



**HAL**  
open science

# A multi-seasonal model of the dynamics of the African maize stalk borer

Blériot S Tchienkou Tchiengang, Israël Tankam, Jean Jules Tewa

► **To cite this version:**

Blériot S Tchienkou Tchiengang, Israël Tankam, Jean Jules Tewa. A multi-seasonal model of the dynamics of the African maize stalk borer. CARI 2022, Oct 2022, Tunis, Tunisia. hal-03718556

**HAL Id: hal-03718556**

**<https://hal.science/hal-03718556>**

Submitted on 8 Jul 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A multi-seasonal model of the dynamics of the African maize stalk borer

Blériot S. Tchienkou Tchiengang<sup>\*1</sup>, Israël Tankam<sup>2,3</sup>, Jean Jules Tewa<sup>2,3</sup>

<sup>1</sup>Department of Mathematics, University of Yaounde I, PO Box 812 Yaounde, Cameroon

<sup>2</sup>[UMMISCO]UMI 209 IRD - UPMC UMMISCO, Bondy, France

<sup>3</sup>National Advanced School of Engineering University of Yaounde I, P.O. Box 8390 Yaounde, Cameroon

\*Corresponding author: [bleriottchiengang@yahoo.fr](mailto:bleriottchiengang@yahoo.fr)

---

## Abstract

In this paper, we build a simple, multi-seasonal mathematical model describing the dynamics of the maize stalk borer *Busseola fusca* (Lepidoptera: Noctuidae). We consider in this model the main harmful stages of the pests and the control methods used by farmers to reduce pest populations from one maize cropping season to another. Firstly, immature stages are controlled during the maize cropping season by applying pesticides to destroy the eggs and immature larvae present on the plant leaves. secondly, crop residues harboring the larvae are burnt or buried in the soil after the harvest. The semi-discrete model obtained is studied and the basic reproduction number of the pests is computed. The numerical simulations carried out illustrate the theoretical results and also allow us to find the minimum quantity of larvae to be destroyed at the end of each maize cropping season to eradicate pest populations in the fields.

## Keywords

Multi-seasonal model, Semi-discrete model, Stability, Maize stalk borer, Pest management.

---

## I INTRODUCTION

Maize (*Zea mays* L.), also known as corn, is one of the most cultivated crops in the world along with rice and wheat. The total world production of maize was estimated to be 1147.6 million tons in 2018, feeding a billion people on earth. Thus, maize production provides a safety valve against food insecurity in some parts of the world. In a large part of Cameroon regions, maize is the main staple food crop and represents an important part of the caloric intake of the population. It provides a sustainable and secure food supply in terms of high yields and a significant increase in farmers' incomes since the maize yield is often sold in local markets. However, this crop is threaten in particular by insects that bore its stalk and ear. Maize stalk borer insects in sub-Saharan Africa all belong to the Lepidopteran order, they are the only group of pests that have a real economic impact. Several species of maize pests have been listed in sub-Saharan Africa but only three of these species have appeared to be economically important: the European moths *Eldana saccharina* (Walker) and *Mussidia nigricornis* (Ragonot) and the moth *Busseola fusca* (Fuller) [(9)]. The maize stalk borer (MSB) *B. fusca* is the most widespread and harmful pest to maize crops in Cameroon and the feeding habits of its larvae on maize lead to yield losses of up to 59% depending on several factors that include the maize cultivar, the development stage

of the plant and the infestation rate [(5)]. Faced with the extent of the damage, several control programs and methods have been developed by maize producers, such as growing resistant varieties, improving cultivation practices, chemical and biological control and trapping [(4)]. In this paper, we build a simple multi-seasonal model describing the dynamics of the African maize stalk borer *B. fusca* and taking into account two means of control used by farmers to manage the pests. The latter are based first on the application of pesticides on maize plants during their growth and secondly on the management of crop residues harbouring *B. fusca*'s larvae after the harvest. Our aim is to show the importance of crop residue management in controlling pest populations. After the formulation of the model, we carry out its analysis and provide some numerical simulations to illustrate the theoretical results.

## II BUSSEOLA FUSCA CYCLE AND MODELING

The maize stalk borer *B. fusca* is an African lepidoptera, an insect whose adult form is a butterfly, the larva, a caterpillar, and the immobile nymph is a pupa. Its life cycle (see [(4; 9; 10)]) includes the various classic stages of the development of moths, namely: eggs, caterpillars or larvae, pupae, imagos. The duration of these stages, in particular that of the larvae, depends on ecological conditions. In short, the life cycle starts with mating between a female and a male, then an oviposition takes place on the plant between the leaf sheath and the maize stalk. The eggs laid in a pile hatch about a week later and the young larvae go through a phyllophagous period of about ten days in the horn before entering the stem on which they keep feeding. At the last stage of their evolution they become pupae: this is the diapause phenomenon [(12)]. Adult butterflies then raise 9 to 11 days after the beginning of the diapause and the cycle starts again. When weather conditions are unfavorable, the diapause phenomenon can last up to 6 months or more. A normal cycle lasts about 45 days for a female and 50 days for a male.

