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FINITE ELEMENTS FOR WASSERSTEIN W_p GRADIENT FLOWS

CLÉMENT CANCÈS, DANIEL MATTHES, FLORE NABET, AND EVA-MARIA ROTT

ABSTRACT. Convergence of a finite element discretization of a degenerate parabolic equation of q -Laplace type with an additional external potential is considered. The main novelty of our approach is that we use the underlying gradient flow structure in the L^p -Wasserstein metric: from the abstract machinery of metric gradient flows, convergence of scheme is obtained solely on the basis of estimates that result naturally from the equation's variational structure. In particular, the limit is identified as the unique gradient flow solution without reference to monotonicity methods.

1. INTRODUCTION

We are interested in the numerical approximation of non-negative unit mass solutions to the following degenerate parabolic equation

$$(1) \quad \partial_t \rho - \nabla \cdot \left(\rho |\nabla(\eta'(\rho) + \Psi)|^{q-2} \nabla(\eta'(\rho) + \Psi) \right) = 0, \quad \text{in } Q = \mathbb{R}_{\geq 0} \times \Omega,$$

on a bounded convex polyhedral domain $\Omega \subset \mathbb{R}^d$ with unit outward normal n , complemented with no-flux boundary conditions

$$(2) \quad - \left(\rho |\nabla(\eta'(\rho) + \Psi)|^{q-2} \nabla(\eta'(\rho) + \Psi) \right) \cdot n = 0 \quad \text{on } \mathbb{R}_{\geq 0} \times \partial\Omega,$$

and initial conditions ρ^0 at time $t = 0$. The nonlinearity $\eta : [0, +\infty) \rightarrow \mathbb{R}_{\geq 0}$ is continuous, smooth on $(0, \infty)$, strictly convex with superlinear growth at infinity, and satisfies McCann's condition [29] for geodesic convexity (see (11) later on). We discuss both the case where η is differentiable up to zero, and where $\lim_{r \downarrow 0} \eta'(r) = -\infty$. The external potential $\Psi \geq 0$ is Lipschitz continuous and does not depend on time. Finally, the exponent q is larger than one. Models of the form (1) appear as diffusive approximations of the Euler or shallow water equations with high (but non-necessarily linear w.r.t. the velocity) friction, see for instance [4, 26, 13, 32].

1.1. Gradient flow structure. The definition of our numerical method and the proof of convergence heavily use the variational structure of the boundary value problem (1)&(2): it is the metric p -gradient flow of the energy functional

$$(3) \quad \mathcal{E}(\rho) = \int_{\Omega} (\eta(\rho) + \rho\Psi) \, dx$$

with respect to the L^p -Wasserstein distance \mathbb{W}_p on the space $\mathcal{P}(\Omega)$ of probability measures on Ω . We briefly recall some key elements of the corresponding theory developed in [5]; more detailed information is given in Section 2 below

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Define p as q 's dual exponent, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. A curve $(\rho_t)_{t \geq 0}$ in the space $\mathcal{P}(\Omega)$ is a p -gradient flow iff it satisfies the following energy inequality

$$(4) \quad -\frac{d}{dt} \mathfrak{E}(t) \geq \frac{1}{p} |\rho'|^p(t) + \frac{1}{q} |\partial \mathcal{E}|^q(\rho_t)$$

for almost every $t > 0$, where \mathfrak{E} is a non-increasing function with $\mathfrak{E}(t) = \mathcal{E}(\rho_t)$ at almost every $t > 0$. Formally (rigorous definitions are postponed to Section 2), the metric velocity $|\rho'|$ and the local slope $|\partial \mathcal{E}|$ along a general curve $(\rho_t)_{t \geq 0}$ amount to

$$|\rho'|^p(t) = \int_{\Omega} |\mathbf{v}_t|^p \rho(t) \, dx, \quad |\partial \mathcal{E}|^q(\rho_t) = \int_{\Omega} \rho_t |\nabla[\eta'(\rho_t) + \Psi]|^q \, dx,$$

where \mathbf{v}_t is the minimizing vector field for the above integral among all \mathbf{v} satisfying

$$(5) \quad \partial_t \rho_t = -\nabla \cdot (\rho_t \mathbf{v}).$$

The key point is that the evolution equation (1)&(2), which is a relation on an infinite-dimensional Banach space, is equivalently expressed as the single relation (4) between the three scalar quantities $\mathcal{E}(\rho_t)$, $|\rho'|^p(t)$ and $|\partial \mathcal{E}|^q(\rho_t)$. On a formal level, that equivalence is easily derived by substituting the chain rule $-\partial_t \mathfrak{E}(t) \leq |\partial \mathcal{E}|^q(\rho_t) |\rho'|^p(t)$ on the left-hand side, and then using Young's inequality $a \cdot b \leq \frac{1}{p} |a|^p + \frac{1}{q} |b|^q$ for vectors $a, b \in \mathbb{R}^d$, which is sharp precisely if $a = |b|^{q-2} b$, with the choices $a = \mathbf{v}_t$ and $b = \nabla[\eta'(\rho_t) + \Psi]$ on the right-hand side.

The gradient flow structure of (1)&(2) has been used in the seminal paper [2] to construct weak solutions as limits of a variational time discretization, and in [5] to prove qualitative properties of solutions, like energy dissipation and instantaneous regularization. Unfortunately, the theory for metric p -gradient flows for $p \neq 2$ is much weaker than the one for $p = 2$, where e.g. \mathcal{E} 's geodesic convexity would have significant consequences on the long-time asymptotics, as shown in [31].

Here, we use the gradient in a particular way for our needs: we consider a full discretization of (1)&(2) that — albeit not truly variational — dissipates \mathcal{E} in a good way. From that dissipation, we are not only able to conclude convergence of the discrete approximations to some limit curve $\rho^* = (\rho_t^*)_{t \geq 0}$. We are further able to show — essentially just by soft arguments, using lower semi-continuity of $|\partial \mathcal{E}|$ etc. — that ρ^* satisfies the energy inequality (4). In conclusion, we obtain a p -gradient flow solution to (1)&(2), which is energy dissipating and a weak solution to the PDE. In particular, by this approach we circumvent the use of methods à la Browder-Minty to identify the flux in the limit, i.e., the term $|\nabla[\eta'(\rho) + \Psi]|^{q-2} \nabla[\eta'(\rho) + \Psi]$, and we obtain strong convergence results on the gradient expressions for our method. The price we pay is the use of various results from the theory of metric gradient flows as a “black box”, like an explicit characterization of the subdifferential $|\partial \mathcal{E}|$ in terms of differential operators, see Proposition A.2. For a more detailed description of our method, see Section 2.6 further below.

1.2. The discretization. The equation is discretized on a simplicial decomposition of Ω , and a time-step τ . Details are given in Section 3 further below. For a piecewise affine function f_h , let \bar{f}_h be its piecewise constant projection onto the associated Donald mesh, such that $\bar{f}_h(a) = f_h(a)$ at all vertices a . Our fully discrete evolution equation is then given by

$$(6) \quad \int_{\Omega} (\bar{\rho}_h^n - \bar{\rho}_h^{n-1}) \bar{w}_h + \tau^n \int_{\Omega} \rho_h^n |\nabla(\mu_h^n + \Psi_h)|^{q-2} \nabla(\mu_h^n + \Psi_h) \cdot \nabla w_h = 0,$$

for every piecewise affine test function w_h . Above, the piecewise affine μ_h^n is such that $\mu_h^n(a) \in \partial\eta(\rho_h^n(a))$ at all vertices a . So in particular $\mu_h^n(a) = \eta'(\rho_h^n(a))$ if $\rho_h^n(a) > 0$.

Several remarks on the seemingly complicated definition of our scheme are in place. First, unless $p = q = 2$, the term $|\nabla(\eta'(\rho) + \Psi)|^{q-2}$ requires a full reconstruction the gradient of the potential. In space dimension $d \geq 2$, the convenient *Two Point Flux Approximation* (TPFA) [20] are not anymore sufficient for consistency. We refer the reader to [21] for an illustration of the TPFA's failure in the context of \mathbb{W}_p due to hazardous geometrical effects from the mesh, or to [22] where similar effects appear already for the simpler transient q -Laplace equation. Instead, we make use of lowest order Lagrange finite elements as in [35, 8], but with mass-lumping as a key ingredient of our method. Since we do not assume any Delaunay type regularity on the mesh, the method does not have nice monotonicity properties also shared by TPFA approaches, or by the very specific discrete duality finite volume method [6] designed to approach strongly degenerate operators involving some q -Laplacian too.

In view of the gradient flow interpretation, energy stability is a crucial aspect for our scheme. Here we build on various recent contributions [10, 9, 11, 15], where energy dissipative methods for general meshes are designed and analyzed. Unfortunately, these schemes are not positivity preserving, but the latter is required for the gradient flow interpretation. As a remedy, we introduce a Lagrange multiplier for the positivity constraint, and use an interior point method inspired by [30]. We emphasize that the scheme we propose is not variational, but relies on the cheaper backward Euler scheme. For variational discretizations building on the JKO scheme [24] in the much easier case $p = q = 2$, see for instance [12, 27, 14]. Continuous in time but discrete in space dynamics are also considered in [28, 17, 19].

1.3. Results: well-posedness and convergence. We have two main results. The first is on the well-posedness of the fully discrete iteration scheme.

Theorem 1.1. *Let $\bar{\rho}_h^0 \in X_h \cap \mathcal{P}(\Omega)$, then for all $n \geq 1$, there exists $\bar{\rho}_h^n \in X_h \cap \mathcal{P}(\Omega)$ and $\bar{\mu}_h^n \in \partial\eta(\bar{\rho}_h^n)$ such that (6) holds true. Moreover, the scheme is energy diminishing and the following discrete energy / dissipation inequality holds:*

$$(7) \quad \mathcal{E}_h(\bar{\rho}_h^n) + \tau^n \int_{\Omega} \rho_h^n |\nabla(\mu_h^n + \Psi_h)|^q \leq \mathcal{E}_h(\bar{\rho}_h^{n-1}), \quad \forall n \geq 1.$$

The second result concerns the continuous limit.

Theorem 1.2. *For a fixed initial condition ρ^0 of finite energy $\mathcal{E}(\rho^0) < \infty$, consider a family of fully discrete solutions $\rho_{h\tau}$ on uniformly regular meshes. Then, as h and τ tend to zero along a suitable sequence, $\rho_{h\tau}(t)$ converges in \mathbb{W}_p , locally uniformly with respect to $t \geq 0$, and in $L_{loc}^1(\mathbb{R}_{\geq 0} \times \bar{\Omega})$, and the functions $t \mapsto \mathcal{E}(\rho_{h\tau}(t))$ converge pointwise to a non-increasing limit \mathfrak{E} . The limit ρ satisfies the energy identity (4). In particular, it is a weak solution to (1)&(2) in the sense specified in Proposition 2.3.*

1.4. Organization of the paper. In Section 2, we introduce some material related to the gradient flow interpretation of the continuous problem (1)&(2). The finite element scheme is described in Section 3. It is based on conformal Lagrange \mathbb{P}_1 finite elements and mass lumping, while the time discretization relies on the backward Euler scheme. In Section 4, we establish the a priori estimates our analysis

will rely on. These estimates are all consequences of the (global) mass conservativity and the energy diminishing property of the scheme. In Section 5, then we establish Theorem 1.1 by showing that the scheme admits solutions regardless to the time-step. Section 5 is devoted to the proof of Theorem 1.2, which relies on compactness arguments. Finally, numerical experiments are collected in Section 7.

2. HYPOTHESES, SOLUTION CONCEPT, IDEA OF PROOF

2.1. Spaces and distances. For general information on Wasserstein metrics and Wasserstein gradient flow, see [33, 34].

Denote by $\mathcal{P}(\Omega)$ the space of probability measures on Ω . By abuse of notation, we shall identify absolutely continuous measures $\mu \in \mathcal{P}(\Omega)$ with their Lebesgue-density $\rho \in L^1(\Omega)$. The L^p -Wasserstein distance $\mathbb{W}_p(\mu, \nu)$ between $\mu, \nu \in \mathcal{P}(\Omega)$ is defined as

$$(8) \quad \mathbb{W}_p(\mu, \nu) = \left(\inf_{\pi \in \Pi(\mu, \nu)} \iint_{\Omega \times \Omega} |x - y|^p d\pi(x, y) \right)^{1/p},$$

where $\Pi(\mu, \nu)$ is the set of all probability measures on $\Omega \times \Omega$ with respective marginals μ and ν . Since Ω is bounded, convergence in \mathbb{W}_p is equivalent to weak- $*$ -convergence. The metric space $(\mathcal{P}(\Omega), \mathbb{W}_p)$ is complete. Occasionally, we shall use the L^1 -Wasserstein distance \mathbb{W}_1 — also known as bounded Lipschitz distance — on $\mathcal{P}(\Omega)$, which is defined by (8) with $p = 1$, or alternatively in the dual representation

$$\mathbb{W}_1(\mu, \nu) = \sup_{0 \neq \varphi \in C^1(\mathbb{R}^d)} \frac{\int_{\Omega} \varphi d(\mu - \nu)}{\|\nabla \varphi\|_{L^\infty}}.$$

It is easily seen from the definitions that

$$(9) \quad \mathbb{W}_p(\mu, \nu) \leq \text{diam}(\Omega)^{1/q} \mathbb{W}_1(\mu, \nu)^{1/p}.$$

2.2. Energy functional. We assume that $\eta : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ is a continuous and convex function of superlinear growth. We assume further continuous differentiability on $\mathbb{R}_{>0}$, with

$$(10) \quad \lim_{\rho \rightarrow +\infty} \eta'(\rho) = +\infty.$$

Along the paper, η is further assumed to satisfy McCann's condition

$$(11) \quad s \mapsto s^d \eta(s^{-d}) \text{ is convex and non-increasing on } \mathbb{R}_{>0}.$$

Moreover, in terms of the continuous and increasing *pressure* function $\phi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$,

$$(12) \quad \phi(r) = r \eta'(r) - \eta(r) + \eta(0) = \int_0^r s \eta''(s) ds,$$

we assume that

$$(13) \quad \lim_{s \rightarrow +\infty} \frac{\eta \circ \phi^{-1}(s)}{s^\alpha} = 0, \quad \text{for some } \alpha < \min \left(q, \frac{d}{d-1} \right).$$

This is a soft condition, allowing for instance for $\eta(r) = r \log r - r + 1$, or $\eta(r) = r^m$ for $m > 1$, with the latter restriction on m coming from (10) rather than (13).

