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## ON THE RELATIVE KUO CONDITION AND THE SECOND RELATIVE KUO CONDITION

KARIM BEKKA AND SATOSHI KOIKE

ABSTRACT. The Kuo condition and the the second Kuo condition are known as criteria for an  $r$ -jet to be  $V$ -sufficient in  $C^r$  mappings and  $C^{r+1}$  mappings, respectively. In [2] we considered the notions of  $V$ -sufficiency of jets and these Kuo conditions in the relative case to a given closed set  $\Sigma$ , and showed that the relative Kuo condition is a criterion for a relative  $r$ -jet to be  $\Sigma$ - $V$ -sufficient in  $C^r$  mappings and the second relative Kuo condition is a sufficient condition for a relative  $r$ -jet to be  $\Sigma$ - $V$ -sufficient in  $C^{r+1}$  mappings. In this paper we discuss several conditions equivalent to the relative Kuo condition or the second relative Kuo condition.

### 1. INTRODUCTION

Tzee-Char Kuo formulated in [11] necessary and sufficient conditions, called the Kuo condition and the second Kuo condition, for an  $r$ -jet to be  $V$ -sufficient in  $C^r$  mappings and  $C^{r+1}$  mappings, respectively. The notion of sufficiency of jets was introduced by René Thom in relation to the structural stability problem; that of  $V$ -sufficiency is for the topological stability of the zero set of a mapping.

Let  $\Sigma$  be the germ of a closed subset of  $\mathbb{R}^n$  at  $0 \in \mathbb{R}^n$  such that  $0 \in \Sigma$ . In [2] we generalised the notions of  $V$ -sufficiency of jets and the Kuo conditions to the relative case with respect to a given  $\Sigma$ . Then we proved that the relative Kuo condition is a criterion for a relative  $r$ -jet to be  $\Sigma$ - $V$ -sufficient in  $C^r$  mappings (see §3.1), and that the second relative Kuo condition is a sufficient condition for a relative  $r$ -jet to be  $\Sigma$ - $V$ -sufficient in  $C^{r+1}$  mappings (see §3.2).

In §4 we show the main results of this paper on equivalent conditions to the above two relative Kuo conditions. In §4.1 we show a result on conditions equivalent to the relative Kuo condition (Theorem 4.5). In §4.2 we show two kinds of results on conditions equivalent to the second relative Kuo condition (Theorems 4.7, 4.10).

We give the definition of  $\Sigma$ - $V$ -sufficiency of jet, namely  $V$ -sufficiency of the relative jet to  $\Sigma$  in §2, and the definitions of the relative Kuo condition and the second relative Kuo condition in §3.

Throughout this paper, we denote by  $\mathbb{N}$  the set of positive integers.

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## 2. PRELIMINARIES

Let  $s \in \mathbb{N} \cup \{\infty, \omega\}$ . Let  $\mathcal{E}_{[s]}(n, p)$  denote the set of  $C^s$  map germs  $: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ , let  $j^r f(0)$  denote the  $r$ -jet of  $f$  at  $0 \in \mathbb{R}^n$  for  $f \in \mathcal{E}_{[s]}(n, p)$ ,  $s \geq r$ , and let  $J^r(n, p)$  denote the set of  $r$ -jets in  $\mathcal{E}_{[s]}(n, p)$ .

Throughout this paper, let  $\Sigma$  denote the germ of a closed subset of  $\mathbb{R}^n$  at  $0 \in \mathbb{R}^n$  such that  $0 \in \Sigma$ . Then we denote by  $d(x, \Sigma)$  the distance from a point  $x \in \mathbb{R}^n$  to the subset  $\Sigma$ .

We consider on  $\mathcal{E}_{[s]}(n, p)$  the following equivalence relation:

Two map germs  $f, g \in \mathcal{E}_{[s]}(n, p)$  are  $r$ - $\Sigma$ -equivalent, denoted by  $f \sim g$ , if there exists a neighbourhood  $U$  of  $0$  in  $\mathbb{R}^n$  such that the  $r$ -jet extensions of  $f$  and  $g$  satisfy  $j^r f(\Sigma \cap U) = j^r g(\Sigma \cap U)$ .

We denote by  $j^r f(\Sigma; 0)$  the equivalence class of  $f$ , and by  $J_\Sigma^r(n, p)$  the quotient set  $\mathcal{E}_{[s]}(n, p) / \sim$ .

*Remark 2.1.* Any  $r$ -jet,  $r \in \mathbb{N}$ , has a unique polynomial realisation of degree not exceeding  $r$  in the non-relative case. But relative-jets do not always have a  $C^\omega$  realisation. See Remark 2.1 in [2] for the details.

Let us introduce an equivalence for elements of  $\mathcal{E}_{[s]}(n, p)$ . We say that  $f, g \in \mathcal{E}_{[s]}(n, p)$  are  $\Sigma$ -V-equivalent, if  $f^{-1}(0)$  is homeomorphic to  $g^{-1}(0)$  as germs at  $0 \in \mathbb{R}^n$  by a homeomorphism which fixes  $f^{-1}(0) \cap \Sigma$ .

