Continuum mechanics in finite deformation

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Chapitre 1

Some elements of linear algebra

1.1 Notation

- λ, μ, α, β, γ etc., or a, b, c etc. lower case greek or latin letters : scalars, which are here real numbers.
- $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, $\vec{e}$, $\vec{f}$ etc. lower case latin letters with an overlain arrow : vectors of a $\mathbb{R}$-vector space E whose dimension is 3.
- $\widetilde{\boldsymbol{A}}$, $\widetilde{\boldsymbol{B}}$, $\widetilde{\boldsymbol{C}}$, $\widetilde{\boldsymbol{D}}$, $\widetilde{\boldsymbol{E}}$ etc. blod capital latin letters with an overlain tilde : second-order tensors, elements of space $E \otimes E$ (a dimension-9 vector space), that is to say also, linear map from E to E (endomorphism).
- $\overset{\sim 4}{\boldsymbol{A}}$, $\overset{\sim 4}{\boldsymbol{B}}$, $\overset{\sim 4}{\boldsymbol{C}}$, $\overset{\sim 4}{\boldsymbol{D}}$, $\overset{\sim 4}{\boldsymbol{E}}$ etc. bold capital greek or latin letters with an overlain tilde and a 4 figure : fourth-order tensors, elements of space $E \otimes E \otimes E \otimes E$ (a dimension-81 vector space), that is to say also, linear map from $E \otimes E$ to $E \otimes E$.

1.2 Operator « » (nothing) or «.» – product with a scalar

$$\lambda.\vec{a} = \vec{b} = \lambda\vec{a} \tag{1.1}$$

$$\lambda.\widetilde{\boldsymbol{A}} = \widetilde{\boldsymbol{B}} = \lambda\widetilde{\boldsymbol{A}} \tag{1.2}$$

1.3 Operator «+» – addition

$$\lambda + \mu = \alpha \tag{1.3}$$

$$\vec{a} + \vec{b} = \vec{c} \tag{1.4}$$

$$\widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{B}} = \widetilde{\boldsymbol{C}} \tag{1.5}$$

Properties –

$$\lambda.\left(\vec{a} + \vec{b}\right) = \lambda.\vec{a} + \lambda.\vec{b} \tag{1.6}$$

$$(\lambda + \mu).\vec{a} = \lambda.\vec{a} + \mu.\vec{a} \tag{1.7}$$

$$\lambda.\left(\widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{B}}\right) = \lambda.\widetilde{\boldsymbol{A}} + \lambda.\widetilde{\boldsymbol{B}} \tag{1.8}$$

$$(\lambda + \mu).\widetilde{\boldsymbol{A}} = \lambda.\widetilde{\boldsymbol{A}} + \mu.\widetilde{\boldsymbol{A}} \tag{1.9}$$

1.4 Operator «.» – scalar product

$$\vec{a} \centerdot \vec{b} = \lambda \tag{1.10}$$

By definition, this operator fulfills the following relations :

1. $\left(\lambda.\vec{a} + \mu.\vec{b}\right) \centerdot \vec{c} = \lambda.\left(\vec{a} \centerdot \vec{c}\right) + \mu.\left(\vec{b} \centerdot \vec{c}\right)$,
2. $\vec{a} \centerdot \vec{b} = \vec{b} \centerdot \vec{a}$,
3. $\forall \vec{a} \neq \vec{0}$, $\vec{a} \centerdot \vec{a} > 0$,
4. si $\vec{a} = \vec{0}$, $\vec{a} \centerdot \vec{a} = 0$.

Properties –

$$\text{If} \quad \vec{a} \centerdot \vec{a} = 0 \quad \text{Then} \quad \vec{a} = \vec{0}\ . \tag{1.11}$$

This is derived simply from properties 3 and 4.

Properties –

$$\text{If} \quad \forall \vec{c} \quad \vec{a} \centerdot \vec{c} = 0 \quad \text{Then} \quad \vec{a} = \vec{0}\ . \tag{1.12}$$

Indeed, because this is true $\forall \vec{c}$, let us choose $\vec{c} = \vec{a}$. Then, we have $\vec{a} \centerdot \vec{a} = 0$. From property 1.11, it derives that $\vec{a} = \vec{0}$.

1.5 Operator «.» – a map “applied to”

$$\widetilde{\boldsymbol{A}} \centerdot \vec{a} = \vec{b}\ , \tag{1.13}$$

where $\vec{b}$ is the result of $\vec{a}$ according to linear map $\widetilde{\boldsymbol{A}}$.

Properties –

$$\widetilde{\boldsymbol{A}}\boldsymbol{.}\left(\lambda.\vec{a}+\mu.\vec{b}\right)=\lambda.\left(\widetilde{\boldsymbol{A}}\boldsymbol{.}\vec{a}\right)+\mu.\left(\widetilde{\boldsymbol{A}}\boldsymbol{.}\vec{b}\right) \tag{1.14}$$

$$\left(\lambda.\widetilde{\boldsymbol{A}}+\mu.\widetilde{\boldsymbol{B}}\right)\boldsymbol{.}\vec{a}=\lambda.\left(\widetilde{\boldsymbol{A}}\boldsymbol{.}\vec{a}\right)+\mu.\left(\widetilde{\boldsymbol{B}}\boldsymbol{.}\vec{a}\right) \tag{1.15}$$

1.6 Operator «.» – linear map composition

$$\widetilde{\boldsymbol{A}}\boldsymbol{.}\widetilde{\boldsymbol{B}}=\widetilde{\boldsymbol{C}} \tag{1.16}$$

such that $\forall\vec{a}$

$$\widetilde{\boldsymbol{C}}\boldsymbol{.}\vec{a}=\widetilde{\boldsymbol{A}}\boldsymbol{.}\left(\widetilde{\boldsymbol{B}}\boldsymbol{.}\vec{a}\right) \tag{1.17}$$

Properties –

$$\widetilde{\boldsymbol{A}}\boldsymbol{.}\left(\lambda.\widetilde{\boldsymbol{B}}+\mu.\widetilde{\boldsymbol{C}}\right)=\lambda.\widetilde{\boldsymbol{A}}\boldsymbol{.}\widetilde{\boldsymbol{B}}+\mu.\widetilde{\boldsymbol{A}}\boldsymbol{.}\widetilde{\boldsymbol{C}} \tag{1.18}$$

$$\left(\lambda.\widetilde{\boldsymbol{A}}+\mu.\widetilde{\boldsymbol{B}}\right)\boldsymbol{.}\widetilde{\boldsymbol{C}}=\lambda.\widetilde{\boldsymbol{A}}\boldsymbol{.}\widetilde{\boldsymbol{C}}+\mu.\widetilde{\boldsymbol{B}}\boldsymbol{.}\widetilde{\boldsymbol{C}} \tag{1.19}$$

1.7 Operator «$*^{-1}$» – inverse

Properties – *If* $\det\widetilde{\boldsymbol{A}}\neq 0$*, then tensor* $\widetilde{\boldsymbol{A}}$ *is invertible, that is to say it exists one tensor* $\widetilde{\boldsymbol{A}}^{\boldsymbol{-1}}$ *such that*

$$\widetilde{\boldsymbol{A}}\boldsymbol{.}\widetilde{\boldsymbol{A}}^{\boldsymbol{-1}}=\widetilde{\boldsymbol{A}}^{\boldsymbol{-1}}\boldsymbol{.}\widetilde{\boldsymbol{A}}=\widetilde{\boldsymbol{I}}\ , \tag{1.20}$$

$$\left(\widetilde{\boldsymbol{A}}\boldsymbol{.}\widetilde{\boldsymbol{B}}\right)^{\boldsymbol{-1}}=\widetilde{\boldsymbol{B}}^{\boldsymbol{-1}}\boldsymbol{.}\widetilde{\boldsymbol{A}}^{\boldsymbol{-1}}, \tag{1.21}$$

and

$$\det\left(\widetilde{\boldsymbol{A}}^{\boldsymbol{-1}}\right)=\frac{1}{\det\widetilde{\boldsymbol{A}}}\ . \tag{1.22}$$

Property – *Let be* $(\vec{e_1},\vec{e_2},\vec{e_3})$ ***an arbitrary basis*** *and* $\widetilde{\boldsymbol{A}}$ *an invertible second-order tensor,*

$$Mat_{\vec{e_i}}\left(\widetilde{\boldsymbol{A}}^{\boldsymbol{-1}}\right)=\left[Mat_{\vec{e_i}}\left(\widetilde{\boldsymbol{A}}\right)\right]^{\boldsymbol{-1}}\ . \tag{1.23}$$

1.8 Operator «$*^{\mathbf{T}}$» – transpose

$$\widetilde{\boldsymbol{A}}^{\mathbf{T}} = \widetilde{\boldsymbol{C}} \tag{1.24}$$

such that $\forall \vec{a}$ and $\forall \vec{b}$,

$$\vec{a} \cdot \left(\widetilde{\boldsymbol{A}} \cdot \vec{b}\right) = \vec{b} \cdot \left(\widetilde{\boldsymbol{A}}^{\mathbf{T}} \cdot \vec{a}\right) \tag{1.25}$$

Properties –

$$\left(\lambda.\widetilde{\boldsymbol{A}} + \mu.\widetilde{\boldsymbol{B}}\right)^{\mathbf{T}} = \lambda.\widetilde{\boldsymbol{A}}^{\mathbf{T}} + \mu.\widetilde{\boldsymbol{B}}^{\mathbf{T}} \tag{1.26}$$

$$\left(\widetilde{\boldsymbol{A}} \cdot \widetilde{\boldsymbol{B}}\right)^{\mathbf{T}} = \widetilde{\boldsymbol{B}}^{\mathbf{T}} \cdot \widetilde{\boldsymbol{A}}^{\mathbf{T}} \tag{1.27}$$

If tensor $\widetilde{\boldsymbol{A}}$ is invertible, then

$$\left(\widetilde{\boldsymbol{A}}^{-1}\right)^{\mathbf{T}} = \left(\widetilde{\boldsymbol{A}}^{\mathbf{T}}\right)^{-1} \overset{\text{written}}{=} \widetilde{\boldsymbol{A}}^{-\mathbf{T}} \tag{1.28}$$

Property – *If $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ is an* ***orthonormal basis*** *and $\widetilde{\boldsymbol{A}}$ a second-order tensor,*

$$Mat_{\vec{e_i}}\left(\widetilde{\boldsymbol{A}}^{\mathbf{T}}\right) = \left[Mat_{\vec{e_i}}\left(\widetilde{\boldsymbol{A}}\right)\right]^{\mathbf{T}} . \tag{1.29}$$

If $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ is not orthonormal,

$$Mat_{\vec{e_i}}\left(\widetilde{\boldsymbol{A}}^{\mathbf{T}}\right) \neq \left[Mat_{\vec{e_i}}\left(\widetilde{\boldsymbol{A}}\right)\right]^{\mathbf{T}} . \tag{1.30}$$

Definition – *A second-order tensor is* ***symmetric****, if and only if :*

$$\widetilde{\boldsymbol{A}} = \widetilde{\boldsymbol{A}}^{\mathbf{T}} . \tag{1.31}$$

Definition – *A second-order tensor is* ***skew-symmetric****, if and only if :*

$$\widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{A}}^{\mathbf{T}} = \widetilde{\mathbf{0}} . \tag{1.32}$$

Definitions – *The* ***symmetric part*** *$\widetilde{\boldsymbol{A}}^{S}$ of a second-order tensor $\widetilde{\boldsymbol{A}}$ is defined as*

$$\widetilde{\boldsymbol{A}}^{S} = \frac{1}{2}\left(\widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{A}}^{\mathbf{T}}\right) . \tag{1.33}$$

*The **skew-symmetric part** $\widetilde{\boldsymbol{A}}^A$ of a second-order tensor $\widetilde{\boldsymbol{A}}$ is defined as*

$$\widetilde{\boldsymbol{A}}^A = \frac{1}{2}\left(\widetilde{\boldsymbol{A}} - \widetilde{\boldsymbol{A}}^{\mathbf{T}}\right) . \tag{1.34}$$

Definition – *A second-order tensor is **orthogonal**, if and only if;*

$$\widetilde{\boldsymbol{A}} \boldsymbol{.} \widetilde{\boldsymbol{A}}^{\mathbf{T}} = \widetilde{\boldsymbol{A}}^{\mathbf{T}} \boldsymbol{.} \widetilde{\boldsymbol{A}} = \widetilde{\boldsymbol{I}} . \tag{1.35}$$

Property – *The determinant of an orthogonal tensor $\widetilde{\boldsymbol{A}}$ is*

$$\det \widetilde{\boldsymbol{A}} = \pm 1 \tag{1.36}$$

Property – *If tensor $\widetilde{\boldsymbol{Q}}$ is the one of a rotation, then it is orthogonal, and*

$$\det \widetilde{\boldsymbol{Q}} = +1 \tag{1.37}$$

Property – *If tensor $\widetilde{\boldsymbol{S}}$ is the one of a symmetry, then it is symmetric and orthogonal, and*

$$\det \widetilde{\boldsymbol{S}} = -1 \tag{1.38}$$

1.9 Symmetric second-order tensors

Definition – *A second-order tensor is **symmetric**, if and only if :*

$$\widetilde{\boldsymbol{A}} = \widetilde{\boldsymbol{A}}^{\mathbf{T}} . \tag{1.39}$$

Eigendecomposition of symmetric tensors – *In our context, any symmetric second-ordre tensor is diagonalizable into an orthonormal basis.*

Product of two symmetric tensors – *If $\widetilde{\boldsymbol{A}}$ and $\widetilde{\boldsymbol{B}}$ are two symmetric second-order tensors :*

$$\widetilde{\boldsymbol{A}} \boldsymbol{.} \widetilde{\boldsymbol{B}} = \widetilde{\boldsymbol{B}} \boldsymbol{.} \widetilde{\boldsymbol{A}} \tag{1.40}$$

if and only if, the 3 eigendirections of $\widetilde{\boldsymbol{A}}$ are parallel to the ones of $\widetilde{\boldsymbol{B}}$.

Properties – *Let $\widetilde{\boldsymbol{A}}$ be a symmetric second-order tensor. Let $(\vec{a_1}, \vec{a_2}, \vec{a_3})$ be an orthonormal basis of eigenvectors of $\widetilde{\boldsymbol{A}}$. Let (a_1, a_2, a_3) be the associated eigenvalues.*

If

$$\widetilde{\boldsymbol{A}} \centerdot \dot{\widetilde{\boldsymbol{A}}} = \dot{\widetilde{\boldsymbol{A}}} \centerdot \widetilde{\boldsymbol{A}} \tag{1.41}$$

Then

$$\dot{\widetilde{\boldsymbol{A}}} = \sum_i \dot{a}_i (\vec{a}_i \otimes \vec{a}_i) \tag{1.42}$$

If

$$\widetilde{\boldsymbol{A}} \centerdot \dot{\widetilde{\boldsymbol{A}}} = \dot{\widetilde{\boldsymbol{A}}} \centerdot \widetilde{\boldsymbol{A}} \tag{1.43}$$

and all (a_1, a_2, a_3) are each distinct from the others, Then

$$\forall i \in [1, 2, 3] \quad \dot{\vec{a}}_i = \vec{0} \tag{1.44}$$

1.10 Invariants of a second-order tensor

[Truesdell-2004] defined the principal invariants of a second-ordre tensor $\widetilde{\boldsymbol{A}}$ (here in the particular case of a dimension of 3 for the vector space), as the coefficients of the following polynomial in λ :

$$\det\left(\lambda\widetilde{\boldsymbol{I}} + \widetilde{\boldsymbol{A}}\right) = \lambda^3 + \mathrm{Tr}\widetilde{\boldsymbol{A}}.\lambda^2 + \sec\widetilde{\boldsymbol{A}}.\lambda + \det\widetilde{\boldsymbol{A}} \tag{1.45}$$

This could be also written as :

$$\det\left(\widetilde{\boldsymbol{A}} - x\widetilde{\boldsymbol{I}}\right) = -x^3 + \mathrm{Tr}\widetilde{\boldsymbol{A}}.x^2 - \sec\widetilde{\boldsymbol{A}}.x + \det\widetilde{\boldsymbol{A}} \tag{1.46}$$

that is called the *characteristic polynomial* of tensor $\widetilde{\boldsymbol{A}}$.

[Truesdell-2004] defined other important invariants, called *moments* :

$$\mathrm{Tr}\widetilde{\boldsymbol{A}} \qquad \mathrm{Tr}(\widetilde{\boldsymbol{A}} \centerdot \widetilde{\boldsymbol{A}}) \qquad \mathrm{Tr}(\widetilde{\boldsymbol{A}} \centerdot \widetilde{\boldsymbol{A}} \centerdot \widetilde{\boldsymbol{A}}) \tag{1.47}$$

The eigenvalues (a_1, a_2, a_3) of tensor $\widetilde{\boldsymbol{A}}$ are solutions of the characteristic polynomial of tensor $\widetilde{\boldsymbol{A}}$:

$$\det\left(\widetilde{\boldsymbol{A}} - x\widetilde{\boldsymbol{I}}\right) = k.(x - a_1).(x - a_2).(x - a_3) \tag{1.48}$$

where k is a non-null scalar. So, the relations between the principal invariants and the eigenvalues are :

$$\mathrm{Tr}\widetilde{\boldsymbol{A}} = a_1 + a_2 + a_3 \tag{1.49}$$

$$\sec\widetilde{\boldsymbol{A}} = a_1.a_2 + a_1.a_3 + a_2.a_3 \tag{1.50}$$

$$\det\widetilde{\boldsymbol{A}} = a_1.a_2.a_3 \tag{1.51}$$

By calculating $\det\left(\widetilde{\boldsymbol{A}} - x\widetilde{\boldsymbol{I}}\right)$ using the matrix components A_{ij} of $\widetilde{\boldsymbol{A}}$ into an arbitrary basis, and the characteristic polynomial we obtain :

$$\mathrm{Tr}\widetilde{\boldsymbol{A}} = A_{11} + A_{22} + A_{33} \tag{1.52}$$

$$\sec\widetilde{\boldsymbol{A}} = A_{11}.A_{22} + A_{11}.A_{33} + A_{22}.A_{33} - A_{12}.A_{21} - A_{13}.A_{31} - A_{23}.A_{32} \tag{1.53}$$

$$\det\widetilde{\boldsymbol{A}} = \det\left[A_{ij}\right] \tag{1.54}$$

As an example, the principal invariants of a rotation tensor $\widetilde{\boldsymbol{Q}}$, of angle θ, are :

$$\mathrm{Tr}\widetilde{\boldsymbol{Q}} = 1 + 2\cos\theta \tag{1.55}$$

$$\sec\widetilde{\boldsymbol{Q}} = 1 + 2\cos\theta = \mathrm{Tr}\widetilde{\boldsymbol{Q}} \tag{1.56}$$

$$\det\widetilde{\boldsymbol{Q}} = +1 \tag{1.57}$$

and its eigenvalues are

$$q_1 = +1 \qquad q_2 = \cos\theta + i.\sin\theta \qquad q_3 = \cos\theta - i.\sin\theta \tag{1.58}$$

As a second example, the principal invariants of a symmetry tensor $\widetilde{\boldsymbol{S}}$, are :

$$\mathrm{Tr}\widetilde{\boldsymbol{S}} = 1 \qquad \sec\widetilde{\boldsymbol{S}} = -1 \qquad \det\widetilde{\boldsymbol{S}} = -1 \tag{1.59}$$

and its eigenvalues are

$$s_1 = +1 \qquad s_2 = +1 \qquad s_3 = -1 \tag{1.60}$$

1.10.1 First principal invariant or operator «Tr»

The result is a scalar :

$$\mathrm{Tr}\widetilde{\boldsymbol{A}} = \lambda \tag{1.61}$$

If the matrix of $\widetilde{\boldsymbol{A}}$ from an arbitrary basis $(\vec{e_1}, \vec{e_2}, \vec{e_3})$, is

$$Mat_{\vec{e_i}}\left(\widetilde{\boldsymbol{A}}\right) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \tag{1.62}$$

then

$$\mathrm{Tr}\widetilde{\boldsymbol{A}} = A_{11} + A_{22} + A_{33} \ . \tag{1.63}$$

Properties –

$$\mathrm{Tr}\left(\lambda.\widetilde{\boldsymbol{A}} + \mu.\widetilde{\boldsymbol{B}}\right) = \lambda.\mathrm{Tr}\widetilde{\boldsymbol{A}} + \mu.\mathrm{Tr}\widetilde{\boldsymbol{B}} \tag{1.64}$$

$$\mathrm{Tr}\widetilde{\boldsymbol{I}} = 3 \tag{1.65}$$

$$\mathrm{Tr}\left(\widetilde{\boldsymbol{A}} \boldsymbol{.} \widetilde{\boldsymbol{B}}\right) = \mathrm{Tr}\left(\widetilde{\boldsymbol{B}} \boldsymbol{.} \widetilde{\boldsymbol{A}}\right) \tag{1.66}$$

$$\mathrm{Tr}\left(\widetilde{\boldsymbol{A}}^{\mathbf{T}}\right) = \mathrm{Tr}\widetilde{\boldsymbol{A}} \tag{1.67}$$

Definition – *A second-order tensor is* ***isotropic*** *, if and only if :*

$$\widetilde{\boldsymbol{A}} = k.\widetilde{\boldsymbol{I}} \ , \tag{1.68}$$

where k is a scalar.

Definition – *A second-order tensor is* ***deviatoric*** *, if and only if :*

$$\mathrm{Tr}\widetilde{\boldsymbol{A}} = 0 \ . \tag{1.69}$$

Definition – *The* ***isotropic part*** $\widetilde{\boldsymbol{A}}^{I}$ *of a second-order tensor* $\widetilde{\boldsymbol{A}}$ *is defined as*

$$\widetilde{\boldsymbol{A}}^{I} = \frac{\mathrm{Tr}\widetilde{\boldsymbol{A}}}{3}.\widetilde{\boldsymbol{I}} \ . \tag{1.70}$$

Properties –

$$\mathrm{Tr}\left(\widetilde{\boldsymbol{A}}^{I}\right) = \mathrm{Tr}\widetilde{\boldsymbol{A}} \tag{1.71}$$

$$\left(\lambda\widetilde{\boldsymbol{A}} + \mu\widetilde{\boldsymbol{B}}\right)^{I} = \lambda.\widetilde{\boldsymbol{A}}^{I} + \mu.\widetilde{\boldsymbol{B}}^{I} \tag{1.72}$$

$$\left(\widetilde{\boldsymbol{A}}^{\mathbf{T}}\right)^{I} = \left(\widetilde{\boldsymbol{A}}^{I}\right)^{\mathbf{T}} = \widetilde{\boldsymbol{A}}^{I} \tag{1.73}$$

Definition – *The* ***deviatoric part*** $\underline{\widetilde{\boldsymbol{A}}}$ *of a second-order tensor* $\widetilde{\boldsymbol{A}}$ *is defined as*

$$\underline{\widetilde{\boldsymbol{A}}} = \widetilde{\boldsymbol{A}} - \frac{\mathrm{Tr}\widetilde{\boldsymbol{A}}}{3}.\widetilde{\boldsymbol{I}} = \widetilde{\boldsymbol{A}} - \widetilde{\boldsymbol{A}}^{I} \ . \tag{1.74}$$

Properties –

$$\mathrm{Tr}\underline{\widetilde{\boldsymbol{A}}} = 0 \tag{1.75}$$

$$\underline{\lambda\widetilde{\boldsymbol{A}} + \mu\widetilde{\boldsymbol{B}}} = \lambda\underline{\widetilde{\boldsymbol{A}}} + \mu\underline{\widetilde{\boldsymbol{B}}} \tag{1.76}$$

$$\underline{\left(\widetilde{\boldsymbol{A}}^{\mathbf{T}}\right)} = (\underline{\widetilde{\boldsymbol{A}}})^{\mathbf{T}} \tag{1.77}$$

$$\underline{\left(\widetilde{\boldsymbol{A}}^{S}\right)} = (\underline{\widetilde{\boldsymbol{A}}})^{S} \tag{1.78}$$

$$\underline{\left(\widetilde{\boldsymbol{A}}^{A}\right)} = (\underline{\widetilde{\boldsymbol{A}}})^{A} \tag{1.79}$$

1.10.2 Second principal invariant or operator «sec»

The result is a scalar :

$$\sec\widetilde{\boldsymbol{A}} = \lambda = \frac{1}{2}\left[\mathrm{Tr}\widetilde{\boldsymbol{A}}.\mathrm{Tr}\widetilde{\boldsymbol{A}} - \mathrm{Tr}\left(\widetilde{\boldsymbol{A}} \centerdot \widetilde{\boldsymbol{A}}\right)\right] , \tag{1.80}$$

If the matrix of $\widetilde{\boldsymbol{A}}$ from an arbitrary basis $(\vec{e_1}, \vec{e_2}, \vec{e_3})$, is

$$Mat_{\vec{e_i}}\left(\widetilde{\boldsymbol{A}}\right) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} , \tag{1.81}$$

then

$$\begin{aligned}\sec\widetilde{\boldsymbol{A}} = A_{11}.A_{22} + A_{11}.A_{33} + A_{22}.A_{33} \\ - A_{12}.A_{21} - A_{13}.A_{31} - A_{23}.A_{32} .\end{aligned} \tag{1.82}$$

Warning –

$$\sec\left(\lambda.\widetilde{\boldsymbol{A}} + \mu.\widetilde{\boldsymbol{B}}\right) \neq \lambda.\sec\widetilde{\boldsymbol{A}} + \mu.\sec\widetilde{\boldsymbol{B}} \tag{1.83}$$

Properties –

$$\sec\left(\lambda.\widetilde{\boldsymbol{A}}\right) = \lambda^2.\sec\widetilde{\boldsymbol{A}} \tag{1.84}$$

$$\sec\widetilde{\boldsymbol{I}} = 3 \tag{1.85}$$

$$\sec\left(\widetilde{\boldsymbol{A}} \centerdot \widetilde{\boldsymbol{B}}\right) = \sec\left(\widetilde{\boldsymbol{B}} \centerdot \widetilde{\boldsymbol{A}}\right) \tag{1.86}$$

$$\sec\left(\widetilde{\boldsymbol{A}}^{\mathbf{T}}\right) = \sec\widetilde{\boldsymbol{A}} \tag{1.87}$$

1.10.3 Third principal invariant or operator «det»

The result is a scalar :

$$\begin{aligned}\det\widetilde{\boldsymbol{A}} = \lambda &= \frac{1}{3}\left\{\mathrm{Tr}\widetilde{\boldsymbol{A}} \cdot \left[\sec\widetilde{\boldsymbol{A}} - \mathrm{Tr}\left(\widetilde{\boldsymbol{A}} \cdot \widetilde{\boldsymbol{A}}\right)\right] + \mathrm{Tr}\left(\widetilde{\boldsymbol{A}} \cdot \widetilde{\boldsymbol{A}} \cdot \widetilde{\boldsymbol{A}}\right)\right\} \\ &= \frac{1}{6}\left[\mathrm{Tr}\widetilde{\boldsymbol{A}}.\mathrm{Tr}\widetilde{\boldsymbol{A}}.\mathrm{Tr}\widetilde{\boldsymbol{A}} - 3\mathrm{Tr}\widetilde{\boldsymbol{A}}.\mathrm{Tr}\left(\widetilde{\boldsymbol{A}} \cdot \widetilde{\boldsymbol{A}}\right) + 2\mathrm{Tr}\left(\widetilde{\boldsymbol{A}} \cdot \widetilde{\boldsymbol{A}} \cdot \widetilde{\boldsymbol{A}}\right)\right] \quad (1.88)\end{aligned}$$

These two equations are derived from the trace operator applied to the Cayley-Hamilton theorem 1.149.

