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► **To cite this version:**

Clément Foucart, Matija Vidmar. CONTINUOUS-STATE BRANCHING PROCESSES WITH COLLISIONS: FIRST PASSAGE TIMES AND DUALITY. 2022. hal-03719207

HAL Id: hal-03719207

<https://hal.science/hal-03719207>

Preprint submitted on 11 Jul 2022

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CONTINUOUS-STATE BRANCHING PROCESSES WITH COLLISIONS: FIRST PASSAGE TIMES AND DUALITY

CLÉMENT FOUCART AND MATIJA VIDMAR

ABSTRACT. We introduce a class of one-dimensional positive Markov processes generalizing continuous-state branching processes (CBs), by taking into account a phenomenon of random collisions. Besides branching, characterized by a general mechanism Ψ , at a constant rate in time two particles are sampled uniformly in the population, collide and leave a mass of particles governed by a (sub)critical mechanism Σ . Such CB processes with collisions (CBCs) are shown to be the only Feller processes without negative jumps satisfying a Laplace duality relationship with one-dimensional diffusions on the half-line. This generalizes the duality observed for logistic CBs in Foucart [18]. Via time-change, CBCs are also related to an auxiliary class of Markov processes, called CB processes with spectrally positive migration (CBMs), recently introduced in Vidmar [52]. We find necessary and sufficient conditions for the boundaries 0 or ∞ to be attracting and for a limiting distribution to exist. The Laplace transform of the latter is provided. Under the assumption that the CBC process does not explode, the Laplace transforms of the first passage times below arbitrary levels are represented with the help of the solution of a second-order differential equation, whose coefficients are given in terms of the Lévy-Khintchine functions Σ and Ψ . Sufficient conditions for non-explosion are given.

1. INTRODUCTION

1.1. Motivation. Imagine a set of particles evolving according to the following two rules: branching occurs at random, and whenever a particle splits (dies), it begets $k \in \mathbb{Z}_+ := \{0, 1, \dots\}$ new particles with probability $p_b(k)$. In the time interval $(t, t + \Delta t)$, the probability for a branching of any given particle to occur is of order $b\Delta t + o(\Delta t)$. Multiple branching events occur in this time interval with probability $o(\Delta t)$. Particles are also allowed to collide. Whenever a collision between two particles occurs, the pair is replaced by $k + 1$ new particles with probability $p_c(k)$, $k \in \mathbb{Z}_+$. The average number of particles left after a collision is assumed to be less than or equal to two: $\sum_{k \in \mathbb{Z}_+} k p_c(k) \leq 1$. The probability of a single collision to occur for any given pair of individuals during $(t, t + \Delta t)$ is of order $c\Delta t + o(\Delta t)$, and for multiple collisions it is $o(\Delta t)$. All collisions and branching events are assumed to occur independently from each other. The infinitesimal generator \mathcal{L} of the continuous-time Markov chain recording the number of particles then takes the form

$$\mathcal{L}f(n) := c \binom{n}{2} \sum_{k=0}^{\infty} (f(n+k-1) - f(n)) p_c(k) + bn \sum_{k=0}^{\infty} (f(n+k-1) - f(n)) p_b(k),$$

for bounded $f : \mathbb{Z}_+ \rightarrow \mathbb{R}$ and $n \in \mathbb{Z}_+$, with $\binom{n}{2} = 0$ if $n \in \{0, 1\}$. Processes with generator \mathcal{L} have been considered by Chen et al. in [6, 7, 8], and they recently reappeared in the works of González-Casanova et al. [23] and Berzunza Ojeda and Pardo [2]. The study in the two latter works makes use of a moment-duality relationship with some generalized Wright-Fisher diffusions.

2020 *Mathematics Subject Classification.* Primary: 60J80. Secondary: 60J50, 60J60, 60J70, 60G53, 92D25.

Key words and phrases. Continuous-state branching process, branching process with interactions, first passage time, Laplace duality, Lamperti time-change, one-dimensional diffusion, Lévy-Khintchine function.

foucart@math.univ-paris13.fr, Université Paris 13, Laboratoire Analyse, Géométrie & Applications UMR 7539 Institut Galilée.

matija.vidmar@fmf.uni-lj.si, Department of Mathematics, Faculty of Mathematics and Physics, University of Ljubljana.

Our aim is to introduce and initiate the study of the continuous-state space counterparts of these processes and to shed light on some new duality relationships with certain diffusions on the positive halfline whose drift and diffusion functions are of the Lévy-Khintchine form. We adopt here the terminology *collision* coined in [8]. The terms negative cooperation or pairwise interactions are also used in the literature.

1.2. A stochastic equation. In order to explain how collisions are encoded in the continuous-state space setting we shall work in Dawson and Li's framework of SDEs, see [11], and consider the following generalisation of the stochastic equation solved by a continuous-state branching process (CB) with branching mechanism Ψ (a $\text{CB}(\Psi)$), for a starting value $z \in [0, \infty)$,

$$\begin{aligned} Z_t = & z + \sigma \int_0^t \sqrt{Z_s} dB_s + b \int_0^t Z_s ds + \int_0^t \int_0^{Z_{s-}} \int_0^1 h \bar{\mathcal{N}}(ds, du, dh) + \int_0^t \int_0^{Z_{s-}} \int_1^\infty h \mathcal{N}(ds, du, dh) \\ & + a \int_0^t Z_s dW_s - \frac{c}{2} \int_0^t Z_s^2 ds + \int_0^t \int_0^{Z_{s-}} \int_0^{Z_{s-}} \int_0^\infty h \bar{\mathcal{M}}(ds, du_1, du_2, dh). \end{aligned} \quad (1.1)$$

In (1.1) we adhere to the convention that the lower delimiters are excluded from the integration, while the upper delimiters are included (except for ∞ , which is of course excluded). The individual ingredients of (1.1) are specified as follows.

The first line in Eq. (1.1) represents the branching dynamics and forms the classical stochastic equation solved by a CB (without the need for a finite first moment, see e.g. Ji and Li [29, Theorem 3.1]): the parameters $b \in \mathbb{R}$, $\sigma \in [0, \infty)$ are the diffusive coefficients, B is a Brownian motion, finally $\mathcal{N}(ds, du, dh)$ is an independent Poisson random measure on $[0, \infty)^2 \times (0, \infty)$ with intensity $ds du \pi(dh)$, π being a measure on $(0, \infty)$ satisfying $\int_0^\infty 1 \wedge h^2 \pi(dh) < \infty$ (a Lévy measure) and $\bar{\mathcal{N}}$ stands for the compensated random measure, $\bar{\mathcal{N}}(ds, du, dh) := \mathcal{N}(ds, du, dh) - ds du \pi(dh)$.¹ Heuristically, prior to an atom of time t of \mathcal{N} , an individual u is chosen uniformly in $[0, Z_{t-}]$ and reproduces or dies. The branching part is governed by the Lévy-Khintchine function

$$\Psi(x) := \frac{\sigma^2}{2} x^2 - bx + \int_0^\infty (e^{-xh} - 1 + xh \mathbb{1}_{\{h \leq 1\}}) \pi(dh), \quad x \in [0, \infty). \quad (1.2)$$

The second line in Eq. (1.1) represents *collisions*: again the parameters $a \in [0, \infty)$, $c \in [0, \infty)$ are the diffusive coefficients, W is a Brownian motion, finally $\mathcal{M}(ds, du_1, du_2, dh)$ is an independent Poisson random measure on $[0, \infty)^3 \times (0, \infty)$ with intensity $ds du_1 du_2 \eta(dh)$, η being a Lévy measure on $(0, \infty)$ satisfying $\int_0^\infty h \wedge h^2 \eta(dh) < \infty$. The stochastic drivers (B, \mathcal{N}) and (W, \mathcal{M}) are defined under a common probability \mathbb{P} and are independent of one-another, i.e. collisions are independent of the branching. Heuristically, prior to an atom of time t of \mathcal{M} , two individuals u_1 and u_2 are picked uniformly in the population, they collide and are replaced by an amount h of new individuals. The collision part is governed by the Lévy-Khintchine function

$$\Sigma(x) := \frac{a^2}{2} x^2 + \frac{c}{2} x + \int_0^\infty (e^{-xh} - 1 + xh) \eta(dh), \quad x \in [0, \infty). \quad (1.3)$$

We assume the collision mechanism Σ is subcritical or critical (i.e. $\Sigma'(0+) = \frac{c}{2} \geq 0$, which dovetails with $\sum_{k \in \mathbb{Z}_+} k p_c(k) \leq 1$ above) but not zero. Thus collisions are either diminishing the number of individuals or keeping it the same on average. One might expect some phenomenon of regulation of the population size when the latter reaches large values. Collisions may for instance prevent or not the growth of the population induced by supercritical branching dynamics.

We stress that the compensated versions of the Poisson random measures $\mathcal{M}(ds, du_1, du_2, dh)$ and $\mathcal{N}(ds, du, dh)$, the latter on $h \in (0, 1]$, are used in (1.1). The two stochastic integrals with

¹Also in the continuation of this text \bar{M} will always designate the compensated measure of M , whatever the Poisson random measure M may be.

respect to the independent Brownian motions in (1.1) can be rewritten respectively as follows:

$$\int_0^t \sqrt{Z_s} dB_s = \int_0^t \int_{[0, Z_s]} \tilde{B}(ds, du) \quad \text{and} \quad \int_0^t Z_s dW_s = \int_0^t \int_{[0, Z_s] \times [0, Z_s]} \tilde{W}(ds, du_1, du_2),$$

with $\tilde{B}(ds, du)$ and $\tilde{W}(ds, du_1, du_2)$ independent Gaussian time-space white noises on $(0, \infty) \times (0, \infty)$ and $(0, \infty) \times (0, \infty)^2$ based on the Lebesgue measures $dsdu$ and $dsdu_1du_2$, respectively. This allows one to interpret also both diffusive parts in terms of branching and collision. We refer to Li and Ma [40, page 940] for the representation of the Feller's branching diffusion part with the white-noise $\tilde{B}(ds, du)$, see also Pardoux [44, Section 4.1] and El Karoui and Méléard [13] for a more general framework.

1.3. Highlights. The structure of continuous-state branching processes with collisions (CBCs) is of specific theoretical interest, since, as we shall see, they form a class of Markov processes with no negative jumps, whose long-term behavior and first passage times can be linked in a dual way to those of certain one-dimensional diffusions. We give now a brief panorama of this.

Once unique existence of $\text{CBC}(\Sigma, \Psi)$, minimal solution to the stochastic equation (1.1), has been established, we study three classical problems for this process. First, in Theorem 2.3, we give necessary and sufficient conditions on the branching and collision mechanisms for the $\text{CBC}(\Sigma, \Psi)$ process to converge as time goes to infinity towards its boundaries 0 or ∞ with positive probability (attracting boundaries). We next work under the assumption that the $\text{CBC}(\Sigma, \Psi)$ process does not explode (for which sufficient conditions will be given in Proposition 2.6) and obtain, see Theorem 2.8, a representation of the Laplace transform, at argument $\theta \in (0, \infty)$, of the first passage time below a given level with the help of an increasing positive function h_θ , solution $h \in C^2((0, \infty))$ to the second-order linear differential equation

$$\mathcal{G}h := \Sigma h'' + (\Sigma' + \Psi)h' = \theta h \quad \text{on } (0, \infty). \quad (1.4)$$

Fundamental results of Feller, see [15, 16, 17], on equations of the type (1.4) apply and allow one for instance to understand h in terms of the first passage time of a diffusion V with generator \mathcal{G} . In particular, we identify in Theorem 2.11 the law of the extinction time of the $\text{CBC}(\Sigma, \Psi)$ started from z with that of the explosion time of V started from an independent exponential random variable with parameter z . Lastly, we find necessary and sufficient conditions on the mechanisms Σ and Ψ for the $\text{CBC}(\Sigma, \Psi)$ to have a limiting distribution, in which case we procure an explicit formula for its Laplace transform, see Theorem 2.14.

An important feature of CBCs lies in the fact that their generator \mathcal{L} satisfies a Laplace duality (also called exponential duality) with the diffusion generator \mathcal{A} given by, for $g \in C^2([0, \infty))$,

$$\mathcal{A}g := \Sigma g'' - \Psi g' \quad \text{on } [0, \infty); \quad (1.5)$$

namely, for $\{x, z\} \subset [0, \infty)$,

$$\mathcal{L}_z e^{-xz} = \mathcal{A}_x e^{-xz}$$

(the subscripts in the operators indicate which variable they are acting on). When the $\text{CBC}(\Sigma, \Psi)$ process $Z := (Z_t, t \geq 0)$ does not explode we will establish that also its semigroup satisfies a Laplace duality relationship; to wit, for $\{t, x, z\} \subset [0, \infty)$:

$$\mathbb{E}_z [e^{-xZ_t}] = \mathbb{E}_x [e^{-zU_t}],$$

U being the diffusion on $[0, \infty)$ with 0 an absorbing state and generator \mathcal{A} (U , as it emerges, does not explode [either]), see Proposition 2.18. Conversely, it will be established in Theorem 2.21 that under a mild assumption on the domains of their generators, CBCs are the only positive Feller processes with no negative jumps and zero an absorbing state, whose generators are in Laplace duality with those of one-dimensional diffusions.

The generator \mathcal{A} of (1.5) in turn is in so-called Siegmund duality with the generator \mathcal{G} of (1.4). Under certain conditions, which shall be specified later on, and which entail that Z does not explode, we get it again at the level of the semigroups: for $t \in [0, \infty)$ and $\{x, y\} \subset (0, \infty)$,

$$\mathbb{P}_x(U_t < y) = \mathbb{P}_y(V_t > x),$$

see Siegmund [51], Cox and Rösler [10], [19, Section 6] and the forthcoming Proposition 2.20. It will emerge (Remark 6.3) that under the assumption of non-explosion of the CBC the boundary 0, like ∞ , of U is also never regular, and in that case there is therefore in fact no need to stipulate boundary conditions for U and V .

The preceding duality relationships explain somehow why the study of the boundaries of Z is linked to those of the processes U and V (for which the general theory of diffusions applies). At the level of the semigroups they will actually be used mainly for studying the existence of the limiting distribution of Z and for its characterization. At the level of the generators they lie however at the core of our study for all CBCs, and are summarized by the following diagram, in which, for the reader's convenience, we also note the corresponding processes:

$$(Z, \mathcal{L}) \xleftrightarrow{\text{Laplace dual}} (U, \mathcal{A}) \xleftrightarrow{\text{Siegmund dual}} (V, \mathcal{G}).$$

The reader is referred to Kurt and Jansen's survey [28] for a recent general account of duality.

1.4. Literature overview and available examples. Attention has recently been paid to the role of duality in the study of eigenfunctions of generators, see for instance Griffiths [24], Foucart and Möhle [20] and Redig and Sau [48]. We should also like to point out that there is a relatively vast and developing literature on exit problems of Markov processes with one-sided jumps to which our study can be connected. In this vein we may mention, restricting to continuous space and time, Duhalde et al. [12] for CBs with immigration (CBIs), Kuznetsov et al. [33] for spectrally negative Lévy processes, Borovkov and Novikov [4] and Patie [45] for generalized Ornstein-Uhlenbeck processes, Patie [46, 47] and Vidmar [54] for positive self-similar Markov processes, see also Landriault et al. [36] and Avram et al. [1] for some general drawdown/drawup results.

A few examples of CBCs have already appeared in the literature. The pure drift collision mechanism, namely the case $\Sigma(x) = \frac{c}{2}x$, $x \in [0, \infty)$, corresponds to logistic CBs, see Lambert [35] and Foucart [18, 19]. Duality relationships with the processes U and V were observed in the two latter works, however their role in the problem of characterizing first passage times of the logistic CB Z was not understood therein. In the pure quadratic collision mechanism, when $\Sigma(x) = \frac{a^2}{2}x^2$, $x \in [0, \infty)$, CBCs match with CBs in Brownian environment, see Palau and Pardo [42] and He et al. [25]. The case $\Sigma(x) = \frac{a^2}{2}x^2 + \frac{c}{2}x$, $x \in [0, \infty)$, has also been recently studied by Leman and Pardo [37], by adapting Lambert's method in [35], which relies on the study of some Ricatti-type nonlinear equations. The duality was not used in the latter article and will simplify the study for us at several levels, especially for the limiting distribution. Lastly, the pure continuous-state collision process, for which $\Psi = 0$ and the first line in Eq. (1.1) vanishes, corresponds to a polynomial CB process, defined in Li [38], with power $\theta = 2$. In this case [38, Theorem 1.8] ensures that there is no extinction in finite time of the process (i.e. 0 is inaccessible).

In the subcritical collision case, i.e. $c > 0$, the process behaves in many aspects as the logistic CB. The critical collision case when $c = 0$ is however more involved to study and many different new behaviors in comparison to the subcritical one may exist. This is merely due to the fact that fluctuations of the martingale part in the second line of Eq. (1.1) are now involved and not the deterministic quadratic drift. We will not address here the complete classification of the boundary ∞ of CBCs. It seems indeed to require a study of its own since all types (natural, entrance, exit, regular) may occur. We may refer the interested reader however to [18] where the case of logistic CBs is treated.

1.5. Article structure. Main results are gathered in Section 2, their proofs deferred to the continuation of the text. Specifically, in Section 3 we study the stochastic equation (1.1) and show that its solution is related, via Lamperti time-change, to a class of processes, called CB processes with spectrally positive migration (CBMs) in [52]. Then, in Section 4 we study the attraction of the boundaries, in Section 5 the first passage times, and in Section 6 the duality relationships and the limiting distribution, as indicated above.