For further discussion, we identify (by abuse of notation) η with its convex and lower semi-continuous extension to \mathbb{R} , such that $\eta(r) = +\infty$ for $r < 0$. Notice that $\partial \eta(r) = \{\eta'(r)\}$ for all $r > 0$ by differentiability. At $r = 0$, there are two alternatives:

- (A1) either η' is singular at 0, i.e. $\lim_{r \searrow 0} \eta'(r) = -\infty$, and $\partial\eta(0) = \emptyset$,
 (A2) or $\eta'(0) := \lim_{r \searrow 0} \eta'(r)$ is finite, and $\partial\eta(0) = (-\infty, \eta'(0)]$.

Accordingly, the energy functional $\mathcal{E} : \mathcal{P}(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined by formula (3) from the introduction.

2.3. Metric slope. For the abstract definition of the metric (local) slope $|\partial\mathcal{E}|$ of \mathcal{E} , see [5, Section 1.2]. In the situation at hand, it is characterized as follows, see [5, Theorem 10.4.6].

Proposition 2.1. *Let $\rho \in \mathcal{P}_r(\Omega)$ be such that $\eta(\rho) \in L^1(\Omega)$ and such that $\phi(\rho) \in W^{1,1}(\Omega)$, and assume that there exists a measurable $\mathbf{u} : \Omega \rightarrow \mathbb{R}^d$ with $\rho|\mathbf{u}|^q \in L^1(\Omega)$, such that*

$$(14) \quad \rho\mathbf{u} = \nabla\phi(\rho) + \rho\nabla\Psi .$$

Then

$$(15) \quad |\partial\mathcal{E}(\rho)|^q = \int_{\Omega} \rho|\mathbf{u}|^q dx.$$

In view of the definition of ϕ , there holds $\nabla\phi(\rho) = \rho\nabla\eta'(\rho)$ as soon as ρ is regular enough (here, ρ bounded away from 0 is sufficient to justify the chain rule). Equation (14) then gives a weak sense to the expected definition of \mathbf{u} , namely $\mathbf{u} = \nabla(\eta'(\rho) + \Psi)$. Note that unless $p = q = 2$, the velocity $\mathbf{v} = |\mathbf{u}|^{q-2}\mathbf{u}$ transporting ρ differs from \mathbf{u} and

$$|\partial\mathcal{E}(\rho)|^q = \int_{\Omega} \rho|\mathbf{v}|^p.$$

2.4. Metric velocity. For the abstract definitions of p -absolute continuity and the metric velocity $|\rho'|$ of a curve $(\rho_t)_{t \geq 0}$ in $\mathcal{P}(\Omega)$ with respect to \mathbb{W}_p (or in any other metric space), see [5, Section 1.1]. Informally, $|\rho'|$ is the smallest function $m \in L^1_{\text{loc}}(\mathbb{R}_{\geq 0})$ with the property that

$$\mathbb{W}_p(\rho_{t_1}, \rho_{t_2}) \leq \int_{t_1}^{t_2} m(t) dt \quad \text{for all } 0 \leq t_1 \leq t_2,$$

and $(\rho_t)_{t \geq 0} \in \text{AC}^p$, i.e., the curve is p -absolutely continuous, if $|\rho'| \in L^p(\mathbb{R}_{> 0})$. Here, we only need the following property, see [5, Theorem 8.3.1].

Proposition 2.2. *If $\mathbf{v} : \mathbb{R}_{> 0} \times \Omega \rightarrow \mathbb{R}^d$ with $\int_0^\infty \int_{\Omega} \rho_t |\mathbf{v}_t|^p dx dt < \infty$ is such that the continuity equation*

$$\partial_t \rho_t + \nabla \cdot \rho_t \mathbf{v}_t = 0$$

is satisfied in the distributional sense, then $(\rho_t)_{t \geq 0} \in \text{AC}^p$, and

$$|\rho'|^p(t) \leq \int_{\Omega} \rho_t |\mathbf{v}_t|^p dx \quad \text{for a.e. } t > 0.$$

2.5. p -gradient flows. A curve $(\rho_t)_{t \geq 0} \in \text{AC}^p$ is a p -gradient flow for \mathcal{E} in \mathbb{W}_p if the energy dissipation inequality (4) is satisfied with some non-increasing function $\mathfrak{E} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ such that $\mathcal{E}(\rho_t) = \mathfrak{E}(t)$ for a.e. $t \geq 0$. The following proposition combines Theorem 11.1.3 with Theorem 10.4.6 and Proposition 10.4.2 from [5].

Proposition 2.3. *If $(\rho_t)_{t \geq 0}$ is a p -gradient flow, then it is also a weak solution to (1)&(2) in the following sense: at a.e. $t \geq 0$, one has $\phi(\rho_t) \in W^{1,1}(\Omega)$, and $\rho_t \left| \frac{\nabla \phi(\rho_t)}{\rho_t} + \nabla \Psi \right|^q \in L^1(\Omega)$; further, for all $\varphi \in C_c^\infty(\mathbb{R}_{>0} \times \bar{\Omega})$*

$$0 = \int_0^\infty \int_\Omega \left[\rho \partial_t \varphi - \left| \frac{\nabla \phi(\rho)}{\rho} + \nabla \Psi \right|^{q-2} (\nabla \phi(\rho) + \rho \nabla \Psi) \cdot \nabla \varphi \right] dx dt.$$

2.6. Strategy of proof. The proof of our convergence result in Theorem 1.2 rests on the strong entropy dissipation estimate (7) for fully discrete solutions. From that, we derive the existence of a limit curve $\rho^* = (\rho_t^*)_{t \geq 0}$ that satisfies (the integral form of) the energy inequality (4), for all $t_2 \geq t_1 \geq 0$:

$$(16) \quad \mathfrak{E}(t_1) - \mathfrak{E}(t_2) \geq \frac{1}{p} \int_{t_1}^{t_2} |(\rho^*)'|^p(t) dt + \frac{1}{q} \int_{t_1}^{t_2} |\partial \mathcal{E}|^q(\rho_t^*) dt.$$

Mere existence of a limit $(\rho_t^*)_{t \geq 0}$ is obtained by means of the Arzela-Ascoli theorem, see Proposition 6.2. With the help of the entropy dissipation estimate, and thanks to certain favourable properties of the mass-lumping procedure, we are further able to identify the limit of $\nabla \phi_{h\tau}$, the affine interpolation of the values $\phi(\rho_{h\tau})$ at vertices, as $\nabla \phi(\rho^*)$, see Proposition 6.3. A little more work is required to show that also

$$(17) \quad \tilde{\rho}_{h\tau} \nabla [\mu_{h\tau} + \Psi] \rightarrow \nabla \phi(\rho^*) + \rho^* \nabla \Psi \quad \text{weakly in } L^1,$$

where $\tilde{\rho}_{h\tau}$ is piecewise constant on the simplicial mesh (not the Donald mesh), see Lemmas 6.5&6.6. Note that in the much easier special case $p = q = 2$, it would now follow almost immediately that ρ is indeed a weak solution to (1)&(2).

The core part of the convergence proof is to conclude (16) from here. There are three ingredients.

(i) *Convergence of the slope:* In Lemma 6.7, we conclude that

$$(18) \quad \int_{t_1}^{t_2} |\partial \mathcal{E}|^q(\rho_t^*) dt = \int_{t_1}^{t_2} \int_\Omega \rho_t^* |\mathbf{u}_t^*|^q dx dt \leq \liminf_{h,\tau \rightarrow 0} \int_{t_1}^{t_2} \int_\Omega \tilde{\rho}_{h,\tau} |\nabla [\mu_{h\tau} + \Psi]|^q dx dt,$$

where \mathbf{u}^* is such that $\tilde{\rho}_{h\tau} \nabla [\mu_{h\tau} + \Psi] \rightarrow \rho^* \mathbf{u}^*$ in the distributional sense. The existence of such an \mathbf{u}^* and the liminf-estimate both follow from the entropy dissipation (7) by means of the abstract convergence result in Proposition A.2. The equality on the left is then a consequence of the characterization of $|\partial \mathcal{E}|(\rho_t^*)$ in Proposition 2.1 above, since $\rho_t^* \mathbf{u}_t^*$ has already been identified as the right-hand side of (15) by means of (17).

(ii) *Convergence of the metric velocity:* In Lemma 6.8, we show that

$$(19) \quad \int_{t_1}^{t_2} |(\rho^*)'|^p(t) dt \leq \int_{t_1}^{t_2} \int_\Omega \rho_t^* |\mathbf{v}_t^*|^p dx dt \leq \liminf_{h\tau \rightarrow 0} \int_{t_1}^{t_2} \int_\Omega \tilde{\rho}_{h\tau} |\mathbf{v}_{h\tau}|^p dx dt,$$

where $\mathbf{v}_{h\tau} := |\nabla [\mu_{h\tau} + \Psi]|^{q-2} \nabla [\mu_{h\tau} + \Psi]$, so that $|\mathbf{v}_{h\tau}|^p = |\nabla [\mu_{h\tau} + \Psi]|^q$, i.e., the right-hand sides of (18) and (19) are identical. And \mathbf{v}^* is such that $\tilde{\rho}_{h\tau} \mathbf{v}_{h\tau} \rightarrow \rho^* \mathbf{v}^*$ as distributions. Existence of \mathbf{v}^* and the liminf-estimate follow just as before. The inequality on the left is a direct consequence of the metric slope's characterization.

(iii) *Convergence of the energy:* In Lemma 6.9, we show that $t \mapsto \mathcal{E}_h(\bar{\rho}_{h\tau}(t))$ converges at almost every $t \geq 0$ to a non-increasing function \mathfrak{E} , and that $\mathfrak{E}(t) = \mathcal{E}(\rho_t^*)$ at almost every $t \geq 0$. The first part is an easy consequence of

Helly's theorem, the second part is established by means of strong convergence of $\bar{\phi}_{h\mathcal{T}}$ and properties of the discretization.

3. ENERGY STABLE FINITE ELEMENTS

We are interested in the approximation thanks to finite elements of solutions ρ to (1)&(2). Our approach relies on lowest order conforming finite elements with mass lumping. The mesh as well as the reconstruction operators are defined in Section 3.1. We then present the scheme in Section 3.2.

3.1. $\mathbb{P}1$ finite elements and mass-lumped reconstructions. The domain Ω is assumed to be tessellated into a simplicial conformal discretization \mathcal{T} (made of triangles if $d = 2$ and of tetrahedra if $d = 3$). Denoting by h_T the diameter of an element $T \in \mathcal{T}$ and by \mathfrak{d}_T the diameter of the largest sphere included in T , the size and regularity of \mathcal{T} are respectively defined by

$$h_{\mathcal{T}} = \max_{T \in \mathcal{T}} h_T, \quad \zeta_{\mathcal{T}} = \max_{T \in \mathcal{T}} \frac{h_T}{\mathfrak{d}_T}.$$

We denote by \mathcal{V}_T the set of the $(d + 1)$ vertices of $T \in \mathcal{T}$, and by $\mathcal{V} = \bigcup_{T \in \mathcal{T}} \mathcal{V}_T$ the set of all the vertices of \mathcal{T} .

We also denote by V_h the usual $\mathbb{P}1$ Lagrange finite elements space corresponding to the mesh \mathcal{T} , i.e.

$$V_h = \{v \in C(\bar{\Omega}) \mid v|_T \text{ is affine for all } T \in \mathcal{T}\},$$

and by \tilde{V}_h the set of piecewise constant functions on \mathcal{T} , i.e.

$$\tilde{V}_h = \{v \in L^\infty(\Omega) \mid v|_T \text{ is constant for all } T \in \mathcal{T}\}.$$

In particular, the gradient ∇v_h of an element $v_h \in V_h$ belongs to $(\tilde{V}_h)^d$.

Mass lumping is frequently presented in the literature as a simple quadrature rule for computing integrals, the quadrature points being located at the vertices, similarly to the degrees of freedom. Here, we rather build on another classical approach for representing mass lumping, which consists in introducing the so-called Donald (or dual barycentric) mesh. To each $a \in \mathcal{V}$ we associate a cell ω_a containing a and delimited by the hyperplans joining

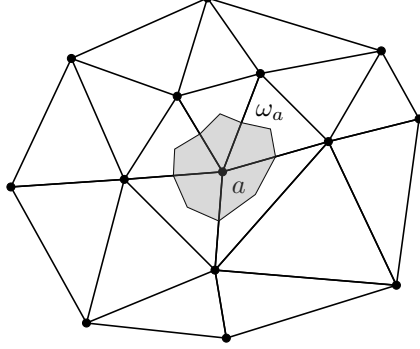
- the barycenters x_T and x_e of all triangle $T \in \mathcal{V}$ and edge e sharing a as a vertex if $d = 2$;
- the barycenters x_T , x_F and x_e of all triangle $T \in \mathcal{V}$, face F and edge e sharing a as a vertex if $d = 3$.

With this construction, one has $|\omega_a \cap T| = \frac{1}{d+1}|T|$ for all $a \in \mathcal{V}_T$, $T \in \mathcal{T}$. An illustration in the two-dimensional case is given on Figure 1. We denote by X_h the set of piecewise constant functions on the Donald mesh, i.e.

$$X_h = \{v \in L^\infty(\Omega) \mid v|_{\omega_a} \text{ is constant for all } a \in \mathcal{V}\}.$$

Now, given a piecewise linear function $v_h \in V_h$, there exists a unique element of X_h denoted by \bar{v}_h such that

$$\bar{v}_h(a) = v_h(a), \quad \forall a \in \mathcal{V}.$$

FIGURE 1. Construction of the Donald mesh cell ω_a for some $a \in \mathcal{V}$.