If the matrix of $\widetilde{\boldsymbol{A}}$ from an arbitrary basis $(\vec{e_1}, \vec{e_2}, \vec{e_3})$, is

$$Mat_{\vec{e_i}}\left(\widetilde{\boldsymbol{A}}\right) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \quad (1.89)$$

then $\det\widetilde{\boldsymbol{A}}$ is the determinant of this matrix ;

$$\begin{aligned}\det\widetilde{\boldsymbol{A}} &= A_{11}.(A_{22}.A_{33} - A_{32}.A_{23}) \\ &\quad -A_{21}.(A_{12}.A_{33} - A_{32}.A_{13}) \\ &\quad +A_{31}.(A_{12}.A_{23} - A_{22}.A_{13}) \;. \quad (1.90)\end{aligned}$$

Warning –

$$\det\left(\lambda.\widetilde{\boldsymbol{A}} + \mu.\widetilde{\boldsymbol{B}}\right) \neq \lambda.\det\widetilde{\boldsymbol{A}} + \mu.\det\widetilde{\boldsymbol{B}} \quad (1.91)$$

Properties –

$$\det\left(\lambda.\widetilde{\boldsymbol{A}}\right) = \lambda^3.\det\widetilde{\boldsymbol{A}} \quad (1.92)$$

$$\det\widetilde{\boldsymbol{I}} = 1 \quad (1.93)$$

$$\det\left(\widetilde{\boldsymbol{A}} \cdot \widetilde{\boldsymbol{B}}\right) = \det\widetilde{\boldsymbol{A}}.\det\widetilde{\boldsymbol{B}} = \det\left(\widetilde{\boldsymbol{B}} \cdot \widetilde{\boldsymbol{A}}\right) \quad (1.94)$$

$$\det\left(\widetilde{\boldsymbol{A}}^{\mathbf{T}}\right) = \det\widetilde{\boldsymbol{A}} \quad (1.95)$$

1.11 Operator «:» – double dot product

Definition – *The double dot product between two second-order tensors is defined as*

$$\widetilde{\boldsymbol{A}} : \widetilde{\boldsymbol{B}} = \mathrm{Tr}\left(\widetilde{\boldsymbol{A}} \cdot \widetilde{\boldsymbol{B}}^{\mathbf{T}}\right) \tag{1.96}$$

Property – *If $\widetilde{\boldsymbol{A}}$ is symmetric and $\widetilde{\boldsymbol{B}}$ is skew-symmetric, then :*

$$\widetilde{\boldsymbol{A}} : \widetilde{\boldsymbol{B}} = 0\ . \tag{1.97}$$

Indeed,

$$\begin{aligned}
\widetilde{\boldsymbol{A}} : \widetilde{\boldsymbol{B}} &= \mathrm{Tr}\left(\widetilde{\boldsymbol{A}} \cdot \widetilde{\boldsymbol{B}}^{\mathbf{T}}\right) && (1.98)\\
&= \mathrm{Tr}\left(\widetilde{\boldsymbol{A}}^{\mathbf{T}} \cdot (-\widetilde{\boldsymbol{B}})\right) && (1.99)\\
&= -\mathrm{Tr}\left(\widetilde{\boldsymbol{B}} \cdot \widetilde{\boldsymbol{A}}^{\mathbf{T}}\right) && (1.100)\\
&= -\widetilde{\boldsymbol{B}} : \widetilde{\boldsymbol{A}} && (1.101)\\
&= -\widetilde{\boldsymbol{A}} : \widetilde{\boldsymbol{B}} && (1.102)
\end{aligned}$$

Property – *If $\widetilde{\boldsymbol{A}}$ is isotropic and $\widetilde{\boldsymbol{B}}$ is deviatoric, then :*

$$\widetilde{\boldsymbol{A}} : \widetilde{\boldsymbol{B}} = 0\ . \tag{1.103}$$

Indeed,

$$\begin{aligned}
\widetilde{\boldsymbol{A}} : \widetilde{\boldsymbol{B}} &= \mathrm{Tr}\left(k.\widetilde{\boldsymbol{I}} \cdot \widetilde{\boldsymbol{B}}^{\mathbf{T}}\right) && (1.104)\\
&= k.\mathrm{Tr}\left(\widetilde{\boldsymbol{B}}^{\mathbf{T}}\right) && (1.105)\\
&= k.\mathrm{Tr}\left(\widetilde{\boldsymbol{B}}\right) && (1.106)\\
&= 0 && (1.107)
\end{aligned}$$

Property – *The double dot product is a scalar product of the vector space of second-order tensors :*

1. $\left(\lambda.\widetilde{\boldsymbol{A}} + \mu.\widetilde{\boldsymbol{B}}\right) : \widetilde{\boldsymbol{C}} = \lambda.\left(\widetilde{\boldsymbol{A}} : \widetilde{\boldsymbol{C}}\right) + \mu.\left(\widetilde{\boldsymbol{B}} : \widetilde{\boldsymbol{C}}\right)$,
2. $\widetilde{\boldsymbol{A}} : \widetilde{\boldsymbol{B}} = \widetilde{\boldsymbol{B}} : \widetilde{\boldsymbol{A}}$,
3. $\forall \widetilde{\boldsymbol{A}} \neq \widetilde{\mathbf{0}},\ \widetilde{\boldsymbol{A}} : \widetilde{\boldsymbol{A}} > 0$,

4. *si* $\widetilde{\boldsymbol{A}} = \widetilde{\boldsymbol{0}}$, $\widetilde{\boldsymbol{A}} : \widetilde{\boldsymbol{A}} = 0$.

Property –

$$\text{If} \quad \widetilde{\boldsymbol{A}} : \widetilde{\boldsymbol{A}} = 0 \quad \text{Then} \quad \widetilde{\boldsymbol{A}} = \widetilde{\boldsymbol{0}}\ . \tag{1.108}$$

Property –

$$\text{If} \quad \forall \widetilde{\boldsymbol{C}} \quad \widetilde{\boldsymbol{A}} : \widetilde{\boldsymbol{C}} = 0 \quad \text{Then} \quad \widetilde{\boldsymbol{A}} = \widetilde{\boldsymbol{0}}\ . \tag{1.109}$$

Property –

$$\text{If} \quad \forall \widetilde{\boldsymbol{C}} \ \textit{symmetric second-order tensor} \quad \widetilde{\boldsymbol{A}} : \widetilde{\boldsymbol{C}} = 0$$
$$\textit{Then } \widetilde{\boldsymbol{A}} \textit{ is skew-symmetric, i.e.} \quad \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{A}}^{\mathbf{T}} = \widetilde{\boldsymbol{0}}\ . \tag{1.110}$$

Indeed, any second-order tensor $\widetilde{\boldsymbol{A}}$ *can be divided into its symmetric and skew-symmetric parts :*

$$\widetilde{\boldsymbol{A}} = \widetilde{\boldsymbol{A}}^{S} + \widetilde{\boldsymbol{A}}^{A} \tag{1.111}$$

Then, in the present case,

$$\begin{aligned} \widetilde{\boldsymbol{A}} : \widetilde{\boldsymbol{C}} &= (\widetilde{\boldsymbol{A}}^{S} + \widetilde{\boldsymbol{A}}^{A}) : \widetilde{\boldsymbol{C}} && (1.112) \\ &= \widetilde{\boldsymbol{A}}^{S} : \widetilde{\boldsymbol{C}} && (1.113) \\ & \quad \forall \widetilde{\boldsymbol{C}} \ \textit{symmetric second-order tensor} && (1.114) \end{aligned}$$

then $\widetilde{\boldsymbol{A}}^{S} = 0$.

Property –

$$\text{If} \quad \forall \widetilde{\boldsymbol{C}} \ \textit{deviatoric second-order tensor} \quad \widetilde{\boldsymbol{A}} : \widetilde{\boldsymbol{C}} = 0$$
$$\textit{Then } \widetilde{\boldsymbol{A}} \textit{ is isotropic, i.e.} \quad \widetilde{\boldsymbol{A}} = k.\widetilde{\boldsymbol{I}} \tag{1.115}$$

where $k \in \mathcal{R}$ *and* $k = \frac{\mathrm{Tr}\widetilde{\boldsymbol{A}}}{3}$.

Indeed, any second-order tensor $\widetilde{\boldsymbol{A}}$ *can be divided into its isotropic and deviatoric parts :*

$$\widetilde{\boldsymbol{A}} = \underline{\widetilde{\boldsymbol{A}}} + \frac{\mathrm{Tr}\widetilde{\boldsymbol{A}}}{3}\widetilde{\boldsymbol{I}} \tag{1.116}$$

Then, in the present case,

$$\begin{aligned}
\widetilde{\boldsymbol{A}} : \widetilde{\boldsymbol{C}} &= \left(\underline{\widetilde{\boldsymbol{A}}} + \frac{\mathrm{Tr}\widetilde{\boldsymbol{A}}}{3} \widetilde{\boldsymbol{I}} \right) : \widetilde{\boldsymbol{C}} && (1.117) \\
&= \underline{\widetilde{\boldsymbol{A}}} : \widetilde{\boldsymbol{C}} && (1.118) \\
& \quad \forall \widetilde{\boldsymbol{C}} \text{ deviatoric second-order tensor} && (1.119)
\end{aligned}$$

then $\underline{\widetilde{\boldsymbol{A}}} = 0$.

Property –

If $\forall \widetilde{\boldsymbol{C}}$ *deviatoric and symmetric second-order tensor* $\widetilde{\boldsymbol{A}} : \widetilde{\boldsymbol{C}} = 0$

Then the symmetric part of $\widetilde{\boldsymbol{A}}$ *is isotropic,*

$$\text{i.e.} \quad \widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{A}}^{\mathbf{T}} = 2k.\widetilde{\boldsymbol{I}} \quad (1.120)$$

where $k \in \mathcal{R}$ *and* $k = \frac{\mathrm{Tr}\widetilde{\boldsymbol{A}}}{3}$.

Indeed, any second-order tensor $\widetilde{\boldsymbol{A}}$ *can be divided into its skew-symmetric part, the deviatoric part of its symmetric part and its isotropic part :*

$$\widetilde{\boldsymbol{A}} = \widetilde{\boldsymbol{A}}^{A} + \underline{\widetilde{\boldsymbol{A}}^{S}} + \frac{\mathrm{Tr}\widetilde{\boldsymbol{A}}}{3} \widetilde{\boldsymbol{I}} \quad (1.121)$$

Then, in the present case,

$$\begin{aligned}
\widetilde{\boldsymbol{A}} : \widetilde{\boldsymbol{C}} &= \left(\widetilde{\boldsymbol{A}}^{A} + \underline{\widetilde{\boldsymbol{A}}^{S}} + \frac{\mathrm{Tr}\widetilde{\boldsymbol{A}}}{3} \widetilde{\boldsymbol{I}} \right) : \widetilde{\boldsymbol{C}} && (1.122) \\
&= \underline{\widetilde{\boldsymbol{A}}^{S}} : \widetilde{\boldsymbol{C}} && (1.123) \\
& \quad \forall \widetilde{\boldsymbol{C}} \text{ deviatoric and symmetric second-order tensor} && (1.124)
\end{aligned}$$

then $\underline{\widetilde{\boldsymbol{A}}^{S}} = 0$.

1.12 Gradient of invariants and Cayley-Hamilton's theorem

Let

$$\epsilon(\widetilde{\boldsymbol{A}}) \quad (1.125)$$

be a scalar-valued tensor function, the result of which is scalar, and its variable is a second-order tensor $\widetilde{\boldsymbol{A}}$. [Truesdell-2004] defined the gradient of this function

$$\widetilde{\frac{\boldsymbol{\partial\epsilon(\widetilde{A})}}{\boldsymbol{\partial\widetilde{A}}}} \tag{1.126}$$

as, for any second-order tensor $\widetilde{\boldsymbol{C}}$,

$$\forall\widetilde{\boldsymbol{C}} \qquad \widetilde{\frac{\boldsymbol{\partial\epsilon(\widetilde{A})}}{\boldsymbol{\partial\widetilde{A}}}} : \widetilde{\boldsymbol{C}} = \frac{d\left[\epsilon(\widetilde{\boldsymbol{A}} + s.\widetilde{\boldsymbol{C}})\right]}{ds}(s=0) \tag{1.127}$$

In the case of an invertible second-order tensor $\widetilde{\boldsymbol{A}}$, [Truesdell-2004] calculated the gradient of $\det(\widetilde{\boldsymbol{A}})$ as follows. One can prove that

$$\forall\widetilde{\boldsymbol{C}} \qquad \det\left(\widetilde{\boldsymbol{A}} + s.\widetilde{\boldsymbol{C}}\right) = s^3.\det\widetilde{\boldsymbol{A}}.\det\left(\frac{1}{s}\widetilde{\boldsymbol{I}} + \widetilde{\boldsymbol{A}}^{\boldsymbol{-1}} \boldsymbol{.} \widetilde{\boldsymbol{C}}\right) \tag{1.128}$$

Moreover (characteristic polynomial 1.45),

$$\forall\widetilde{\boldsymbol{C}} \qquad \det\left(\frac{1}{s}\widetilde{\boldsymbol{I}} + \widetilde{\boldsymbol{A}}^{\boldsymbol{-1}} \boldsymbol{.} \widetilde{\boldsymbol{C}}\right) =$$
$$\left(\frac{1}{s}\right)^3 + \mathrm{Tr}\left(\widetilde{\boldsymbol{A}}^{\boldsymbol{-1}} \boldsymbol{.} \widetilde{\boldsymbol{C}}\right).\left(\frac{1}{s}\right)^2 + \sec\left(\widetilde{\boldsymbol{A}}^{\boldsymbol{-1}} \boldsymbol{.} \widetilde{\boldsymbol{C}}\right).\left(\frac{1}{s}\right) + \det\left(\widetilde{\boldsymbol{A}}^{\boldsymbol{-1}} \boldsymbol{.} \widetilde{\boldsymbol{C}}\right) \tag{1.129}$$

Then

$$\forall\widetilde{\boldsymbol{C}} \qquad \det\left(\widetilde{\boldsymbol{A}} + s.\widetilde{\boldsymbol{C}}\right) = \det\widetilde{\boldsymbol{A}}.$$
$$\left[1 + \mathrm{Tr}\left(\widetilde{\boldsymbol{A}}^{\boldsymbol{-1}} \boldsymbol{.} \widetilde{\boldsymbol{C}}\right).s + \sec\left(\widetilde{\boldsymbol{A}}^{\boldsymbol{-1}} \boldsymbol{.} \widetilde{\boldsymbol{C}}\right).s^2 + \det\left(\widetilde{\boldsymbol{A}}^{\boldsymbol{-1}} \boldsymbol{.} \widetilde{\boldsymbol{C}}\right).s^3\right] \tag{1.130}$$

So,

$$\forall\widetilde{\boldsymbol{C}} \qquad \frac{d\left[\det\left(\widetilde{\boldsymbol{A}} + s.\widetilde{\boldsymbol{C}}\right)\right]}{ds}(s=0) = \det\widetilde{\boldsymbol{A}}\left[\mathrm{Tr}\left(\widetilde{\boldsymbol{A}}^{\boldsymbol{-1}} \boldsymbol{.} \widetilde{\boldsymbol{C}}\right)\right] \tag{1.131}$$
$$= \left[\det\widetilde{\boldsymbol{A}} \;.\; \widetilde{\boldsymbol{A}}^{\boldsymbol{-T}}\right] : \widetilde{\boldsymbol{C}} \tag{1.132}$$

Finally, this proves the following property.

Gradient of the determinant – *If $\widetilde{\boldsymbol{A}}$ is an invertible second-order tensor, then*

$$\frac{\widetilde{\partial \det(\widetilde{\boldsymbol{A}})}}{\partial \widetilde{\boldsymbol{A}}} = \det\widetilde{\boldsymbol{A}} \;.\; \widetilde{\boldsymbol{A}}^{-\mathbf{T}} \tag{1.133}$$

[Truesdell-2004] proved the theorem of Cayley-Hamilton as follows. Since the characteristic polynomial 1.45, the following gradient is

$$\frac{\widetilde{\partial \det(\lambda\widetilde{\boldsymbol{I}} + \widetilde{\boldsymbol{A}})}}{\partial \widetilde{\boldsymbol{A}}} = \frac{\widetilde{\partial \mathrm{Tr}\widetilde{\boldsymbol{A}}}}{\partial \widetilde{\boldsymbol{A}}}.\lambda^2 + \frac{\widetilde{\partial \sec\widetilde{\boldsymbol{A}}}}{\partial \widetilde{\boldsymbol{A}}}.\lambda + \frac{\widetilde{\partial \det\widetilde{\boldsymbol{A}}}}{\partial \widetilde{\boldsymbol{A}}} \tag{1.134}$$

Because 1.133, this is also equal to

$$\frac{\widetilde{\partial \det(\lambda\widetilde{\boldsymbol{I}} + \widetilde{\boldsymbol{A}})}}{\partial \widetilde{\boldsymbol{A}}} = \frac{\widetilde{\partial \det(\lambda\widetilde{\boldsymbol{I}} + \widetilde{\boldsymbol{A}})}}{\partial (\lambda\widetilde{\boldsymbol{I}} + \widetilde{\boldsymbol{A}})} : \left[\frac{\overset{\sim 4}{\partial (\lambda\widetilde{\boldsymbol{I}} + \widetilde{\boldsymbol{A}})}}{\partial \widetilde{\boldsymbol{A}}}\right] \tag{1.135}$$

$$= \frac{\widetilde{\partial \det(\lambda\widetilde{\boldsymbol{I}} + \widetilde{\boldsymbol{A}})}}{\partial (\lambda\widetilde{\boldsymbol{I}} + \widetilde{\boldsymbol{A}})} : \overset{\sim 4}{\boldsymbol{I}} \tag{1.136}$$

$$= \det\left(\lambda\widetilde{\boldsymbol{I}} + \widetilde{\boldsymbol{A}}\right) \;.\; \left(\lambda\widetilde{\boldsymbol{I}} + \widetilde{\boldsymbol{A}}\right)^{-\mathbf{T}} \tag{1.137}$$

This last equation is true only if tensor $(\lambda\widetilde{\boldsymbol{I}} + \widetilde{\boldsymbol{A}})$ is invertible. That is the case when

$$\lambda \notin \{-a_1, -a_2, -a_3\} \tag{1.138}$$

where a_1, a_2 and a_3 are the eigenvalues of tensor $\widetilde{\boldsymbol{A}}$.