2. MAIN RESULTS

2.1. Introduction of CBCs and first properties. For the Lévy-Khintchine function Ψ we denote by L^Ψ the infinitesimal generator of a spectrally positive Lévy process with Laplace exponent Ψ . It acts on a $C_0^2(\mathbb{R})$ (i.e. twice continuously differentiable $f : \mathbb{R} \rightarrow \mathbb{R}$ with f, f', f'' all vanishing at infinity) function f as follows:

$$L^\Psi f(z) = \frac{\sigma^2}{2} f''(z) + b f'(z) + \int_0^\infty (f(z+h) - f(z) - h f'(z) \mathbb{1}_{(0,1]}(h)) \pi(dh), \quad z \in \mathbb{R}. \quad (2.1)$$

Analogously for Σ we have

$$L^\Sigma f(z) = \frac{a^2}{2} f''(z) - \frac{c}{2} f'(z) + \int_0^\infty (f(z+h) - f(z) - h f'(z)) \eta(dh), \quad z \in \mathbb{R}. \quad (2.2)$$

It is clear that if $f : I \rightarrow \mathbb{R}$ is of class $C_b^2(I)$ (b stands for f being bounded²), defined (only) on some interval I of \mathbb{R} unbounded above, then L^Ψ and L^Σ are (still) naturally defined on I by the right-hand sides of the preceding displays. We take this for granted in the continuation of the text.

Below, for notions such as adaptedness, martingale etc. we work with the augmented natural filtration \mathcal{F} of $(W, B, \mathcal{N}, \mathcal{M})$, unless explicitly noted otherwise.

Theorem 2.1. *For each starting value $z \in [0, \infty)$ there is an a.s. unique $[0, \infty]$ -valued càdlàg adapted process $Z = (Z_t, t \geq 0)$ such that*

(i) $Z = \infty$ on $[\uparrow\text{-}\lim_{n \rightarrow \infty} \zeta_n^+, \infty)$ a.s., where

$$\zeta_n^+ := \inf\{t \in [0, \infty) : Z_t \geq n\}, \quad n \in [0, \infty) \quad (2.3)$$

[Z is absorbed at ∞ after its first explosion]

and such that

(ii) $\zeta_\infty := \uparrow\text{-}\lim_{n \rightarrow \infty} \zeta_n^+ = \inf\{t \in [0, \infty) : Z_t = \infty\} > 0$ and Z satisfies (1.1) for $t \in [0, \zeta_\infty)$ a.s., [Z satisfies (1.1) up to first explosion].

The law of the process Z is uniquely determined by the triplet (Σ, Ψ, z) . Furthermore, the process Z is a.s. without negative jumps, has 0 as an absorbing state, is quasi-left continuous and strong Markov, finally, for all $f \in C_b^2([0, \infty))$, setting

$$\mathcal{L}f(z) := z^2 L^\Sigma f(z) + z L^\Psi f(z), \quad z \in [0, \infty), \quad (2.4)$$

then for all $\alpha \in [0, \infty)$ and for all $n \in [0, \infty)$, the process

$$f(Z_{t \wedge \zeta_n^+}) e^{-\alpha(t \wedge \zeta_n^+)} - \int_0^{t \wedge \zeta_n^+} e^{-\alpha s} (\mathcal{L}f(Z_s) - \alpha f(Z_s)) ds, \quad t \in [0, \infty), \quad (2.5)$$

is a local martingale.

²but its first and second derivative need not be bounded!

We call the process Z the (minimal) $\text{CBC}(\Sigma, \Psi)$. We shall not have occasion to deal with extensions of the minimal process in this paper, so the qualification “minimal” will be largely omitted, except for emphasis. We retain the notation introduced in the theorem and set further

$$\zeta_a := \inf\{t \in [0, \infty) : Z_t \leq a\}, \quad a \in [0, \infty). \quad (2.6)$$

In order to stress the initial value $z \in [0, \infty)$ we write \mathbb{P}_z instead of just \mathbb{P} and correspondingly \mathbb{E}_z rather than just \mathbb{E} , being somewhat lax about holding z fixed or variable. In view of the martingale claim surrounding (2.5) (with $\alpha = 0$) we call \mathcal{L} the generator of the $\text{CBC}(\Sigma, \Psi)$ process Z . Glancing at (2.4), likewise as for L^Ψ and L^Σ , so too may we (and shall) consider \mathcal{L} as being capable of taking as input any $f : I \rightarrow \mathbb{R}$ that is $C_b^2(I)$, defined (only) on some interval I of $[0, \infty)$ unbounded above, returning in this case a map defined on I according to the right-hand side of (2.4).

The branching mechanism Ψ may be of two fundamentally different forms: either $\Psi(x) \leq 0$ for all $x \geq 0$ so that $-\Psi$ is the Laplace exponent of a subordinator (we include under this designation the constant process) and in which case we say that we are in the subordinator case; or $\Psi(x) > 0$ for some $x > 0$ so that Ψ is the Laplace exponent of a spectrally positive Lévy process (in the narrow sense) or of a negative linear drift. In the subordinator case, no particle dies after a branching event and the pure $\text{CB}(\Psi)$ has non-decreasing sample paths. Unsurprisingly this case is (can be) a little singular also in the present context as the next proposition demonstrates.

Proposition 2.2. *Put*

$$z^* := \left(\limsup_{\infty} \frac{-\Psi}{\Sigma} \right) \vee 0 < \infty.$$

If $z^ > 0$ and the starting value z of Z satisfies $z > z^*$, then a.s. $Z_t > z^*$ for all $t \in [0, \infty)$. Furthermore, one has $z^* > 0$ if and only if $\Sigma(x)/x \rightarrow D < \infty$ and $\Psi(x)/x \rightarrow -\mu < 0$ (so that $-\Psi$ is the Laplace exponent of subordinator with drift μ and the Lévy process with Lévy-Khintchine function Σ has finite variation), in which case $z^* = \frac{\mu}{D}$.*

2.2. Classification of attracting boundaries. Recall the definition of \mathcal{G} in (1.4) and notice that $\Sigma > 0$ on $(0, \infty)$ and that the local operator \mathcal{G} is the generator of a certain regular diffusion on $(0, \infty)$. Let then $V := (V_t)_{t \geq 0}$ be the minimal diffusion with generator \mathcal{G} , namely with boundaries 0 and ∞ absorbing if they are accessible. Fix an arbitrary $x_0 \in (0, \infty)$. Set $S_V := ((0, \infty) \ni x \mapsto \int_{x_0}^x \frac{1}{\Sigma(u)} e^{\int_u^{x_0} \frac{\Psi(v)}{\Sigma(v)} dv} du \in (-\infty, \infty))$ for the scale function of V , see e.g. Karlin and Taylor [32, Chapter 14, Section 6, page 227]. By abuse of notation denote by S_V also its associated Lebesgue-Stieltjes measure on $(0, \infty)$; to wit, for $\mathbf{a} < \mathbf{b}$ from $(0, \infty)$,

$$S_V(\mathbf{a}, \mathbf{b}] = S_V(\mathbf{b}) - S_V(\mathbf{a}) = \int_{\mathbf{a}}^{\mathbf{b}} \frac{1}{\Sigma(x)} e^{\int_x^{x_0} \frac{\Psi(u)}{\Sigma(u)} du} dx \in (0, \infty), \quad (2.7)$$

which determines S_V uniquely. The measure S_V being locally finite, note that if $S_V(0, \mathbf{b}]$ is infinite for some $\mathbf{b} \in (0, \infty)$ then it is so for all $\mathbf{b} \in (0, \infty)$; similarly for $S_V(\mathbf{b}, \infty)$.

Our next theorem provides necessary and sufficient conditions for the boundaries 0 and ∞ to be attracting, by which we mean that the process tends towards the boundary with positive probability. These conditions are those of the diffusion V for the boundaries ∞ and 0, respectively.

Theorem 2.3 (Attracting boundaries). *Let $\{z, \mathbf{a}\} \subset (z^*, \infty)$, $\mathbf{a} < z$.*

(i) *If $S_V(0, x_0] = \infty$ then $\mathbb{P}_z(\zeta_{\mathbf{a}} < \zeta_{\infty}) = 1$.*

(ii) *If $S_V(0, x_0] < \infty$ then $\mathbb{P}_z(\zeta_{\mathbf{a}} < \zeta_{\infty}) = \frac{S_Z(z)}{S_Z(\mathbf{a})} \in (0, 1)$ with*

$$S_Z(z) := \int_0^{\infty} e^{-xz} S_V(dx) = \int_0^{\infty} \frac{e^{-xz}}{\Sigma(x)} e^{\int_x^{x_0} \frac{\Psi(u)}{\Sigma(u)} du} dx.$$

- (iii) $Z_t \xrightarrow[t \rightarrow \zeta_\infty^-]{} \infty$ with positive \mathbb{P}_z -probability if and only if $S_V(0, x_0] < \infty$.
- (iv) If $\Psi \neq 0$, then $Z_t \xrightarrow[t \rightarrow \infty]{} 0$ with positive \mathbb{P}_z -probability (respectively, \mathbb{P}_z -almost surely) if and only if $S_V(x_0, \infty) < \infty$ (respectively, $S_V(x_0, \infty) < \infty$ and $S_V(0, x_0] = \infty$); when $S_V(0, \infty) < \infty$, then, moreover, $\mathbb{P}_z(Z_t \xrightarrow[t \rightarrow \infty]{} 0) = \frac{S_Z(z)}{S_Z(0)} \in (0, 1)$, where S_Z is given by the same expression as in (ii). If $\Psi = 0$ then \mathbb{P}_z -almost surely $Z_t \xrightarrow[t \rightarrow \infty]{} 0$.

As we have mentioned Theorem 2.3 actually states the following correspondences:

Condition	Boundary of Z	Boundary of V
$S_V(0, x_0] < \infty$	∞ attracting	0 attracting
$\Psi = 0$ or $S_V(x_0, \infty) < \infty$	0 attracting	∞ attracting

TABLE 1. Attracting boundaries of Z and V

Remark 2.4. The convergence towards ∞ in Theorem 2.3(iii), when $S_V(0, x_0] < \infty$, hides two different possibilities: the process can either be transient (∞ is attracting, but not accessible) or can explode (∞ is accessible). Indeed the condition $S_V(0, x_0] = \infty$ is not necessary in general for the process to be non-explosive, see Example 2.5(1) below. In the case $\Sigma(x) = \frac{c}{2}x$, $x \in [0, \infty)$, however, the condition $S_V(0, x_0] = \infty$ turns out to be also necessary for non-explosion [18, Theorem 3.1]. No transience phenomenon can occur in logistic CBs [18, Remark 4.9]; they can only converge to ∞ by reaching it.

In the non-subordinator case, one can easily check that $S_V(x_0, \infty) < \infty$ always holds. So, by Theorem 2.3(iv), the necessary and sufficient condition for almost sure convergence towards 0 is then $S_V(0, x_0] = \infty$. In the subordinator case, the condition $S_V(x_0, \infty) < \infty$ may or may be not satisfied. In other words, collisions can be strong enough ($S_V(x_0, \infty) < \infty$) or not ($S_V(x_0, \infty) = \infty$) for the event of convergence towards 0 to have positive probability or not. Lastly, in the (sub)critical branching case, one always has $S_V(0, x_0] = \infty$.

Example 2.5.

- (1) Let $a > 0$ and $b \in \mathbb{R}$. One of the simplest CBCs is the process with mechanisms

$$\Sigma(x) = \frac{a^2}{2}x^2 \text{ and } \Psi(x) = -bx, x \in [0, \infty).$$

It satisfies the SDE

$$dZ_t = aZ_t dW_t + bZ_t dt, Z_0 = z,$$

which corresponds to a geometric Brownian motion, namely for all $t \geq 0$,

$$Z_t = z \exp \left(\left(b - \frac{a^2}{2} \right) t + aW_t \right).$$

One can directly check that $S_V(0, x_0] < \infty$ if and only if $b > \frac{a^2}{2}$, in which case the process $(Z_t, t \geq 0)$ is transient (and does not explode). We also see that Brownian collisions regulate the deterministic growth, that is to say, $Z_t \xrightarrow[t \rightarrow \infty]{} 0$ a.s., when $\frac{a^2}{2} > b$.

- (2) More generally if $\Psi'(0+) =: -b \in \mathbb{R}$ and $\Sigma(x) \underset{x \rightarrow 0}{\sim} \frac{a}{2}x^2$, then $S_V(0, x_0] = \infty$ if and only if $b \leq \frac{a^2}{2}$. If in addition to the latter condition $\Psi(x) > 0$ for some $x > 0$, then $S_V(x_0, \infty) < \infty$ and thus $Z_t \xrightarrow[t \rightarrow \infty]{} 0$ a.s.. These results are reminiscent of properties of a CB process in a Brownian environment, see Palau and Pardo [42, Proposition 2].
- (3) Consider $\Sigma(x) = dx^\alpha$ with $\alpha \in (1, 2)$ and $\Psi(x) = -d'x^\beta =: -\Phi(x)$ with $\beta \in (0, 1)$, $x \in [0, \infty)$. Then we have as follows.
- If $\beta > \alpha - 1$, neither 0 nor ∞ is attracting.

- If $\beta < \alpha - 1$, 0 and ∞ are both attracting.
- If $\beta = \alpha - 1$, ∞ is attracting if and only if $d'/d > \alpha - 1$, while 0 is attracting if and only if $d'/d < \alpha - 1$. In the case of equality, $d'/d = \alpha - 1$, neither 0 nor ∞ are attracting.

(4) Finally, consider the case when $\Sigma(x) = dx^\alpha$ for all $x \in [0, \infty)$, with $\alpha \in (1, 2)$, and a branching mechanism Ψ such that $\Psi'(0+) \in (-\infty, \infty)$. Then 0 is attracting and if, moreover, $\Psi(x) \geq 0$ for some $x > 0$, then $Z_t \xrightarrow[t \rightarrow \infty]{} 0$ a.s..

2.3. First passage times and extinction. We turn to the study of the law of the first passage time of the CBC(Σ, Ψ) process Z below a given level. We first state a sufficient condition ensuring that Z does not explode in finite time (i.e. its boundary ∞ is inaccessible).

Proposition 2.6. *If $S_V(0, x_0] = \infty$ or $\int_{0+} \frac{dx}{-\Psi(x)} = \infty$, then the CBC(Σ, Ψ) process does not explode.*

Remark 2.7. The fact that when $S_V(0, x_0] = \infty$ the process does not explode is a direct consequence of Theorem 2.3(iii). Note that if $\int_0^{x_0} \frac{\Psi(u)}{\Sigma(u)} du \in (-\infty, \infty)$ then $S_V(0, x_0] = \infty$. The condition $\int_{0+} \frac{dx}{-\Psi(x)} = \infty$ (called Dynkin's condition) is necessary and sufficient for non-explosion of the CB(Ψ), see e.g. Kyprianou [34, Theorem 12.3]. In other words, and it is not surprising in view of their dynamics, collisions are never causing explosion of CBCs.

We will find a representation of the decreasing θ -invariant function of Z with the help of the increasing one of the diffusion V . This enables us to get an expression for the Laplace transforms of the first passage times $\zeta_{\mathbf{a}}$, $\mathbf{a} \in (z^*, \infty)$.

Theorem 2.8. *Assume that the CBC(Σ, Ψ) process does not explode and let $\theta \in (0, \infty)$. Put*

$$f_\theta(z) := z \int_0^\infty e^{-zv} h_\theta(v) dv, \quad z \in (z^*, \infty), \quad (2.8)$$

the function $h_\theta \in C^2((0, \infty))$ being the unique (up to a multiplicative constant³) nonnegative, not zero, nondecreasing solution h on $(0, \infty)$ to

$$\mathcal{G}h = \Sigma h'' + (\Sigma' + \Psi)h' = \theta h. \quad (2.9)$$

Then, for $\mathbf{a} \leq z$ from (z^*, ∞) ,

$$\mathbb{E}_z[e^{-\theta \zeta_{\mathbf{a}}}] = \frac{f_\theta(z)}{f_\theta(\mathbf{a})}. \quad (2.10)$$

Remark 2.9. When there is no collision, $\Sigma = 0$, and we are not in the subordinator case, the ordinary differential equation (o.d.e.) in (2.9) is of first order and there is a possible singularity at $\rho := \sup\{x \in [0, \infty) : \Psi(x) = 0\}$, the largest zero of Ψ . Solving the o.d.e. gives for $v \in (\rho, \infty)$, $h_\theta(v) = e^{\int_{x_0}^v \frac{\theta}{\Psi(u)} du}$, where (still) $x_0 \in (0, \infty)$ is fixed (and arbitrary). In turn we get

$$f_\theta(z) = z \int_\rho^\infty e^{-zv} e^{\int_{x_0}^v \frac{\theta}{\Psi(u)} du} dv, \quad z \in (0, \infty),$$

and recover then through Formula (2.10) the Laplace transform of the first passage time of the CB(Ψ), see [12, Section 6, page 4192].

Remark 2.10. A simple application of Tonelli's theorem ensures that the so-called scale function f_θ in (2.8) satisfies $f_\theta(z) = h_\theta(0+) + \int_0^\infty e^{-zv} h'_\theta(v) dv$, $z \in (0, \infty)$. In particular, f_θ is completely monotone. This phenomenon of "complete monotonicity at first passage" was recently noted and explored for time-changed spectrally positive Lévy processes by Vidmar [53].