In particular, the mapping $v_h \mapsto \bar{v}_h$ is one-to-one and onto from V_h to X_h . Due to the mass-lumping procedure, the mass matrix is diagonal since

$$(20) \quad \int_{\Omega} \bar{v}_h \bar{w}_h = \sum_{a \in \mathcal{V}} |\omega_a| v_h(a) w_h(a), \quad \forall v_h, w_h \in V_h.$$

Another remarkable property of mass-lumped finite elements is its compatibility with nonlinearities. More precisely, there holds

$$(21) \quad \overline{f(v_h)} = f(\bar{v}_h), \quad \text{for all } f \in C(\mathbb{R}) \text{ and all } v_h \in V_h.$$

The combination of (20) and (21) is key to establish the energy stability of our scheme. More generally, mass-lumping enhances the stability of finite elements, see for instance [16]. In our analysis, one also needs to reconstruct elements $\tilde{v}_h \in \tilde{V}_h$ from elements $v_h \in V_h$ by setting

$$\tilde{v}_h(x_T) = \frac{1}{d+1} \sum_{a \in \mathcal{V}_T} v_h(a), \quad \forall T \in \mathcal{T}.$$

For the time discretization, one defines an increasing sequence $(t^n)_{n \geq 0}$ with $t^0 = 0$ and $t^n \rightarrow +\infty$ as $n \rightarrow +\infty$. We denote by $\tau^n = t^n - t^{n-1}$ the n^{th} time step, and by $\tau = \max_{n \geq 1} \tau^n$. Then to a sequence $(v_h^n)_{n \geq 0} \subset V_h$, we associate the piecewise constant w.r.t. time functions by defining time-and-space discretization spaces $V_{h\tau}$, $X_{h\tau}$ and $\tilde{V}_{h\tau}$ by

$$\begin{aligned} V_{h\tau} &= \left\{ v_{h\tau} \in D(\mathbb{R}_{\geq 0}; V_h) : v_{h\tau}(t, \cdot) = v_{h\tau}(0, \cdot) \mathbf{1}_{\{0\}}(t) + \sum_{n \geq 1} v_{h\tau}(t_n, \cdot) \mathbf{1}_{(t^{n-1}, t^n]}(t) \right\}, \\ X_{h\tau} &= \left\{ \bar{v}_{h\tau} \in D(\mathbb{R}_{\geq 0}; X_h) : \bar{v}_{h\tau}(t, \cdot) = \bar{v}_{h\tau}(0, \cdot) \mathbf{1}_{\{0\}}(t) + \sum_{n \geq 1} \bar{v}_{h\tau}(t_n, \cdot) \mathbf{1}_{(t^{n-1}, t^n]}(t) \right\}, \\ \tilde{V}_{h\tau} &= \left\{ \tilde{v}_{h\tau} \in D(\mathbb{R}_{\geq 0}; \tilde{V}_h) : \tilde{v}_{h\tau}(t, \cdot) = \tilde{v}_{h\tau}(0, \cdot) \mathbf{1}_{\{0\}}(t) + \sum_{n \geq 1} \tilde{v}_{h\tau}(t_n, \cdot) \mathbf{1}_{(t^{n-1}, t^n]}(t) \right\}, \end{aligned}$$

where $D(I; E)$ denotes the set of the left-continuous with right-limit (càglàd) functions from the interval I to the space E .

3.2. Definition of the scheme. The initial data ρ^0 is discretized into $\bar{\rho}_h^0 \in X_h$ characterized by

$$(22) \quad \bar{\rho}_h^0(a) := \frac{1}{|\omega_a|} \int_{\omega_a} \rho^0 = \fint_{\omega_a} \rho^0, \quad \forall a \in \mathcal{V}.$$

Similarly, the discretized external potential $\Psi_h \in V_h$ is defined by setting

$$(23) \quad \Psi_h(a) = \fint_{\omega_a} \Psi, \quad \forall a \in \mathcal{V}.$$

With this choice, the existence of some $C_1 > 0$ depending only on the regularity factor $\zeta_{\mathcal{T}}$ and on the dimension d such that

$$(24) \quad \|\nabla \Psi_h\|_{L^\infty(\Omega)^d} \leq C_1 \|\nabla \Psi\|_{L^\infty(\Omega)^d}$$

has been established in the proof of [11, Theorem A.1]. We define the approximate energy $\mathcal{E}_h : \mathcal{P}(\Omega) \cap X_h \rightarrow \mathbb{R}_{\geq 0}$ by setting

$$\mathcal{E}_h(\bar{v}_h) = \int_{\Omega} (\eta(\bar{v}_h) + \bar{\Psi}_h \bar{v}_h), \quad \bar{v}_h \in X_h, v_h \geq 0.$$

The regularity of Ψ implies that $\|\Psi - \bar{\Psi}_h\|_{\infty} \leq C_2 h$ with $C_2 = 2\|\nabla \Psi\|_{\infty}$, hence

$$(25) \quad |\mathcal{E}_h(\bar{v}_h) - \mathcal{E}(\bar{v}_h)| \leq C_2 h, \quad \forall \bar{v}_h \in \mathcal{P}(\Omega) \cap X_h.$$

Therefore since η is convex and in view of the definition (22) of $\bar{\rho}_h^0$, it results from Jensen's inequality that

$$(26) \quad \mathcal{E}_h(\bar{\rho}_h^0) \leq \mathcal{E}(\rho^0) + C_2 h \leq \mathcal{E}(\rho^0) + C_2 \text{diam}(\Omega) < +\infty.$$

Now, given $\bar{\rho}_h^{n-1} \in \mathcal{P}(\Omega)$ with $\mathcal{E}(\bar{\rho}_h^{n-1}) < +\infty$, the mass-lumped finite element scheme consists in finding $\rho_h^n \in V_h$ with $\rho_h^n \in \text{Dom}(\partial\eta)$ in $\bar{\Omega}$ such that, for all $w_h \in V_h$, there holds

$$(27) \quad \int_{\Omega} (\bar{\rho}_h^n - \bar{\rho}_h^{n-1}) \bar{w}_h + \tau^n \int_{\Omega} \rho_h^n |\nabla(\mu_h^n + \Psi_h)|^{q-2} \nabla(\mu_h^n + \Psi_h) \cdot \nabla w_h = 0,$$

where μ_h^n denotes an element of V_h such that $\mu_h^n(a) \in \partial\eta(\rho_h^n(a))$ for all $a \in \mathcal{V}$, or equivalently $\bar{\mu}_h^n \in \partial\eta(\bar{\rho}_h^n)$.

In the case (A1) where $\partial\eta$ is single valued, the above definition of the scheme (27) requires the density ρ_h^n to be (strictly) positive on $\bar{\Omega}$, and the function of $(\rho_h^n(a))_{a \in \mathcal{V}}$ of which we look for a zero is continuous. The alternative case (A2) is more intricate since ρ_h^n is now allowed to vanish at some nodes $a \in \mathcal{V}$. Since $\partial\eta(0)$ is multivalued, $\mu_h^n(a)$ can a priori take arbitrary values in $(-\infty, \eta'(0)]$, and in any case can no longer be expressed as a function of $\rho_h^n(a)$. The problem (27) cannot be interpreted as the Euler-Lagrange condition for the minimization of some functional (our time discretization relies on the backward Euler scheme rather than on a minimizing movement scheme), but roughly speaking, the gap between $\eta'(\rho_h^n(a))$ and $\mu_h^n(a)$ can be thought as a Lagrange multiplier for the non-negativity constraint on $\rho_h^n(a)$. The fact that this gap is multiplied by some mobility ρ_h^n makes the calculations involving, motivating some regularization via the introduction of barrier functions both in the analysis and for the practical computation of the solutions to (27).

4. A PRIORI ESTIMATES

Lemma 4.1. *Let $\rho_h^n \in V_h$ be a solution of the scheme (27), then $\rho_h^n, \bar{\rho}_h^n$ and $\tilde{\rho}_h^n$ all belong to $\mathcal{P}(\Omega)$.*

Proof. Since the positivity of ρ_h^n is assumed for solutions to (27), one only has to check that

$$\int_{\Omega} \tilde{\rho}_h^n = \int_{\Omega} \rho_h^n = \int_{\Omega} \bar{\rho}_h^n = \int_{\Omega} \bar{\rho}_h^{n-1} = \int_{\Omega} \bar{\rho}_h^0 = \int_{\Omega} \rho^0 = 1.$$

The first two equalities are provided by the exactness of quadrature rules, while the third one directly follows from testing (27) by $w_h \equiv 1$. A straightforward induction yields the fourth equality, and the fifth one stems from the definition (22) of the discretized initial condition $\bar{\rho}_h^0$. The last equality holds since $\rho^0 \in \mathcal{P}(\Omega)$. \square

Proposition 4.2. *Let $(\rho_h^n)_{n \geq 1}$ be a solution to (27), then for all $n_* \geq 1$, there holds*

$$(28) \quad \mathcal{E}_h(\bar{\rho}_h^{n_*}) + \sum_{n=1}^{n_*} \tau^n \int_{\Omega} \rho_h^n |\nabla(\mu_h^n + \Psi_h)|^q \leq \mathcal{E}_h(\bar{\rho}_h^0) \leq \mathcal{E}(\rho^0) + C_2 h.$$

Proof. Choosing $w_h = \mu_h^n + \Psi_h$ in (27), then since mass-lumping was imposed on the accumulation term, one can make use of the convexity (in the usual sense) of \mathcal{E} to deduce the one-step energy / dissipation (7). Summing over $n = 1, \dots, n_*$ and using (26) yields the desired inequality. \square

A first consequence of the energy estimate is the following $L^\infty(\mathbb{R}_{\geq 0}; L^1(\Omega))$ on $\eta(\bar{\rho}_{h\tau})$.

Corollary 4.3. *There exists C_3 depending only on Ω , $\mathcal{E}(\rho^0)$ and Ψ such that*

$$(29) \quad \int_{\Omega} \eta(\tilde{\rho}_h^n) \leq \int_{\Omega} \eta(\bar{\rho}_h^n) \leq C_3, \quad \forall n \geq 1.$$

Proof. We infer from Proposition 4.2 that

$$\mathcal{E}_h(\bar{\rho}_h^n) \leq \mathcal{E}(\rho^0) + C_2 \text{diam}(\Omega) =: C_3, \quad \forall n \geq 0.$$

Bearing the definition (3) of the energy and the non-negativity of Ψ in mind, one gets that

$$\int_{\Omega} \eta(\bar{\rho}_h^n) \leq \mathcal{E}_h(\bar{\rho}_h^n) \leq C_3, \quad \forall n \geq 0.$$

The first inequality in (29) is a direct consequence of Jensen's inequality. \square

We now state a second direct consequence of Proposition 4.2.

Corollary 4.4. *There exists C_4 depending only on Ω , $\mathcal{E}(\rho^0)$, Ψ , q , $\zeta\tau$ and d such such that*

$$\sum_{n=1}^{n_*} \tau^n \int_{\Omega} \rho_h^n |\nabla \mu_h^n|^q \leq C_4 \left(1 + \sum_{n=1}^{n_*} \tau^n \right), \quad \forall n_* \geq 1.$$

Proof. Using the elementary inequality $(a+b)^q \leq 2^{q-1}(a^q + b^q)$ if $a, b \geq 0$, one gets that

$$\sum_{n=1}^{n_*} \tau^n \int_{\Omega} \rho_h^n |\nabla \mu_h^n|^q \leq 2^{q-1} \left(\sum_{n=1}^{n_*} \tau^n \int_{\Omega} \rho_h^n |\nabla(\mu_h^n + \Psi_h)|^q + \sum_{n=1}^{n_*} \tau^n \int_{\Omega} \rho_h^n |\nabla \Psi_h|^q \right).$$

Since $\mathcal{E}_h(\bar{\rho}_h^n) \geq 0$ for all $n \geq 1$, one deduces from (28) that the first term in the right-hand side is bounded by $2^{q-1}(\mathcal{E}(\rho^0) + C_2 \text{diam}(\Omega))$. On the other hand, we infer from (24) and from Lemma 4.1 that

$$\int_{\Omega} \rho_h^n |\nabla \Psi_h|^q \leq (C_1)^q \|\nabla \Psi\|_{\infty}^q,$$

concluding the proof of Corollary 4.4. \square

In the singular case (A1) where $\eta' : (0, +\infty) \rightarrow \mathbb{R}$ is a (strictly) monotone onto (thus invertible) function, one needs to bound ρ_h^n away from 0. This is the purpose of the following proposition.

Proposition 4.5. *Assume that (A1) holds, then there exists $\epsilon_n > 0$ depending on the data of the continuous problem (among which η') as well as on \mathcal{T} and on $(\tau^\ell)_{1 \leq \ell \leq n}$ such that $\rho_h^n \geq \epsilon_n$ for all $n \geq 1$.*

Proof. As a consequence of (A1), $\mu_h^n(a) = \eta'(\rho_h^n(a))$ for all $a \in \mathcal{T}$. Since ρ_h^n belongs to $\mathcal{P}(\Omega)$ owing to Lemma 4.1, one knows a priori that there exists $a_0 \in \mathcal{V}$ such that

$$\rho_h^n(a_0) \geq \int_{\Omega} \rho_h^n = |\Omega|^{-1} =: \epsilon_n^{(0)} \quad \Leftrightarrow \quad \mu_h^n(a_0) \geq \eta'(|\Omega|^{-1}) =: M_n^{(0)}.$$

Now, let $a_1 \in \mathcal{V}$ be a vertex sharing an edge with a_0 , i.e. such that there exists $T_0 \in \mathcal{T}$ having a_0 and a_1 as vertices. Then we deduce from Corollary 4.4 that

$$\int_{T_0} \rho_h^n |\nabla \mu_h^n|^q \leq \frac{C_4}{\tau^n} \left(1 + \sum_{\ell=1}^n \tau^\ell \right) =: R_n.$$

Since $\nabla \mu_h^n$ is constant on T_0 , the above integrand is piecewise linear, hence we can replace ρ_h^n by $\tilde{\rho}_h^n$, for which we can use

$$\tilde{\rho}_h^n(x) = \frac{1}{d+1} \sum_{a \in \mathcal{V}_{T_0}} \rho_h^n(a) \geq \frac{1}{d+1} \rho_h^n(a_0) \geq \frac{\epsilon_n^{(0)}}{(d+1)}, \quad x \in T_0.$$

On the other hand,

$$|\nabla \mu_h^n| \geq \left| \frac{\mu_h^n(a_0) - \mu_h^n(a_1)}{h_{T_0}} \right|,$$

hence, setting $A_{\mathcal{T}} = \min_{T \in \mathcal{T}} |T|$, one has

$$\mu_h^n(a_1) \geq M_n^{(0)} - h_{\mathcal{T}} \left(\frac{R_n(d+1)}{A_{\mathcal{T}} \epsilon_n^{(0)}} \right)^{\frac{1}{q}} =: M_n^{(1)},$$

or equivalently

$$\rho_h^n(a_1) \geq (\eta')^{-1} \left(M_n^{(1)} \right) =: \epsilon_n^{(1)} > 0$$

since η' is one-to-one and onto from $(0, +\infty)$ to \mathbb{R} . Iterating the process provides a lower bound $M_n^{(k)}$ on $\mu_h^n(a_k)$ at a vertex located k edges away from a_0 with

$$M_n^{(j+1)} = M_n^{(j)} - h_{\mathcal{T}} \left(\frac{R_n(d+1)}{A_{\mathcal{T}} (\eta')^{-1}(M_n^{(j)})} \right)^{\frac{1}{q}}, \quad j \geq 0.$$

Since Ω is connected, all the vertices are separated by a finite number of edges, hence the existence of a finite M_n depending on the data of the continuous problem as well as on $h_{\mathcal{T}}$ and on the time discretization such that

$$\mu_h^n(a) \geq M_n, \quad \forall a \in \mathcal{V}.$$

Then Proposition 4.5 holds with $\epsilon_n = (\eta')^{-1}(M_n)$. \square

Denote by ϕ_h^n the element of V_h such that $\phi_h^n(a) = \phi(\rho_h^n(a))$ for all $a \in \mathcal{V}$, where ϕ is defined by (12).

