



HAL
open science

Finite elements for Wasserstein W_p gradient flows

Clément Cancès, Daniel Matthes, Flore Nabet, Eva-Maria Rott

► **To cite this version:**

Clément Cancès, Daniel Matthes, Flore Nabet, Eva-Maria Rott. Finite elements for Wasserstein W_p gradient flows. 2024. hal-03719189v3

HAL Id: hal-03719189

<https://hal.science/hal-03719189v3>

Preprint submitted on 13 Feb 2025

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

FINITE ELEMENTS FOR WASSERSTEIN \mathbb{W}_p GRADIENT FLOWS

CLÉMENT CANCÈS, DANIEL MATTHES, FLORE NABET, AND EVA-MARIA ROTT

ABSTRACT. Wasserstein \mathbb{W}_p gradient flows for nonlinear integral functionals of the density yield degenerate parabolic equations involving diffusion operators of q -Laplacian type, with q being p 's conjugate exponent. We propose a finite element scheme building on conformal \mathbb{P}_1 Lagrange elements with mass lumping and a backward Euler time discretization strategy. Our scheme preserves mass and positivity while energy decays in time. Building on the theory of gradient flows in metric spaces, we further prove convergence towards a weak solution of the PDE that satisfies the energy dissipation equality. The analytical results are illustrated by numerical simulations.

1. INTRODUCTION

We are interested in the numerical approximation of non-negative unit mass solutions to the following degenerate parabolic equation

$$(1) \quad \partial_t \rho - \nabla \cdot \left(\rho |\nabla(\eta'(\rho) + \Psi)|^{q-2} \nabla(\eta'(\rho) + \Psi) \right) = 0, \quad \text{in } \mathbb{R}_{\geq 0} \times \Omega,$$

on a bounded convex polyhedral domain $\Omega \subset \mathbb{R}^d$ (where $d = 2$ or 3), with unit outward normal n , complemented with no-flux boundary conditions

$$(2) \quad - \left(\rho |\nabla(\eta'(\rho) + \Psi)|^{q-2} \nabla(\eta'(\rho) + \Psi) \right) \cdot n = 0 \quad \text{on } \mathbb{R}_{\geq 0} \times \partial\Omega,$$

and initial condition

$$(3) \quad \rho = \rho^0 \quad \text{on } \{0\} \times \Omega.$$

Here and throughout the paper, we denote by $\mathbb{R}_{\geq 0} = \{t \in \mathbb{R} : t \geq 0\}$ and $\mathbb{R}_{> 0} = \{t \in \mathbb{R} : t > 0\}$. The exponent $q > 1$ determines the degeneracy of the equation. Models of the form (1) appear as diffusive approximations of the Euler or shallow water equations with high and possibly non-linear friction, see for instance [2, 33, 16, 40].

We consider two sets of assumptions on the nonlinearity $\eta : [0, +\infty) \rightarrow \mathbb{R}_{\geq 0}$ and the external potential $\Psi : \Omega \rightarrow \mathbb{R}$. Our *basic hypotheses* for well-definedness of the scheme are:

Hypothesis 1. η is non-negative, continuous, smooth on $(0, \infty)$, and strictly convex with superlinear growth at infinity. We consider both cases that

(A1) η' is singular at 0, i.e. $\lim_{r \searrow 0} \eta'(r) = -\infty$, or

(A2) $\eta'(0) := \lim_{r \searrow 0} \eta'(r)$ is a non-negative real number.

Moreover, Ψ is non-negative, independent of time, and Lipschitz continuous on $\overline{\Omega}$.

2010 *Mathematics Subject Classification.* 65M12, 65M60, 35K65, 35K92 .

Key words and phrases. Wasserstein gradient flow; finite elements; nonlinear stability; convergence analysis .

Actually, non-negativity of $\eta'(0)$ in [A2] is no significant hypothesis: one can always replace η by $\tilde{\eta}$ with $\tilde{\eta}(r) := \eta(r) - \eta(0) - \eta'(0)r$, which satisfies $\tilde{\eta}'(0) = 0$, without changing the evolution equation (1)–(3). Similarly, changing a potentially negative potential Ψ into the non-negative $\tilde{\Psi} := \Psi - \min_{\bar{\Omega}} \Psi$ does not affect (1)–(3).

In the potential free case $\Psi \equiv 0$, one can choose η such that (1) turns into the transient q -Laplace equation $\partial_t \rho = \nabla(|\nabla \rho|^{q-2} \nabla \rho)$, see (86) and (87) for details. The corresponding functions η satisfy either [A1] of [A2], depending on the value of q . For the convergence analysis using variational methods, we need the following *additional hypotheses*. These ensure that the corresponding energy functional \mathcal{E} introduced in (4) below is λ -uniformly geodesically semi-convex in the sense of [4, Definition 9.1.1], which in turn implies that \mathcal{E} 's local slope is a strong upper gradient. Note that here and below, we distinguish between *semi-convex*, meaning that the convexity modulus λ might be negative, and *convex*, when only $\lambda \geq 0$ is allowed.

Hypothesis 2. η satisfies McCann's condition for geodesic convexity and the doubling condition, see (13) and (14) below. Moreover, Ψ is semi-convex if $q \leq 2$, or convex if $q > 2$.

1.1. Gradient flow structure. The motivation for our numerical method and the proof of convergence heavily rely on the variational structure of the boundary value problem (1)&(2): it is the metric p -gradient flow of the energy functional

$$(4) \quad \mathcal{E}(\rho) = \int_{\Omega} (\eta(\rho) + \rho\Psi) \, dx$$

with respect to the L^p -Wasserstein distance \mathbb{W}_p on the space $\mathcal{P}_{ac}(\Omega)$ of probability densities on Ω , where p is q 's dual exponent, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. We briefly recall some key elements of the corresponding theory developed in [4]; more detailed information is given in Section 2 below.

A curve $(\rho(t))_{t \geq 0}$ of probability densities $\rho(t)$ on Ω is a p -gradient flow with respect to a strong upper gradient $|\partial\mathcal{E}|$ iff the energy $t \mapsto \mathcal{E}(\rho(t))$ is an absolutely continuous function that satisfies the energy identity

$$(5) \quad -\frac{d}{dt} \mathcal{E}(\rho(t)) = \frac{1}{p} |\rho'|^p(t) + \frac{1}{q} |\partial\mathcal{E}|^q(\rho(t))$$

for almost every $t > 0$. Formally (see Propositions 2.1&2.2 for rigorous statements), the metric velocity $|\rho'|$ and the local slope $|\partial\mathcal{E}|$ along a general curve $(\rho_t)_{t \geq 0}$ amount to

$$(6) \quad |\rho'|^p(t) = \int_{\Omega} |\mathbf{v}(t)|^p \rho(t) \, dx, \quad |\partial\mathcal{E}|^q(\rho(t)) = \int_{\Omega} |\nabla[\eta'(\rho(t)) + \Psi]|^q \rho(t) \, dx,$$

where $\mathbf{v}(t)$ is a minimizing vector field for the above integral among all \mathbf{v} satisfying the continuity equation $\partial_t \rho(t) + \nabla \cdot (\rho(t)\mathbf{v}) = 0$ at a given instance of time $t > 0$.

The key point is that the evolution equation (1)&(2), which is a relation on an infinite-dimensional Banach space, is equivalently expressed as the single relation (5) between the three scalar quantities $\mathcal{E}(\rho(t))$, $|\rho'|^p(t)$ and $|\partial\mathcal{E}|^q(\rho(t))$. On a formal level, that equivalence is easily derived by estimating inside the chain rule $-(d/dt)\mathcal{E}(\rho(t)) \leq |\partial\mathcal{E}|(\rho(t)) |\rho'|^p(t)$ on the left-hand side, and then using Young's inequality $a \cdot b \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q$ for vectors $a, b \in \mathbb{R}^d$, which is sharp precisely if $a = |b|^{q-2}b$, with the choices $a = \mathbf{v}(t)$ and $b = \nabla[\eta'(\rho(t)) + \Psi]$ on the right-hand side.

The gradient flow structure of (1)&(2) has been used in the seminal paper [1] to construct weak solutions as limits of a variational time discretization, and in [4] to prove

qualitative properties of solutions, like energy dissipation and instantaneous regularization. Unfortunately, the theory for metric p -gradient flows with $p \neq 2$ is much weaker than the one for $p = 2$, where e.g. \mathcal{E} 's geodesic convexity would have significant consequences on the long-time asymptotics, as shown in [39].

Here, we use the gradient flow structure in a particular way for our needs: our discretization (26) — although not truly variational — dissipates \mathcal{E} in a good way. With that dissipation at hand, we conclude via compactness and variational arguments the convergence of the discrete approximations to a limit curve $\rho_* = (\rho_*(t))_{t \geq 0}$ that satisfies the energy inequality (5). We thus obtain a p -gradient flow solution to (1)&(2), which is energy dissipating and a weak solution of the PDE in the sense detailed below. By this approach, we circumvent the use of methods à la Browder-Minty to identify the flux in the limit, i.e., the term $|\nabla[\eta'(\rho) + \Psi]|^{q-2} \nabla[\eta'(\rho) + \Psi]$, and we obtain strong convergence results on the gradient expressions for our method. The price we pay is the use of various results from the theory of metric gradient flows as a “black box”, like an explicit characterization of the subdifferential $|\partial\mathcal{E}|$ in terms of differential operators, see Proposition 2.1. For a more detailed description of our method, see Section 3.6 further below.

1.2. Solution concept. The aforementioned gradient flow representation of (1)&(2) suggests a particular weak formulation. First, introduce the non-negative, continuous and increasing *pressure* function $\phi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ via

$$(7) \quad \phi(r) = r\eta'(r) - \eta(r) + \eta(0) = \int_0^r s\eta''(s) \, ds.$$

Since $\phi'(r) = r\eta''(r)$ for all $r > 0$, we formally have that $\nabla\phi(\rho) = \rho \nabla\eta'(\rho)$. The key point is that, while weak differentiability of $\eta'(\rho)$ is a delicate issue on the set where ρ vanishes, differentiability of $\phi(\rho)$ is much better accessible.

Definition 1.1. $\rho : \mathbb{R}_{\geq 0} \times \Omega \rightarrow \mathbb{R}_{\geq 0}$ is a weak solution to the initial boundary value problem (1)&(2)&(3) if

- (1) $\rho(t, \cdot)$ is a probability density at each $t \geq 0$;
- (2) $\rho(t, \cdot)$ converges to ρ^0 weakly-* as $t \searrow 0$;
- (3) $\phi(\rho) \in L^1_{\text{loc}}(\mathbb{R}_{>0}, W^{1,1}(\Omega))$;
- (4) the time-dependent vector field $\mathbf{u} : \mathbb{R}_{>0} \times \Omega \rightarrow \mathbb{R}^d$ defined by

$$(8) \quad \mathbf{u} = \frac{\nabla\phi(\rho)}{\rho} + \nabla\Psi \quad \text{on } \{\rho > 0\}$$

and by $\mathbf{u} \equiv 0$ on $\{\rho = 0\}$ satisfies $\rho|\mathbf{u}|^q \in L^1_{\text{loc}}(\mathbb{R}_{>0} \times \overline{\Omega})$;

- (5) (the distributional form of) the following nonlinear continuity equation

$$(9) \quad \int_0^\infty \int_\Omega (\rho \partial_t \varphi + \rho |\mathbf{u}|^{q-2} \mathbf{u} \cdot \nabla \varphi) \, dx \, dt = 0$$

holds for arbitrary $\varphi \in C_c^\infty(\mathbb{R}_{>0} \times \overline{\Omega})$.

Notice that the no-flux boundary condition (2) is encoded in by validity of the continuity equation (9) for test functions φ that do not necessarily vanish on $\partial\Omega$.

1.3. Organization of the paper. In Section 2, we introduce some material related to the gradient flow interpretation of the continuous problem (1)&(2). The finite element scheme is described in Section 3. It is based on conformal Lagrange \mathbb{P}_1 finite elements and mass lumping, while the time discretization relies on the backward Euler scheme. The main results of the paper are set out in Section 3.5 and we give the sketch of proof in Section 3.6.

In Section 4, then we establish Theorem 3.1 by showing that the scheme admits solutions for any time step size. In Section 5, we establish the a priori estimates our analysis will rely on. These estimates are all consequences of the (global) mass conservativity and the energy diminishing property of the scheme. Section 6 is devoted to the proof of the convergence of approximate solutions (Theorem 3.2), which relies on compactness arguments. The proof of strong convergence (Theorem 3.3) is the purpose of Section 7. Finally, numerical experiments are collected in Section 8.

2. VARIATIONAL CONCEPTS

2.1. Spaces and distances. For general information on Wasserstein metrics and Wasserstein gradient flow, see [42, 43].

Denote by $\mathcal{P}_{\text{ac}}(\Omega)$ the space of absolutely continuous probability measures on Ω . By abuse of notation, we shall identify elements in $\mathcal{P}_{\text{ac}}(\Omega)$ with their Lebesgue-density $\rho \in L^1(\Omega)$, which is non-negative and of unit mass. From the various ways to define the L^p -Wasserstein distance $\mathbb{W}_p(\rho_0, \rho_1)$ between $\rho_0, \rho_1 \in \mathcal{P}_{\text{ac}}(\Omega)$, the dynamical one from [8] is closest to the heart of this work,

$$(10) \quad \mathbb{W}_p(\rho_0, \rho_1) = \left(\inf \left\{ \int_0^1 \int_{\Omega} |\mathbf{v}|^p \rho \, dx \, ds \mid \partial_s \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \right\} \right)^{1/p},$$

where the pair $(\rho_s, \mathbf{v}_s)_{s \in [0,1]}$ runs over all distributional solutions of the continuity equation that connect $\rho(0) := \rho_0$ to $\rho(1) := \rho_1$. Since Ω is bounded, convergence in \mathbb{W}_p is equivalent to weak- $*$ -convergence. Occasionally, we shall use the L^1 -Wasserstein distance \mathbb{W}_1 — also known as bounded Lipschitz distance — on $\mathcal{P}_{\text{ac}}(\Omega)$, which is defined by (10) with $p = 1$, or alternatively in the dual representation

$$\mathbb{W}_1(\rho_0, \rho_1) = \sup_{0 \neq \varphi \in C^1(\mathbb{R}^d)} \frac{\int_{\Omega} \varphi(\rho_1 - \rho_0) \, dx}{\|\nabla \varphi\|_{L^\infty}}.$$

We will make use in the sequel of the following classical identity which follows from Kantorovitch's dual representation of the optimal transportation problem [42]

$$(11) \quad \mathbb{W}_p(\rho_0, \rho_1) \leq \text{diam}(\Omega)^{1/q} \mathbb{W}_1(\rho_0, \rho_1)^{1/p}.$$

2.2. Energy functional. Recall from Hypothesis 1 that $\eta : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ is continuous on $\mathbb{R}_{\geq 0}$, smooth on $\mathbb{R}_{> 0}$, convex function of superlinear growth, specifically

$$(12) \quad \lim_{\rho \rightarrow +\infty} \eta'(\rho) = +\infty.$$

For simplicity (and by abuse of notation), we identify η with its extension to \mathbb{R} by $\eta(r) = +\infty$ for $r < 0$. This extension is convex and lower semi-continuous, so that the subdifferential $\partial\eta$ can be considered. Specifically, we have $\partial\eta(r) = \{\eta'(r)\}$ for all $r > 0$, and $\partial\eta(r) = \emptyset$ for all $r < 0$. At $r = 0$, there are two alternatives:

- in case (A1), when $\lim_{r \searrow 0} \eta'(r) = -\infty$, we have $\partial\eta(0) = \emptyset$;
- in case (A2), when $\eta'(0) := \lim_{r \searrow 0} \eta'(r) \in \mathbb{R}_{\geq 0}$, we have $\partial\eta(0) = (-\infty, \eta'(0)]$.

In Hypothesis 2, we further assume *McCann's condition* [36] on geodesic convexity,

$$(13) \quad s \mapsto s^d [\eta(s^{-d}) - \eta(0)] \quad \text{is convex and non-increasing on } \mathbb{R}_{> 0}$$

and the *doubling condition*:

$$(14) \quad \exists C > 0 \text{ s.t. } \eta(s + \check{s}) \leq C(1 + \eta(s) + \eta(\check{s})), \quad \forall s, \check{s} \in \mathbb{R}_{> 0}.$$

The energy functional $\mathcal{E} : \mathcal{P}_{\text{ac}}(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined by formula (4) from the introduction.

2.3. Metric slope. For the abstract definition of the metric (local) slope $|\partial\mathcal{E}|$ of \mathcal{E} , see [4, Section 1.2]. In the situation at hand, it is characterized as follows, see [4, Theorem 10.4.13].

Proposition 2.1. *Let $\rho \in \mathcal{P}_{ac}(\Omega)$ be such that $\mathcal{E}(\rho) < \infty$. Then $|\partial\mathcal{E}|^q(\rho) < \infty$ if and only if $\phi(\rho) \in W^{1,1}(\Omega)$ with*

$$(15) \quad \rho \mathbf{u} = \nabla \phi(\rho) + \rho \nabla \Psi \quad \text{a.e. on } \Omega$$

for a measurable $\mathbf{u} : \Omega \rightarrow \mathbb{R}^d$ satisfying $\rho |\mathbf{u}|^q \in L^1(\Omega)$. In that case,

$$(16) \quad |\partial\mathcal{E}|^q(\rho) = \int_{\Omega} \rho |\mathbf{u}|^q dx.$$

Notice that any \mathbf{u} satisfying (15) is ρ -almost everywhere uniquely defined as the quotient that appears in (8). Actually, in view of the definition of ϕ , there holds $\nabla \phi(\rho) = \rho \nabla \eta'(\rho)$ as soon as ρ is regular enough (here, ρ bounded away from 0 is sufficient to justify the chain rule). Equation (15) thus gives a rigorous meaning to the formally expected definition of \mathbf{u} , namely $\mathbf{u} = \nabla(\eta'(\rho) + \Psi)$. Note that expression (16) is consistent with (6) provided (8) holds true.

2.4. Metric velocity. Recall that $p = q/(q-1)$ is the Hölder conjugate of q . For the abstract definitions of p -absolute continuity and the metric velocity $|\rho'|$ of a curve $(\rho(t))_{t \geq 0}$ in $\mathcal{P}_{ac}(\Omega)$ with respect to \mathbb{W}_p (or in any other metric space), see [4, Section 1.1]. In short: the curve is p -absolutely continuous on $(0, \infty)$, denoted by $(\rho(t))_{t \geq 0} \in AC^p$, if there exists a function $m \in L^p(\mathbb{R}_{>0})$ with the property that

$$\mathbb{W}_p(\rho(t_1), \rho(t_2)) \leq \int_{t_1}^{t_2} m(t) dt \quad \text{for all } 0 \leq t_1 \leq t_2.$$

If this is the case, then there is a (pointwise a.e.) smallest such function m ; it is called metric derivative and denoted by $|\rho'| \in L^p(\mathbb{R}_{>0})$. Here, we only need the following property, see [4, Theorem 8.3.1].

Proposition 2.2. *If $\mathbf{v} : \mathbb{R}_{>0} \times \Omega \rightarrow \mathbb{R}^d$ with $\int_0^\infty \int_{\Omega} \rho |\mathbf{v}|^p dx dt < \infty$ is such that the continuity equation*

$$(17) \quad \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$$

is satisfied in the distributional sense, then $(\rho(t))_{t \geq 0} \in AC^p$, and

$$|\rho'|^p(t) \leq \int_{\Omega} |\mathbf{v}(t)|^p \rho(t) dx \quad \text{for a.e. } t > 0.$$

2.5. p -gradient flows. Below, we need to assume in addition also Hypothesis 2: then, the metric slope $|\partial\mathcal{E}|$ is a *strong* upper gradient in the sense of [4, Definition 1.2.1], which follows from [4, Corollary 2.4.10] thanks to the geodesic convexity of \mathcal{E} in \mathbb{W}_p , see [4, Proposition 9.3.2] and [4, Proposition 9.3.9]. The conditions stated in Hypothesis 2, including Ψ 's (semi-)convexity, are essential here.

In the context of gradient flows in metric spaces, see [4, Section 1.3], an absolutely continuous curve $(\rho(t))_{t \geq 0} \in AC^p$ is a p -gradient flow for the functional \mathcal{E} in the metric \mathbb{W}_p with respect to the strong upper gradient $|\partial\mathcal{E}|$ if and only if $t \mapsto \mathcal{E}(\rho(t))$ is an absolutely continuous function, and the following time-integrated version of the energy identity (5) holds at every $t^* > 0$:

$$(18) \quad \mathcal{E}(\rho(t^*)) + \frac{1}{p} \int_0^{t^*} |\rho'|^p(t) dt + \frac{1}{q} \int_0^{t^*} |\partial\mathcal{E}|^q(\rho(t)) dt = \mathcal{E}(\rho^0).$$

Of the two inequalities behind this equation, the one with \geq is already contained in the definition of the strong upper gradient, while the one with \leq characterizes the curve $(\rho(t))_{t \geq 0}$ as gradient flow. Validity of (18) for every $t^* > 0$ implies the distributional formulation (5), and in particular also

$$(19) \quad -\partial_t \mathcal{E}(\rho(t)) = |\rho'|^p(t) = |\partial \mathcal{E}|^q(\rho(t))$$

at almost every $t \geq 0$, see [4, Remark 1.3.3]. Moreover, for the specific choice (4) of \mathcal{E} , the p -gradient flow also satisfies the weak form of the initial boundary value problem (1)&(2)&(3) given in Definition (1.1). This follows by combination of Theorem 11.1.3 with Theorem 10.4.13 from [4], see also [4, Theorem 11.3.4] for the corresponding existence result.