Let  $w \in J_\Sigma^r(n, p)$ . We call the relative jet  $w$   $\Sigma$ -V-sufficient in  $\mathcal{E}_{[s]}(n, p)$ ,  $s \geq r$ , if any two realisations  $f, g \in \mathcal{E}_{[s]}(n, p)$  of  $w$ , namely  $j^r f(\Sigma; 0) = j^r g(\Sigma; 0) = w$ , are  $\Sigma$ -V-equivalent.

We next prepare notations concerning large and small relation and equivalence between two non-negative functions.

**Definition 2.2.** Let  $f, g : U \rightarrow \mathbb{R}$  be non-negative functions, where  $U \subset \mathbb{R}^N$  is an open neighbourhood of  $0 \in \mathbb{R}^N$ . If there are real numbers  $K > 0$ ,  $\delta > 0$  with  $B_\delta(0) \subset U$  such that

$$f(x) \leq Kg(x) \text{ for any } x \in B_\delta(0),$$

where  $B_\delta(0)$  is a closed ball in  $\mathbb{R}^N$  of radius  $\delta$  centred at  $0 \in \mathbb{R}^N$ , then we write  $f \lesssim g$  (or  $g \gtrsim f$ ). If  $f \lesssim g$  and  $f \gtrsim g$ , we write  $f \approx g$ .

At the end of this section, let us recall a useful lemma to treat the  $C^{r+1}$  case.

**Lemma 2.3.** ([2]) *Let  $r \in \mathbb{N}$ , and let  $\Sigma$  be the germ at  $0 \in \mathbb{R}^n$  of a closed set. Let  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be a  $C^r$  map germ,  $r \geq 1$ , such that  $j^r f(\Sigma; 0) = \{0\}$ . Then  $\|f(x)\| = o(d(x, \Sigma)^r)$ . If moreover  $f$  is of classe  $C^{r+1}$ , then  $\|f(x)\| \lesssim d(x, \Sigma)^{r+1}$ .*

3. CRITERIA FOR  $\Sigma$ -V-SUFFICIENCY OF JETS

**3.1. The relative Kuo condition.** Let  $\Sigma$  be given. We now define the Kuo condition relative to  $\Sigma$ . A criterion for an  $r$ -jet to be  $C^0$ -sufficient or  $V$ -sufficient in

$C^r$  functions (resp. in  $C^{r+1}$  functions) is known as the Kuiper-Kuo condition (resp. the second Kuiper-Kuo condition) in the non-relative, function case (see N. Kuiper [8], T.-C. Kuo [9] and J. Bochnak - S. Lojasiewicz [4]). In this case the Kuiper-Kuo condition is equivalent to the Kuo condition ([4]).

Let  $v_1, \dots, v_p$  be  $p$  vectors in  $\mathbb{R}^n$  where  $n \geq p$ . The *Kuo distance*  $\kappa$  ([11]) is defined by  $\kappa(v_1, \dots, v_p) = \min_i \{\text{distance of } v_i \text{ to } V_i\}$ , where  $V_i$  is the span of the  $v_j$ 's,  $j \neq i$ . In the case where  $p = 1$ ,  $\kappa(v) = \|v\|$ .

We first recall the notion of the relative Kuo condition. The original condition was introduced by T.-C. Kuo [11] as a criterion of  $V$ -sufficiency of jets in the mapping case.

**Definition 3.1** (The relative Kuo condition). A map germ  $f \in \mathcal{E}_{[r]}(n, p)$ ,  $n \geq p$ , satisfies the *relative Kuo condition* ( $K_\Sigma$ ) if there are strictly positive numbers  $C, \alpha$  and  $\bar{w}$  such that

$$\kappa(df(x)) \geq Cd(x, \Sigma)^{r-1} \text{ in } \mathcal{H}_r^\Sigma(f; \bar{w}) \cap \{\|x\| < \alpha\},$$

namely,  $\kappa(df(\cdot)) \gtrsim d(\cdot, \Sigma)^{r-1}$  on a set of points where  $\|f\| \lesssim d(\cdot, \Sigma)^r$ .

In the definition 3.1,  $\mathcal{H}_r^\Sigma(f; \bar{w})$  denotes the *horn-neighbourhood of  $f^{-1}(0)$  relative to  $\Sigma$  of degree  $r$  and width  $\bar{w}$* ,

$$\mathcal{H}_r^\Sigma(f; \bar{w}) := \{x \in \mathbb{R}^n : \|f(x)\| \leq \bar{w} d(x, \Sigma)^r\}.$$

The horn-neighbourhood was originally introduced in [10] in the non-relative case.

By definition, it is easy to see that the relative Kuo condition ( $K_\Sigma$ ) is equivalent to the following condition.

**Definition 3.2** (Condition  $(\tilde{K}_\Sigma)$ ). A map germ  $f \in \mathcal{E}_{[r]}(n, p)$ ,  $n \geq p$ , satisfies *condition  $(\tilde{K}_\Sigma)$*  if

$$d(x, \Sigma)\kappa(df(x)) + \|f(x)\| \gtrsim d(x, \Sigma)^r$$

holds in some neighbourhood of  $0 \in \mathbb{R}^n$ .