Lemma 4.6. *There exists C_5 depending only on the mesh regularity $\zeta_{\mathcal{T}}$ and the dimension d such that*

$$(30) \quad |\nabla \phi_h^n| \leq C_5 \tilde{\rho}_h^n |\nabla \mu_h^n|, \quad \forall n \geq 1.$$

Proof. Let $T \in \mathcal{T}$, the vertices of which being denoted by a_0^T, \dots, a_d^T , let \hat{T} be the reference elements, with $a_0^{\hat{T}} = 0_{\mathbb{R}^d}$ and $a_i^{\hat{T}} = e_i$ the i^{th} vector of the canonical basis of \mathbb{R}^d , let F_T be the unique affine map sending $a_i^{\hat{T}}$ on a_i^T , and let J_T denote its Jacobian matrix. Then classical results from the theory of finite elements (see for instance [18, Lemma 1.100]) combined with elementary calculations show that

$$(31) \quad |\det(J_T)| = d!|T|, \quad |J_T|_2 \leq \frac{d + \sqrt{d}}{2} h_T \quad \text{and} \quad |J_T^{-1}|_2 \leq \frac{\sqrt{2}}{\mathfrak{d}_T} \leq \sqrt{2} \frac{\zeta_{\mathcal{T}}}{h_T},$$

with $|\cdot|_2$ standing for the matrix norm inherited from the Euclidian norm of \mathbb{R}^d . We deduce from a change of variable that for all $n \geq 1$ and all $T \in \mathcal{T}$, there holds

$$(32) \quad |\nabla \phi_h^n(x)| = \int_T |\nabla \phi_h^n| = d! |J_T^{-1} \delta^T \phi_h^n| \leq \sqrt{2} d! \frac{\zeta_{\mathcal{T}}}{h_T} |\delta^T \phi_h^n|, \quad \forall x \in T,$$

where $\delta_T \phi_h^n$ is the vector of the variations of ϕ_h^n in T defined by

$$\delta_T \phi_h^n = \begin{pmatrix} \phi_h^n(a_1^T) - \phi_h^n(a_0^T) \\ \vdots \\ \phi_h^n(a_d^T) - \phi_h^n(a_0^T) \end{pmatrix}.$$

Let R_T^n be the diagonal matrix with positive entries defined by

$$(R_T^n)_{i,i} = \begin{cases} \frac{\phi_h^n(a_i^T) - \phi_h^n(a_0^T)}{\mu_h^n(a_i^T) - \mu_h^n(a_0^T)} & \text{if } \mu_h^n(a_i^T) \neq \mu_h^n(a_0^T), \\ \rho_h^n(a_i^T) & \text{otherwise,} \end{cases}$$

so that, setting

$$\delta_T \mu_h^n = \begin{pmatrix} \mu_h^n(a_1^T) - \mu_h^n(a_0^T) \\ \vdots \\ \mu_h^n(a_d^T) - \mu_h^n(a_0^T) \end{pmatrix},$$

there holds $\delta_T \phi_h^n = R_T^n \delta_T \mu_h^n$. Assume that $\rho_h^n(a_i^T) > \rho_h^n(a_0^T)$. Then the definition (12) of ϕ and the fact that $\bar{\mu}_h^n \in \partial\eta(\bar{\rho}_h^n)$ respectively provide that

$$\phi_h^n(a_i^T) - \phi_h^n(a_0^T) = \int_{\rho_h^n(a_0^T)}^{\rho_h^n(a_i^T)} s \eta''(s) ds$$

and

$$\mu_h^n(a_i^T) - \mu_h^n(a_0^T) \geq \eta'(\rho_h^n(a_i^T)) - \eta'(\rho_h^n(a_0^T)) = \int_{\rho_h^n(a_0^T)}^{\rho_h^n(a_i^T)} \eta''(s) ds.$$

The above inequality is an equality if $\rho_h^n(a_0^T) > 0$, so that

$$\rho_h^n(a_0^T) \leq (R_T^n)_{i,i} \leq \rho_h^n(a_i^T),$$

which clearly still holds when $\rho_h^n(a_0^T) = 0$. Proceeding similarly if $\rho_h^n(a_i^T) < \rho_h^n(a_0^T)$, one gets that

$$(33) \quad \min(\rho_h^n(a_i^T), \rho_h^n(a_0^T)) \leq (R_T^n)_{i,i} \leq \max(\rho_h^n(a_i^T), \rho_h^n(a_0^T)) \leq \sum_{i=0}^d \rho_h^n(a_i^T),$$

hence

$$(34) \quad |\delta_T \phi_h^n| \leq |\delta_T \mu_h^n| \left(\sum_{i=0}^d \rho_h^n(a_i^T) \right) = (d+1) \bar{\rho}_h^n(x) |\delta_T \mu_h^n|, \quad \forall x \in T.$$

On the other hand, using (31) once again provides

$$(35) \quad |\delta_T \mu_h^n| \leq |J_T|_2 |J_T^{-1} \delta_T \mu_h^n| \leq \frac{d + \sqrt{d}}{2d!} h_T \int_T |\nabla \mu_h^n| = \frac{d + \sqrt{d}}{2d!} h_T |\nabla \mu_h^n(x)|$$

holds for all x in T . Combining (32), (34) and (35), one finally gets that (30) holds with $C_5 = (d+1) \frac{d + \sqrt{d}}{\sqrt{2}} \zeta_{\mathcal{T}}$. \square

The control on the gradient of ϕ_h^n derived in previous lemma also provides a control on ϕ_h^n itself as shows next lemma.

Lemma 4.7. *There exists C_6 depending only on η , Ω , and $\zeta_{\mathcal{T}}$ such that*

$$(36) \quad \int_{\Omega} |\phi_h^n| \leq C_6 \left(1 + \int_{\Omega} |\nabla \phi_h^n| \right).$$

Proof. Denoting by

$$U^n = \{x \in \Omega \mid \bar{\rho}_h^n(x) \leq 2|\Omega|^{-1}\}, \quad n \geq 1,$$

then it follows from Markov inequality that $|U^n| \geq |\Omega|/2$. One writes

$$\bar{\phi}_h^n \leq \phi(2|\Omega|^{-1}) + \left(\bar{\phi}_h^n - \phi(2|\Omega|^{-1}) \right)^+,$$

the second term in the right-hand side above vanishing on U^n since ϕ is increasing. Then [3, Lemma A.3] gives that

$$\int_{\Omega} \left(\bar{\phi}_h^n - \phi(2|\Omega|^{-1}) \right)^+ \leq C_7 \int_{\Omega} \nabla (\phi_h^n - \phi(2|\Omega|^{-1}))^+ \leq C_7 \int_{\Omega} |\nabla \phi_h^n|$$

for some C_7 depending only on Ω and $\zeta_{\mathcal{T}}$, hence

$$\int_{\Omega} \phi_h^n = \int_{\Omega} \bar{\phi}_h^n \leq |\Omega| \phi(2|\Omega|^{-1}) + C_7 \int_{\Omega} |\nabla \phi_h^n|.$$

Then the triangle inequality and Poincaré-Wirtinger inequality lead to

$$\int_{\Omega} |\phi_h^n| \leq \left| \phi_h^n - \int_{\Omega} \phi_h^n \right| + \left| \int_{\Omega} \phi_h^n \right| \leq |\Omega| \phi(2|\Omega|^{-1}) + (C_7 + C_P) \int_{\Omega} |\nabla \phi_h^n|$$

where C_P is the constant in Poincaré-Wirtinger inequality in L^1 , which only depends on Ω . Note that for Ω convex, $C_P \leq \frac{\text{diam}(\Omega)}{2}$ owing to [1]. \square

5. EXISTENCE OF A SOLUTION TO THE SCHEME

The goal of this section is to show that the nonlinear system (27) always admits (at least) a solution ρ_h^n and μ_h^n with $\bar{\mu}_h^n \in \partial\eta(\bar{\rho}_h^n)$. Combining such an existence result with Lemma 4.1 and Proposition 4.2 gives Theorem 1.1. The proof splits into two steps. The first step deals with the case (A1) of a singular η near 0, where $\text{Dom}(\partial\eta) = (0, +\infty)$.

Proposition 5.1. *Assume that (A1) holds, then there exists (at least) one solution $\rho_h^n > 0$ to the scheme (27), and $\bar{\mu}_h^n = \eta'(\bar{\rho}_h^n)$ since $\partial\eta$ is single-valued.*

The proof of Proposition 5.1 can be carried out thanks to a topological degree argument. Since it is very similar to the one of [10, Proposition 3.8], we do not detail it here.

The situation in the case (A2) is more intricate. To show the existence of a solution (ρ_h^n, μ_h^n) in this case, the idea is to approximate η fulfilling (A2) by a sequence $(\eta_\varepsilon)_{\varepsilon>0}$ of functions entering the framework of (A1). More precisely, define for $\varepsilon > 0$ the function $\beta_\varepsilon \in C(\mathbb{R}_{\geq 0}) \cap C^1((0, +\infty))$ by setting

$$(37) \quad \beta_\varepsilon(\rho) = \int_\varepsilon^\rho \min\{0, \log(a/\varepsilon)\} da = \begin{cases} \rho \log \rho - \rho(\log \varepsilon + 1) + \varepsilon & \text{if } \rho \leq \varepsilon, \\ 0 & \text{else,} \end{cases}$$

and by

$$(38) \quad \eta_\varepsilon = \eta + \beta_\varepsilon, \quad \varepsilon > 0.$$

The function η_ε enters the framework (A1) and converges uniformly towards η as ε goes to 0 since $0 \leq \beta_\varepsilon \leq \varepsilon$. Proposition 5.1 provides a sequence $(\rho_{\varepsilon,h}^n)_{\varepsilon>0}$ of solutions to (27) where μ_h^n has been replaced by $\mu_{\varepsilon,h}^n$ with $\bar{\mu}_{\varepsilon,h}^n = \eta'_\varepsilon(\bar{\rho}_{\varepsilon,h}^n)$. Since $V_h \cap \mathcal{P}(\Omega)$ to which $\rho_{\varepsilon,h}^n$ belongs is closed and bounded in the finite dimensional space V_h , there exists $\rho_h^n \in V_h \cap \mathcal{P}(\Omega)$ such that, up to a subsequence,

$$(39) \quad \rho_{\varepsilon,h}^n \rightarrow \rho_h^n \quad \text{as } \varepsilon \rightarrow 0.$$

Since we are in a finite dimensional setting, the above convergence holds true for any norm. The convergence of $\eta'_\varepsilon(\bar{\rho}_{\varepsilon,h}^n)$ is unclear. However, we can still prove the following result.

Proposition 5.2. *Let ρ_h^n be defined by (39), then there exists μ_h^n with $\bar{\mu}_h^n \in \partial\eta(\bar{\rho}_h^n)$ such that (27) holds true.*

Proof. Let us split the set \mathcal{V} of the vertices into three disjointed subsets

$$\mathcal{V} = \mathcal{V}_+^n \cup \mathcal{V}_0^n \cup \mathcal{V}_\partial^n, \quad n \geq 1,$$

where we have set

$$\begin{aligned} \mathcal{V}_+^n &= \{a \in \mathcal{V} \mid \rho_h^n(a) > 0\}, \\ \mathcal{V}_0^{\varepsilon,n} &= \{a \in \mathcal{V} \mid \rho_{\varepsilon,h}^n(a) = 0 \text{ and } \rho_h^n(a') = 0 \text{ for all } a' \in \mathcal{V} \text{ sharing an edge with } a\}, \\ \mathcal{V}_\partial^{\varepsilon,n} &= \{a \in \mathcal{V} \mid \rho_{\varepsilon,h}^n(a) = 0 \text{ and } \rho_h^n(a') > 0 \text{ for some } a' \in \mathcal{V} \text{ sharing an edge with } a\}. \end{aligned}$$

Since η'_ε converges uniformly towards η' on every compact subset of $(0, +\infty)$, one deduces that

$$(40) \quad \mu_{\varepsilon,h}^n(a) \xrightarrow{\varepsilon \rightarrow 0} \eta'(\rho_h^n(a)) \quad \forall a \in \mathcal{V}_+^n.$$