Based on the biological background, we build a multi-seasonal compartmental model illustrating the dynamics of *B. fusca*. We work on the scale of a whole maize growing area. The following hypotheses are considered.

- We assume that there host plants are always available for the pests during the maize cropping season and therefore we are only interested in the dynamics of the pests.
- We summarize all the non-adult stages into one compartment named the compartment of immature stages, and we denote it by the state variable  $I$ . We also consider a compartment of female adults, that we denote by the state variable  $A$ . The units of these state variables are "larva" and "pest". This implies that all emerging immature individuals will become either males or females. For a matter of simplicity, we assume that when pests are presents, they are always males and females available in enough proportion for mating and reproduction to occur.
- We assume that the time needed for eggs emergence is  $\frac{1}{\nu}$  days during the cropping season, thus the transfer rate from  $I$  to  $A$  is  $\nu$ .
- Females feed the immature compartment logistically at a rate  $r_I \left(1 - \frac{I}{K}\right)$  where  $r_I$  is the average number of eggs provided by fertilized females (the intrinsic egg laying rate) during the cultivation period and during the off season,  $K$  its carrying capacity. We assume this carrying capacity to be constant, even though it depends on the availability of the hosts, the amount of crop residues and the density of alternative hosts.
- We term by  $u$  a parameter representing the efforts made to reduce larval populations in maize plantations. This can be seen as a chemical control through the use of pesticides that destroy the eggs and larvae that haven't yet penetrated the plant stem for feeding.

- We name  $p$  the probability that a larva gives an adult female,  $\mu_I$  and  $\mu_A$  respectively the mortality of immature pests and adult pests.
- We name  $D$  the duration of a maize cropping season and  $\tau$  the duration of the off-season, when maize plants are unavailable. We let  $T = D + \tau$  the duration of maize cropping season plus the off-season and we set  $t_n = nT, \forall n \in \mathbb{N}$ .

From the hypothesis above, during maize cropping seasons, i.e.  $t \in (t_n, t_n + D]$ , the dynamics of pests is given by the following equations:

$$\begin{cases} \dot{I}(t) &= r_I A(t) \left(1 - \frac{I(t)}{K}\right) - (\nu + \mu_I + u)I(t), \\ \dot{A}(t) &= p\nu I(t) - \mu_A A(t). \end{cases} \quad (1)$$

The initial conditions at the beginning of the first season are:  $I(0^+) = I_0 \geq 0, A(0^+) = A_0 \geq 0$ ; where the "+" superscript denotes the instant that directly follows.

At the end of a maize cropping season, after the harvest of corn, plant stems are either cut and buried in the ground or burnt to reduce pest reservoirs and therefore decrease the pest population. Depending on the effectiveness of these practices, we assume that a proportion  $q$  ( $q \in ]0, 1[$ ) of larvae is destroyed by the control efforts. We illustrate this switching in pest populations by the following difference equation when  $t = t_n + D$ :

$$\begin{cases} I(t_n + D^+) &= (1 - q)I(t_n + D), \\ A(t_n + D^+) &= A(t_n + D). \end{cases} \quad (2)$$

The case  $q = 0$  stands for the lack of burning or burying operation.

During the off-season, i.e. when  $t \in (t_n + D, t_{n+1}]$ , there are no maize plants and immature pests grow logistically, but nevertheless:

- Their limiting capacity  $K_0$  is reduced as it depends only on the amount of residues and the density alternative hosts ( $K_0 \leq K$ ).
- When climatic conditions are unfavourable for the development of immature pests, they enter in the diapause state [(12)] that reduces the transfer rate of larvae to moths. We therefore note  $\nu_0 \leq \nu$  the average transfer rate from larvae to moths.

The dynamic of evolution of the pests during this period is given by the following system:

$$\begin{cases} \dot{I}(t) &= r_I A(t) \left(1 - \frac{I(t)}{K_0}\right) - (\nu_0 + \mu_I)I(t), \\ \dot{A}(t) &= p\nu_0 I(t) - \mu_A A(t). \end{cases} \quad (3)$$

At the beginning of the new maize cropping season, for  $t = t_{n+1}$ , the same quantity of adult pests is kept and we obtain the following switching rule between seasons:

$$\begin{cases} I(t_{n+1}^+) &= I(t_{n+1}), \\ A(t_{n+1}^+) &= A(t_{n+1}). \end{cases} \quad (4)$$

The mathematical model formed by systems (1-2-3-4) represents our semi-discrete model of the dynamics of the African maize stalk borer *B. fusca* with a cultural control of pests, in a multi-seasonal framework.

### III THEORETICAL RESULTS

The well-posedness of our models relies on the existence, the uniqueness, the positivity and the boundedness of their solutions given the initial values.