*Remark 3.3.* 1) Condition  $(\tilde{K}_\Sigma)$  was introduced in [1], in the non-relative case, namely  $\Sigma = \{0\}$ .

2) The relative Kuo condition ( $K_\Sigma$ ) and condition  $(\tilde{K}_\Sigma)$  are invariant under rotations.

As a sufficient condition for an  $r$ -jet to be  $\Sigma$ - $V$ -sufficient in  $\mathcal{E}_{[r]}(n, p)$ , we have the following result.

**Theorem 3.4.** ([2]) *Let  $r \in \mathbb{N}$ , and let  $f \in \mathcal{E}_{[r]}(n, p)$ ,  $n \geq p$ . If  $f$  satisfies condition  $(\tilde{K}_\Sigma)$ , then the relative  $r$ -jet,  $j^r f(\Sigma; 0)$ , is  $\Sigma$ - $V$ -sufficient in  $\mathcal{E}_{[r]}(n, p)$ .*

On the other hand, we have the following criteria for an  $r$ -jet to be  $\Sigma$ - $V$ -sufficient in  $\mathcal{E}_{[r]}(n, p)$ .

**Theorem 3.5.** ([2]) *Let  $r \in \mathbb{N}$ , and let  $f \in \mathcal{E}_{[r]}(n, p)$ ,  $n > p$ . Then the following conditions are equivalent.*

(1)  *$f$  satisfies the relative Kuo condition  $(K_\Sigma)$ .*

- (2)  $f$  satisfies condition  $(\tilde{K}_\Sigma)$ .  
(3) The relative  $r$ -jet  $j^r f(\Sigma; 0)$  is  $\Sigma$ - $V$ -sufficient in  $\mathcal{E}_{[r]}(n, p)$ .

**Theorem 3.6.** ([2]) *Let  $r \in \mathbb{N}$ , and let  $f \in \mathcal{E}_{[r]}(n, n)$ . Suppose that  $j^r f(\Sigma; 0)$  has a subanalytic  $C^r$ -realisation and that  $\Sigma$  is the germ of a subanalytic closed subset of  $\mathbb{R}^n$  such that  $0 \in \Sigma$ . Then the following conditions are equivalent.*

- (1)  $f$  satisfies the relative Kuo condition  $(K_\Sigma)$ .  
(2)  $f$  satisfies condition  $(\tilde{K}_\Sigma)$ .  
(3) The relative  $r$ -jet  $j^r f(\Sigma; 0)$  is  $\Sigma$ - $V$ -sufficient in  $\mathcal{E}_{[r]}(n, n)$ .

For the subanalyticity, see H. Hironaka [5].

Related to the Kuo condition, we proved in [3] the equivalence of Kuo quantity and Thom quantity for analytic functions.

**3.2. The second relative Kuo condition.** We next recall the notion of the second relative Kuo condition. The original condition was introduced also by T.-C. Kuo [11] as a criterion of  $V$ -sufficiency of  $r$ -jets in  $C^{r+1}$  mappings.

**Definition 3.7** (The second relative Kuo condition). A map germ  $f \in \mathcal{E}_{[r+1]}(n, p)$ ,  $n \geq p$ , satisfies the *second relative Kuo condition*  $(K_\Sigma^\delta)$  if for any map germ  $g \in \mathcal{E}_{[r+1]}(n, p)$  satisfying  $j^r g(\Sigma; 0) = j^r f(\Sigma; 0)$  there are strictly positive numbers  $C, \alpha, \delta$  and  $\bar{w}$  (depending on  $g$ ), such that

$$\kappa(df(x)) \geq Cd(x, \Sigma)^{r-\delta} \text{ in } \mathcal{H}_{r+1}^\Sigma(g; \bar{w}) \cap \{\|x\| < \alpha\},$$

namely,  $\kappa(df(\cdot)) \gtrsim d(\cdot, \Sigma)^{r-\delta}$  on a set of points where  $\|g(\cdot)\| \lesssim d(\cdot, \Sigma)^{r+1}$ .

*Remark 3.8.* 1) For a map germ  $f \in \mathcal{E}_{[r]}(n, p)$  satisfying the relative Kuo condition or the second relative Kuo condition, we have

$$\text{Sing}(f) \cap \mathcal{H}_r^\Sigma(f; \bar{w}) \subset \Sigma,$$

in a neighbourhood of  $0 \in \mathbb{R}^n$ , where  $\text{Sing}(f)$  is the singular set of  $f$ . In particular, in a neighbourhood of  $0 \in \mathbb{R}^n$ ,  $\text{grad} f_1(x), \dots, \text{grad} f_p(x)$  are linearly independent on  $f^{-1}(0) \setminus \Sigma$ .