Consider now a vertex $a \in \mathcal{V}_\partial^n$, then there exists $T \in \mathcal{T}$ with $a \in \mathcal{V}_T$ and $a' \in \mathcal{V}_+^n \cap \mathcal{V}_T$. As a consequence, $\tilde{\rho}_h^n \geq \frac{\rho_h^n(a')}{d+1} > 0$ on T , and $\tilde{\rho}_{\varepsilon,h}^n$ is uniformly bounded away from 0. Therefore, using Corollary 4.4 and (31), one deduces that

$$|\delta^T \mu_{\varepsilon,n}|^q \leq |J_T|_2 |J_T^{-1} \delta^T \mu_{\varepsilon,n}|^q = |J_T|_2 \int_T |\nabla \mu_{\varepsilon,h}^n|^q \leq C, \quad \forall \varepsilon > 0,$$

with C depending on the data of the continuous problem and on the discretization, but not on ε . Therefore, we deduce that

$$\mu_{\varepsilon,h}^n(a) \geq \mu_{\varepsilon,h}^n(a') - C \xrightarrow{\varepsilon \rightarrow 0} \eta'(\rho_h^n(a')) - C > \eta'(0) - C,$$

while $\limsup_{\varepsilon \rightarrow 0} \mu_{\varepsilon,h}^n(a) \leq \eta'(0)$. So the sequence $(\mu_{\varepsilon,h}^n(a))_{\varepsilon > 0}$ is bounded, and we can extract a converging subsequence, hence

$$(41) \quad \mu_{\varepsilon,h}^n(a) \xrightarrow{\varepsilon \rightarrow 0} \mu_h^n(a) \in \partial\eta(0) = (-\infty, \eta'(0)], \quad \forall a \in \mathcal{V}_\partial^n.$$

We proceed differently for $a \in \mathcal{V}_0^n$ since we do not build $\mu_h^n(a)$ as the limit of $\mu_{\varepsilon,h}^n(a)$. Instead, we show that the choice of $\mu_h^n(a) \in \partial\eta(0)$ does not affect the scheme (27). Denote by

$$\mathcal{T}_0^n = \{T \in \mathcal{T} \mid \mathcal{V}_T \subset \mathcal{V}_0^n\} \quad \text{and} \quad \mathcal{T}_+^n = \mathcal{T} \setminus \mathcal{T}_0^n.$$

The elements $T \in \mathcal{T}_0^n$ are those where $\tilde{\rho}_{\varepsilon,h}^n$ tends to 0 with ε . Given $w_h \in V_h$, the term

$$\Lambda_\varepsilon^n = \tau^n \sum_{T \in \mathcal{T}_0^n} \int_T \rho_{\varepsilon,h}^n |\nabla(\mu_{\varepsilon,h}^n + \Psi)|^{q-2} \nabla(\mu_{\varepsilon,h}^n + \Psi) \cdot \nabla w_h$$

appearing in (27) applied for the augmented energy profile η_ε defined by (38) instead of η tends to 0 since Hölder's inequality shows that

$$(42) \quad \Lambda_\varepsilon^n \leq \left(\int_\Omega \tau^n \tilde{\rho}_{\varepsilon,h}^n |\nabla(\mu_{\varepsilon,h}^n + \Psi)|^q \right)^{\frac{1}{p}} \left(\sum_{T \in \mathcal{T}_0^n} \int_T |\nabla w_h|^q \tilde{\rho}_{\varepsilon,h}^n \right)^{\frac{1}{q}} \xrightarrow{\varepsilon \rightarrow 0} 0$$

since the first integral is bounded thanks to Proposition 4.2 and the second tends to 0. Therefore, owing to (39)–(42), passing to the limit $\varepsilon \rightarrow 0$ in the scheme (27) yields

$$\int_\Omega (\bar{\rho}_h^n - \bar{\rho}_h^{n-1}) \bar{w}_h + \tau^n \sum_{T \in \mathcal{T}_+^n} \int_T \tilde{\rho}_h^n |\nabla(\mu_h^n + \Psi_h)|^{q-2} \nabla(\mu_h^n + \Psi_h) \cdot \nabla w_h = 0,$$

while $\tilde{\rho}_h^n = 0$ on $\bigcup_{T \in \mathcal{T}_0^n} T$. Therefore we can fix $\mu_h^n(a)$, $a \in \mathcal{V}_T^0$ arbitrarily in $\partial\eta(0)$ without affecting the scheme. In particular, (27) holds if one sets $\mu_h^n(a) = \eta'(0)$ for all $a \in \mathcal{V}_0^n$. This concludes the proof of Proposition 5.2, and then of Theorem 1.1. \square

6. CONVERGENCE TOWARDS AN EDI SOLUTION

The goal of this section is to establish Theorem 1.2 that is the convergence of approximate solutions towards a solution of the continuous problem when the discretization parameters h, τ tend to 0. Our proof is based on compactness. Since we investigate the limit $h, \tau \rightarrow 0$, we can assume without loss of generality that $\tau \leq 1$.

6.1. Compactness properties. Our first lemma is about controlling the time variations of $\rho_{h\tau}$. For this purpose, define the piecewise linear in time and piecewise constant in space reconstruction $\hat{\rho}_{h\tau}$ defined by

$$(43) \quad \hat{\rho}_{h\tau}(t) = \bar{\rho}_h^{n-1} + \frac{t - t^{n-1}}{\tau^n} (\bar{\rho}_h^n - \bar{\rho}_h^{n-1}) \quad \text{if } t \in [t^{n-1}, t^n].$$

Lemma 6.1. *There exists C_8 depending only on $\xi_{\mathcal{T}}$, $\mathcal{E}(\rho^0)$, Ψ , Ω and p such that, for all $\varphi_{h\tau}$ in $V_{h\tau}$ such that $\varphi_{h\tau}(t) = 0$ if $t \geq t_*$ for some $t_* > 0$, there holds*

$$(44) \quad \int_{\mathbb{R}_{\geq 0}} \int_{\Omega} \partial_t \hat{\rho}_{h\tau} \bar{\varphi}_{h\tau} \leq C_8 \|\nabla \varphi_{h\tau}\|_{\infty}.$$

In particular, for all $\varphi \in W^{1,\infty}(\Omega)$, there holds

$$(45) \quad \int_{\Omega} (\bar{\rho}_{h\tau}(t) - \bar{\rho}_{h\tau}(s)) \varphi \leq C_1 C_8 (|t - s| + \tau)^{\frac{1}{q}} \|\nabla \varphi\|_{\infty}, \quad \forall \varphi \in W^{1,\infty}(\Omega).$$

Proof. Due to the definition (43) of $\hat{\rho}_{h\tau}$, the left-hand side in (44) rewrites

$$\begin{aligned} \int_{\mathbb{R}_{\geq 0}} \int_{\Omega} \partial_t \hat{\rho}_{h\tau} \bar{\varphi}_{h\tau} &= \sum_{n \geq 1} \int_{\Omega} (\bar{\rho}_h^n - \bar{\rho}_h^{n-1}) \bar{\varphi}_h^n \\ &= \sum_{n \geq 1} \tau^n \int_{\Omega} \rho_h^n |\nabla(\mu_h^n + \Psi_h)|^{q-2} \nabla(\mu_h^n + \Psi_h) \cdot \nabla \varphi_h^n. \end{aligned}$$

Applying Hölder's inequality then provides

$$\int_{\mathbb{R}_{\geq 0}} \int_{\Omega} \partial_t \hat{\rho}_{h\tau} \bar{\varphi}_{h\tau} \leq \left(\sum_{n \geq 1} \tau^n \int_{\Omega} \rho_h^n |\nabla(\mu_h^n + \Psi_h)|^q \right)^{\frac{1}{p}} \left(\sum_{n \geq 1} \tau^n \int_{\Omega} \rho_h^n |\nabla \varphi_h^n|^q \right)^{\frac{1}{q}}.$$

The first term in the right-hand side is bounded by $C_9 = (\mathcal{E}(\rho^0) + C_2 \text{diam}(\Omega))^{\frac{1}{p}}$ owing to Proposition 4.2. Using moreover Lemma 4.1 and the fact that $\nabla \varphi_{h\tau}$ is compactly supported to estimate the second term, we obtain

$$\int_{\mathbb{R}_{\geq 0}} \int_{\Omega} \partial_t \hat{\rho}_{h\tau} \bar{\varphi}_{h\tau} \leq C_9 \|\nabla \varphi_{h\tau}\|_{\infty} (t_* + \tau).$$

One recovers (44) if one further uses that $\tau \leq 1$.

Let us now focus on (45). Let $0 \leq s \leq t$, and let $m \leq n$ be such that $t^{m-1} \leq s \leq t^m$ and $t^{n-1} \leq t \leq t^n$. Given $\varphi \in W^{1,\infty}(\Omega)$, define $\bar{\varphi}_h$ as the L^2 projection of φ on X_h , i.e. $\varphi_h(a) = \int_{\omega_a} \varphi$ for all $a \in \mathcal{V}$ and $\varphi_{h\tau}(t, x) = \varphi_h(x) \mathbf{1}_{(t^{m-1}, t^n]}(t)$. Then

$$\int_{\Omega} (\bar{\rho}_{h\tau}(t) - \bar{\rho}_{h\tau}(s)) \varphi = \int_{\Omega} (\bar{\rho}_{h\tau}(t) - \bar{\rho}_{h\tau}(s)) \bar{\varphi}_h = \iint_{\mathbb{R}_{\geq 0} \times \Omega} \partial_t \hat{\rho}_{h\tau} \bar{\varphi}_{h\tau}.$$

Reproducing the above calculations, we obtain that

$$\int_{\Omega} (\bar{\rho}_{h\tau}(t) - \bar{\rho}_{h\tau}(s)) \varphi \leq C_9 \|\nabla \varphi_h\|_{\infty} \left(\sum_{k=m+1}^n \tau^k \right)^{\frac{1}{q}} \leq C_9 \|\nabla \varphi_h\|_{\infty} (t - s + \tau)^{\frac{1}{q}}.$$

Then (45) follows from estimate $\|\nabla \varphi_h\|_{\infty} \leq C_1 \|\nabla \varphi\|_{\infty}$ established in the proof of [11, Theorem A.1]. \square

Our next result is about the following uniform in time but weak in space compactness property.

Lemma 6.2. *There exists $\rho \in C(\mathbb{R}_{\geq 0}; (\mathcal{P}(\Omega), \mathbb{W}_p))$ with $\int_{\Omega} \eta(\rho(t)) \leq C_3$ for all $t \geq 0$ such that, as $h, \tau \rightarrow 0$, up to a subsequence,*

$$\mathbb{W}_p(\bar{\rho}_{h\tau}(t), \rho(t)) \rightarrow 0 \quad \text{locally uniformly with respect to } t \geq 0.$$

Proof. First, Corollary 4.3 implies that

$$\int_{\Omega} \eta(\rho_{h\tau}(t)) \leq C_3, \quad \forall t \geq 0,$$

so that owing to de La Vallée Poussin theorem, $\bar{\rho}_{h\tau}(t)$ is tight and hence pre-compact in $(\mathcal{P}(\Omega), \mathbb{W}_p)$ for all $t \geq 0$. Second, Lemma 6.1 implies the following estimate in the L^1 -Wasserstein metric \mathbb{W}_1 :

$$\mathbb{W}_1(\bar{\rho}_{h\tau}(t), \bar{\rho}_{h\tau}(s)) = \sup_{\varphi \in W^{1,\infty}(\Omega)} \frac{\int_{\Omega} (\bar{\rho}_{h\tau}(t) - \bar{\rho}_{h\tau}(s)) \varphi}{\|\nabla \varphi\|_{\infty}} \leq C_1 C_8 (|t - s| + \tau)^{\frac{1}{q}}$$

for all $t, s \geq 0$. By means of (9), this implies

$$\mathbb{W}_p(\bar{\rho}_{h\tau}(t), \bar{\rho}_{h\tau}(s)) \leq C_1 C_8 (\text{diam}(\Omega))^{1/q} (|t - s| + \tau)^{1/(pq)}.$$

This is sufficient to apply the Arzelà-Ascoli theorem in metric spaces (see e.g. [5, Theorem 3.3.1]) and conclude local uniform convergence to a (Hölder-)continuous limit ρ . \square

One can further establish the point-wise convergence of the densities, as well as some weak convergence on the gradient of $\phi_{h\tau}$.

Proposition 6.3. *Up to the extraction of yet another subsequence, there holds*

$$(46) \quad \bar{\rho}_{h\tau} \xrightarrow{h, \tau \rightarrow 0} \rho \quad \text{a.e. in } \mathbb{R}_{\geq 0} \times \Omega \quad \text{and in } L^1_{loc}(\mathbb{R}_{\geq 0} \times \bar{\Omega}).$$

Moreover, $\phi(\rho) \in L^q_{loc}(\mathbb{R}_{\geq 0}; \mathbf{BV}(\Omega)) \cap L^1_{loc}(\mathbb{R}_{\geq 0}; W^{1,1}(\Omega))$ since

$$(47) \quad \nabla \phi_{h\tau} \xrightarrow{h, \tau \rightarrow 0} \nabla \phi(\rho) \quad \text{weakly in } L^1_{loc}(\mathbb{R}_{\geq 0} \times \bar{\Omega})^d.$$

and

$$(48) \quad \nabla \phi_{h\tau} \xrightarrow{h, \tau \rightarrow 0} \nabla \phi(\rho) \quad \text{in the } L^q_{loc}(\mathbb{R}_{\geq 0}; \mathcal{M}(\bar{\Omega}))^d \text{ weak-}\star \text{ sense.}$$

Proof. The first step of the proof consists in proving the relative compactness of the sequence $(\phi_{h\tau})_{h,\tau}$ in $L^1_{loc}(\mathbb{R}_{\geq 0}; W^{1,1}(\Omega))$ equipped with its weak topology. In view of Lemma 4.7, it is sufficient to show that the sequence $(\nabla \phi_{h\tau})_{h,\tau}$ is uniformly equi-integrable in $L^1((0, t_{\star}) \times \Omega)^d$ whatever $t_{\star} > 0$. To this end, one remarks that Lemma 4.6 implies

$$(49) \quad |\nabla \phi_{h\tau}| \leq C_5 (\tilde{\rho}_{h\tau})^{\frac{1}{p}} (\tilde{\rho}_{h\tau})^{\frac{1}{q}} |\nabla \mu_{h\tau}|.$$

It follows from Corollary 4.4 that $(\tilde{\rho}_{h\tau})^{\frac{1}{q}} |\nabla \mu_{h\tau}|$ is bounded in $L^q((0, t_{\star}) \times \Omega)$, whereas $(\tilde{\rho}_{h\tau})^{\frac{1}{p}}$ is bounded in $L^{\infty}((0, t_{\star}); L^p(\Omega))$ owing to Lemma 4.1. Thus

$$\|\nabla \phi_{h\tau}\|_{L^q((0, t_{\star}); L^1(\Omega))^d} \leq C_{10}$$

with $C_{10} = C_5 C_4^{\frac{1}{q}} (1 + t_{\star} + \tau)^{\frac{1}{q}}$. Using Corollary 4.3 together with de La Vallée Poussin theorem, one checks that $(\tilde{\rho}_{h\tau}(t, \cdot))_{h,\tau}$ is uniformly equi-integrable on Ω .