*Proposition III.1:*

**Existence, uniqueness, positivity and the boundedness of the solutions.**

1. The model (1-2-3-4) admits a unique solution that is continuous on  $(t_n, t_n + D]$  and on  $(t_n + D, t_{n+1}]$ .
2. The positive set  $\mathbb{R}_+^2$  is positively invariant by the flow of the model (1-2-3-4).
3. For  $n \in \mathbb{N}$ , the set

$$\mathcal{D} = \left\{ (I(t), A(t)) \in \mathbb{R}^2 : 0 \leq I(t) \leq K, 0 \leq A(t) \leq \frac{p\nu}{\mu_A} K \right\}$$

is a positively invariant compact by the flow of model (1-2-3-4).

The proof is given in Appendix A.

In the following proposition, we find the transition laws of pest populations from one season to another around the pest-free solution (*PFS*). This finally allows us to compute the basic reproduction number of pests. In a pest-free context,  $I = 0, A = 0$  thus  $(0, 0)$  is a pest-free solution of model (1-2-3-4).

*Proposition III.2:*

**Basic reproduction number of pests.**

1. For all  $n \in \mathbb{N}$ , in the neighbourhood of the pest-free solution  $PFS = (I^*(t), A^*(t)) = (0, 0)$ , the discrete pest dynamics of the linearized model are defined by:

$$\begin{pmatrix} I(t_n^+) \\ A(t_n^+) \end{pmatrix} = \begin{pmatrix} \Phi_{1,1} & \Phi_{1,2} \\ \Phi_{2,1} & \Phi_{2,2} \end{pmatrix}^n \begin{pmatrix} I(0^+) \\ A(0^+) \end{pmatrix}, \quad (5)$$

with the explicit values of  $\Phi_{i,j}$  ( $i, j \in \{1, 2\}$ ) given in the proof.

2. Let  $\Phi = \begin{pmatrix} \Phi_{1,1} & \Phi_{1,2} \\ \Phi_{2,1} & \Phi_{2,2} \end{pmatrix}$ . The basic pest reproduction number for model (1-2-3-4) is given by:

$$\mathcal{R}_0 = \max |\phi_i|, \quad i \in \{1, 2\}. \quad (6)$$

where  $\phi_i$  are the eigenvalues of the matrices  $\Phi$ .

3. The *PFS* of model (1-2-3-4) is locally asymptotically stable if  $\mathcal{R}_0 < 1$ .

The proof is given in Appendix B.

*Remark III.3:* 1. When the basic reproduction number  $\mathcal{R}_0$  is smaller than 1, the pests tend to disappear over seasons, and when it is larger than 1, they may persist.

2. The number  $\mathcal{R}_0$  can be defined as the absolute value of the largest factor of change in the size of a pest populations (pair of larvae and moths) from one season to the next in a pest-free context.

## IV NUMERICAL SIMULATIONS

In this section, we perform numerical simulations to illustrate our theoretical results obtained above (Proposition III.2). Since we are in a multi-seasonal framework, the simulations are made over a period of 20 seasons. The length of a maize cropping season is  $D = 120$  days and the length of the off-season is  $\tau = 245$  days ( $T + \tau \approx 1 \text{ year}$ ). We consider a large cropping field for which  $I(0^+) = 150 \text{ larvae}$ ,  $A(0^+) = 15 \text{ insects}$ . The values of the different parameters are given in Table 1. We illustrate in these simulations how, when the basic reproduction number of pests  $\mathcal{R}_0$  is strictly less than 1, pest populations gradually decrease over the seasons to negligible levels. Conversely, if these threshold are greater than 1, then pest populations persist throughout the seasons. Subsequently, through simulations we show how the basic reproduction number evolves as a function of the proportion  $q$  of larvae destroyed after each harvest. We find a linear relationship between  $\mathcal{R}_0$  and  $q$ .

Table 1: Table of parameters and values.

Parameters	Literature values	value(s)
$r_I$	4 – 116 <i>pests.day</i> <sup>-1</sup> [(3)]	20 <i>pests day</i> <sup>-1</sup>
$K$	/	5000 <i>pests</i> <sup>(1)</sup>
$K_0$	/	2500 <i>pests</i> <sup>(1)</sup>
$\nu$	1/45 <i>day</i> <sup>-1</sup> [(2)]	1/45 <i>day</i> <sup>-1</sup>
$\nu_0$	/	0.0177 <i>day</i> <sup>-1</sup> (2)
$\mu_I$	0.35 <i>day</i> <sup>-1</sup> [(10)]	0.35 <i>day</i> <sup>-1</sup>
$\mu_A$	0.5 <i>day</i> <sup>-1</sup> [(10)]	0,5 <i>day</i> <sup>-1</sup>
$p$	0, 3939 – 0, 6078 [(6)]	0.48
$u$	/	0.0025 <i>day</i> <sup>-1</sup> (3)
$q$	/	0.75

(1)  $K$  and  $K_0$  are estimated in decent ranges according to what could be a field size.