- 2) For a map germ  $f \in \mathcal{E}_{[r]}(n, p)$  satisfying the second relative Kuo, we have for any map germ  $g \in \mathcal{E}_{[r+1]}(n, p)$  satisfying  $j^r g(\Sigma; 0) = j^r f(\Sigma; 0)$ , in a neighbourhood of  $0 \in \mathbb{R}^n$ , the intersection of the singular set of  $f$ ,  $\text{Sing}(f)$ , and the horn neighbourhood  $\mathcal{H}_{r+1}^\Sigma(g; \bar{w})$  is contained in  $\Sigma$ , namely  $\text{Sing}(f) \cap \mathcal{H}_{r+1}^\Sigma(g; \bar{w}) \subset \Sigma$ . Since  $\|(f - g)(x)\| \lesssim d(x, \Sigma)^{r+1}$ , we have  $f^{-1}(0) \subset \mathcal{H}_{r+1}^\Sigma(g; \bar{w})$ , then, in a neighbourhood of  $0 \in \mathbb{R}^n$ ,  $\text{grad} f_1(x), \dots, \text{grad} f_p(x)$  are linearly independent on  $f^{-1}(0) \setminus \Sigma$ .

We next consider a condition of type condition  $(\tilde{K}_\Sigma)$  in the  $C^{r+1}$  case.

**Definition 3.9** (Condition  $(\tilde{K}_\Sigma^\delta)$ ). A map germ  $f \in \mathcal{E}_{[r+1]}(n, p)$ ,  $n \geq p$ , satisfies *condition*  $(\tilde{K}_\Sigma^\delta)$  if for any map germ  $g \in \mathcal{E}_{[r+1]}(n, p)$  satisfying  $j^r g(\Sigma; 0) = j^r f(\Sigma; 0)$  there exists  $\delta > 0$  (depending on  $g$ ), such that

$$d(x, \Sigma)\kappa(df(x)) + \|g(x)\| \gtrsim d(x, \Sigma)^{r+1-\delta}$$

holds in some neighbourhood of  $0 \in \mathbb{R}^n$ .

*Remark 3.10.* (1) The second relative Kuo condition  $(K_\Sigma^\delta)$ , and condition  $(\tilde{K}_\Sigma^\delta)$  are invariant under rotations.

(2) Condition  $(\tilde{K}_\Sigma^\delta)$  can be equivalently written as: for any map germ  $g \in \mathcal{E}_{[r+1]}(n, p)$  satisfying  $j^r g(\Sigma; 0) = j^r f(\Sigma; 0)$  there exists  $\delta > 0$  (depending on  $g$ ), such that

$$d(x, \Sigma)\kappa(dg(x)) + \|g(x)\| \gtrsim d(x, \Sigma)^{r+1-\delta}$$

holds in some neighbourhood of  $0 \in \mathbb{R}^n$ .

As a sufficient condition for an  $r$ -jet to be  $\Sigma$ - $V$ -sufficient in  $\mathcal{E}_{[r+1]}(n, p)$ , we have the following result.

**Theorem 3.11.** ([2]) *Let  $r$  be a positive integer, and let  $f \in \mathcal{E}_{[r+1]}(n, p)$ ,  $n \geq p$ . If  $f$  satisfies condition  $(K_\Sigma^\delta)$ , then the relative  $r$ -jet,  $j^r f(\Sigma; 0)$  is  $\Sigma$ - $V$ -sufficient in  $\mathcal{E}_{[r+1]}(n, p)$ .*

As a corollary of Theorem 3.11 and Lemma 2.3, we have the following.

**Corollary 3.12.** ([2]) *Let  $r$  be a positive integer, and let  $f \in \mathcal{E}_{[r+1]}(n, p)$ ,  $n \geq p$ . If there exists  $\delta > 0$  such that*

$$d(x, \Sigma)\kappa(df(x)) + \|f(x)\| \gtrsim d(x, \Sigma)^{r+1-\delta}$$

*holds in some neighbourhood of  $0 \in \mathbb{R}^n$ , then  $j^r f(\Sigma; 0)$  is  $\Sigma$ - $V$ -sufficient in  $\mathcal{E}_{[r+1]}(n, p)$ .*

## 4. MAIN RESULTS

**4.1. Conditions equivalent to the relative Kuo condition.** We first recall Rabier's function. Let  $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$  denote the space of linear mappings from  $\mathbb{R}^n$  to  $\mathbb{R}^p$ . For  $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ , we denote by  $T^*$  the adjoint map in  $\mathcal{L}(\mathbb{R}^p, \mathbb{R}^n)$ .

**Definition 4.1.** Let  $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ . Set

$$\nu(T) = \inf\{\|T^*(v)\| : v \in \mathbb{R}^p, \|v\| = 1\},$$

where  $T^*$  is the dual operator. This function is called *Rabier's function* ([14]).

