}{(\rho e^{-t})^\nu} \frac{\partial v}{\partial r}(\rho e^{-t}, \theta) \in L^p((0, +\infty) \times (0, \omega)),$$

and

$$\begin{aligned} \int_0^{+\infty} \int_0^\omega \left| \frac{1}{(\rho e^{-t})^\nu} \frac{\partial v}{\partial r}(\rho e^{-t}, \theta) \right|^p d\theta dt &= \int_0^\rho \int_0^\omega \left| \frac{1}{r^{\nu+\frac{1}{p}}} \frac{\partial v}{\partial r}(r, \theta) \right|^p d\theta dr \\ &= \int_0^\rho \int_0^\omega r^{-3p+1} \left| \frac{\partial v}{\partial r}(r, \theta) \right|^p d\theta dr. \end{aligned}$$

So, we obtain

$$\frac{\partial v}{\partial r} \in L_{\gamma_1}^p(S_{\omega, \rho}), \quad (16)$$

where  $\gamma_1 = -3 + \frac{1}{p}$ . Furthermore, we have

$$\begin{aligned} \frac{\partial^2 V_1}{\partial t^2}(t, \theta) &= \frac{\nu+1}{\rho^{\nu+1}} \frac{\partial}{\partial t} \left( v(\rho e^{-t}, \theta) e^{(\nu+1)t} \right) - \frac{\nu}{\rho^\nu} e^{\nu t} \frac{\partial v}{\partial r}(\rho e^{-t}, \theta) + \frac{\rho}{\rho^\nu} e^{-t} e^{\nu t} \frac{\partial^2 v}{\partial r^2}(\rho e^{-t}, \theta) \\ &= \frac{\nu+1}{\rho^{\nu+1}} \left( (\nu+1) e^{(\nu+1)t} v(\rho e^{-t}, \theta) - \rho e^{\nu t} \frac{\partial v}{\partial r}(\rho e^{-t}, \theta) \right) - \frac{\nu}{\rho^\nu} e^{\nu t} \frac{\partial v}{\partial r}(\rho e^{-t}, \theta) \\ &\quad + \frac{1}{\rho^{\nu-1}} e^{(\nu-1)t} \frac{\partial^2 v}{\partial r^2}(\rho e^{-t}, \theta) \\ &= \frac{(\nu+1)^2}{(\rho e^{-t})^{\nu+1}} v(\rho e^{-t}, \theta) - \frac{2\nu+1}{(\rho e^{-t})^\nu} \frac{\partial v}{\partial r}(\rho e^{-t}, \theta) + \frac{1}{(\rho e^{-t})^{\nu-1}} \frac{\partial^2 v}{\partial r^2}(\rho e^{-t}, \theta), \end{aligned}$$

hence

$$\frac{1}{(\rho e^{-t})^{\nu-1}} \frac{\partial^2 v}{\partial r^2}(\rho e^{-t}, \theta) = \frac{\partial^2 V_1}{\partial t^2}(t, \theta) + \frac{2\nu+1}{(\rho e^{-t})^\nu} \frac{\partial v}{\partial r}(\rho e^{-t}, \theta) - \frac{(\nu+1)^2}{(\rho e^{-t})^{\nu+1}} v(\rho e^{-t}, \theta).$$

Thus, in virtue of (14), (15) and (16), it follows that

$$(t, \theta) \mapsto \frac{1}{(\rho e^{-t})^{\nu-1}} \frac{\partial^2 v}{\partial r^2}(\rho e^{-t}, \theta) \in L^p((0, +\infty) \times (0, \omega)),$$

and

$$\begin{aligned} \int_0^{+\infty} \int_0^\omega \left| \frac{1}{(\rho e^{-t})^{\nu-1}} \frac{\partial^2 v}{\partial r^2}(\rho e^{-t}, \theta) \right|^p d\theta dt &= \int_0^\rho \int_0^\omega \left| \frac{1}{r^{\nu-1+\frac{1}{p}}} \frac{\partial^2 v}{\partial r^2}(r, \theta) \right|^p d\theta dr \\ &= \int_0^\rho \int_0^\omega r^{-2p+1} \left| \frac{\partial^2 v}{\partial r^2}(r, \theta) \right|^p d\theta dr. \end{aligned}$$

So, we obtain

$$\frac{\partial^2 v}{\partial r^2} \in L_{\gamma_2}^p(S_{\omega, \rho}),$$

where  $\gamma_2 = -2 + \frac{1}{p}$ .