³A multiplicative constant we intend always to be from $(0, \infty)$.

Theorem 2.8 deals with first passage times below (accessible) positive levels. As for the first passage time to zero of Z and the event of extinction we offer

Theorem 2.11. *Assume that the $\text{CBC}(\Sigma, \Psi)$ process does not explode. Let $z \in (z^*, \infty)$. The following equivalence holds true.*

$$\mathbb{P}_z(\zeta_0 < \infty) > 0 \text{ if and only if } \Psi(\infty) = \infty \text{ and } \int_0^\infty \frac{du}{\Psi(u)} < \infty. \quad (2.11)$$

Furthermore, the Laplace transform of the extinction time of Z satisfies:

$$\mathbb{E}_z[e^{-\theta\zeta_0}] = \int_0^\infty z e^{-xz} \frac{h_\theta(x)}{h_\theta(\infty)} dx = \mathbb{E}[e^{-\theta\tau_\infty^{ez}}], \quad \theta \in (0, \infty), \quad (2.12)$$

where τ_∞^{ez} denotes the explosion time of V when the latter starts from an independent exponential random variable e_z with rate z .

Remark 2.12. The integrability condition $\Psi(\infty) = \infty$, $\int_0^\infty \frac{du}{\Psi(u)} < \infty$ (called Grey's condition) is necessary and sufficient for the $\text{CB}(\Psi)$ process to become extinct with positive probability, see e.g. [39, Theorem 3.1.3]. Collisions are therefore also never causing extinction in finite time of the population.

Remark 2.13. Identity (2.12) reveals that under the assumption of non-explosion of Z , the boundary 0 is accessible for Z if and only if ∞ is accessible for V . It was established via different arguments for logistic CBs and their extensions in [19, Theorem 3.2].

2.4. Stationary distribution. As observed in Example 2.5(3), in the subordinator case, some phenomenon of recurrence can occur and a stationary regime may exist. Let M_V be the speed measure of V on $(0, \infty)$: for $\mathbf{a} < \mathbf{b}$ from $(0, \infty)$,

$$M_V(\mathbf{a}, \mathbf{b}) = \int_{\mathbf{a}}^{\mathbf{b}} e^{\int_{x_0}^x \frac{\Psi(u)}{\Sigma(u)} du} dx \in (0, \infty), \quad (2.13)$$

where, still, $x_0 \in (0, \infty)$ is arbitrary but fixed.

Theorem 2.14 (Stationary distribution and long-term behavior). *Assume that $S_V(0, x_0] = \infty$ and $S_V(x_0, \infty) = \infty$. Let $z \in (0, \infty)$. Then the minimal CBC process converges in law towards a non-degenerate random variable Z_∞ on (z^*, ∞) if and only if $M_V(0, \infty) < \infty$. Moreover, the Laplace transform of the latter is then given by*

$$\mathbb{E}_z[e^{-xZ_\infty}] = \frac{M_V(x, \infty)}{M_V(0, \infty)}, \quad x \in [0, \infty). \quad (2.14)$$

The case $M_V(0, \infty) = \infty$ covers three different possibilities:

- (i) If $M_V(0, x_0] < \infty$ and $M_V(x_0, \infty) = \infty$, then $Z_t \xrightarrow[t \rightarrow \infty]{} 0$ in probability.
- (ii) If $M_V(0, x_0] = \infty$ and $M_V(x_0, \infty) < \infty$, then $Z_t \xrightarrow[t \rightarrow \infty]{} \infty$ in probability.
- (iii) If $M_V(0, x_0] = \infty$ and $M_V(x_0, \infty) = \infty$, then Z has no limiting distribution.

Remark 2.15. Plainly, if $-\Psi$ is not the Laplace exponent of a subordinator, then $M_V(x_0, \infty) = \infty$. Here are two simple conditions ensuring, between them, that $M_V(0, \infty) < \infty$ and hence that a limiting distribution exists. If $\Sigma'(0+) = c/2 > 0$, then $M_V(0, x_0] < \infty$ (without further assumptions on Ψ). If $-\Psi$ is the Laplace exponent of a subordinator with drift d , i.e. $-\Psi(x)/x \xrightarrow[x \rightarrow \infty]{} d > 0$, such that $\frac{2d}{a^2} > 1$ (with $a \geq 0$ the diffusive coefficient in (1.3) and by convention $1/0 = \infty$), then $M_V(x_0, \infty) < \infty$.

Remark 2.16. One verifies easily from (2.14) that the limiting distribution of the $\text{CBC}(\Sigma, \Psi)$ admits a first moment if and only if $\int_0^{x_0} \frac{-\Psi(u)}{\Sigma(u)} du < \infty$.

Example 2.17.

- (1) Consider the $\text{CBC}(\Sigma, \Psi)$ process with collisions and branching mechanisms satisfying, for $x \in [0, \infty)$, $\Sigma(x) = \frac{a^2}{2}x^2 + \frac{c}{2}x$ with $a, c \in (0, \infty)$ and $\Psi(x) = -\mu x$ with $\mu \in \mathbb{R}$. In other words, $(Z_t, t \geq 0)$ satisfies the SDE, called stochastic Verhulst equation

$$dZ_t = aZ_t dW_t + \left(\mu Z_t - \frac{c}{2}Z_t^2\right)dt, Z_0 = z.$$

See Giet et al. [22] for a deep study of this diffusion (including its first passage times). The $\text{CBC}(\Sigma, \Psi)$ process Z admits a limiting distribution if and only if $\mu > \frac{a^2}{2}$. When it exists, the latter has for its Laplace transform:

$$\mathbb{E}[e^{-xZ_\infty}] = \left(\frac{a^2}{c}x + 1\right)^{-\left(\frac{2\mu}{a^2}-1\right)}, \quad x \in [0, \infty),$$

which is the Laplace transform of a gamma distribution with density

$$(0, \infty) \ni u \mapsto \frac{\beta^\alpha}{\Gamma(\alpha)} u^{\alpha-1} e^{-\beta u},$$

its parameters being $\alpha := \frac{2\mu}{a^2} - 1$ and $\beta := \frac{c}{a^2}$.

- (2) Assume that, for $x \in [0, \infty)$, $\Sigma(x) = dx^\alpha$ with $\alpha \in (1, 2)$ and $\Psi(x) = -d'x^\beta =: -\Phi(x)$ with $\beta > \alpha - 1$ and $d, d' \in (0, \infty)$. Then the $\text{CBC}(\Sigma, \Psi)$ process Z admits a limiting distribution with Laplace transform:

$$\mathbb{E}[e^{-xZ_\infty}] = \frac{\int_x^\infty e^{-\frac{d'}{d}u^{\beta-\alpha+1}} du}{\int_0^\infty e^{-\frac{d'}{d}u^{\beta-\alpha+1}} du} = \frac{\Gamma\left(\frac{1}{\beta-\alpha+1}, \frac{d'}{d}x^{\beta-\alpha+1}\right)}{\Gamma\left(\frac{1}{\beta-\alpha+1}\right)}, \quad x \in [0, \infty),$$

where $\Gamma(s, x) := \int_x^\infty u^{s-1} e^{-u} du$, for $s > 0$ and $x \geq 0$, is the incomplete Gamma function.

- (3) Assume that, for $x \in [0, \infty)$, $\Sigma(x) = dx^\alpha$ with $\alpha \in (1, 2)$, $d \in (0, \infty)$ and $\Psi(x) = -dx^{\alpha-1}$. Then if $d'/d < 1$, $M_V(x_0, \infty) = \infty$ and $M_V(0, x_0] < \infty$, thus Z goes to 0 in probability. If $d'/d > 1$, $M_V(x_0, \infty) < \infty$ and $M_V(0, x_0] = \infty$, and Z goes to ∞ in probability. In the case $d'/d = 1$, Z has no limiting distribution.

2.5. The role of Laplace and Siegmund dualities. The second order differential operator \mathcal{G} , defined in (1.4), will first appear as an analytical trick in the quest for an eigenfunction of \mathcal{L} , see the proof of Theorem 2.8, especially the forthcoming Lemma 5.4. The link between the generators \mathcal{G} , \mathcal{A} and \mathcal{L} hinges in fact on two duality relationships, known as *Laplace duality* and *Siegmund duality*. We explore now these dualities, which will for instance allow us to represent, under certain conditions, the semigroup of the process Z with that of U and in turn V .

From (1.5) and (2.4) one checks by direct computation the key identity

$$\mathcal{L}_z e^{-xz} = \Sigma(x)z^2 e^{-xz} + \Psi(x)z e^{-xz} = \mathcal{A}_x e^{-xz}, \quad \{x, z\} \subset [0, \infty). \quad (2.15)$$

We say that Laplace duality (2.15) holds at the *level of the generators*. Under the assumption of non-explosion of Z we have moreover the following duality relationship at the *level of the semigroups*.

Proposition 2.18. *Let Z be the $\text{CBC}(\Sigma, \Psi)$ and U the diffusion with generator \mathcal{A} and 0 an absorbing state. Assume that Z does not explode. Then*

$$\mathbb{E}_z[e^{-xZ_t}] = \mathbb{E}_x[e^{-zU_t}], \quad \{t, x, z\} \subset [0, \infty). \quad (2.16)$$

Here, on the right-hand side, x is of course the initial value of U .

Remark 2.19. Proposition 2.18 requires the non-explosiveness of the process Z ; this assumption will play an important role in the proof. On the other hand, the diffusion U is automatically non-explosive, as we shall prove in due course (Lemma 6.1). The duality allows one to represent the semigroup of Z with the help of that of U . In particular, one can check from (2.16) that under non-explosion the semigroup of the CBC process Z is Feller, see [18, Lemma 6.3].

Under extra conditions, which guarantee that V has no attracting boundaries, the diffusion U in turn is in Siegmund duality with the diffusion V , in the following precise sense.

Proposition 2.20. *Assume that $S_V(0, x_0] = \infty$, $S_V(x_0, \infty) = \infty$, and recall U is the diffusion with generator \mathcal{A} . For all $x, y \in (0, \infty)$,*

$$\mathbb{P}_x(U_t < y) = \mathbb{P}_y(V_t > x), \quad (2.17)$$

where V is the diffusion with generator \mathcal{G} . Moreover, for any $z \in (0, \infty)$, one has

$$\mathbb{E}_z[e^{-xZ_t}] = \int_0^\infty ze^{-zy} \mathbb{P}_y(V_t > x) dy, \quad x \in [0, \infty).$$

The final substantial result on which we report here establishes that, in a sense that shall be made precise presently, Laplace duality with a diffusion at the level of the generators (2.15) actually characterizes CBCs. In order to formulate this with ease we suspend temporarily all meaning attached hitherto to Z , \mathcal{L} , \mathcal{A} , Σ and Ψ (and indeed just all the notation introduced thusfar).

Theorem 2.21. *Let \mathcal{L} be the infinitesimal generator of a positive (possibly explosive) Feller process $(Z_t, t \geq 0)$ without negative jumps and 0 an absorbing state, whose domain includes⁴*

$$\mathcal{S} := \{f \in \mathbb{R}^{[0, \infty)} : (\exists \lim_{\infty} f) \ \& \ (f - f(\infty) \in \text{Schwartz space of rapidly decaying functions})\}$$

and for which $\mathcal{L}1 = 0$ (the latter is always satisfied when Z is not explosive). Suppose further \mathcal{L} is in Laplace duality with the conservative generator of a diffusion process on $[0, \infty)$, more precisely suppose that

$$\mathcal{L}_z e^{-xz} = \Sigma(x)z^2 e^{-xz} + \Psi(x)z e^{-xz} =: \mathcal{A}_x e^{-xz}, \quad \{x, z\} \subset [0, \infty), \quad (2.18)$$

holds true for some $\Sigma : [0, \infty) \rightarrow [0, \infty)$, not zero, and some $\Psi : [0, \infty) \rightarrow \mathbb{R}$, both continuous at zero. Then Ψ and Σ are Lévy-Khintchine functions of the spectrally positive type as in (1.2)-(1.3) and \mathcal{L} acts on $C_c^\infty([0, \infty))$ according to (2.4).

We shall see later in Corollary 3.1 that CBCs actually meet the property assumed on the generator of Z above, so, together with the Feller property noted in Remark 2.19, this really is a characterization of non-explosive CBCs through Laplace duality with diffusions. Remark also that in the general theory of Feller processes the infinitesimal generator is usually only defined on (a subset of) continuous maps vanishing at infinity. For the complete formulation of the Laplace duality it is however convenient to include in the domain the constants, hence our slight departure from this convention.

3. CONSTRUCTION AND LAMPERTI REPRESENTATION OF CBCS

3.1. Study of stochastic equation (1.1): proof of Theorem 2.1. Stochastic equations of the form (1.1) fall into the general class of certain SDEs with jumps studied by Dawson and Li

⁴Rapidly decaying functions are those $f \in C^\infty([0, \infty))$ (admitting a C^∞ extension to a neighborhood of $[0, \infty)$) such that $\lim_{z \rightarrow \infty} P(z)f^{(k)}(z) = 0$ for any polynomial P and any $k \in \mathbb{N}_0$.

[11], Fu and Li [21], and Palau and Pardo [43]. Here we are able to apply directly the result [43, Proposition 1], recognizing that (1.1) is just [43, Eq. (5)]

$$\begin{aligned} Z_t = & z + \int_0^t \underline{b}(Z_s) ds + \int_0^t \int_{\underline{E}} \underline{\sigma}(Z_s, u) \underline{W}(ds, du) \\ & + \int_0^t \int_{\underline{U}} \underline{g}(Z_{s-}, u) \underline{M}(ds, du) + \int_0^t \int_{\underline{V}} \underline{h}(Z_{s-}, v) \underline{N}(ds, dv), \end{aligned}$$

with the following input data of [43] in which we underline the objects of [43] to avoid confusion with our own:

$$\begin{aligned} \underline{b}(z) &= bz - \frac{c}{2}z^2, \quad z \in [0, \infty) \\ \underline{E} &= \{1, 2\} \\ \underline{\sigma}(z, 1) &= \sigma\sqrt{z}, \quad z \in [0, \infty) \\ \underline{\sigma}(z, 2) &= az, \quad z \in [0, \infty) \\ \underline{W}(ds, de) &= B(ds)\delta_1(de) + W(ds)\delta_2(de) \\ &\quad [\text{white noise on } (0, \infty) \times E \text{ with intensity } ds\underline{\pi}(de)] \\ \underline{\pi} &= \delta_1 + \delta_2 \\ \underline{U} &= [0, \infty) \times (1, \infty) \\ \underline{g}(z, (u, h)) &= h\mathbb{1}_{(0, z]}(u), \quad (z, (u, h)) \in [0, \infty) \times \underline{U} \\ \underline{M}(ds, du, dh) &= \mathcal{N}|_{[0, \infty) \times ([0, \infty) \times (1, \infty))}(ds, du, dh) \\ &\quad [\text{Poisson random measure with intensity } ds\underline{\mu}(d(u, h))] \\ \underline{\mu}(d(u, h)) &= du\underline{\pi}(dh) \\ \underline{V} &= ([0, \infty) \times (0, 1]) \cup ([0, \infty) \times [0, \infty) \times (0, \infty)) \\ \underline{h}(z, (u, h)) &= h\mathbb{1}_{(0, z]}(u), \quad (z, (u, h)) \in [0, \infty) \times ([0, \infty) \times (0, 1]) \\ \underline{h}(z, (u_1, u_2, h)) &= h\mathbb{1}_{(0, z] \times (0, z]}(u_1, u_2), \quad (z, (u_1, u_2, h)) \in [0, \infty) \times ([0, \infty) \times [0, \infty) \times (0, \infty)) \\ \underline{N}(ds, du, dh) &= \mathcal{N}(ds, du, dh) \text{ on } [0, \infty) \times ([0, \infty) \times (0, 1]) \\ \underline{N}(ds, du_1, du_2, dh) &= \mathcal{M}(ds, du_1, du_2, dh) \text{ on } [0, \infty) \times ([0, \infty) \times [0, \infty) \times (0, \infty)) \\ \underline{\nu}(d(u, h)) &= du\underline{\pi}(dh) \text{ and } \underline{\nu}(d(u_1, u_2, h)) = du_1 du_2 \eta(dh) \\ &\quad [\text{characteristic measure of } \underline{N}]. \end{aligned}$$

Then the admissibility conditions (i)-(iv) of [43, p. 60] are met evidently: (i) b is continuous nonnegative; (ii) σ is continuous and vanishing at zero in the first entry; (iii) g is Borel and majorizing minus the identity in the first entry; (iv) h is Borel, vanishing at zero and majorizing minus the identity in the first entry. Choosing $\tilde{U} = U$ we have $\underline{\mu}(U \setminus \tilde{U}) = 0$ trivially but also

$$\int_{\tilde{U}} |g(z, (u, h))| \wedge \mathbb{1}_{\underline{\mu}(d(u, h))} \leq z\underline{\pi}(1, \infty), \quad z \in [0, \infty),$$

which verifies [is] (a) of [43, p. 60]. Choosing $\underline{b}_1(z) = bz$ and $\underline{b}_2(z) = \frac{c}{2}z^2$ and putting $\underline{r}_n(z) = [1 \vee (|b| + \int n \wedge h\underline{\pi}(dh))]z$ for $n \in \mathbb{N}_0$ we get (b) of [43, p. 60]: $\underline{b} = \underline{b}_1 - \underline{b}_2$, \underline{b}_1 continuous, \underline{b}_2 nondecreasing; \underline{r}_n nondecreasing concave, $\int_{0+} \underline{r}_n^{-1} = \infty$ and (since, for $u \in [0, \infty)$, $|(\mathbb{1}_{(0, x]}(u)h) \wedge n - (\mathbb{1}_{(0, y]}(u)h) \wedge n| = (h \wedge n) \mathbb{1}_{(0, x] \Delta (0, y]}(u)$)

$$|\underline{b}_1(x) - \underline{b}_2(y)| + \int_{\tilde{U}} |g(x, (u, h)) \wedge n - g(y, (u, h)) \wedge n| \underline{\mu}(d(u, h)) \leq \underline{r}_n(|x - y|)$$

for all $n \in \mathbb{N}_0$ and $\{x, y\} \subset [0, \infty)$. One also easily verifies (c) of [43, p. 61] taking into account that $\int h \wedge h^2 \eta(dh) < \infty$ and $\int_{(0,1]} h^2 \pi(dh) < \infty$: $z \mapsto \underline{h}(z, v) + z$ is nondecreasing; and, for each $n \in \mathbb{N}_0$, there is a $\underline{B}_n < \infty$ such that for $\{x, y\} \subset [0, n]$ we have

$$\int_{\underline{E}} |\underline{\sigma}(x, u) - \underline{\sigma}(y, u)| \underline{\pi}(du) + \int_{\underline{V}} |\underline{l}(x, y, v)| \wedge \underline{l}(x, y, v)^2 \underline{\nu}(dv) \leq \underline{B}_n |x - y|$$

with $\underline{l}(x, y, v) := \underline{h}(x, v) - \underline{h}(y, v)$ for $v \in \underline{V}$.