(2)  $\nu_0$  is taken so that  $\nu_0 \leq \nu$ .

(3)  $u$  is taken such that chemical control destroys 30% of the larvae during the maize cropping season.

/ No data available in the literature.

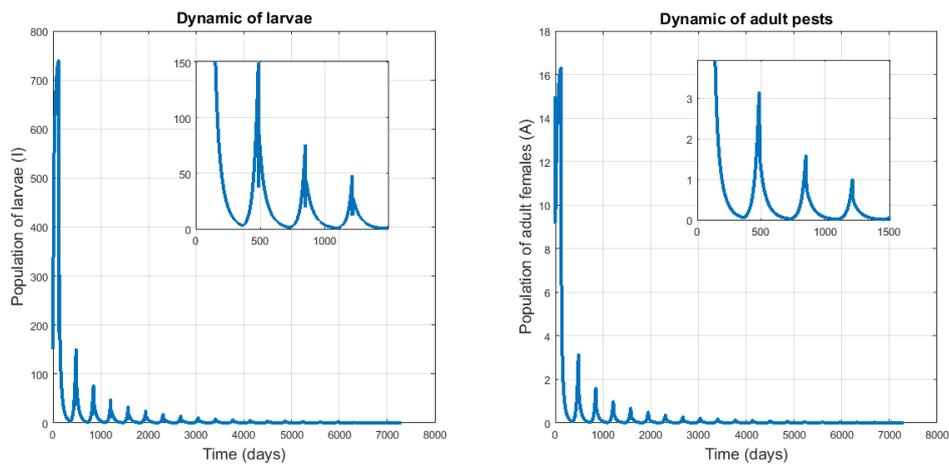


Figure 1: Long-term dynamics of pest for model (1-2-3-4) with 20 cropping seasons, when  $q = 0.75$ . This parameter means that 75% of pest larvae are destroyed at the end of each maize cropping season. This leads to a basic reproduction number  $\mathcal{R}_0 = 0.84$ .

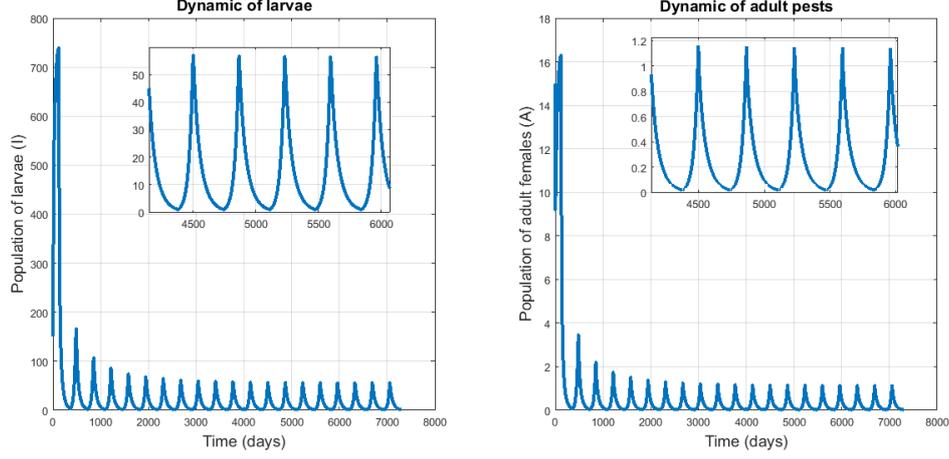


Figure 2: Long-term dynamics of pest for model (1-2-3-4) with 20 cropping seasons, when  $q = 0$ . This means that no pest larvae are destroyed at the end of each maize cropping season. This leads to basic reproduction number  $\mathcal{R}_0 = 1.41$ .

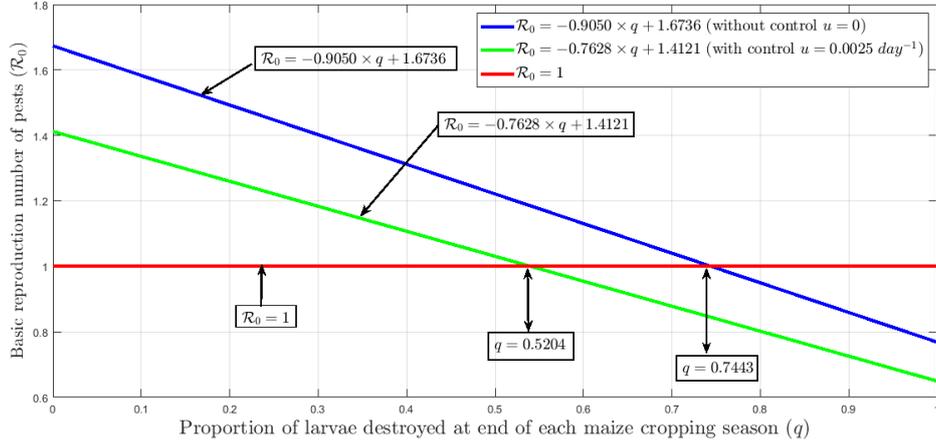


Figure 3: Linear relationship between the proportion  $q$  of larvae destroyed at the end of each maize cropping season and the basic reproduction number of pests  $\mathcal{R}_0$ .