Then so does $(\nabla\phi_{h\tau})_{h,\tau}$ in $L^1((0, t_\star)\times\Omega)^d$ thanks to (49). Therefore we deduce from Dunford-Pettis theorem that there exists some F such that, up to a subsequence,

$$(50) \quad \nabla\phi_{h\tau} \xrightarrow{h,\tau\rightarrow 0} F \quad \text{weakly in } L^1((0, t_\star)\times\Omega)^d.$$

On the other hand, since $\nabla\phi_{h\tau}$ is bounded in $L^q((0, t_\star); L^1(\Omega))^d$, we deduce that

$$(51) \quad \nabla\phi_{h\tau} \xrightarrow{h,\tau\rightarrow 0} F \quad \text{in the } L^q((0, t_\star), \mathcal{M}(\overline{\Omega}))^d \text{ weak-}\star \text{ sense.}$$

Let $\theta : \mathbb{R} \rightarrow \mathbb{R}$ be Lipschitz continuous, increasing and bounded, then denoting by $\psi = \theta \circ \phi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$, one has

$$(52) \quad \|\overline{\psi}_{h\tau}\|_{L^\infty((0, t_\star)\times\Omega)} \leq \|\theta\|_\infty, \quad \|\nabla\psi_{h\tau}\|_{L^1((0, t_\star)\times\Omega)^d} \leq \|\theta'\|_\infty C_{10} T^{\frac{1}{p}}.$$

As a consequence, there exists $\psi_\star \in L^\infty((0, t_\star)\times\Omega) \cap L^1((0, t_\star); W^{1,1}(\Omega))$ such that

$$\overline{\psi}_{h\tau} \xrightarrow{h,\tau\rightarrow 0} \psi_\star \quad \text{in the } L^\infty((0, t_\star)\times\Omega) \text{ weak-}\star \text{ sense.}$$

Combining the estimates (52) with Lemma 6.1 and Proposition 6.2, and recalling that $\overline{\psi}_{h\tau} = \psi(\overline{\rho}_{h\tau})$ thanks to the mass-lumping procedure, one can make use of [7, Theorem 3.9] (see also [11, Theorem A.1]) which, since ψ is continuous and increasing, shows that $\overline{\rho}_{h\tau}$ converges pointwise towards ρ . As a consequence, $\overline{\phi}_{h\tau}$ as well as $\phi_{h\tau}$ tend to $\phi(\rho)$, so that one can identify F as $\nabla\phi(\rho)$. Moreover, the sequence $(\overline{\rho}_{h\tau})_{h,\tau}$ being uniformly equi-integrable owing to Corollary 4.3 combined with de La Vallée Poussin theorem, it converges strongly in $L^1((0, t_\star)\times\Omega)$. \square

The next lemma shows that any reasonable reconstruction of the density converges towards the same limit ρ as the one exhibited in Propositions 6.2 and 6.3.

Lemma 6.4. *Define $\check{\rho}_{h\tau}, \hat{\rho}_{h\tau} \in \tilde{X}_{h\tau}$ respectively by*

$$(53) \quad \begin{cases} \check{\rho}_{h\tau}(t, x) = \max_{y \in T} \rho_h^n(y), \\ \hat{\rho}_{h\tau}(t, x) = \min_{y \in T} \rho_h^n(y), \end{cases} \quad (t, x) \in (t_{n-1}, t_n] \times T, \quad T \in \mathcal{T},$$

then, up to a subsequence,

$$\check{\rho}_{h\tau} \xrightarrow{h,\tau\rightarrow 0} \rho \quad \text{and} \quad \hat{\rho}_{h\tau} \xrightarrow{h,\tau\rightarrow 0} \rho \quad \text{in } L^1_{loc}(\mathbb{R}_{\geq 0} \times \overline{\Omega}).$$

Proof. Define $\check{\phi}_{h\tau} = \phi(\check{\rho}_{h\tau})$ and $\hat{\phi}_{h\tau} = \phi(\hat{\rho}_{h\tau})$, then $\check{\phi}_{h\tau}, \hat{\phi}_{h\tau} \in \tilde{X}_{h\tau}$ are such that

$$(54) \quad \hat{\phi}_{h\tau} \leq \overline{\phi}_{h\tau} \leq \check{\phi}_{h\tau}.$$

Using in the mean value inequality, one gets that

$$(55) \quad 0 \leq \check{\phi}_{h\tau} - \hat{\phi}_{h\tau} \leq |\nabla\phi_{h\tau}| h_T \quad \text{a.e. in } \mathbb{R}_{\geq 0} \times T.$$

Besides, it follows from Proposition 6.3 and Lemma 4.7 that

$$(56) \quad \overline{\phi}_{h\tau} \xrightarrow{h,\tau\rightarrow 0} \phi(\rho) \quad \text{in } L^1_{loc}(\mathbb{R}_{\geq 0} \times \overline{\Omega}).$$

We deduce from (54)–(56) that $\check{\phi}_{h\tau}$ and $\hat{\phi}_{h\tau}$ converge towards $\phi(\rho)$ in $L^1_{loc}(\mathbb{R}_{\geq 0} \times \overline{\Omega})$ too, so that the convergence also holds point-wise almost everywhere up to the extraction of yet another subsequence. Since ϕ^{-1} is continuous, we obtain that

$$\check{\rho}_{h\tau} \xrightarrow{h,\tau\rightarrow 0} \rho \quad \text{and} \quad \hat{\rho}_{h\tau} \xrightarrow{h,\tau\rightarrow 0} \rho \quad \text{a.e. in } \mathbb{R}_{\geq 0} \times \Omega.$$

Finally, remark that

$$0 \leq \hat{\rho}_{h\tau} \leq \check{\rho}_{h\tau} \leq (d+1)\tilde{\rho}_{h\tau},$$

so that $\hat{\rho}_{h\tau}$ and $\check{\rho}_{h\tau}$ are tight in $L^1((0, t_\star) \times \Omega)$ whatever $t_\star > 0$ thanks to Corollary 4.3. Then one can apply Vitali's convergence theorem and conclude the proof of Lemma 6.4. \square

6.2. Identification of the limit. The purpose of this section is to identify the limiting curve $t \mapsto \rho(t)$ exhibited at Proposition 6.2 as a p -gradient flow in the sense of (4).

Lemma 6.5. *Up to a subsequence, there holds*

$$\tilde{\rho}_{h\tau} \nabla \mu_{h\tau} \xrightarrow{h, \tau \rightarrow 0} \nabla \phi(\rho) \quad \text{weakly in } L^1_{\text{loc}}(\mathbb{R}_{\geq 0} \times \bar{\Omega})^d.$$

Proof. In view of (47), it is (more than) sufficient to establish that

$$(57) \quad \tilde{\rho}_{h\tau} \nabla \mu_{h\tau} - \nabla \phi_{h\tau} \xrightarrow{h, \tau \rightarrow 0} 0 \quad \text{in } L^1_{\text{loc}}(\mathbb{R}_{\geq 0} \times \bar{\Omega})^d.$$

Let $T \in \mathcal{T}$ and $n \geq 1$, then, building on the material introduced in the proof of Lemma 4.6, one has

$$\int_T |\tilde{\rho}_h^n \nabla \mu_h^n - \nabla \phi_h^n| = d! |T| |J_T^{-1} (\tilde{\rho}_T^n I_d - R_T^n) \delta^T \mu_h^n|,$$

where I_d is the identity matrix of $R^{d \times d}$, and where

$$\tilde{\rho}_T^n = \rho_h^n(x_T) = \frac{1}{d+1} \sum_{a \in \mathcal{V}_T} \rho_h^n(a).$$

Then we deduce from (31) that

$$\begin{aligned} \int_T |\tilde{\rho}_h^n \nabla \mu_h^n - \nabla \phi_h^n| &\leq d! |T| |J_T^{-1}|_2 |J_T|_2 \left| (\tilde{\rho}_T^n)^{\frac{1}{p}} I_d - (\tilde{\rho}_T^n)^{-\frac{1}{q}} R_T^n \right|_2 \left| (\tilde{\rho}_T^n)^{\frac{1}{q}} J_T^{-1} \delta^T \mu_h^n \right| \\ &\leq \zeta_{\mathcal{T}} d! \frac{d + \sqrt{d}}{\sqrt{2}} \int_T \left| (\tilde{\rho}_T^n)^{\frac{1}{p}} I_d - (\tilde{\rho}_T^n)^{-\frac{1}{q}} R_T^n \right|_2 \left| (\tilde{\rho}_h^n)^{\frac{1}{q}} \nabla \mu_h^n \right|. \end{aligned}$$

Note that the term $(\tilde{\rho}_T^n)^{-\frac{1}{q}} R_T^n$ vanishes if $\tilde{\rho}_T^n = 0$ thanks to (33) since $q > 1$. Multiplying the above estimate by τ^n , summing over $T \in \mathcal{T}$ and $n = 1, \dots, n_\star$, applying Hölder inequality and Corollary 4.4 yields

$$(58) \quad \|\tilde{\rho}_{h\tau} \nabla \mu_{h\tau} - \nabla \phi_{h\tau}\|_{L^1((0, t_{n_\star}) \times \Omega)} \leq \zeta_{\mathcal{T}} d! \frac{d + \sqrt{d}}{\sqrt{2}} C_4^{\frac{1}{q}} (1 + t_{n_\star})^{\frac{1}{q}} \|\tilde{\gamma}_{h\tau}\|_{L^p((0, t_{n_\star}) \times \Omega)}$$

with $t_{n_\star} = \sum_{n=1}^{n_\star} \tau^n$ and

$$\tilde{\gamma}_{h\tau}(t, x) = (\tilde{\rho}_T^n)^{\frac{1}{p}} I_d - (\tilde{\rho}_T^n)^{-\frac{1}{q}} R_T^n \quad \text{if } (t, x) \in (t_{n-1}, t_n) \times T.$$

Owing to (33), to $\tilde{\rho}_{h\tau} \leq \check{\rho}_{h\tau}$, to the concavity and to the subadditivity of $u \mapsto p u^{\frac{1}{p}}$ there holds

$$|\tilde{\gamma}_{h\tau}|_2 \leq (\hat{\rho}_{h\tau})^{-\frac{1}{q}} |\check{\rho}_{h\tau} - \hat{\rho}_{h\tau}| \leq p \left| (\check{\rho}_{h\tau})^{\frac{1}{p}} - (\hat{\rho}_{h\tau})^{\frac{1}{p}} \right| \leq p |\check{\rho}_{h\tau} - \hat{\rho}_{h\tau}|^{\frac{1}{p}}.$$

Therefore, Lemma 6.4 implies that $|\tilde{\gamma}_{h\tau}|_2$ tends to 0 in $L^p_{\text{loc}}(\mathbb{R}_{\geq 0} \times \bar{\Omega})$, so that (57) holds thanks to (58). \square

Lemma 6.6. *Up to a subsequence, there holds*

$$\tilde{\rho}_{h\tau} \nabla \Psi_h \xrightarrow{h, \tau \rightarrow 0} \rho \nabla \Psi \quad \text{weakly in } L^1_{\text{loc}}(\mathbb{R}_{\geq 0} \times \overline{\Omega})^d.$$

Proof. Since $\hat{\rho}_{h\tau} \leq \tilde{\rho}_{h\tau} \leq \check{\rho}_{h\tau}$ with $\hat{\rho}_{h\tau}$ and $\check{\rho}_{h\tau}$ defined by (53), we deduce from Lemma 6.4 that

$$\tilde{\rho}_{h\tau} \xrightarrow{h, \tau \rightarrow 0} \rho \quad \text{in } L^1_{\text{loc}}(\mathbb{R}_{\geq 0} \times \overline{\Omega}).$$

On the other hand, because of (24), $\nabla \Psi_h$ is uniformly bounded in $L^\infty(\Omega)^d$, hence it converges (up to a subsequence) towards some U in the $L^\infty(\Omega)^d$ -weak- \star sense. Finally one readily check that $U = \nabla \Psi$ in the distributional sense since Ψ_h converges uniformly towards Ψ . \square

We are now in the position to estimate the slope term in the energy inequality for ρ .

Lemma 6.7. *There holds*

$$(59) \quad \int_{t_1}^{t_2} |\partial \mathcal{E}(\rho)|^q(t) dt \leq \liminf_{h, \tau \rightarrow 0} \int_{t_1}^{t_2} \int_{\Omega} \rho_{h\tau} |\nabla(\mu_{h\tau} + \Psi_h)|^q, \quad 0 \leq t_1 < t_2.$$

Proof. For each $T > 0$, define probability measures $\nu_{h\tau} \in \mathcal{P}(\mathbb{R}^{d+1})$ and vector fields $\xi_{h\tau} : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1}$ by

$$(60) \quad \nu_{h\tau} = \begin{cases} \frac{1}{T} \tilde{\rho}_{h\tau} & \text{on } (0, T) \times \Omega, \\ 0 & \text{outside,} \end{cases}, \quad \xi_{h\tau} = \begin{cases} (0, \mathbf{u}_{h\tau}) & \text{on } (0, T) \times \Omega, \\ 0 & \text{outside,} \end{cases}$$

with $\mathbf{u}_{h\tau} = \nabla(\mu_{h\tau} + \Psi_h) : \mathbb{R}_{\geq 0} \times \Omega \rightarrow \mathbb{R}^d$. The basic energy estimate in Proposition 4.2 implies a (h, τ) -uniform bound

$$(61) \quad \iint_{\mathbb{R}^{d+1}} \nu_{h\tau} |\xi_{h\tau}|^q \leq C_{??}.$$

Further, by Proposition 6.2, the measures $\nu_{h\tau}$ converge narrowly to $\nu = \frac{1}{T} \rho$ as h, τ tend to 0. Now apply Proposition A.2 from the appendix: there exists a limiting vector field $\xi : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1}$ with $\rho |\xi|^q \in L^1((0, T) \times \Omega)$, such that $\nu_{h\tau} \xi_{h\tau}$ converges narrowly towards $\frac{1}{T} \rho \xi$. One readily checks that $\xi = (0, \mathbf{u})$ on $(0, T) \times \Omega$. Taking a sequence $T \rightarrow \infty$, one extends \mathbf{u} to $\mathbb{R}_{\geq 0} \times \Omega$ by the usual diagonal argument.