All in all the preceding allows us to infer the conclusion of [43, Proposition 1], which is, that for each starting value $z \in [0, \infty)$ there is an a.s. unique $[0, \infty]$ -valued càdlàg process Z , adapted to the natural filtration of $(\underline{W}, \underline{M}, \underline{N})$, that is to say, of $(W, B, \mathcal{N}, \mathcal{M})$, such that (i)-(ii) of Theorem 2.1 hold true.

It is clear from (1.1) that Z a.s. has no negative jumps and that 0 is an absorbing state for Z . Quasi left-continuity also follows directly from (1.1) because the jump times of a homogenous Poisson process are not announcable, while the integrals against the Brownian motions and the Lebesgue integrals are anyway continuous. The proof of the strong Markov property is essentially the same as for the CBM processes [52, Theorem 2.1(iii)] and boils down to the strong Markov property for the Brownian and Poisson drivers of (1.1); we omit the details. Finally, by Itô's formula, see e.g. Ikeda and Watanabe [26, Theorem II.5.1], and (1.1) again, the martingale conclusion of Theorem 2.1 follows (to see how such a computation evolves on a technical level the reader may again consult the CBM case [52, Theorem 2.1(v)], there is no fundamental difference).

The fact that the law of Z is uniquely determined by the triplet (Σ, Ψ, z) follows from the observation that pathwise uniqueness implies uniqueness in law for SDEs, which completes the proof of Theorem 2.1. \square

Corollary 3.1. *Suppose (a) $f \in C^2([0, \infty))$ has a finite limit at infinity and (b) $\mathcal{L}f$ is vanishing at infinity. Then the process $(f(Z_t) - \int_0^t \mathcal{L}f(Z_s) ds, t \geq 0)$ is a martingale and $\mathcal{L}f$ gives the action of the infinitesimal generator of Z on f (here we understand $f(\infty) := \lim_{\infty} f$ and $\mathcal{L}f(\infty) = 0$, of course). Any function from the set*

$$\mathcal{D} := \left\{ f \in C^2([0, \infty)) : \exists \lim_{z \rightarrow \infty} f(z) \in \mathbb{R}, \text{ and } \lim_{z \rightarrow \infty} z^2 (|f(z) - f(\infty)| + |f'(z)| + |f''(z)|) = 0 \right\}$$

meets the properties (a)-(b).

Example 3.2. *The Schwartz space of rapidly decaying functions, a fortiori $C_c^\infty([0, \infty))$, is a subset of \mathcal{D} . In particular, for $x \in [0, \infty)$ the exponential map $([0, \infty) \ni z \mapsto e^{-xz})$ belongs to \mathcal{D} ; moreover, $\mathcal{C} := \{\int e^{-x} \nu(dx) : \nu \text{ a finite signed measure on } \mathcal{B}_{[0, \infty)}\} \subset \mathcal{D}$.*

Remark 3.3. With a view towards the Laplace duality of (2.15) it is perhaps worth noting explicitly that for $f \in C_b^2([0, \infty))$ with $\mathcal{A}f$ bounded, for the exponential maps in particular, likewise the process $(f(U_t) - \int_0^t \mathcal{A}f(U_s) ds, t \geq 0)$ is a martingale and $\mathcal{A}f$ gives the action of the infinitesimal generator of U on f .

Proof. Note that f and $\mathcal{L}f$ are both bounded and continuous. Let S be a bounded stopping time. The local martingale of (2.5) with $\alpha = 0$ is bounded up to every bounded time, therefore a martingale. Sampling this martingale at S we get

$$\mathbb{E}_z \left[f(Z_{S \wedge \zeta_n^+}) - \int_0^{S \wedge \zeta_n^+} \mathcal{L}f(Z_s) ds \right] = f(z), \quad n \in [0, \infty).$$

Letting $n \rightarrow \infty$, $f(Z_{S \wedge \zeta_n^+}) \rightarrow f(Z_{S \wedge \zeta_\infty})$ boundedly (since $f(\infty) \in \mathbb{R}$) and $\int_0^{S \wedge \zeta_n^+} \mathcal{L}f(Z_s) ds \rightarrow \int_0^{S \wedge \zeta_\infty} \mathcal{L}f(Z_s) ds$ boundedly by bounded convergence (for the Lebesgue integral). By bounded

convergence in the displayed formula we infer that

$$\mathbb{E}_z \left[f(Z_{S \wedge \zeta_\infty}) - \int_0^{S \wedge \zeta_\infty} \mathcal{L}f(Z_s) ds \right] = f(z).$$

Since $Z = \infty$ on $[\zeta_\infty, \infty)$ and since $\mathcal{L}f(\infty) = 0$ we may get rid of “ $\wedge \zeta_\infty$ ”. It being true for arbitrary S entails that $(f(Z_t) - \int_0^t \mathcal{L}f(Z_s) ds, t \geq 0)$ is a martingale that is even bounded up to every bounded time. The second claim now follows easily:

$$\lim_{t \downarrow 0} \frac{\mathbb{E}_z[f(Z_t)] - f(z)}{t} = \lim_{t \downarrow 0} \frac{\mathbb{E}_z[\int_0^t \mathcal{L}f(Z_s) ds]}{t} = \mathcal{L}f(z),$$

by bounded convergence and the continuity of $\mathcal{L}f$ (and the right-continuity of Z at time zero).

Take now $f \in \mathcal{D}$ and we check that $\lim_{z \rightarrow \infty} \mathcal{L}f(z) = 0$. Since \mathcal{L} annihilates the constants we may and do assume that $\lim_{\infty} f = 0$. Since $f \in C^2([0, \infty))$ one has for it the following Taylor formula with integral form of remainder, see e.g. Zorich [55, page 363],

$$f(z+h) - f(z) - hf'(z) = h^2 \int_0^1 f''(z+hv)(1-v)dv, \quad \{z, h\} \subset [0, \infty).$$

Recalling (2.1)-(2.4) we estimate

$$\begin{aligned} z |L^\Psi f(z)| &\leq \frac{\sigma^2}{2} \cdot z |f''(z)| + |b| \cdot z |f'(z)| \\ &\quad + z \left| \int_0^1 \pi(dh) h^2 \int_0^1 f''(z+hv)(1-v)dv \right| + z \left| \int_1^\infty \pi(dh) (f(z+h) - f(z)) \right| \\ &\leq \frac{\sigma^2}{2} \cdot z |f''(z)| + |b| \cdot z |f'(z)| + \int_0^1 \pi(dh) h^2 \int_0^1 (z+hv) |f''(z+hv)|(1-v)dv \\ &\quad + \int_1^\infty \pi(dh) (z+h) |f(z+h)| + \pi(1, \infty) z |f(z)|, \quad z \in [0, \infty). \end{aligned}$$

Now, the terms $(z+hv)f''(z+hv)$ and $(z+h)f(z+h)$ converge towards 0 as z goes to ∞ uniformly for positive h, v , and in particular are bounded. Hence by dominated convergence, both integrals in the upper bound above vanish when z goes to ∞ . The same is true for the other terms. Similar calculations entail that $z^2 |L^\Sigma f(z)|$ converges to 0 as z goes to ∞ , which then allows us to conclude. \square

3.2. CBCs as time-changes of CBMs. CBMs are a kind-of generalization of CBIs in which, roughly speaking, the immigration subordinator is replaced by a spectrally positive Lévy process (this will be the process X in (3.1) below, Y being the CBM). Though, CBMs are stopped when reaching 0, while CBIs are not. The Brownian part, and drift when it is negative, of X are interpreted as migration (emigration/immigration) in the population. Such processes were defined and studied by Vidmar in [52].

CBCs may be connected to CBMs via time-change. On an heuristic level this is clear from the form of their generators. Indeed, comparing (2.4) with [52, Eq. (2.1)], which gives the action of the generator \mathcal{L}' of a CBM Y with branching mechanism Σ and migration mechanism Ψ on a $C_b^2([0, \infty))$ map g satisfying $L^\Psi g(0) = 0$ as

$$\mathcal{L}'g(y) := L^\Psi g(y) + yL^\Sigma g(y), \quad y \in [0, \infty),$$

strongly suggests that Z should be just a Lamperti-type transform of a such a CBM by the inverse of $\int_0^{\cdot} \frac{du}{Y_u}$. It is indeed so:

Theorem 3.4. Put $\kappa := \int_0^{\zeta_\infty} Z_t dt$, define the additive functional $\gamma := \int_0^\cdot Z_t dt$ and let γ^{-1} be its inverse on $[0, \kappa)$, extended by $\zeta_0 \wedge \zeta_\infty$ on $[\kappa, \infty)$. Set $Y := Z_{\gamma^{-1}}$, defined on $[0, \zeta)$, $\zeta := \int_0^{\zeta_\infty} Z_u du + \infty \mathbb{1}_{\{\zeta_0 < \zeta_\infty\}}$. Then $Y = (Y_u)_{u \in [0, \zeta)}$ is a CBM process with branching mechanism Σ , migration mechanism Ψ (and initial value z), $\zeta = \infty$ a.s. (non-explosivity) and letting ω be the right-continuous inverse of $\int_0^\cdot \frac{du}{Y_u}$ on $[0, \int_0^\zeta \frac{du}{Y_u})$ [with the understanding $1/0 = \infty$] we have a.s.

$$\zeta_\infty = \int_0^\zeta \frac{du}{Y_u} \text{ and } Z_t = Y_{\omega(t)} \text{ for } t \in [0, \zeta_\infty).$$

Thus the CBC Z may be viewed as driving along the sample paths of the (non-explosive) CBM Y with a velocity that is given by its position.

Proof. We time-change (1.1) into an SDE for the process Y .

By definition γ^{-1} is a continuous time-change for the filtration \mathcal{F} . Possibly by enlarging the underlying probability space we grant ourselves access to the following mutually independent stochastic items, independent also of $(B, W, \mathcal{M}, \mathcal{N})$: Brownian motions \tilde{B}, \tilde{W} ; Poisson random measures $\tilde{\mathcal{M}}(ds, du, dh)$ with intensity $ds du \eta(dh)$, $\tilde{\mathcal{N}}(ds, dh)$ with intensity $ds \pi(dh)$. Let \mathcal{F}' be $\mathcal{F}_{\gamma^{-1}}$ enlarged by the natural filtration of $(\tilde{B}, \tilde{W}, \tilde{\mathcal{M}}, \tilde{\mathcal{N}})$ and augmented.

Put $B' := \int_0^{\gamma^{-1}(\cdot)} \sqrt{Z_s} dB_s$ on $[0, \kappa)$ and extend it by the increments of \tilde{B} after κ . By the martingale characterization of Brownian motion [26, Theorem II.6.1] it follows that B' is an \mathcal{F}' -Brownian motion. In the same manner we procure an \mathcal{F}' -Brownian motion W' . The covariation process of W' with B' vanishes; thus W' and B' are actually independent \mathcal{F}' -Brownian motions.

Next define $\mathcal{M}'([0, t] \times L \times A) := \int_0^{\gamma^{-1}(t)} \int_0^{Z_{s-}} \int_L \int_A \mathcal{M}(ds, du_1, du_2, dh)$ for $t \in [0, \kappa)$ and Borel L, A . The measure \mathcal{M}' is extended in the first coordinate from $[0, \kappa)$ to $[0, \infty)$ by using $\tilde{\mathcal{M}}$ on $[\kappa, \infty)$. From the martingale characterization of Poisson point processes [26, Theorem II.6.2] it follows that $\mathcal{M}'(ds, du, dh)$ is an \mathcal{F}' -Poisson random measure with intensity measure $ds du \eta(dh)$. In an analogous way we avail ourselves of an \mathcal{F}' -Poisson random measure $\mathcal{N}'(ds, dh)$ of intensity $ds \pi(dh)$. The Poisson point processes (corresponding to) \mathcal{M}' and \mathcal{N}' a.s. have no jumps in common; therefore are actually independent.

Being defined in the common filtration \mathcal{F}' , the Brownian pair (W', B') and Poisson pair $(\mathcal{N}', \mathcal{M}')$ are also automatically independent. Thus $W', B', \mathcal{N}', \mathcal{M}'$ are jointly independent.

Rewriting (1.1) in terms of Y we get a.s.

$$Y_t = X_{t \wedge \sigma_0} + a \int_0^t \sqrt{Y_s} dW'_s - \frac{c}{2} \int_0^t Y_s ds + \int_0^t \int_0^{Y_{s-}} \int_0^\infty h \bar{\mathcal{M}}'(ds, du, dh), \quad t \in [0, \zeta), \quad (3.1)$$

where

$$X_t := z + \sigma B'_t + bt + \int_0^t \int_0^1 h \bar{\mathcal{N}}'(ds, dh) + \int_0^t \int_1^\infty h \mathcal{N}'(ds, dh), \quad t \in [0, \infty),$$

and where $\sigma_0 := \inf\{t \in [0, \zeta) : Y_t = 0\}$; also $\sup_{[0, \zeta)} Y = \infty$ a.s. on $\{\zeta < \infty\}$, by construction. It follows from [52, Theorem 2.1] that Y is a CBM with branching mechanism Σ , migration mechanism Ψ (and initial value z that of Z), which is non-explosive because Σ is (sub)critical [52, Corollary 2.5]. The proof of Theorem 3.4 is completed by pathwise arguments to go back from Y to Z . \square

When $-\Psi$ is the Laplace exponent of a subordinator, the CBM process Y is a CBI process with immigration mechanism $-\Psi$, stopped at its first hitting time of 0.

Proof of Proposition 2.2: Recall that by definition $z^* = (\limsup \frac{-\Psi}{\Sigma}) \vee 0$. Notice first that $z^* > 0$ if and only if we are in the subordinator case with $\mu := \lim_{u \rightarrow \infty} \frac{-\Psi(u)}{u} > 0$ and Σ is of

finite variation type, i.e. $D := \lim_{u \rightarrow \infty} \frac{\Sigma(u)}{u} < \infty$ (the two limits exist a priori in $[0, \infty)$ and $(0, \infty]$, respectively), in which case, $z^* = \limsup_{\infty} \frac{-\Psi}{\Sigma} = \lim_{\infty} \frac{-\Psi}{\Sigma} = \frac{\mu}{D}$. According to [12, Proposition 5], when $Y_0 = z > z^* > 0$, then $Y_t \geq e^{-Dt}z + z^*(1 - e^{-Dt}) > z^*$ for all $t \in [0, \infty)$; here Y is as in Theorem 3.4. Hence by the time-change representation of the CBC process Z , one also has $Z_t > z^*$ for all $t \in [0, \infty)$ as soon as its starting value z lies in (z^*, ∞) . \square

4. ATTRACTION TO THE BOUNDARIES: PROOF OF THEOREM 2.3

The proof of Theorem 2.3 is based on the time-change representation of CBCs via CBMs. Let then Y be the CBM of Theorem 3.4. We state several lemmas, the combined conclusion of which will be Theorem 2.3.

Lemma 4.1. *Let $\{z, \mathbf{a}\} \subset (z^*, \infty)$, $\mathbf{a} < z$. If $S_V(0, x_0] < \infty$ then $\mathbb{P}_z(\zeta_{\mathbf{a}} < \zeta_{\infty}) = \frac{S_Z(z)}{S_Z(\mathbf{a})} \in (0, 1)$ with*

$$S_Z(z) := \int_0^{\infty} e^{-xz} \frac{1}{\Sigma(x)} e^{-\int_{x_0}^x \frac{\Psi(u)}{\Sigma(u)} du} dx. \quad (4.1)$$

If $S_V(0, x_0] = \infty$ then $\mathbb{P}_z(\zeta_{\mathbf{a}} < \zeta_{\infty}) = 1$.