We have the following facts on Rabier's function  $\nu(T)$  (see [12], [14] and [6] for instance):

- (1) Let  $\mathcal{S}$  be the set of non surjective linear map in  $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ . Then for all  $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ , we have:
  - (a)  $\nu(T) = d(T, \mathcal{S}) := \inf_{T' \in \mathcal{S}} \|T - T'\|$ .
  - (b)  $\nu(T) = \sup\{r > 0 : B(0, r) \subseteq T(B(0, r))\}$ .
  - (c) If  $T \in GL_n(\mathbb{R})$ , then  $\nu(T) = \frac{1}{\|T^{-1}\|}$ .
- (2) For  $T, T' \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$  we have  $\nu(T + T') \geq \nu(T) - \|T'\|$ .
- (3) The relationship  $\nu \approx \kappa$  holds between the Rabier's function and the Kuo distance. More precisely, for  $T = (T_1, \dots, T_p) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ , we have

$$\nu(T) \leq \kappa(T) \leq \sqrt{p} \nu(T).$$

**Definition 4.2.** Let  $A = [a_{ij}]$  be a matrix in  $\mathcal{M}_{n,p}(\mathbb{R})$ ,  $n \geq p$ . By  $M_I(A)$ , we denote a  $p \times p$  minor of  $A$  indexed by  $I$ , where  $I = (i_1, \dots, i_p)$  is any subsequence of  $(1, \dots, n)$ . Moreover, if  $J = (j_1, \dots, j_{p-1})$  is any subsequence of  $(1, \dots, n)$  and  $j \in \{1, \dots, p\}$ , then by  $M_J(j)(A)$  we denote an  $(p-1) \times (p-1)$  minor of  $A$  given by columns indexed by  $J$  and with deleted  $j$ -th row (if  $p = 1$  we put  $M_J(j)(A) = 1$ ). We define  $\eta : \mathcal{M}_{n,p}(\mathbb{R}) \rightarrow \mathbb{R}_+$  by

$$\eta(A) := \left( \frac{\sum_I |M_I(A)|^2}{\sum_{J,j} |M_J(j)(A)|^2} \right)^{\frac{1}{2}}.$$

It is easy to see that the above  $\eta$  is equivalent to the following non-negative function  $\tilde{\eta}$  in the sense that  $\eta \approx \tilde{\eta}$ :

$$\tilde{\eta}(A) := \max_I \frac{|M_I(A)|}{h_I(A)},$$

where  $h_I(A) = \max\{|M_J(j)(A)| : J \subset I, j = 1, \dots, p\}$ , with the convention that  $\frac{0}{0} = 0$ .

**Lemma 4.3.** *The relationship  $\eta \approx \kappa$  holds. More precisely, for  $T = (T_1, \dots, T_p) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ , we have*

$$\eta(T) \leq \kappa(T) \leq \sqrt{p} \eta(T).$$

*Remark 4.4.* The functions  $\nu$ ,  $\kappa$ ,  $\eta$  and  $\tilde{\eta}$  are continuous. In order to see these facts, it suffices to show that  $\eta$  is continuous at  $A \in \mathcal{M}_{p,p}(\mathbb{R})$ . It is obvious if the denominator is bigger than 0. Let us assume that the denominator is equal to 0. Then  $\eta(A) = 0$  and thus  $A \in \mathcal{S}$ . Let  $\{A_k\}$  be a sequence of elements of  $\mathcal{M}_{p,p}(\mathbb{R})$  which tend to  $A$ . Then there exists  $C > 0$  such that as  $k \rightarrow \infty$ , we have

$$\eta(A_k) \leq C\nu(A_k) = Cd(A_k, \mathcal{S}) \rightarrow 0.$$

We have the following equivalent conditions to the relative Kuo condition.

**Theorem 4.5.** *Let  $\Sigma$  be the germ at 0 of a closed subset of  $\mathbb{R}^n$ . For  $f \in \mathcal{E}_{[r]}(n, p)$ ,  $n \geq p$ , the following conditions are equivalent:*

- (1)  *$f$  satisfies condition  $(K_\Sigma)$ .*
- (2)  *$f$  satisfies condition  $(\tilde{K}_\Sigma)$ .*
- (3) *The inequality*

$$d(x, \Sigma) \left( \frac{\Gamma((\text{grad } f_i(x))_{1 \leq i \leq p})}{\sum_{j=1}^p \Gamma((\text{grad } f_i(x))_{i \neq j})} \right)^{\frac{1}{2}} + \|f(x)\| \gtrsim d(x, \Sigma)^r$$

*holds in some neighbourhood of  $0 \in \mathbb{R}^n$ , where*

$$\Gamma(v_1, \dots, v_k) := \det(\langle v_i, v_j \rangle_{\{i,j \in \{1, \dots, k\}\}})$$

*is the Gram determinant.*

- (4) *The inequality*

$$d(x, \Sigma) \|df^*(x)y\| + \|f(x)\| \gtrsim d(x, \Sigma)^r$$

holds for  $x$  in some neighbourhood of  $0 \in \mathbb{R}^n$ , uniformly for all  $y \in \mathbb{S}^{p-1}$ , where  $df^*(x)$  is the dual map of  $df(x)$ , and  $\mathbb{S}^{p-1}$  denotes the unit sphere in  $\mathbb{R}^p$  centred at  $0 \in \mathbb{R}^p$ .

*Proof.* As mentioned above, it is easy to see the equivalence (1)  $\iff$  (2).