Proposition A.2 thus yields further that

$$(62) \quad \int_{t_1}^{t_2} \int_{\Omega} \rho |\mathbf{u}|^q dx dt \leq \liminf_{h\tau \rightarrow 0} \int_{t_1}^{t_2} \int_{\Omega} \tilde{\rho}_{h\tau} |\mathbf{u}_{h\tau}|^q dx dt.$$

By the above, $\rho \mathbf{u}$ is the narrow limit of $\tilde{\rho}_{h\tau} \nabla(\mu_{h\tau} + \Psi_h)$. In Lemmas 6.5 and 6.6 above, this limit has been identified as $\nabla \phi + \rho \nabla \Psi$. We can thus use the characterization of the metric slope given in Proposition 2.1: at almost every $t \in (t_1, t_2)$, we have

$$|\partial \mathcal{E}|^q(\rho(t)) = \int_{\Omega} \rho(t, x) |\mathbf{u}(t, x)|^q dx,$$

and thanks to (62), this implies (59), since $\tilde{\rho}_{h\tau}$ and $\rho_{h\tau}$ yield the same integral values. \square

Next, we identify the metric velocity.

Lemma 6.8. ρ is an absolutely continuous curve from $\mathbb{R}_{\geq 0}$ to $\mathbb{W}_p(\Omega)$, and its metric velocity $|\rho'| : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ satisfies

$$(63) \quad \int_{t_1}^{t_2} |\rho'|^p \leq \liminf_{h, \tau \rightarrow 0} \int_{t_1}^{t_2} \int_{\Omega} \tilde{\rho}_{h\tau} |\nabla(\mu_{h\tau} + \Psi_h)|^q dx, \quad 0 \leq t_1 < t_2.$$

Proof. Proceed as in the proof of Lemma 6.7, with

$$\mathbf{v}_{h\tau} = |\nabla(\mu_{h\tau} + \Psi_h)|^{q-2} \nabla(\mu_{h\tau} + \Psi_h)$$

instead of $\mathbf{u}_{h\tau}$ in the definition (60) of $\xi_{h\tau}$. Observe that $|\mathbf{v}_{h\tau}|^p = |\mathbf{u}_{h\tau}|^q$ by construction, and since both are constant on triangular cells, the bound (61) remains the same, upon replacing p by q . Proposition A.2 eventually implies narrow convergence of $\tilde{\rho}_{h\tau} \mathbf{v}_{h\tau}$ to a limit of the form $\rho \mathbf{v}$, with $\rho |\mathbf{v}|^p \in L^1$. Additionally, still from Proposition A.2, one obtains

$$\int_{t_1}^{t_2} \int_{\Omega} \rho |\mathbf{v}|^p \leq \liminf_{h, \tau \rightarrow 0} \int_{t_1}^{t_2} \int_{\Omega} \tilde{\rho}_{h\tau} |\mathbf{v}_{h\tau}|^p.$$

In order to estimate the metric velocity with the left-hand side above by means of Proposition 2.2, we need to show that ρ and \mathbf{v} satisfy the continuity equation

$$(64) \quad \partial_t \rho = \nabla \cdot \rho \mathbf{v}$$

in the distributional sense on $\mathbb{R}_{\geq 0} \times \bar{\Omega}$. We shall derive (64) directly from the weak formulation (27) of the scheme. To that end, let $\varphi \in C_c^\infty(\mathbb{R}_{>0} \times \bar{\Omega})$ be given, and define accordingly $\varphi_{h\tau}$ as the piecewise-constant-in-time and piecewise-affine-in-space function such that for all $t \in (t^{n-1}, t_n]$, the function $\varphi_{h\tau}(t; \cdot)$ is given by affine interpolation on the triangles of the values $\varphi(t_{n-1}, a)$ at the vertices $a \in \mathcal{V}$. Introduce further the temporal difference quotient

$$\delta_t \bar{\varphi}_{h\tau}(t, x) := \frac{\varphi(t_n, a) - \varphi_{h\tau}(t_{n-1}, a)}{\tau_n} \quad \text{for all } t \in (t_{n-1}, t_n] \text{ and } x \in \omega_a.$$

By smoothness of φ ,

$$(65) \quad \nabla \varphi_{h\tau} \rightarrow \nabla \varphi, \quad \delta_t \bar{\varphi}_{h\tau} \rightarrow \partial_t \varphi \quad \text{uniformly a.e. on } \mathbb{R}_{>0} \times \Omega,$$

where the *almost* everywhere simply accounts for the fact that $\nabla \varphi_{h\tau}$ cannot be evaluated on edges.

Now substitute $w_h^n = \varphi_{h\tau}(t^{n-1}, \cdot)$ this into (27) and sum over $n = 1, 2, \dots$ — the sum is actually finite — to obtain

$$- \int_0^\infty \int_{\Omega} \bar{\rho}_{h\tau} \delta_t \bar{\varphi}_{h\tau} dx dt + \int_0^\infty \int_{\Omega} \tilde{\rho}_{h\tau} \mathbf{v}_{h\tau} \cdot \nabla \varphi_{h\tau} dx dt = 0.$$

By (65) above, by strong convergence of $\bar{\rho}_{h\tau}$ to ρ , and by weak convergence of $\tilde{\rho}_{h\tau} \mathbf{v}_{h\tau}$ to $\rho \mathbf{v}$, we may pass to the limit and obtain

$$- \int_0^\infty \int_{\Omega} \rho \partial_t \varphi dx dt + \int_0^\infty \int_{\Omega} \rho \mathbf{v} \cdot \nabla \varphi dx dt = 0,$$

which is just the distributional formulation of (64). The claim (63) now follows from Proposition (2.2). \square

The last element we need to recover (4) — and thus to complete the proof of Theorem 1.2 — is the following convergence of the energy $\mathcal{E}_h(\rho_{h\tau})$.

Lemma 6.9. *There exists a decreasing (thus with bounded variations) function $\mathfrak{E} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, with $\mathfrak{E} = \mathcal{E}(\rho)$ a.e. on $\mathbb{R}_{\geq 0}$ such that, up to a subsequence,*

$$(66) \quad \mathcal{E}_h(\bar{\rho}_{h\tau}(t)) \xrightarrow{h,\tau \rightarrow 0} \mathfrak{E}(t), \quad \text{for a.e. } t \geq 0.$$

Proof. As a direct consequence of the one-step energy estimate (7), the function $t \mapsto \mathcal{E}_h(\bar{\rho}_{h\tau}(t))$ is non-increasing, thus in $L^\infty \cap \text{BV}(\mathbb{R}_{\geq 0})$ since $\mathcal{E}_h(\bar{\rho}_h^0)$ is bounded after (26) and nonnegative. Helly's compactness criterion then provides the existence of $\mathfrak{E} \in L^\infty \cap \text{BV}(\mathbb{R}_{\geq 0})$ such that (66) holds.

It remains to check that $\mathfrak{E} = \mathcal{E}(\rho)$ a.e. in $\mathbb{R}_{\geq 0}$. To this end, let us decompose

$$(67) \quad \|\mathcal{E}_h(\bar{\rho}_{h\tau}) - \mathcal{E}(\rho)\|_{L^1(0,t^*)} \leq I_{h\tau} + J_{h\tau} + K_{h\tau},$$

with

$$I_{h\tau} = \|\mathcal{E}_h(\bar{\rho}_{h\tau}) - \mathcal{E}(\bar{\rho}_{h\tau})\|_{L^1(0,t^*)}, \quad J_{h\tau} = \|\eta(\bar{\rho}_{h\tau}) - \eta(\rho)\|_{L^1((0,t^*) \times \Omega)},$$

and

$$K_{h\tau} = \|\Psi\|_\infty \|\bar{\rho}_{h\tau} - \rho\|_{L^1((0,t^*) \times \Omega)}.$$

We infer from (25) that

$$(68) \quad I_{h\tau} \xrightarrow{h,\tau \rightarrow 0} 0.$$

Besides, it follows from Proposition 6.3 that

$$(69) \quad K_{h\tau} \xrightarrow{h,\tau \rightarrow 0} 0.$$

Let us now turn to the term $J_{h\tau}$. In view of Proposition 6.3, and thanks to the Sobolev injection of $\text{BV}(\Omega)$ in $L^{\frac{d}{d-1}}(\Omega)$, it holds that

$$\phi_{h\tau} \xrightarrow{h,\tau \rightarrow 0} \phi(\rho) \quad \text{weakly in } L^q_{\text{loc}}(\mathbb{R}_{\geq 0}; L^{\frac{d}{d-1}}(\Omega)).$$

Then applying [10, Lemma A.6] one gets that a similar convergence result holds for the piecewise constant reconstruction $\bar{\phi}_{h\tau}$:

$$\bar{\phi}_{h\tau} = \phi(\bar{\rho}_{h\tau}) \xrightarrow{h,\tau \rightarrow 0} \phi(\rho) \quad \text{weakly in } L^q_{\text{loc}}(\mathbb{R}_{\geq 0}; L^{\frac{d}{d-1}}(\Omega)).$$

Since $\bar{\phi}_{h\tau}$ converges pointwise towards $\phi(\rho)$ due to the continuity of ϕ , we get that

$$(70) \quad \bar{\phi}_{h\tau} \xrightarrow{h,\tau \rightarrow 0} \phi(\rho) \quad \text{strongly in } L^r_{\text{loc}}(\mathbb{R}_{\geq 0} \times \bar{\Omega}), \quad 1 \leq r < \min(q, \frac{d}{d-1}).$$

On the other hand, $\eta(\rho_{h\tau})$ tends almost everywhere in $(0, t^*) \times \Omega$ towards $\eta(\rho)$ since η is continuous. Using Assumption (13), one can readily show (see for instance [10, Lemma 3.3]) that

$$\eta(\bar{\rho}_{h\tau}) \leq \phi(\bar{\rho}_{h\tau})^\alpha + C \quad \text{for some } \alpha < \min\left(q, \frac{d}{d-1}\right) \text{ and } C > 0.$$

The above right-hand side is uniformly equi-integrable thanks to (70), whence

$$(71) \quad J_{h\tau} \xrightarrow{h,\tau \rightarrow 0} 0.$$

Combining (68), (69) and (71) in (67), we get that $\mathcal{E}_h(\bar{\rho}_{h\tau})$ converges in $L^1(0, t^*)$ towards $\mathcal{E}(\rho)$. It also converges almost everywhere up to the extraction of a subsequence, whence (66). \square

7. NUMERICAL RESULTS

7.1. On the effective resolution. We present in the next Subsection several numerical experiments using the scheme (27). All numerical results were performed with FreeFem++ (see [23]). Since the method to solve the problem is not fully classical we detail here some important steps of the algorithm.

At the n^{th} iteration, we know ρ_h^{n-1} and we search for ρ_h^n solution to the scheme (27), and so to a nonlinear system $F^n(\rho_h^n) = 0$, which is equivalent to find

$$(72) \quad \rho_h^n = \operatorname{argmin}_{\rho} \frac{1}{2} \|F^n(\rho)\|^2.$$

As explain in Section 3.2, when ρ_h^n vanishes at some nodes $a \in \mathcal{V}$, the quantity $\mu_h^n(a)$ can no longer be written in terms of $\rho_h^n(a)$. But we can see it as a minimization problem with a non-negative constraint on $\rho_h^n(a)$. Thus, we add a barrier function $\bar{\mu}_h^n$ to the derivative of the density which is increasing, convex and singular in zero. More precisely, we replace μ_h^n by $\mu(\rho_h^n) + \varepsilon \bar{\mu}(\rho_h^n)$ where $\varepsilon > 0$ is a small parameter. We then solve this problem by paying attention to the fact that the smaller the perturbation parameter ε is, the more difficult the problem is to solve. The idea is to construct a sequence of solution $(\rho_h^{n,k})_{k \geq 0}$ to the scheme (27) for a sequence of parameter $(\varepsilon_k)_{k \geq 0}$ which tends to zero. More precisely, we begin with $\varepsilon_0 > 0$ and we solve the problem using the iterative strategy described above. At iteration k we choose as initial value the solution $\rho_h^{n,k-1}$ obtained in the previous step. At the end of the step k , we stiffen the problem by setting $\varepsilon_{k+1} = \varpi \varepsilon_k$ with $\varpi < 1$. We stop the algorithm when $\varepsilon_k < \varepsilon_*$ for a given $\varepsilon_* > 0$. Then we set $\rho_h^n = \rho_h^{n,k}$.

We detail now the iterative algorithm. The main idea is to use a line search strategy. Thus, at the iteration n and step k , we know $\rho_h^{n,k-1}$ and we look for for $\rho_h^{n,k}$. It is computed as follows:

- first compute $d\rho^{n,k}$ solution to $DF(\rho_h^{n,k})d\rho^{n,k} = -F(\rho_h^{n,k})$;
- then choose $d\rho^{n,k}$ as descent direction and set

$$G(s) = \frac{d}{dt} \left(\frac{1}{2} \|F(\rho_h^{n,k} + sd\rho^{n,k})\|^2 \right), \quad s \in \mathbb{R}_{\geq 0}.$$

Thus, noticing that $G(s) = \langle DF(\rho_h^{n,k} + sd\rho^{n,k})d\rho, F(\rho_h^{n,k} + sd\rho^{n,k}) \rangle$ we are able to compute $G(s)$ for any s .