Proof. Set $\sigma_{\mathbf{a}} := \inf\{t \in [0, \zeta) : Y_t \leq \mathbf{a}\}$. By [52, Theorem 3.1] for the non-subordinator case, respectively by [12, Theorem 1] for the subordinator case, we have for any $\theta \in (0, \infty)$,

$$\mathbb{E}_z[e^{-\theta\sigma_{\mathbf{a}}}] = \frac{\Phi_{\theta}(z)}{\Phi_{\theta}(\mathbf{a})}, \quad (4.2)$$

where

$$\Phi_{\theta}(z) := \int_0^{\infty} \frac{dx}{\Sigma(x)} e^{-zx - \int_{x_0}^x \frac{\Psi(u) - \theta}{\Sigma(u)} du} = \int_0^{\infty} S_V(dx) e^{-zx + \int_{x_0}^x \frac{\theta}{\Sigma(u)} du}.$$

Therefore, by the time-change representation:

$$\mathbb{P}_z(\zeta_{\mathbf{a}} < \infty) = \mathbb{P}_z(\sigma_{\mathbf{a}} < \zeta) = \lim_{\theta \rightarrow 0} \frac{\Phi_{\theta}(z)}{\Phi_{\theta}(\mathbf{a})} = \lim_{\theta \rightarrow 0} \frac{\int_0^{\infty} \frac{dx}{\Sigma(x)} e^{-zx - \int_{x_0}^x \frac{\Psi(u) - \theta}{\Sigma(u)} du}}{\int_0^{\infty} \frac{dx}{\Sigma(x)} e^{-\mathbf{a}x - \int_{x_0}^x \frac{\Psi(u) - \theta}{\Sigma(u)} du}}. \quad (4.3)$$

Assume first $S_V(0, x_0] = \int_0^{x_0} \frac{dx}{\Sigma(x)} e^{-\int_{x_0}^x \frac{\Psi(u)}{\Sigma(u)} du} < \infty$. Then, since $z > \mathbf{a} > z^*$,

$$\int_0^{\infty} e^{-zx} S_V(dx) < \int_0^{\infty} e^{-\mathbf{a}x} S_V(dx) < \infty.$$

Moreover, splitting the integrals in (4.3) in two pieces, according to the domains $(0, x_0]$ and (x_0, ∞) , and applying monotone convergence on $(0, x_0]$ and dominated convergence on (x_0, ∞) , we get the convergence as θ goes to 0 of the right-hand side of (4.3) and obtain

$$\mathbb{P}_z(\zeta_{\mathbf{a}} < \zeta_{\infty}) = \frac{\int_0^{\infty} \frac{dx}{\Sigma(x)} e^{-zx - \int_{x_0}^x \frac{\Psi(u)}{\Sigma(u)} du}}{\int_0^{\infty} \frac{dx}{\Sigma(x)} e^{-\mathbf{a}x - \int_{x_0}^x \frac{\Psi(u)}{\Sigma(u)} du}} = \frac{S_Z(z)}{S_Z(\mathbf{a})} \in (0, 1). \quad (4.4)$$

Assume now $S_V(0, x_0] = \infty$. Then we see from (4.3) that

$$\begin{aligned} \mathbb{P}_z(\zeta_{\mathbf{a}} < \zeta_{\infty}) &\geq \lim_{\theta \rightarrow 0} \frac{\int_0^{x_0} e^{-xz} \frac{1}{\Sigma(x)} e^{-\int_{x_0}^x \frac{\Psi(u) - \theta}{\Sigma(u)} du} dx}{\int_0^{x_0} e^{-\mathbf{a}x} \frac{1}{\Sigma(x)} e^{-\int_{x_0}^x \frac{\Psi(u) - \theta}{\Sigma(u)} du} dx + \int_{x_0}^{\infty} e^{-\mathbf{a}x} \frac{1}{\Sigma(x)} e^{-\int_{x_0}^x \frac{\Psi(u) - \theta}{\Sigma(u)} du} dx} \\ &\geq \lim_{\theta \rightarrow 0} \frac{e^{-x_0 z} \int_0^{x_0} \frac{1}{\Sigma(x)} e^{-\int_{x_0}^x \frac{\Psi(u) - \theta}{\Sigma(u)} du} dx}{\int_0^{x_0} \frac{1}{\Sigma(x)} e^{-\int_{x_0}^x \frac{\Psi(u) - \theta}{\Sigma(u)} du} dx + \int_{x_0}^{\infty} e^{-\mathbf{a}x} \frac{1}{\Sigma(x)} e^{-\int_{x_0}^x \frac{\Psi(u) - \theta}{\Sigma(u)} du} dx} \end{aligned}$$

$$= \lim_{\theta \rightarrow 0} \frac{e^{-x_0 z}}{1 + \int_{x_0}^{\infty} e^{-x a} \frac{1}{\Sigma(x)} e^{-\int_{x_0}^x \frac{\Psi(u) - \theta}{\Sigma(u)} du} dx / \int_0^{x_0} \frac{1}{\Sigma(x)} e^{-\int_{x_0}^x \frac{\Psi(u) - \theta}{\Sigma(u)} du} dx}.$$

By monotone convergence and the fact that

$$\int_0^{x_0} \frac{1}{\Sigma(x)} e^{-\int_{x_0}^x \frac{\Psi(u) - \theta}{\Sigma(u)} du} dx \xrightarrow{\theta \rightarrow 0+} S_V(0, x_0] = \infty,$$

we obtain that $\mathbb{P}_z(\zeta_{\mathbf{a}} < \zeta_{\infty}) \geq e^{-z x_0}$ and since x_0 can be chosen arbitrarily small, we get finally $\mathbb{P}_z(\zeta_{\mathbf{a}} < \zeta_{\infty}) = 1$. \square

Lemma 4.2. *Let $z \in (z^*, \infty)$. Then $Z_t \xrightarrow[t \rightarrow \infty]{} \infty$ a.s.- \mathbb{P}_z off $\{\inf_{t \in [0, \infty)} Z_t \leq z^*\}$. Furthermore, $\mathbb{P}_z(Z_t \xrightarrow[t \rightarrow \infty]{} \infty) > 0$ if and only if $S_V(0, x_0] < \infty$.*

Proof. Given the first statement, according to Lemma 4.1, when $S_V(0, x_0] < \infty$, then $\mathbb{P}_z(\zeta_{\mathbf{a}} = \infty) > 0$ for $\mathbf{a} \in (z^*, z)$, which yields $\mathbb{P}_z(Z_t \xrightarrow[t \rightarrow \infty]{} \infty) > 0$ (by the first statement). Conversely, when the process escapes to ∞ with positive \mathbb{P}_z -probability, the \mathbb{P}_z -probability of staying above level \mathbf{a} is positive for some $\mathbf{a} \in (z^*, \infty)$ and $S_V(0, x_0]$ has to be finite, since otherwise by Lemma 4.1 the latter probability would be zero.

As for the first statement, suppose, per absurdum, that $\liminf_{t \rightarrow \infty} Z_t < \infty$ and $\inf_{t \in [0, \infty)} Z_t > z^*$ with positive \mathbb{P}_z -probability. There are therefore levels $N \in [0, \infty)$ and $\mathbf{a} \in (z^*, \infty)$ such that on an event A of positive \mathbb{P}_z -probability $Z_t \leq N$ at arbitrarily large times $t \in [0, \infty)$ (in particular, necessarily $\zeta_{\infty} = \infty$) but $Z_t > \mathbf{a}$ for all $t \in [0, \infty)$. For sure $p_0 := \mathbb{P}_N(\zeta_{\mathbf{a}} < \zeta_{\infty}) > 0$ (since $\mathbf{a} > z^*$), hence there is $r \in (0, \infty)$ such that $\mathbb{P}_y(\zeta_{\mathbf{a}} < r) \geq p_0$ for $y = N$, a fortiori for all $y \in [0, N]$ (by the absence of negative jumps). Put, inductively,

$$S_k := \inf\{t \in [S_{k-1}, \zeta_{\infty}) : Z_t \in [0, N]\} + r, \quad k \in \mathbb{N},$$

with the convention $S_0 := 0$. Thus, in plainer tongue, $S_1 =$ (the first time Z enters $[0, N]) + r$; $S_2 :=$ (the first time Z enters $[0, N]$ after $S_1) + r$; and so on. Perhaps $S_k = \infty$ at some $k \in \mathbb{N}$, in which case $S_{l+1} = \infty$ for all $l \in \mathbb{N}$, $l \geq k$. But anyway

$$\{Z_t \leq N \text{ at arbitrarily large times } t \in [0, \infty)\} \subset \{S_k < \infty \text{ for all } k \in \mathbb{N}\}.$$

Now, Z always has (at least) the strictly positive chance p_0 to hit \mathbf{a} before a time of length r has elapsed, no matter where in $[0, N]$ it starts. On A it must fail to do so infinitely many times over. By an inductive application of the strong Markov property at the stopping times $S_k - r$, $k \in \mathbb{N}$, one gets that $\mathbb{P}_z(A) \leq (1 - p_0)^m$ for all $m \in \mathbb{N}$ (strong Markov property is done backwards here: first condition on $\mathcal{F}_{S_m - r}$ and estimate not hitting \mathbf{a} on $[S_m - r, S_m)$, you get a factor of $1 - p_0$; then condition on $\mathcal{F}_{S_{m-1} - r}$, and so on), hence on letting $m \rightarrow \infty$, $\mathbb{P}_z(A) = 0$, which is a contradiction. \square

Lemma 4.3. *Let $z \in (z^*, \infty)$. If $\Psi \neq 0$, then the following equivalences hold: $Z_t \xrightarrow[t \rightarrow \infty]{} 0$ with positive \mathbb{P}_z -probability (respectively \mathbb{P}_z -almost surely) if and only if $S_V(x_0, \infty) < \infty$ (respectively $S_V(x_0, \infty) < \infty$ and $S_V(0, x_0] = \infty$); when $S_V(0, \infty) < \infty$, then, moreover, $\mathbb{P}_z(Z_t \xrightarrow[t \rightarrow \infty]{} 0) = \frac{S_Z(z)}{S_Z(0)} \in (0, 1)$, where S_Z is as in (4.1). If $\Psi = 0$, then \mathbb{P}_z -almost surely $Z_t \xrightarrow[t \rightarrow \infty]{} 0$.*

Proof. Recall that by Theorem 3.4, the CBC process Z is the Lamperti time-change of the CBM process Y . If $\Psi = 0$, then Y is a CB process with branching mechanism Σ . Since Σ is (sub)critical, one then has $Y_t \xrightarrow[t \rightarrow \infty]{} 0$ a.s.- \mathbb{P}_z , always, see e.g. [34, Theorem 12.7], therefore $Z_t \xrightarrow[t \rightarrow \infty]{} 0$ a.s.- \mathbb{P}_z .

Assume now $\Psi \neq 0$. Recall σ_0 is the first hitting time of 0 by the CBM Y . CBMs and CBIs (that are not CBs) cannot converge towards 0 without hitting it (said another way, they do not extinguish), i.e. we have that $\{Y_t \xrightarrow[t \rightarrow \infty]{} 0\} = \{\sigma_0 < \infty\}$ a.s.. For CBMs (that are not CBIs) this

is noted in [52, Corollary 3.5], for CBIs (that are not CBs), it follows at once from [12, Eq. (18)] which states that such a process has infinite superior limit. One has therefore the following almost sure equality of events $\{Z_t \xrightarrow[t \rightarrow \infty]{} 0\} = \{\sigma_0 < \infty\}$. By letting θ go to 0 in (4.2) when $z^* = 0$, trivially by Proposition 2.2 for the case when $z^* > 0$, we see that

$$\mathbb{E}_z[e^{-\theta\sigma_0}; \sigma_0 < \zeta] = \frac{\Phi_\theta(z)}{\Phi_\theta(0)}, \quad (4.5)$$

and $\Phi_\theta(0) = \int_0^\infty \frac{dx}{\Sigma(x)} e^{-\int_{x_0}^x \frac{\Psi(u)-\theta}{\Sigma(u)} du} < \infty$ if and only if $S_V(x_0, \infty) = \int_{x_0}^\infty \frac{dx}{\Sigma(x)} e^{-\int_{x_0}^x \frac{\Psi(u)}{\Sigma(u)} du} < \infty$ (note that $S_V(x_0, \infty) < \infty$ entails $\int_{x_0}^\infty \frac{du}{\Sigma(u)} < \infty$ which in turn ensures that $\Phi_\theta(0) < \infty$ when $S_V(x_0, \infty) < \infty$). Thus $\sigma_0 < \infty$ with positive \mathbb{P}_z -probability if and only if $S_V(x_0, \infty) < \infty$. This establishes the equivalence for convergence towards 0 with positive probability. For the almost sure convergence, we may and do assume $S_V(x_0, \infty) < \infty$. Then the same reasoning as in the proof of Lemma 4.1 yields

$$\mathbb{P}_z(\sigma_0 < \zeta) \geq \lim_{\theta \rightarrow 0} \frac{e^{-x_0 z}}{1 + \int_{x_0}^\infty \frac{1}{\Sigma(x)} e^{-\int_{x_0}^x \frac{\Psi(u)-\theta}{\Sigma(u)} du} dx / \int_0^{x_0} \frac{1}{\Sigma(x)} e^{-\int_{x_0}^x \frac{\Psi(u)-\theta}{\Sigma(u)} du} dx}.$$

If $S_V(0, x_0] = \infty$ the denominator above converges to 1 as $\theta \rightarrow 0$ and we have $\mathbb{P}_z(\sigma_0 < \zeta) \geq e^{-x_0 z}$; since x_0 can be chosen arbitrarily small we get $\mathbb{P}_z(\sigma_0 < \zeta) = 1$. Conversely, if $S_V(0, x_0] < \infty$, i.e. (together with $S_V(x_0, \infty) < \infty$) $S_V(0, \infty) < \infty$, then by dominated convergence in (4.5) we get $\mathbb{P}_z(Z_t \xrightarrow[t \rightarrow \infty]{} 0) = \frac{S_Z(z)}{S_Z(0)}$. \square

Proof of Theorem 2.3: Statements (i) and (ii) follow from Lemma 4.1. Parts (iii) and (iv) follow from Lemmas 4.2 and 4.3. \square

5. STUDY OF EXPLOSION, FIRST PASSAGE TIMES & EXTINCTION

5.1. A sufficient condition for non-explosion: proof of Proposition 2.6. We know already that if $S_V(0, x_0] = \infty$ then ∞ is not attracting for Z and therefore Z does not explode, see Remark 2.7. In particular if Ψ is (sub)critical then $S_V(0, x_0] = \infty$. We finish the proof by establishing through a series of lemmas that when $\Psi'(0+) \in [-\infty, 0)$ and $\int_{0+} \frac{dx}{-\Psi(x)} = \infty$, then the CBC(Σ, Ψ) cannot explode.

The first lemma provides an increasing invariant function for supercritical CBs. We state it separately as it can be of independent interest for other generalisations of CBs. Call \mathcal{L}^b the generator of the CB(Ψ), viz. for $f \in C_b^2([0, \infty))$ and $z \in [0, \infty)$, $\mathcal{L}^b f(z) := zL^\Psi f(z)$. Assume $\Psi'(0+) \in [-\infty, 0)$ and put $\rho := \sup\{x \in (0, \infty) : \Psi(x) < 0\} \in (0, \infty]$. Pick $x_0 \in (0, \rho)$ and let $\theta \in (0, \infty)$. Set

$$\bar{f}_\theta(z) := \int_0^\rho (1 - e^{-xz}) \frac{\theta}{-\Psi(x)} e^{\int_{x_0}^x \frac{\theta}{-\Psi(u)} du} dx \in (0, \infty], \quad z \in [0, \infty). \quad (5.1)$$

Lemma 5.1 (Increasing eigenfunction of CB(Ψ)). *Assume $\theta \in (0, -\Psi'(0+))$. Then*

$$\bar{f}_\theta(z) = z \int_0^\rho e^{-xz} e^{\int_{x_0}^x \frac{\theta}{-\Psi(u)} du} dx < \infty, \quad z \in [0, \infty); \quad (5.2)$$

furthermore

- (i) \bar{f}_θ is an increasing solution to $\mathcal{L}^b \bar{f}_\theta = \theta \bar{f}_\theta$ and
- (ii) \bar{f}_θ is bounded if and only if $\int_{0+} \frac{du}{-\Psi(u)} < \infty$.