On the other hand

$$\frac{\partial V_1}{\partial \theta}(\rho e^{-t}, \theta) = \frac{1}{(\rho e^{-t})^{\nu+1}} \frac{\partial v}{\partial \theta}(\rho e^{-t}, \theta).$$

Thus, it follows that

$$\begin{aligned} \int_0^{+\infty} \int_0^\omega \left| \frac{\partial V_1}{\partial \theta}(\rho e^{-t}, \theta) \right|^p d\theta dt &= \int_0^{+\infty} \int_0^\omega \frac{1}{(\rho e^{-t})^{p(\nu+1)}} \left| \frac{\partial v}{\partial \theta}(\rho e^{-t}, \theta) \right|^p d\theta dt \\ &= \int_0^\rho \int_0^\omega r^{-4p+1} \left| \frac{\partial v}{\partial \theta}(r, \theta) \right|^p d\theta dr. \end{aligned}$$

So, we obtain

$$\frac{\partial v}{\partial \theta} \in L_{\gamma_0}^p(S_{\omega, \rho}).$$

In the same way, we deduce that

$$\frac{\partial^2 v}{\partial \theta^2} \in L_{\gamma_0}^p(S_{\omega, \rho}) \quad \text{and} \quad \frac{\partial^2 v}{\partial r \partial \theta} \in L_{\gamma_1}^p(S_{\omega, \rho}).$$

Now, we explicit the fact that

$$\frac{\partial^3 V_1}{\partial t \partial \theta^2}, \frac{\partial^3 V_1}{\partial t^2 \partial \theta}, \frac{\partial^4 V_1}{\partial t^2 \partial \theta^2} \in L^p((0, +\infty) \times (0, \omega))$$

We have

$$\frac{\partial^3 V_1}{\partial t^2 \partial \theta}(t, \theta) = \frac{(\nu+1)^2}{(\rho e^{-t})^{\nu+1}} \frac{\partial v}{\partial \theta}(\rho e^{-t}, \theta) - \frac{2\nu+1}{(\rho e^{-t})^\nu} \frac{\partial^2 v}{\partial r \partial \theta}(\rho e^{-t}, \theta) + \frac{1}{(\rho e^{-t})^{\nu-1}} \frac{\partial^3 v}{\partial r^2 \partial \theta}(\rho e^{-t}, \theta),$$

hence

$$\frac{1}{(\rho e^{-t})^{\nu-1}} \frac{\partial^3 v}{\partial r^2 \partial \theta}(\rho e^{-t}, \theta) = \frac{\partial^3 V_1}{\partial t^2 \partial \theta}(t, \theta) - \frac{(\nu+1)^2}{(\rho e^{-t})^{\nu+1}} \frac{\partial v}{\partial \theta}(\rho e^{-t}, \theta) + \frac{2\nu+1}{(\rho e^{-t})^\nu} \frac{\partial^2 v}{\partial r \partial \theta}(\rho e^{-t}, \theta).$$

Thus, we obtain

$$\frac{\partial^3 v}{\partial r^2 \partial \theta} \in L_{\gamma_2}^p(S_{\omega, \rho}),$$

and in the same way, we have also

$$\frac{\partial^3 v}{\partial r \partial \theta^2} \in L_{\gamma_1}^p(S_{\omega, \rho}) \quad \text{and} \quad \frac{\partial^4 v}{\partial r^2 \partial \theta^2} \in L_{\gamma_2}^p(S_{\omega, \rho}).$$

Now, we explicit the fact that

$$V_1 \in L^p\left((0, +\infty); W^{4,p}(0, \omega)\right),$$

that is, for all  $i = 1, 2, 3, 4$

$$\int_0^{+\infty} \left\| \frac{\partial^i V_1}{\partial \theta^i}(t, \theta) \right\|_{L^p(0, \omega)}^p dt < +\infty.$$

Then, we have

$$\begin{aligned} \int_0^{+\infty} \int_0^\omega \left| \frac{\partial^i V_1}{\partial \theta^i}(t, \theta) \right|^p d\theta dt &= \int_0^{+\infty} \int_0^\omega \left| \frac{1}{(\rho e^{-t})^{\nu+1}} \frac{\partial^i v}{\partial \theta^i}(\rho e^{-t}, \theta) \right|^p d\theta dt \\ &= \int_0^\rho \int_0^\omega r^{-4p+1} \left| \frac{\partial^i v}{\partial \theta^i}(r, \theta) \right|^p d\theta dr, \end{aligned}$$

which gives

$$\frac{\partial^i v}{\partial \theta^i} \in L_{\gamma_0}^p(S_{\omega, \rho}), \quad \text{for } i = 1, 2, 3, 4.$$

#### 4.2.2 Regularity of $V_2$

In the same way, we explicit the fact that

$$V_2 \in W^{2,p}((0, +\infty) \times (0, \omega)),$$

where

$$V_2(t, \theta) = \left(r \frac{\partial}{\partial r}\right)^2 \left(\frac{v}{r}\right)(\rho e^{-t}, \theta) = r \frac{\partial^2 v}{\partial r^2}(\rho e^{-t}, \theta) - \frac{\partial v}{\partial r}(\rho e^{-t}, \theta) + \frac{v}{r}(\rho e^{-t}, \theta).$$

It is clear, from subsection 4.2.1, that

$$r \frac{\partial^2 v}{\partial r^2}, \frac{\partial v}{\partial r}, \frac{v}{r} \in L_{\gamma_1}^p(S_{\omega, \rho}).$$