The equivalence (2)  $\iff$  (3) follows from Lemma 4.3 and

$$\Delta(f, x) := \sum_{1 \leq i_1 < \dots < i_p \leq n} \left( \det \frac{D(f_1, \dots, f_p)}{D(x_{i_1}, \dots, x_{i_p})} \right)^2 = \Gamma((\text{grad } f_i(x))_{1 \leq i \leq p}).$$

Lastly we show the equivalence (2)  $\iff$  (4). For a given set of vectors  $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ , let  $\tilde{\kappa}(v_1, v_2, \dots, v_p)$  be defined as follows:

$$\tilde{\kappa}(v_1, v_2, \dots, v_p) := \min \left\{ \left| \sum_{i=1}^p \lambda_i v_i \right| : \lambda_i \in \mathbb{R}, \sum_{i=1}^p \lambda_i^2 = 1 \right\}.$$

Then the equivalence between conditions (2) and (4) follows from the equivalence between  $\kappa$  and  $\tilde{\kappa}$  (see Lemma 4.3) and the fact that

$$\|df^*(x)y\| = \left\| \sum_{i=1}^p y_i \text{grad } f_i(x) \right\| \gtrsim d(x, \Sigma)^{r-1} \quad \text{for all } y \in \mathbb{S}^{p-1}$$

is equivalent to

$$\tilde{\kappa}(\text{grad } f_1(x), \dots, \text{grad } f_p(x)) \gtrsim d(x, \Sigma)^{r-1}.$$

□

**4.2. Equivalent conditions to the second relative Kuo condition.** For condition  $(\tilde{K}_\Sigma^\delta)$ , we have the following equivalent conditions.

**Proposition 4.6.** *Let  $\Sigma$  be the germ at  $0$  of a closed subset of  $\mathbb{R}^n$ . For a map germ  $f \in \mathcal{E}_{[r+1]}(n, p)$ ,  $n \geq p$ , the following conditions are equivalent:*

- (1)  $f$  satisfies condition  $(\tilde{K}_\Sigma^\delta)$ .
- (2) For any map germ  $g \in \mathcal{E}_{[r+1]}(n, p)$  satisfying  $j^r g(\Sigma; 0) = j^r f(\Sigma; 0)$ , there exists  $\delta > 0$  (depending on  $g$ ), such that the inequality

$$d(x, \Sigma) \left( \frac{\Gamma((\text{grad } g_i(x))_{1 \leq i \leq p})}{\sum_{j=1}^p \Gamma((\text{grad } g_i(x))_{i \neq j})} \right)^{\frac{1}{2}} + \|g(x)\| \gtrsim d(x, \Sigma)^{r+1-\delta}$$

holds in some neighbourhood of  $0 \in \mathbb{R}^n$ .

- (3) For any map germ  $g \in \mathcal{E}_{[r+1]}(n, p)$  satisfying  $j^r g(\Sigma; 0) = j^r f(\Sigma; 0)$ , there exists  $\delta > 0$  (depending on  $g$ ), such that the inequality

$$d(x, \Sigma) \|dg^*(x)y\| + \|g(x)\| \gtrsim d(x, \Sigma)^{r+1-\delta}$$

holds for some  $\delta > 0$ , for  $x$  in some neighbourhood of  $0 \in \mathbb{R}^n$  and uniformly for all  $y \in \mathbb{S}^{p-1}$ .



(4) For any map germ  $g \in \mathcal{E}_{[r+1]}(n, p)$  satisfying  $j^r g(\Sigma; 0) = j^r f(\Sigma; 0)$ , there exists  $\delta > 0$  (depending on  $g$ ), such that the inequality

$$(4.1) \quad d(x, \Sigma) \|df^*(x)y\| + \|g(x)\| \gtrsim d(x, \Sigma)^{r+1-\delta}$$

holds in a neighbourhood of the origin, (uniformly) for all  $y \in \mathbb{S}^{p-1}$ .

*Proof.* The proofs of the equivalences between (1) (2), and (3) are identical to the corresponding ones in Theorem 4.5.

The equivalence between (3) and (4) is an application of Lemma 2.3.  $\square$

In the next theorem, we establish the equivalence between the second relative Kuo condition  $(K_\Sigma^\delta)$  and condition  $(\tilde{K}_\Sigma^\delta)$  for subanalytic maps and  $\Sigma$ .

**Theorem 4.7.** *Let  $\Sigma$  be the germ of a closed subanalytic set at  $0 \in \mathbb{R}^n$ , such that  $0$  is an accumulation point of  $\Sigma$  and  $\mathbb{R}^n \setminus \Sigma$ . For a subanalytic map germ  $f \in \mathcal{E}_{[r+1]}(n, p)$ ,  $n \geq p$ , the following conditions are equivalent:*

(1)  *$f$  satisfies the second relative Kuo condition  $(K_\Sigma^\delta)$  : for any subanalytic map germ  $g \in \mathcal{E}_{[r+1]}(n, p)$  with  $j^r g(\Sigma; 0) = j^r f(\Sigma; 0)$ , there are strictly positive numbers  $C, \alpha, \delta$  and  $\bar{w}$  (depending on  $g$ ) such that*

$$(4.2) \quad \kappa(df(x)) \geq Cd(x, \Sigma)^{r-\delta} \text{ in } \mathcal{H}_{r+1}^\Sigma(g; \bar{w}) \cap \{\|x\| < \alpha\}.$$