Proof. First we check that $\bar{f}_\theta(z) < \infty$ for all $z \in [0, \infty)$. Recall that $1 - e^{-xz} \leq (xz) \wedge 1$ for $x \in [0, \infty)$ and that $\frac{-\Psi(x)}{x} \xrightarrow[x \rightarrow 0]{} -\Psi'(0+) \in (0, \infty]$. Let $c \in (\theta, -\Psi'(0+))$; there exists then x_0

(which we may vary to our convenience changing \bar{f}_θ only by a multiplicative constant) close enough to 0 such that for all $u \in (0, x_0]$, $\frac{-\Psi(u)}{u} \geq c$, thus $\frac{u}{-\Psi(u)} \leq \frac{1}{c}$ and

$$\begin{aligned} \bar{f}_\theta(z) &\leq \theta \int_0^{x_0} \frac{xz}{-\Psi(x)} e^{\int_x^{x_0} \frac{\theta}{cu} du} dx + \int_{x_0}^\rho \frac{\theta}{-\Psi(x)} e^{\int_x^{x_0} \frac{\theta}{-\Psi(u)} du} dx \\ &\leq \theta \frac{z}{c} \int_0^{x_0} \left(\frac{x_0}{x}\right)^{\theta/c} dx - e^{-\int_{x_0}^\rho \frac{\theta}{-\Psi(u)} du} \Big|_{x=x_0}^{x=\rho} < \infty, \text{ since } \theta/c < 1. \end{aligned}$$

(i). It is plain that \bar{f}_θ is increasing. Notice that $\mathcal{L}_z^b(1 - e^{-xz}) = -z\Psi(x)e^{-xz}$ for $x \in [0, \infty)$, $z \in [0, \infty)$. Differentiation under the integral sign and Tonelli's theorem, then integration by parts yield

$$\begin{aligned} \mathcal{L}^b \bar{f}_\theta(z) &= \int_0^\rho z(-\Psi(x))e^{-xz} \frac{\theta}{-\Psi(x)} e^{\int_x^{x_0} \frac{\theta}{-\Psi(u)} du} dx = \theta \int_0^\rho z e^{-xz} e^{\int_x^{x_0} \frac{\theta}{-\Psi(u)} du} dx \\ &= \left(\theta(1 - e^{-xz}) e^{\int_x^{x_0} \frac{\theta}{-\Psi(u)} du} \right) \Big|_{x=0}^\rho + \theta \bar{f}_\theta(z) = \theta \bar{f}_\theta(z), \end{aligned}$$

where the last equality uses again the estimates $1 - e^{-xz} \leq (xz) \wedge 1$ and $e^{\int_x^{x_0} \frac{\theta}{-\Psi(u)} du} \leq \left(\frac{x_0}{x}\right)^{\theta/c}$, so that $\lim_{x \downarrow 0} (1 - e^{-xz}) e^{\int_x^{x_0} \frac{\theta}{-\Psi(u)} du} = 0$, but also the fact that $\Psi(x)$ behaves like a linear function vanishing at ρ and with strictly positive slope around ρ when $\rho < \infty$, respectively that $-\Psi$ is bounded in linear growth when $\rho = \infty$, which renders $\lim_{x \uparrow \rho} (1 - e^{-xz}) e^{\int_x^{x_0} \frac{\theta}{-\Psi(u)} du} = 0$. En route we have checked the equality in (5.2).

(ii). Note that by definition, as z goes to ∞ , $\bar{f}_\theta(z)$ tends by monotone convergence to

$$\int_0^\rho \frac{\theta}{-\Psi(x)} e^{\int_x^{x_0} \frac{\theta}{-\Psi(u)} du} dx = -e^{-\int_{x_0}^\rho \frac{\theta}{-\Psi(u)} du} \Big|_{x=0}^{x=\rho} = e^{\int_0^{x_0} \frac{\theta}{-\Psi(u)} du} \in (0, \infty].$$

So \bar{f}_θ is bounded according to whether $\int_{0+} \frac{du}{-\Psi(u)} < \infty$ or not. \square

We now return to CBCs. Recall \mathcal{L} of (2.4).

Lemma 5.2. *Assume $\theta \in (0, -\Psi'(0+))$. Then $\mathcal{L}\bar{f}_\theta \leq \theta\bar{f}_\theta$.*

Proof. Set $\mathcal{L}^c f(z) := z^2 L^\Sigma f(z)$ for $f \in C_b^2([0, \infty))$, $z \in [0, \infty)$, so that $\mathcal{L} = \mathcal{L}^c + \mathcal{L}^b$. For $z \in [0, \infty)$ we estimate

$$\begin{aligned} \mathcal{L}^c \bar{f}_\theta(z) &= \int_0^\rho \mathcal{L}_z^c(1 - e^{-xz}) \frac{\theta}{-\Psi(x)} e^{\int_x^{x_0} \frac{\theta}{-\Psi(u)} du} dx \\ &= \int_0^\rho (-z^2 \Sigma(x) e^{-xz}) \frac{\theta}{-\Psi(x)} e^{\int_x^{x_0} \frac{\theta}{-\Psi(u)} du} dx \leq 0. \end{aligned}$$

Thus $\mathcal{L}\bar{f}_\theta = \mathcal{L}^c \bar{f}_\theta + \mathcal{L}^b \bar{f}_\theta \leq \mathcal{L}^b \bar{f}_\theta = \theta\bar{f}_\theta$ on using Lemma 5.1. \square

The next lemma concludes the argument for Proposition 2.6.

Lemma 5.3. *Assume $\Psi'(0+) \in [-\infty, 0)$. If $\int_{0+} \frac{du}{-\Psi(u)} = \infty$ then the CBC(Σ, Ψ) process Z does not explode.*

Proof. Pick a $\theta \in (0, -\Psi'(0+))$. Since $\int_{0+} \frac{dx}{-\Psi(x)} = \infty$, we have $\uparrow\text{-}\lim_{z \rightarrow \infty} \bar{f}_\theta(z) = \infty$. Fix also an $r \in (0, \infty)$. For $c \in [r, \infty)$ let \bar{f}_θ^c be any nonnegative $C_b^2([0, \infty))$ function which agrees with \bar{f}_θ on $[0, c)$ and minorizes \bar{f}_θ everywhere, e.g. one such function is obtained by taking any nonnegative $C^1([0, \infty))$ map h that agrees with \bar{f}'_θ on $[0, c)$, minorizes \bar{f}'_θ everywhere and which vanishes on a neighborhood of infinity (clearly it exists), and then putting $\bar{f}_\theta^c(x) := \bar{f}_\theta(c) + \int_c^x h(y) dy$, $x \in [c, \infty)$

(note that \bar{f}_θ itself is nonnegative $C^2([0, \infty))$, which follows by differentiation under the integral sign in (5.1)). Since $\bar{f}_\theta \geq \bar{f}_\theta^c$ for the first inequality, and by Lemma 5.2 for the second,

$$\mathcal{L}\bar{f}_\theta^c(z) \leq \mathcal{L}\bar{f}_\theta(z) \leq \theta\bar{f}_\theta(z) = \theta\bar{f}_\theta^c(z), \quad z \in [0, r).$$

Taking into account that \bar{f}_θ^c is bounded, it follows from the statement surrounding (2.5) that the process $(e^{-\theta(t \wedge \zeta_r^+)} \bar{f}_\theta^c(Z_{t \wedge \zeta_r^+}), t \geq 0)$ is a supermartingale; hence, for all $t \in [0, \infty)$,

$$\mathbb{E}_z[e^{-\theta(t \wedge \zeta_r^+)} \bar{f}_\theta^c(Z_{t \wedge \zeta_r^+})] \leq \bar{f}_\theta^c(z).$$

Let now $c \rightarrow \infty$, consider the event $\{\zeta_r^+ \leq t\}$ and recall that \bar{f}_θ is nondecreasing; one gets

$$\bar{f}_\theta(z) \geq \mathbb{E}_z[e^{-\theta(t \wedge \zeta_r^+)} \bar{f}_\theta(Z_{t \wedge \zeta_r^+})] \geq \mathbb{E}_z[e^{-\theta\zeta_r^+} \bar{f}_\theta(Z_{\zeta_r^+}) \mathbb{1}_{\{\zeta_r^+ \leq t\}}] \geq \bar{f}_\theta(r) \mathbb{E}_z[e^{-\theta\zeta_r^+} \mathbb{1}_{\{\zeta_r^+ \leq t\}}].$$

Letting next $t \rightarrow \infty$ we get the bound $\mathbb{E}_z[e^{-\theta\zeta_r^+}] \leq \frac{\bar{f}_\theta(z)}{\bar{f}_\theta(r)}$. Effecting finally the limit $r \rightarrow \infty$ yields

$$\mathbb{E}_z[e^{-\theta\zeta_\infty}] \leq \lim_{r \rightarrow \infty} \frac{\bar{f}_\theta(z)}{\bar{f}_\theta(r)} = 0,$$

which means that $\zeta_\infty = \infty$ a.s., as required. \square

5.2. A decreasing eigenfunction of Z : proof of Theorem 2.8. The proof will again proceed in several steps. We start by linking nondecreasing eigenfunctions of \mathcal{G} to decreasing ones for \mathcal{L} . Recall the form (2.4) of \mathcal{L} , the Laplace duality $\mathcal{L}_z e^{-vz} = z^2 \Sigma(v) e^{-zv} + z \Psi(v) e^{-zv} = \mathcal{A}_v e^{-zv}$, and the action $\mathcal{G}h = \Sigma h'' + (\Sigma' + \Psi)h' = (\Sigma h')' + \Psi h'$. Observe also that the equation $\mathcal{G}h = \theta h$ admits at least one strictly increasing solution $h : (0, \infty) \rightarrow (0, \infty)$, see e.g. Mandl [41, #3, Chapter II, page 28].

Lemma 5.4 (A decreasing eigenfunction). *Let $\theta \in (0, \infty)$, and suppose $h_\theta \in C^2((0, \infty))$ is nonnegative, not zero, nondecreasing and satisfies $\mathcal{G}h_\theta = \theta h_\theta$ on $(0, \infty)$. Put*

$$f_\theta(z) := z \int_0^\infty e^{-zv} h_\theta(v) dv = h_\theta(0+) + \int_0^\infty e^{-zv} h'_\theta(v) dv, \quad z \in (0, \infty). \quad (5.3)$$

Then $\mathcal{L}f_\theta = \theta f_\theta$ on the interior of $\{f_\theta < \infty\}$.

Proof. The equality in (5.3) follows by Tonelli. For z from the interior of $\{f_\theta < \infty\}$, we compute by differentiating under the integral sign & using Tonelli, then via per partes:

$$\begin{aligned} \mathcal{L}f_\theta(z) &= \mathcal{L}_z \int_0^\infty e^{-zv} h'_\theta(v) dv = \int_0^\infty \mathcal{L}_z e^{-zv} h'_\theta(v) dv = \int_0^\infty (z^2 \Sigma(v) e^{-zv} + z \Psi(v) e^{-zv}) h'_\theta(v) dv \\ &= \lim_{\epsilon \downarrow 0} \Sigma(\epsilon) h'_\theta(\epsilon) z e^{-\epsilon z} + \lim_{n \rightarrow \infty} -\Sigma(n) h'_\theta(n) z e^{-zn} \\ &\quad + \int_\epsilon^n \left(\frac{d}{dv} (\Sigma(v) h'_\theta(v)) + \Psi(v) h'_\theta(v) \right) z e^{-zv} dv \\ &= \lim_{\epsilon \downarrow 0} \Sigma(\epsilon) h'_\theta(\epsilon) z e^{-\epsilon z} + \lim_{n \rightarrow \infty} -\Sigma(n) h'_\theta(n) z e^{-zn} + \int_\epsilon^n \mathcal{G}h_\theta(v) z e^{-zv} dv \\ &= \lim_{\epsilon \downarrow 0} \Sigma(\epsilon) h'_\theta(\epsilon) z e^{-\epsilon z} + \lim_{n \rightarrow \infty} -\Sigma(n) h'_\theta(n) z e^{-zn} + \theta \int_\epsilon^n h_\theta(z) z e^{-zv} dv \\ &= \theta z \int_0^\infty h_\theta(z) e^{-zv} dv + \lim_{\epsilon \downarrow 0} \Sigma(\epsilon) h'_\theta(\epsilon) z e^{-\epsilon z} + \lim_{n \rightarrow \infty} -\Sigma(n) h'_\theta(n) z e^{-zn} \\ &= \theta f_\theta(z) + \lim_{\epsilon \downarrow 0} \Sigma(\epsilon) h'_\theta(\epsilon) z e^{-\epsilon z} + \lim_{n \rightarrow \infty} -\Sigma(n) h'_\theta(n) z e^{-zn} \\ &= \theta f_\theta(z) + \lim_{\epsilon \downarrow 0} \Sigma(\epsilon) h'_\theta(\epsilon) z e^{-\epsilon z} + \lim_{n \rightarrow \infty} -\Sigma(n) h'_\theta(n) z e^{-zn}. \end{aligned}$$

In particular the limits $\lim_{n \rightarrow \infty} -\Sigma(n)h'_\theta(n)ze^{-zn}$ and $\lim_{\epsilon \downarrow 0} \Sigma(\epsilon)h'_\theta(\epsilon)ze^{-\epsilon z}$ both exist in \mathbb{R} for all z from the interior of $\{f_\theta < \infty\}$. The first limit must in fact be zero, since such z can always be made a little smaller. As for the second limit, it is (modulo z) $\lim_{\epsilon \downarrow 0} \Sigma(\epsilon)h'_\theta(\epsilon)$. Suppose per absurdum that this limit is not zero, hence, from $(0, \infty)$. Since $\int_{0+} \frac{1}{\Sigma(x)} dx = \infty$, we see that $\int_{0+} h'_\theta(x) dx$ diverges, hence $h_\theta(x)$ would be infinite for all $x > 0$, which is a contradiction. \square

We now check that the function defined in (5.3) is finite on (z^*, ∞) , where we may recall that $z^* = (\limsup_{\infty} \frac{-\Psi}{\Sigma}) \vee 0 < \infty$ and that actually $z^* = 0$ except possibly in the subordinator case, see Proposition 2.2.

Lemma 5.5. *For all $\theta \in (0, \infty)$ the function f_θ in (5.3) is finite on (z^*, ∞) .*

Proof. Write $h := h_\theta$ for short and consider $g := \Sigma h'$. Let $z \in (z^*, \infty)$. Pick a $c \in (0, \infty)$ such that $A_c + \sqrt{\frac{\theta}{\Sigma(c)}} < z$, where $A_c := (\sup_{[c, \infty)} \frac{-\Psi}{\Sigma}) \vee 0$. We have $g' = -\frac{\Psi}{\Sigma}g + \theta \int_c^{\cdot} \frac{g}{\Sigma} + \theta h(c) \leq A_c g + \frac{\theta}{\Sigma(c)} \int_c^{\cdot} g + \theta h(c)$ on $[c, \infty)$. Then let \tilde{g} be the $C^2([c, \infty))$ solution to $\tilde{g}' = A_c \tilde{g} + \frac{\theta}{\Sigma(c)} \int_c^{\cdot} \tilde{g} + \theta h(c)$ with initial condition $\tilde{g}(c) = g(c) + 1$; in other words, solution of the second order o.d.e. with constant coefficients $\tilde{g}'' = A_c \tilde{g}' + \frac{\theta}{\Sigma(c)} \tilde{g}$, $\tilde{g}(c) = g(c) + 1$, $\tilde{g}'(c) = A_c \tilde{g}(c) + \theta h(c)$. The function \tilde{g} is a linear combination of (at most) two exponentials with absolute rate $\leq A_c + \sqrt{\frac{\theta}{\Sigma(c)}} < z$ (using the elementary estimate $\sqrt{a^2 + b^2} \leq a + b$, $\{a, b\} \subset [0, \infty)$, to get a bound on the roots of the characteristic polynomial). Furthermore $\zeta := \tilde{g} - g$ satisfies $\zeta(c) = 1$ and $\zeta' \geq A_c \zeta + \frac{\theta}{\Sigma(c)} \int_c^{\cdot} \zeta$; therefore $\zeta \geq 0$ (even ≥ 1), i.e. $g \leq \tilde{g}$ throughout $[c, \infty)$. Consequently $h' = \frac{g}{\Sigma} \leq \Sigma(c)^{-1}g \leq \Sigma(c)^{-1}\tilde{g}$ on $[c, \infty)$. The derivative of h' being bounded (up to a multiplicative constant) on a neighborhood of infinity with an exponential of rate $< z$, the same is true of h itself. The claim follows. \square

Under the assumption of non-explosion the next lemma characterizes the Laplace transforms of the first-passage times via the maps f_θ , $\theta \in (0, \infty)$.

Lemma 5.6. *Assume that the process Z does not explode. Let $\theta \in (0, \infty)$ and let f_θ be defined as in (5.3). Then for $a \leq z$ from (z^*, ∞) ,*

$$\mathbb{E}_z[e^{-\theta \zeta_a}] = \frac{f_\theta(z)}{f_\theta(a)}. \quad (5.4)$$

Proof. By Lemmas 5.4 and 5.5 the map f_θ is finite and $\mathcal{L}f_\theta = \theta f_\theta$, both on (z^*, ∞) . Since h_θ is not zero, f_θ is strictly positive everywhere. Besides, $Z_{\zeta_a} = a$ a.s.- \mathbb{P}_z on $\{\zeta_a < \infty\}$, because there are no negative jumps. By Theorem 2.1 and the non-explosiveness of Z , the process $(e^{-\theta(t \wedge \zeta_a)} f_\theta(Z_{t \wedge \zeta_a}), t \geq 0)$ is a local martingale, which is bounded by $f_\theta(a)$ (since f_θ is decreasing), hence a martingale. Therefore

$$\mathbb{E}_z[e^{-\theta(\zeta_a \wedge t)} f_\theta(Z_{\zeta_a \wedge t})] = f_\theta(z), \quad t \in [0, \infty).$$

Letting t tend to ∞ gives the target identity (5.4). \square

Uniqueness of the solution h_θ up to a multiplicative constant is settled by

Lemma 5.7. *Assume that the CBC(Σ, Ψ) does not explode. Then, up to a multiplicative constant, there is a unique nondecreasing, not zero, nonnegative function h_θ , solution h to $\mathcal{G}h = \theta h$.*

Proof. Up to a multiplicative constant the function $f_\theta : (z^*, \infty) \rightarrow (0, \infty)$ satisfying $\mathbb{E}_z[e^{-\theta \zeta_a}] = \frac{f_\theta(z)}{f_\theta(a)}$ for $a \leq z$ from (z^*, ∞) is unique evidently. In turn this guarantees the same kind of uniqueness of the nondecreasing, not zero, nonnegative solution h to $\mathcal{G}h = \theta h$, as if there were two different solutions, Lemma 5.6 would provide two different (in the preceding sense) functions f_θ (since finite Laplace transforms on a neighborhood of infinity determine continuous functions uniquely). \square

All in all, under non-explosion of Z the function h_θ of Lemma 5.4 exists uniquely (up to a multiplicative constant) and is strictly increasing and strictly positive everywhere. The proof of Theorem 2.8 follows straightforwardly by combining the above lemmas.