So, $\forall\lambda \notin \{-a_1, -a_2, -a_3\}$:

$$\det\left(\lambda\widetilde{\boldsymbol{I}} + \widetilde{\boldsymbol{A}}\right) \;.\; \widetilde{\boldsymbol{I}}$$

$$= \left(\lambda\widetilde{\boldsymbol{I}} + \widetilde{\boldsymbol{A}}\right)^{\mathbf{T}} \boldsymbol{.} \left(\frac{\widetilde{\partial \mathrm{Tr}\widetilde{\boldsymbol{A}}}}{\partial \widetilde{\boldsymbol{A}}}.\lambda^2 + \frac{\widetilde{\partial \sec\widetilde{\boldsymbol{A}}}}{\partial \widetilde{\boldsymbol{A}}}.\lambda + \frac{\widetilde{\partial \det\widetilde{\boldsymbol{A}}}}{\partial \widetilde{\boldsymbol{A}}}\right)$$

$$= \left(\lambda\widetilde{\boldsymbol{I}} + \widetilde{\boldsymbol{A}}^{\mathbf{T}}\right) \boldsymbol{.} \left(\frac{\widetilde{\partial \mathrm{Tr}\widetilde{\boldsymbol{A}}}}{\partial \widetilde{\boldsymbol{A}}}.\lambda^2 + \frac{\widetilde{\partial \sec\widetilde{\boldsymbol{A}}}}{\partial \widetilde{\boldsymbol{A}}}.\lambda + \frac{\widetilde{\partial \det\widetilde{\boldsymbol{A}}}}{\partial \widetilde{\boldsymbol{A}}}\right)$$

$$= \frac{\widetilde{\partial \mathrm{Tr}\widetilde{\boldsymbol{A}}}}{\partial \widetilde{\boldsymbol{A}}}.\lambda^3 + \frac{\widetilde{\partial \sec\widetilde{\boldsymbol{A}}}}{\partial \widetilde{\boldsymbol{A}}}.\lambda^2 + \frac{\widetilde{\partial \det\widetilde{\boldsymbol{A}}}}{\partial \widetilde{\boldsymbol{A}}}.\lambda$$

$$+ \widetilde{\boldsymbol{A}}^{\mathbf{T}} \boldsymbol{.} \frac{\widetilde{\partial \mathrm{Tr}\widetilde{\boldsymbol{A}}}}{\partial \widetilde{\boldsymbol{A}}}.\lambda^2 + \widetilde{\boldsymbol{A}}^{\mathbf{T}} \boldsymbol{.} \frac{\widetilde{\partial \sec\widetilde{\boldsymbol{A}}}}{\partial \widetilde{\boldsymbol{A}}}.\lambda + \widetilde{\boldsymbol{A}}^{\mathbf{T}} \boldsymbol{.} \frac{\widetilde{\partial \det\widetilde{\boldsymbol{A}}}}{\partial \widetilde{\boldsymbol{A}}} \tag{1.139}$$

Because 1.45, this is also equal to

$$\det\left(\lambda\widetilde{\boldsymbol{I}}+\widetilde{\boldsymbol{A}}\right)\ .\ \widetilde{\boldsymbol{I}}=\left(\lambda^3+\mathrm{Tr}\widetilde{\boldsymbol{A}}.\lambda^2+\sec\widetilde{\boldsymbol{A}}.\lambda+\det\widetilde{\boldsymbol{A}}\right)\widetilde{\boldsymbol{I}} \tag{1.140}$$

From these two expressions, the two polynomials in λ are equal, then their associated coefficients are equal :

$$\widetilde{\boldsymbol{I}}=\widetilde{\frac{\partial\mathrm{Tr}\widetilde{\boldsymbol{A}}}{\partial\widetilde{\boldsymbol{A}}}} \tag{1.141}$$

$$\mathrm{Tr}\widetilde{\boldsymbol{A}}\ .\ \widetilde{\boldsymbol{I}}=\widetilde{\frac{\partial\sec\widetilde{\boldsymbol{A}}}{\partial\widetilde{\boldsymbol{A}}}}+\widetilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{.}\widetilde{\frac{\partial\mathrm{Tr}\widetilde{\boldsymbol{A}}}{\partial\widetilde{\boldsymbol{A}}}} \tag{1.142}$$

$$\sec\widetilde{\boldsymbol{A}}\ .\ \widetilde{\boldsymbol{I}}=\widetilde{\frac{\partial\det\widetilde{\boldsymbol{A}}}{\partial\widetilde{\boldsymbol{A}}}}+\widetilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{.}\widetilde{\frac{\partial\sec\widetilde{\boldsymbol{A}}}{\partial\widetilde{\boldsymbol{A}}}} \tag{1.143}$$

$$\det\widetilde{\boldsymbol{A}}\ .\ \widetilde{\boldsymbol{I}}=\widetilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{.}\widetilde{\frac{\partial\det\widetilde{\boldsymbol{A}}}{\partial\widetilde{\boldsymbol{A}}}} \tag{1.144}$$

So, the gradients of all principal invariants are :

Properties –

$$\widetilde{\frac{\partial\mathrm{Tr}\widetilde{\boldsymbol{A}}}{\partial\widetilde{\boldsymbol{A}}}}=\widetilde{\boldsymbol{I}} \tag{1.145}$$

$$\widetilde{\frac{\partial\sec\widetilde{\boldsymbol{A}}}{\partial\widetilde{\boldsymbol{A}}}}=\mathrm{Tr}\widetilde{\boldsymbol{A}}\ .\ \widetilde{\boldsymbol{I}}-\widetilde{\boldsymbol{A}}^{\mathrm{T}} \tag{1.146}$$

$$\widetilde{\frac{\partial\det\widetilde{\boldsymbol{A}}}{\partial\widetilde{\boldsymbol{A}}}}=\sec\widetilde{\boldsymbol{A}}\ .\ \widetilde{\boldsymbol{I}}-\mathrm{Tr}\widetilde{\boldsymbol{A}}\ .\ \widetilde{\boldsymbol{A}}^{\mathrm{T}}+\widetilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{.}\widetilde{\boldsymbol{A}}^{\mathrm{T}} \tag{1.147}$$

From equation 1.147 and 1.144, we have :

$$\det\widetilde{\boldsymbol{A}}\ .\ \widetilde{\boldsymbol{I}}=\sec\widetilde{\boldsymbol{A}}\ .\ \widetilde{\boldsymbol{A}}^{\mathrm{T}}-\mathrm{Tr}\widetilde{\boldsymbol{A}}\ .\ \widetilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{.}\widetilde{\boldsymbol{A}}^{\mathrm{T}}+\widetilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{.}\widetilde{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{.}\widetilde{\boldsymbol{A}}^{\mathrm{T}} \tag{1.148}$$

By calculating the transpose of this equation, we obtain the theorem of Cayley-Hamilton.

Theorem of Cayley-Hamilton – *Any second-order tensor $\widetilde{\boldsymbol{A}}$ is solution of its own characteristic polynomial :*

$$-\widetilde{\boldsymbol{A}}\boldsymbol{.}\widetilde{\boldsymbol{A}}\boldsymbol{.}\widetilde{\boldsymbol{A}}+\left(\mathrm{Tr}\widetilde{\boldsymbol{A}}\right).\widetilde{\boldsymbol{A}}\boldsymbol{.}\widetilde{\boldsymbol{A}}-\left(\sec\widetilde{\boldsymbol{A}}\right).\widetilde{\boldsymbol{A}}+\left(\det\widetilde{\boldsymbol{A}}\right).\widetilde{\boldsymbol{I}}\ =\ \widetilde{\boldsymbol{0}} \tag{1.149}$$

in the case of a 3-dimension vector space E.

This theorem gives an expression of the inverse of a tensor, if its determinant is not null :

$$\widetilde{\boldsymbol{A}}^{-1} = \frac{1}{\det \widetilde{\boldsymbol{A}}} . \left[\widetilde{\boldsymbol{A}} \cdot \widetilde{\boldsymbol{A}} - \left(\mathrm{Tr} \widetilde{\boldsymbol{A}} \right) . \widetilde{\boldsymbol{A}} + \left(\sec \widetilde{\boldsymbol{A}} \right) . \widetilde{\boldsymbol{I}} \right] \ . \tag{1.150}$$

1.13 Positive-definite second-order tensor

Definition – *A **symmetric** second-order tensor is **positive-definite**, if and only if :*

$$\forall \vec{x} \neq \vec{0} \qquad \vec{x} \cdot \left(\widetilde{\boldsymbol{A}} \cdot \vec{x} \right) > 0 \ . \tag{1.151}$$

Eigenvalues of symmetric positive-definite second-order tensors – *Any symmetric second-order tensor is positive-definite, if and only if, its eigenvalues are positive (non null).*

Polar decomposition – *Let $\widetilde{\boldsymbol{A}}$ be a invertible second-order tensor. It exists $\widetilde{\boldsymbol{B}}$ orthogonal second-order tensor, and $\widetilde{\boldsymbol{C}}$ symmetric positive-definite second-order tensor, such that [Truesdell-2004]*

$$\widetilde{\boldsymbol{A}} = \widetilde{\boldsymbol{B}} \cdot \widetilde{\boldsymbol{C}} \ . \tag{1.152}$$

Moreover, if $\det \widetilde{\boldsymbol{A}} > 0$, then $\det \widetilde{\boldsymbol{B}} = +1$ and so $\widetilde{\boldsymbol{B}}$ is a rotation tensor.

1.14 Some fourth-order tensors

The jacobian (or rigidity or tangent) *matrices* of the constitutive laws could be described by fourth-order tensors. For example, a jacobian matrix $\overset{\sim 4}{\boldsymbol{J}}$ may be defined as the linear map, which, applied to the stretching tensor $\widetilde{\boldsymbol{D}}$, gives the Jaumann objective rate of the Kirchhoff stress tensor ;

$$\overset{\circ j}{\overline{J\widetilde{\boldsymbol{T}}}} = J . \overset{\sim 4}{\boldsymbol{J}} : \widetilde{\boldsymbol{D}} \ . \tag{1.153}$$

This is the definition of the jacobian matrices according to the finite element code Abaqus.

We may define the following fourth-order tensors, useful for calculating the jacobian matrices :

– Fourth-order identity tensor $\overset{\sim 4}{\boldsymbol{I}}$

$$\forall \widetilde{\boldsymbol{X}} \qquad \overset{\sim 4}{\boldsymbol{I}} : \widetilde{\boldsymbol{X}} = \widetilde{\boldsymbol{X}} \; . \tag{1.154}$$

– The tensor $\overset{\sim 4}{\boldsymbol{P}}$ of the projection on the space of the deviatoric second-order tensors

$$\forall \widetilde{\boldsymbol{X}} \qquad \overset{\sim 4}{\boldsymbol{P}} : \widetilde{\boldsymbol{X}} = \underline{\widetilde{\boldsymbol{X}}} = \widetilde{\boldsymbol{X}} - \frac{\mathrm{Tr}\,\widetilde{\boldsymbol{X}}}{3} \widetilde{\boldsymbol{I}} \; . \tag{1.155}$$

– Fourth-order tensor $\overset{\sim 4}{\boldsymbol{M}}_{(\widetilde{\boldsymbol{Y}}, \widetilde{\boldsymbol{Z}})}$

$$\forall \widetilde{\boldsymbol{X}} \qquad \overset{\sim 4}{\boldsymbol{M}}_{(\widetilde{\boldsymbol{Y}}, \widetilde{\boldsymbol{Z}})} : \widetilde{\boldsymbol{X}} = \mathrm{Tr}\left(\widetilde{\boldsymbol{X}} \boldsymbol{\cdot} \widetilde{\boldsymbol{Z}}^{\mathbf{T}} \right) . \widetilde{\boldsymbol{Y}} \; . \tag{1.156}$$

– Fourth-order tensor $\overset{\sim 4}{\boldsymbol{N}}_{(\widetilde{\boldsymbol{Y}})}$

$$\forall \widetilde{\boldsymbol{X}} \qquad \overset{\sim 4}{\boldsymbol{N}}_{(\widetilde{\boldsymbol{Y}})} : \widetilde{\boldsymbol{X}} = \widetilde{\boldsymbol{X}} \boldsymbol{\cdot} \widetilde{\boldsymbol{Y}} + \widetilde{\boldsymbol{Y}} \boldsymbol{\cdot} \widetilde{\boldsymbol{X}} \; . \tag{1.157}$$

– Fourth-order tensor $\overset{\sim 4}{\boldsymbol{Q}}_{(\widetilde{\boldsymbol{Y}})}$

$$\forall \widetilde{\boldsymbol{X}} \qquad \overset{\sim 4}{\boldsymbol{Q}}_{(\widetilde{\boldsymbol{Y}})} : \widetilde{\boldsymbol{X}} = \widetilde{\boldsymbol{Y}} \boldsymbol{\cdot} \widetilde{\boldsymbol{X}} \boldsymbol{\cdot} \widetilde{\boldsymbol{Y}} \; . \tag{1.158}$$

– Fourth-order tensor $\overset{\sim 4}{\boldsymbol{R}}_{(\widetilde{\boldsymbol{Y}})}$

$$\forall \widetilde{\boldsymbol{X}} \qquad \overset{\sim 4}{\boldsymbol{R}}_{(\widetilde{\boldsymbol{Y}})} : \widetilde{\boldsymbol{X}} = \widetilde{\boldsymbol{X}} \boldsymbol{\cdot} \widetilde{\boldsymbol{Y}} - \widetilde{\boldsymbol{Y}} \boldsymbol{\cdot} \widetilde{\boldsymbol{X}} \; . \tag{1.159}$$

Above, $\widetilde{\boldsymbol{Y}}$ and $\widetilde{\boldsymbol{Z}}$ are second-order tensors.

Definition – *Fourth-order tensor $\overset{\sim 4}{\gamma}_{(\widetilde{\boldsymbol{A}})}$ is a function of a symmetric positive-definite second-order tensor $\widetilde{\boldsymbol{A}}$.*

The following matrix of this tensor $\overset{\sim 4}{\gamma}_{(\widetilde{\boldsymbol{A}})}$ is expressed in Voigt-Mandel notation from a orthonormal basis $(\vec{a_1}, \vec{a_2}, \vec{a_3})$ of eigenvectors of $\widetilde{\boldsymbol{A}}$. The sequence of the components is $(11, 22, 33, 12, 13, 23)$, for the Voigt-Mandel notation.

$$Mat_{\mathrm{VoigtMandel}(\vec{a_i})} \left(\overset{\sim 4}{\gamma}_{(\widetilde{\boldsymbol{A}})} \right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & g(a_1, a_2) & 0 & 0 \\ 0 & 0 & 0 & 0 & g(a_1, a_3) & 0 \\ 0 & 0 & 0 & 0 & 0 & g(a_2, a_3) \end{bmatrix} \tag{1.160}$$

where (a_1, a_2, a_3) are the positive eigenvalues of $\widetilde{\boldsymbol{A}}$ associated respectively to eigenvectors $(\vec{a_1}, \vec{a_2}, \vec{a_3})$, and function $g(x, y)$ is :

$$g(x,y) = \frac{2xy\ln(x/y)}{(x^2-y^2)} \quad \text{si} \quad x \neq y \tag{1.161}$$
$$= 1 \quad \text{si} \quad x = y\ . \tag{1.162}$$

with x and y are positive real (non-null).

Properties – *Let us assume $\widetilde{\boldsymbol{A}}$ be a symmetric positive-definite tensor. For all symmetric tensors $\widetilde{\boldsymbol{B}}$ and $\widetilde{\boldsymbol{C}}$,*

$$\overset{\sim 4}{\gamma}_{(\widetilde{\boldsymbol{A}})} : \widetilde{\boldsymbol{I}} = \widetilde{\boldsymbol{I}} \tag{1.163}$$

$$\overset{\sim 4}{\gamma}_{(\widetilde{\boldsymbol{A}})} : \widetilde{\boldsymbol{A}} = \widetilde{\boldsymbol{A}} \tag{1.164}$$

$$\overset{\sim 4}{\gamma}_{(\widetilde{\boldsymbol{A}})} : (\widetilde{\boldsymbol{A}} \,.\, \widetilde{\boldsymbol{A}}) = \widetilde{\boldsymbol{A}} \,.\, \widetilde{\boldsymbol{A}} \tag{1.165}$$

$$\forall n \in \mathbb{N} \quad \overset{\sim 4}{\gamma}_{(\widetilde{\boldsymbol{A}})} : (\widetilde{\boldsymbol{A}}^n) = \widetilde{\boldsymbol{A}}^n = \widetilde{\boldsymbol{A}} \,.\, \cdots \,.\, \widetilde{\boldsymbol{A}} \tag{1.166}$$

$$\overset{\sim 4}{\gamma}_{(\widetilde{\boldsymbol{A}})} : \widetilde{\boldsymbol{LnA}} = \widetilde{\boldsymbol{LnA}} \tag{1.167}$$

$$\mathrm{Tr}\left[\overset{\sim 4}{\gamma}_{(\widetilde{\boldsymbol{A}})} : \widetilde{\boldsymbol{B}}\right] = \mathrm{Tr}\widetilde{\boldsymbol{B}} \tag{1.168}$$

$$\underline{\left[\overset{\sim 4}{\gamma}_{(\widetilde{\boldsymbol{A}})} : \widetilde{\boldsymbol{B}}\right]} = \overset{\sim 4}{\gamma}_{(\widetilde{\boldsymbol{A}})} : \underline{\widetilde{\boldsymbol{B}}} \tag{1.169}$$

$$\widetilde{\boldsymbol{C}} : \left[\overset{\sim 4}{\gamma}_{(\widetilde{\boldsymbol{A}})} : \widetilde{\boldsymbol{B}}\right] = \widetilde{\boldsymbol{B}} : \left[\overset{\sim 4}{\gamma}_{(\widetilde{\boldsymbol{A}})} : \widetilde{\boldsymbol{C}}\right] \tag{1.170}$$

Property – *For all rotation tensor $\widetilde{\boldsymbol{Q}}$, and all symmetric tensors $\widetilde{\boldsymbol{A}}$ and $\widetilde{\boldsymbol{B}}$,*

$$\widetilde{\boldsymbol{Q}} \,.\left[\overset{\sim 4}{\gamma}_{(\widetilde{\boldsymbol{A}})} : \widetilde{\boldsymbol{B}}\right] .\, \widetilde{\boldsymbol{Q}}^{\mathrm{T}} = \overset{\sim 4}{\gamma}_{(\widetilde{\boldsymbol{Q}}.\widetilde{\boldsymbol{A}}.\widetilde{\boldsymbol{Q}}^{\mathrm{T}})} : \left(\widetilde{\boldsymbol{Q}} \,.\, \widetilde{\boldsymbol{B}} \,.\, \widetilde{\boldsymbol{Q}}^{\mathrm{T}}\right) \tag{1.171}$$

Since the last property, from equation 7.98, we can derive that

$$\overset{\bullet}{\widetilde{\boldsymbol{LnU}}} = \overset{\sim 4}{\gamma}_{(\widetilde{\boldsymbol{U}})} : \widetilde{\boldsymbol{D}}^{\backslash rp} = \frac{1}{2} \overset{\sim 4}{\gamma}_{(\widetilde{\boldsymbol{U}})} : \left(\overset{\bullet}{\widetilde{\boldsymbol{U}}} \,.\widetilde{\boldsymbol{U}}^{-1} + \widetilde{\boldsymbol{U}}^{-1} .\, \overset{\bullet}{\widetilde{\boldsymbol{U}}}\right) \tag{1.172}$$

Definition – *Fourth-order tensor* $\overset{\sim 4}{\boldsymbol{\delta}}_{(\widetilde{\boldsymbol{A}})}$ *is a function of a symmetric positive-definite second-order tensor* $\widetilde{\boldsymbol{A}}$.

The following matrix of this tensor $\overset{\sim 4}{\boldsymbol{\delta}}_{(\widetilde{\boldsymbol{A}})}$ *is expressed in Voigt-Mandel notation from a orthonormal basis* $(\vec{a_1}, \vec{a_2}, \vec{a_3})$ *of eigenvectors of* $\widetilde{\boldsymbol{A}}$. *The sequence of the components is* $(11, 22, 33, 12, 13, 23)$, *for the Voigt-Mandel notation.*

$$Mat_{\mathrm{VoigtMandel}(\vec{a_i})}\left(\overset{\sim 4}{\boldsymbol{\delta}}_{(\widetilde{\boldsymbol{A}})}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & h(a_1, a_2) & 0 & 0 \\ 0 & 0 & 0 & 0 & h(a_1, a_3) & 0 \\ 0 & 0 & 0 & 0 & 0 & h(a_2, a_3) \end{bmatrix} \tag{1.173}$$

where (a_1, a_2, a_3) *are the positive eigenvalues of* $\widetilde{\boldsymbol{A}}$ *associated respectively to eigenvectors* $(\vec{a_1}, \vec{a_2}, \vec{a_3})$, *and function* $h(x, y)$ *is :*

$$h(x, y) = \frac{(x^2 + y^2)\ln(x/y)}{(x^2 - y^2)} \quad \text{si} \quad x \neq y \tag{1.174}$$

$$= 1 \quad \text{si} \quad x = y\ . \tag{1.175}$$

with x *and* y *are positive real (non-null).*

Properties – *For all symmetric tensors* $\widetilde{\boldsymbol{A}}$, $\widetilde{\boldsymbol{B}}$ *and* $\widetilde{\boldsymbol{C}}$,

$$\overset{\sim 4}{\boldsymbol{\delta}}_{(\widetilde{\boldsymbol{A}})} : \widetilde{\boldsymbol{I}} = \widetilde{\boldsymbol{I}} \tag{1.176}$$

$$\overset{\sim 4}{\boldsymbol{\delta}}_{(\widetilde{\boldsymbol{A}})} : \widetilde{\boldsymbol{A}} = \widetilde{\boldsymbol{A}} \tag{1.177}$$

$$\overset{\sim 4}{\boldsymbol{\delta}}_{(\widetilde{\boldsymbol{A}})} : (\widetilde{\boldsymbol{A}} \centerdot \widetilde{\boldsymbol{A}}) = \widetilde{\boldsymbol{A}} \centerdot \widetilde{\boldsymbol{A}} \tag{1.178}$$

$$\forall n \in \mathbb{N} \quad \overset{\sim 4}{\boldsymbol{\delta}}_{(\widetilde{\boldsymbol{A}})} : (\widetilde{\boldsymbol{A}}^n) = \widetilde{\boldsymbol{A}}^n = \widetilde{\boldsymbol{A}} \centerdot \cdots \centerdot \widetilde{\boldsymbol{A}} \tag{1.179}$$

$$\overset{\sim 4}{\boldsymbol{\delta}}_{(\widetilde{\boldsymbol{A}})} : \widetilde{\boldsymbol{LnA}} = \widetilde{\boldsymbol{LnA}} \tag{1.180}$$

$$\mathrm{Tr}\left[\overset{\sim 4}{\boldsymbol{\delta}}_{(\widetilde{\boldsymbol{A}})} : \widetilde{\boldsymbol{B}}\right] = \mathrm{Tr}\widetilde{\boldsymbol{B}} \tag{1.181}$$

$$\widetilde{\boldsymbol{C}} : \left[\overset{\sim 4}{\boldsymbol{\delta}}_{(\widetilde{\boldsymbol{A}})} : \widetilde{\boldsymbol{B}}\right] = \widetilde{\boldsymbol{B}} : \left[\overset{\sim 4}{\boldsymbol{\delta}}_{(\widetilde{\boldsymbol{A}})} : \widetilde{\boldsymbol{C}}\right] \tag{1.182}$$

Property – *For all rotation tensor $\widetilde{\boldsymbol{Q}}$, and all symmetric tensors $\widetilde{\boldsymbol{A}}$ and $\widetilde{\boldsymbol{B}}$,*

$$\widetilde{\boldsymbol{Q}} \cdot \left[\overset{\sim 4}{\boldsymbol{\delta}}_{(\widetilde{\boldsymbol{A}})} : \widetilde{\boldsymbol{B}} \right] \cdot \widetilde{\boldsymbol{Q}}^{\mathrm{T}} = \overset{\sim 4}{\boldsymbol{\delta}}_{(\widetilde{\boldsymbol{Q}}.\widetilde{\boldsymbol{A}}.\widetilde{\boldsymbol{Q}}^{\mathrm{T}})} : \left(\widetilde{\boldsymbol{Q}} \cdot \widetilde{\boldsymbol{B}} \cdot \widetilde{\boldsymbol{Q}}^{\mathrm{T}} \right) \tag{1.183}$$

♣ `Règle de Leibnitz`
`pour la dérivation sous le signe d'intégration:`

$$\begin{aligned} \frac{d}{d\alpha} \int_{\phi_1(\alpha)}^{\phi_2(\alpha)} F(x,\alpha) dx &= \int_{\phi_1(\alpha)}^{\phi_2(\alpha)} \frac{\partial F}{\partial \alpha}(x,\alpha) dx \\ &+ F(\phi_2(\alpha),\alpha) \frac{d\phi_1}{d\alpha}(\alpha) - F(\phi_1(\alpha),\alpha) \frac{d\phi_2}{d\alpha}(\alpha) \end{aligned} \tag{1.184}$$

Chapitre 2

Finite deformations and strain measures

Let us consider a thin steel sheet, of a flat disk shape, that will undergo a deepdrawing (**FIG.*D*1**).

At initial time t_0, the placement or the position of the sheet is called configuration Ω_0, while after deformation, at time t (current time), it is called configuration Ω_t.

Here, two participants are required :

The observed object $=$ all material points M of the sheet.

The observer $=$ us *sat* on a reference frame, chosen here fixed to the rigid frame of the deepdrawing machine. Subsequently, we will consider the observer as its reference frame. So, a reference frame is like an observer.

The reference frame, chosen by the observer, will be called *base reference frame* and a will be its symbol. This frame may be chosen arbitrarily.

This reference frame is sometimes called *initial* or *fixed*.

2.1 Position of material point M

As in mechanics of material point, the position of point along time has to be given by specifying from which reference frame it is observed. So, we will use the following notation :

- position(M)/ref.a = $\vec{x}^{\backslash a} = \vec{x}$ (with simplified notation).
- $\vec{x}^{\backslash a}$(at t_0, in configuration Ω_0) = $\vec{X}^{\backslash a} = \vec{X}$,
- $\vec{x}^{\backslash a}$(at t, in configuration Ω_t) = $\vec{x}^{\backslash a} = \vec{x}$.

2.2 Reference configuration

♣ `The concept of strain in a actual configuration` Ω_t `requires the choice of a reference configuration` Ω_0`.`

By comparing the position of the material points between the actual configuration Ω_t and the chosen reference configuration Ω_0, we will be able to produce a strain measure.

Obviously, the choice of the reference configuration Ω_0 has a great impact on the current value of the strain. We are totally free for this choice. But, because of its importance on the strain values, the choice of Ω_0 must not be forgotten, when using the strain based on it.