3. ENERGY STABLE FINITE ELEMENTS

3.1. Discretization in space. The goal of this section is to introduce the basic material on finite element to define the scheme and to initiate the analysis. Our approach relies on lowest order conforming finite elements with mass lumping. The domain Ω is assumed to be tessellated into a simplicial conformal discretization \mathcal{T}_h (made of triangles if $d = 2$ and of tetrahedra if $d = 3$). Denoting by h_T the diameter of an element $T \in \mathcal{T}_h$ and by \mathfrak{d}_T the diameter of the largest sphere included in T , the size and regularity of \mathcal{T}_h are respectively defined by

$$(20) \quad h_{\mathcal{T}} = \max_{T \in \mathcal{T}_h} h_T, \quad \zeta_{\mathcal{T}_h} = \max_{T \in \mathcal{T}_h} \frac{h_T}{\mathfrak{d}_T}.$$

We denote by \mathcal{V}_T the set of the $(d + 1)$ vertices of $T \in \mathcal{T}_h$, and by $\mathcal{V} = \bigcup_{T \in \mathcal{T}_h} \mathcal{V}_T$ the set of all the vertices of \mathcal{T}_h .

Define the usual $\mathbb{P}1$ Lagrange finite elements space corresponding to the mesh \mathcal{T}_h by

$$V_h = \left\{ v \in C(\overline{\Omega}) \mid v|_T \text{ is affine for all } T \in \mathcal{T}_h \right\}.$$

The mass matrix \mathbb{A} corresponding to classical $\mathbb{P}1$ Lagrange finite elements is given by

$$(21) \quad \mathbb{A}_{aa'} = \int_{\Omega} v_a v_{a'},$$

where v_a is the piecewise linear (hat) function that is equal to 1 at vertex a and 0 at all the other vertices. Clearly, \mathbb{A} is not diagonal. Mass lumping means to replace the non-diagonal mass matrix \mathbb{A} by the diagonal matrix \mathbb{M} that is obtained by using the line sums of \mathbb{A} as diagonal entries:

$$\mathbb{M}_{aa} = \sum_{a' \in \mathcal{V}} \int_{\Omega} v_a v_{a'} \text{ for } a \in \mathcal{V}, \quad \mathbb{M}_{aa'} = 0 \text{ for } a' \neq a.$$

Alternatively, \mathbb{M} is obtained by approximation of the integral expressions for $\mathbb{A}_{aa'}$ in (21), using a quadrature method with quadrature points located at the vertices \mathcal{V} . We refer for instance to [44] for a presentation of mass lumping in more generality.

Here, we rather build on another classical approach for representing mass lumping, which consists in introducing the so-called Donald (or dual barycentric) mesh. To each $a \in \mathcal{V}$ we associate a cell ω_a containing a and delimited by the hyperplans joining

- in $d = 2$: the barycenters x_T and x_e of all triangle $T \in \mathcal{T}_h$ and edge e sharing a as a vertex.
- in $d = 3$: the barycenters x_T , x_F and x_e of all triangle $T \in \mathcal{T}_h$, face F and edge e sharing a as a vertex.

With this construction, one has $|\omega_a \cap T| = \frac{1}{d+1}|T|$ for all $a \in \mathcal{V}_T$, $T \in \mathcal{T}_h$. An illustration in the two-dimensional case is given on Figure 1. In what follows, we denote the set of

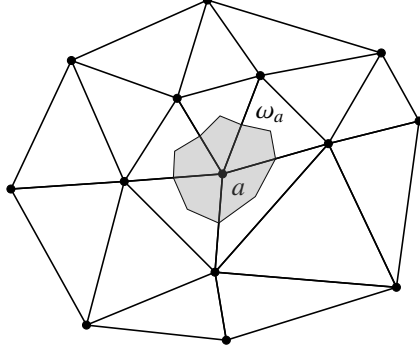


FIGURE 1. Construction of the Donald mesh cell ω_a for some $a \in \mathcal{V}$.

piecewise constant functions on the Donald mesh by

$$X_h = \{v \in L^\infty(\Omega) \mid v|_{\omega_a} \text{ is constant for all } a \in \mathcal{V}\}.$$

Further, introduce the set of piecewise constant functions on \mathcal{T}_h by

$$\tilde{V}_h = \{v \in L^\infty(\Omega) \mid v|_T \text{ is constant for all } T \in \mathcal{T}_h\}.$$

In particular, the gradient ∇v_h of an element $v_h \in V_h$ belongs to $(\tilde{V}_h)^d$.

3.2. Reconstruction operators. Both V_h and X_h are isomorphic to $\mathbb{R}^{\mathcal{V}}$ as the elements of V_h and X_h are fully determined by their value at the vertices. We denote by $\bar{\varphi}_h$ the unique element of X_h having the same nodal values as $\varphi_h \in V_h$. The mapping $\varphi_h \mapsto \bar{\varphi}_h$ is one-to-one and onto from V_h to X_h , and makes the mass matrix diagonal since

$$(22) \quad \int_{\Omega} \bar{v}_h \bar{w}_h = \sum_{a \in \mathcal{V}} |\omega_a| v_h(a) w_h(a), \quad \forall v_h, w_h \in V_h.$$

Another remarkable property of mass-lumped finite elements is its compatibility with nonlinearities. More precisely, given $f \in C(\mathbb{R})$ and $v_h \in V_h$, and denoting by f_h the element of V_h such that $f_h(a) = f(v_h(a))$, there holds

$$(23) \quad \bar{f}_h = f(\bar{v}_h),$$

The combination of (22) and (23) is key to establish the energy stability of our scheme. More generally, mass-lumping enhances the stability of finite elements, see for instance [19]. In our analysis, one also needs to reconstruct elements $\tilde{v}_h \in \tilde{V}_h$ from elements $v_h \in V_h$ by setting

$$(24) \quad \tilde{v}_h(x_T) = \frac{1}{d+1} \sum_{a \in \mathcal{V}_T} v_h(a), \quad \forall T \in \mathcal{T}_h.$$

3.3. Discretization in time. For the time discretization, one defines an increasing sequence $(t^n)_{n \geq 0}$ with $t^0 = 0$ and $t^n \rightarrow +\infty$ as $n \rightarrow +\infty$. We denote by $\tau^n = t^n - t^{n-1}$ the n^{th} time step, and by $\tau = \max_{n \geq 1} \tau^n$.

3.4. Definition of the scheme. The initial boundary value problem (1)&(2)&(3) is discretized using a simplicial decomposition \mathcal{T}_h of Ω and a temporal mesh $0 = t^0 < t^1 < t^2 < \dots$, with time steps $\tau^k = t^k - t^{k-1}$. The sought solution ρ to (1)–(3) is approximated by a piecewise affine in space and piecewise constant in time function $\rho_{h\tau}$.

The weak formulation of our time-iterative scheme uses mass-lumping: to the simplicial decomposition \mathcal{T}_h , we associate a Donald mesh, see Figure 1 for illustration. And to a piecewise affine (w.r.t. \mathcal{T}_h) function $f_h \in V_h$, we associate its mass-lumped reconstruction $\bar{f}_h \in X_h$ that is piecewise constant on the Donald mesh as prescribed in Section 3.2

Let Ψ_h be the piecewise affine approximation of Ψ that agrees with the original values on the vertices, that is $\Psi_h \in V_h$ and

$$\Psi_h(a) = \int_{\omega_a} \Psi \, dx := \frac{1}{|\omega_a|} \int_{\omega_a} \Psi \, dx, \quad \forall a \in \mathcal{V}.$$

With this choice, the existence of some $C_1 > 0$ depending only on the regularity factor $\zeta_{\mathcal{T}_h}$ and on the dimension d such that

$$(25) \quad \|\nabla \Psi_h\|_{L^\infty(\Omega)^d} \leq C_1 \|\nabla \Psi\|_{L^\infty(\Omega)^d}$$

has been established in the proof of [14, Theorem A.1].

For a given non-negative and of unit mass $\rho_{h\tau}^{n-1} \in V_h$, the piecewise affine functions $\rho_{h\tau}^n := \rho_{h\tau}(t^n, \cdot)$ are subject to

$$(26) \quad \int_{\Omega} (\bar{\rho}_{h\tau}^n - \bar{\rho}_{h\tau}^{n-1}) \bar{w}_h + \tau^n \int_{\Omega} \rho_{h\tau}^n \left| \nabla (\mu_{h\tau}^n + \Psi_h) \right|^{q-2} \nabla (\mu_{h\tau}^n + \Psi_h) \cdot \nabla w_h = 0,$$

for every piecewise affine test function $w_h \in V_h$. Above, the piecewise affine potentials $\mu_{h\tau}^n$ are such that

$$(27) \quad \mu_{h\tau}^n(a) \in \partial \eta(\rho_{h\tau}^n(a)) \quad \text{at all vertices } a,$$

where $\partial \eta(r) \subset \mathbb{R}$ denotes the subdifferential of the convex function η at r . More explicitly, under Assumption (A1), one has $\rho_{h\tau}^n(a) > 0$ and $\mu_{h\tau}^n(a) = \eta'(\rho_{h\tau}^n(a))$ at every vertex a , and under Assumption (A2), also $\rho_{h\tau}^n(a) = 0$ and $\mu_{h\tau}^n(a) \leq \eta'(0)$ are admissible.

Several remarks on the seemingly complicated definition of our scheme are in order. Unless $q = 2$, the term $|\nabla(\eta'(\rho) + \Psi)|^{q-2}$ requires a full reconstruction of the gradient of the potential. In space dimension $d \geq 2$, the convenient *Two Point Flux Approximation* (TPFA) [25] is not anymore sufficient for consistency. We refer the reader to [28] for an illustration of the TPFA's failure in the context of \mathbb{W}_p due to hazardous geometrical effects from the mesh, or to [29] where similar effects appear already for the simpler transient q -Laplace equation. Instead, we make use of lowest order Lagrange finite elements as in [45, 7], but with mass-lumping as a key ingredient of our method.

The TPFA approach with Delaunay regular meshes, or very specific finite volume methods [5] designed to approach strongly degenerate operators involving also a q -Laplacian are able to preserve the monotonicity of the problem. Our method will *not* be monotone in general. Instead, stability is obtained by energy methods, which is natural in view of the gradient flow interpretation described below. Here we build on various recent contributions [12, 10, 14, 18], where energy dissipative methods for general meshes are designed and analyzed. Unfortunately, these schemes are not positivity preserving under Assumption (A2), but the latter is required for the gradient flow interpretation. As a remedy, we introduce a Lagrange multiplier for the positivity constraint, and use an interior point method inspired by [38]. We emphasize that the scheme we propose is not variational, but relies on the cheaper backward Euler scheme. For variational discretizations building on the JKO

scheme [31] in the much easier case $q = 2$, see for instance [15, 34, 17]. Continuous in time but discrete in space dynamics are also considered in [35, 20, 24].

3.5. Results: well-posedness and convergence. We have two main results. The first is on the well-posedness of the fully discrete iteration scheme and the essential energy estimate. For concise formulation, introduce the modified energy functional

$$\mathcal{E}_h(\bar{\rho}_h) = \int_{\Omega} [\eta(\bar{\rho}_h) + \bar{\rho}_h \bar{\Psi}_h] dx,$$

defined on densities $\bar{\rho}_h \in X_h$ that are piecewise constant on the Donald mesh introduced in Section 3.1.

Theorem 3.1. *Assume Hypothesis 1. Let a triangulation \mathcal{T}_h of Ω and a time step τ^n be given, and consider a piecewise affine datum $\rho_{h\tau}^{n-1} \in V_h$, that is non-negative and of unit mass. Then there exists a piecewise affine solution $\rho_{h\tau}^n \in V_h$ to (26)&(27), non-negative and of unit mass. Moreover, the following discrete energy–dissipation inequality holds:*

$$(28) \quad \mathcal{E}_h(\bar{\rho}_{h\tau}^n) + \tau^n \int_{\Omega} \rho_{h\tau}^n \left| \nabla (\mu_{h\tau}^n + \Psi_h) \right|^q \leq \mathcal{E}_h(\bar{\rho}_{h\tau}^{n-1}), \quad \forall n \geq 1.$$

The energy estimate (28) above is the key ingredient for the entire analysis that follows. Its proof uses in an essential way properties of the discretization by mass lumped finite elements. This makes mass lumping an indispensable element of our approach.

Theorem 3.1 gives rise to global-in-time fully discrete solutions in the usual way: assume that simplicial meshes \mathcal{T}_h and time steps (τ^n) are given, parametrized by (h, τ) . For each (h, τ) , consider the non-negative, unit mass and piecewise constant initial datum $\bar{\rho}_{h\tau}^0 \in X_h$ defined by

$$(29) \quad \bar{\rho}_{h\tau}^0(a) = \int_{\omega_a} \rho^0, \quad \forall a \in \mathcal{V},$$

where ρ^0 is positive, of unit mass and such that $\int_{\Omega} \eta(\rho^0) < +\infty$.

Inductively applying Theorem 3.1 above, one obtains a sequence $(\rho_{h\tau}^n)_{n=0}^{\infty}$. Denote the advanced piecewise constant-in-time “interpolation” by $\bar{\rho}_{h\tau}$, i.e.,

$$(30) \quad \bar{\rho}_{h\tau}(t) = \bar{\rho}_{h\tau}^n \quad \text{for } t^{n-1} < t \leq t^n.$$

For these, we prove the following convergence result.

Theorem 3.2. *Assume Hypotheses 1 and 2, and let a non-negative and unit mass initial datum ρ^0 of finite energy $\mathcal{E}(\rho^0) < \infty$ be given. Assume that the simplicial decompositions \mathcal{T}_h are of uniform regularity $\zeta_{\mathcal{T}_h}$, see (20), and that the initial data are given by (29). Define the discrete solutions $\rho_{h\tau}$ as above.*

Then, as $(h, \tau) \rightarrow (0, 0)$ along a suitable sequence, $\rho_{h\tau}(t)$ converges in \mathbb{W}_p , locally uniformly with respect to $t \geq 0$, and also in $L_{loc}^1(\mathbb{R}_{\geq 0} \times \bar{\Omega})$ to a limit ρ_ . Moreover, the functions $t \mapsto \mathcal{E}_h(\bar{\rho}_{h\tau}(t))$ converge pointwise w.r.t. $t \geq 0$ to the absolutely continuous limit $t \mapsto \mathcal{E}(\rho_*(t))$. Finally, the limit ρ_* satisfies the energy identity (5), and in particular, it is a weak solution to (1)&(2)&(3) in the sense of Definition 1.1.*

In the statement of the theorem, the phrase “along a suitable sequence” is important. The family $\rho_{h\tau}$ may contain a multitude of accumulation points for $(h, \tau) \rightarrow (0, 0)$. Indeed, uniqueness of the solutions to (1)&(2) is not guaranteed, not even in the class of gradient flow solutions — this is in sharp contrast to the case $p = 2$ addressed in [9]. In our numerical experiments, however, we always observed convergence to one and only one limiting solution.

A third result concerns strong convergence of the flux vector field. For a concise statement, define

$$(31) \quad \mathbf{u}_{h\tau} := \nabla[\mu_{h\tau} + \Psi_h], \quad \mathbf{v}_{h\tau} := |\mathbf{u}_{h\tau}|^{q-2} \mathbf{u}_{h\tau},$$

so that $|\mathbf{v}_{h\tau}|^p = |\nabla[\mu_{h\tau} + \Psi_h]|^q$, and the “limiting vector fields” \mathbf{u}_* and \mathbf{v}_* in accordance with (8) by

$$(32) \quad \mathbf{u}_* = \frac{\nabla\phi(\rho_*)}{\rho_*} + \nabla\Psi, \quad \mathbf{v}_* = |\mathbf{u}_*|^{q-2} \mathbf{u}_* \quad \text{on } \{\rho_* > 0\},$$

and $\mathbf{u}_* = \mathbf{v}_* = 0$ on $\{\rho_* = 0\}$.

Denoting by $\widetilde{\rho}_{h\tau}$ the constant-in-time “interpolation” of the constant-in-space reconstruction of $\rho_{h\tau}$, i.e.,

$$\widetilde{\rho}_{h\tau}(t) = \widetilde{\rho}_{h\tau}^n \quad \text{for } t^{n-1} < t \leq t^n,$$

where $\widetilde{\rho}_{h\tau}^n \in \widetilde{V}_h$ is defined by (24), we have the following result.

Theorem 3.3. *Under the same assumptions and with the same notations as in Theorem 3.2 above, the vector fields $\mathbf{u}_{h\tau}$ from (31) approximate \mathbf{u}_* from (32) in the following sense: for each $r \in [1, q]$, the product $|\mathbf{u}_{h\tau}|^{r-1} \mathbf{u}_{h\tau} \widetilde{\rho}_{h\tau}^{r/q}$ converges to $|\mathbf{u}_*|^{r-1} \mathbf{u}_* \rho_*^{r/q}$ strongly in $L_{loc}^{q/r}(\mathbb{R}_{\geq 0} \times \overline{\Omega})$. Consequently, the fluxes converge strongly in L^1 , that is*

$$|\mathbf{u}_{h\tau}|^{q-2} \mathbf{u}_{h\tau} \widetilde{\rho}_{h\tau} \rightarrow |\mathbf{u}_*|^{q-2} \mathbf{u}_* \rho_* \quad \text{in } L_{loc}^1(\mathbb{R}_{\geq 0} \times \overline{\Omega}),$$

or equivalently, $\mathbf{v}_{h\tau} \widetilde{\rho}_{h\tau} \rightarrow \mathbf{v}_* \rho_*$ in $L_{loc}^1(\mathbb{R}_{\geq 0} \times \overline{\Omega})$.

The convergence of a related numerical method for degenerate equations involving q -Laplace operators is also addressed in [21], see also [22] in the special case $p = q = 2$. The approach here, however, is not directly comparable to the one in [21], since the general assumptions here and there are mutually incompatible, and the two notions of weak solution are different. Specifically for those problems of our type (1)–(3) that also fit the framework of [21], the discrete energy dissipation inequality (28) does not follow under the hypotheses therein. In [13], a discretization closely related to ours is proposed, and in [22] a so-called gradient schemes related to the one from [21]. A numerical comparison between the two is carried out in [11] on the example of the porous medium equation.

3.6. Strategy of proof. The central point in the proof of our convergence result in Theorem 3.2 is to first establish a relatively weak convergence of the $\rho_{h\tau}$ to a limit curve ρ_* and then to obtain the “significant half” of the integral form (18) of the energy identity (5) for ρ_* , that is

$$(33) \quad \mathcal{E}(\rho_*(t^*)) + \frac{1}{p} \int_0^{t^*} |\rho_*'|^p dt + \frac{1}{q} \int_0^{t^*} |\partial \mathcal{E}|^q(\rho_*) dt \leq \mathcal{E}(\rho^0).$$

All estimates for both parts — convergence and energy identity — rest on the strong energy dissipation estimate (28) for fully discrete solutions.

Concerning the convergence: weak convergence of $\widetilde{\rho}_{h\tau}(t)$ to a limit $\rho_*(t)$ at every $t \geq 0$ is a standard result from gradient flow theory, using the implied uniform (almost) Hölder continuity of the curves $\rho_{h\tau}$. Stronger convergence in $L^1([0, t^*] \times \Omega)$ uses that the expression $\rho_{h\tau} \nabla[\mu_{h\tau} + \Psi_h]$ in the dissipation term in (18) controls $\nabla\phi_{h\tau}$, where $\phi_{h\tau}$ is a piecewise affine approximation of the concatenation $\phi(\rho_{h\tau})$. The proof of this control requires detailed estimates for piecewise affine functions on simplicial meshes. The bound on $\nabla\phi_{h\tau}$ then provides sufficient compactness to apply the Aubin-Lions lemma, see Lemma 6.4. A posteriori, we are able to identify both the limit of $\nabla\phi_{h\tau}$ and the limit of $\widetilde{\rho}_{h\tau} \nabla\mu_{h\tau}$ with

$\nabla\phi(\rho_*)$, see Lemma 6.7. This identification is crucial for the identification of the subdifferential later.

Concerning the proof of (33): there are four ingredients.