Figure 1 illustrates the case where the basic reproduction number of pests is less than 1. In this case, in addition to the larval control carried out during the maize cropping season, 75% of larvae are destroyed at the end of each maize cropping season and therefore pest populations gradually decrease over seasons. In Figure 2, no proportion of larvae is destroyed at the end of the maize cropping season and therefore the basic reproduction number is therefore greater than 1. The pests hence persist over seasons. These two figures illustrate the importance of crop residue management to control pest populations from one season to another. Figure 3 illustrates the linear relationship between the basic reproduction number  $\mathcal{R}_0$  and  $q$ :  $\mathcal{R}_0 = -0.9050 \times q + 1.6736$  (without control  $u$ ) and  $\mathcal{R}_0 = -0.7628 \times q + 1.4121$  (with control  $u$ ). This relationship allows us to find the minimum proportion of larvae to destroy after each harvest to have a decrease of the pest populations over the seasons. From this figure, it is noticeable that if no effort is made during each maize cropping season to reduce larvae, at least 74.43% of larvae must be destroyed after each harvest for pest populations to decrease over the seasons. However, if larvae are regularly controlled by pesticides, only 54.02% would need to be destroyed.

## V CONCLUSION AND PERSPECTIVES

In this paper, we formulated a simple model for the dynamics of the African maize stalk borer *B. fusca*. Since the host of the pest is available seasonally, we have implicitly integrated into the model its seasonal availability and the means of control used by farmers to control pest populations from one season to another. The semi-discrete model obtained was analyzed and the basic reproduction number of pests was calculated. Numerical simulations were carried out over several seasons to illustrate the theoretical results. Namely, when the basic reproduction number of pests is less than 1, pest populations gradually decrease over the seasons and when this number is greater than 1, pests persist. Using numerical simulations, we also illustrated the importance of larval control during the maize cropping season, and the importance of crop residue management after harvest to control pest populations. We found the minimum proportion of larvae to be destroyed at the end of each maize cropping season for the control based on crop residue management to be effective. This work can therefore open the debate on the possibilities for biologists to carry out field experiments, and these could help us to calibrate the model. It is desirable that field experiments can be done in order to estimate some precise parameters for these pests and to be able to validate our model.

## References

- [1] P. AUGERC. LETTJ.-C. POGGIALE, Modélisation Mathématique en Ecologie, *Dunod, Paris, 2010*.
- [2] P. A. CALATAYUDB. P LE RÜJ. VAN DEN BERGF. SCHULTHESS, Ecology of the african maize stalk borer, *Busseola fusca* (Lepidoptera: Noctuidae) with special reference to insect-plant interactions, *Insect 2014*, 5:539-563, doi:10.3390/insect5030539.
- [3] P. A. CALATAYUDH. GUÉNÉGOB. P LE RÜJ.-F. SILVAINB. FRÉROT, Temporal patterns of emergence, calling behaviour and oviposition period of the maize stem borer, *Busseola fusca* (Fuller) (Lepidoptera: Noctuidae), *Ann.soc.entomol.Fr. (n.s)*, 43(1) 63-68, 2007. doi:10.1080/00379271.2007.10697495.
- [4] K. M. HARRISK. F. NWANZE, *Busseola fusca* (Fuller), the African maize stalk borer: a handbook of information., *International Crops Research Institute for the Semi-Arid Tropics, and Wallingford*, Information Bulletin 33.
- [5] R. KFIRW. A. OVERHOLTA. POLASZEK, Biology and management of economically important lepidopteran cereal stem borers in Africa, *Annual review of entomology*, 47(1) 701-731, 2002. doi:10.1146/annurev.ento.47.091201.145254.
- [6] M. KRUGERJ. B. J. VAN RENSBERGJ. VAN DEN BERG, Reproductive biology of bt-resistant and susceptible field-collected larvae of the maize stem borer, *Busseola fusca* (Lepidoptera: Noctuidae), *African Entomology*, 20(1) 35-43, 2012. doi:10.4001/003.020.0105.
- [7] L. MAILLERETV. LEMESLE, A note on semi-discrete modelling in the life sciences, 1. *INRA, UR 880, URIH, 06903 Sophia Antipolis, France* 2. *ENS Lyon, UMPA, 69364 Lyon, France*.
- [8] L. MAILLERETM. CASTELJ. MONTARRYF. M. HAMELIN, From elaborate to compact seasonal plant epidemic models and back: is competitive exclusion in the details?, *Theor Ecol (2012)* 5 pp. 311-324, 2011. doi: 10.1007/s12080-011-0126-0
- [9] P. MOYAL, Les foreurs du maïs en zone des savannes en côte-d'ivoire. Données Morphologiques, Biologique, Ecologique. Essai de lutte et relation plante-insecte, *Editions de l'ORSTOM Institut Francais de recherche scientifique pour le développement en coopération. Collection ÉTUDES et THÈSES PARIS 1988*,
- [10] J. P. NTAHOMVUKIYE. A. TEMGOUAS. BOWONG, Study of the Population Dynamics of *Busseola fusca*, Maize Pest, *Acta Biotheoretica* <https://doi.org/10.1007/s10441-018-9335-x> 2018.
- [11] I TANKAM-CHEDJOUS. TOUZEAU. L. MAILLERETJ. J. TEWAF. GROGNARD, Modelling and control of a banana soilborne pest in a multi-seasonal framework, *Mathematical Biosciences; Elsevier*, 322, 2020. doi:10.1016/j.mbs.2020.155883.
- [12] E. J. USUA, Diapause in maize stemborer, *School of Biological Sciences, University of Iagos, Nigeria, 1970. cari 2018*.