Discutons de quelques choix possibles :

1. On peut choisir la position à l'instant initial t_0, pour la tôle en acier .
   C'est souvent ce qui est annoncé, mais pourquoi choisir cet instant t_0 ? Est-il particulier en ce qui concerne le comportement mécanique de la matière ? On appelle cet instant *initial* car il se situe avant l'emboutissage. Mais la tôle a subi avant t_0 une histoire thermomécanique ; coulée, laminage et découpe. Cette tôle subira d'autres chargements après t_0. Cet instant t_0 a-t-il vraiment quelquechose de particulier dans l'histoire mécanique de cette tôle ?
2. On pourrait aussi guider ce choix ainsi ; choisir la position pour laquelle la structure est «au repos», c'est-à-dire sans application d'efforts extérieurs.
   Mais en réalité la matière, qui constitue la structure, est très rarement libre de toute contrainte. En effet du fait du procédé de fabrication, qui inclut souvent une histoire thermomécanique complexe, la structure connaît des contraintes internes ou résiduelles. Une partie de la structure est en traction, lorsqu'une autre est en compression. Ces deux efforts s'annulant, de sorte qu'en surface de la structure, il n'y ait aucun vecteur contrainte non nul. Ainsi, la structure n'est que très rarement rigoureusement «au repos».
3. Une autre approche consiste à choisir la configuration de référence selon une géométrie parfaite ; un plan, un cylindre etc.
   Là aussi, ce choix est souvent fait en pratique et, si l'on recherche la rigueur, il est discutable. En effet, une tôle laminée n'est jamais rigoureusement plane et même elle présente une courbure connue du fait de son procédé d'obtention. Ainsi, la configuration de référence choisie n'est ni une position passée de la tôle, ni future. Elle est une position fictive, qui n'a pas et n'existera jamais.

Ceci étant dit, il faut bien faire un choix. Ce choix est souvent celui d'une géométrie parfaite associée à un champ de contrainte parfaitement nul au sein de la structure. Le but de la discussion, qui précède, est d'alerter l'apprenti modélisateur mécanicien sur les hypothèses, bien sûr nécessaires, qu'il fera lors du choix de la configuration de référence.

2.3 Deformation or movement of the matter

Each material point M has a position in the reference configuration Ω_0 and also an other position in the actual one Ω_t. Because material points do neither appear nor disappear from one configuration to an other, and because two distinct material points can not be on the same position in one configuration, then it exists a bijective map from the set of all material point positions in the reference configuration to the one in the actual configuration :

$$\forall \vec{X} \in \Omega_0 \quad \vec{x} = \chi_{0t}\left(\vec{X}\right) \qquad \forall \vec{x} \in \Omega_t \quad \vec{X} = \chi_{0t}^{-\mathbf{1}}(\vec{x}) \tag{2.1}$$

or, with the time t as a variable,

$$\forall \vec{X} \in \Omega_0 \quad \vec{x} = \chi_0\left(\vec{X}, t\right) \qquad \forall \vec{x} \in \Omega_t \quad \vec{X} = \chi_0^{-\mathbf{1}}(\vec{x}, t) \tag{2.2}$$

Let us assume that, at time $t = 0$, the configuration is the reference one :

$$\forall \vec{X} \in \Omega_0 \quad \vec{x} = \chi_0\left(\vec{X}, 0\right) = \vec{X} \qquad \forall \vec{x} \in \Omega_t \quad \vec{X} = \chi_0^{-\mathbf{1}}(\vec{x}, 0) = \vec{x} \tag{2.3}$$

So, $\chi_0\left(\vec{X}, 0\right)$ and $\chi_0^{-\mathbf{1}}(\vec{x}, 0)$ are identity functions.

2.4 Deformation gradient

Let us differentiate function $\chi_0\left(\vec{X}, t\right)$:

$$d\vec{x} = \left(\widetilde{\frac{\boldsymbol{\partial \chi_0}}{\boldsymbol{\partial \vec{X}}}}\right)_{/t}\left(\vec{X}, t\right) \cdot d\vec{X} + \left(\overrightarrow{\frac{\partial \chi_0}{\partial t}}\right)_{/\vec{X}}\left(\vec{X}, t\right) .dt \tag{2.4}$$

The vector

$$\left(\overrightarrow{\frac{\partial \chi_0}{\partial t}}\right)_{/\vec{X}}\left(\vec{X}, t\right) = \left(\overrightarrow{\frac{\partial \vec{x}}{\partial t}}\right)_{/\vec{X}}\left(\vec{X}, t\right) = \dot{\vec{x}}\left(\vec{X}, t\right) = \vec{v}\left(\vec{X}, t\right) \tag{2.5}$$

is the velocity of particle M at time t.

Definition – *The second-order tensor*

$$\widetilde{\left(\frac{\partial \chi_0}{\partial \vec{X}}\right)}_{/t}\left(\vec{X},t\right) = \widetilde{\left(\frac{\partial \vec{x}}{\partial \vec{X}}\right)}_{/t}\left(\vec{X},t\right) = \widetilde{\boldsymbol{F}}\left(\vec{X},t\right) \tag{2.6}$$

is the ***deformation gradient*** *of particle M at time t.*

At a given and fixed time t, let M' be a material point in the vicinity of material point M (**FIG.*D*2**) :

$$\begin{aligned} d\vec{X} &= \vec{X}'(t) - \vec{X}(t) & (2.7)\\ d\vec{x} &= \vec{x}'(t) - \vec{x}(t)\ . & (2.8) \end{aligned}$$

Then, at a given and fixed time t, the deformation gradient $\widetilde{\boldsymbol{F}}$ is the linear map such that

$$\forall d\vec{X}\ ,\quad d\vec{x} = \widetilde{\boldsymbol{F}} \centerdot d\vec{X}\ . \tag{2.9}$$

Because function $\chi_0\left(\vec{X},t\right)$ is invertible, the deformation gradient is also invertible :

$$\forall d\vec{x}\ ,\quad d\vec{X} = \widetilde{\boldsymbol{F}}^{-1} \centerdot d\vec{x}\ . \tag{2.10}$$

The deformation gradient describes the deformation of the matter between configurations Ω_0 and Ω_t, only for the vicinity of a material point M.

Remark – $\widetilde{\boldsymbol{F}}$ does not contain any information about the translation rigid movement of the vicinity of point M.

For the calculation of the deformation gradient tensor $\widetilde{\boldsymbol{F}}$ from a given current position field $\vec{x}\left(\vec{X}\right)$ as a function of the reference position field :

$$\widetilde{\boldsymbol{F}} = \widetilde{\left(\frac{\partial \vec{x}}{\partial \vec{X}}\right)}_{/t} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \tag{2.11}$$

if the coordinate system is cartesian (neither cylindrical, nor spherical).

If $\vec{U}$ is the displacement vector, defined as :

$$\vec{U} = \vec{x} - \vec{X} \tag{2.12}$$

Then

$$\widetilde{\boldsymbol{F}} = \widetilde{\boldsymbol{I}} + \widetilde{\left(\frac{\partial \vec{U}}{\partial \vec{X}}\right)}_{/t} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{\partial U_1}{\partial X_1} & \frac{\partial U_1}{\partial X_2} & \frac{\partial U_1}{\partial X_3} \\ \frac{\partial U_2}{\partial X_1} & \frac{\partial U_2}{\partial X_2} & \frac{\partial U_2}{\partial X_3} \\ \frac{\partial U_3}{\partial X_1} & \frac{\partial U_3}{\partial X_2} & \frac{\partial U_3}{\partial X_3} \end{bmatrix} \tag{2.13}$$

Because $\widetilde{\boldsymbol{F}}$ is invertible, i.e.

$$\forall\, \vec{X} \text{ and } \forall t \qquad \det \widetilde{\boldsymbol{F}}\left(\vec{X}, t\right) \neq 0 \tag{2.14}$$

and because at time 0, $\widetilde{\boldsymbol{F}} = \widetilde{\boldsymbol{I}}$, i.e.

$$\forall\, \vec{X} \qquad \det \widetilde{\boldsymbol{F}}\left(\vec{X}, 0\right) = 1 \tag{2.15}$$

then [Truesdell-2004]

Property –

$$\forall\, \vec{X} \text{ and } \forall t \qquad \det \widetilde{\boldsymbol{F}}\left(\vec{X}, t\right) > 0 \tag{2.16}$$

The determinant of the deformation gradient is always positive (non null).

2.5 Polar decomposition

According to the polar decomposition theorem 1.152[p23], the deformation gradient $\widetilde{\boldsymbol{F}}$ may be decomposed in two ways as follows :

1$^{\text{rst}}$ way – a rotation followed by a strain,

2$^{\text{nd}}$ way – a strain followed by a rotation.

This is called the polar decomposition of the deformation gradient $\widetilde{\boldsymbol{F}}$, and is expressed as :

$$\widetilde{\boldsymbol{F}} = \widetilde{\boldsymbol{V}} \boldsymbol{.} \widetilde{\boldsymbol{R}} = \widetilde{\boldsymbol{R}} \boldsymbol{.} \widetilde{\boldsymbol{U}} \ , \tag{2.17}$$

where $\widetilde{\boldsymbol{U}}$ and $\widetilde{\boldsymbol{V}}$ are symmetric positive-definite second-order tensors. Because of their symmetry, they are diagonalizable into orthonormal basis ; it exists two orthonormal bases $(\vec{u_1}, \vec{u_2}, \vec{u_3})$ and $(\vec{v_1}, \vec{v_2}, \vec{v_3})$ of eigenvectors of $\widetilde{\boldsymbol{U}}$ and $\widetilde{\boldsymbol{V}}$, respectively.

Definition – *Tensors $\widetilde{\boldsymbol{U}}$ and $\widetilde{\boldsymbol{V}}$, resulting from the polar decomposition of $\widetilde{\boldsymbol{F}}$, are called* right and left stretch *tensors, respectively.*

According to the polar decomposition 1.152[p23], and because $\det \widetilde{\boldsymbol{F}} > 0$, $\widetilde{\boldsymbol{R}}$ is a rotation tensor, i.e. such that

$$\widetilde{\boldsymbol{R}} \boldsymbol{.} \widetilde{\boldsymbol{R}}^{\mathbf{T}} = \widetilde{\boldsymbol{R}}^{\mathbf{T}} \boldsymbol{.} \widetilde{\boldsymbol{R}} = \widetilde{\boldsymbol{I}} \tag{2.18}$$

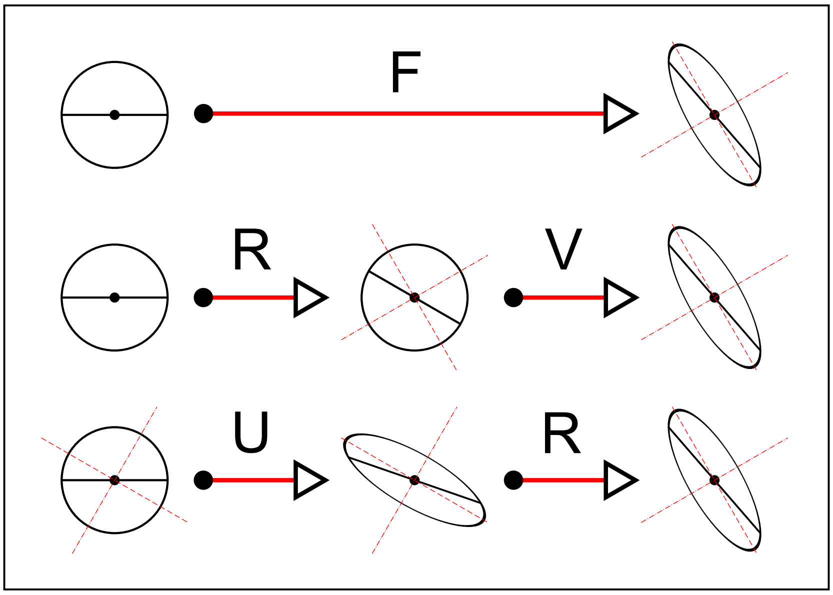

FIGURE 2.1 – Polar decomposition of the deformation gradient $\widetilde{\boldsymbol{F}}$.

and

$$\det\widetilde{\boldsymbol{R}} = +1 \ . \tag{2.19}$$

Definition – *Tensor $\widetilde{\boldsymbol{R}}$ is called the* proper rotation *tensor.*

Figure 2.1 illustrates this decomposition, in the two-dimensional case.

$\vec{u_i}$ and $\vec{v_i}$ are two orthonormal bases. If we assume that they are also right-handed, then the linear map, that transforms $\vec{u_i}$ into $\vec{v_i}$, is a rotation tensor and this rotation is exactly equal to the proper rotation :

$$(\vec{u_1}, \vec{u_2}, \vec{u_3}) \xrightarrow{\widetilde{\boldsymbol{R}}} (\vec{v_1}, \vec{v_2}, \vec{v_3}) \ . \tag{2.20}$$

Indeed, from the polar decomposition, we derive :

$$\widetilde{\boldsymbol{V}} \boldsymbol{.} \left(\widetilde{\boldsymbol{R}} \boldsymbol{.} \vec{u_i}\right) = \widetilde{\boldsymbol{R}} \boldsymbol{.} \widetilde{\boldsymbol{U}} \boldsymbol{.} \vec{u_i} = \lambda_i \left(\widetilde{\boldsymbol{R}} \boldsymbol{.} \vec{u_i}\right) \ . \tag{2.21}$$

Morevover, we can derive from the above equation, that the eigenvalues of $\widetilde{\boldsymbol{U}}$ and $\widetilde{\boldsymbol{V}}$ are the same. They are called *principal stretches* and are usually written λ_1, λ_2 et λ_3. The principal stretches are always **positive, non null**, because $\widetilde{\boldsymbol{U}}$ and $\widetilde{\boldsymbol{V}}$ are symmetric positive-definite second-order tensor (section 1.13[p23]).

Remark – In the case of a simple tensile test, if L and l are respectively the initial and deformed lengths, then the tensile principal stretch λ_1 is :

$$\lambda_1 = \frac{l}{L} \; . \tag{2.22}$$

A classical method for calculating the stretch tensor $\widetilde{\boldsymbol{U}}$ (respectively $\widetilde{\boldsymbol{V}}$) from the deformation gradient $\widetilde{\boldsymbol{F}}$ is :

1. Calcul of symmetric tensor $\widetilde{\boldsymbol{F}}^{\mathbf{T}} \cdot \widetilde{\boldsymbol{F}}$ (resp. $\widetilde{\boldsymbol{F}} \cdot \widetilde{\boldsymbol{F}}^{\mathbf{T}}$), that is also equal to $\widetilde{\boldsymbol{U}} \cdot \widetilde{\boldsymbol{U}}$ (resp. $\widetilde{\boldsymbol{V}} \cdot \widetilde{\boldsymbol{V}}$).
2. Eigendecomposition of this tensor (diagonalization). The resulting eigenvalues are the square of the principal stretches $(\lambda_1^2, \lambda_2^2, \lambda_3^2)$ and the unit eigenvectors (chosen of unit norm), are $(\vec{u_1}, \vec{u_2}, \vec{u_3})$, and are also eigenvectors of $\widetilde{\boldsymbol{U}}$ (resp. $(\vec{v_1}, \vec{v_2}, \vec{v_3})$, eigenvectors of $\widetilde{\boldsymbol{V}}$).
3. tensor $\widetilde{\boldsymbol{U}}$ (resp. $\widetilde{\boldsymbol{V}}$) is then given by :

$$Mat_{\vec{u_i}}\left(\widetilde{\boldsymbol{U}}\right) = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \; , \tag{2.23}$$

respectively

$$Mat_{\vec{v_i}}\left(\widetilde{\boldsymbol{V}}\right) = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \; . \tag{2.24}$$

An other method, a numerical one, aims first to calculate the proper rotation tensor $\widetilde{\boldsymbol{R}}$ by the following iterative algorithm [Curnier-2005] :

1. $\widetilde{\boldsymbol{R}_0} = \widetilde{\boldsymbol{F}}$,
2. $\widetilde{\boldsymbol{R}_{i+1}} = \frac{1}{2}\left(\widetilde{\boldsymbol{R}_i} + \widetilde{\boldsymbol{R}_i}^{-\mathbf{T}}\right)$.

The convergence towards $\widetilde{\boldsymbol{R}}$ is quadratic.

Once, the proper rotation is calculated with sufficient precision, the stretch tensors are calculated as :

$$\widetilde{\boldsymbol{U}} = \widetilde{\boldsymbol{R}}^{\mathbf{T}} \cdot \widetilde{\boldsymbol{F}} \qquad , \qquad \widetilde{\boldsymbol{V}} = \widetilde{\boldsymbol{F}} \cdot \widetilde{\boldsymbol{R}}^{\mathbf{T}} \; . \tag{2.25}$$

2.6 The rotation of the material particles ?

The answer to this apparently simple question is not clear.

Today, several answers were proposed to this question. Here, is two of these answers :

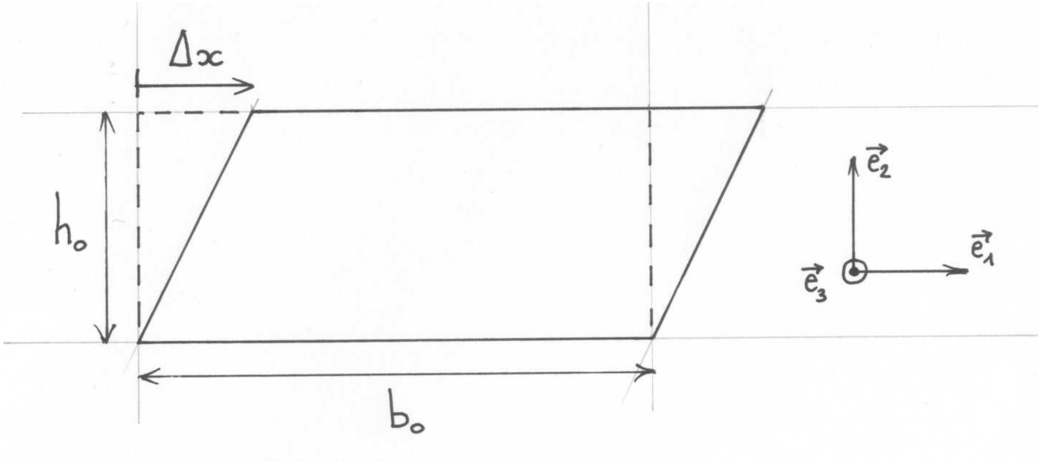

FIGURE 2.2 – Simple shear deformation ; the reference configuration Ω_0 is in dotted line, and the deformed one Ω_t is in continuous line.

- the proper rotation $\widetilde{\boldsymbol{R}}$, resulting of the polar decomposition of $\widetilde{\boldsymbol{F}}$,
- the rotation $\widetilde{\boldsymbol{R}^{aj}}$ (eq. 6.60[p105] and 6.61) associated with the Jaumann objective rate, we will see later (chapter 7).

For example, let us consider the case of the simple shear deformation (fig. 2.2).

2.7 Several strain measures

All following tensors are symmetric second-order tensors.

They gather into two families. The first family is the set of the **right strain tensors** based on the right stretch tensor $\widetilde{\boldsymbol{U}}$:

- $\widetilde{\boldsymbol{U}}$ right stretch tensor (no strain when $\widetilde{\boldsymbol{U}} = \widetilde{\boldsymbol{I}}$).
- $\widetilde{\boldsymbol{P}} = (\widetilde{\boldsymbol{U}} - \widetilde{\boldsymbol{I}})$ first Biot strain tensor (no strain when $\widetilde{\boldsymbol{P}} = \widetilde{\boldsymbol{0}}$).
- $\widetilde{\boldsymbol{C}} = (\widetilde{\boldsymbol{U}} \cdot \widetilde{\boldsymbol{U}})$ right Green or Cauchy-Greenstrain tensor (no strain when $\widetilde{\boldsymbol{C}} = \widetilde{\boldsymbol{I}}$). Let us underline that the calculus of this tensor is very simple :

$$\widetilde{\boldsymbol{C}} = \widetilde{\boldsymbol{F}}^{\mathbf{T}} \cdot \widetilde{\boldsymbol{F}} \ . \tag{2.26}$$

- $\widetilde{\boldsymbol{E}} = \frac{1}{2}(\widetilde{\boldsymbol{U}} \cdot \widetilde{\boldsymbol{U}} - \widetilde{\boldsymbol{I}})$ (right) Green-Lagrange or Green-S^{t}Venant-Boussinesq strain tensor (no strain when $\widetilde{\boldsymbol{E}} = \widetilde{\boldsymbol{0}}$). Let us underline that the calculus of this tensor is very simple :

$$\widetilde{\boldsymbol{E}} = \frac{1}{2}(\widetilde{\boldsymbol{F}}^{\mathbf{T}} \cdot \widetilde{\boldsymbol{F}} - \widetilde{\boldsymbol{I}}) \ . \tag{2.27}$$

- $\widetilde{\boldsymbol{K}} = \frac{1}{2}\left(\widetilde{\boldsymbol{I}} - \widetilde{\boldsymbol{U}}^{-1} \cdot \widetilde{\boldsymbol{U}}^{-1}\right)$ Karni strain tensor (no strain when $\widetilde{\boldsymbol{K}} = \widetilde{\boldsymbol{0}}$).

– $\widetilde{\boldsymbol{E_n}} = \frac{1}{n}(\underbrace{\widetilde{\boldsymbol{U}}\,\boldsymbol{.}\,\widetilde{\boldsymbol{U}}\,\boldsymbol{.}\cdots\boldsymbol{.}\,\widetilde{\boldsymbol{U}}}_{n\text{ fois}} - \widetilde{\boldsymbol{I}})$ where $n \in \mathbb{Z} - \{0\}$; first Hill strain tensors (no strain when $\widetilde{\boldsymbol{E_n}} = \widetilde{\boldsymbol{0}}$). If n is negative, the formula becomes :

$$\widetilde{\boldsymbol{E_n}} = \frac{1}{-n}(\widetilde{\boldsymbol{I}} - \underbrace{\widetilde{\boldsymbol{U}}^{-1}\boldsymbol{.}\,\widetilde{\boldsymbol{U}}^{-1}\boldsymbol{.}\cdots\boldsymbol{.}\,\widetilde{\boldsymbol{U}}^{-1}}_{|n|\text{ fois}}) \ . \tag{2.28}$$

Remark : $\widetilde{\boldsymbol{E_2}} = \widetilde{\boldsymbol{E}}$.

– $\widetilde{\boldsymbol{C_a}} = \left(\frac{2+a}{8}\widetilde{\boldsymbol{C}} - \frac{a}{4}\widetilde{\boldsymbol{I}} - \frac{2-a}{8}\widetilde{\boldsymbol{C}}^{-1}\right)$ where $a \in [-2;+2] \subset \mathbb{R}$; strain tensors proposed by [Curnier-2005] (no strain when $\widetilde{\boldsymbol{C_a}} = \widetilde{\boldsymbol{0}}$). Remarks : $\widetilde{\boldsymbol{C_2}} = \widetilde{\boldsymbol{E}}$ and $\widetilde{\boldsymbol{C_{-2}}} = \widetilde{\boldsymbol{K}}$.

– $\widetilde{\boldsymbol{QLnU}} = \frac{1}{4}\left(\widetilde{\boldsymbol{C}} - \widetilde{\boldsymbol{C}}^{-1}\right)$; right quasilogarithmic strain tensor proposed by [Curnier-2005] (no strain when $\widetilde{\boldsymbol{QLnU}} = \widetilde{\boldsymbol{0}}$). Remark : $\widetilde{\boldsymbol{QLnU}} = \widetilde{\boldsymbol{C_0}}$.

– $\widetilde{\boldsymbol{LnU}}$ (or $\widetilde{\boldsymbol{E_0}}$) right Hill or Hencky or logarithmic strain tensor (no strain when $\widetilde{\boldsymbol{LnU}} = \widetilde{\boldsymbol{0}}$).

Definition of $\widetilde{\boldsymbol{LnA}}$ – *Let $\widetilde{\boldsymbol{A}}$ be a symmetric positive-definite second-order tensor (i.e. its eigenvalues are all positive, non null). The logarithm $\widetilde{\boldsymbol{LnA}}$ of tensor $\widetilde{\boldsymbol{A}}$ is the symmetric second-order tensor, with the same eigenvectors than $\widetilde{\boldsymbol{A}}$ and with eigenvalues $(\ln A)_i$:*

$$(\ln A)_i = \ln a_i \qquad i = 1, 2, 3 \ , \tag{2.29}$$

where a_i are the eigenvalues of $\widetilde{\boldsymbol{A}}$. That is to say, if the matrix of $\widetilde{\boldsymbol{A}}$ into its eigenvector orthonormal basis $(\vec{a_1}, \vec{a_2}, \vec{a_3})$ is :

$$Mat_{\vec{a_i}}\left(\widetilde{\boldsymbol{A}}\right) = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} , \tag{2.30}$$

then the matrix of its logarithm $\widetilde{\boldsymbol{LnA}}$, into the same basis, is :

$$Mat_{\vec{a_i}}\left(\widetilde{\boldsymbol{LnA}}\right) = \begin{bmatrix} \ln a_1 & 0 & 0 \\ 0 & \ln a_2 & 0 \\ 0 & 0 & \ln a_3 \end{bmatrix} . \tag{2.31}$$

The second family is the set of the **left strain tensors** based on the left stretch tensor $\widetilde{\boldsymbol{V}}$:

– $\widetilde{\boldsymbol{V}}$ left stretch tensor (no strain when $\widetilde{\boldsymbol{V}} = \widetilde{\boldsymbol{I}}$).