- (i) *Discrete energy dissipation inequality*: By direct iteration of (28), the piecewise constant interpolation in time, the definitions of $\mathbf{u}_{h\tau}$ and $\mathbf{v}_{h\tau}$ in (31), and finally that $\frac{1}{p} + \frac{1}{q} = 1$, it follows immediately that

$$\mathcal{E}_h(\bar{\rho}_{h\tau}(t^*)) + \frac{1}{p} \int_0^{t^*} \int_{\Omega} |\mathbf{v}_{h\tau}|^p \bar{\rho}_{h\tau} \, dx \, dt + \frac{1}{q} \int_0^{t^*} \int_{\Omega} |\mathbf{u}_{h\tau}|^q \bar{\rho}_{h\tau} \, dx \, dt \leq \mathcal{E}_h(\bar{\rho}_{h\tau}^0).$$

The remaining steps are concerned with convergence of the three terms on the left-hand side, to obtain the left-hand side of (33) as \liminf .

- (ii) *Convergence of the energy*: In Lemma 6.11, we show that

$$\mathcal{E}(\rho_*(t^*)) \leq \liminf_{h,\tau \rightarrow 0} \mathcal{E}_h(\bar{\rho}_{h\tau}(t^*)) \quad \text{at every } t^* \geq 0.$$

This is actually an immediate consequence of $\bar{\rho}_{h\tau}(t^*)$'s weak convergence and \mathcal{E} 's weak lower semi-continuity.

- (iii) *Convergence of the slope*: In Lemma 6.13, we conclude that

$$(34) \quad \int_0^{t^*} |\partial\mathcal{E}|^q(\rho_*) \, dt = \int_0^{t^*} \int_{\Omega} \rho_* |\mathbf{u}_*|^q \, dx \, dt \leq \liminf_{h,\tau \rightarrow 0} \int_0^{t^*} \int_{\Omega} \bar{\rho}_{h\tau} |\mathbf{u}_{h\tau}|^q \, dx \, dt.$$

The existence of a limiting vector field \mathbf{u}_* such that $\bar{\rho}_{h\tau} \mathbf{u}_{h\tau} \rightarrow \rho_* \mathbf{u}_*$ weakly follows from step (i) above by means of the abstract convergence result in Proposition A.2, and so does the \liminf -estimate. The fact that \mathbf{u}_* has the form (32) relies on the convergence of $\bar{\rho}_{h\tau} \nabla \mu_{h\tau}$ to $\nabla \phi(\rho_*)$, hence the characterization of $|\partial\mathcal{E}|(\rho_*)$ in Proposition 2.1 yields the equation above.

- (iv) *Convergence of the metric velocity*: In Lemma 6.14, we show that

$$(35) \quad \int_0^{t^*} |\rho_*'|^p(t) \, dt \leq \int_0^{t^*} \int_{\Omega} \rho_* |\mathbf{v}_*|^p \, dx \, dt \leq \liminf_{h,\tau \rightarrow 0} \int_0^{t^*} \int_{\Omega} \bar{\rho}_{h\tau} |\mathbf{v}_{h\tau}|^p \, dx \, dt.$$

In view of (31) and (32), the right inequality in (35) just rephrases the respective inequality in (34) above. The left inequality in (35) follows from Proposition 2.2 on the metric velocity; the challenge here is to establish the continuity equation for ρ_* and \mathbf{v}_* .

4. EXISTENCE OF A SOLUTION TO THE SCHEME

The goal in this section is to show that the nonlinear system (26) always admits at least one solution $\rho_{h\tau}^n \geq 0$ with $\mu_{h\tau}^n$ such that $\mu_{h\tau}^n(a) \in \partial\eta(\rho_{h\tau}^n(a))$ at every $a \in \mathcal{V}$. Our strategy is to first consider the easier case (A1), where η' diverges at 0 and thus is a bijection from $\mathbb{R}_{>0}$ to \mathbb{R} . The case (A2) of a regular η is then treated by approximation with singular η 's. Throughout this section, the spatial discretization \mathcal{T}_h , the time step τ^n , and the datum $\rho_{h\tau}^{n-1} \in V_h$ are fixed. Most of the constants appearing below will depend on these parameters.

We start by proving the a priori estimate (28) in a slightly more general form.

Lemma 4.1. *Suppose that $\rho_h^+ \in V_h$ is non-negative, and $\mu_h^+ \in V_h$ is such that $\mu_h^+(a) \in \partial\eta(\rho_h^+(a))$ at every $a \in \mathcal{V}$. Assume further that, for every $w_h \in V_h$,*

$$(36) \quad \int_{\Omega} (\bar{\rho}_h^+ - \bar{\rho}_{h\tau}^{n-1}) \bar{w}_h + \tau^n \int_{\Omega} \gamma |\nabla(\mu_h^+ + \Psi_h)|^{q-2} \nabla(\mu_h^+ + \Psi_h) \cdot \nabla w_h = 0,$$

where $\gamma \in L^1(\Omega)$. Then

$$(37) \quad \tau^n \int_{\Omega} \gamma |\nabla(\mu_h^+ + \Psi_h)|^q \leq \mathcal{E}_h(\bar{\rho}_{h\tau}^{n-1}) - \mathcal{E}_h(\bar{\rho}_h^+).$$

The estimate (28) is a particular consequence of this lemma; simply use $\gamma = \rho_h^+ = \rho_{h\tau}^n$.

Proof. Thanks to the convexity of η , and by definition of the subdifferential $\partial\eta$, we have at every $a \in \mathcal{V}$ that

$$(38) \quad \mu_h^+(a)(\rho_{h\tau}^{n-1}(a) - \rho_h^+(a)) \leq \eta(\rho_{h\tau}^{n-1}(a)) - \eta(\rho_h^+(a)).$$

Therefore, using $w_h := \mu_h^+ + \Psi_h$ in (36) yields

$$\begin{aligned} \tau^n \int_{\Omega} \gamma |\nabla(\mu_h^+ + \Psi_h)|^q &= \int_{\Omega} \bar{\mu}_h^+(\bar{\rho}_{h\tau}^{n-1} - \bar{\rho}_h^+) + \int_{\Omega} \bar{\rho}_{h\tau}^{n-1} \bar{\Psi}_h - \int_{\Omega} \bar{\rho}_h^+ \bar{\Psi}_h \\ &\leq \int_{\Omega} \eta(\bar{\rho}_{h\tau}^{n-1}) - \int_{\Omega} \eta(\bar{\rho}_h^+) + \int_{\Omega} \bar{\rho}_{h\tau}^{n-1} \bar{\Psi}_h - \int_{\Omega} \bar{\rho}_h^+ \bar{\Psi}_h \\ &= \mathcal{E}_h(\bar{\rho}_{h\tau}^{n-1}) - \mathcal{E}_h(\bar{\rho}_h^+), \end{aligned}$$

which is (37). Note that mass-lumping allows us to transfer the inequality (38) at each vertex to the piecewise constant $\bar{\rho}_{h\tau}^n$. \square

Our first existence result is:

Proposition 4.2. *Assume that (A1) holds. If $\rho_{h\tau}^{n-1} \in V_h$ is a positive probability density, then there exists a solution to (26), which is such that $\rho_{h\tau}^n$ is a positive probability density, and $\mu_{h\tau}^n(a) = \eta'(\rho_{h\tau}^n(a))$ at every $a \in \mathcal{V}$. Moreover, the a priori estimate (28) is satisfied.*

The main technical tool for proving Proposition (4.2) is the following positivity result about an approximation of (26).

Lemma 4.3. *There exists a constant $\varepsilon_{h\tau}^n > 0$ such that for every $\alpha > 0$, the following is true: if $\rho_{h\tau}^{n,\alpha} \in V_h$ is a positive probability density, if $\mu_{h\tau}^{n,\alpha} \in V_h$ is such that $\mu_{h\tau}^{n,\alpha}(a) = \eta'(\rho_{h\tau}^{n,\alpha}(a))$ for all $a \in \mathcal{V}$, and if*

$$(39) \quad \int_{\Omega} (\bar{\rho}_{h\tau}^{n,\alpha} - \bar{\rho}_{h\tau}^{n-1}) \bar{w}_h + \tau^n \int_{\Omega} \max(\alpha, \rho_{h\tau}^{n,\alpha}) |\nabla(\mu_{h\tau}^{n,\alpha} + \Psi_h)|^{q-2} \nabla(\mu_{h\tau}^{n,\alpha} + \Psi_h) \cdot \nabla w_h = 0$$

for every $w_h \in V_h$, then $\rho_{h\tau}^{n,\alpha} \geq \varepsilon_{h\tau}^n$.

Remark 4.4. *The significance of this lemma is that, although the solution $\rho_{h\tau}^{n,\alpha}$ will depend on $\alpha > 0$ in general, the positive lower bound $\varepsilon_{h\tau}^n$ is uniform in α . In particular, if $\alpha \leq \varepsilon_{h\tau}^n$, then $\max(\alpha, \rho_{h\tau}^{n,\alpha}) = \rho_{h\tau}^{n,\alpha}$, and thus any solution to the modified problem (39) is actually a solution to the original problem (26).*

Proof of Lemma 4.3. To begin with, apply Lemma 4.1 with $\rho_h^+ = \rho_{h\tau}^{n,\alpha}$ and $\gamma = \max(\alpha, \rho_h^+)$. It follows that

$$\begin{aligned} \int_{\Omega} \rho_{h\tau}^{n,\alpha} |\nabla \mu_{h\tau}^{n,\alpha}|^q &\leq 2^{q-1} \left(\int_{\Omega} \rho_{h\tau}^{n,\alpha} |\nabla(\mu_{h\tau}^{n,\alpha} + \Psi_h)|^q + \int_{\Omega} \rho_{h\tau}^{n,\alpha} |\nabla \Psi_h|^q \right) \\ &\leq 2^{q-1} \left(\frac{1}{\tau^n} (\mathcal{E}_h(\bar{\rho}_{h\tau}^{n-1}) - \mathcal{E}_h(\bar{\rho}_h^+)) + \|\nabla \Psi_h\|_{\infty}^q \int_{\Omega} \rho_{h\tau}^{n,\alpha} \right). \end{aligned}$$

Since $\rho_{h\tau}^{n,\alpha}$ is positive and of unit mass and using the estimate (25) on the external potential, there is a constant R — independent of α — such that

$$(40) \quad \int_{\Omega} \rho_{h\tau}^{n,\alpha} |\nabla \mu_{h\tau}^{n,\alpha}|^q \leq R.$$

Moreover, since $\rho_{h\tau}^{n,\alpha}$ is positive and of unit mass, there exists a node $a_0 \in \mathcal{V}$ such that

$$\rho_{h\tau}^{n,\alpha}(a_0) \geq \int_{\Omega} \rho_{h\tau}^{n,\alpha} = |\Omega|^{-1}, \quad \text{hence} \quad \mu_{h\tau}^{n,\alpha}(a_0) \geq M_0 := \eta'(|\Omega|^{-1}).$$

Now, let $a_1 \in \mathcal{V}$ be a vertex sharing an edge with a_0 , i.e., there exists $T_0 \in \mathcal{T}_h$ having a_0 and a_1 as vertices. Since the integrand in (40) is piecewise linear, we can replace $\rho_{h\tau}^{n,\alpha}$ by $\bar{\rho}_{h\tau}^n$, and it then follows that

$$R \geq \int_{T_0} \rho_{h\tau}^{n,\alpha} |\nabla \mu_{h\tau}^{n,\alpha}|^q = |T_0| \bar{\rho}_{h\tau}^{n,\alpha}(x) |\nabla \mu_{h\tau}^{n,\alpha}(x)|^q,$$

for all $x \in T_0$. On the one hand,

$$\bar{\rho}_{h\tau}^{n,\alpha}(x) = \frac{1}{d+1} \sum_{a \in \mathcal{V}_{T_0}} \rho_{h\tau}^{n,\alpha}(a) \geq \frac{1}{d+1} \rho_{h\tau}^{n,\alpha}(a_0) \geq \frac{|\Omega|^{-1}}{(d+1)}.$$

And on the other hand,

$$|\nabla \mu_{h\tau}^{n,\alpha}(x)| \geq \left| \frac{\mu_{h\tau}^{n,\alpha}(a_0) - \mu_{h\tau}^{n,\alpha}(a_1)}{h_{T_0}} \right|.$$

Combining the above, and setting $A_{\mathcal{T}} = \min_{T \in \mathcal{T}_h} |T|$, one has

$$\mu_{h\tau}^{n,\alpha}(a_1) \geq M_1 := M_0 - h_{\mathcal{T}} \left(\frac{R(d+1)}{A_{\mathcal{T}} |\Omega|^{-1}} \right)^{\frac{1}{q}}, \quad \text{hence} \quad \rho_{h\tau}^{n,\alpha}(a_1) \geq (\eta')^{-1}(M_1) > 0,$$

since η' is one-to-one and onto from $(0, +\infty)$ to \mathbb{R} . Iterating the process provides a lower bound $M_j > 0$ on $\mu_{h\tau}^{n,\alpha}(a_j)$ at any vertex located j edges away from a_0 , where M_j is defined by means of the recursion

$$M_{j+1} = M_j - h_{\mathcal{T}} \left(\frac{R(d+1)}{A_{\mathcal{T}} (\eta')^{-1}(M_j)} \right)^{\frac{1}{q}}, \quad j = 0, 1, 2, \dots$$

Since Ω is connected and made of a finite number of vertices, a finite number of iterations allows to reach all the vertices of the mesh. The lower bound on $\rho_{h\tau}^{n,\alpha}$ follows. \square

With the lower bound at hand, we proceed to the proof of Proposition 4.2.

Proof of Proposition 4.2. In view of Remark 4.4 above, it suffices to prove that there exists a solution to the modified problem (39) for every $\alpha > 0$. Therefore, let $\alpha > 0$ be arbitrary but fixed in the following.

We prove solvability of (39) by combining a variational with a fixed point argument. Given $\rho_h^- \in V_h$, define the functional $\Upsilon_{\rho_h^-}^{\alpha} : V_h \rightarrow \mathbb{R}$

$$\Upsilon_{\rho_h^-}^{\alpha}(\mu_h) = \int_{\Omega} (\eta^*(\bar{\mu}_h) - \bar{\rho}_{h\tau}^{n-1} \bar{\mu}_h) + \frac{\tau}{q} \int_{\Omega} \max(\alpha, \rho_h^-) |\nabla(\mu_h + \Psi_h)|^q,$$

where $\eta^* : \mathbb{R} \rightarrow \mathbb{R}$ is the Legendre transform of η ,

$$(41) \quad \eta^*(m) = \sup_{r \geq 0} [mr - \eta(r)].$$

Let $M \in \mathbb{R}$ be such that the M -sublevel set $S_M = \{\mu_h \in V_h \mid \Upsilon_{\rho_h^-}^{\alpha}(\mu) \leq M\}$ is not empty, e.g., $M := \Upsilon_{\rho_h^-}^{\alpha}(0)$. Since

$$\eta^*(\bar{\mu}_h) \geq \bar{\mu}_h \bar{\rho}_{h\tau}^{n-1} - \eta(\bar{\rho}_{h\tau}^{n-1})$$

by (41), it follows for each $\mu_h \in S_M$ that

$$M \geq - \int_{\Omega} \eta(\bar{\rho}_{h\tau}^{n-1}) + \frac{\tau}{q} \int_{\Omega} \max(\alpha, \rho_h^-) |\nabla(\mu_h + \Psi_h)|^q.$$

Recalling that $\alpha > 0$, it follows further that

$$\int_{\Omega} |\nabla(\mu_h + \Psi_h)|^q \leq B := \frac{q}{\alpha\tau} \left[M + \int_{\Omega} \eta(\bar{\rho}_{h\tau}^{n-1}) \right].$$

In particular, for any simplex $T \in \mathcal{T}_h$,

$$|T| |\nabla(\mu_h + \Psi_h)(x)|^q = \int_T |\nabla(\mu_h + \Psi_h)|^q \leq B$$

at each $x \in T$, and hence for arbitrary vertices a and a' of T , thanks to estimate (25) one has

$$|\mu_h(a') - \mu_h(a)| \leq h_T (C_1 \|\nabla\Psi\|_{L^\infty(\Omega)^d} + (B/|T|)^{1/q}).$$

As there is a finite number of vertices, this means that there is some constant R (recall that the mesh is fixed) such that each $\mu_h \in S_M$ satisfies

$$(42) \quad \max \mu_h - \min \mu_h \leq R.$$

Directly from the definition (41), we also obtain that

$$\eta^*(\bar{\mu}_h) \geq 2\bar{\mu}_h \bar{\rho}_{h\tau}^{n-1} - \eta(2\bar{\rho}_{h\tau}^{n-1}) \quad \text{and} \quad \eta^*(\bar{\mu}_h) \geq -\eta(0),$$

and therefore, recalling that $\rho_{h\tau}^{n-1}$ is a probability density,

$$M \geq - \int_{\Omega} \eta(2\bar{\rho}_{h\tau}^{n-1}) + \min \mu_h \quad \text{and} \quad M \geq -|\Omega|\eta(0) - \max \mu_h.$$

In combination with (42), this implies the existence of bounds \hat{m} and \check{m} such that $\check{m} \leq \mu \leq \hat{m}$ for all $\mu_h \in S_M$. In particular, S_M is compact, and so the continuous functional $\Upsilon_{\rho_h}^{\alpha}$ attains its minimum at some $\mu_h^+ \in S_M$. By strict convexity of $\Upsilon_{\rho_h}^{\alpha}$, it is clear that the minimizer μ_h^+ is unique, and depends continuously on ρ_h^- . Define accordingly $\rho_h^+ \in V_h$ by $\rho_h^+(a) = (\eta')^{-1}(\mu_h^+(a))$ at every $a \in \mathcal{V}$; this function satisfies $\check{r} \leq \rho_h^+ \leq \hat{r}$ with bounds given by $\eta'(\check{r}) = \check{m}$ and $\eta'(\hat{r}) = \hat{m}$. Considering variations in μ_h around μ_h^+ , and recalling that $(\eta^*)' = (\eta')^{-1}$, it is easily seen that ρ_h^+ and μ_h^+ satisfy the Euler-Lagrange equation

$$(43) \quad \int_{\Omega} (\bar{\rho}_h^+ - \bar{\rho}_{h\tau}^{n-1}) \bar{w}_h + \tau \int_{\Omega} \max(\alpha, \rho_h^-) |\nabla(\mu_h^+ + \Psi_h)|^{q-2} \nabla(\mu_h^+ + \Psi_h) \cdot \nabla w_h = 0$$

for all $w_h \in V_h$. Using in particular $w_h \equiv 1$, it follows that ρ_h^+ has the same mass as $\rho_{h\tau}^{n-1}$, so ρ_h^+ is a probability density.

We are finally in the position to define the fixed point operator. On the compact and convex set

$$U := \{\rho_h \in V_h \mid \check{r} \leq \rho_h \leq \hat{r}\},$$

define the continuous map $S^\alpha : U \rightarrow U$ via $S^\alpha(\rho_h^-) := \rho_h^+$, where ρ_h^+ is the unique minimizer of $\Upsilon_{\rho_h}^{\alpha}$. By Brouwer's fixed point theorem, S^α possesses a fixed point $\rho_h^* \in U$. As noted above, ρ_h^* is a probability density. Recalling the Euler-Lagrange equation (43), the functions $\rho_{h\tau}^{n,\alpha} := \rho_h^*$ and $\mu_{h\tau}^{n,\alpha} \in V_h$ such that $\mu_{h\tau}^{n,\alpha}(a) = \eta'(\rho_{h\tau}^{n,\alpha}(a))$ for all $a \in \mathcal{V}$ satisfy the relations (39). \square

To show the existence of a solution $(\rho_{h\tau}^n, \mu_{h\tau}^n)$ in the case (A2) of a regular η , we first approximate η by a sequence $(\eta_\varepsilon)_{\varepsilon>0}$ of functions η_ε satisfying (A1) as follows. For $\varepsilon > 0$, let $\beta_\varepsilon \in C(\mathbb{R}_{\geq 0}) \cap C^1(\mathbb{R}_{> 0})$ be given by

$$(44) \quad \beta_\varepsilon(\rho) = \begin{cases} \rho \log \rho - \rho(\log \varepsilon + 1) + \varepsilon & \text{if } \rho \leq \varepsilon, \\ 0 & \text{else,} \end{cases}$$

and define

$$(45) \quad \eta_\varepsilon = \eta + \beta_\varepsilon, \quad \varepsilon > 0.$$

The functions η_ε satisfies (A1) and converge uniformly towards η as $\varepsilon \rightarrow 0$ since $0 \leq \beta_\varepsilon \leq \varepsilon$. Proposition 4.2 guarantees the existence of strictly positive solutions $\rho_{\varepsilon, h\tau}^n \in V_h$ to (26). Since V_h is finite dimensional, and since $V_h \cap \mathcal{P}_{\text{ac}}(\Omega)$ is a closed bounded (hence compact) subset, there exists an accumulation point $\rho_{h\tau}^n \in V_h \cap \mathcal{P}_{\text{ac}}(\Omega)$ of $(\rho_{\varepsilon, h\tau}^n)_{\varepsilon>0}$. That is, up to a subsequence,

$$(46) \quad \rho_{\varepsilon, h\tau}^n \rightarrow \rho_{h\tau}^n \quad \text{as } \varepsilon \rightarrow 0$$

uniformly on Ω . It remains to prove the existence of an accompanying $\mu_{h\tau}^n \in V_h$ with

$$(47) \quad \mu_{h\tau}^n(a) \in \partial\eta(\rho_{h\tau}^n(a)) \quad \text{at every } a \in \mathcal{V},$$

such that $\rho_{h\tau}^n$ and $\mu_{h\tau}^n$ satisfy (26).