- Since $G(0) = -\langle F, F \rangle < 0$, we have the following alternative:

(1) either $G(1) \leq 0$, then we use a classical Newton step and we set

$$\rho_h^{n,k+1/2} = \rho_h^{n,k} + d\rho^{n,k};$$

(2) or $G(1) > 0$, and we search for $s^{n,k} \in (0, 1)$ such that $G(s) \leq 0$. Then we set

$$\rho_h^{n,k+1/2} = \rho_h^{n,k} + s^{n,k} d\rho^{n,k}.$$

- To make sure that the solution remains away from 0, we set

$$\rho_h^{n,k+1} = \max(\rho_h^{n,k+1/2}, 10^{-8}).$$

7.2. Barenblatt solution of the q -Laplace equation. We begin by a convergence study for an analytical solution to

$$(73) \quad \partial_t \rho - \nabla \cdot (|\nabla \rho|^{q-2} \nabla \rho) = \partial_t \rho - \nabla \cdot (\rho |\nabla \eta'(\rho)|^{q-2} \nabla \eta'(\rho)) = 0, \quad \text{in } [0, t_f] \times \Omega,$$

where the internal energy density η is given by

$$(74) \quad \eta(\rho) = \begin{cases} \rho \log \rho - \rho + 1 & \text{if } q = 2, \\ \rho - 1 - \log(\rho) & \text{if } q = \frac{3}{2}, \\ \frac{q-1}{q-2} \left(\frac{q-1}{2q-3} \left(\rho^{\frac{2q-3}{q-1}} - 1 \right) - \rho + 1 \right) & \text{else.} \end{cases}$$

The corresponding chemical potential is given by

$$\eta'(\rho) = \begin{cases} \log(\rho) & \text{if } p = q = 2, \\ \frac{1}{2-p} (\rho^{2-p} - 1) & \text{else.} \end{cases}$$

The function η of (74) satisfies Assumptions (10) and (13) provided $q \geq 2$. Assume that $q > 2$ (or equivalently that $p < 2$), so that η' does not blow-up near 0, as prescribed in the case (A2). The Barenblatt profile

$$(75) \quad \rho(t, x) = (t + t_0)^{-k} \left((M - \alpha |\xi|^p)^+ \right)^{\frac{1}{2-p}}$$

solves (73), cf. [25]. In formula (75), $t_0 > 0$ is a parameter set to avoid blow-up at $t = 0$, whereas

$$(76) \quad k = \frac{1}{q-2 + \frac{q}{d}}, \quad \alpha = \frac{q-2}{q} \left(\frac{k}{d} \right)^{\frac{1}{q-1}}, \quad \xi = x(t + t_0)^{-\frac{k}{d}}.$$

The quantity M has to be set in order to ensure that $\rho(t, \cdot) \in \mathcal{P}(\Omega)$. The Barenblatt profile (75) is compactly supported, hence the no-flux boundary conditions (2) is satisfied for a finite time provided $0 \in \Omega$ and $\text{dist}(0, \partial\Omega)$ is large enough.

We study the convergence of the approximate solution towards the exact solution in the 1-dimensional case for $q = 5$ (and so $p = \frac{5}{4}$). In this case we have $M \simeq 0.6868$, the final time is fixed at $t_f = 0.25$ and we choose $t_0 = 0.05$, the barrier function is $\bar{\mu}(x) = -\frac{1}{x}$, $\varepsilon_0 = 0.5$ and $\varpi = 0.7$. Two types of convergence are studied here:

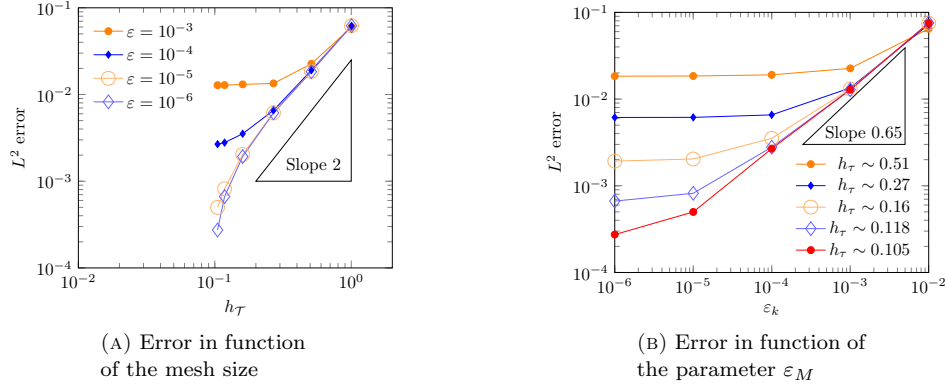
- we fix ε_* and make the mesh size tend to 0 (see Fig. 2a);
- we fix the mesh size and make the parameter ε_* tend to 0 (see Fig. 2b).

We begin with a coarse mesh containing 10 vertices and a mesh size approximately equal to 1, then we refine five times the mesh to obtain a fine mesh with 258 vertices and a mesh size approximately equal to 0.105. Furthermore, the time step associated with the coarsest mesh is 10^{-2} and when we refine the mesh, we divide the time step by 4.

In each case we plot the relative error between the approximate solution and the exact solution for the $L^2((0, t_f) \times \Omega)$ norm.

We observe on Figure 2 a second-order convergence in space and an order of convergence approximately equal to 0.65 for ε_k . As expected, we observe a saturation in the convergence w.r.t. $h_{\mathcal{T}}$ or ε_k if ε_k or $h_{\mathcal{T}}$ are respectively chosen too coarsely.

7.3. Solution of the q -Laplace equation with a non-zero potential. In this section we focus on qualitative numerical results in 2 space dimensions of the q -Laplacian equation described in (1) for a non-zero potential Ψ . For the numerical tests presented here, the energy density is given by $\eta(\rho) = \frac{1}{2}\rho^2$ and the initial density $\rho_0 \in \mathcal{P}(\Omega)$ is a cross as described on Fig. 3a. The external potential is

FIGURE 2. Relative error for the $L^2((0, t_f) \times \Omega)$ norm

$\Psi(x) = k\|x - \frac{1}{2}\|^2$ and the stationary state $\rho_\infty(x) = (M - \Psi(x))_+$ (see Fig. 3b), where the parameters M and k are chosen such that $\rho_\infty \in \mathcal{P}(\Omega)$ and such that the support of ρ_∞ , which is the ball of radius $\sqrt{\frac{M}{k}}$ centered at the point $(\frac{1}{2}, \frac{1}{2})$, is included in $\Omega = (0, 1)^2$ (here we choose $\sqrt{\frac{M}{k}} = 0.4$).

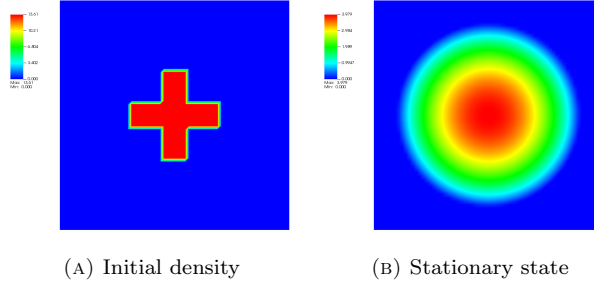


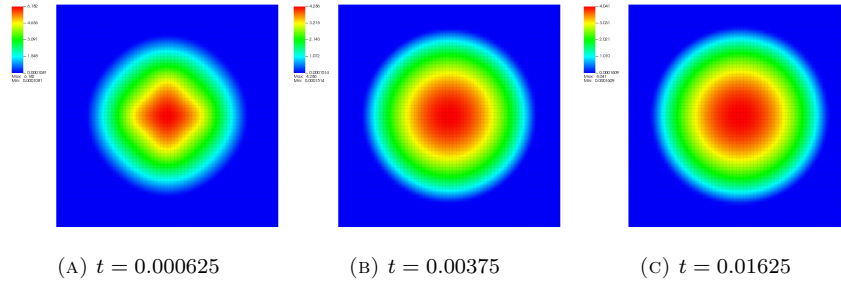
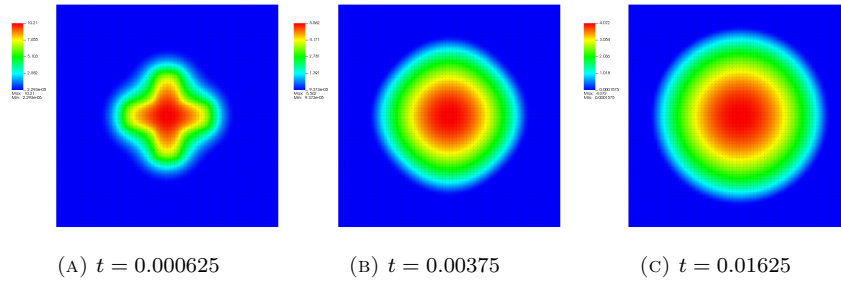
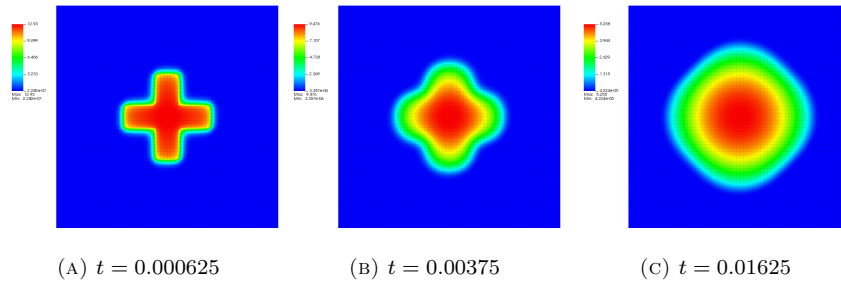
FIGURE 3. Initial density and stationary state

We choose as final time $t_f = 0.25$, the time step is $\tau = 0.000625$, the mesh has 8192 triangles and the mesh size is approximately $h_T \sim 0.022$. We observe the behavior of the solution for 3 different values of p (and q):

- $p = \frac{3}{2}$ (and so $q = 3$) on Fig. 4;
- $p = q = 2$ on Fig. 5;
- $p = 3$ (and so $q = \frac{3}{2}$) on Fig. 6.

In each case the barrier function is $\bar{\mu}(x) = -\frac{1}{x}$, $\varepsilon_\star = 10^{-3}$ but $\varepsilon_0 = 0.1$ for $p = \frac{3}{2}$ and $p = 2$ and $\varepsilon_0 = 0.2$ for $p = 3$. Note that the color scale varies from one snapshot to another in order to better visualize the behavior of the solution over time.

We also plot on Fig 7 the evolution of the energy $E(\bar{\rho}_h^n) - E(\rho_\infty)$ where $E(\rho_\infty) = \frac{2}{3}M$.

FIGURE 4. Evolution of the density for $p = \frac{3}{2}$ FIGURE 5. Evolution of the density for $p = 2$ FIGURE 6. Evolution of the density for $p = 3$

As can be seen from the Fig. 7, the energy decreases faster for $p = \frac{3}{2}$ and therefore the steady state is reached faster in this case. The saturation around 10^{-3} is related to the choice of ε_x .

APPENDIX A. SOME RESULTS BY AMBROSIO, GIGLI AND SAVARÉ

Our first proposition in this appendix is a version tuned for the proof of Proposition 6.2 of the *refined version of the Ascoli-Arzelà theorem* presented in [5, Proposition 3.3.1].

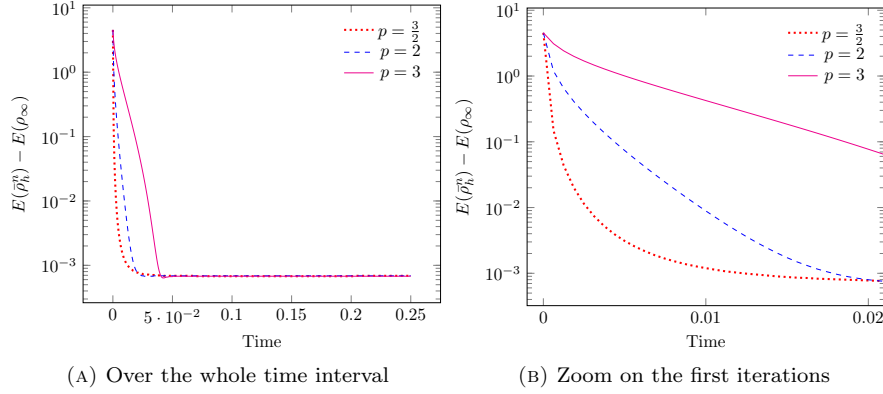


FIGURE 7. Energy dissipation

Proposition A.1. *Let $T > 0$, and for $\ell \geq 1$ let $\rho_\ell : [0, T] \rightarrow \mathcal{P}(\Omega)$ be such that $(\rho_\ell(t))_{\ell \geq 1}$ is tight for all $t \in [0, T]$ and such that*

$$\limsup_{\ell \rightarrow +\infty} \text{dist}(\rho_\ell(t), \rho_\ell(s)) \leq \omega(|t - s|), \quad \forall 0 \leq t, s \leq T,$$

with dist being a distance metrizing the narrow convergence on $\mathcal{P}(\Omega)$, and $\omega \in C(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ satisfying $\omega(0) = 0$, then there exists $\rho \in C([0, T], \mathcal{P}(\Omega))$ with

$$\text{dist}(\rho(t), \rho(s)) \leq \omega(|t - s|)$$

such that, up to a subsequence, $\rho_\ell(t)$ converges narrowly towards $\rho(t)$ as ℓ tends to $+\infty$ for all $t \in [0, T]$.

Next proposition, a more general version of which can be found in [5, Theorem 5.4.4], is about (weak) compactness for the momentum.

Proposition A.2. *Let $k \geq 1$ be an integer, let $(\nu_\ell)_{\ell \geq 1} \subset \mathcal{P}(\mathbb{R}^k)$ be such that ν_ℓ converges narrowly towards $\nu \in \mathcal{P}(\mathbb{R}^k)$ as ℓ tends to $+\infty$, and let $(\xi_\ell)_{\ell \geq 1}$ be a sequence of measurable vector fields mapping \mathbb{R}^k to \mathbb{R}^k such that there exists $C_{11} > 0$ and $r > 1$ such that*

$$\int_{\mathbb{R}^k} \nu_\ell |\xi_\ell|^r \leq C_{11}, \quad \forall \ell \geq 1.$$

Then there exists a measurable $\xi : \mathbb{R}^k \rightarrow \mathbb{R}^k$ such that $\nu_\ell \xi_\ell$ converges narrowly towards $\nu \xi$. Moreover, one has

$$\int_{\mathbb{R}^k} \nu |\xi|^r \leq \liminf_{\ell \geq 1} \int_{\mathbb{R}^k} \nu_\ell |\xi_\ell|^r.$$

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