Remark 5.8. If the existence of a strictly increasing solution $h : (0, \infty) \rightarrow (0, \infty)$ to $\mathcal{G}h = \theta h$ is never in question, several (differing by more than a multiplicative constant) such solutions exist when the boundary 0 of \mathcal{G} is regular, see e.g. Borodin and Salminen [3, Chapter II, Section 1, Paragraph 10]. Thus, when Z does not explode, since there is a unique such solution to $\mathcal{G}h = \theta h$, then the boundary 0 of V cannot be regular. At this stage though we cannot as yet specify whether the boundary 0 is natural, entrance or exit, see the forthcoming Remark 6.3. Note however that under the assumption of Proposition 2.6 the process Z does not explode and it can be checked from Feller's tests on the other hand that the boundary 0 of V is inaccessible in this case (hence either entrance or natural).

The solution h_θ may be represented with the help of τ_y , the first hitting time of y by the diffusion V . Namely we have, for $v < y$ from $(0, \infty)$,

$$\mathbb{E}_v[e^{-\theta\tau_y}] = \frac{h_\theta(v)}{h_\theta(y)}. \quad (5.5)$$

Here, as usual, the subscript v in the expectation indicates the starting value of V .

5.3. Extinction: proof of Theorem 2.11. We focus here on extinction under the assumption of non-explosion. We first verify (2.12) in case $z^* = 0$. For sure $\zeta_{0+} := \uparrow\text{-lim}_{a \downarrow 0} \zeta_a \leq \zeta_0$. On $\{\zeta_{0+} = \infty\}$ trivially $\zeta_0 = \infty = \zeta_{0+}$; on $\{\zeta_{0+} < \infty\}$, due to quasi-left continuity and the absence of negative jumps, a.s.

$$Z_{\zeta_{0+}} = \lim_{a \downarrow 0} Z_{\zeta_a} = \lim_{a \downarrow 0} a = 0,$$

and thus $\zeta_{0+} \geq \zeta_0$, which ensures that (again) $\zeta_{0+} = \zeta_0$. Hence, by letting a go to 0 in (5.4), we have

$$\mathbb{E}_z[e^{-\theta\zeta_0}] = \frac{f_\theta(z)}{f_\theta(0+)}.$$

Besides, from (5.3), $f_\theta(0+) = h_\theta(\infty)$. Therefore

$$\mathbb{E}_z[e^{-\theta\zeta_0}] = \int_0^\infty z e^{-xz} \frac{h_\theta(x)}{h_\theta(\infty)} dx. \quad (5.6)$$

Thanks to (5.5) we may indeed rewrite this as

$$\mathbb{E}_z[e^{-\theta\zeta_0}] = \mathbb{E}[e^{-\theta\tau_\infty^{ez}}],$$

where τ_∞^{ez} is the explosion time of the diffusion V started from an independent exponential random variable with parameter z .

We proceed to study accessibility of the boundary 0 of the CBC. If $z^* = 0$ then by letting θ go to 0 in (5.6), we see that it is accessible if and only if ∞ is accessible for the diffusion V . Recall the scale and speed measures of V , S_V and M_V , given in (2.7) and (2.13) respectively. Define

$$\mathcal{I} := \int_{x_0}^\infty S_V(x, \infty) dM_V(x) = \int_{x_0}^\infty e^{Q(u)} \left(\int_u^\infty \frac{e^{-Q(x)}}{\Sigma(x)} dx \right) du, \quad (5.7)$$

where

$$Q(u) := \int_{x_0}^u \frac{\Psi(v)}{\Sigma(v)} dv, \quad u \in (0, \infty).$$

Feller's classification ensures that $h_\theta(\infty) < \infty$ (i.e. ∞ accessible for V) iff $\mathcal{I} < \infty$ see e.g. [32, Lemma 6.2, page 230]. We are left to show that $\mathcal{I} < \infty$ if and only if $\Psi(\infty) = \infty$ and $\int^\infty \frac{du}{\Psi(u)} < \infty$

(Grey's condition), indeed thanks to Proposition 2.2 this will handle also (2.11)-(2.12) for the case when $z^* > 0$.

Assume $\Psi(\infty) < \infty$ in the first instance, so that $-\Psi$ is the Laplace exponent of a subordinator. Since $-\Psi(v) \geq 0$ for all $v \geq 0$, we get the following lower bound,

$$\mathcal{I} = \int_{x_0}^{\infty} \left(\int_u^{\infty} \frac{e^{-\int_u^x \frac{\Psi(v)}{\Sigma(v)} dv}}{\Sigma(x)} dx \right) du \geq \int_{x_0}^{\infty} \left(\int_u^{\infty} \frac{dx}{\Sigma(x)} \right) du = \int_{x_0}^{\infty} \frac{u - x_0}{\Sigma(u)} du = \infty,$$

where in the penultimate equality we have applied Tonelli's theorem, and we recall that $\Sigma(u) \leq Cu^2$ for all $u \in [x_0, \infty)$ for some constant $C < \infty$, which gives finally the divergence of the integral.

Now assume that $\Psi(\infty) = \infty$. Recall that Ψ is positive increasing on (ρ, ∞) , where $\rho \in [0, \infty)$ is the largest zero of Ψ ; moreover, $(0, \infty) \ni u \mapsto \Psi(u)/u$ is nondecreasing. We may and do insist that $x_0 \in (\rho, \infty)$. There exists $c > 0$ such that for all $u \in [x_0, \infty)$, $\Psi(u) \geq cu$. Then, for all $x \in [x_0, \infty)$, $Q(x) = \int_{x_0}^x \frac{\Psi(u)}{\Sigma(u)} du \geq c \int_{x_0}^x \frac{u}{\Sigma(u)} du$. Therefore, $Q(x) \geq \frac{c}{C} \log x$ for all $x \in [x_0, \infty)$, in particular $Q(\infty) = \infty$.

For typographical ease set also

$$\varphi(u) := \int_u^{\infty} \frac{1}{\Sigma(x)} e^{-Q(x)} dx \leq \int_u^{\infty} \frac{dx}{x^{c/C} \Sigma(x)} < \infty, \quad u \in [x_0, \infty).$$

Note that

$$\begin{aligned} (e^{Q(u)} \varphi(u))' &= Q'(u) e^{Q(u)} \varphi(u) + e^{Q(u)} \varphi'(u) \\ &= Q'(u) e^{Q(u)} \varphi(u) - e^{Q(u)} \frac{1}{\Sigma(u)} e^{-Q(u)} \\ &= (\Psi(u) e^{Q(u)} \varphi(u) - 1) \frac{1}{\Sigma(u)}, \quad u \in [x_0, \infty). \end{aligned}$$

Hence, for $x \in [x_0, \infty)$,

$$\int_{x_0}^x \varphi(u) e^{Q(u)} du = \int_{x_0}^x \frac{du}{\Psi(u)} + \int_{x_0}^x (e^{Q(u)} \varphi(u))' \frac{\Sigma(u)}{\Psi(u)} du. \quad (5.8)$$

Furthermore, since $\Psi(u) \leq \Psi(x)$ for all $x \geq u \geq x_0$ and $Q(\infty) = \infty$,

$$\Psi(u) e^{Q(u)} \varphi(u) = \Psi(u) e^{\int_{x_0}^u \frac{\Psi(v)}{\Sigma(v)} dv} \int_u^{\infty} \frac{1}{\Sigma(x)} e^{-\int_{x_0}^x \frac{\Psi(v)}{\Sigma(v)} dv} dx \leq e^{\int_{x_0}^u \frac{\Psi(v)}{\Sigma(v)} dv} \int_u^{\infty} \frac{\Psi(x)}{\Sigma(x)} e^{-\int_{x_0}^x \frac{\Psi(v)}{\Sigma(v)} dv} dx = 1.$$

Thus $(e^{Q(u)} \varphi(u))' \leq 0$ for all $u \in [x_0, \infty)$ and

$$\int_{x_0}^x \varphi(u) e^{Q(u)} du \leq \int_{x_0}^x \frac{du}{\Psi(u)}, \quad x \in [x_0, \infty).$$

Hence, if Grey's condition holds, namely $\int_{x_0}^{\infty} \frac{du}{\Psi(u)} < \infty$, then $\mathcal{I} < \infty$ and the process Z goes extinct. We now study the other direction of the equivalence and assume $\mathcal{I} < \infty$, the question being whether collisions can cause extinction in CBC processes for which Grey's condition is not fulfilled. With c and C as above, $\frac{\Sigma(u)}{\Psi(u)} \leq \frac{C}{c} u$ for all $u \in [x_0, \infty)$; by (5.8),

$$\int_{x_0}^x \varphi(u) e^{Q(u)} du \geq \int_{x_0}^x \frac{du}{\Psi(u)} + \int_{x_0}^x (e^{Q(u)} \varphi(u))' \frac{C}{c} u du, \quad x \in [x_0, \infty).$$

Via integration by parts:

$$\int_{x_0}^x (e^{Q(u)} \varphi(u))' u du = [e^{Q(u)} \varphi(u) u]_{u=x_0}^{u=x} - \int_{x_0}^x e^{Q(u)} \varphi(u) du, \quad x \in [x_0, \infty).$$

Combining the preceding two displays we get

$$(1 + \mathbf{C}/\mathbf{c}) \int_{x_0}^x \varphi(u) e^{\mathcal{Q}(u)} du \geq \int_{x_0}^x \frac{du}{\Psi(u)} - \frac{\mathbf{C}}{\mathbf{c}} \varphi(x_0) x_0, \quad x \in [x_0, \infty).$$

Thus, on letting x tend to ∞ , by monotone convergence, if $\mathcal{I} < \infty$, i.e. $\int_{x_0}^{\infty} \varphi(u) e^{\mathcal{Q}(u)} du < \infty$, then also $\int_{x_0}^{\infty} \frac{dx}{\Psi(x)} < \infty$. \square

6. LAPLACE/SIEGMUND DUALITY AND & LIMITING DISTRIBUTION

6.1. Laplace duality at the level of semigroups: proof of Proposition 2.18. We start with a lemma ensuring that the diffusion U does not explode, as was previously announced. Together with the assumed non-explosivity of Z it will play a key role in establishing the Laplace duality. Recall $\mathcal{A}g = \Sigma g'' - \Psi g'$ for $g \in C^2([0, \infty))$, the generator of U .

Lemma 6.1. *The boundary ∞ of U is inaccessible.*

Proof. Consider the Feller test for the boundary ∞ of U to be accessible. Set

$$\mathcal{J} := \int_{x_0}^{\infty} \frac{dx}{\Sigma(x)} \int_x^{\infty} \exp\left(\int_x^y \frac{\Psi(u)}{\Sigma(u)} du\right) dy.$$

Then the boundary ∞ is inaccessible for U if and only if $\mathcal{J} = \infty$. The non-subordinator case for which $\Psi > 0$ in a neighbourhood of ∞ satisfies clearly $\mathcal{J} = \infty$. Assume now that $-\Psi$ is the Laplace exponent of a subordinator. Note that $\frac{\Psi(u)}{u} \xrightarrow{u \rightarrow \infty} \mathbf{d} \in (-\infty, 0]$. Let $\gamma \in (-\infty, \mathbf{d})$; then there is a large enough $u_0 \in [0, \infty)$ such that for all $u \in [u_0, \infty)$ we have $\Psi(u) \geq \gamma u$. Furthermore, since Σ is convex the map $(0, \infty) \ni u \mapsto \frac{\Sigma(u)}{u}$ is non-decreasing and thus for $u \geq x$ from $(0, \infty)$ we get $\frac{u}{\Sigma(u)} \leq \frac{x}{\Sigma(x)}$. Hence, for $x \in [u_0, \infty)$, since $\gamma < 0$,

$$\exp\left(\int_x^y \frac{\Psi(u)}{\Sigma(u)} du\right) \geq \exp\left(\int_x^y \frac{\gamma u}{\Sigma(u)} du\right) \geq \exp\left(\frac{\gamma x}{\Sigma(x)}(y - x)\right).$$

Besides,

$$\int_x^{\infty} \exp\left(\frac{\gamma x}{\Sigma(x)} y\right) dy = \frac{\Sigma(x)}{-\gamma x} \exp\left(\frac{\gamma x^2}{\Sigma(x)}\right)$$

and therefore

$$\begin{aligned} \mathcal{J} &\geq \int_{x_0}^{\infty} \frac{dx}{\Sigma(x)} \left(\int_x^{\infty} \exp\left(\frac{\gamma x}{\Sigma(x)} y\right) dy \right) \exp\left(\frac{-\gamma x^2}{\Sigma(x)}\right) \\ &= \int_{x_0}^{\infty} \frac{dx}{\Sigma(x)} \frac{\Sigma(x)}{-\gamma x} \exp\left(\frac{\gamma x^2}{\Sigma(x)}\right) \exp\left(\frac{-\gamma x^2}{\Sigma(x)}\right) \\ &= \int_{x_0}^{\infty} \frac{dx}{-\gamma x} dx = \infty, \end{aligned}$$

which concludes the argument. \square

Proof of Proposition 2.18: We may and do assume $z > 0$, $x > 0$. We work under a probability under which Z and U are independent processes starting at z and x respectively and apply the duality result of Ethier and Kurtz [14, Corollary 4.4.15]. Recall Z does not explode by assumption, while U does not explode by Lemma 6.1.

Let $\mathbf{a} < x < \mathbf{b}$ be from $(0, \infty)$ and put $\sigma_{\mathbf{a}, \mathbf{b}} := \sigma_{\mathbf{a}}^- \wedge \sigma_{\mathbf{b}}^+$, where $\sigma_{\mathbf{c}}^{\pm} := \inf\{t \in [0, \infty) : \pm U_t \geq \pm \mathbf{c}\}$, $\mathbf{c} \in [0, \infty)$. In the notation of [14, Corollaries 4.4.14 and 4.4.15] take then $E_1 := E_2 := [0, \infty)$, $X := Z$, $Y := U$, \mathcal{F} the natural filtration of X , \mathcal{G} the natural filtration of Y , $\alpha := \beta := 0$, $\tau := \infty$, $\sigma := \sigma_{\mathbf{a}, \mathbf{b}}$ and, for $(x', y') \in [0, \infty) \times [0, \infty)$,

$$f(x', y') := e^{-x'y'},$$

$$h(x', y') := g(x', y') := \mathcal{L}_{x'} f(x', y') = (x'^2 \Sigma(y') + x' \Psi(y')) e^{-x' y'} = \mathcal{A}_{y'} f(x', y').$$

Note that f is bounded. The function g ($= h$) is bounded separately in each coordinate, but in general not globally; nevertheless it is bounded on $[0, \infty) \times [a, b]$ (since Ψ and Σ are continuous on $(0, \infty)$ and the maps $[0, \infty) \ni u \mapsto u e^{-u}$ and $[0, \infty) \ni u \mapsto u^2 e^{-u}$ are bounded), which is why we (have had to) employ $\sigma_{a,b}$. Furthermore, from Corollary 3.1 it follows that, for each $y' \in [0, \infty)$, the process

$$\left(f(X_t, y') - \int_0^t g(X_s, y') ds, t \geq 0 \right)$$

is an \mathcal{F} -martingale. On the other hand the process

$$\left(f(x', Y_{t \wedge \sigma_{a,b}}) - \int_0^{t \wedge \sigma_{a,b}} h(x', Y_s) ds, t \geq 0 \right)$$

is a \mathcal{G} -martingale for all $x' \in [0, \infty)$. Combining all of the preceding we infer that the conditions of [14, Corollary 4.15] are met and we get that for all $t \in [0, \infty)$,

$$\begin{aligned} \mathbb{E}_z[e^{-xZ_t}] - \mathbb{E}_x[e^{-xU_{t \wedge \sigma_{a,b}}}] &= \mathbb{E}[e^{-xX_t}] - \mathbb{E}[e^{-zY_{t \wedge \sigma_{a,b}}}] \\ &= \int_0^t \mathbb{E} \left[(1 - \mathbb{1}_{[0, \sigma_{a,b}]}(t-s)) e^{-X_s Y_{(t-s) \wedge \sigma_{a,b}}} (X_s^2 \Sigma(Y_{(t-s) \wedge \sigma_{a,b}}) + X_s \Psi(Y_{(t-s) \wedge \sigma_{a,b}})) \right] ds \\ &= \int_0^t \mathbb{E} \left[e^{-X_s Y_{\sigma_{a,b}}} (X_s^2 \Sigma(Y_{\sigma_{a,b}}) + X_s \Psi(Y_{\sigma_{a,b}})) ; \sigma_{a,b} < t-s \right] ds \\ &= \mathbb{E} \left[\int_0^{t - \sigma_{a,b} \wedge t} e^{-Z_s Y_{\sigma_{a,b}}} (Z_s^2 \Sigma(Y_{\sigma_{a,b}}) + Z_s \Psi(Y_{\sigma_{a,b}})) ds \right] \\ &= \mathbb{E}[e^{-aZ_{t - \sigma_a^-}} - e^{-aZ_0}; \sigma_a^- \leq \sigma_b^+ \wedge t] + \mathbb{E}[e^{-bZ_{t - \sigma_b^+}} - e^{-bZ_0}; \sigma_b^+ \leq \sigma_a^- \wedge t], \end{aligned}$$

where the last equality follows from the constancy in time of the expectation of the \mathcal{F} -martingales noted above and by independence of U from Z . In the preceding display we may now let $\mathbf{b} \rightarrow \infty$ and $\mathbf{a} \downarrow 0$ (in this order) and using the fact that neither Z (for the first term) nor U (for the second term) explode, we get the Laplace duality between the non-explosive CBC(Σ, Ψ) process Z and the minimal diffusion U (absorbed at 0):

$$\mathbb{E}_z[e^{-xZ_t}] = \mathbb{E}_x[e^{-zU_t}]. \quad \square \tag{6.1}$$

Remark 6.2. Laplace duality of Z with U on the level of the semigroups entails that the set $\mathcal{C} \cap C_0([0, \infty))$ of Example 3.2 is invariant (in the sense that it is closed under the action of the semigroup of Z); it is also dense in $C_0([0, \infty))$ by Stone-Weierstrass. Therefore, see e.g. Kallenberg [30, Proposition 19.9], it is a core for the infinitesimal generator of Z on $C_0([0, \infty))$.