Proposition 4.5. *Assume that (A2) holds, and let $\rho_{h\tau}^n$ be given by (46). Then there exists an accompanying $\mu_{h\tau}^n$ satisfying (47) such that (26) holds.*

Proof. For brevity, let $\mu_{h\tau, \varepsilon}^n \in V_h$ be given by $\mu_{h\tau, \varepsilon}^n(a) = \eta'_\varepsilon(\rho_{h\tau, \varepsilon}^n(a))$ at all $a \in \mathcal{V}$. The proof uses an estimate on $\nabla \mu_{h\tau, \varepsilon}^n$ of the form (28), uniformly in $\varepsilon > 0$. Indeed, the approximate solutions satisfy (28), with $\rho_{h\tau, \varepsilon}^n$ and $\mu_{h\tau, \varepsilon}^n$ in place of $\rho_{h\tau}^n$ and $\mu_{h\tau}^n$, respectively, and with the modified energy

$$\mathcal{E}_{h, \varepsilon}(\bar{\rho}_h) = \int_{\Omega} [\eta_\varepsilon(\bar{\rho}_h) + \bar{\Psi}_h \bar{\rho}_h].$$

Since η_ε converges to η uniformly, the difference on the right-hand side of (28) is bounded independently of $\varepsilon \in (0, 1)$. In analogy to (40), we obtain an ε -independent bound R such that

$$(48) \quad \int_{\Omega} \rho_{h\tau, \varepsilon}^n |\nabla \mu_{h\tau, \varepsilon}^n|^q \leq R.$$

Split the set \mathcal{V} of vertices into three disjoint subsets, $\mathcal{V} = \mathcal{V}_+ \cup \mathcal{V}_0 \cup \mathcal{V}_\partial$, where

$$\mathcal{V}_+ = \{a \in \mathcal{V} \mid \rho_{h\tau}^n(a) > 0\},$$

$$\mathcal{V}_0 = \{a \in \mathcal{V} \mid \rho_{h\tau}^n(a) = 0 \text{ and } \rho_{h\tau}^n(a') = 0 \text{ for all } a' \in \mathcal{V} \text{ sharing an edge with } a\},$$

$$\mathcal{V}_\partial = \{a \in \mathcal{V} \mid \rho_{h\tau}^n(a) = 0 \text{ and } \rho_{h\tau}^n(a') > 0 \text{ for some } a' \in \mathcal{V} \text{ sharing an edge with } a\}.$$

Let $a \in \mathcal{V}_+$. Since η'_ε converges locally uniformly to η' on $\mathbb{R}_{> 0}$,

$$(49) \quad \mu_{h\tau, \varepsilon}^n(a) \rightarrow \eta'(\rho_{h\tau}^n(a)) \quad \text{as } \varepsilon \rightarrow 0.$$

Next, let $a \in \mathcal{V}_\partial$. Then there exist $T \in \mathcal{T}_h$ and $a' \in \mathcal{V}_+$ such that $a, a' \in \mathcal{V}_T$. We argue similar as in the proof of Lemma 4.3: on the one hand, by (48),

$$R \geq \int_T \rho_{h\tau, \varepsilon}^n |\nabla \mu_{h\tau, \varepsilon}^n|^q = |T| \bar{\rho}_{h\tau, \varepsilon}^n(x) |\nabla \mu_{h\tau, \varepsilon}^n(x)|^q$$

at every $x \in T$; on the other hand,

$$\bar{\rho}_{h\tau,\varepsilon}^n(x) = \frac{1}{d+1} \sum_{a'' \in \mathcal{V}_T} \rho_{h\tau,\varepsilon}^n(a'') \geq \frac{\rho_{h\tau,\varepsilon}^n(a')}{d+1};$$

and consequently,

$$|\mu_{h\tau,\varepsilon}^n(a) - \mu_{h\tau,\varepsilon}^n(a')|^q \leq h_T^q |\nabla \mu_{h\tau,\varepsilon}^n(x)|^q \leq \frac{(d+1)h_T^q R}{|T| \rho_{h\tau,\varepsilon}^n(a')}.$$

Since both $\rho_{h\tau,\varepsilon}^n(a')$ and $\mu_{h\tau,\varepsilon}^n(a')$ converge as $\varepsilon \rightarrow 0$, we conclude boundedness of $\mu_{h\tau,\varepsilon}^n(a)$, and thus convergence along a subsequence to a limit $\mu_{h\tau}^n(a)$. Observe further that η'_ε converges to η' from below, and so $\limsup_{\varepsilon \rightarrow 0} \mu_{h\tau,\varepsilon}^n(a) \leq \eta'(0)$. Hence

$$(50) \quad \mu_{h\tau,\varepsilon}^n(a) \rightarrow \mu_{h\tau}^n(a) \in \partial\eta(0) = (-\infty, \eta'(0)] \quad \text{as } \varepsilon \rightarrow 0.$$

Finally, let $a \in \mathcal{V}_0^n$. In this case, we do not expect that $\mu_{h\tau,\varepsilon}^n(a)$ possesses an accumulation point. Instead, we show that *any* choice of $\mu_{h\tau}^n(a) \in \partial\eta(0)$ is compatible with (26). Denote by

$$\mathcal{T}_0 = \{T \in \mathcal{T}_h \mid \mathcal{V}_T \subset \mathcal{V}_0\}$$

the set of simplices T for which $\rho_{h\tau,\varepsilon}^n(a)$ converges to zero at *every* vertex $a \in \mathcal{V}_T$, then $\mathcal{V}_0 = \bigcup_{T \in \mathcal{T}_0} \mathcal{V}_T$ as the mesh is made of simplices. Given $w_h \in V_h$, the term

$$\Lambda_\varepsilon := \tau^n \sum_{T \in \mathcal{T}_0} \int_T \rho_{h\tau,\varepsilon}^n |\nabla(\mu_{h\tau,\varepsilon}^n + \Psi_h)|^{q-2} \nabla(\mu_{h\tau,\varepsilon}^n + \Psi_h) \cdot \nabla w_h,$$

that is part of the left-hand side in (26), tends to 0 since, by Hölder's inequality and estimate (48),

$$(51) \quad \Lambda_\varepsilon \leq \left(\int_\Omega \tau^n \bar{\rho}_{h\tau,\varepsilon}^n |\nabla(\mu_{h\tau,\varepsilon}^n + \Psi_h)|^q \right)^{\frac{1}{p}} \left(\sum_{T \in \mathcal{T}_0} \int_T |\nabla w_h|^q \bar{\rho}_{h\tau,\varepsilon}^n \right)^{\frac{1}{q}} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Summarizing (46)–(51), we can pass to the limit $\varepsilon \rightarrow 0$ in (26) and obtain

$$\int_\Omega (\bar{\rho}_h^n - \bar{\rho}_{h\tau}^{n-1}) \bar{w}_h + \tau^n \sum_{T \notin \mathcal{T}_0} \int_T \bar{\rho}_{h\tau}^n |\nabla(\mu_{h\tau}^n + \Psi_h)|^{q-2} \nabla(\mu_{h\tau}^n + \Psi_h) \cdot \nabla w_h = 0,$$

while $\bar{\rho}_h^n = 0$ on $\bigcup_{T \in \mathcal{T}_0} T$. □

Combining Propositions 4.2 and 4.5 with Lemma 4.1, we obtain Theorem 3.1.

5. UNIFORM ESTIMATES

Throughout this section, we assume that a simplicial decomposition \mathcal{T}_h and a time discretization τ are given. The initial data ρ^0 is discretized into $\bar{\rho}_{h\tau}^0 \in X_h$ following (29). From Section 4 above, we know that the scheme (26) admits a solution at each time step τ^n . The goal of this section is to derive uniform estimates on a sequence $(\rho_{h\tau}^n)_{n=0}^\infty$ satisfying (26) that are independent of the discretization parameters h and τ . Any constant C that may depend on the parameters Ω , η , Ψ , the initial energy $\mathcal{E}(\rho^0)$, or the mesh regularity $\zeta_{\mathcal{T}_h}$, but is otherwise independent of (h, τ) will be called *uniform*.

Lemma 5.1. $\rho_{h\tau}^n, \bar{\rho}_{h\tau}^n$ and $\bar{\rho}_{h\tau}^n$ are probability densities, at each n .

Proof. Since non-negativity of $\rho_{h\tau}^n$ is assumed for solutions to (26), one only has to check that

$$\int_{\Omega} \bar{\rho}_{h\tau}^n = \int_{\Omega} \rho_{h\tau}^n = \int_{\Omega} \bar{\rho}_{h\tau}^n = \int_{\Omega} \bar{\rho}_{h\tau}^{n-1} = \int_{\Omega} \bar{\rho}_{h\tau}^0 = \int_{\Omega} \rho^0 = 1.$$

The first two equalities are provided by the exactness of quadrature rules, while the third one directly follows from testing (26) by $w_h \equiv 1$. A straightforward induction yields the fourth equality, and the fifth one simply uses the definition of $\bar{\rho}_{h\tau}^0$. The last equality holds since $\rho^0 \in \mathcal{P}_{ac}(\Omega)$. \square

Proposition 5.2. *For all $N \geq 1$, there holds*

$$(52) \quad \mathcal{E}_h(\bar{\rho}_{h\tau}^N) + \sum_{n=1}^N \tau^n \int_{\Omega} \rho_{h\tau}^n |\nabla(\mu_{h\tau}^n + \Psi_h)|^q \leq \mathcal{E}_h(\bar{\rho}_{h\tau}^0).$$

Proof. Simply sum the a priori estimates (28) over $n = 1, \dots, N$ to obtain (52). \square

A first consequence of the energy estimate is a time-uniform estimate on $\eta(\bar{\rho}_{h\tau})$ in $L^1(\Omega)$.

Corollary 5.3. *There exists a uniform constant C_2 such that*

$$(53) \quad \int_{\Omega} \eta(\bar{\rho}_{h\tau}^n) \leq \int_{\Omega} \eta(\bar{\rho}_{h\tau}^0) \leq C_2, \quad \forall n \geq 1.$$

Proof. Bearing the definition (4) of the energy and the non-negativity of Ψ in mind, one gets that

$$\int_{\Omega} \eta(\bar{\rho}_{h\tau}^n) \leq \mathcal{E}_h(\bar{\rho}_{h\tau}^n) \leq \mathcal{E}_h(\bar{\rho}_{h\tau}^0) \quad \forall n \geq 0,$$

and $\mathcal{E}_h(\bar{\rho}_{h\tau}^0) \leq \mathcal{E}(\rho^0) + Ch_{\mathcal{T}} \leq C_2$ owing to Jensen's inequality applied on the Donald mesh and to (25). This yields the second inequality in (53), and the first is a direct consequence of Jensen's inequality applied on the simplicial mesh. \square

Another direct consequence of Proposition 5.2 is this.

Corollary 5.4. *There exists a uniform constant C_3 such that*

$$\sum_{n=1}^N \tau^n \int_{\Omega} \rho_{h\tau}^n |\nabla \mu_{h\tau}^n|^q \leq C_3(1 + t^N) \quad \text{for every } N \geq 1.$$

Proof. Using the elementary inequality $(a + b)^q \leq 2^{q-1}(a^q + b^q)$ for $a, b \geq 0$, one obtains

$$\sum_{n=1}^N \tau^n \int_{\Omega} \rho_{h\tau}^n |\nabla \mu_{h\tau}^n|^q dx \leq 2^{q-1} \left(\sum_{n=1}^N \tau^n \int_{\Omega} \rho_{h\tau}^n |\nabla(\mu_{h\tau}^n + \Psi_h)|^q dx + \sum_{n=1}^N \tau^n \int_{\Omega} \rho_{h\tau}^n |\nabla \Psi_h|^q dx \right).$$

The first summation is estimated by (52) above, in the second summation, the n th term is bounded by a multiple of τ^n — recall that Ψ is Lipschitz. Since $\sum_{n=1}^N \tau^n = t^N$, this yields the desired estimate above. \square

Recall the definition of $\phi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ from (7), and introduce accordingly $\phi_{h\tau}^n \in V_h$ as the piecewise affine function with $\phi_{h\tau}^n(a) = \phi(\rho_h^n(a))$ for all $a \in \mathcal{V}$.

Lemma 5.5. *There exists a uniform constant C_4 such that*

$$(54) \quad |\nabla \phi_{h\tau}^n| \leq C_4 \bar{\rho}_{h\tau}^n |\nabla \mu_{h\tau}^n|, \quad \forall n \geq 1.$$

Proof. Let $T \in \mathcal{T}_h$, the vertices of which being denoted by a_0^T, \dots, a_d^T , let \widehat{T} be the reference elements, with $a_0^{\widehat{T}} = 0_{\mathbb{R}^d}$ and $a_i^{\widehat{T}} = e_i$ the i^{th} vector of the canonical basis of \mathbb{R}^d , let F_T be the unique affine map sending $a_i^{\widehat{T}}$ on a_i^T , and let J_T denote its Jacobian matrix. Then classical results from the theory of finite elements (see for instance [23, Lemma 1.100]) combined with elementary calculations show that

$$(55) \quad |\det(J_T)| = d!|T|, \quad |J_T|_2 \leq \frac{d + \sqrt{d}}{2} h_T \quad \text{and} \quad |J_T^{-1}|_2 \leq \frac{\sqrt{2}}{d_T} \leq \sqrt{2} \frac{\zeta_{\mathcal{T}_h}}{h_T},$$

with $|\cdot|_2$ standing for the matrix norm inherited from the Euclidian norm of \mathbb{R}^d . We deduce from a change of variable that for all $n \geq 1$ and all $T \in \mathcal{T}_h$, there holds

$$(56) \quad |\nabla \phi_{h\tau}^n(x)| = \int_T |\nabla \phi_{h\tau}^n| = d! |J_T^{-1}| \delta_T \phi_{h\tau}^n \leq \sqrt{2} d! \frac{\zeta_{\mathcal{T}_h}}{h_T} |\delta_T \phi_{h\tau}^n|, \quad \forall x \in T,$$

where $\delta_T \phi_{h\tau}^n$ is the vector of the variations of $\phi_{h\tau}^n$ in T defined by

$$\delta_T \phi_{h\tau}^n = \begin{pmatrix} \phi_{h\tau}^n(a_1^T) - \phi_{h\tau}^n(a_0^T) \\ \vdots \\ \phi_{h\tau}^n(a_d^T) - \phi_{h\tau}^n(a_0^T) \end{pmatrix}.$$

Let R_T^n be the diagonal matrix with positive entries defined by

$$(R_T^n)_{i,i} = \begin{cases} \frac{\phi_{h\tau}^n(a_i^T) - \phi_{h\tau}^n(a_0^T)}{\mu_{h\tau}^n(a_i^T) - \mu_{h\tau}^n(a_0^T)} & \text{if } \mu_{h\tau}^n(a_i^T) \neq \mu_{h\tau}^n(a_0^T), \\ \rho_{h\tau}^n(a_i^T) & \text{otherwise,} \end{cases}$$

so that, setting

$$\delta_T \mu_{h\tau}^n = \begin{pmatrix} \mu_{h\tau}^n(a_1^T) - \mu_{h\tau}^n(a_0^T) \\ \vdots \\ \mu_{h\tau}^n(a_d^T) - \mu_{h\tau}^n(a_0^T) \end{pmatrix},$$

there holds $\delta_T \phi_{h\tau}^n = R_T^n \delta_T \mu_{h\tau}^n$. Assume that $\rho_{h\tau}^n(a_i^T) > \rho_{h\tau}^n(a_0^T)$. Then the definition (7) of ϕ and the fact that $\mu_{h\tau}^n \in \partial\eta(\bar{\rho}_{h\tau}^n)$ respectively provide that

$$\phi_{h\tau}^n(a_i^T) - \phi_{h\tau}^n(a_0^T) = \int_{\rho_{h\tau}^n(a_0^T)}^{\rho_{h\tau}^n(a_i^T)} s \eta''(s) ds$$

and

$$\mu_{h\tau}^n(a_i^T) - \mu_{h\tau}^n(a_0^T) \geq \eta'(\rho_{h\tau}^n(a_i^T)) - \eta'(\rho_{h\tau}^n(a_0^T)) = \int_{\rho_{h\tau}^n(a_0^T)}^{\rho_{h\tau}^n(a_i^T)} \eta''(s) ds.$$

The above inequality is an equality if $\rho_{h\tau}^n(a_0^T) > 0$, so that

$$\rho_{h\tau}^n(a_0^T) \leq (R_T^n)_{i,i} \leq \rho_{h\tau}^n(a_i^T),$$

which clearly still holds when $\rho_{h\tau}^n(a_0^T) = 0$. Proceeding similarly if $\rho_{h\tau}^n(a_i^T) < \rho_{h\tau}^n(a_0^T)$, one gets that

$$(57) \quad \min(\rho_{h\tau}^n(a_i^T), \rho_{h\tau}^n(a_0^T)) \leq (R_T^n)_{i,i} \leq \max(\rho_{h\tau}^n(a_i^T), \rho_{h\tau}^n(a_0^T)) \leq \sum_{i=0}^d \rho_{h\tau}^n(a_i^T),$$

hence

$$(58) \quad |\delta_T \phi_{h\tau}^n| \leq |\delta_T \mu_{h\tau}^n| \left(\sum_{i=0}^d \rho_{h\tau}^n(a_i^T) \right) = (d+1) \bar{\rho}_{h\tau}^n(x) |\delta_T \mu_{h\tau}^n|, \quad \forall x \in T.$$

On the other hand, using (55) once again provides

$$(59) \quad |\delta_T \mu_{h\tau}^n| \leq |J_T|_2 |J_T^{-1} \delta_T \mu_{h\tau}^n| \leq \frac{d + \sqrt{d}}{2d!} h_T \int_T |\nabla \mu_{h\tau}^n| = \frac{d + \sqrt{d}}{2d!} h_T |\nabla \mu_{h\tau}^n(x)|$$

holds for all x in T . Combining (56), (58) and (59), one finally gets that (54) holds with $C_4 = (d + 1) \frac{d + \sqrt{d}}{\sqrt{2}} \zeta_{\mathcal{T}_h}$. \square

The control on the gradient of $\phi_{h\tau}^n$ from the previous lemma also provides a control on $\phi_{h\tau}^n$ itself.

Lemma 5.6. *There exists a uniform constant C_5 such that*

$$(60) \quad \int_{\Omega} \phi_{h\tau}^n \leq C_5 \left(1 + \int_{\Omega} |\nabla \phi_{h\tau}^n| dx \right).$$

Proof. Define the following set of “small” values of $\rho_{h\tau}^n$,

$$U^n = \{x \in \Omega \mid \bar{\rho}_{h\tau}^n(x) \leq 2|\Omega|^{-1}\}, \quad n \geq 1,$$

with $|U^n| \geq |\Omega|/2$ thanks to Markov’s inequality. By monotonicity of ϕ , we have $\bar{\phi}_{h\tau}^n \leq \phi(2|\Omega|^{-1})$ on U^n . The Poincaré-Wirtinger inequality and the equivalence of norms for affine and mass interpolated functions (see e.g. [12, Lemma A.6]) now imply that

$$\int_{\Omega} \left| \bar{\phi}_{h\tau}^n - \int_{\Omega} \bar{\phi}_{h\tau}^n dx' \right| dx \leq C_{\mathcal{T}} \int_{\Omega} \left| \phi_{h\tau}^n - \int_{\Omega} \phi_{h\tau}^n dx' \right| dx \leq C_P C_{\mathcal{T}} \int_{\Omega} |\nabla \phi_{h\tau}^n| dx,$$

where the Poincaré constant C_P depends on Ω ’s geometry, and $C_{\mathcal{T}}$ depends on the quality $\zeta_{\mathcal{T}_h}$ of the mesh. Further, by the triangle inequality, and recalling that $|U^n| \geq |\Omega \setminus U^n|$,

$$\begin{aligned} \int_{\Omega} \left| \bar{\phi}_{h\tau}^n - \int_{\Omega} \bar{\phi}_{h\tau}^n dx' \right| dx &= \int_{\Omega \setminus U^n} \left| \bar{\phi}_{h\tau}^n - \int_{\Omega} \bar{\phi}_{h\tau}^n dx' \right| dx + \int_{U^n} \left| \bar{\phi}_{h\tau}^n - \int_{\Omega} \bar{\phi}_{h\tau}^n dx' \right| dx \\ &\geq (|U^n| - |\Omega \setminus U^n|) \int_{\Omega \setminus U^n} \bar{\phi}_{h\tau}^n dx' + \int_{\Omega \setminus U^n} \bar{\phi}_{h\tau}^n dx - \int_{U^n} \bar{\phi}_{h\tau}^n dx \geq \int_{\Omega \setminus U^n} \bar{\phi}_{h\tau}^n dx - |\Omega| \phi(2|\Omega|^{-1}). \end{aligned}$$

By combination of the above, we obtain

$$\int_{\Omega} \bar{\phi}_{h\tau}^n dx \leq |\Omega| \phi(2|\Omega|^{-1}) + \int_{\Omega \setminus U^n} \bar{\phi}_{h\tau}^n dx \leq 2|\Omega| \phi(2|\Omega|^{-1}) + C_P C_{\mathcal{T}} \int_{\Omega} |\nabla \phi_{h\tau}^n| dx,$$

which proves (60). \square

6. CONVERGENCE TOWARDS A GRADIENT FLOW SOLUTION

The goal of this section is to establish Theorem 3.2, that is the convergence of approximate solutions towards a solution of the continuous problem when the discretization parameters h, τ tend to 0. Our proof is based on compactness.