- $\widetilde{\boldsymbol{J}} = (\widetilde{\boldsymbol{V}} - \widetilde{\boldsymbol{I}})$ second Biot strain tensor (no strain when $\widetilde{\boldsymbol{J}} = \widetilde{\mathbf{0}}$).
- $\widetilde{\boldsymbol{B}} = (\widetilde{\boldsymbol{V}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{V}})$ left Cauchy-Green strain tensor (no strain when $\widetilde{\boldsymbol{B}} = \widetilde{\boldsymbol{I}}$). Let us underline that the calculus of this tensor is very simple :
$$\widetilde{\boldsymbol{B}} = \widetilde{\boldsymbol{F}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{F}}^{\mathbf{T}} \,. \tag{2.32}$$
- $\widetilde{\boldsymbol{A}} = \frac{1}{2}(\widetilde{\boldsymbol{I}} - \widetilde{\boldsymbol{V}}^{-\mathbf{1}} \boldsymbol{.} \widetilde{\boldsymbol{V}}^{-\mathbf{1}})$ Almansi-Euler strain tensor (no strain when $\widetilde{\boldsymbol{A}} = \widetilde{\mathbf{0}}$). Its calculus is :
$$\widetilde{\boldsymbol{A}} = \frac{1}{2}(\widetilde{\boldsymbol{I}} - \widetilde{\boldsymbol{F}}^{-\mathbf{T}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{F}}^{-\mathbf{1}}) \,. \tag{2.33}$$
- $\widetilde{\boldsymbol{e_n}} = \frac{1}{n}(\underbrace{\widetilde{\boldsymbol{V}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{V}} \,\boldsymbol{.}\, \cdots \,\boldsymbol{.}\, \widetilde{\boldsymbol{V}}}_{n\,\text{fois}} - \widetilde{\boldsymbol{I}})$ where $n \in \mathbb{Z} - \{0\}$; Seth or second Hill strain tensors (no strain when $\widetilde{\boldsymbol{e_n}} = \widetilde{\mathbf{0}}$). If n is negative, the formula becomes
$$\widetilde{\boldsymbol{e_n}} = \frac{1}{-n}(\widetilde{\boldsymbol{I}} - \underbrace{\widetilde{\boldsymbol{V}}^{-\mathbf{1}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{V}}^{-\mathbf{1}} \,\boldsymbol{.}\, \cdots \,\boldsymbol{.}\, \widetilde{\boldsymbol{V}}^{-\mathbf{1}}}_{|n|\,\text{fois}}) \,. \tag{2.34}$$
Remark : $\widetilde{\boldsymbol{e_{-2}}} = \widetilde{\boldsymbol{A}}$.
- $\widetilde{\boldsymbol{QLnV}} = \frac{1}{4}\left(\widetilde{\boldsymbol{B}} - \widetilde{\boldsymbol{B}}^{-\mathbf{1}}\right)$; left quasilogarithmic strain tensor (no strain when $\widetilde{\boldsymbol{QLnV}} = \widetilde{\mathbf{0}}$).
- $\widetilde{\boldsymbol{LnV}}$ (or $\widetilde{\boldsymbol{e_0}}$) left Hill or Hencky or logarithmic strain tensor (no strain when $\widetilde{\boldsymbol{LnV}} = \widetilde{\mathbf{0}}$).

Properties – *All right strain tensors have same eigendirections, that are parallel to the orthonormal basis $(\vec{u_1}, \vec{u_2}, \vec{u_3})$.*

All left strain tensors have same eigendirections, that are parallel to the orthonormal basis $(\vec{v_1}, \vec{v_2}, \vec{v_3})$.

Vocabulary –

The right strain tensors are called ***lagrangian****, because they give the strain measure into the reference configuration orientation, before applying the proper rotation. The left strain tensors are called* ***eulerian****, because they give the strain measure into the deformed configuration orientation, after applying the proper rotation.*

2.8 Volume change strain

Let us consider an infinitesimal volume of matter around a material point M (**FIG.*D*4**). The reference volume is written $d\mathrm{V}$, while the deformed one is written $d\mathrm{v}$.

The volume change between configurations Ω_0 and Ω_t is usually written J and is defined as :

Definition and properties – *The volume change J is*

$$J = \frac{\mathrm{dv}}{\mathrm{dV}} \ . \tag{2.35}$$

It is also equal to

$$J = \det\widetilde{\boldsymbol{F}} = \frac{\rho_0}{\rho_t} \ , \tag{2.36}$$

where ρ_0 and ρ_t are the reference and deformed density at point M, respectively.

Exercise 2.1. *Prove that*

$$J = \det\widetilde{\boldsymbol{U}} = \det\widetilde{\boldsymbol{V}} = \sqrt{\det\widetilde{\boldsymbol{C}}} = \sqrt{\det\widetilde{\boldsymbol{B}}} \ . \tag{2.37}$$

Exercise 2.2. *Prove that*

$$\mathrm{Tr}\widetilde{\boldsymbol{LnV}} = \mathrm{Tr}\widetilde{\boldsymbol{LnU}} = \ln\det\widetilde{\boldsymbol{U}} = \ln\det\widetilde{\boldsymbol{V}} = \ln\det\widetilde{\boldsymbol{F}} = \ln J \ . \tag{2.38}$$

Exercise 2.3. *Strain measures in the case of simple tensile* **FIG.*D*10** *:*

1. *Calculate the following strain tensors ;*
   - $\widetilde{\boldsymbol{E}}$ *Green-Lagrange tensor,*
   - $\widetilde{\boldsymbol{A}}$ *Almansi-Euler tensor,*
   - $\widetilde{\boldsymbol{P}}$, $\widetilde{\boldsymbol{J}}$ *first and second Biot strain tensors,*
   - $\widetilde{\boldsymbol{LnU}}$, $\widetilde{\boldsymbol{LnV}}$ *right and left logarithmic strain tensors,*
   - $\widetilde{\boldsymbol{\varepsilon}}_{HPP}$ *the strain tensor in the case of small displacements.*

   For that, give their matrix in basis $(\vec{e_1}, \vec{e_2}, \vec{e_3})$.
2. *Give the axial or tensile strain for all these strain measures, as a function of lengths L and l.*
3. *Calculate the change of volume, if D and d are the reference and deformed diameters, respectively.*

Exercise 2.4. *Strain measures in the case of simple shear* **FIG.*D*11** *:*

1. *Calculate the following strain tensors ;*
   - $\widetilde{\boldsymbol{E}}$ *Green-Lagrange tensor,*
   - $\widetilde{\boldsymbol{A}}$ *Almansi-Euler tensor,*
   - $\widetilde{\boldsymbol{\varepsilon}}_{HPP}$ *the strain tensor in the case of small displacements.*

   For that, give their matrix in basis $(\vec{e_1}, \vec{e_2}, \vec{e_3})$.
2. *Calculate the change of volume.*

2.9 Strain with small-displacement approximation

The strain tensor according to the approximation of small displacements may be defined as :

Definition – *The small-displacement strain tensor is*

$$\widetilde{\varepsilon_{HPP}} = \frac{1}{2}\left[\widetilde{\left(\frac{\partial \vec{U}}{\partial \vec{X}}\right)_{/t}} + \widetilde{\left(\frac{\partial \vec{U}}{\partial \vec{X}}\right)_{/t}}^{\mathbf{T}}\right] , \tag{2.39}$$

where $\widetilde{\left(\frac{\partial \vec{U}}{\partial \vec{X}}\right)_{/t}} = (\widetilde{\boldsymbol{F}} - \widetilde{\boldsymbol{I}})$ *is the displacement gradient tensor.*

Properties A –

$$\widetilde{\varepsilon_{HPP}} + \frac{1}{2}\widetilde{\left(\frac{\partial \vec{U}}{\partial \vec{X}}\right)_{/t}}^{\mathbf{T}} \boldsymbol{.} \widetilde{\left(\frac{\partial \vec{U}}{\partial \vec{X}}\right)_{/t}} = \frac{1}{2}(\widetilde{\boldsymbol{U}} \boldsymbol{.} \widetilde{\boldsymbol{U}} - \widetilde{\boldsymbol{I}}) = \widetilde{\boldsymbol{E}} , \tag{2.40}$$

$$\widetilde{\varepsilon_{HPP}} + \frac{1}{2}\widetilde{\left(\frac{\partial \vec{U}}{\partial \vec{X}}\right)_{/t}} \boldsymbol{.} \widetilde{\left(\frac{\partial \vec{U}}{\partial \vec{X}}\right)_{/t}}^{\mathbf{T}} = \frac{1}{2}(\widetilde{\boldsymbol{V}} \boldsymbol{.} \widetilde{\boldsymbol{V}} - \widetilde{\boldsymbol{I}}) . \tag{2.41}$$

Exercise 2.5. *Prove these two last properties.*

Properties B –

$$\widetilde{\varepsilon_{HPP}} = \frac{1}{2}\left(\widetilde{\boldsymbol{V}} \boldsymbol{.} \widetilde{\boldsymbol{R}} + \widetilde{\boldsymbol{R}}^{\mathbf{T}} \boldsymbol{.} \widetilde{\boldsymbol{V}}\right) - \widetilde{\boldsymbol{I}} \tag{2.42}$$

$$= \frac{1}{2}\left(\widetilde{\boldsymbol{U}} \boldsymbol{.} \widetilde{\boldsymbol{R}}^{\mathbf{T}} + \widetilde{\boldsymbol{R}} \boldsymbol{.} \widetilde{\boldsymbol{U}}\right) - \widetilde{\boldsymbol{I}} . \tag{2.43}$$

Exercise 2.6. *Prove these two last equalities.*

2.9.1 Deformation without any proper rotation

In the case of no proper rotation,

$$\widetilde{\boldsymbol{R}} = \widetilde{\boldsymbol{I}} \iff \widetilde{\boldsymbol{F}}^{\mathbf{T}} = \widetilde{\boldsymbol{F}} \iff \widetilde{\boldsymbol{U}} = \widetilde{\boldsymbol{V}} = \widetilde{\boldsymbol{F}} . \tag{2.44}$$

Examples of this case are :

- mechanical test of uniaxial compression-tension (simple compression and tension),
- mechanical test of biaxial tension,

if we choose as reference frame a the one fixed with the test machine rigid frame.

In this non-proper-rotation case, because of relations 2.42 and 2.43,

$$\widetilde{\varepsilon_{\mathsf{HPP}}} = (\widetilde{\boldsymbol{U}} - \widetilde{\boldsymbol{I}}) = \widetilde{\boldsymbol{P}} \tag{2.45}$$

$$= (\widetilde{\boldsymbol{V}} - \widetilde{\boldsymbol{I}}) = \widetilde{\boldsymbol{J}} \,, \tag{2.46}$$

which are the first and second Biot strain tensors. Therefore,

♣ `In the case of deformation without any proper rotation, and even in large displacements or strains, the approximated strain tensor` $\widetilde{\varepsilon_{\mathsf{HPP}}}$ `could be used as strain tensor, without any approximation issue.`

2.9.2 Deformation with proper rotation

In the case of proper rotation,

$$\widetilde{\boldsymbol{R}} \neq \widetilde{\boldsymbol{I}} \,. \tag{2.47}$$

♣ `In the case of deformation with proper rotation, and if the displacements are too large, the use of the approximated strain tensor` $\widetilde{\varepsilon_{\mathsf{HPP}}}$ `has to be avoided.`

Indeed, let us consider the following example of deformation equal to a rigid rotation of $90°$, without any strain :

$$\widetilde{\boldsymbol{F}} = \widetilde{\boldsymbol{R}}_{(90°)} \qquad \widetilde{\boldsymbol{U}} = \widetilde{\boldsymbol{V}} = \widetilde{\boldsymbol{I}} \,. \tag{2.48}$$

In this case, the small-displacement strain tensor is (rel. 2.42, 2.43),

$$\widetilde{\varepsilon_{\mathsf{HPP}}} = \frac{1}{2}\left(\widetilde{\boldsymbol{R}} + \widetilde{\boldsymbol{R}}^{\mathbf{T}}\right) - \widetilde{\boldsymbol{I}} \,. \tag{2.49}$$

Let us choose an orthonormal basis $(\vec{e_1}, \vec{e_2}, \vec{e_3})$, such that vector $\vec{e_3}$ is along the rotation axis and in the positive-rotation direction. That is to say that the rotation matrix into this basis is :

$$Mat_{\vec{e_i}}\left(\widetilde{\boldsymbol{R}}_{(90°)}\right) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \,. \tag{2.50}$$

Then the matrix of strain $\widetilde{\varepsilon_{\mathsf{HPP}}}$ is

$$Mat_{\vec{e_i}}\left(\widetilde{\varepsilon_{\mathsf{HPP}}}\right) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \tag{2.51}$$

This is highly problematic, because according to the polar decomposition the matter does not undergo any strains. This strain matrix has to be null, in this case.

2.10 Simple time derivative

Definition – *The simple derivative by time is defined for a vector, as*

$$\overset{\bullet}{\vec{a}}_{(t)}= \lim_{dt\to 0}\left[\frac{\vec{a}_{(t+dt)} - \vec{a}_{(t)}}{dt}\right] \tag{2.52}$$

for a second-order tensor, as

$$\overset{\bullet}{\widetilde{\boldsymbol{A}}}_{(t)}= \lim_{dt\to 0}\left[\frac{\widetilde{\boldsymbol{A}}_{(t+dt)} - \widetilde{\boldsymbol{A}}_{(t)}}{dt}\right] \tag{2.53}$$

and for a fourth-order tensor, as

$$\overset{\bullet}{\overset{\sim 4}{\boldsymbol{A}}}_{(t)}= \lim_{dt\to 0}\left[\frac{\overset{\sim 4}{\boldsymbol{A}}_{(t+dt)} - \overset{\sim 4}{\boldsymbol{A}}_{(t)}}{dt}\right] \tag{2.54}$$

Properties –

$$\begin{aligned} &\overset{\bullet}{\left[\vec{x}+\vec{y}\right]}=\overset{\bullet}{\vec{x}}+\overset{\bullet}{\vec{y}} \qquad && \overset{\cdot}{\left[\vec{x}\,.\,\vec{y}\right]}=\overset{\bullet}{\vec{x}}\,.\vec{y}+\vec{x}.\,\overset{\bullet}{\vec{y}} \\ &\overset{\bullet}{\left[\widetilde{\boldsymbol{A}}\,.\,\vec{x}\right]}=\overset{\bullet}{\widetilde{\boldsymbol{A}}}\,.\vec{x}+\widetilde{\boldsymbol{A}}.\,\overset{\bullet}{\vec{x}} && \end{aligned} \tag{2.55}$$

$$\overset{\bullet}{\widetilde{\boldsymbol{I}}}= \widetilde{\boldsymbol{0}} \tag{2.56}$$

$$\left[\overset{\bullet}{\widetilde{\boldsymbol{A}}}\right]^{\mathbf{T}} = \left[\overset{\bullet}{\widetilde{\boldsymbol{A}}^{\mathbf{T}}}\right] \qquad \mathrm{Tr}\left[\overset{\bullet}{\widetilde{\boldsymbol{A}}}\right] = \left[\overset{\bullet}{\mathrm{Tr}\widetilde{\boldsymbol{A}}}\right]$$

$$\left[\overset{\bullet}{\lambda.\widetilde{\boldsymbol{A}}}\right] = \dot{\lambda}.\widetilde{\boldsymbol{A}} + \lambda.\,\overset{\bullet}{\widetilde{\boldsymbol{A}}} \qquad \left[\overset{\bullet}{\underline{\widetilde{\boldsymbol{A}}}}\right] = \underline{\left[\overset{\bullet}{\widetilde{\boldsymbol{A}}}\right]} \tag{2.57}$$

$$\left[\overset{\bullet}{\widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{B}}}\right] = \overset{\bullet}{\widetilde{\boldsymbol{A}}} + \overset{\bullet}{\widetilde{\boldsymbol{B}}} \qquad \left[\overset{\bullet}{\widetilde{\boldsymbol{A}} \boldsymbol{.} \widetilde{\boldsymbol{B}}}\right] = \overset{\bullet}{\widetilde{\boldsymbol{A}}} \boldsymbol{.}\widetilde{\boldsymbol{B}} + \widetilde{\boldsymbol{A}}\boldsymbol{.}\,\overset{\bullet}{\widetilde{\boldsymbol{B}}}$$

Then, we have :

$$\left[\overset{\bullet}{\mathrm{Tr}\widetilde{\boldsymbol{A}}}\right] = \widetilde{\boldsymbol{I}} : \overset{\bullet}{\widetilde{\boldsymbol{A}}} \tag{2.58}$$

Because of relation 1.146[p22], we have :

$$\left[\overset{\bullet}{\mathrm{sec}\widetilde{\boldsymbol{A}}}\right] = \left[\mathrm{Tr}\widetilde{\boldsymbol{A}}\ .\ \widetilde{\boldsymbol{I}} - \widetilde{\boldsymbol{A}}^{\mathbf{T}}\right] : \overset{\bullet}{\widetilde{\boldsymbol{A}}} \tag{2.59}$$

Because of relation 1.147[p22], we have :

$$\left[\overset{\bullet}{\det\widetilde{\boldsymbol{A}}}\right] = \left[\left((\mathrm{sec}\widetilde{\boldsymbol{A}})\widetilde{\boldsymbol{I}} - (\mathrm{Tr}\widetilde{\boldsymbol{A}})\widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{A}} \boldsymbol{.} \widetilde{\boldsymbol{A}}\right)^{\mathbf{T}}\right] : \overset{\bullet}{\widetilde{\boldsymbol{A}}} \tag{2.60}$$

and, if $\widetilde{\boldsymbol{A}}$ is invertible (relation 1.133[p21]), we have :

$$\left[\overset{\bullet}{\det\widetilde{\boldsymbol{A}}}\right] = (\det\widetilde{\boldsymbol{A}})\ .\ \widetilde{\boldsymbol{A}}^{-\mathbf{T}} : \overset{\bullet}{\widetilde{\boldsymbol{A}}} \tag{2.61}$$

On the other hand,

$$\overset{\bullet}{\overset{\sim 4}{\boldsymbol{I}}} = \overset{\sim 4}{\mathbf{0}} \tag{2.62}$$

$$\left[\overset{\bullet}{\lambda.\ \overset{\sim 4}{\boldsymbol{A}}}\right] = \dot{\lambda}.\ \overset{\sim 4}{\boldsymbol{A}} + \lambda.\ \overset{\bullet}{\overset{\sim 4}{\boldsymbol{A}}} \tag{2.63}$$

$$\left(\overset{\bullet}{\overset{\sim 4}{\boldsymbol{A}} + \overset{\sim 4}{\boldsymbol{B}}}\right) = \overset{\bullet}{\overset{\sim 4}{\boldsymbol{A}}} + \overset{\bullet}{\overset{\sim 4}{\boldsymbol{B}}} \tag{2.64}$$

$$\left(\overset{\bullet}{\overset{\sim 4}{\boldsymbol{A}} : \widetilde{\boldsymbol{B}}}\right) = \overset{\bullet}{\overset{\sim 4}{\boldsymbol{A}}} : \widetilde{\boldsymbol{B}} + \overset{\sim 4}{\boldsymbol{A}} : \overset{\bullet}{\widetilde{\boldsymbol{B}}} \tag{2.65}$$

$$\dot{\left(\overset{\sim 4}{\boldsymbol{A}}:\overset{\sim 4}{\boldsymbol{B}}\right)} = \dot{\overset{\sim 4}{\boldsymbol{A}}}:\overset{\sim 4}{\boldsymbol{B}} + \overset{\sim 4}{\boldsymbol{A}}:\dot{\overset{\sim 4}{\boldsymbol{B}}} \tag{2.66}$$

Warning –

$$\left[\dot{\widetilde{\boldsymbol{A}}}\right]^{-1} \neq \left[\dot{\widetilde{\boldsymbol{A}}^{-1}}\right] = -\widetilde{\boldsymbol{A}}^{-1} \,.\, \dot{\widetilde{\boldsymbol{A}}} \,.\, \widetilde{\boldsymbol{A}}^{-1} \tag{2.67}$$

2.11 Velocities, stretching and spin tensors

Let us differentiate the particle velocity $\vec{v}\left(\vec{X},t\right)$:

$$d\vec{v} = \widetilde{\left(\frac{\boldsymbol{\partial \vec{v}}}{\boldsymbol{\partial \vec{X}}}\right)}_{/t}\left(\vec{X},t\right) . \, d\vec{X} + \overrightarrow{\left(\frac{\partial \vec{v}}{\partial t}\right)}_{/\vec{X}}\left(\vec{X},t\right) .dt \tag{2.68}$$

The vector

$$\overrightarrow{\left(\frac{\partial \vec{v}}{\partial t}\right)}_{/\vec{X}}\left(\vec{X},t\right) = \dot{\vec{v}}\left(\vec{X},t\right) = \vec{a}\left(\vec{X},t\right) \tag{2.69}$$

is the acceleration of particle M at time t.

Definition – *The second-order tensor*

$$\widetilde{\left(\frac{\boldsymbol{\partial \vec{v}}}{\boldsymbol{\partial \vec{X}}}\right)}_{/t}\left(\vec{X},t\right) = \widetilde{\boldsymbol{L}_0}\left(\vec{X},t\right) \tag{2.70}$$

is the ***lagrangian velocity gradient*** *of particle M at time t.*

Definition – *The second-order tensor*

$$\widetilde{\boldsymbol{L}}\left(\vec{X},t\right) = \widetilde{\boldsymbol{L}_0}\left(\vec{X},t\right) . \, \widetilde{\boldsymbol{F}}^{-1}\left(\vec{X},t\right) \tag{2.71}$$

is the ***eulerian velocity gradient*** *of particle M at time t.*

For a better understanding of these two velocity gradients, let us consider a material point M' in the vicinity of material point M (**FIG.*R*4**) ;

$$d\vec{x} = \vec{x}' - \vec{x} = \vec{x}(M') - \vec{x}(M) \tag{2.72}$$

$$d\vec{v} = \vec{v}' - \vec{v} = \vec{v}(M') - \vec{v}(M) \tag{2.73}$$

Property – *The **velocity gradients** are such that, at a fixed time t :*

$$\forall d\vec{x}, \qquad \vec{dv} = \widetilde{\boldsymbol{L_0}}_{(t)} \,\boldsymbol{.}\, d\vec{X} = \widetilde{\boldsymbol{L}}_{(t)} \,\boldsymbol{.}\, d\vec{x} \; . \tag{2.74}$$

Tensor $\widetilde{\boldsymbol{L_0}}$ is called *lagrangian* because it has to be applied to the increment $d\vec{X}$ in the reference configuration Ω_0. While tensor $\widetilde{\boldsymbol{L}}$ is called *eulerian* because it has to be applied to the increment $d\vec{x}$ in the deformed configuration Ω_t.

Definition – *The* stretching tensor $\widetilde{\boldsymbol{D}}$ *is defined as :*

$$\widetilde{\boldsymbol{D}} = \frac{1}{2}\left(\widetilde{\boldsymbol{L}} + \widetilde{\boldsymbol{L}}^{\mathrm{T}}\right) \; . \tag{2.75}$$

$\widetilde{\boldsymbol{D}}$ *is the **symmetric part of the eulerian velocity gradient**.*

Definition – *The* spin tensor $\widetilde{\boldsymbol{W}}$ *is defined as :*

$$\widetilde{\boldsymbol{W}} = \frac{1}{2}\left(\widetilde{\boldsymbol{L}} - \widetilde{\boldsymbol{L}}^{\mathrm{T}}\right) \; . \tag{2.76}$$

$\widetilde{\boldsymbol{W}}$ *is the **skew-symmetric part of the eulerian velocity gradient**.*

Let us recall the differentiate of function $\chi_0\left(\vec{X}, t\right)$ (eq. 2.4[p31]) :

$$d\vec{x} = \widetilde{\boldsymbol{F}}\left(\vec{X}, t\right) \boldsymbol{.}\, d\vec{X} + \vec{v}\left(\vec{X}, t\right) .dt \tag{2.77}$$

According to Schwarz's theorem (or Clairaut's or Young's theorem) :

$$\left(\widetilde{\frac{\partial \widetilde{\boldsymbol{F}}}{\partial \boldsymbol{t}}}\right)_{/\vec{X}}\left(\vec{X}, t\right) = \left(\widetilde{\frac{\partial \vec{\boldsymbol{v}}}{\partial \vec{\boldsymbol{X}}}}\right)_{/t}\left(\vec{X}, t\right) \tag{2.78}$$

$$\dot{\widetilde{\boldsymbol{F}}}\left(\vec{X}, t\right) = \widetilde{\boldsymbol{L_0}}\left(\vec{X}, t\right) \tag{2.79}$$

So, we have the following property :

Property – *The simple time derivative of the deformation gradient is related to the eulerian velocity gradient as ;*

$$\dot{\widetilde{\boldsymbol{F}}} = \widetilde{\boldsymbol{L}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{F}} \; . \tag{2.80}$$

2.12 Stretching tensor $\widetilde{\boldsymbol{D}}$ and time derivative of strain measures

According to its definition 2.75, stretching tensor $\widetilde{\boldsymbol{D}}$ is not a strain velocity, because it is not the result of a time derivative of a strain measure tensor. But tensor $\widetilde{\boldsymbol{D}}$ is related to the strain measure velocities and we can prove this by means of relations between $\widetilde{\boldsymbol{D}}$ and time derivatives of strain measures. For instance, the following relations.