(2) *For any subanalytic map germ  $g \in \mathcal{E}_{[r+1]}(n, p)$  with  $j^r g(\Sigma; 0) = j^r f(\Sigma; 0)$ ,*

$$(4.3) \quad \frac{d(x, \Sigma)\kappa(df(x)) + \|g(x)\|}{d(x, \Sigma)^{r+1}} \rightarrow \infty \quad \text{as } d(x, \Sigma) \rightarrow 0, d(x, \Sigma) \neq 0.$$

(3)  *$f$  satisfies condition  $(\tilde{K}_\Sigma^\delta)$  : for any subanalytic map germ  $g \in \mathcal{E}_{[r+1]}(n, p)$  with  $j^r g(\Sigma; 0) = j^r f(\Sigma; 0)$ , there exists  $\delta > 0$  (depending on  $g$ ) such that*

$$(4.4) \quad d(x, \Sigma)\kappa(df(x)) + \|g(x)\| \gtrsim d(x, \Sigma)^{r+1-\delta}$$

holds in some neighbourhood of  $0 \in \mathbb{R}^n$ .

*Proof.* We first show the implication (1)  $\implies$  (2). If condition (2) does not hold, then there exist a realisation  $g$  of  $j^r f(\Sigma; 0)$ , and an analytic arc  $\gamma : I \rightarrow \mathbb{R}^n$  where  $I = [0, \beta)$ ,  $\beta > 0$ , such that  $\gamma(0) = 0 \in \mathbb{R}^n$  and for  $t \in I$

$$(4.5) \quad \|g(\gamma(t))\| \lesssim d(\gamma(t), \Sigma)^{r+1}, \quad \kappa(df(\gamma(t))) \lesssim d(\gamma(t), \Sigma)^r.$$

In particular, for any sequence  $\{x_i\}$  where  $x_i = \gamma(t_i)$ ,  $t_i \rightarrow 0$ ,  $t_i \neq 0$ , and for any  $\delta > 0$  we have

$$(4.6) \quad \|g(x_i)\| = o(d(x_i, \Sigma)^{r+1-\delta}), \quad \kappa(df(x_i)) = o(d(x_i, \Sigma)^{r-\delta}).$$

Then (4.6) implies that for any choice of the positive numbers  $C, \alpha, \delta$  and  $\bar{w}$ , the inequality

$$\kappa(df(x)) \geq Cd(x, \Sigma)^{r-\delta}$$

cannot hold in  $\mathcal{H}_{r+1}^\Sigma(g; \bar{w}) \cap \{\|x\| < \alpha\}$ , namely condition  $(K_\Sigma^\delta)$  is not satisfied. Therefore the implication (1)  $\implies$  (2) is shown.

We next show the implication (2)  $\implies$  (3). By condition (2), the continuous subanalytic function germ on  $(\mathbb{R}^n \setminus \Sigma, 0)$ , defined by

$$h(x) := \frac{d(x, \Sigma)^{r+1}}{d(x, \Sigma)\kappa(df(x)) + \|g(x)\|},$$

can be extended continuously across 0 on  $\Sigma$ . Since  $\Sigma = h^{-1}(0)$ , by the Lojasiewicz inequality ([13] §18), there is some  $\delta > 0$  such that

$$0 \leq h(x) \lesssim d(x, \Sigma)^\delta$$

in some neighbourhood of  $0 \in \mathbb{R}^n$ . Thus  $(\tilde{K}_\Sigma^\delta)$  is satisfied.

We lastly show the implication (3)  $\implies$  (1). Suppose that  $f$  satisfies condition  $(\tilde{K}_\Sigma^\delta)$ . Let  $g \in \mathcal{E}_{[r+1]}(n, p)$  with  $j^r g(\Sigma; 0) = j^r f(\Sigma; 0)$  which satisfies the condition that there are positive constants  $\delta, C$  and  $\alpha$  such that

$$(4.7) \quad d(x, \Sigma)\kappa(df(x)) + \|g(x)\| \geq Cd(x, \Sigma)^{r+1-\delta}$$

for  $x \in \mathbb{R}^n$ ,  $\|x\| < \alpha$ . If  $x$  is in the horn-neighbourhood  $\mathcal{H}_{r+1}^\Sigma(g; \frac{C}{2}) \cap \{\|x\| < \alpha\}$ , then

$$\|g(x)\| \leq \frac{C}{2}d(x, \Sigma)^{r+1} \leq \frac{C}{2}d(x, \Sigma)^{r+1-\delta}$$

and by (4.7)  $\kappa(df(x)) \geq \frac{C}{2}d(x, \Sigma)^{r-\delta}$ . Therefore condition  $(K_\Sigma^\delta)$  is satisfied; which shows the implication (3)  $\implies$  (1).  $\square$

*Remark 4.8.* From the proof, we can see that without the assumption of subanalyticity, the implication (3)  $\implies$  (1) in Theorem 4.7 holds, namely condition  $(\tilde{K}_\Sigma^\delta)$  implies the second relative Kuo condition  $(K_\Sigma^\delta)$ .