In agreement with the statement of Theorem 3.2, we denote — by slight abuse of notation — by $\rho_{h\tau} : \mathbb{R}_{\geq 0} \times \Omega \rightarrow \mathbb{R}$ the time-dependent density obtained by the time-advanced piecewise constant interpolation (30); accordingly, $\bar{\rho}_{h\tau}$ and $\widetilde{\rho}_{h\tau}$ are the corresponding time-dependent densities that are piecewise constant on the Donald and on the simplicial mesh, respectively. Similarly, $\phi_{h\tau}$, $\bar{\phi}_{h\tau}$ and $\widetilde{\phi}_{h\tau}$ are the respective reconstructions of $\phi_{h\tau}^n$.

6.1. Compactness in time. Our first lemma is about controlling variations of $\rho_{h\tau}$ in time. To this end, we define the piecewise constant function $\delta_{h\tau} : \mathbb{R}_{\geq 0} \times \Omega \rightarrow \mathbb{R}$ by

$$\delta_{h\tau}(t) = \frac{\bar{\rho}_{h\tau}^n - \bar{\rho}_{h\tau}^{n-1}}{\tau^n} \quad \text{if } t \in (t^{n-1}, t^n].$$

Lemma 6.1. *There is a constant C such that*

$$(61) \quad \sum_{n'=m+1}^n \mathbb{W}_1(\bar{\rho}_{h\tau}^{n'}, \bar{\rho}_{h\tau}^{n'-1}) \leq C(t^n - t^m)^{1/q}$$

and consequently,

$$(62) \quad \mathbb{W}_1(\bar{\rho}_{h\tau}(t), \bar{\rho}_{h\tau}(s)) \leq C[|t - s| + \tau]^{1/q}.$$

Moreover, there exists C_{t^*} depending on t^* (but uniform w.r.t. h and τ) such that, for any $\varphi \in C^1([0, t^*] \times \bar{\Omega})$, there holds

$$(63) \quad \int_0^{t^*} \int_{\Omega} \delta_{h\tau} \varphi \leq C_{t^*} \|\nabla \varphi\|_{\infty}.$$

Proof. For a Lipschitz continuous test function $\varphi : \Omega \rightarrow \mathbb{R}$ with piecewise affine interpolation $\varphi_h \in V_h$ such that for any vertex $a \in \mathcal{V}$, $\varphi_h(a) = \int_{\omega_a} \varphi \, dx$, the weak formulation (26) of the scheme provides

$$\begin{aligned} \int_{\Omega} (\bar{\rho}_{h\tau}^n - \bar{\rho}_{h\tau}^{n-1}) \varphi \, dx &= \int_{\Omega} (\bar{\rho}_{h\tau}^n - \bar{\rho}_{h\tau}^{n-1}) \bar{\varphi}_h \, dx \\ &= -\tau^n \int_{\Omega} \rho_{h\tau}^n |\nabla(\mu_{h\tau}^n + \Psi_h)|^{q-2} \nabla(\mu_{h\tau}^n + \Psi_h) \cdot \nabla \varphi_h \, dx \\ &\leq \tau^n \|\nabla \varphi_h\|_{L^\infty(\Omega)^d} \left(\int_{\Omega} \rho_{h\tau}^n \, dx \right)^{1/q} \left(\int_{\Omega} \rho_{h\tau}^n |\nabla(\mu_{h\tau}^n + \Psi_h)|^q \, dx \right)^{1/p}. \end{aligned}$$

Using the conservation of mass from Lemma 5.1, the energy inequality from Proposition 5.2 and the fact that $\|\nabla \varphi_h\|_{L^\infty(\Omega)} \leq C \|\nabla \varphi\|_{L^\infty(\Omega)}$, see e.g. the proof of [14, Theorem A.1], we readily recover (63). We also arrive at

$$\mathbb{W}_1(\bar{\rho}_{h\tau}^n, \bar{\rho}_{h\tau}^{n-1}) \leq C \tau^n \left(\int_{\Omega} \rho_{h\tau}^n |\nabla(\mu_{h\tau}^n + \Psi_h)|^q \, dx \right)^{1/p}.$$

By Hölder's inequality for sums, and by the fundamental energy estimate in (52),

$$\begin{aligned} \sum_{n'=m+1}^n \mathbb{W}_1(\bar{\rho}_{h\tau}^{n'}, \bar{\rho}_{h\tau}^{n'-1}) &\leq C \left(\sum_{n'=m+1}^n \tau^{n'} \int_{\Omega} \rho_{h\tau}^{n'} |\nabla(\mu_{h\tau}^{n'} + \Psi_h)|^q \, dx \right)^{1/p} \left(\sum_{n'=m+1}^n \tau^{n'} \right)^{1/q} \\ &\leq C [\mathcal{E}_h(\bar{\rho}_{h\tau}^0)]^{1/p} (t^n - t^m)^{1/q}, \end{aligned}$$

showing (61). By the triangle inequality, this immediately implies

$$\mathbb{W}_1(\bar{\rho}_{h\tau}^n, \bar{\rho}_{h\tau}^m) \leq C(t^n - t^m)^{1/q}.$$

To derive (62) from here, assume $t > s$ and observe that the largest t^k with $t^k \leq s$ is at most τ away from s . \square

Our next result is about the following uniform in time but weak in space compactness property.

Lemma 6.2. *There exists a weak- $*$ -continuous curve $(\rho_*(t))_{t \geq 0}$ in the space \mathcal{P}_{ac} such that $\mathbb{W}_p(\bar{\rho}_{h\tau}(t), \rho_*(t)) \rightarrow 0$ and $\mathbb{W}_p(\bar{\rho}_{h\tau}(t), \rho_*(t)) \rightarrow 0$ locally uniformly with respect to $t \geq 0$ along a suitable subsequence as $h, \tau \rightarrow 0$.*

Proof. Estimate (62) above is sufficient to apply the Arzelà-Ascoli theorem in metric spaces (see Proposition A.1 from the Appendix) and conclude local uniform convergence of $\bar{\rho}_{h\tau}$ to a Hölder-continuous limit ρ_* . Thanks to the triangle inequality and to (11), it suffices to check that

$$(64) \quad \mathbb{W}_1(\bar{\rho}_{h\tau}(t), \tilde{\rho}_{h\tau}(t)) \rightarrow 0$$

to ensure that the convergence also holds for $\bar{\rho}_{h\tau}$.

Let φ be a Lipschitz continuous function on Ω , and let $\tilde{\varphi}_h$ be defined by

$$\tilde{\varphi}_h(x) = \int_T \varphi \, dx, \quad \forall x \in T, T \in \mathcal{T}_h.$$

then since $\int_T \bar{\rho}_{h\tau}(t) \, dx = \int_T \tilde{\rho}_{h\tau}(t) \, dx$, one has

$$\int_{\Omega} (\bar{\rho}_{h\tau}(t) - \tilde{\rho}_{h\tau}(t)) \varphi \, dx = \int_{\Omega} (\bar{\rho}_{h\tau}(t) - \tilde{\rho}_{h\tau}(t)) (\varphi - \tilde{\varphi}_h) \, dx.$$

As $\|\varphi - \tilde{\varphi}_h\|_{\infty} \leq h_{\mathcal{T}} \|\nabla \varphi\|_{\infty}$, and as $\bar{\rho}_{h\tau}(t)$ and $\tilde{\rho}_{h\tau}(t)$ are both absolutely continuous probability densities, one gets that

$$\int_{\Omega} (\bar{\rho}_{h\tau}(t) - \tilde{\rho}_{h\tau}(t)) \varphi \, dx \leq 2h_{\mathcal{T}} \|\nabla \varphi\|_{\infty}, \quad \forall \varphi \in W^{1,\infty}(\Omega),$$

whence (64). \square

6.2. Compactness in space-time. In this section, we establish strong compactness of $\rho_{h\tau}$ both in space and time. We start by an implication of the estimates on $\phi_{h\tau}$ from Lemmas 5.5 and 5.6

Lemma 6.3. *The functions $\phi_{h\tau}$ are uniformly bounded in $L^q([0, t^*]; W^{1,1}(\Omega))$ and in $L^r([0, t^*] \times \Omega)$, for each $t^* > 0$. Here $r := \min(q, d/(d-1))$.*

Proof. Thanks to (54), we can apply Hölder's inequality to obtain

$$\int_{\Omega} |\nabla \phi_{h\tau}^n| \, dx \leq C_4 \int_{\Omega} (\bar{\rho}_{h\tau}^n)^{1/p} (\tilde{\rho}_{h\tau}^n)^{1/q} |\nabla \mu_{h\tau}^n| \, dx \leq C_4 \left(\int_{\Omega} \bar{\rho}_{h\tau}^n \, dx \right)^{1/p} \left(\int_{\Omega} \tilde{\rho}_{h\tau}^n |\nabla \mu_{h\tau}^n|^q \, dx \right)^{1/q}.$$

By combination with Lemma 5.1 and Corollary 5.4 (since the integrand in the last integral is piecewise linear), we obtain

$$\int_0^{t^*} \left[\int_{\Omega} |\nabla \phi_{h\tau}^n| \, dx \right]^q \, dt \leq C.$$

A combination with Lemma 5.6 yields the desired bound in $L^q([0, t^*]; W^{1,1}(\Omega))$.

For the other claim, we use the Sobolev inequality

$$\|f\|_{L^{d/(d-1)}(\Omega)} \leq C \|f\|_{W^{1,1}(\Omega)}$$

for arbitrary $f \in W^{1,1}(\Omega)$. In combination with the estimate above, we obtain a bound on $\phi_{h\tau}$ in $L^r([0, t^*] \times \Omega)$. \square

The next result is the decisive one about space-time convergence of the densities.

Lemma 6.4. *Up to a subsequence, $\bar{\rho}_{h\tau} \rightarrow \rho_*$ almost everywhere on $[0, t^*] \times \Omega$ and in $L^1([0, t^*] \times \Omega)$, for every $t^* > 0$.*

Proof. This is a consequence of the Aubin-Lions compactness principle. As an alternative to the classical approach consisting in estimating the time and space translates of the solution (see [3] in the continuous setting and [26] in the discrete one), we build on ideas introduced in [37] and adapted to the discrete setting in [6]. For alternative approaches, see [41, Theorem 2] (for quasi-uniform time discretizations) or [27].

To begin with, define the increasing and bounded function $\pi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ by $\pi(r) = \phi(r)/(1 + \phi(r))$, then since $z \mapsto z/(1 + z)$ is 1-Lipschitz continuous on $\mathbb{R}_{\geq 0}$, one has

$$\mathrm{TV}[\bar{\pi}_h^n] \leq \mathrm{TV}[\bar{\phi}_{h\tau}^n], \quad \forall n \geq 1.$$

Here

$$\mathrm{TV}[f] = \sup \left\{ \int_{\Omega} f \nabla \cdot \xi \, dx \mid \xi \in C^1(\bar{\Omega}; \mathbb{R}^d), \sup_{\Omega} |\xi| \leq 1 \right\}$$

is the total variation of a function $f \in L^1(\Omega)$, and $\bar{\pi}_{h\tau}^n \in X_h$ stands for $\pi(\bar{\rho}_{h\tau}^n)$.

For such a piecewise constant function on Ω , the total variation equals to a finite sum: $\mathrm{TV}[\bar{\phi}_{h\tau}^n] = \sum_{T \in \mathcal{T}_h} G_T$, where the sum G_T runs over the modulus of the differences of the two values across interior $(d - 1)$ -dimensional interfaces, times the hypervolume of the respective interfaces. On a given simplex T , with two vertices a, b and an interior point x , one clearly has

$$|\phi(\bar{\rho}_{h\tau}^n(b)) - \phi(\bar{\rho}_{h\tau}^n(a))| = |\phi_{h\tau}^n(b) - \phi_{h\tau}^n(a)| \leq |b - a| |\nabla \phi_{h\tau}^n(x)|.$$

Further, if $H_{a,b,T} := \bar{\omega}_a \cap \bar{\omega}_b \cap T$ is the part of the hyperplane in T associated to a and b , then one has $|b - a| \mathcal{H}^{d-1}(H_{a,b,T}) \leq C|T|$, with a uniform constant C thanks to the regularity hypothesis on \mathcal{T}_h . The contribution V_T on T to the total variation of $\phi(\bar{\rho}_{h\tau}^n)$ thus amounts to

$$\begin{aligned} G_T &= \sum_{a,b \in \mathcal{V}_T} |\phi(\bar{\rho}_{h\tau}^n(b)) - \phi(\bar{\rho}_{h\tau}^n(a))| \mathcal{H}^{d-1}(H_{a,b,T}) \\ &\leq |\nabla \phi_{h\tau}^n(x)| \sum_{a,b \in T} |b - a| \mathcal{H}^{d-1}(H_{a,b,T}) \leq C \frac{d(d+1)}{2} |T| |\nabla \phi_{h\tau}^n(x)|, \end{aligned}$$

where the sum runs over all $d(d+1)/2$ many pairs $\{a, b\}$ of distinct vertices of T . As a consequence,

$$(65) \quad \int_0^{t^*} \mathrm{TV}[\bar{\pi}_{h\tau}(t)] \, dt \leq \int_0^{t^*} \mathrm{TV}[\bar{\phi}_{h\tau}(t)] \, dt \leq C \|\nabla \phi_{h\tau}\|_{L^1((0,t^*) \times \Omega)},$$

the latter being bounded thanks to Lemma 6.3. This yields some compactness with respect to the space variable: extending $\bar{\pi}_{h\tau}$ by 0 for x outside of Ω and using the fact that $0 \leq \bar{\pi}_{h\tau} \leq 1$, one infers from (65) that

$$(66) \quad \int_0^{t^*} \int_{\Omega} |\bar{\pi}_{h\tau}(t, x+y) - \bar{\pi}_{h\tau}(t, x)| \, dx \, dt \leq C|y|, \quad \forall y \in \mathbb{R}^d.$$

Besides, since $0 \leq \bar{\pi}_{h\tau} \leq 1$, one can extract a subsequence such that

$$(67) \quad \bar{\pi}_{h\tau} \xrightarrow{h,\tau \rightarrow 0} \pi_* \quad \text{in the } L^\infty((0, t^*) \times \Omega)\text{-weak-}^* \text{ sense}$$

for some π_* to be characterized. The combination of estimates (63), (66) and (67) allows to make use of [6, Proposition 3.8], which gives

$$(68) \quad \bar{\rho}_{h\tau} \bar{\pi}_{h\tau} \xrightarrow{h,\tau \rightarrow 0} \rho_* \pi_* \quad \text{weakly in } L^1((0, t^*) \times \Omega).$$

As a consequence, for all $\kappa \geq 0$ and all $\varphi \in L^\infty(\mathbb{R}_{\geq 0} \times \Omega)$, $\varphi \geq 0$, one has

$$\iint_{(0,t^*) \times \Omega} (\rho_* - \kappa)(\pi_* - \pi(\kappa)) \varphi \, dx \, dt = \lim_{h,\tau \rightarrow 0} \iint_{(0,t^*) \times \Omega} (\bar{\rho}_{h\tau} - \kappa)(\pi(\bar{\rho}_{h\tau}) - \pi(\kappa)) \varphi \, dx \, dt \geq 0$$

as π is increasing. Since φ is arbitrary, this implies that $(\rho_* - \kappa)(\pi_* - \pi(\kappa)) \geq 0$ a.e. in $\mathbb{R}_{\geq 0} \times \Omega$ for any $\kappa \geq 0$, ensuring that

$$(69) \quad \pi_* = \pi(\rho_*) \quad \text{almost everywhere in } \mathbb{R}_{\geq 0} \times \Omega.$$

To conclude the proof, we still make use of the strict monotonicity of π . As a consequence of (67), (68) and (69), one has

$$\lim_{h, \tau \rightarrow 0} \iint_{(0, t^*] \times \Omega} (\bar{\rho}_{h\tau} - \rho_*)(\pi(\bar{\rho}_{h\tau}) - \pi(\rho_*)) = 0.$$

Since the integrand is non-negative, it converges in L^1 towards 0, thus almost everywhere (up to yet another subsequence). As π is increasing, we obtain the desired pointwise convergence of $\bar{\rho}_{h\tau}$ towards ρ_* . \square

The strong convergence of $\bar{\rho}_{h\tau}$ implies the strong convergence of essentially *all* reasonable reconstructions of ρ . For a rigorous statement, define the piecewise constant in space functions $\hat{\rho}_{h\tau}$ and $\check{\rho}_{h\tau}$ via

$$\hat{\rho}_{h\tau}(t, x) = \max_{x' \in T} \rho_{h\tau}(t, x'), \quad \check{\rho}_{h\tau}(t, x) = \min_{x' \in T} \rho_{h\tau}(t, x'), \quad \text{for all } x \in T \text{ and } T \in \mathcal{T}_h,$$

which makes sense since $\rho_{h\tau}(t, x)$ is continuous in x by construction. Then we have:

Lemma 6.5. *Along the chosen sequence $(h, \tau) \rightarrow (0, 0)$, the functions $\rho_{h\tau}$, $\bar{\rho}_{h\tau}$, $\hat{\rho}_{h\tau}$ and $\check{\rho}_{h\tau}$ all converge to ρ_* in $L^1([0, t^*] \times \Omega)$, for each $t^* > 0$.*

Proof. We start by proving convergence of $\bar{\rho}_{h\tau}$: for each h , consider the linear operator J_h on $L^1(\Omega)$ that averages over cells of \mathcal{T}_h ,

$$J_h f(x) = \int_T f(x') dx' \quad \text{for all } x \in T \text{ and } T \in \mathcal{T}_h.$$

Then $\bar{\rho}_{h\tau} = J_h \bar{\rho}_{h\tau}$. Note that J_h is 1-Lipschitz continuous in $L^1(\Omega)$, and that $J_h f \rightarrow f$ in $L^1(\Omega)$ as $h \rightarrow 0$. Thus, the convergence $\bar{\rho}_{h\tau} - \rho_* \rightarrow 0$ in $L^1([0, t^*] \times \Omega)$ implies the convergence $\bar{\rho}_{h\tau} - J_h \rho_* = J_h(\bar{\rho}_{h\tau} - \rho_*) \rightarrow 0$ in $L^1([0, t^*] \times \Omega)$.

Next, we turn to $\hat{\rho}_{h\tau}$. The inequality

$$\|\hat{\rho}_{h\tau} - \bar{\rho}_{h\tau}\|_{L^1(\Omega)} \leq (d+1) \|\bar{\rho}_{h\tau} - \rho_*\|_{L^1(\Omega)},$$

is immediately verified by elementary considerations on each simplex of \mathcal{T}_h . Convergence of $\bar{\rho}_{h\tau}$ and $\bar{\rho}_{h\tau}$ to the same limit ρ_* implies convergence of $\hat{\rho}_{h\tau}$ to the same limit as $\bar{\rho}_{h\tau}$, that is ρ_* . The argument for $\check{\rho}_{h\tau}$ is analogous.

Finally, convergence of the spatially affine interpolation $\rho_{h\tau}$ to ρ_* follows since $\check{\rho}_{h\tau} \leq \rho_{h\tau} \leq \hat{\rho}_{h\tau}$. \square

The convergence of ρ has immediate consequences on weak and strong convergence of the concatenation $\phi(\rho)$ in various spaces.

Lemma 6.6. *Up to a subsequence, we have for every $t^* > 0$:*

$$(70) \quad \phi_{h\tau} \rightarrow \phi(\rho_*) \quad \text{strongly in } L^1([0, t^*] \times \Omega),$$

$$(71) \quad \nabla \phi_{h\tau} \rightharpoonup \nabla \phi(\rho_*) \quad \text{weakly in } L^1([0, t^*] \times \Omega).$$

Proof. A comparison of the piecewise affine hat functions and the indicator functions on the Donald cells associated to \mathcal{T}_h leads to $\bar{\phi}_{h\tau} \leq (d+1)\phi_{h\tau}$. Therefore, since $\phi_{h\tau}$ is uniformly integrable in L^1 thanks to the bound in $L^r([0, t^*] \times \Omega)$ with $r > 1$ from Lemma 6.3, also $\bar{\phi}_{h\tau}$ is uniformly integrable. On the other hand, continuity of ϕ implies convergence of

$\bar{\phi}_{h\tau} = \phi(\bar{\rho}_{h\tau})$ to $\phi(\rho_*)$ a.e. on $[0, t^*] \times \Omega$. So $\bar{\phi}_{h\tau} \rightarrow \phi(\rho_*)$ in $L^1([0, t^*] \times \Omega)$ by dominated convergence.