Property –

$$\dot{\widetilde{\boldsymbol{E}}} = \widetilde{\boldsymbol{F}}^{\mathrm{T}} \boldsymbol{.}\, \widetilde{\boldsymbol{D}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{F}} \tag{2.81}$$

Exercise 2.7. *Prove relation 2.81.*

Properties –

$$\widetilde{\boldsymbol{D}} = \frac{1}{2} \widetilde{\boldsymbol{R}} \boldsymbol{.} \left(\dot{\widetilde{\boldsymbol{U}}} \boldsymbol{.} \widetilde{\boldsymbol{U}}^{-1} + \widetilde{\boldsymbol{U}}^{-1} \boldsymbol{.} \dot{\widetilde{\boldsymbol{U}}} \right) \boldsymbol{.} \widetilde{\boldsymbol{R}}^{\mathrm{T}} \tag{2.82}$$

$$\widetilde{\boldsymbol{W}} = \dot{\widetilde{\boldsymbol{R}}} \boldsymbol{.} \widetilde{\boldsymbol{R}}^{\mathrm{T}} + \frac{1}{2} \widetilde{\boldsymbol{R}} \boldsymbol{.} \left(\dot{\widetilde{\boldsymbol{U}}} \boldsymbol{.} \widetilde{\boldsymbol{U}}^{-1} - \widetilde{\boldsymbol{U}}^{-1} \boldsymbol{.} \dot{\widetilde{\boldsymbol{U}}} \right) \boldsymbol{.} \widetilde{\boldsymbol{R}}^{\mathrm{T}} \tag{2.83}$$

[Truesdell-2004]

Exercise 2.8. *Prove relations 2.82 and 2.83.*

Property –

$$\dot{\widetilde{\boldsymbol{LnU}}} = \overset{\sim 4}{\gamma}_{(\widetilde{\boldsymbol{U}})} : (\widetilde{\boldsymbol{R}}^{\mathrm{T}} \boldsymbol{.} \widetilde{\boldsymbol{D}} \boldsymbol{.} \widetilde{\boldsymbol{R}}) = \overset{\sim 4}{\gamma}_{(\widetilde{\boldsymbol{U}})} : \left(\widetilde{\boldsymbol{U}}^{-1} \boldsymbol{.} \dot{\widetilde{\boldsymbol{E}}} \boldsymbol{.} \widetilde{\boldsymbol{U}}^{-1} \right) \tag{2.84}$$

$$\widetilde{\boldsymbol{R}} \boldsymbol{.} \overline{\dot{\widetilde{\boldsymbol{LnU}}}} \boldsymbol{.} \widetilde{\boldsymbol{R}}^{\mathrm{T}} = \overset{\sim 4}{\gamma}_{(\widetilde{\boldsymbol{V}})} : \widetilde{\boldsymbol{D}} \tag{2.85}$$

See equation 1.160[p24] for a definition of $\overset{\sim 4}{\gamma}_{(\widetilde{\boldsymbol{U}})}$.

Proprerty –

$$\mathrm{Tr}\widetilde{\boldsymbol{D}} = \mathrm{Tr}\widetilde{\boldsymbol{L}} = \frac{\dot{J}}{J}\,. \tag{2.86}$$

Indeed, according to the conservation of mass

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho.\vec{v}) &= 0 && (2.87)\\
\frac{\partial \rho}{\partial t} + \left(\overrightarrow{\mathrm{grad}\rho}\right) \cdot \vec{v} + \rho.\mathrm{div}\vec{v} &= 0 && (2.88)\\
\frac{D\rho}{Dt} + \rho.\mathrm{div}\vec{v} &= 0 && (2.89)\\
\dot{\rho}_t + \rho_t.\mathrm{div}\vec{v} &= 0 && (2.90)\\
\dot{\left(\frac{\rho_0}{J}\right)} + \frac{\rho_0}{J}.\mathrm{div}\vec{v} &= 0 && (2.91)\\
-\rho_0.\frac{\dot{J}}{J^2} + \frac{\rho_0}{J}.\mathrm{div}\vec{v} &= 0 && (2.92)
\end{aligned}
$$

This may be simplified by dividing by ρ_0 and by multiplying by J

$$
\begin{aligned}
-\frac{\dot{J}}{J} + \mathrm{div}\vec{v} &= 0 && (2.93)\\
-\frac{\dot{J}}{J} + \mathrm{Tr}\left(\widetilde{\boldsymbol{grad}(\vec{\boldsymbol{v}})}\right) &= 0 && (2.94)\\
-\frac{\dot{J}}{J} + \mathrm{Tr}\widetilde{\boldsymbol{L}} &= 0\,. && (2.95)\\
& && (2.96)
\end{aligned}
$$

This property may also be proved by relation 2.61.

Property –

$$
\mathrm{Tr}\widetilde{\boldsymbol{D}} \;=\; \mathrm{Tr}\left(\dot{\widetilde{\boldsymbol{LnV}}}\right) \;=\; \mathrm{Tr}\left(\dot{\widetilde{\boldsymbol{LnU}}}\right) \qquad (2.97)
$$

2.13 Déduction de la rotation propre dans des cas simples de transformation

Exercise 2.9. *Que peut-on affirmer au sujet de $\widetilde{\boldsymbol{R}}$, $\widetilde{\boldsymbol{U}}$ et $\widetilde{\boldsymbol{V}}$ sans calculs dans les cas de chargements suivants ?*

1. ***Essai mécanique en traction uniaxiale*** *: le référentiel de base a est choisi lié au bâti de la machine, la configuration de référence Ω_0 est choisie comme celle avant l'essai avec on le suppose une contrainte nulle, et le point matériel M considéré est au centre de l'éprouvette dans la partie centrale (***FIG.*D*5***).*
2. ***Essai mécanique en bi-traction*** *: le référentiel de base a est choisi lié au bâti de la machine, la configuration de référence Ω_0 est choisie comme celle avant l'essai avec on le suppose des contraintes nulles, et le point matériel M considéré est au centre de l'éprouvette dans la partie centrale (***FIG.*D*6***).*
3. ***Mouton Charpy*** *: le référentiel de base a est choisi lié au bâti de la machine, la configuration de référence Ω_0 est choisie comme celle avant la chute du pendule, lorsqu'il est en position haute, et le point matériel M considéré est comme sur la* ***FIG.*D*7***. *La configuration actuelle Ω_t est choisie comme celle durant la chute du pendule mais avant l'impact avec l'éprouvette.*
4. ***Pression de fluide sur une tôle*** *: soit une tôle d'acier en forme de disque serrée sur le bord annulaire, un fluide sous pression pousse la tôle d'un côté afin de former une cloque (***FIG.*D*8***). Le référentiel de base a est choisi lié au bâti de la machine, la configuration de référence Ω_0 est choisie comme celle avant la mise en pression, lorsque la tôle est plane, le point matériel M est choisi au centre de la tôle.*
5. ***Essai mécanique de cisaillement pur*** *: le référentiel de base a est choisi lié au mors fixe de la machine, la configuration de référence Ω_0 est choisie comme celle avant l'essai avec on le suppose une contrainte nulle, et le point matériel M considéré est au centre de l'éprouvette dans la partie centrale (***FIG.*D*9***).*
6. ***Mechanical test of simple shear*** *: the base reference frame a is chosen as the fixed jaw of the machine, the reference configuration Ω_0 is chosen as the one before the loading for which the stress is assumed null, and the material point M is at the sample center.*

Une **méthode pour déduire des informations sur la rotation propre** et les déformations pures $\widetilde{\boldsymbol{U}}$ et $\widetilde{\boldsymbol{V}}$ est la suivante :

♣ Il faut trouver trois axes matériels perpendiculaires entre eux, qui restent perpendiculaires lorsqu'ils sont entrainés par la transformation. Par exemple, dans le cas de la déformation d'une tôle dans son plan (sans pliage), il s'agit de tracer au feutre deux droites perpendiculaires sur la surface de la tôle, de sorte qu'après la transformation

`ou déformation, ces droites restent toujours perpendiculaires.`
`Si ces trois droites sont trouvées alors elles donnent`
`dans la configuration de référence` Ω_0 `les trois directions`
`principales du tenseur` $\widetilde{\boldsymbol{U}}$`, et dans la configuration actuelle`
Ω_t `celle du tenseur` $\widetilde{\boldsymbol{V}}$`. De plus, il existe une rotation`
`qui change les trois droites de` Ω_0 `vers` Ω_t`, et elle est`
`identique à la rotation propre de la transformation.`

Sur la figure 2.1[p34], qui est un cas en deux dimensions, les droites sont celles en traits fins pointillés et mixtes.

Exercise 2.10. *Velocities in case of simple shear (***FIG.*D*11***) : calculate the following tensors ;*
- $\widetilde{\boldsymbol{D}}$ *stretching tensor,*
- $\widetilde{\boldsymbol{W}}$ *spin tensor.*

Give their matrix in basis $(\vec{e_1}, \vec{e_2}, \vec{e_3})$.

Chapitre 3

Stress tensors and associated strains

3.1 Cauchy stress tensor $\widetilde{\boldsymbol{T}}$

It is a symmetric second-order tensor. It describes the state of effort within the matter, at a material point. This is the classical stress tensor, usually presented into continuum mechanics books.

Let us redefine this concept. We consider an infinitesimal surface element within a continuum body (**FIG.*C*1*a***) in the actual configuration.

dS will be the area amount of this surface element in the reference configuration Ω_0, while ds is this amount in the actual configuration Ω_t.

This surface element is surrounded by matter. Let us choose arbitrarily one side of this matter, called *exterior*, and the other side, called *interior*, separated by the surface element ds.

We represent this choice by the direction of a unit vector $\vec{n}$, normal to the plane of the surface element and **from the interior to the exterior** (**FIG.*C*1*b***).

We define vector $\vec{ds}$ (**FIG.*C*1*c***), as ;

$$\vec{ds} = ds.\vec{n} \ . \tag{3.1}$$

Through this surface element ds, the exterior side applies a force on the interior side. Let us write it $\vec{df}$ (**FIG.*C*1*d***).

Let us consider a material particle arround point M, in the actual configuration Ω_t (**FIG.*C*2**). Let us consider all possible surface elements $\vec{ds}\,(t)$ containing point M, and also all associated infinitesimal forces $\vec{df}\,(t)$. Then, it is possible to prove that the relation between $\vec{ds}\,(t)$ and $\vec{df}\,(t)$ is a linear map :

Definition – *The Cauchy stress tensor is defined as the following linear map :*

$$\forall \vec{ds}\,(t) \ni M, \qquad \vec{df}\,(t) = \widetilde{\boldsymbol{T}}\,(M,t) \cdot \vec{ds}\,(t) \ . \tag{3.2}$$

or

$$\forall \vec{ds}\,(t) \ni \vec{x}\left(\vec{X},t\right), \qquad \vec{df}\,(t) = \widetilde{\boldsymbol{T}}\left(\vec{X},t\right) \cdot \vec{ds}\,(t) \ . \tag{3.3}$$

For example, durign a uniaxial or simple tensile test (**FIG.$C3$**) :

$$T_{(t)} = \frac{f_{(t)}}{s} \qquad \left(\neq \frac{f_{(t)}}{S}\right) \ , \tag{3.4}$$

This is sometimes called the **true stress**.

3.2 Kirchhoff stress tensor $\widetilde{\boldsymbol{\tau}}$

It is a symmetric second-order tensor.

Definition – *The Kirchhoff stress tensor is defined as*

$$\widetilde{\boldsymbol{\tau}} = J.\widetilde{\boldsymbol{T}} \ = (\det\widetilde{\boldsymbol{F}}).\widetilde{\boldsymbol{T}} \ = \frac{\rho_0}{\rho_t}.\widetilde{\boldsymbol{T}} \ . \tag{3.5}$$

3.3 Specific stress tensor $\widetilde{\boldsymbol{\Sigma}}$

It is a symmetric second-order tensor.

Definition – *The specific stress tensor is defined as*

$$\widetilde{\boldsymbol{\Sigma}} = \frac{\widetilde{\boldsymbol{T}}}{\rho_t} \ . \tag{3.6}$$

This quantity has a unusual unit. It is not, as the others, pascal Pa or megapascal MPa.

Its unit is joule per gram $\mathrm{J/g}$ or newton-meter per gram $\mathrm{N.m/g}$.

3.4 First Piola-Kirchhoff stress tensor $\widetilde{\mathbf{\Pi}}$

It is sometimes called *Boussinesq* stress, or *PK1* stress.

Be careful, this tensor is not symmetric in the general case. Its transposed tensor $\widetilde{\mathbf{\Pi}}^{\mathbf{T}}$ is sometimes called *nominal stress*.

Definition – *The first Piola-Kirchhoff stress tensor is defined as*

$$\widetilde{\mathbf{\Pi}} = (\det\widetilde{\boldsymbol{F}}).\left(\widetilde{\boldsymbol{T}}\,\boldsymbol{.}\,\widetilde{\boldsymbol{F}}^{-\mathbf{T}}\right) \ . \tag{3.7}$$

Because the surface $\vec{ds}$ is convected by (or attached to) the matter, it exists a relation between $\vec{dS}$ and $\vec{ds}$ via the deformation gradient $\widetilde{\boldsymbol{F}}$; this relation is called **NANSON formula** :

$$\vec{ds} = \det\widetilde{\boldsymbol{F}}.\left(\widetilde{\boldsymbol{F}}^{-\mathbf{T}}\,\boldsymbol{.}\,\vec{dS}\right) \ . \tag{3.8}$$

Let us prove this relation.

FIG.$C4$ illustrates the two surfaces $\vec{dS}$ and $\vec{ds}$. Let us consider a material point P in the vicinity of material point M. So, vector $\overrightarrow{MP} = \vec{dl}$ is convected by the matter, i.e. :

$$\vec{dl} = \widetilde{\boldsymbol{F}}\,\boldsymbol{.}\,\vec{dL} \ . \tag{3.9}$$

Let us consider the volume $d\mathrm{v}$ resulting from the extrusion of surface ds along vector $\vec{dl}$. Since surface ds and vector $\vec{dl}$ are convected by the matter, so is volume $d\mathrm{v}$. This leads to :

$$\frac{d\mathrm{v}}{d\mathrm{V}} = J = \det\widetilde{\boldsymbol{F}} \ . \tag{3.10}$$

By definition of volume $d\mathrm{v}$;

$$d\mathrm{v} = \vec{ds}\,\boldsymbol{.}\,\vec{dl} \ . \tag{3.11}$$

So,

$$\begin{aligned}
d\mathrm{v} &= \det\widetilde{\boldsymbol{F}}.d\mathrm{V} && (3.12)\\
\vec{ds}\,\boldsymbol{.}\,\vec{dl} &= \det\widetilde{\boldsymbol{F}}.\ \vec{dS}\,\boldsymbol{.}\,\vec{dL} && (3.13)\\
\vec{ds}\,\boldsymbol{.}\left(\widetilde{\boldsymbol{F}}\,\boldsymbol{.}\,\vec{dL}\right) &= && (3.14)\\
\vec{dL}\,\boldsymbol{.}\left(\widetilde{\boldsymbol{F}}^{\mathbf{T}}\,\boldsymbol{.}\,\vec{ds}\right) &= && (3.15)
\end{aligned}$$

$$\Longleftrightarrow \left(\widetilde{\boldsymbol{F}}^{\mathbf{T}}\,\boldsymbol{.}\,\vec{ds} - \det\widetilde{\boldsymbol{F}}.\vec{dS}\right)\boldsymbol{.}\,\vec{dL} = 0 \ . \tag{3.16}$$

This is true for all other choices of $\vec{dL}$, without changing neither $\widetilde{\boldsymbol{F}}$, nor $\vec{dS}$, nor $\vec{ds}$. Then,

$$\forall \vec{dL}, \qquad \left(\widetilde{\boldsymbol{F}}^{\mathbf{T}} \cdot \vec{ds} - \det\widetilde{\boldsymbol{F}}.\vec{dS}\right) \cdot \vec{dL} = 0 \tag{3.17}$$

$$\Longleftrightarrow \left(\widetilde{\boldsymbol{F}}^{\mathbf{T}} \cdot \vec{ds} - \det\widetilde{\boldsymbol{F}}.\vec{dS}\right) = 0 \tag{3.18}$$

$$\Longleftrightarrow \vec{ds} = \det\widetilde{\boldsymbol{F}}. \left(\widetilde{\boldsymbol{F}}^{-\mathbf{T}} \cdot \vec{dS}\right) \ . \tag{3.19}$$

In order to better understand the meaning of this stress tensor, let us calculate the term :

$$\begin{aligned}
\widetilde{\boldsymbol{\Pi}} \cdot \vec{dS} &= (\det\widetilde{\boldsymbol{F}}). \left(\widetilde{\boldsymbol{T}} \cdot \widetilde{\boldsymbol{F}}^{-\mathbf{T}}\right) \cdot \vec{dS} && (3.20)\\
&= (\det\widetilde{\boldsymbol{F}}). \ \widetilde{\boldsymbol{T}} \cdot \left(\widetilde{\boldsymbol{F}}^{-\mathbf{T}} \cdot \vec{dS}\right) && (3.21)\\
&= \widetilde{\boldsymbol{T}} \cdot \left(\det\widetilde{\boldsymbol{F}}. \ \widetilde{\boldsymbol{F}}^{-\mathbf{T}} \cdot \vec{dS}\right) && (3.22)\\
&= \widetilde{\boldsymbol{T}} \cdot \left(\vec{ds}\right) && (3.23)\\
&= \vec{df}_{(t)} \ . && (3.24)
\end{aligned}$$

So, this leads to the following property :

Property –

$$\vec{df}_{(t)} = \widetilde{\boldsymbol{\Pi}}_{(t)} \cdot \vec{dS} \ . \tag{3.25}$$

Exercise 3.1. *Prove that stress $\widetilde{\boldsymbol{\Pi}}$ is not always symmetric.*

As an example, during a uniaxial or simple tensile test (**FIG.C5**) :

$$\Pi_{(t)} = \frac{f_{(t)}}{S} \ , \tag{3.26}$$

This is called often **nominal stress** or **conventional stress**.

3.5 Second Piola-Kirchhoff stress tensor $\widetilde{\boldsymbol{S}}$

It is sometimes called *Piola-Lagrange* stress, and also *PK2* stress. It is a symmetric second-order tensor.

Definition – *The second Piola-Kirchhoff stress tensor is defined as*

$$\widetilde{\boldsymbol{S}} = (\det\widetilde{\boldsymbol{F}}).\left(\widetilde{\boldsymbol{F}}^{-1} \boldsymbol{.} \widetilde{\boldsymbol{T}} \boldsymbol{.} \widetilde{\boldsymbol{F}}^{-\mathrm{T}}\right) \quad . \tag{3.27}$$

Current vector transformed back to the reference configuration In order to better understand the meaning of this stress tensor, let us define this transformation of a current vector from the actual configuration to the reference one.

Let $\vec{v}$ be a vector at current time t, in the current configuration Ω_t ; $\vec{v_{(t)}}$. **FIG.$C6$** illustrates a vicinity around a material point M in the reference and actual configuration.

Imagine that we stick the arrow representing vector $\vec{v_{(t)}}$ to the matter, and we deform back the matter from the actual configuration Ω_t to the reference configuration Ω_0 :

$$\vec{v_{(t)}} \quad \overset{\widetilde{\boldsymbol{F}}^{-1}}{\longmapsto} \quad \vec{v}_{t\to 0} \tag{3.28}$$

et so

$$\vec{v}_{t\to 0} = \widetilde{\boldsymbol{F}}^{-1} \boldsymbol{.} \vec{v_{(t)}} \quad . \tag{3.29}$$

Let us apply this process to $\vec{v_{(t)}} = \vec{df_{(t)}}$ the force applied on the surface element $\vec{ds}$ at time t. We define then $\vec{df}_{t\to 0}$ as :

$$\vec{df}_{t\to 0} = \widetilde{\boldsymbol{F}}^{-1} \boldsymbol{.} \vec{df_{(t)}} \quad . \tag{3.30}$$

In order to put forward a property of stress tensor $\widetilde{\boldsymbol{S}}$, let us calculate the term

$$\begin{aligned}
\widetilde{\boldsymbol{S}} \boldsymbol{.} \vec{dS} &= (\det\widetilde{\boldsymbol{F}}).\left(\widetilde{\boldsymbol{F}}^{-1} \boldsymbol{.} \widetilde{\boldsymbol{T}} \boldsymbol{.} \widetilde{\boldsymbol{F}}^{-\mathrm{T}}\right) \boldsymbol{.} \vec{dS} && (3.31)\\
&= \left(\widetilde{\boldsymbol{F}}^{-1} \boldsymbol{.} \widetilde{\boldsymbol{\Pi}}\right) \boldsymbol{.} \vec{dS} && (3.32)\\
&= \widetilde{\boldsymbol{F}}^{-1} \boldsymbol{.} \vec{df_{(t)}} && (3.33)\\
&= \vec{df}_{t\to 0} \quad . && (3.34)
\end{aligned}$$

Then the following property is obtained :

Property –

$$\vec{df}_{t\to 0} = \widetilde{\boldsymbol{S}}_{(t)} \boldsymbol{.} \vec{dS} \quad . \tag{3.35}$$

So we understand, that the second Piola-Kirchhoff stress tensor $\widetilde{\boldsymbol{S}}_{(t)}$ contains the same effort information as the Cauchy stress tensor $\widetilde{\boldsymbol{T}}$, but we have access to this information in the context of the reference configuration Ω_0. While, with the Cauchy stress $\widetilde{\boldsymbol{T}}$, we have access to this information in the context of the actual configuration Ω_t.

Because of that, the Cauchy stress tensor $\widetilde{\boldsymbol{T}}$ is sometimes called **eulerian**, while the PK2 stress tensor $\widetilde{\boldsymbol{S}}$ is called **lagrangian**.

Exercise 3.2. *Prove that the second Piola-Kirchhoff stress tensor $\widetilde{\boldsymbol{S}}$ is symmetric.*

As an example, during a uniaxial or simple tesile test (**FIG.*C*5**) :

$$\vec{f}_{t\to 0} = \widetilde{\boldsymbol{F}}^{-1} \cdot \vec{f}_{(t)} \tag{3.36}$$

$$\Longleftrightarrow Col_{\vec{e}_i}\left(\vec{f}_{t\to 0}\right) = \begin{bmatrix} \frac{D}{d} & 0 & 0 \\ 0 & \frac{D}{d} & 0 \\ 0 & 0 & \frac{L}{l} \end{bmatrix} \times \begin{pmatrix} 0 \\ 0 \\ f_{(t)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{L.f_{(t)}}{l} \end{pmatrix} . \tag{3.37}$$

Then

$$S_{(t)} = \frac{L.f_{(t)}}{l.S} , \tag{3.38}$$

This tensile PK2 stress is close to a Cauchy stress assuming that the volume is constant :

$$\frac{f_{(t)}}{S}.(1+\varepsilon) = \frac{f_{(t)}}{S}.\left(1+\frac{\Delta l}{L}\right) = \frac{l.f_{(t)}}{L.S} , \tag{3.39}$$

but actually the Pk2 stress is not equal to this last one.

3.6 Summary of the stress tensors

Table 3.1 gives a short summary of the presented stress tensors.