Moreover, we have

$$\frac{\partial V_2}{\partial t}(t, \theta) = -r \rho e^{-t} \frac{\partial^3 v}{\partial r^3}(\rho e^{-t}, \theta) + \rho e^{-t} \frac{\partial^2 v}{\partial r^2}(\rho e^{-t}, \theta) - \frac{\rho e^{-t}}{r} \frac{\partial v}{\partial r}(\rho e^{-t}, \theta),$$

hence

$$r \rho e^{-t} \frac{\partial^3 v}{\partial r^3}(\rho e^{-t}, \theta) = \rho e^{-t} \frac{\partial^2 v}{\partial r^2}(\rho e^{-t}, \theta) - \frac{\rho e^{-t}}{r} \frac{\partial v}{\partial r}(\rho e^{-t}, \theta) - \frac{\partial V_2}{\partial t}(t, \theta).$$

Then

$$\begin{aligned} \int_0^{+\infty} \int_0^\omega \left| \rho e^{-t} \frac{\partial^3 v}{\partial r^3}(\rho e^{-t}, \theta) \right|^p d\theta dt &= \int_0^\rho \int_0^\omega \frac{r^{3p-1}}{r} r^{-3p+1} \left| r \frac{\partial^2 v}{\partial r^2}(r, \theta) \right|^p d\theta dr \\ &= \int_0^\rho \int_0^\omega r^{3p-2} r^{-3p+1} \left| r \frac{\partial^2 v}{\partial r^2}(r, \theta) \right|^p d\theta dr \\ &\leq \rho^{3p-2} \int_0^\rho \int_0^\omega r^{-3p+1} \left| r \frac{\partial^2 v}{\partial r^2}(r, \theta) \right|^p d\theta dr < +\infty, \end{aligned}$$

and

$$\begin{aligned} \int_0^{+\infty} \int_0^\omega \left| \frac{\rho e^{-t}}{r} \frac{\partial v}{\partial r}(\rho e^{-t}, \theta) \right|^p d\theta dt &= \int_0^\rho \int_0^\omega \frac{r^{3p-1}}{r} r^{-3p+1} \left| \frac{\partial v}{\partial r}(r, \theta) \right|^p d\theta dr \\ &\leq \rho^{3p-2} \int_0^\rho \int_0^\omega r^{-3p+1} \left| \frac{\partial v}{\partial r}(r, \theta) \right|^p d\theta dr < +\infty. \end{aligned}$$

It follows that

$$\int_0^{+\infty} \int_0^\omega \left| r \rho e^{-t} \frac{\partial^3 v}{\partial r^3}(\rho e^{-t}, \theta) \right|^p d\theta dt = \int_0^\rho \int_0^\omega r^{2p-1} \left| \frac{\partial^3 v}{\partial r^3}(r, \theta) \right|^p d\theta dr < +\infty,$$

which means that

$$\frac{\partial^3 v}{\partial r^3} \in L_{\gamma_3}^p(S_{\omega, \rho}),$$

where  $\gamma_3 = 2 - \frac{1}{p}$ .

In the same way, we have

$$\frac{\partial^4 v}{\partial r \partial \theta^3} \in L_{\gamma_1}^p(S_{\omega, \rho}) \quad \text{and} \quad \frac{\partial^4 v}{\partial r^3 \partial \theta} \in L_{\gamma_3}^p(S_{\omega, \rho}).$$

Furthermore, we have

$$\begin{aligned} \frac{\partial^2 V_2}{\partial t^2}(t, \theta) &= r \rho e^{-t} \frac{\partial^3 v}{\partial r^3}(\rho e^{-t}, \theta) + r (\rho e^{-t})^2 \frac{\partial^4 v}{\partial r^4}(\rho e^{-t}, \theta) - \rho e^{-t} \frac{\partial^2 v}{\partial r^2}(\rho e^{-t}, \theta) \\ &\quad - (\rho e^{-t})^2 \frac{\partial^3 v}{\partial r^3}(\rho e^{-t}, \theta) + \frac{\rho e^{-t}}{r} \frac{\partial v}{\partial r}(\rho e^{-t}, \theta) + \frac{(\rho e^{-t})^2}{r} \frac{\partial^2 v}{\partial r^2}(\rho e^{-t}, \theta), \end{aligned}$$

hence

$$\int_0^{+\infty} \int_0^\omega \left| r (\rho e^{-t})^2 \frac{\partial^4 v}{\partial r^4}(\rho e^{-t}, \theta) \right|^p d\theta dt = \int_0^\rho \int_0^\omega r^{3p-1} \left| \frac{\partial^4 v}{\partial r^4}(r, \theta) \right|^p d\theta dr < +\infty.$$

Then

$$\frac{\partial^4 v}{\partial r^4} \in L_{\gamma_4}^p(S_{\omega, \rho}),$$

where  $\gamma_4 = 3 - \frac{1}{p}$ .

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