Next, by the intermediate value theorem we have that

$$|\phi_{h\tau} - \bar{\phi}_{h\tau}| \leq h_{\mathcal{T}} |\nabla \phi_{h\tau}|$$

on each simplex, and therefore

$$\|\phi_{h\tau} - \bar{\phi}_{h\tau}\|_{L^1([0, t^*] \times \Omega)} \leq h_{\mathcal{T}} \int_0^{t^*} \int_{\Omega} |\nabla \phi_{h\tau}| \, dx \, dt,$$

which goes to zero thanks to Lemma 6.3. Thus also $\phi_{h\tau} \rightarrow \phi(\rho_*)$ in $L^1([0, t^*] \times \Omega)$.

Finally, observe that $|\nabla \phi_{h\tau}| \leq C_4 \bar{\rho}_{h\tau} |\nabla \mu_{h\tau}|$ by Lemma 5.5. Therefore, for any measurable $\sigma \subset [0, t^*] \times \Omega$,

$$\iint_{\sigma} |\nabla \phi_{h\tau}| \, dx \, dt \leq C \left(\iint_{\sigma} \bar{\rho}_{h\tau} \right)^{1/p} \left(\int_0^{t^*} \int_{\Omega} \bar{\rho}_{h\tau} |\nabla \mu_{h\tau}|^q \right)^{1/q},$$

by Hölder's inequality. The second integral on the right-hand side above is uniformly bounded by Corollary 5.4 — observe that for integration against the piecewise constant function $|\nabla \mu_{h\tau}|$ on Ω , one can replace $\bar{\rho}_{h\tau}$ by $\rho_{h\tau}$. Concerning the first integral on the right-hand side: since $\bar{\rho}_{h\tau}$ converges in $L^1([0, t^*] \times \Omega)$ by Lemma 6.5 above, it is uniformly integrable. The estimate above thus provides uniform integrability for $|\nabla \phi_{h\tau}|$ on the left-hand side. Consequently, $\nabla \phi_{h\tau}$ converges weakly in $L^1([0, t^*] \times \Omega)$ to some limit ξ . By the above, ξ is easily identified, since for any test vector field $\theta \in C_c^\infty((0, t^*) \times \Omega; \mathbb{R}^d)$:

$$\begin{aligned} \int_0^{t^*} \int_{\Omega} \xi \cdot \theta \, dx \, dt &= \lim_{(h,\tau) \rightarrow (0,0)} \int_0^{t^*} \int_{\Omega} \nabla \phi_{h\tau} \cdot \theta \, dx \, dt \\ &= - \lim_{(h,\tau) \rightarrow (0,0)} \int_0^{t^*} \int_{\Omega} \phi_{h\tau} \nabla \cdot \theta \, dx \, dt = - \int_0^{t^*} \int_{\Omega} \phi(\rho_*) \nabla \cdot \theta \, dx \, dt, \end{aligned}$$

that is, ξ is the distributional derivative of $\phi(\rho_*)$. \square

6.3. Identification of the limit. The main ingredient for the convergence proof is the identification of the weak limit of $\bar{\rho}_{h\tau} \nabla \mu_{h\tau}$ as $\nabla \phi(\rho_*)$. This will then allow to characterize the local slope $|\partial \mathcal{E}|(\rho_*)$ in the final step of the proof.

Lemma 6.7. *Up to a subsequence, there holds*

$$\bar{\rho}_{h\tau} \nabla \mu_{h\tau} \xrightarrow{h,\tau \rightarrow 0} \nabla \phi(\rho_*) \quad \text{weakly in } L_{loc}^1(\mathbb{R}_{\geq 0} \times \bar{\Omega})^d.$$

Proof. In view of (71), it is (more than) sufficient to establish that

$$(72) \quad \bar{\rho}_{h\tau} \nabla \mu_{h\tau} - \nabla \phi_{h\tau} \xrightarrow{h,\tau \rightarrow 0} 0 \quad \text{in } L_{loc}^1(\mathbb{R}_{\geq 0} \times \bar{\Omega})^d.$$

Let $T \in \mathcal{T}_h$ and $n \geq 1$, then, building on the material introduced in the proof of Lemma 5.5, one has

$$\int_T |\bar{\rho}_{h\tau}^n \nabla \mu_{h\tau}^n - \nabla \phi_{h\tau}^n| = d! |T| |J_T^{-1} (\bar{\rho}_T^n I_d - R_T^n) \delta_T \mu_{h\tau}^n|,$$

where I_d is the identity matrix of $\mathbb{R}^{d \times d}$, and where

$$\bar{\rho}_T^n = \rho_{h\tau}^n(x_T) = \frac{1}{d+1} \sum_{a \in \mathcal{V}_T} \rho_{h\tau}^n(a).$$

Then we deduce from (55) that

$$\begin{aligned} \int_T |\bar{\rho}_{h\tau}^n \nabla \mu_{h\tau}^n - \nabla \phi_{h\tau}^n| &\leq d! |T| |J_T^{-1}|_2 |J_T|_2 \left| (\bar{\rho}_T^n)^{\frac{1}{p}} I_d - (\bar{\rho}_T^n)^{-\frac{1}{q}} R_T^n \right|_2 \left| (\bar{\rho}_T^n)^{\frac{1}{q}} J_T^{-1} \delta_T \mu_{h\tau}^n \right| \\ &\leq \zeta_{\mathcal{T}_h} \frac{d + \sqrt{d}}{\sqrt{2}} \int_T \left| (\bar{\rho}_T^n)^{\frac{1}{p}} I_d - (\bar{\rho}_T^n)^{-\frac{1}{q}} R_T^n \right|_2 \left| (\bar{\rho}_{h\tau}^n)^{\frac{1}{q}} \nabla \mu_{h\tau}^n \right|. \end{aligned}$$

Note that the term $(\bar{\rho}_T^n)^{-\frac{1}{q}} R_T^n$ vanishes if $\bar{\rho}_T^n = 0$ thanks to (57) since $q > 1$. Multiplying the above estimate by τ^n , summing over $T \in \mathcal{T}_h$ and $n = 1, \dots, n_*$, applying Hölder's inequality and Corollary 5.4 yields

$$(73) \quad \|\bar{\rho}_{h\tau} \nabla \mu_{h\tau} - \nabla \phi_{h\tau}\|_{L^1((0, t^{n_*}) \times \Omega)} \leq \zeta_{\mathcal{T}_h} \frac{d + \sqrt{d}}{\sqrt{2}} C_3^{\frac{1}{q}} (1 + t^{n_*})^{\frac{1}{q}} \|\bar{\gamma}_{h\tau}\|_{L^p((0, t^{n_*}) \times \Omega)}$$

with $t^{n_*} = \sum_{n=1}^{n_*} \tau^n$ and

$$\bar{\gamma}_{h\tau}(t, x) = (\bar{\rho}_T^n)^{\frac{1}{p}} I_d - (\bar{\rho}_T^n)^{-\frac{1}{q}} R_T^n \quad \text{if } (t, x) \in (t^{n-1}, t^n) \times T.$$

Owing to (57), to $\bar{\rho}_{h\tau} \geq \check{\rho}_{h\tau}$, to the concavity and to the subadditivity of $u \mapsto p u^{\frac{1}{p}}$ there holds

$$|\bar{\gamma}_{h\tau}|_2 \leq (\check{\rho}_{h\tau})^{-\frac{1}{q}} |\check{\rho}_{h\tau} - \hat{\rho}_{h\tau}| \leq p \left| (\check{\rho}_{h\tau})^{\frac{1}{p}} - (\hat{\rho}_{h\tau})^{\frac{1}{p}} \right| \leq p |\check{\rho}_{h\tau} - \hat{\rho}_{h\tau}|^{\frac{1}{p}}.$$

Therefore, Lemma 6.5 implies that $|\bar{\gamma}_{h\tau}|_2$ tends to 0 in $L^p_{\text{loc}}(\mathbb{R}_{\geq 0} \times \bar{\Omega})$, so that (72) holds thanks to (73). \square

Lemma 6.8. *Along a subsequence $(h, \tau) \rightarrow (0, 0)$, there holds for every $t^* > 0$:*

$$\bar{\rho}_{h\tau} \nabla \Psi_h \rightharpoonup \rho_* \nabla \Psi \quad \text{weakly in } L^1([0, t^*] \times \Omega)^d.$$

Proof. Recall that Ψ is Lipschitz on Ω , and so is each affine approximation Ψ_h owing to (25). Thus, $\nabla \Psi_h$ converges weakly-* in $L^\infty([0, t^*] \times \Omega)$, and the limit is easily identified as $\nabla \Psi$ since by construction, $\Psi_h \rightarrow \Psi$ uniformly on Ω . Combine this with the strong convergence $\bar{\rho}_{h\tau} \rightarrow \rho_*$ in $L^1([0, t^*] \times \Omega)$ from Lemma 6.5 to obtain the claim. \square

6.4. Proof of the energy identity. The starting point of this section is the form (33) of the energy inequality for the discrete solution. We recall it for the reader's convenience:

Lemma 6.9. *At every $t^* > 0$, we have*

$$(74) \quad \mathcal{E}_h(\bar{\rho}_{h\tau}(t^*)) + \frac{1}{p} \int_0^{t^*} \int_\Omega |\mathbf{v}_{h\tau}|^p \bar{\rho}_{h\tau} \, dx \, dt + \frac{1}{q} \int_0^{t^*} \int_\Omega |\mathbf{u}_{h\tau}|^q \bar{\rho}_{h\tau} \, dx \, dt \leq \mathcal{E}_h(\bar{\rho}_{h\tau}^0).$$

Proof. Since $|\mathbf{v}_{h\tau}|^p = |\mathbf{u}_{h\tau}|^q$ by definition, and since $1/p + 1/q = 1$, the inequality (74) is identical to the fundamental energy estimate (52) if $t^* = t^N$ is a point on the time mesh. Note that (52) is formulated in terms of the piecewise affine $\rho_{h\tau}$, but since $\mathbf{v}_{h\tau}$ and $\mathbf{u}_{h\tau}$ are piecewise constant, $\rho_{h\tau}$ can be replaced by $\bar{\rho}_{h\tau}$. By the time-advanced piecewise constant interpolation, and since $\mathcal{E}_h(\bar{\rho}_{h\tau}^n)$ is non-increasing in n , the inequality (74) is also true for t^* 's in between mesh points. \square

Below, we shall pass to the limit (inferior, on the left-hand side) in (74) along a suitable sequence $(h, \tau) \rightarrow (0, 0)$ to obtain

$$(75) \quad \mathcal{E}(\rho_*(t^*)) + \frac{1}{p} \int_0^{t^*} |\rho_*'|^p \, dt + \frac{1}{q} \int_0^{t^*} |\partial \mathcal{E}|^q(\rho_*) \, dt \leq \mathcal{E}(\rho^0).$$

Our first lemma establishes the convergence of the initial energy.

Lemma 6.10. *Let ρ^0 be a probability density such that $\eta(\rho^0) \in L^1(\Omega)$. Let $\bar{\rho}_{h\tau}^0$ be defined by (29), then*

$$(76) \quad \|\eta(\bar{\rho}_{h\tau}^0) - \eta(\rho^0)\|_{L^1(\Omega)} \xrightarrow{h \rightarrow 0} 0.$$

As a consequence,

$$(77) \quad \mathcal{E}_h(\bar{\rho}_{h\tau}^0) \xrightarrow{h \rightarrow 0} \mathcal{E}(\rho^0).$$

Proof. The function $\eta(\rho^0)$, as any element of $L^1(\Omega)$ is uniformly integrable, so by the de La Vallée Poussin theorem, there exists a nondecreasing, convex and superlinear function $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\int_{\Omega} \gamma \circ \eta(\rho^0) \, dx < +\infty.$$

The function $\gamma \circ \eta$ being convex, it follows from the definition (29) of $\bar{\rho}_{h\tau}^0$ and from Jensen's inequality that

$$\int_{\Omega} \gamma \circ \eta(\bar{\rho}_{h\tau}^0) \, dx \leq \int_{\Omega} \gamma \circ \eta(\rho^0) \, dx < +\infty.$$

In particular, $(\eta(\bar{\rho}_{h\tau}^0))_{h>0}$ is uniformly equi-integrable.

Moreover, $\bar{\rho}_{h\tau}^0$ converges in $L^1(\Omega)$ towards ρ^0 : this is easily shown for a smooth ρ^0 and then for ρ^0 in $L^1(\Omega)$ by a density argument and the non-expansivity in $L^1(\Omega)$ of the mapping $\rho^0 \mapsto \bar{\rho}_{h\tau}^0$. In particular, one can extract a subsequence such that $\bar{\rho}_{h\tau}^0$ converges almost everywhere towards ρ^0 . Since η is continuous, so does $\eta(\bar{\rho}_{h\tau}^0)$ towards $\eta(\rho^0)$. Applying Vitali's convergence theorem gives (76) along the subsequence. The convergence of the full sequence follows from the uniqueness of its limit value.

The convergence of the energy (77) follows from the above convergences for $\bar{\rho}_{h\tau}^0$ and $\eta(\bar{\rho}_{h\tau}^0)$ together with the uniform convergence of $\bar{\Psi}_h$ towards Ψ . \square

The liminf of the energy at positive times is easy.

Lemma 6.11. *Along the considered sequence $(h, \tau) \rightarrow (0, 0)$, we have for each $t^* > 0$:*

$$\liminf_{(h, \tau) \rightarrow (0, 0)} \mathcal{E}_h(\bar{\rho}_{h\tau}(t^*)) \geq \mathcal{E}(\rho_*(t^*)).$$

Proof. Recall that $\bar{\rho}_{h\tau}(t^*)$ converges weakly-* to $\rho_*(t^*)$ by Lemma 6.2. On the one hand, we have thanks to convexity of η that

$$\liminf_{(h, \tau) \rightarrow (0, 0)} \int_{\Omega} \eta(\bar{\rho}_{h\tau}(t^*, x)) \, dx \geq \int_{\Omega} \eta(\rho_*(t^*, x)) \, dx.$$

And on the other hand, we have thanks to the uniform convergence of the continuous approximations Ψ_h to the limit Ψ that

$$\lim_{(h, \tau) \rightarrow (0, 0)} \int_{\Omega} \bar{\rho}_{h\tau}(t^*, x) \Psi_h(x) \, dx = \int_{\Omega} \rho_*(t^*, x) \Psi(x) \, dx.$$

This gives the desired result. \square

The limits of the time integrals are slightly more complicated to obtain. We start with an auxiliary result.

Lemma 6.12. *The functions $\mathbf{u}_{h\tau}$ and $\mathbf{v}_{h\tau}$ defined in (31) are globally integrable in time:*

$$(78) \quad \int_0^\infty \int_{\Omega} |\mathbf{v}_{h\tau}|^p \bar{\rho}_{h\tau} \, dx \, dt \leq \mathcal{E}_h(\bar{\rho}_{h\tau}^0), \quad \int_0^\infty \int_{\Omega} |\mathbf{u}_{h\tau}|^q \bar{\rho}_{h\tau} \, dx \, dt \leq \mathcal{E}_h(\bar{\rho}_{h\tau}^0).$$

Proof. By definition, for $t \in (t^{n-1}, t^n]$,

$$|\mathbf{v}_{h\tau}(t)|^p = |\mathbf{u}_{h\tau}(t)|^q = |\nabla(\mu_{h\tau}^n + \Psi_h)|^q.$$

Now simply pass to the limit $n \rightarrow \infty$ in the energy inequality (52), recalling that $\mathcal{E}_h \geq 0$, to obtain (78). \square

We are now in the position to estimate the slope term.

Lemma 6.13. *Passing to a further subsequence for $(h, \tau) \rightarrow 0$, one has for every $t^* > 0$:*

$$(79) \quad \int_0^{t^*} |\partial\mathcal{E}(\rho)|^q(t) dt \leq \liminf_{h,\tau \rightarrow 0} \int_0^{t^*} \int_{\Omega} \bar{\rho}_{h\tau} |\mathbf{u}_{h\tau}|^q dx dt.$$

Proof. In view of (78) and of Lemma 6.2, Proposition A.2 from the Appendix is applicable. It yields the existence of a subsequence (not relabelled) and a limiting field $\mathbf{u}_* : \mathbb{R}_{\geq 0} \times \Omega \rightarrow \mathbb{R}^d$ such that $\bar{\rho}_{h\tau} \mathbf{u}_{h\tau} \rightarrow \rho_* \mathbf{u}_*$ weakly-* in $L^1(\mathbb{R}_{\geq 0} \times \bar{\Omega})$, and

$$(80) \quad \int_0^{t^*} \int_{\Omega} \rho_* |\mathbf{u}_*|^q dx dt \leq \liminf_{h,\tau \rightarrow 0} \int_0^{t^*} \int_{\Omega} \bar{\rho}_{h\tau} |\mathbf{u}_{h\tau}|^q dx dt$$

for each $t^* > 0$. In Lemmas 6.7 and 6.8 above, the weak limit of $\bar{\rho}_{h\tau} \mathbf{u}_{h\tau}$ has been identified as $\nabla\phi(\rho_*) + \rho_* \nabla\Psi$. This means that at almost every $t > 0$, we have $\rho_*(t) \mathbf{u}_*(t) = \nabla\phi(\rho_*(t)) + \rho_*(t) \nabla\Psi$, and also that $\rho_*(t) |\mathbf{u}_*(t)|^q \in L^1(\Omega)$. Thus, at those $t > 0$, we have

$$|\partial\mathcal{E}|^q(\rho_*(t)) = \int_{\Omega} \rho_*(t) |\mathbf{u}_*(t)|^q dx$$

thanks to Proposition 2.1. Substitute this into (80) above to obtain (79). \square

Finally, we estimate the metric velocity.

Lemma 6.14. *ρ_* is an absolutely continuous curve from $\mathbb{R}_{\geq 0}$ to $\mathbb{W}_p(\Omega)$, and passing to a further subsequence for $(h, \tau) \rightarrow 0$, one has for every $t^* > 0$:*

$$(81) \quad \int_0^{t^*} |\rho_*'|^p(t) dt \leq \liminf_{h,\tau \rightarrow 0} \int_0^{t^*} \int_{\Omega} \bar{\rho}_{h\tau} |\mathbf{v}_{h\tau}|^p dx dt.$$

Proof. In full analogy to the proof of Lemma 6.13 above, we obtain from (78) by means of Proposition A.2 the existence of a further subsequence (not relabelled) and a limit velocity field \mathbf{v}_* such that $\bar{\rho}_{h\tau} \mathbf{v}_{h\tau} \rightarrow \rho_* \mathbf{v}_*$ weakly-*, with

$$(82) \quad \int_0^{t^*} \int_{\Omega} \rho_* |\mathbf{v}_*|^p dx dt \leq \liminf_{h,\tau \rightarrow 0} \int_0^{t^*} \int_{\Omega} \bar{\rho}_{h\tau} |\mathbf{v}_{h\tau}|^p dx dt$$

for every $t^* > 0$. The proof is finished by applying Proposition 2.2, which directly yields absolute continuity as well as the estimate (81) from (82). For that, we need to verify that (ρ_*, \mathbf{v}_*) satisfies the continuity equation (17).

We shall derive (17) directly from the weak formulation (26) of the scheme. To that end, let $\varphi \in C_c^\infty(\mathbb{R}_{>0} \times \bar{\Omega})$ be given, and define accordingly $\varphi_{h\tau}$ as the piecewise-constant-in-time and piecewise-affine-in-space function such that for all $t \in (t^{n-1}, t^n]$, the function $\varphi_{h\tau}(t, \cdot)$ is given by affine interpolation on the triangles of the values $\varphi(t^{n-1}, a)$ at the vertices $a \in \mathcal{V}$. Introduce further the temporal difference quotient

$$\delta_t \bar{\varphi}_{h\tau}(t, x) := \frac{\varphi_{h\tau}(t^n, a) - \varphi_{h\tau}(t^{n-1}, a)}{\tau^n} \quad \text{for all } t \in (t^{n-1}, t^n] \text{ and } x \in \omega_a.$$

By smoothness of φ ,

$$(83) \quad \nabla\varphi_{h\tau} \rightarrow \nabla\varphi, \quad \delta_t \bar{\varphi}_{h\tau} \rightarrow \partial_t \varphi \quad \text{uniformly a.e. on } \mathbb{R}_{>0} \times \Omega,$$

where the *almost everywhere* simply accounts for the fact that $\nabla\varphi_{h\tau}$ cannot be evaluated on edges.