Remark – If we assume small displacements, i.e. the deformation gradient tensor $\widetilde{\boldsymbol{F}}$ is close identity tensor $\widetilde{\boldsymbol{I}}$

$$\widetilde{\boldsymbol{F}} \simeq \widetilde{\boldsymbol{I}} , \tag{3.40}$$

Then the stress tensors becomes very close with each other

$$\widetilde{\boldsymbol{S}} \simeq \widetilde{\boldsymbol{\Pi}} \simeq \widetilde{\boldsymbol{\tau}} \simeq \widetilde{\boldsymbol{T}} . \tag{3.41}$$

Remark – If we assume small strains, but possibly large rotations, i.e. the deformation gradient tensor $\widetilde{\boldsymbol{F}}$ is close to a rotation

$$\widetilde{\boldsymbol{F}} \simeq \widetilde{\boldsymbol{R}} , \tag{3.42}$$

Stress tensor	Symmetric ?	Definition	Relation force-surface	In tension	Eulerian, lagrangian ?	Unit
Cauchy $\widetilde{\boldsymbol{T}}$	sym.	classical stress	$d\vec{f}_{(t)} = \widetilde{\boldsymbol{T}} \cdot \vec{ds}$	$\frac{f_{(t)}}{s}$	eulerian	MPa
Kirchhoff $\widetilde{\boldsymbol{\tau}}$	sym.	$\widetilde{\boldsymbol{\tau}} = \det\widetilde{\boldsymbol{F}}.\widetilde{\boldsymbol{T}}$			eulerian	MPa
Specific $\widetilde{\boldsymbol{\Sigma}}$	sym.	$\widetilde{\boldsymbol{\Sigma}} = \frac{\widetilde{\boldsymbol{T}}}{\rho_t}$		$\frac{f_{(t)}}{\eta_t}$ η_t : current mass per unit length of the sample.	eulerian	J/g
First Piola-Kirchhoff $\widetilde{\boldsymbol{\Pi}}$	not sym.	$\widetilde{\boldsymbol{\Pi}} = \widetilde{\boldsymbol{\tau}} \cdot \widetilde{\boldsymbol{F}}^{-\mathbf{T}}$	$d\vec{f}_{(t)} = \widetilde{\boldsymbol{\Pi}} \cdot \vec{dS}$	$\frac{f_{(t)}}{S}$	mixed, neither eulerian nor lagrangian	MPa
Second Piola-Kirchhoff $\widetilde{\boldsymbol{S}}$	sym.	$\widetilde{\boldsymbol{S}} = \widetilde{\boldsymbol{F}}^{-\mathbf{1}} \cdot \widetilde{\boldsymbol{\Pi}}$	$\vec{df}_{t\to 0} = \widetilde{\boldsymbol{S}} \cdot \vec{dS}$	$\frac{L.f_{(t)}}{l.S}$	lagrangian	MPa

TABLE 3.1 – Summary of the stress tensors.

then the two eulerian stress tensors are close

$$\tilde{\boldsymbol{\tau}} \simeq \tilde{\boldsymbol{T}} \,, \tag{3.43}$$

and the second Piola-Kirchhoff stress tensor is close to the eulerian ones regardless of the proper rotation

$$\tilde{\boldsymbol{S}} \simeq \tilde{\boldsymbol{R}}^{\mathrm{T}} \boldsymbol{.} \tilde{\boldsymbol{\tau}} \boldsymbol{.} \tilde{\boldsymbol{R}} \simeq \tilde{\boldsymbol{R}}^{\mathrm{T}} \boldsymbol{.} \tilde{\boldsymbol{T}} \boldsymbol{.} \tilde{\boldsymbol{R}} \,. \tag{3.44}$$

3.7 What stress-strain couple ?

For proposing a constitutive law, we write a relation between a chosen stress tensor and a chosen strain tensor. How make this stress and strain choices ?

Our recommendation : use one of this two ways

- a strain tensor based on $\tilde{\boldsymbol{V}}$ (left) and a lagrangian stress tensor (for example $\tilde{\boldsymbol{S}}$).
- a strain tensor based on $\tilde{\boldsymbol{U}}$ (right) and an eulerian stress tensor (for example $\tilde{\boldsymbol{T}}$ or $\tilde{\boldsymbol{\tau}}$ or $\tilde{\boldsymbol{\Sigma}}$).

For proposing a law in agreement with the thermodynamic principles, it is compulsory to choose a couple such that :

- for a chosen right strain tensor $\tilde{\boldsymbol{\varepsilon}}_{\boldsymbol{U}}$

$$\forall \tilde{\boldsymbol{D}} \text{ symmetric,} \quad \frac{-1}{\rho_0} . \mathrm{Tr} \left(\tilde{\boldsymbol{\sigma}} \boldsymbol{.} \overset{\bullet}{\tilde{\boldsymbol{\varepsilon}}_{\boldsymbol{U}}} \right) = \frac{-1}{\rho_t} . \mathrm{Tr} \left(\tilde{\boldsymbol{T}} \boldsymbol{.} \tilde{\boldsymbol{D}} \right) (= p_{\mathrm{int}}) \,, \tag{3.45}$$

- for a chosen left strain tensor $\tilde{\boldsymbol{\varepsilon}}_{\boldsymbol{V}}$

$$\forall \tilde{\boldsymbol{D}} \text{ symmetric,} \quad \frac{-1}{\rho_t} . \mathrm{Tr} \left(\tilde{\boldsymbol{\sigma}} \boldsymbol{.} \overset{\circ}{\tilde{\boldsymbol{\varepsilon}}_{\boldsymbol{V}}} \right) = \frac{-1}{\rho_t} . \mathrm{Tr} \left(\tilde{\boldsymbol{T}} \boldsymbol{.} \tilde{\boldsymbol{D}} \right) (= p_{\mathrm{int}}) \,, \tag{3.46}$$

where p_{int} is the specific power of internal forces and where $\left(\overset{\circ}{*}\right)$ is an objective rate (chapter 7). Tensor $\tilde{\boldsymbol{D}}$ is the stretching tensor.

Exercise 3.3. *Quelle contrainte lagrangienne doit-on choisir pour la déformation* $\tilde{\boldsymbol{E}}$ *de Green-Lagrange afin qu'elle respecte l'équation 3.45 ?*

Afin de répondre à la question de l'exercice 3.3, nous recherchons tout d'abord une relation entre la vitesse $\dot{\widetilde{\boldsymbol{E}}}$ et le tenseur $\widetilde{\boldsymbol{D}}$. Dérivons par rapport au temps la propriété suivante :

$$\widetilde{\boldsymbol{E}} = \frac{1}{2}\left(\widetilde{\boldsymbol{F}}^{\mathbf{T}} \cdot \widetilde{\boldsymbol{F}} - \widetilde{\boldsymbol{I}}\right) \; . \tag{3.47}$$

Ainsi,

$$\dot{\widetilde{\boldsymbol{E}}} = \frac{1}{2}\left[\left(\dot{\widetilde{\boldsymbol{F}}^{\mathbf{T}}}\right) \cdot \widetilde{\boldsymbol{F}} + \widetilde{\boldsymbol{F}}^{\mathbf{T}} \cdot \dot{\widetilde{\boldsymbol{F}}} - \dot{\widetilde{\boldsymbol{I}}}\right] \; , \tag{3.48}$$

or $\left(\dot{\widetilde{\boldsymbol{F}}^{\mathbf{T}}}\right) = \left(\dot{\widetilde{\boldsymbol{F}}}\right)^{\mathbf{T}}$ et $\dot{\widetilde{\boldsymbol{I}}} = \widetilde{\mathbf{0}}$, donc

$$\dot{\widetilde{\boldsymbol{E}}} = \frac{1}{2}\left[\left(\dot{\widetilde{\boldsymbol{F}}}\right)^{\mathbf{T}} \cdot \widetilde{\boldsymbol{F}} + \widetilde{\boldsymbol{F}}^{\mathbf{T}} \cdot \dot{\widetilde{\boldsymbol{F}}}\right] \; . \tag{3.49}$$

Une relation importante et toujours vraie est utile à cette étape du raisonnement :

$$\dot{\widetilde{\boldsymbol{F}}} = \widetilde{\boldsymbol{L}} \cdot \widetilde{\boldsymbol{F}} \; , \tag{3.50}$$

où $\widetilde{\boldsymbol{L}}$ est le tenseur gradient du champ de vitesse (chap. 6). Par conséquent,

$$\begin{aligned} \dot{\widetilde{\boldsymbol{E}}} &= \frac{1}{2}\left[\widetilde{\boldsymbol{F}}^{\mathbf{T}} \cdot \widetilde{\boldsymbol{L}}^{\mathbf{T}} \cdot \widetilde{\boldsymbol{F}} + \widetilde{\boldsymbol{F}}^{\mathbf{T}} \cdot \widetilde{\boldsymbol{L}} \cdot \widetilde{\boldsymbol{F}}\right] && (3.51) \\ &= \widetilde{\boldsymbol{F}}^{\mathbf{T}} \cdot \left[\frac{1}{2}\left(\widetilde{\boldsymbol{L}}^{\mathbf{T}} + \widetilde{\boldsymbol{L}}\right)\right] \cdot \widetilde{\boldsymbol{F}} \; . && (3.52) \end{aligned}$$

Or, par définition, $\widetilde{\boldsymbol{D}} = \frac{1}{2}\left(\widetilde{\boldsymbol{L}} + \widetilde{\boldsymbol{L}}^{\mathbf{T}}\right)$, donc

$$\dot{\widetilde{\boldsymbol{E}}} = \widetilde{\boldsymbol{F}}^{\mathbf{T}} \cdot \widetilde{\boldsymbol{D}} \cdot \widetilde{\boldsymbol{F}} \; . \tag{3.53}$$

Voilà, notre relation entre la vitesse de $\widetilde{\boldsymbol{E}}$ et le tenseur $\widetilde{\boldsymbol{D}}$. Ainsi, si $\widetilde{\boldsymbol{\sigma}}$ est le tenseur de contrainte recherché ;

$$\begin{aligned} \mathrm{Tr}\left(\widetilde{\boldsymbol{\sigma}} \cdot \dot{\widetilde{\boldsymbol{E}}}\right) &= \mathrm{Tr}\left(\widetilde{\boldsymbol{\sigma}} \cdot \widetilde{\boldsymbol{F}}^{\mathbf{T}} \cdot \widetilde{\boldsymbol{D}} \cdot \widetilde{\boldsymbol{F}}\right) && (3.54) \\ &= \mathrm{Tr}\left(\widetilde{\boldsymbol{F}} \cdot \widetilde{\boldsymbol{\sigma}} \cdot \widetilde{\boldsymbol{F}}^{\mathbf{T}} \cdot \widetilde{\boldsymbol{D}}\right) \; . && (3.55) \end{aligned}$$

Par là, pour satisfaire la relation 3.45, il faut que

$$\forall \widetilde{\boldsymbol{D}} \text{ symétrique}, \quad \frac{-1}{\rho_0}.\mathrm{Tr}\left(\widetilde{\boldsymbol{F}}\,.\,\widetilde{\boldsymbol{\sigma}}\,.\,\widetilde{\boldsymbol{F}}^{\mathbf{T}}.\,\widetilde{\boldsymbol{D}}\right) = \frac{-1}{\rho_t}.\mathrm{Tr}\left(\widetilde{\boldsymbol{T}}\,.\,\widetilde{\boldsymbol{D}}\right) \tag{3.56}$$

$$\iff \mathrm{Tr}\left(\widetilde{\boldsymbol{F}}\,.\,\widetilde{\boldsymbol{\sigma}}\,.\,\widetilde{\boldsymbol{F}}^{\mathbf{T}}.\,\widetilde{\boldsymbol{D}}\right) = \mathrm{Tr}\left(\frac{\rho_0}{\rho_t}\widetilde{\boldsymbol{T}}\,.\,\widetilde{\boldsymbol{D}}\right) \tag{3.57}$$

$$= \mathrm{Tr}\left(\widetilde{\boldsymbol{\tau}}\,.\,\widetilde{\boldsymbol{D}}\right) \tag{3.58}$$

$$\iff \mathrm{Tr}\left[\left(\widetilde{\boldsymbol{F}}\,.\,\widetilde{\boldsymbol{\sigma}}\,.\,\widetilde{\boldsymbol{F}}^{\mathbf{T}} - \widetilde{\boldsymbol{\tau}}\right).\,\widetilde{\boldsymbol{D}}\right] = 0 \tag{3.59}$$

Comme cette dernière égalité doit être vérifiée pour tout tenseur $\widetilde{\boldsymbol{D}}$ symétrique, alors nous pouvons affirmer que le tenseur $\left(\widetilde{\boldsymbol{F}}\,.\,\widetilde{\boldsymbol{\sigma}}\,.\,\widetilde{\boldsymbol{F}}^{\mathbf{T}} - \widetilde{\boldsymbol{\tau}}\right)$ est antisymétrique :

$$\left(\widetilde{\boldsymbol{F}}\,.\,\widetilde{\boldsymbol{\sigma}}\,.\,\widetilde{\boldsymbol{F}}^{\mathbf{T}} - \widetilde{\boldsymbol{\tau}}\right)^{\mathbf{T}} = -\left(\widetilde{\boldsymbol{F}}\,.\,\widetilde{\boldsymbol{\sigma}}\,.\,\widetilde{\boldsymbol{F}}^{\mathbf{T}} - \widetilde{\boldsymbol{\tau}}\right) . \tag{3.60}$$

$$\iff \frac{1}{2}\left(\widetilde{\boldsymbol{\sigma}} + \widetilde{\boldsymbol{\sigma}}^{\mathbf{T}}\right) = \widetilde{\boldsymbol{F}}^{-\mathbf{1}}.\,\widetilde{\boldsymbol{\tau}}\,.\,\widetilde{\boldsymbol{F}}^{-\mathbf{T}} = \widetilde{\boldsymbol{S}} . \tag{3.61}$$

Faisons l'hypothèse que le tenseur de contrainte recherché soit symétrique

$$\widetilde{\boldsymbol{\sigma}}^{\mathbf{T}} = \widetilde{\boldsymbol{\sigma}} , \tag{3.62}$$

alors nous concluons ainsi,

♣ `le tenseur de contrainte lagrangien et symétrique accouplé à la déformation de Green-Lagrange` $\widetilde{\boldsymbol{E}}$ `selon l'équation 3.45 est le second tenseur de Piola-Kirchhoff` $\widetilde{\boldsymbol{S}}$.

D'autres couple contrainte-déformation construits selon l'équation 3.46 sont les suivants :

- $\left(\widetilde{\boldsymbol{\sigma}_{\boldsymbol{Jrp}}}, \widetilde{\boldsymbol{J}}\right)$ où $\widetilde{\boldsymbol{\sigma}_{\boldsymbol{Jrp}}} = \frac{1}{2}\left(\widetilde{\boldsymbol{V}}^{-\mathbf{1}}.\,\widetilde{\boldsymbol{T}} + \widetilde{\boldsymbol{T}}\,.\,\widetilde{\boldsymbol{V}}^{-\mathbf{1}}\right)$, en effet

$$\frac{-1}{\rho_t}\mathrm{Tr}\left(\widetilde{\boldsymbol{\sigma}_{\boldsymbol{Jrp}}}.\,\overset{\circ rp}{\widetilde{\boldsymbol{J}}}\right) = p_{\mathrm{int}} , \tag{3.63}$$

- $\left(\widetilde{\boldsymbol{\sigma}_{\boldsymbol{Arp}}}, \widetilde{\boldsymbol{A}}\right)$ où $\widetilde{\boldsymbol{\sigma}_{\boldsymbol{Arp}}} = \left(\widetilde{\boldsymbol{V}}\,.\,\widetilde{\boldsymbol{T}}\,.\,\widetilde{\boldsymbol{V}}\right)$, en effet

$$\frac{-1}{\rho_t}\mathrm{Tr}\left(\widetilde{\boldsymbol{\sigma}_{\boldsymbol{Arp}}}.\,\overset{\circ rp}{\widetilde{\boldsymbol{A}}}\right) = p_{\mathrm{int}} , \tag{3.64}$$

– $\left(\widetilde{\boldsymbol{\sigma}_{\boldsymbol{e_1 rp}}}, \widetilde{\boldsymbol{e_1}}\right)$ où $\widetilde{\boldsymbol{\sigma}_{\boldsymbol{e_1 rp}}} = \left(\widetilde{\boldsymbol{V}}^{-1} \cdot \widetilde{\boldsymbol{T}} \cdot \widetilde{\boldsymbol{V}}^{-1}\right)$, en effet

$$\frac{-1}{\rho_t}\mathrm{Tr}\left(\widetilde{\boldsymbol{\sigma}_{\boldsymbol{e_1 rp}}} \cdot \overset{\circ rp}{\widetilde{\boldsymbol{e_1}}}\right) = p_{\mathrm{int}} \ , \tag{3.65}$$

– $\left(\widetilde{\boldsymbol{\sigma}_{\boldsymbol{Aj}}}, \widetilde{\boldsymbol{A}}\right)$ où $\widetilde{\boldsymbol{\sigma}_{\boldsymbol{Aj}}} = 2\left[\overset{\sim 4}{\boldsymbol{N}}_{\left(\widetilde{\boldsymbol{B}}^{-1}\right)}\right]^{-1} : \widetilde{\boldsymbol{T}}$, en effet

$$\frac{-1}{\rho_t}\mathrm{Tr}\left(\widetilde{\boldsymbol{\sigma}_{\boldsymbol{Aj}}} \cdot \overset{\circ j}{\widetilde{\boldsymbol{A}}}\right) = p_{\mathrm{int}} \ , \tag{3.66}$$

– $\left(\widetilde{\boldsymbol{\sigma}_{\boldsymbol{e_1 j}}}, \widetilde{\boldsymbol{e_1}}\right)$ où $\widetilde{\boldsymbol{\sigma}_{\boldsymbol{e_1 j}}} = 2\left[\overset{\sim 4}{\boldsymbol{N}}_{(\widetilde{\boldsymbol{B}})}\right]^{-1} : \widetilde{\boldsymbol{T}}$, en effet

$$\frac{-1}{\rho_t}\mathrm{Tr}\left(\widetilde{\boldsymbol{\sigma}_{\boldsymbol{e_1 j}}} \cdot \overset{\circ j}{\widetilde{\boldsymbol{e_1}}}\right) = p_{\mathrm{int}} \ , \tag{3.67}$$

Dans les expressions ci-dessus, $\overset{\circ rp}{(.)}$ et $\overset{\circ j}{(.)}$ désignent respectivement les dérivées temporelles objectives de Green-Naghdi-McInnis et de Jaumann. Ces deux dérivées objectives seront vues au chapitre 7. Le tenseur d'ordre quatre $\overset{\sim 4}{\boldsymbol{N}}_{(\widetilde{\boldsymbol{Y}})}$, qui est fonction d'un tenseur d'ordre deux $\widetilde{\boldsymbol{Y}}$, est défini par l'égalité suivante pour tout tenseur d'ordre deux $\widetilde{\boldsymbol{X}}$:

$$\overset{\sim 4}{\boldsymbol{N}}_{(\widetilde{\boldsymbol{Y}})} : \widetilde{\boldsymbol{X}} = \widetilde{\boldsymbol{X}} \cdot \widetilde{\boldsymbol{Y}} + \widetilde{\boldsymbol{Y}} \cdot \widetilde{\boldsymbol{X}} \ . \tag{3.68}$$

3.8 Example of constitutive law

At this point of the present lecture, we may use strain and stress tensors. We can now propose linear or non-linear **elastic isotropic constitutive laws**, by proposing a bijective map between a stress and a strain tensors.

For example, let us choose the Cauchy stress $\widetilde{\boldsymbol{T}}$ and the left logarithmique strain $\widetilde{\boldsymbol{LnV}}$. A possible law is

$$\widetilde{\boldsymbol{T}} \ = \ f(\widetilde{\boldsymbol{LnV}}).\widetilde{\boldsymbol{I}} \ + \ g(\widetilde{\boldsymbol{LnV}}).\widetilde{\boldsymbol{LnV}} \ + \ h(\widetilde{\boldsymbol{LnV}}).\widetilde{\boldsymbol{LnV}} \cdot \widetilde{\boldsymbol{LnV}} \ , \tag{3.69}$$

where f, g and h are scalar-value function of the strain tensor, and these functions are invariant, that is to say that they are function scalar invariants of the strain $\widetilde{\boldsymbol{LnV}}$ (section 1.10[p12]).

Actually, it is possible to prove that this law includes all elastic isotropic constitutive laws. For that, you may read books or articles about the representation theory [Zheng-1994].

Chapitre 4

Thermodynamique des solides déformables

Considérons un système fermé, c'est-à-dire qui n'échange pas de matière avec son environnement.

Ce système sera appelé *corps* $\mathcal{B}$. Il n'est pas isolé, c'est-à-dire qu'il peut échanger de la chaleur et du travail mécanique avec son environnement, appelé parfois *extérieur* du corps.

Nous supposerons que le corps $\mathcal{B}$ possède une surface extérieure $\mathcal{S}_{\mathcal{B}}$ qui enveloppe complètement le corps.

Précisons que ce corps n'est pas un ensemble de positions dans l'espace tri-dimensionnel, il est un ensemble de points ou particules matérielles. Nous supposons une connexion entre ces points matériels, indépendamment de leur position dans l'espace 3D, tout comme il est possible de définir une connexion entre éléments d'un maillage éléments finis, sans connaître la position des nœuds.

Comme conséquence de cette connexion, une surface extérieure à ce corps peut être définie. Elle sera l'ensemble des points matériels à la surface du corps.

Les échanges de travail du corps avec l'extérieur pourront se faire à travers la surface extérieure par le moyen de forces surfaciques ou de déplacement des points de cette surface. Mais ils pourront aussi se faire par la matière à l'intérieur du corps via des forces à distance, comme la gravité, des forces électromagnétiques ou des forces d'inertie.

Sur le même schéma, la chaleur échangée par le corps peut passer à travers sa surface extérieure, par conduction thermique, mais également de la chaleur pourra être apporter à la matière à l'intérieur du corps, par micro-ondes, courant électrique et effet Joule etc.

Nous utiliserons également des parties de ce corps $\mathcal{B}$. Nous les appellerons $\mathcal{D}$. La surface extérieure d'une partie $\mathcal{D}$ du corps sera notée $\mathcal{S}$. Selon le choix de la partie, sa surface extérieure $\mathcal{S}$ pourra être complètement incluse dans le corps $\mathcal{B}$.

L'état thermodynamique pourra être complètement défini par un jeu de variables, appelées variables d'état. Le choix de ces variables d'état est crucial dans un modèle thermodynamique, puisqu'elles ont des conséquences fortes sur la construction du modèle. Elles doivent définir tout ce qui est variable lorsque les phénomènes, que l'on souhaite modéliser, se déroulent. Et cela sans redondance. Par exemple, la température des particules matérielles est une variable d'état incontournable. Nous pouvons aussi avoir une déformation comme variable d'état, ou bien une déformation élastique du modèle mécanique. Les phénomènes mécaniques plastiques nécessiteront le choix de variables comme par exemple une déformation plastique cumulée ou un tenseur de déformation plastique.

4.1 Positions ou configurations

Nous adoptons un référentiel de base a afin de décrire les positions des points matériels.

Comme précédemment, nous appelons configuration Ω_0 les positions de référence des points du corps $\mathcal{B}$ et configuration Ω_t ces positions actuelles.

Les positions de référence et actuelle de la surface extérieure du corps seront notées respectivement $\mathcal{S}_{0\mathcal{B}}$ et $\mathcal{S}_{t\mathcal{B}}$.

Les positions de référence et actuelle d'une partie $\mathcal{D}$ du corps seront notées respectivement $\mathcal{D}_0$ et $\mathcal{D}_t$.

Les positions de référence et actuelle de la surface extérieure de cette partie $\mathcal{D}$ seront notées respectivement $\mathcal{S}_0$ et $\mathcal{S}_t$.

4.2 Énergie cinétique

La vitesse des points matériels depuis le référentiel a, et dans la configuration actuelle Ω_t, est :

$$\vec{v}^{\backslash a} = \overset{\bullet}{\vec{x}}{}^{\backslash a} = \vec{v} \tag{4.1}$$

L'accélération des points matériels depuis le référentiel a, et dans la configuration actuelle Ω_t, est :

$$\vec{a}^{\backslash a} = \overset{\bullet}{\vec{v}}^{\backslash a} = \vec{a} \tag{4.2}$$

The velocity and acceleration vectors are neither invariant nor objective.

L'énergie cinétique de la partie $\mathcal{D}$ depuis le référentiel a, et dans la configuration actuelle Ω_t, est :

$$K[\mathcal{D}] = \iiint_{\mathcal{D}} \frac{1}{2} \cdot v^2(M) \ . \ dm \tag{4.3}$$

où M est une particule matérielle de la partie $\mathcal{D}$, dm sa masse et

$$v^2 = \vec{v} \centerdot \vec{v} \ . \tag{4.4}$$

The kinetic energy is not invariant.