## APPENDIX A :

### Proof of proposition III.1

1. The system (3) with initial condition given by (2) satisfies the Cauchy-Lipschitz conditions. Then it admits a unique continuous solution in the interval  $(t_n + D, t_{n+1}]$ . The system (1) has  $(I_0, A_0)$  as initial condition when  $n = 0$  and for  $n \geq 1$  the initial condition are given by (4). This system satisfies also the Cauchy-Lipschitz conditions so it admits a unique continuous solution in the interval  $(t_n, t_n + D]$ . Therefore the model is well-posed.
2. First, we suppose  $n = 0$  and let  $X(t) = (I(t), A(t))$ . Since these variables represent biological quantities then  $X(0^+) \geq 0$ . We consider the system (1) with the initial condition  $X(0^+)$ . We have:

$$\dot{I}|_{I=0} = r_I A \geq 0, \dot{A}|_{A=0} = p\nu I \geq 0.$$

Therefore,  $\forall t \in ]0; D]$  all solutions of system (1) with initial positive conditions stay in the non-negative set  $\mathbb{R}_+^2$ , so  $\mathbb{R}_+^2$  is positively invariant by the flow of system (1).

Now, if we consider the system (3), it has as initial conditions  $X(D^+)$ .

Using discrete equation (2)  $X(D^+) \geq 0$ . Indeed, according to (2)

$$X(D^+) = ((1 - q)I(D), A(D)).$$

From the above,  $I(D) \geq 0, A(D) \geq 0$  hence  $X(D^+) \geq 0$ .

$$\dot{I}|_{I=0} = r_I A \geq 0, \dot{A}|_{A=0} = p\nu_0 I(t) \geq 0.$$

The same reasoning as above leads us to conclude that  $\mathbb{R}_+^2$  is positively invariant by the flow of system (3). Thus  $\mathbb{R}_+^2$  is positively invariant by the flow of the model (1-2-3-4) at season 1. Furthermore, if  $\mathbb{R}_+^2$  is positively invariant by the flow of the model (1-2-3-4) at season  $n$  then the initial conditions for season  $n + 1$  given by (4) will always be positive and therefore  $\mathbb{R}_+^2$  will be positively invariant by the flow of the model (1-2-3-4) for season  $n + 1$ .

3. We first consider the model (1-2-3-4). We use the barrier theorem to bound the solutions of the continuous systems of our model. Consider the system (1),  $\forall n \in \mathbb{N}, \forall t \in ]t_n, t_n + D]$

$$\dot{I}|_{I=K} = -(\nu + \mu_I + u)K \leq 0 \text{ thus } I(t) \leq K,$$

$$\dot{A}|_{A=\frac{p\nu}{\mu_A}K} = p\nu I(t) - p\nu K = p\nu(I(t) - K) \leq 0, \text{ thus } A(t) \leq \frac{p\nu}{\mu_A}K.$$

Thus, according to the above,  $\forall t \in ]t_n, t_n + D]$  all solutions of system (1) are bounded. At time  $t = nT + D$ , all these solutions remain bounded. In fact, according to (2) we have:

$$\begin{aligned} I(t_n + D^+) &\leq (1 - q)K \\ A(t_n + D^+) &\leq \frac{p\nu}{\mu_A}K. \end{aligned}$$

For  $t \in (t_n + D, t_{n+1}]$ , according to (3) we show in the same way as before, using the barrier theorem that:

$$\begin{aligned} I(t) &\leq K_0 \\ A(t) &\leq \frac{p\nu_0}{\mu_A}K_0. \end{aligned}$$

For  $t = t_{n+1}$ , according to (4), we have:

$$\begin{aligned} I(t_{n+1}^+) &\leq K_0 \\ A(t_{n+1}^+) &\leq \frac{p\nu_0}{\mu_A} K_0. \end{aligned}$$

So for  $t = t_{n+1}$ , the solutions of the model (1-2-3-4) also remain bounded and therefore, from the above we finally get:  $\forall t \in ]t_n, t_{n+1}]$ ,

$$\begin{aligned} I(t) &\leq \max \{K, (1-q)K, K_0\} = K \\ A(t) &\leq \max \left\{ \frac{p\nu}{\mu_A} K, \frac{p\nu_0}{\mu_A} K_0, \right\} = \frac{p\nu}{\mu_A} K. \end{aligned}$$

This allows us to show that all solutions of model (1-2-3-4) are bounded, hence the construction of the set  $\mathcal{D}$ .