Let us now introduce a (ostensibly) weaker condition in terms of  $f$  only (namely, not using all the realisations of the jet  $j^r f(\Sigma; 0)$ ), to be compared to V. Kozyakin [7].

**Definition 4.9.** A map germ  $f \in \mathcal{E}_{[r+1]}(n, p)$ ,  $n \geq p$ , satisfies condition  $(KZ_\Sigma)$  if

$$(4.8) \quad \frac{d(x, \Sigma)\|df^*(x)y\| + \|f(x)\|}{d(x, \Sigma)^{r+1}} \rightarrow \infty \quad \text{as } d(x, \Sigma) \rightarrow 0, d(x, \Sigma) \neq 0$$

in a neighbourhood of the origin, (uniformly) for all  $y \in \mathbb{S}^{p-1}$ .

We have another equivalent condition to the second relative Kuo condition.

**Theorem 4.10.** *Let  $\Sigma$  be the germ at 0 of a closed subanalytic subset of  $\mathbb{R}^n$ , and let  $f \in \mathcal{E}_{[r+1]}(n, p)$ ,  $n \geq p$ , be a subanalytic map germ. Then  $f$  satisfies the second relative Kuo condition, equivalently for any subanalytic map germ  $g \in \mathcal{E}_{[r+1]}(n, p)$  with  $j^r g(\Sigma; 0) = j^r f(\Sigma; 0)$ , there are positive constants  $\delta, C$  and  $\alpha$  such that*

$$(4.9) \quad d(x, \Sigma)\kappa(dg(x)) + \|g(x)\| \geq Cd(x, \Sigma)^{r+1-\delta},$$

for  $\|x\| < \alpha$  if and only if  $f$  satisfies condition  $(KZ_\Sigma)$ .

*Proof.* By Theorem 4.6, condition (4.9) is equivalent to condition (4), which implies  $(KZ_\Sigma)$  for any map germ  $g \in \mathcal{E}_{[r+1]}(n, p)$  such that  $j^r g(\Sigma; 0) = j^r f(\Sigma; 0)$ , in particular, for  $f$ .

To prove the converse, we will use the subanalyticity. Suppose that condition  $(KZ_\Sigma)$  is satisfied. Let  $g \in \mathcal{E}_{[r+1]}(n, p)$  be a subanalytic map germ such that  $j^r g(\Sigma; 0) = j^r f(\Sigma; 0)$ , and set  $h(x) := g(x) - f(x)$ . By Lemma 2.3,  $\|h(x)\| \lesssim d(x, \Sigma)^{r+1}$  for sufficiently small values of  $\|x\|$ . Then, for all  $y \in \mathbb{S}^{p-1}$

$$d(x, \Sigma)\|df^*(x)y\| + \|g(x)\| = d(x, \Sigma)\|df^*(x)y\| + \|f(x) + h(x)\|,$$

and

$$\frac{d(x, \Sigma)\|df^*(x)y\| + \|g(x)\|}{d(x, \Sigma)^{r+1}} \geq \frac{d(x, \Sigma)\|df^*(x)y\| + \|f(x)\|}{d(x, \Sigma)^{r+1}} - \frac{\|h(x)\|}{d(x, \Sigma)^{r+1}}.$$

Since  $\frac{\|h(x)\|}{d(x, \Sigma)^{r+1}}$  is bounded,  $\lim_{d(x, \Sigma) \rightarrow 0} \frac{d(x, \Sigma)\|df^*(x)y\| + \|f(x)\|}{d(x, \Sigma)^{r+1}} = \infty$  which implies

$\lim_{d(x, \Sigma) \rightarrow 0} \frac{d(x, \Sigma)\|df^*(x)y\| + \|g(x)\|}{d(x, \Sigma)^{r+1}} = \infty$ . Therefore condition (4.1) is satisfied.

Now the continuous subanalytic function germ on  $(\mathbb{R}^n \setminus \Sigma, 0) \times \mathbb{S}^{p-1}$ , defined by

$$q(x, y) := \frac{d(x, \Sigma)^{r+1}}{d(x, \Sigma)\|df^*(x)y\| + \|g(x)\|},$$

can be extended continuously across 0 to  $\Sigma \times \mathbb{S}^{p-1}$ . Since  $\Sigma \times \mathbb{S}^{p-1} = q^{-1}(0)$ , by the Lojasiewicz inequality, there is some  $\delta > 0$  such that

$$|q(x, y)| \lesssim d((x, y), \Sigma \times \mathbb{S}^{p-1})^\delta = d(x, \Sigma)^\delta$$

in some neighbourhood of  $0 \in \mathbb{R}^n$ . This is exactly condition (4) in Proposition 4.6.  $\square$

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INSTITUT DE RECHERCHE MATHÉMATIQUE DE RENNES, UNIVERSITÉ DE RENNES 1, CAMPUS BEAULIEU, 35042 RENNES CEDEX, FRANCE

DEPARTMENT OF MATHEMATICS, HYOGO UNIVERSITY OF TEACHER EDUCATION, KATO, HYOGO 673-1494, JAPAN

*Email address:* karim.bekka@univ-rennes1.fr

*Email address:* koike@hyogo-u.ac.jp