Now substitute $w_h^n = \varphi_{h\tau}(t^{n-1}, \cdot)$ into (26) and sum over $n = 1, 2, \dots$ — the sum is actually finite — to obtain

$$- \int_0^\infty \int_\Omega \bar{\rho}_{h\tau} \delta_t \bar{\varphi}_{h\tau} \, dx \, dt + \int_0^\infty \int_\Omega \bar{\rho}_{h\tau} \mathbf{v}_{h\tau} \cdot \nabla \varphi_{h\tau} \, dx \, dt = 0.$$

By (83) above, by strong convergence of $\bar{\rho}_{h\tau}$ to ρ_* , and by weak convergence of $\bar{\rho}_{h\tau} \mathbf{v}_{h\tau}$ to $\rho_* \mathbf{v}_*$, we may pass to the limit and obtain

$$- \int_0^\infty \int_\Omega \rho_* \partial_t \varphi \, dx \, dt + \int_0^\infty \int_\Omega \rho_* \mathbf{v}_* \cdot \nabla \varphi \, dx \, dt = 0,$$

which is just the distributional formulation of (17), finishing the proof. \square

Conclusion of the proof of Theorem 3.2. Substituting the results from Lemmas 6.10, 6.11, 6.13 and 6.14 into the inequality (74), we directly obtain the energy dissipation inequality (75) for ρ_* .

An immediate consequence of (75) is the integrability of the product $|\rho_*'| |\partial \mathcal{E}|(\rho_*)$ of metric velocity and local slope with respect to time, thanks to Hölder's inequality. It thus follows from $|\partial \mathcal{E}|$ being a *strong* upper gradient, and since ρ_* is an absolutely continuous curve, directly by means of [4, Definition 1.2.1] that the *opposite* inequality holds, that is (75) with \geq in place of \leq . Consequently, the inequality (75) is actually an equality, and so ρ_* is a p -gradient flow for \mathcal{E} in the metric \mathbb{W}_p .

A posteriori, the \liminf used in the derivation of (75) is actually a true limit. That is, along the chosen sequence $(h, \tau) \rightarrow (0, 0)$, we have in particular that $\mathcal{E}_h(\bar{\rho}_{h\tau}(t^*))$ converges to $\mathcal{E}(\rho_*(t^*))$, for every $t^* > 0$. \square

7. FROM WEAK TO STRONG CONVERGENCE

The main ingredient in the proof of Theorem 3.3 is the observation that along the sequence $(h, \tau) \rightarrow (0, 0)$ under consideration,

$$(84) \quad \limsup_{h, \tau \rightarrow 0} \int_0^{t^*} \int_\Omega |\mathbf{u}_{h\tau}|^q \bar{\rho}_{h\tau} \, dx \, dt \leq \int_0^{t^*} \int_\Omega |\mathbf{u}_*|^q \rho_* \, dx \, dt$$

with \mathbf{u}_* from (32), which allows to conclude stronger convergence of $\mathbf{u}_{h\tau}$ by virtue of Proposition A.2.

Lemma 7.1. *Along the chosen subsequence with $h, \tau \rightarrow 0$, the relation (84) holds.*

Proof. From the discrete dissipation estimate (28) and the advanced interpolation in time, we know that

$$\int_0^{t^*} \int_\Omega |\mathbf{u}_{h\tau}|^q \bar{\rho}_{h\tau} \, dx \, dt \leq \mathcal{E}_h(\bar{\rho}_{h\tau}^0) - \mathcal{E}_h(\bar{\rho}_{h\tau}(t^*))$$

holds for each $t^* > 0$. By Theorem 3.2, we know that $\mathcal{E}_h(\bar{\rho}_{h\tau}(t^*)) \rightarrow \mathcal{E}(\rho_*(t^*))$ along the considered sequence $(h, \tau) \rightarrow (0, 0)$, and thus

$$\limsup_{h, \tau \rightarrow 0} \int_0^{t^*} \int_\Omega |\mathbf{u}_{h\tau}|^q \bar{\rho}_{h\tau} \, dx \, dt \leq \mathcal{E}(\rho^0) - \mathcal{E}(\rho_*(t^*)).$$

On the other hand, by the fundamental property (19) of the gradient flow, and since the subdifferential is given by means of Proposition 2.1,

$$\mathcal{E}(\rho^0) - \mathcal{E}(\rho_*(t^*)) = \int_0^{t^*} |\partial\mathcal{E}|^q(\rho_*) dt = \int_0^{t^*} \int_{\Omega} |\mathbf{u}_*|^q \rho_* dx dt.$$

In combination, this proves (84). \square

Proof of Theorem 3.3. The idea is to combine Proposition A.2 with the strong convergence of $\rho_{h\tau}$ in $L^1([0, t^*] \times \Omega)$. Fix an exponent $r \in [1, q]$. For each $M \geq 1$, consider an approximation $\mathbf{U}_M : [0, t^*] \times \Omega \rightarrow \mathbb{R}^n$ of \mathbf{u}_* such that \mathbf{U}_M is continuous, satisfies the uniform bound $\sup_{\Omega} |\mathbf{U}_M| \leq M$, and approximates \mathbf{u}_* in the sense that

$$\int_0^{t^*} \int_{\Omega} \left| |\mathbf{u}_*|^{r-1} \mathbf{u}_* - |\mathbf{U}_M|^{r-1} \mathbf{U}_M \right|^{q/r} \rho_* dx dt \rightarrow 0 \quad \text{as } M \rightarrow \infty.$$

Such an approximation exists since $\int_0^{t^*} \int_{\Omega} |\mathbf{u}_*|^q \rho_* dx dt < \infty$ (so that the integral of $|\mathbf{u}_*|^q \rho_*$ over the set $\{|\mathbf{u}_*| > M\}$ vanishes as $M \rightarrow \infty$), and since ρ_* is an integrable function, not merely a probability measure (so that convergence in measure of continuous approximations of the measurable bounded function $\min(M, |\mathbf{u}_*|) \frac{\mathbf{u}_*}{|\mathbf{u}_*|}$ is sufficient). With \mathbf{U}_M at hand, we define the continuous function $\varphi_M : [0, t^*] \times \Omega \times \mathbb{R}^d$ via

$$\varphi_M(t, x, \xi) = \left| |\xi|^{r-1} \xi - |\mathbf{U}_M(t, x)|^{r-1} \mathbf{U}_M(t, x) \right|^{q/r},$$

which clearly satisfies the estimate $|\varphi_M(t, x, \xi)| \leq C(1 + |\xi|^q)$ for an appropriate C (possibly depending on M and q). Application of the second part of Proposition A.2 with the choice $\varphi := \varphi_M$ for the test function yields that

$$\begin{aligned} \delta(M, h, \tau) := & \left| \int_0^{t^*} \int_{\Omega} \left| |\mathbf{u}_{h\tau}|^{r-1} \mathbf{u}_{h\tau} - |\mathbf{U}_M|^{r-1} \mathbf{U}_M \right|^{q/r} \widetilde{\rho}_{h\tau} dx dt \right. \\ & \left. - \int_0^{t^*} \int_{\Omega} \left| |\mathbf{u}_*|^{r-1} \mathbf{u}_* - |\mathbf{U}_M|^{r-1} \mathbf{U}_M \right|^{q/r} \rho_* dx dt \right| \end{aligned}$$

converges to zero as $h, \tau \rightarrow 0$, for each fixed $M \geq 1$. This implies

$$\begin{aligned} & \left(\int_0^{t^*} \int_{\Omega} \left| |\mathbf{u}_{h\tau}|^{r-1} \mathbf{u}_{h\tau} \widetilde{\rho}_{h\tau}^{r/q} - |\mathbf{u}_*|^{r-1} \mathbf{u}_* \rho_*^{r/q} \right|^{q/r} dx dt \right)^{r/q} \\ & \leq \left(\int_0^{t^*} \int_{\Omega} \left| |\mathbf{u}_{h\tau}|^{r-1} \mathbf{u}_{h\tau} - |\mathbf{U}_M|^{r-1} \mathbf{U}_M \right|^{q/r} \widetilde{\rho}_{h\tau} dx dt \right)^{r/q} \\ & \quad + \left(\int_0^{t^*} \int_{\Omega} \left| |\mathbf{U}_M|^{r-1} \mathbf{U}_M (\widetilde{\rho}_{h\tau}^{r/q} - \rho_*^{r/q}) \right|^{q/r} dx dt \right)^{r/q} \\ & \quad + \left(\int_0^{t^*} \int_{\Omega} \left| |\mathbf{u}_*|^{r-1} \mathbf{u}_* - |\mathbf{U}_M|^{r-1} \mathbf{U}_M \right|^{q/r} \rho_* dx dt \right)^{r/q} \\ & \leq \delta(M, h, \tau)^{r/q} + 2 \left(\int_0^{t^*} \int_{\Omega} \left| |\mathbf{u}_*|^{r-1} \mathbf{u}_* - |\mathbf{U}_M|^{r-1} \mathbf{U}_M \right| \rho_* dx dt \right)^{r/q} + M \left\| \widetilde{\rho}_{h\tau}^{r/q} - \rho_*^{r/q} \right\|_{L^{q/r}}. \end{aligned}$$

Now, given $\varepsilon > 0$, it suffices to choose M sufficiently large to make the first integral in the last line above less than $\varepsilon/4$, and then τ and h sufficiently small to have both $\delta(M, h, \tau)^{r/q} < \varepsilon/4$ and $\left\| \widetilde{\rho}_{h\tau}^{r/q} - \rho_*^{r/q} \right\|_{L^{q/r}} < \varepsilon/(4M)$. For the last step, we have used that thanks to $\widetilde{\rho}_{h\tau} \rightarrow \rho_*$ in L^1 , we also have $\widetilde{\rho}_{h\tau}^{r/q} \rightarrow \rho_*^{r/q}$ in $L^{q/r}$. This shows convergence of $|\mathbf{u}_{h\tau}|^{r-1} \mathbf{u}_{h\tau} \widetilde{\rho}_{h\tau}^{r/q}$ to $|\mathbf{u}_*|^{r-1} \mathbf{u}_* \rho_*^{r/q}$ in $L^{q/r}$.

To prove convergence of the flux, choose $r := q - 1$ above, yielding $|\mathbf{u}_{h\tau}|^{q-2} \mathbf{u}_{h\tau} \widetilde{\rho}_{h\tau}^{1/p} \rightarrow |\mathbf{u}_*|^{q-2} \mathbf{u}_* \rho_*$ in L^p , and combine this with $\widetilde{\rho}_{h\tau}^{1/q} \rightarrow \rho_*^{1/q}$ in L^q , which holds by virtue of $\widetilde{\rho}_{h\tau} \rightarrow \rho_*$ in L^1 . Finally, the equivalent formulation in terms of \mathbf{v}_* instead of \mathbf{u}_* immediately follows from the facts that $\mathbf{v}_{h\tau} = |\mathbf{u}_{h\tau}|^{q-2} \mathbf{u}_{h\tau}$ and also $\mathbf{v}_* = |\mathbf{u}_*|^{q-2} \mathbf{u}_*$ by definition, see (31) and (32), respectively. \square

8. NUMERICAL RESULTS

8.1. On the effective resolution. We present in the next Subsection several numerical experiments using the scheme (26)&(27). All numerical results were performed with FreeFem++ (see [30]). Since the method to solve the problem is not fully classical we detail here some important steps of the algorithm.

Assume first that assumption (A1) holds true. At the n^{th} iteration, we know $\rho_{h\tau}^{n-1}$ and we search for $\rho_{h\tau}^n$ and $\mu_{h\tau}^n$, such that $\mu_{h\tau}^n(a) = \eta'(\rho_{h\tau}^n(a))$ for all $a \in \mathcal{V}$, solution to (26), and so to a nonlinear system $F^n(\rho_{h\tau}^n) = 0$, which is equivalent to find

$$(85) \quad (\rho_{h\tau}^n) \in \underset{\rho}{\operatorname{argmin}} \frac{1}{2} \|F^n(\rho)\|^2.$$

As explained in Section 3.4, when assumption (A2) is satisfied and $\rho_{h\tau}^n$ vanishes at some nodes $a \in \mathcal{V}$, the quantity $\mu_{h\tau}^n(a)$ can no longer be written in terms of $\rho_{h\tau}^n(a)$. To circumvent this difficulty, we take inspiration in the proof of Proposition 4.5 as we add a barrier function $\varepsilon \bar{\mu}$ which is increasing and singular close to zero, so that we come back to the framework of assumption (A1). More precisely, we replace $\mu_{h\tau}^n(a)$ by $\eta'(\rho_{h\tau}^n(a)) + \varepsilon \bar{\mu}(\rho_{h\tau}^n(a))$ where $\varepsilon > 0$ is a small parameter. We then solve this problem by paying attention to the fact that the smaller the perturbation parameter ε is, the more difficult the problem is to solve. The idea is to construct a sequence of solution $(\rho_{h\tau}^{n,k})_{k \geq 0}$ to the scheme (26) with $\mu_{h\tau}^{n,k}(a) = \eta'(\rho_{h\tau}^{n,k}(a)) + \varepsilon_k \bar{\mu}(\rho_{h\tau}^{n,k}(a))$ for a sequence of parameter $(\varepsilon_k)_{k \geq 0}$ which tends to zero. More precisely, we begin with $\varepsilon_0 > 0$ and we solve the problem using the iterative strategy described below. At iteration k we choose as initial value the solution $\rho_{h\tau}^{n,k-1}$ obtained in the previous step. At the end of the step k , we stiffen the problem by setting $\varepsilon_{k+1} = \varpi \varepsilon_k$ with $\varpi < 1$. We stop the algorithm when $\varepsilon_k < \varepsilon_*$ for a given $\varepsilon_* > 0$. Then we set $\rho_{h\tau}^n = \rho_{h\tau}^{n,k}$. For numerical convenience, we choose $\bar{\mu}(z) = -1/z$.

We detail now the iterative algorithm. The main idea is to use a line search strategy. Thus, at the iteration n and step k , we know $\rho_{h\tau}^{n,k-1}$ and we look for for $\rho_{h\tau}^{n,k}$. It is computed as follows:

- first compute $d\rho^{n,k}$ solution to $DF(\rho_{h\tau}^{n,k})d\rho^{n,k} = -F(\rho_{h\tau}^{n,k})$;
- then choose $d\rho^{n,k}$ as descent direction and set

$$G(s) = \frac{d}{ds} \left(\frac{1}{2} \|F^n(\rho_{h\tau}^{n,k} + sd\rho^{n,k})\|^2 \right), \quad s \in \mathbb{R}_{\geq 0}.$$

Thus, noticing that $G(s) = \langle DF^n(\rho_{h\tau}^{n,k} + sd\rho^{n,k})d\rho^{n,k}, F^n(\rho_{h\tau}^{n,k} + sd\rho^{n,k}) \rangle$ we are able to compute $G(s)$ for any s .

- Since $G(0) = -\langle F^n(\rho_{h\tau}^{n,k}), F^n(\rho_{h\tau}^{n,k}) \rangle < 0$, we have the following alternative:
 - (1) either $G(1) \leq 0$, then we use a classical Newton step and we set

$$\rho_{h\tau}^{n,k+1/2} = \rho_{h\tau}^{n,k} + d\rho^{n,k};$$

- (2) or $G(1) > 0$, and we search for $s^{n,k} \in (0, 1)$ such that $G(s) \leq 0$. Then we set

$$\rho_{h\tau}^{n,k+1/2} = \rho_{h\tau}^{n,k} + s^{n,k} d\rho^{n,k}.$$

- To make sure that the solution remains away from 0, we set

$$\rho_{hr}^{n,k+1} = \max(\rho_{hr}^{n,k+1/2}, 10^{-8}).$$

8.2. Barenblatt solution of the q -Laplace equation. We begin by a convergence study for an analytical solution to

$$(86) \quad \partial_t \rho - \nabla \cdot (|\nabla \rho|^{q-2} \nabla \rho) = \partial_t \rho - \nabla \cdot (\rho |\nabla \eta'(\rho)|^{q-2} \nabla \eta'(\rho)) = 0, \text{ in } [0, t_f] \times \Omega,$$

where the internal energy density η is given by

$$(87) \quad \eta(\rho) = \begin{cases} \rho \log \rho - \rho + 1 & \text{if } q = 2, \\ \rho - 1 - \log(\rho) & \text{if } q = \frac{3}{2}, \\ \frac{q-1}{q-2} \left(\frac{q-1}{2q-3} \left(\rho^{\frac{2q-3}{q-1}} - 1 \right) - \rho + 1 \right) & \text{else.} \end{cases}$$

The corresponding chemical potential is given by

$$\eta'(\rho) = \begin{cases} \log(\rho) & \text{if } p = q = 2, \\ \frac{1}{2-p} (\rho^{2-p} - 1) & \text{else.} \end{cases}$$

The function η of (87) satisfies Hypotheses 1&2 provided $q \geq 2$. Assume that $q > 2$ (or equivalently that $p < 2$), so that η' does not blow-up near 0, as prescribed in the case (A2). The Barenblatt profile

$$(88) \quad \rho(t, x) = (t + t_0)^{-k} \left((M - \alpha |\xi|^p)^+ \right)^{\frac{1}{2-p}}$$

solves (86), cf. [32]. In formula (88), $t_0 > 0$ is a parameter set to avoid blow-up at $t = 0$, whereas

$$(89) \quad k = \frac{1}{q-2 + \frac{q}{d}}, \quad \alpha = \frac{q-2}{q} \left(\frac{k}{d} \right)^{\frac{1}{q-1}}, \quad \xi = x(t + t_0)^{-\frac{k}{d}}.$$

The quantity M has to be set in order to ensure that $\rho(t, \cdot) \in \mathcal{P}_{ac}(\Omega)$. The Barenblatt profile (88) is compactly supported, hence the no-flux boundary conditions (2) is satisfied for a finite time provided $0 \in \Omega$ and $\text{dist}(0, \partial\Omega)$ is large enough.

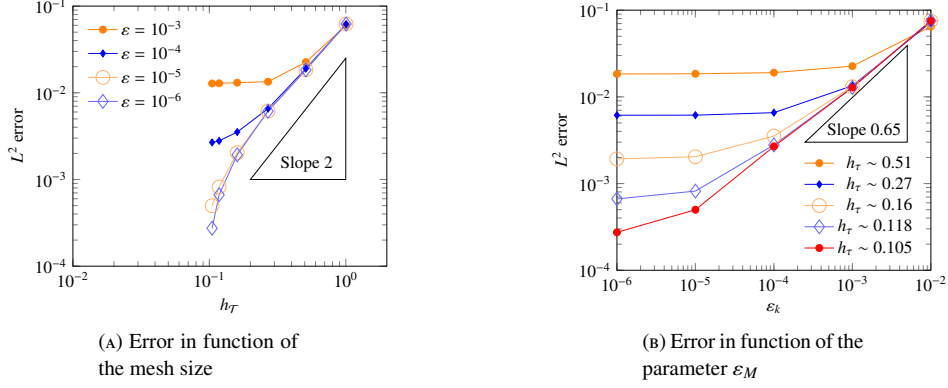
We study the convergence of the approximate solution towards the exact solution in the 1-dimensional case for $q = 5$ (and so $p = \frac{5}{4}$). In this case we have $\Omega =]-2, 2[$, $M \simeq 0.6868$, the final time is fixed at $t_f = 0.25$ and we choose $t_0 = 0.05$, the barrier function is $\bar{\mu}(x) = -\frac{1}{x}$, $\varepsilon_0 = 0.5$ and $\varpi = 0.7$. Two types of convergence are studied here:

- we fix ε_* and make the mesh size tend to 0 (see Fig. 2a);
- we fix the mesh size and make the parameter ε_* tend to 0 (see Fig. 2b).

We begin with a coarse mesh containing 10 vertices and a mesh size approximately equal to 1, then we refine five times the mesh to obtain a fine mesh with 258 vertices and a mesh size approximately equal to 0.105. Furthermore, the time step associated with the coarsest mesh is 10^{-2} and when we refine the mesh, we divide the time step by 4.

In each case we plot the relative error between the approximate solution and the exact solution for the $L^2((0, t_f) \times \Omega)$ norm.

We observe on Figure 2 a second-order convergence in space and an order of convergence approximately equal to 0.65 for ε_k . As expected, we observe a saturation in the convergence w.r.t. h_T or ε_k if ε_k or h_T are respectively chosen too coarsely.

FIGURE 2. Relative error for the $L^2((0, t_f) \times \Omega)$ norm

8.3. Solution of the q -Laplace equation with a non-zero potential. In this section we focus on qualitative numerical results in 2 space dimensions of the q -Laplacian equation described in (1) for a non-zero potential Ψ . For the numerical tests presented here, the energy density is given by $\eta(\rho) = \frac{1}{2}\rho^2$ and the initial density $\rho_0 \in \mathcal{P}_{\text{ac}}(\Omega)$ is a cross as described on Fig. 3a. The external potential is $\Psi(x) = k[(x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2]$ and the stationary state $\rho_\infty(x) = (M - \Psi(x))_+$, see Fig. 3b. The parameters M and k are chosen such that $\rho_\infty \in \mathcal{P}_{\text{ac}}(\Omega)$ and such that the support of ρ_∞ , which is the ball of radius $\sqrt{\frac{M}{k}}$ centered at the point $(\frac{1}{2}, \frac{1}{2})$, is included in $\Omega = (0, 1)^2$ (here we choose $\sqrt{\frac{M}{k}} = 0.4$).