## APPENDIX B :

### Proof of Proposition III.2

1. For all  $t \in (t_n, t_n + D]$ , the jacobian matrix in the neighbourhood of  $PFS = (I^*(t), A^*(t)) = (0, 0)$  of system (1) is given by:

$$J_{(PFS)} = \begin{pmatrix} -(\nu + \mu_I + u) & r_I \\ p\nu & -\mu_A \end{pmatrix}$$

In a neighbourhood of  $PFS$ , the system (1) is equivalent to a following linearised system:

$$\begin{cases} \dot{\tilde{I}}(t) &= r_I \tilde{A}(t) - (\nu + \mu_I + u) \tilde{I}(t), \\ \dot{\tilde{A}}(t) &= p\nu \tilde{I}(t) - \mu_A \tilde{A}(t). \end{cases} \quad (7)$$

Where  $\tilde{I}(t) = I(t) - I^*(t) = I(t)$  and  $\tilde{A}(t) = A(t) - A^*(t) = A(t)$ .  
i.e

$$\begin{cases} \dot{I}(t) &= r_I A(t) - (\nu + \mu_I + u) I(t), \\ \dot{A}(t) &= p\nu I(t) - \mu_A A(t). \end{cases} \quad (8)$$

let's name

$$X(t) = (I(t), A(t))^T, \text{ and } B = \begin{pmatrix} -\alpha & r_I \\ p\nu & -\mu_A \end{pmatrix}, \alpha = \nu + \mu_I + u,$$

the system (8) gives

$$\dot{X}(t) = BX(t)$$

The matrix  $B$  is a Metzler matrix and therefore it admits two distinct real eigenvalues  $\lambda_{1,2} = \frac{tr(B)}{2} \pm \frac{1}{2} \sqrt{tr^2(B) - 4det(B)}$ , and therefore we have:

$$\exp\left(\int_{t_n}^t B dt\right) = e^{B(t-t_n)} = \begin{pmatrix} \Gamma_{1,1}(t-t_n) & \Gamma_{1,2}(t-t_n) \\ \Gamma_{2,1}(t-t_n) & \Gamma_{2,2}(t-t_n) \end{pmatrix}$$

Where

$$\begin{aligned} \Gamma_{1,1}(t) &= \frac{1}{\lambda_2 - \lambda_1} (e^{\lambda_1 t} (\lambda_2 + \alpha) - e^{\lambda_2 t} (\lambda_1 + \alpha)) \\ \Gamma_{1,2}(t) &= \frac{r_I}{\lambda_2 - \lambda_1} (e^{\lambda_2 t} - e^{\lambda_1 t}) \\ \Gamma_{2,1}(t) &= \frac{p\nu}{\lambda_2 - \lambda_1} (e^{\lambda_2 t} - e^{\lambda_1 t}) \\ \Gamma_{2,2}(t) &= \frac{1}{\lambda_2 - \lambda_1} (e^{\lambda_1 t} (\lambda_2 + \mu_A) - e^{\lambda_2 t} (\lambda_1 + \mu_A)) \end{aligned}$$

Therefore (8) has a solution:

$$\begin{cases} I(t) = \Gamma_{1,1}(t-t_n)I(t_n^+) + \Gamma_{1,2}(t-t_n)A(t_n^+) \\ A(t) = \Gamma_{2,1}(t-t_n)I(t_n^+) + \Gamma_{2,2}(t-t_n)A(t_n^+) \end{cases} \quad (9)$$

For  $t = t_n + D$  in (9) we have the following relation:

$$\begin{cases} I(t_n + D) = \Gamma_{1,1}(D)I(t_n) + \Gamma_{1,2}(D)A(t_n) \\ A(t_n + D) = \Gamma_{2,1}(D)I(t_n) + \Gamma_{2,2}(D)A(t_n) \end{cases} \quad (10)$$

For all  $t \in (t_n + D, t_{n+1}]$ , we consider the system (4). The jacobian matrix in the neighbourhood of  $PFS(0, 0)$  of system (3) is given by:

$$J_{PFS} = \begin{pmatrix} -(\nu_0 + \mu_I) & r_I \\ p\nu_0 & -\mu_A \end{pmatrix}$$