6.2. Siegmund duality: proof of Proposition 2.20. Recall that V is the (minimal) diffusion with generator \mathcal{G} . The assumptions $S_V(0, x_0) = \infty$ and $S_V(x_0, \infty) = \infty$, which are in effect, entail that V has boundaries 0 and ∞ inaccessible: in other words either entrance or natural. An application of [19, Theorem 6.1] ensures that the Siegmund dual process U of V , i.e. the process such that for all $x, y \in (0, \infty)$ and $t \geq 0$:

$$\mathbb{P}_x(U_t < y) = \mathbb{P}_y(V_t > x) \tag{6.2}$$

is indeed our diffusion with generator \mathcal{A} . Finally, we know that under the assumption $S_V(0, x_0) = \infty$ the process Z does not explode. Therefore, applying Proposition 2.18, introducing an exponential random variable e_z with parameter z independent of U , and plugging it into (6.2), we get

$$\mathbb{E}_z[e^{-xZ_t}] = \mathbb{E}_x[e^{-zU_t}] = \mathbb{P}_x(e_z > U_t) = \int_0^\infty z e^{-zy} \mathbb{P}_y(V_t > x) dy. \quad \square \tag{6.3}$$

Remark 6.3. Siegmund duality exchanges the nature of the boundaries (the scale and speed measures are interchanged), see [19, Table 5]. We observed in Remark 5.8 that, under the assumption of non-explosion of Z , the boundary 0 of V is not regular, and noted that it can therefore be either natural, entrance or exit. The exit option is precluded, since if Z does not explode and 0 is an exit for V , then by Siegmund duality, 0 is an entrance for U , and letting x tend to 0 in the Laplace duality (6.1) yields $\mathbb{P}_z(Z_t < \infty) = \mathbb{E}_{0^+}[e^{-zU_t}] < 1$, which contradicts the non-explosivity of Z . Together with Remark 5.8, it establishes in fact that inaccessibility of ∞ for Z automatically entails inaccessibility of 0 for V . Establishing whether the latter is even an equivalence does not seem to follow easily from our approach. We have also seen in Lemma 6.1 that the boundary ∞ of U is either natural or entrance. By Siegmund duality, V has therefore its boundary ∞ either natural or exit. In particular, it is important to note that under the assumption of non-explosion of Z the diffusion with generator \mathcal{G} is uniquely specified since neither one of its boundaries is regular.

6.3. Limiting distribution: proof of Theorem 2.14. By assumption $S_V(0, x_0] = \infty$ and $S_V(x_0, \infty) = \infty$, which ensures that V is a regular recurrent diffusion on $(0, \infty)$. In this setting, the only possible invariant measure for V is its speed measure M_V . If the latter is finite on $(0, \infty)$, the diffusion V is positive recurrent and converges in distribution towards the law $\frac{M_V(dw)}{M_V(0, \infty)}$, see e.g. Rogers and Williams [49, Theorem 54.5, page 303]. Under this proviso we see by letting t tend to ∞ in (6.3) that for any $x \in [0, \infty)$ and $z \in (0, \infty)$:

$$\mathbb{E}_z[e^{-xZ_t}] \xrightarrow{t \rightarrow \infty} \frac{M_V(x, \infty)}{M_V(0, \infty)}.$$

We now study the case in which M_V gives an infinite mass to $(0, \infty)$. It is slightly easier to work directly with the dual diffusion U . Recall that, up to a multiplicative constant (we avoid making this reservation explicit below) $M_V = S_U$, where S_U is the scale measure of U . The following three cases may occur, see e.g. Karatzas and Shreve [31, Proposition 5.22 page 345].

- (i) If $S_U(0, x_0] = M_V(0, x_0] < \infty$ and $S_U(x_0, \infty) = M_V(x_0, \infty) = \infty$, then, for all $x \in [0, \infty)$,

$$\mathbb{P}_x(\lim_{t \rightarrow \infty} U_t = 0) = 1;$$

hence, by (6.1), $\lim_{t \rightarrow \infty} \mathbb{E}_z[e^{-xZ_t}] = 1$ and Z_t converges in probability towards 0 as t tends to infinity.

- (ii) If $S_U(0, x_0] = M_V(0, x_0] = \infty$ and $S_U(x_0, \infty) = M_V(x_0, \infty) < \infty$, then, for all $x \in (0, \infty)$,

$$\mathbb{P}_x(\lim_{t \rightarrow \infty} U_t = \infty) = 1;$$

hence, by (6.1), $\lim_{t \rightarrow \infty} \mathbb{E}_z[e^{-xZ_t}] = 0$ and Z_t converges in probability towards ∞ .

- (iii) If $S_U(0, x_0] = M_V(0, x_0] = \infty$ and $S_U(x_0, \infty) = M_V(x_0, \infty) = \infty$ then U is recurrent and by the interchange of scale and speed measures the assumption $S_V(0, x_0] = \infty$ and $S_V(x_0, \infty) = \infty$ implies $M_U(0, x_0] = \infty$ and $M_U(x_0, \infty) = \infty$, where M_U denotes the speed measure of U . We see therefore that U is a null recurrent diffusion without a limiting distribution on $[0, \infty]$. A final application of (6.1) entails that Z cannot have a limiting distribution. \square

6.4. Characterizing CBCs: proof of Theorem 2.21. Recall that the usual meaning attached to Z , \mathcal{L} , \mathcal{A} , Σ and Ψ in this paper is suspended in the context of Theorem 2.21.

We start by establishing a lemma of independent interest, specifying how the generator of a positive Feller process with no negative jumps may act on exponential functions.

Lemma 6.4 (Courrège form on exponentials). *Assume that Z is a positive Feller process with no negative jumps, 0 absorbing, and infinitesimal generator \mathcal{L} , whose domain includes the Schwartz*

space of rapidly decaying functions on $[0, \infty)$. For $x \in (0, \infty)$ let $e_x := ([0, \infty) \ni z \mapsto e^{-xz})$ be the exponential function (of rate x). Then, for any $f \in C_c^\infty([0, \infty)) \cup \{e_x : x \in (0, \infty)\}$, the generator \mathcal{L} acts on f as follows:

$$\mathcal{L}f(z) = \int (f(z+h) - f(z) - hf'(z)\mathbb{1}_{(0,1]}(h)) \nu(z, dh) + a(z)f''(z) + b(z)f'(z) - c(z)f(z), \quad (6.4)$$

with $\nu(z, dh)$ a Lévy measure on $(0, \infty)$, $a(z) \in [0, \infty)$, $b(z) \in \mathbb{R}$, $c(z) \in [0, \infty)$ for $z \in [0, \infty)$ and $a(0) = b(0) = c(0) = \nu(0, dh) = 0$.

Proof. Because 0 is an absorbing state for Z we may and for a moment do extend it to a Feller process on the whole real line by taking it as constant on $(-\infty, 0)$. So extended, its infinitesimal generator includes $C_c^\infty(\mathbb{R})$. From the so-called Courrège form of \mathcal{L} , see e.g. Böttcher et al. [5, Theorem 2.21], it follows that for $f \in C_c^\infty(\mathbb{R})$,

$$\mathcal{L}f(z) = \int (f(z+h) - f(z) - hf'(z)\mathbb{1}_{\{|h|\leq 1\}}) \nu(z, dh) + a(z)f''(z) + b(z)f'(z) - c(z)f(z) \quad (6.5)$$

for certain Lévy measures $\nu(z, dh)$ on \mathbb{R} , diffusion coefficients $a(z) \in [0, \infty)$, drifts $b(z) \in \mathbb{R}$ and killing rates $c(z) \in [0, \infty)$ as z runs over \mathbb{R} . Because Z has no negative jumps we know from applying Dynkin's characteristic operator [5, Theorem 1.39] that, for all $z \in (0, \infty)$, for all $f \in C_c^\infty(\mathbb{R})$ whose support is bounded away on the right from z , $\mathcal{L}f(z) = 0$: the key is simply to note that, for all sufficiently small $r \in (0, \infty)$, at time $\zeta_{z+r}^+ \wedge \zeta_{z-r}$ of first exit from the interval of radius r around z the process Z is either above $z+r$ or equal to $z-r$ by the absence of negative jumps, hence $f(Z_{\zeta_{z+r}^+ \wedge \zeta_{z-r}}) = 0 = f(z)$ a.s.. Therefore, for $z \in (0, \infty)$, $\nu(z, dh)$ is carried by $(0, \infty)$.

We now revert back to the non-extended process and observe that (6.5) then holds true for $f \in C_c^\infty([0, \infty))$. The action of its right-hand side extends naturally to all $f \in C_b^2([0, \infty))$ and we use the symbol $\tilde{\mathcal{L}}$ for the corresponding operator. In short,

$$\mathcal{L} = \tilde{\mathcal{L}} \text{ on } C_c^\infty([0, \infty)), \quad (6.6)$$

the action of $\tilde{\mathcal{L}}$ on $C_b^2([0, \infty))$ being given by the right-hand side of (6.5). Since 0 is absorbing for Z we may and do take $a(0) = b(0) = c(0) = \nu(0, dh) = 0$.

Next, fix $x \in (0, \infty)$ and we show that the equality in (6.6) extends also to the exponential map e_x . Let $(\phi_n)_{n \in \mathbb{N}}$ be a sequence in $C_c^\infty([0, \infty))$ satisfying $\mathbb{1}_{[0,n]} \leq \phi_n \leq 1$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \phi_n = 1$ pointwise (such smooth transition functions exist, see e.g. [9, page 49]). Then, on the one hand, it is clear by bounded convergence that $\tilde{\mathcal{L}}(e_x \phi_n) \rightarrow \tilde{\mathcal{L}}e_x$ pointwise as $n \rightarrow \infty$. On the other hand, for all $z \in [0, \infty)$, for all $n \in \mathbb{N}$ with $n \geq z+1$,

$$|\mathcal{L}(\phi_n e_x) - \mathcal{L}e_x|(z) \leq \limsup_{t \downarrow 0} \frac{\mathbb{E}_z[e^{-xZ_t}; Z_t > n]}{t} \leq e^{-xn/2} \lim_{t \downarrow 0} \frac{\mathbb{E}_z[f(Z_t)]}{t} = e^{-xn/2} \mathcal{L}f(z)$$

by choosing any f from the Schwartz space of rapidly decaying functions on $[0, \infty)$ that majorizes $e_{x/2}\mathbb{1}_{[z+1, \infty)}$ but vanishes at z (it exists, it may depend on x and z , but not on n ; one way to get it is by multiplying $e_{x/2}$ with a smooth transition function that vanishes on $[0, z+1/4)$ and is equal to one on $[z+3/4, \infty)$). Letting $n \rightarrow \infty$ in the preceding display we deduce that also $\mathcal{L}(e_x \phi_n) \rightarrow \mathcal{L}e_x$ pointwise as $n \rightarrow \infty$. But $\mathcal{L}(e_x \phi_n) = \tilde{\mathcal{L}}(e_x \phi_n)$ for all $n \in \mathbb{N}$. Hence $\mathcal{L}e_x = \tilde{\mathcal{L}}e_x$. \square

Lemma 6.5. *Under the assumptions of Lemma 6.4, if the generator \mathcal{L} includes in its domain even*

$$\mathcal{S} := \{f \in \mathbb{R}^{[0, \infty)} : (\exists \lim_{\infty} f) \ \& \ (f - f(\infty)) \in \text{Schwartz space of rapidly decaying functions}\}$$

⁵Notwithstanding the first paragraph of this subsection we use ζ_{z+r}^+ and ζ_{z-r} in their established relation, see (2.3) & (2.6), vis-à-vis the process Z .

and further satisfies a Laplace duality relationship

$$\mathcal{L}_z e^{-xz} = \mathcal{A}_x e^{-xz} := z^2 e^{-xz} \Sigma(x) + z e^{-xz} \Psi(x), \quad \{x, z\} \subset [0, \infty), \quad (6.7)$$

with the generator \mathcal{A} of a one-dimensional diffusion on $[0, \infty)$ having drift $-\Psi$ and non-zero diffusion coefficient Σ , both assumed to be continuous at 0, then Ψ and Σ are Lévy-Khintchine functions of the form (1.2) and (1.3), respectively.

Proof. Since $\mathcal{L}1 = 0$, from (6.7), on setting $x = 0$ we get $\Sigma(0) = 0 = \Psi(0)$. Furthermore, (6.7) and (6.4) tell us that for $\{x, z\} \subset (0, \infty)$,

$$\begin{aligned} z^2 \Sigma(x) + z \Psi(x) &= e^{xz} \mathcal{A}_x e^{-xz} = e^{xz} \mathcal{L}_z e^{-xz} \\ &= \int_0^\infty (e^{-xh} - 1 + xh \mathbb{1}_{\{h \leq 1\}}) \nu(z, dh) - b(z)x + a(z)x^2 - c(z). \end{aligned} \quad (6.8)$$

Letting $x \downarrow 0$ renders $c(z) = 0$ by continuity of Σ and Ψ at 0. Then dividing by z we get

$$z \Sigma(x) + \Psi(x) = \int_0^\infty (e^{-xh} - 1 + xh \mathbb{1}_{\{h \leq 1\}}) z^{-1} \nu(z, dh) - z^{-1} b(z)x + z^{-1} a(z)x^2. \quad (6.9)$$

For any fixed z the right-hand side of (6.9) is analytic in $x \in \mathbb{C}$ with $\Re(x) > 0$ and continuous in $x \in \mathbb{C}$ with $\Re(x) \geq 0$. Considering two different z we deduce that Σ and Ψ admit unique extensions to continuous maps defined on $\{\Re \geq 0\}$, analytic on $\{\Re > 0\}$. By analytic continuation and continuity at fixed x (6.9) then obtains for all $x \in \mathbb{C}$ with $\Re(x) \geq 0$, for imaginary x in particular.

It now follows that the characteristic functions of the infinitely divisible distributions whose Laplace exponents are given by the right-hand sides of (6.9) converge weakly as $z \downarrow 0$ towards a continuous function, namely $(\mathbb{R} \ni x \mapsto e^{\Psi(-ix)})$. By Lévy's continuity theorem it implies weak convergence of the infinitely divisible distributions, and since the latter are sequentially closed under weak convergence, Sato [50, Lemma 2.7.8], we deduce that the limiting distribution is itself infinitely divisible, i.e. $(\mathbb{R} \ni x \mapsto \Psi(-ix))$ is the characteristic exponent of a Lévy process. We also know that convergence of characteristic exponents of Lévy processes implies their weak convergence for the Skorohod topology, see e.g. Jacod and Shiryaev [27, Corollary VII.3.6]. This allows us to infer [27, Corollary VI.2.8] finally that Ψ is Lévy-Khintchine of the spectrally positive type, i.e. it takes the form (1.2).

Similarly, dividing again by z in (6.9) we get

$$\Sigma(x) + z^{-1} \Psi(x) = \int_0^\infty (e^{-xh} - 1 + xh \mathbb{1}_{\{h \leq 1\}}) z^{-2} \nu(z, dh) - z^{-2} b(z)x + z^{-2} a(z)x^2. \quad (6.10)$$

Applying the very same reasoning but this time with $z \rightarrow \infty$ in lieu of $z \downarrow 0$ allows to conclude that Σ is Lévy-Khintchine of the spectrally positive type as per (1.3). \square

Proof of Theorem 2.21: Returning now to (6.6) & (6.8), since a Lévy-Khintchine function determines the associated Lévy triplet uniquely, it follows that \mathcal{L} acts on $C_c^\infty([0, \infty))$ according to (2.4). \square

Acknowledgements. René Schilling has the authors' thanks for a stimulating exchange on the topic of the Courrège form of the infinitesimal generators of Feller processes. MV acknowledges support from the Slovenian Research Agency (project No. N1-0174 and programme No. P1-0402).

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