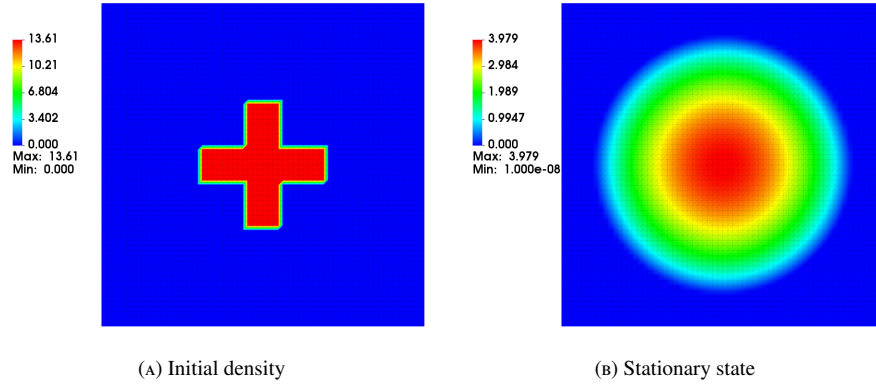


FIGURE 3. Initial density and stationary state

We choose as final time $t_f = 0.25$, the time step is $\tau = 0.000625$, the mesh has 8192 triangles and the mesh size is approximately $h_T \sim 0.022$. We observe the behavior of the solution for 3 different values of p (and q):

- $p = \frac{3}{2}$ (and so $q = 3$) on Fig. 4;
- $p = q = 2$ on Fig. 5;
- $p = 3$ (and so $q = \frac{3}{2}$) on Fig. 6.

In each case the barrier function is $\bar{\mu}(x) = -\frac{1}{x}$, $\varepsilon_\star = 10^{-3}$ but $\varepsilon_0 = 0.1$ for $p = \frac{3}{2}$ and $p = 2$ and $\varepsilon_0 = 0.2$ for $p = 3$. Note that the color scale varies from one snapshot to another in order to better visualize the behavior of the solution over time.

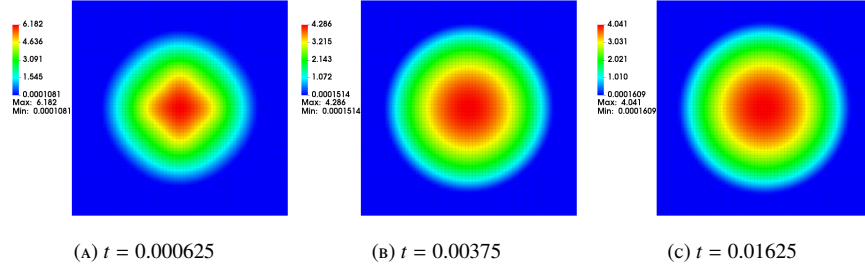


FIGURE 4. Evolution of the density for $p = \frac{3}{2}$

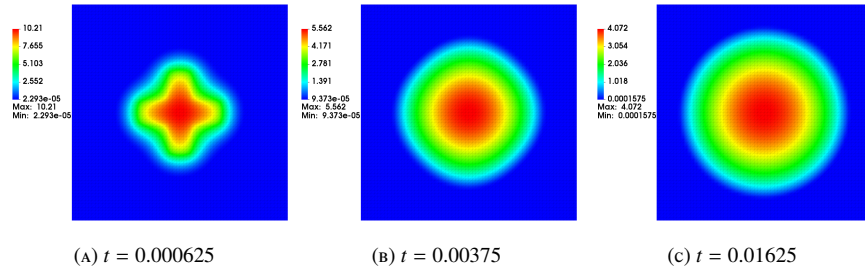


FIGURE 5. Evolution of the density for $p = 2$

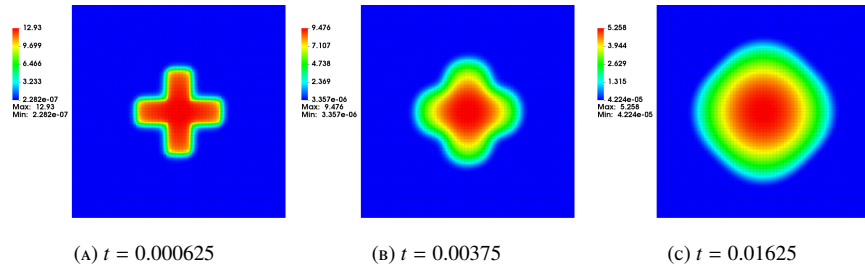


FIGURE 6. Evolution of the density for $p = 3$

We also plot on Fig. 7 the evolution of the energy $E(\bar{\rho}_{h\tau}^n) - E(\rho_\infty)$ where $E(\rho_\infty) = \frac{2}{3}M$.

As can be seen from the Fig. 7, the energy decreases faster for $p = \frac{3}{2}$ and therefore the steady state is reached faster in this case. The saturation around 10^{-3} is related to the choice of ε_\star .

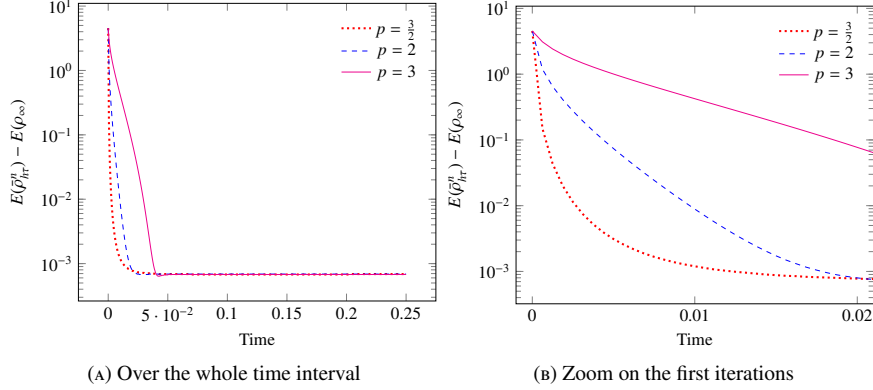


FIGURE 7. Energy dissipation

APPENDIX A. SOME ABSTRACT CONVERGENCE RESULTS

First, we recall the *refined version of the Ascoli-Arzelà theorem* from [4, Proposition 3.3.1], which is tailored to application in the proof of Lemma 6.2.

Proposition A.1. *For $t^* > 0$ and $r \geq 1$ fixed, consider a sequence (ρ_ℓ) of curves $\rho_\ell : [0, t^*] \rightarrow \mathcal{P}_{ac}(\Omega)$ satisfying*

$$\limsup_{\ell \rightarrow +\infty} \mathbb{W}_r(\rho_\ell(t), \rho_\ell(s)) \leq \omega(|t-s|) \quad \text{for all } 0 \leq t, s \leq t^*, \text{ uniformly in } \ell,$$

with a modulus $\omega \in C(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ of continuity, satisfying $\omega(0) = 0$. Then there exists an accumulation point $\rho \in C([0, t^*], \mathcal{P}_{ac}(\Omega))$ with

$$\mathbb{W}_r(\rho(t), \rho(s)) \leq \omega(|t-s|) \quad \text{for all } 0 \leq t, s \leq t^*,$$

such that, along a subsequence with $\ell_k \rightarrow \infty$, one has $\rho_{\ell_k}(t) \rightarrow \rho(t)$ narrowly at each $t \in [0, t^*]$.

Next, we recall a result on weak and strong convergence of sequences in Lebesgue spaces with varying reference density. It extends the following two results that are classical for the space $L^r(\Omega_\infty; \mu)$ with $r > 1$ and $\Omega_\infty := \mathbb{R}_{\geq 0} \times \Omega$ for a *fixed* reference measure μ :

- if a sequence (f_ℓ) is bounded in $L^r(\Omega_\infty; \mu)$, then one has weak convergence $f_{\ell_k} \rightharpoonup f_*$ along a subsequence, and $\|f_*\|_{L^r} \leq \liminf_k \|f_{\ell_k}\|_{L^r}$;
- if additionally $\|f_*\|_{L^r} = \lim_k \|f_{\ell_k}\|_{L^r}$, then the convergence is actually strong, $f_{\ell_k} \rightarrow f_*$.

A generalization to the case where the reference measure varies with ℓ as well is given by the following adapted and reduced version of [4, Theorem 5.4.4]:

Proposition A.2. *Let an exponent $r > 1$ be fixed. Consider a sequence of time-dependent probability densities $\rho_\ell : \mathbb{R}_{\geq 0} \rightarrow \mathcal{P}_{ac}$ that possesses a limit $\rho_* : \mathbb{R}_{\geq 0} \rightarrow \mathcal{P}_{ac}$ with $\rho_\ell(t) \rightarrow \rho_*(t)$ weakly-* for each $t \geq 0$, and consider further a sequence of time-dependent measurable vector fields $\mathbf{w}_\ell : \mathbb{R}_{\geq 0} \times \Omega \rightarrow \mathbb{R}^d$.*

- If $\int_0^\infty \int_\Omega |\mathbf{w}_\ell|^r \rho_\ell \, dx \, dt$ is ℓ -uniformly bounded, then along a subsequence with $\ell_k \rightarrow \infty$, the products $\mathbf{w}_{\ell_k} \rho_{\ell_k}$ converge weakly-* in $L^1(\Omega_\infty)$ to $\mathbf{w}_* \rho_*$ for some time-dependent

vector field $\mathbf{w}_* : \mathbb{R}_{\geq 0} \times \Omega \rightarrow \mathbb{R}^d$, and for every $t^* > 0$,

$$\int_0^{t^*} \int_{\Omega} |\mathbf{w}_*|^r \rho_* \, dx \, dt \leq \liminf_{k \rightarrow \infty} \int_0^{t^*} \int_{\Omega} |\mathbf{w}_{\ell_k}|^r \rho_{\ell_k} \, dx \, dt.$$

- If additionally for every $t^* > 0$

$$\int_0^{t^*} \int_{\Omega} |\mathbf{w}_*|^r \rho_* \, dx \, dt \geq \limsup_{k \rightarrow \infty} \int_0^{t^*} \int_{\Omega} |\mathbf{w}_{\ell_k}|^r \rho_{\ell_k} \, dx \, dt,$$

then for each continuous function $\varphi : \Omega_{\infty} \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying $\varphi(t, x, v) \leq C(1+|v|^r)$ with a C for all $(t, x) \in \mathbb{R}_{\geq 0} \times \Omega$ and $v \in \mathbb{R}^d$, and every $t^* > 0$, one has:

$$\int_0^{t^*} \int_{\Omega} \varphi(t, x, \mathbf{w}_*(t, x)) \rho_*(t, x) \, dx \, dt = \lim_{k \rightarrow \infty} \int_0^{t^*} \int_{\Omega} \varphi(t, x, \mathbf{w}_{\ell_k}(t, x)) \rho_{\ell_k}(t, x) \, dx \, dt.$$

Acknowledgements. The authors thank Pierre-Henri Tournier for his kind help with FreeFEM++. CC acknowledges support from Labex CEMPI (ANR-11-LABX-0007-01).

REFERENCES

- [1] M. Agueh. Existence of solutions to degenerate parabolic equations via the Monge-Kantorovich theory. *Adv. Differential Equations*, 10(3):309–360, 2005.
- [2] R. Alonso, M. Santillana, and C. Dawson. On the diffusive wave approximation of the shallow water equations. *European J. Appl. Math.*, 19:576–606, 2008.
- [3] H. W. Alt and S. Luckhaus. Quasilinear elliptic-parabolic differential equations. *Math. Z.*, 183(3):311–341, 1983.
- [4] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, second edition, 2008.
- [5] B. Andreianov, M. Bendahmane, and K. H. Karlsen. Discrete duality finite volume schemes for doubly nonlinear degenerate hyperbolic-parabolic equations. *J. Hyperbolic Differ. Equ.*, 7(1):1–67, 2010.
- [6] B. Andreianov, C. Cancès, and A. Moussa. A nonlinear time compactness result and applications to discretization of degenerate parabolic–elliptic PDEs. *J. Funct. Anal.*, 273(12):3633–3670, 2017.
- [7] J. W. Barrett and W. B. Liu. Finite element approximation of the parabolic p -Laplacian. *SIAM J. Numer. Anal.*, 31(2):413–428, 1994.
- [8] Jean-David Benamou and Yann Brenier. A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem. *Numer. Math.*, 84(3):375–393, 2000.
- [9] F. Bolley and J. A. Carrillo. Nonlinear diffusion: geodesic convexity is equivalent to Wasserstein contraction. *Comm. Partial Differential Equations*, 39(10):1860–1869, 2014.
- [10] C. Cancès, C. Chainais-Hillairet, and S. Krell. Numerical analysis of a nonlinear free-energy diminishing Discrete Duality Finite Volume scheme for convection diffusion equations. *Comput. Methods Appl. Math.*, 18(3):407–432, 2018.
- [11] C. Cancès, J. Droniou, C. Guichard, G. Manzini, M. Olivares, and I. S. Pop. *Error Estimates for the Gradient Discretisation Method on Degenerate Parabolic Equations of Porous Medium Type*, pages 37–72. Springer International Publishing, Cham, 2021.
- [12] C. Cancès and C. Guichard. Numerical analysis of a robust free energy diminishing finite volume scheme for parabolic equations with gradient structure. *Found. Comput. Math.*, 17(6):1525–1584, 2017.
- [13] C. Cancès and C. Guichard. Numerical analysis of a robust free energy diminishing finite volume scheme for parabolic equations with gradient structure. *Found. Comput. Math.*, 17(6):1525–1584, 2017.
- [14] C. Cancès, F. Nabet, and M. Vohralík. Convergence and a posteriori error analysis for energy-stable finite element approximations of degenerate parabolic equations. *Math. Comp.*, 90(328):517–563, 2021.
- [15] C. Cancès, T. O. Gallouët, and Gabriele Todeschi. A variational finite volume scheme for Wasserstein gradient flows. *Numer. Math.*, 146(3):437–480, 2020.
- [16] J. A. Carrillo, Y.-P. Choi, and O. Tse. Convergence to equilibrium in Wasserstein distance for damped Euler equations with interaction forces. *Comm. Math. Phys.*, 365:329–361, 2019.
- [17] J. A. Carrillo, K. Craig, L. Wang, and C. Wei. Primal dual methods for Wasserstein gradient flows. *Found. Comput. Math.*, 2021.
- [18] C. Chainais-Hillairet, M. Herda, S. Lemaire, and J. Moatti. Long-time behaviour of hybrid finite volume schemes for advection-diffusion equations: linear and nonlinear approaches. *Numer. Math.*, 2022.

- [19] P. Chatzipantelidis, Z. Horváth, and V. Thomée. On preservation of positivity in some finite element methods for the heat equation. *Comput. Methods Appl. Math.*, 15(4):417–437, 2015.
- [20] K. Disser and M. Liero. On gradient structures for Markov chains and the passage to Wasserstein gradient flows. *Netw. Heterog. Media*, 10(2):233–253, 2015.
- [21] J. Droniou and R. Eymard. Uniform-in-time convergence of numerical methods for non-linear degenerate parabolic equations. *Numer. Math.*, 132(4):721–766, 2016.
- [22] J. Droniou, R. Eymard, T. Gallouët, C. Guichard, and R. Herbin. *The Gradient Discretisation Method*, volume 42 of *Mathématiques et Applications*. Springer International Publishing, Cham, 2018.
- [23] A. Ern and J.L. Guermond. *Theory and Practice of Finite Elements*, volume 159 of *Applied Mathematical Series*. Springer, New York, 2004.
- [24] A. Esposito, F. S. Patacchini, A. Schlichting, and D. Slepčev. Nonlocal-interaction equation on graphs: Gradient flow structure and continuum limit. *Arch. Ration. Mech. Anal.*, 240:699–760, 2021.
- [25] R. Eymard, T. Gallouët, C. Guichard, R. Herbin, and R. Masson. TP or not TP, that is the question. *Comput. Geosci.*, 18:285–296, 2014.
- [26] R. Eymard, T. Gallouët, and R. Herbin. Finite volume methods. In Ph Ciarlet and J.L. Lions, editors, *Handbook of numerical analysis, Vol. VII*, Handb. Numer. Anal., VII, pages 715–1022. North-Holland, Amsterdam, 2000.
- [27] T. Gallouët. Discrete functional analysis tools for some evolution equations. *Comput. Methods Appl. Math.*, 18(3):477–493, 2018.
- [28] P. Gladbach, E. Kopfer, J. Maas, and L. Portinale. Homogenisation of dynamical optimal transport on periodic graphs. arXiv:2110.15321, 2021.
- [29] Y. Hafiene, J. Fadili, C. Chesneau, and A. E. Moataz. Continuum limit of the nonlocal p -Laplacian evolution problem on random inhomogeneous graphs. *ESAIM: Math. Model. Numer. Anal.*, 54:565–589, 2020.
- [30] F. Hecht. New development in freefem++. *J. Numer. Math.*, 20(3-4):251–265, 2012.
- [31] R. Jordan, D. Kinderlehrer, and F. Otto. The variational formulation of the Fokker-Planck equation. *SIAM J. Math. Anal.*, 29(1):1–17, 1998.
- [32] S. Kamin and J.-L. Vázquez. Fundamental solutions and asymptotic behaviour for the p -Laplacian equation. *Rev. Mat. Iberoam.*, 4(2):339–354, 1988.
- [33] C. Lattanzio and A. E. Tzavaras. From gas dynamics with large friction to gradient flows describing diffusion theories. *Comm. Partial Differential Equations*, 42(2):261–290, 2017.
- [34] W. Li, J. Lu, and L. Wang. Fisher information regularization schemes for Wasserstein gradient flows. *J. Comput. Phys.*, 416:109449, 2020.
- [35] J. Maas. Gradient flows of the entropy for finite Markov chains. *J. Funct. Anal.*, 261(8):2250–2292, 2011.
- [36] R. J. McCann. A convexity principle for interacting gases. *Adv. Math.*, 128(1):153–179, 1997.
- [37] A. Moussa. Some variants of the classical Aubin-Lions Lemma. *J. Evol. Equ.*, 16(1):65–93, 2016.
- [38] A. Natale and G. Todeschi. TPFA finite volume approximation of Wasserstein gradient flows. In R. Klöfkom, E. Keilegavlen, F. A. Radu, and J. Fuhrmann, editors, *Finite Volumes for Complex Applications IX – Methods, Theoretical Aspects, Examples*, volume 323 of *Springer Proceedings in Mathematics & Statistics*, pages 193–202, 2020.
- [39] F. Otto. The geometry of dissipative evolution equations: the porous medium equation. *Comm. Partial Differential Equations*, 26(1-2):101–174, 2001.
- [40] N. Peton, C. Cancès, D. Granjeon, Q.-H. Tran, and S. Wolf. Numerical scheme for a water flow-driven forward stratigraphic model. *Comput. Geosci.*, 24:37–60, 2020.
- [41] Riccarda Rossi and Giuseppe Savaré. Tightness, integral equicontinuity and compactness for evolution problems in Banach spaces. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 2(2):395–431, 2003.
- [42] F. Santambrogio. *Optimal Transport for Applied Mathematicians: Calculus of Variations, PDEs, and Modeling*. Progress in Nonlinear Differential Equations and Their Applications 87. Birkhäuser Basel, 1 edition, 2015.
- [43] C. Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.
- [44] Y. Voet, E. Sande, and A. Buffa. A mathematical theory for mass lumping and its generalization with applications to isogeometric analysis. *Comput. Methods Appl. Mech. Eng.*, 410:116033, 2023.
- [45] D. Wei. Existence, uniqueness, and numerical analysis of solutions of a quasilinear parabolic problem. *SIAM J. Numer. Anal.*, 29:484–497, 1992.

CLÉMENT CANCES: UNIV. LILLE, INRIA, CNRS, UMR 8524 - LABORATOIRE PAUL PAINLEVÉ, F-59000 LILLE, FRANCE.
Email address: `clement.cances@inria.fr`

DANIEL MATTHES AND EVA-MARIA ROTT: ZENTRUM FÜR MATHEMATIK, TECHNISCHE UNIVERSITÄT MÜNCHEN, 85747 GARCHING, GERMANY.
Email address: `matthes@ma.tum.de`, `eva-maria.rott@tum.de`

FLORE NABET: CMAP, CNRS, ÉCOLE POLYTECHNIQUE, INSTITUT POLYTECHNIQUE DE PARIS, 91120 PALAISEAU, FRANCE.
Email address: `flore.nabet@polytechnique.edu`