If we pose  $Y(t) = (I(t), A(t))^T$ , in a neighbourhood of  $(0, 0)$ , the system (3) is equivalent to a following linearised system :

$$\dot{Y}(t) = J_{PFS}Y(t) \quad (11)$$

As previously, we can extract from (11) the following subsystem:

$$\begin{cases} \dot{I}(t) = r_I A(t) - (\nu_0 + \mu_I)I(t), \\ \dot{A}(t) = p\nu_0 I(t) - \mu_A A(t). \end{cases} \quad (12)$$

The solutions of the system (12) are given by:

$$\begin{cases} I(t) = \Pi_{1,1}(t - (t_n + D))I(t_n + D^+) + \Pi_{1,2}(t - (t_n + D))A(t_n + D^+) \\ A(t) = \Pi_{2,1}(t - (t_n + D))I(t_n + D^+) + \Pi_{2,2}(t - (t_n + D))A(t_n + D^+) \end{cases} \quad (13)$$

Where the  $\Pi_{i,j}$  are given by:

$$\begin{aligned}\Pi_{1,1}(t) &= \frac{1}{\lambda'_2 - \lambda'_1} (e^{\lambda'_1 t} (\lambda'_2 + \alpha_0) - e^{\lambda'_2 t} (\lambda'_1 + \alpha_0)) \\ \Pi_{1,2}(t) &= \frac{r_I}{\lambda'_2 - \lambda'_1} (e^{\lambda'_2 t} - e^{\lambda'_1 t}) \\ \Pi_{2,1}(t) &= \frac{p\nu_0}{\lambda'_2 - \lambda'_1} (e^{\lambda'_2 t} - e^{\lambda'_1 t}) \\ \Pi_{2,2}(t) &= \frac{1}{\lambda'_2 - \lambda'_1} (e^{\lambda'_1 t} (\lambda'_2 + \mu_A) - e^{\lambda'_2 t} (\lambda'_1 + \mu_A))\end{aligned}$$

and  $\lambda'_{1,2}$  are the eigenvalues of the matrix  $B_0 = \begin{pmatrix} -\alpha_0 & r_I \\ p\nu_0 & -\mu_A \end{pmatrix}$ ,  $\alpha_0 = \nu_0 + \mu_I$ .

For  $t = (n+1)T = t_{n+1}$ , taking into account the system (2), the relation (13) gives:

$$\begin{cases} I(t_{n+1}) = (1-q)\Pi_{1,1}(\tau)I(t_n + D) + \Pi_{1,2}(\tau)A(t_n + D) \\ A(t_{n+1}) = (1-q)\Pi_{2,1}(\tau)I(t_n + D) + \Pi_{2,2}(\tau)A(t_n + D) \end{cases} \quad (14)$$

Considering (10) we obtain from (14) the following relation:

$$\begin{cases} I((n+1)T) = \Phi_{1,1}I(nT) + \Phi_{1,2}A(nT) \\ A((n+1)T) = \Phi_{2,1}I(nT) + \Phi_{2,2}A(nT) \end{cases} \quad (15)$$

where

$$\begin{aligned}\Phi_{1,1} &= (1-q)\Pi_{1,1}(\tau)\Gamma_{1,1}(D) + \Pi_{1,2}(\tau)\Gamma_{2,1}(D) \\ \Phi_{1,2} &= (1-q)\Pi_{1,1}(\tau)\Gamma_{1,2}(D) + \Pi_{1,2}(\tau)\Gamma_{2,2}(D) \\ \Phi_{2,1} &= (1-q)\Pi_{2,1}(\tau)\Gamma_{1,1}(D) + \Pi_{2,2}(\tau)\Gamma_{2,1}(D) \\ \Phi_{2,2} &= (1-q)\Pi_{2,1}(\tau)\Gamma_{1,2}(D) + \Pi_{2,2}(\tau)\Gamma_{2,2}(D)\end{aligned}$$

By adopting the following notations  $I(t_n) = I_n$ ,  $X_n = (I_n, A_n)^T$ ,  $\Phi = \begin{pmatrix} \Phi_{1,1} & \Phi_{1,2} \\ \Phi_{2,1} & \Phi_{2,2} \end{pmatrix}$ , the relation (15) gives  $X_{n+1} = \Phi X_n \implies X_n = \Phi^n X_0$  and relation (5) is thus obtained.

2. From relation (15), we obtain the following discrete dynamic system illustrating the pest dynamics in the neighbourhood of the pest-free solution:

$$\begin{cases} I_{n+1} = \Phi_{1,1}I_n + \Phi_{1,2}A_n \\ A_{n+1} = \Phi_{2,1}I_n + \Phi_{2,2}A_n \end{cases} \quad (16)$$

The system of equations defined by relation (16) admits  $(0, 0)$  as a unique fixed point. Let's pose  $\phi_i$  the eigenvalues of the matrix  $\Phi$ . According to the Jury criteria (1),  $(0, 0)$  is locally asymptotically stable if and only if  $\max|\phi_i| < 1$  for  $i = 1, 2$ . Hence the basic reproduction number of pests for model (1-2-3-4) is  $\mathcal{R}_0 = \max|\phi_i|$ .