Cette intégrale peut être calculée comme une intégrale sur le volume occupé par la partie $\mathcal{D}$ dans la configuration actuelle Ω_t :

$$K[\mathcal{D}] = \iiint_{\mathcal{D}_t} \frac{1}{2} \cdot v^2(\vec{x}) \ . \ \rho_t(\vec{x}) \, .d\mathrm{v} \tag{4.5}$$

Opérons le changement de variable suivant :

$$\vec{x} \longrightarrow \vec{X} = \chi_{0t}^{-1}(\vec{x}) \tag{4.6}$$

$$\mathcal{D}_t \longrightarrow \mathcal{D}_0 \tag{4.7}$$

$$d\mathrm{v} = J.d\mathrm{V} \tag{4.8}$$

Alors

$$K[\mathcal{D}] = \iiint_{\mathcal{D}_0} \frac{1}{2} \cdot v^2\left(\chi_{0t}(\vec{X})\right) \ . \ \rho_t\left(\chi_{0t}(\vec{X})\right).J.d\mathrm{V} \tag{4.9}$$

$$= \iiint_{\mathcal{D}_0} \frac{1}{2} \cdot v^2\left(\chi_{0t}(\vec{X})\right) \ . \ \rho_0\left(\vec{X}\right).d\mathrm{V} \tag{4.10}$$

écrit plus succinctement

$$K[\mathcal{D}] = \iiint_{\mathcal{D}_0} \frac{1}{2} \cdot v^2\left(\vec{X}\right) \ . \ \rho_0\left(\vec{X}\right).d\mathrm{V} \tag{4.11}$$

La partie $\mathcal{D}$ ne changeant pas au cours du temps, il est possible de montrer que la dérivée par rapport au temps de cette énergie cinétique vaut :

$$\overset{\bullet}{K}[\mathcal{D}] = \iiint_{\mathcal{D}} \vec{a}(M) \centerdot \vec{v}(M) \ . \ dm \tag{4.12}$$

qui peut être intégrée en volume sur Ω_0 ou Ω_t.

4.3 Puissance des efforts extérieurs

La puissance des efforts extérieurs agissant sur une partie $\mathcal{D}$ est définie naturellement comme suit :

$$P_{\text{ext}}[\mathcal{D}] \;=\; P\phi[\mathcal{D}] \;+\; P\delta[\mathcal{D}] \tag{4.13}$$

avec, par intégration en surface et volume dans la configuration actuelle Ω_t :

$$P\phi[\mathcal{D}] \;=\; \oint_{\mathcal{S}_t} \left(\vec{t}\,(\vec{x})\,.ds\right) \centerdot \vec{v}\,(\vec{x}) \tag{4.14}$$

$$\begin{aligned} P\delta[\mathcal{D}] \;&=\; \iiint_{\mathcal{D}_t} \left(\vec{f}\,(\vec{x})\,.\rho_t\,(\vec{x})\,.d\mathrm{v}\right) \centerdot \vec{v}\,(\vec{x}) && (4.15)\\ &=\; \iiint_{\mathcal{D}} \left(\vec{f}\,(M)\,.dm\right) \centerdot \vec{v}\,(M) && (4.16) \end{aligned}$$

où $\vec{t}\,(\vec{x})$ est une force surfacique (par unité de surface) extérieure à la partie $\mathcal{D}$ agissant sur un élement de surface ds de la surface extérieure de $\mathcal{D}$.

The power of external forces is not invariant.

4.4 Chaleur et échanges

Pour un matériau volumique, nous décrivons la conduction thermique au sein du volume de matière par un vecteur appelé *vecteur courant de chaleur* ou *vecteur densité de flux de chaleur* et noté $\vec{q}$. Sa définition peut être la suivante, pour toute facette de matière $\vec{ds}$, dans la configuration actuelle Ω_t, le débit de chaleur qui passe à travers cette facette, dans le sens de l'intérieur vers l'extérieur, est

$$\frac{\delta Q}{dt} = \vec{q} \centerdot \vec{ds} \tag{4.17}$$

Vector heat flux $\vec{q}$ is objective.

De la chaleur peut être échangée entre une particule de matière dm et son environnement, autrement que par conduction thermique et proportionnellement à sa masse. Nous définissons une densité massique d'apport de chaleur noté r par la relation suivante

$$\frac{\delta Q}{dt} = r.dm \tag{4.18}$$

The specific external heat rate r is invariant.

Ainsi, le débit de chaleur reçue par une partie $\mathcal{D}$ du corps de la part de son environnement est

$$\overset{\bullet}{Q}_{\text{ext}}[\mathcal{D}] = \iiint_{\mathcal{D}} r\left(M\right)\ . \ dm - \oint_{\mathcal{S}_t} \vec{q}\left(\vec{x}\right) \centerdot \vec{ds} \tag{4.19}$$

D'après les formules de Gauss, il est possible de montrer que :

$$\begin{aligned}
\overset{\bullet}{Q}_{\text{ext}}[\mathcal{D}] &= \iiint_{\mathcal{D}_t} \left(r\left(\vec{x}\right) - \frac{1}{\rho_t\left(\vec{x}\right)} \text{div}\vec{q}\left(\vec{x}\right)\right) .\ \rho_t\left(\vec{x}\right).\text{dv} &(4.20)\\
&= \iiint_{\mathcal{D}} \left(r\left(M\right) - \frac{1}{\rho_t\left(M\right)} \text{div}\vec{q}\left(M\right)\right) dm &(4.21)\\
&= \iiint_{\mathcal{D}} \left(r - \frac{1}{\rho_t} \text{div}\vec{q}\right) dm &(4.22)
\end{aligned}$$

où $\text{div}\vec{q}(M)$ est une dérivée du champ de vecteur $\vec{q}$ par rapport à la position $\vec{x}$ dans la configuration actuelle Ω_t.

De la même manière que pour l'énergie cinétique (équation 4.11)

$$\begin{aligned}
\iiint_{\mathcal{D}} r\left(M\right).dm &= \iiint_{\mathcal{D}_t} r\left(\vec{x}\right)\ .\ \rho_t\left(\vec{x}\right).\text{dv} &(4.23)\\
&= \iiint_{\mathcal{D}_0} r\left(\vec{X}\right)\ .\ \rho_0\left(\vec{X}\right).\text{dV} &(4.24)
\end{aligned}$$

D'autre part, pour l'intégrale

$$\oint_{\mathcal{S}_t} \vec{q}\left(\vec{x}\right) \centerdot \vec{ds} \tag{4.25}$$

opérons le changement de variable suivant :

$$\begin{aligned}
\vec{x} &\longrightarrow \vec{X} = \chi_{0t}^{-\mathbf{1}}(\vec{x}) &(4.26)\\
\mathcal{S}_t &\longrightarrow \mathcal{S}_0 &(4.27)
\end{aligned}$$

et rappelons la relation 3.8

$$\vec{ds} = J.\left(\widetilde{\boldsymbol{F}}^{-\mathbf{T}} \centerdot \vec{dS}\right) \tag{4.28}$$

Alors

$$\oint_{\mathcal{S}_t} \vec{q}\,(\vec{x}) \centerdot \vec{ds} = \oint_{\mathcal{S}_0} J.\, \vec{q}\left(\chi_{0t}(\vec{X})\right) \centerdot \left(\widetilde{\boldsymbol{F}}^{-\mathbf{T}} \centerdot \vec{dS}\right) \tag{4.29}$$

$$= \oint_{\mathcal{S}_0} J.\, \vec{dS} \centerdot \left(\widetilde{\boldsymbol{F}}^{-\mathbf{1}} \centerdot \vec{q}\left(\chi_{0t}(\vec{X})\right)\right) \tag{4.30}$$

$$= \oint_{\mathcal{S}_0} \left(J.\widetilde{\boldsymbol{F}}^{-\mathbf{1}} \centerdot \vec{q}\left(\chi_{0t}(\vec{X})\right)\right) \centerdot \vec{dS} \tag{4.31}$$

$$= \oint_{\mathcal{S}_0} \vec{Q}\left(\vec{X}\right) \centerdot \vec{dS} \tag{4.32}$$

où

$$\vec{Q}\left(\vec{X}\right) = J.\, \widetilde{\boldsymbol{F}}^{-\mathbf{1}} \centerdot \vec{q}\left(\chi_{0t}(\vec{X})\right) \tag{4.33}$$

est le *vecteur courant de chaleur lagrangien* car exprimé dans le contexte de la configuration de référence Ω_0.

Ainsi le débit de chaleur reçue par la partie $\mathcal{D}$ s'exprime dans la configuration de référence par

$$\overset{\bullet}{Q}_{\text{ext}}[\mathcal{D}] = \iiint_{\mathcal{D}_0} r\left(\vec{X}\right) \;.\; \rho_0\left(\vec{X}\right).d\mathrm{V} - \oint_{\mathcal{S}_0} \vec{Q}\left(\vec{X}\right) \centerdot \vec{dS} \tag{4.34}$$

qui, d'après les formules de Gauss, devient

$$\overset{\bullet}{Q}_{\text{ext}}[\mathcal{D}] = \iiint_{\mathcal{D}_0} \left(r\left(\vec{X}\right) - \frac{1}{\rho_0\left(\vec{X}\right)}\mathrm{DIV}\vec{Q}\left(\vec{X}\right)\right) \;.\; \rho_0\left(\vec{X}\right).d\mathrm{V} \tag{4.35}$$

où $\mathrm{DIV}\vec{Q}\,(M)$ est une dérivée du champ de vecteur $\vec{Q}$ par rapport à la position $\vec{X}$ dans la configuration de référence Ω_0.

Lagrangian vector heat flux $\vec{Q}$ is invariant.

The rate of external heat $\overset{\bullet}{Q}_{\text{ext}}[\mathcal{D}]$ is invariant.

4.5 État d'équilibre et variables d'état en thermostatique

Pour la description de la thermostatique des systèmes à un nombre fini de paramètres, Paul Germain [Germain-1973] décrit l'état d'un système en équilibre ainsi :

La thermostatique vise à décrire des systèmes en équilibre aussi indépendamment que possible de leurs propriétés physiques. Nous dirons qu'un système est en équilibre s'il est constitué des mêmes corps formant une configuration indépendante du temps, chaque partie ayant une structure physico-chimique invariable dans le temps. Il s'agit évidemment d'une définition expérimentale, à apprécier dans un cadre de schématisation donné, et qui nous est précisé du point de vue mathématique par la définition suivante.

Définition 1. L'état $\mathcal{E}$ d'un système en équilibre est l'ensemble des grandeurs caractéristiques de l'équilibre, – géométriques, mécaniques, physico-chimiques –, s'exprimant à l'aide de nombres réels et qui demeurent invariables. Le système est dit fini si $n+1$ de ces caractéristiques, $\chi_0, \chi_1, \ldots, \chi_n$, constituent un système de variables indépendantes telles que toute autre grandeur caractéristique est une fonction bien définie de $\chi_0, \chi_1, \ldots, \chi_n$, dite fonction d'état. L'ensemble des χ_p constitue un système complet de variables d'état de l'état $\mathcal{E}$. L'ensemble des états possibles d'un système forme une variété différentielle connexe $\mathcal{V}$.

4.6 Premier principe dans le cas de la thermostatique

Nous citons Paul Germain [Germain-1973] : **Définition 2.** Un système est dit fermé si toutes les parois (extérieures et intérieures) sont imperméables. Un système est dit simple s'il n'existe aucune paroi intérieure, toutes les parties du système étant homogènes dans tout état d'équilibre.

Ici, une paroi imperméable interdit les échanges de masses.

Le premier principe est alors énoncé ainsi dans le cas d'un système fermé simple [Germain-1973] : **Premier principe**. Il existe une fonction d'état $E(\mathcal{E})$, appelée énergie interne du système, définie à une constante additive près, telle que, pour tout $\mathcal{F}(\mathcal{E}_1, \mathcal{E}_2)$:

$$\mathcal{T}_\mathcal{F} + \mathcal{Q}_\mathcal{F} = E(\mathcal{E}_2) - E(\mathcal{E}_1) \tag{4.36}$$

Ici, $\mathcal{F}(\mathcal{E}_1, \mathcal{E}_2)$ est une transformation de l'état d'équilibre 1 vers l'état d'équilibre 2, $\mathcal{T}_\mathcal{F}$ et $\mathcal{Q}_\mathcal{F}$ sont respectivement le travail et la chaleur reçus par le système au cours de la transformation.

4.7 Axiome de l'état local

Pour la thermodynamique des milieux continus, en dehors de la thermostatique, Paul Germain écrit [Germain-1973] :

La méthode de l'état local. Le potentiel thermodynamique

Hypothèse de départ. Considérons un milieu dont on connaisse complètement le comportement thermostatique. Ceci implique que tout état d'équilibre de ce milieu est parfaitement défini par la donnée de $n+1$ paramètres $\chi_0, \chi_1, \ldots, \chi_n$, et que les grandeurs thermostatiques de cet état, par exemple son énergie interne spécifique, son entropie spécifique, sa température absolue... sont des fonctions connues de $\chi_0, \chi_1, \ldots, \chi_n$, soit

$$e(\chi_0, \chi_1, \ldots, \chi_n), s(\chi_0, \chi_1, \ldots, \chi_n), T(\chi_0, \chi_1, \ldots, \chi_n) \ldots . \tag{4.37}$$

Supposons maintenant ce milieu en mouvement. L'axiome de l'état local peut s'énoncer comme suit :

Axiome de l'état local A tout instant t fixé, on peut attacher à toute particule $n+1$ paramètres $\chi_0, \chi_1, \ldots, \chi_n$, tels que l'énergie interne spécifique e, l'entropie spécifique s, la température absolue T... de cette particule à l'instant t soient données en fonction de $\chi_0, \chi_1, \ldots, \chi_n$ par les mêmes expressions (4.37) valables en thermostatique.

4.8 Énergie interne et premier principe

Nous supposons qu'il existe, pour chaque particule de matière M de masse dm, une fonction d'état, appelée *énergie interne spécifique* ou massique et notée $e\,(M)$,

$$e\,(M) = e(\theta, \widetilde{\varepsilon}, V_i) \tag{4.38}$$

où les variables d'état sont notées $(\theta, \widetilde{\varepsilon}, V_i)$,

telle que, pour toute partie $\mathcal{D}$ du corps et pour toute transformation,

$$\overset{\bullet}{E}[\mathcal{D}] + \overset{\bullet}{K}[\mathcal{D}] = P_{\text{ext}}[\mathcal{D}] + \overset{\bullet}{Q}_{\text{ext}}[\mathcal{D}] \tag{4.39}$$

où

$$E[\mathcal{D}] \;=\; \iiint_{\mathcal{D}} e\,(M)\;.\,dm \tag{4.40}$$

est l'énergie interne de la partie $\mathcal{D}$.

Tout comme pour l'énergie cinétique et le débit de chaleur reçu par une partie $\mathcal{D}$, l'énergie interne peut être exprimée dans la configuration de référence Ω_0 ou bien dans la configuration actuelle Ω_t :

$$E[\mathcal{D}] = \iiint_{\mathcal{D}} e(M)\ .\ dm \tag{4.41}$$

$$= \iiint_{\mathcal{D}_t} e(\vec{x})\ .\ \rho_t(\vec{x})\,.d\mathrm{v} \tag{4.42}$$

$$= \iiint_{\mathcal{D}_0} e\left(\vec{X}\right)\ .\ \rho_0\left(\vec{X}\right).d\mathrm{V} \tag{4.43}$$

En appliquant la règle de Leibnitz (dérivation sous le signe d'intégration) à l'équation 4.43, nous avons :

$$\overset{\bullet}{E}[\mathcal{D}] = \iiint_{\mathcal{D}_0} \overset{\bullet}{e}\left(\vec{X}\right)\ .\ \rho_0\left(\vec{X}\right).d\mathrm{V} \tag{4.44}$$

qui, par changement de variable, devient

$$\overset{\bullet}{E}[\mathcal{D}] = \iiint_{\mathcal{D}_0} \overset{\bullet}{e}\left(\vec{X}\right)\ .\ \rho_0\left(\vec{X}\right).d\mathrm{V} \tag{4.45}$$

$$= \iiint_{\mathcal{D}_t} \overset{\bullet}{e}(\vec{x})\ .\ \rho_t(\vec{x})\,.d\mathrm{v} \tag{4.46}$$

$$= \iiint_{\mathcal{D}} \overset{\bullet}{e}(M)\ .\ dm \tag{4.47}$$

Rappelons que $\overset{\bullet}{a}$ est la dérivée particulaire de la grandeur a.

4.9 Équation locale du mouvement d'un milieu continu

Nous admettrons cette équation du mouvement.

Pour cela, il est nécessaire que le réferentiel d'observation soit galiléen. Ainsi, nous supposerons que le référentiel a est galiléen.

Cette équation s'écrit alors dans la configuration actuelle Ω_t de la façon suivante :

$$\overrightarrow{\mathrm{div}\vec{\vec{\boldsymbol{T}}}} + \rho_t.\vec{f} = \rho_t.\vec{a} \tag{4.48}$$

où $\vec{f}$ est une force qui agit à distance sur la particule matérielle. Attention, ici la convention pour cette force est qu'elle est définie par unité de masse de la particule et non par unité de volume.

4.10 Puissance des efforts intérieurs

Posons une puissance des forces de contact comme suit :

$$P\sigma[\mathcal{D}] \;=\; \iiint_{\mathcal{D}_t} \left(\overrightarrow{\mathrm{div}\widetilde{\boldsymbol{T}}}\left(\vec{x}\right).\mathrm{dv}\right) \boldsymbol{.}\, \vec{v}\left(\vec{x}\right) \tag{4.49}$$

D'après les formules de Gauss, il est possible de montrer que :

$$\forall\mathcal{D}\subset\mathcal{B}\quad P\phi[\mathcal{D}] \;=\; P\sigma[\mathcal{D}] \;+\; \iiint_{\mathcal{D}_t} \mathrm{Tr}\left(\widetilde{\boldsymbol{T}}\left(\vec{x}\right)\boldsymbol{.}\,\widetilde{\boldsymbol{D}}\left(\vec{x}\right)\right)\mathrm{dv} \tag{4.50}$$

D'après l'équation locale du mouvement, il est possible de montrer que :

$$\forall\mathcal{D}\subset\mathcal{B}\quad \overset{\bullet}{K}[\mathcal{D}] \;=\; P\sigma[\mathcal{D}] \;+\; P\delta[\mathcal{D}] \tag{4.51}$$

D'après le théorème de l'énergie cinétique, dans un référentiel galiléen a :

$$\forall\mathcal{D}\subset\mathcal{B}\quad \overset{\bullet}{K}[\mathcal{D}] \;=\; P_{\mathrm{ext}}[\mathcal{D}] \;+\; P_{\mathrm{int}}[\mathcal{D}] \tag{4.52}$$

Par conséquent, la puissance des efforts intérieurs est égale à

$$P_{\mathrm{int}}[\mathcal{D}] \;=\; P\sigma[\mathcal{D}] \;-\; P\phi[\mathcal{D}] \;=\; -\iiint_{\mathcal{D}_t} \mathrm{Tr}\left(\widetilde{\boldsymbol{T}}\left(\vec{x}\right)\boldsymbol{.}\,\widetilde{\boldsymbol{D}}\left(\vec{x}\right)\right)\mathrm{dv} \tag{4.53}$$

qui est indépendante du choix du référentiel a.

So, the power of internal forces $P_{\mathrm{int}}[\mathcal{D}]$ is invariant.

Donc, la puissance des efforts intérieurs est égale à

$$\forall\mathcal{D}\subset\mathcal{B} \tag{4.54}$$

$$P_{\mathrm{int}}[\mathcal{D}] \;=\; -\iiint_{\mathcal{D}_t} \mathrm{Tr}\left(\widetilde{\boldsymbol{T}}\left(\vec{x}\right)\boldsymbol{.}\,\widetilde{\boldsymbol{D}}\left(\vec{x}\right)\right)\mathrm{dv} \tag{4.55}$$

$$=\; -\iiint_{\mathcal{D}_t} \frac{1}{\rho_t\left(\vec{x}\right)}\mathrm{Tr}\left(\widetilde{\boldsymbol{T}}\left(\vec{x}\right)\boldsymbol{.}\,\widetilde{\boldsymbol{D}}\left(\vec{x}\right)\right)\rho_t\left(\vec{x}\right)\mathrm{dv} \tag{4.56}$$

qui par changement de variable s'exprime aussi selon

$$\forall\mathcal{D}\subset\mathcal{B} \tag{4.57}$$

$$P_{\mathrm{int}}[\mathcal{D}] \;=\; -\iiint_{\mathcal{D}_0} \mathrm{Tr}\left(\widetilde{\boldsymbol{T}}\left(\vec{X}\right)\boldsymbol{.}\,\widetilde{\boldsymbol{D}}\left(\vec{X}\right)\right) J\left(\vec{X}\right).d\mathrm{V} \tag{4.58}$$

$$=\; -\iiint_{\mathcal{D}_0} \frac{1}{\rho_t\left(\vec{X}\right)}\mathrm{Tr}\left(\widetilde{\boldsymbol{T}}\left(\vec{X}\right)\boldsymbol{.}\,\widetilde{\boldsymbol{D}}\left(\vec{X}\right)\right)\rho_0\left(\vec{X}\right).d\mathrm{V} \tag{4.59}$$

$$=\; -\iiint_{\mathcal{D}_0} \frac{1}{\rho_0\left(\vec{X}\right)}\mathrm{Tr}\left(\widetilde{\boldsymbol{S}}\left(\vec{X}\right)\boldsymbol{.}\,\overset{\bullet}{\widetilde{\boldsymbol{E}}}\left(\vec{X}\right)\right)\rho_0\left(\vec{X}\right).d\mathrm{V} \tag{4.60}$$

Introduisons le terme de puissance des efforts intérieurs par unité de masse p_{int}, tel que

$$\begin{aligned}
\forall \mathcal{D} \subset \mathcal{B} \quad P_{\text{int}}[\mathcal{D}] &= \iiint_{\mathcal{D}} p_{\text{int}}(M)\ .\ dm && (4.61)\\
&= \iiint_{\mathcal{D}_t} p_{\text{int}}(\vec{x})\ .\ \rho_t(\vec{x})\,.d\text{v} && (4.62)\\
&= \iiint_{\mathcal{D}_0} p_{\text{int}}\left(\vec{X}\right)\ .\ \rho_0\left(\vec{X}\right).d\text{V} && (4.63)
\end{aligned}$$

Alors cette puissance des efforts intérieurs par unité de masse est égale à :

$$\begin{aligned}
\forall M \in \mathcal{B} \quad p_{\text{int}}(M) &= \frac{-1}{\rho_t(M)}.\text{Tr}\left(\widetilde{\boldsymbol{T}}(M)\,\boldsymbol{.}\,\widetilde{\boldsymbol{D}}(M)\right) && (4.64)\\
&= -\ \text{Tr}\left(\widetilde{\boldsymbol{\Sigma}}(M)\,\boldsymbol{.}\,\widetilde{\boldsymbol{D}}(M)\right) && (4.65)\\
&= \frac{-1}{\rho_0(M)}.\text{Tr}\left(\widetilde{\boldsymbol{S}}(M)\,\boldsymbol{.}\,\overset{\bullet}{\widetilde{\boldsymbol{E}}}(M)\right) && (4.66)
\end{aligned}$$

The specific power of internal forces p_{int} is invariant.

D'après le théorème de l'énergie cinétique, l'équation 4.39 du premier principe devient :

$$\overset{\bullet}{E}[\mathcal{D}] = -P_{\text{int}}[\mathcal{D}] + \overset{\bullet}{Q}_{\text{ext}}[\mathcal{D}] \tag{4.67}$$

pour toute partie $\mathcal{D}$ du corps $\mathcal{B}$.

So, the rate of internal energy $\overset{\bullet}{E}[\mathcal{D}]$ is invariant. And, also the internal energy $E[\mathcal{D}]$ is invariant. And, the same is true for the specific internal energy $e(\vec{x})$.

Cette équation s'exprime aussi par

$$\iiint_{\mathcal{D}_t} \overset{\bullet}{e}(\vec{x})\ .\ \rho_t d\text{v} = -\iiint_{\mathcal{D}_t} p_{\text{int}}(\vec{x})\ .\ \rho_t d\text{v} + \iiint_{\mathcal{D}_t}\left(r(\vec{x}) - \frac{1}{\rho_t}\text{div}\vec{q}(\vec{x})\right)\rho_t d\text{v} \tag{4.68}$$

qui équivaut à

$$\iiint_{\mathcal{D}_t}\left(\overset{\bullet}{e}(\vec{x}) + p_{\text{int}}(\vec{x}) - r(\vec{x}) + \frac{1}{\rho_t}\text{div}\vec{q}(\vec{x})\right)\rho_t d\text{v} = 0 \tag{4.69}$$

pour toute partie $\mathcal{D}$ du corps $\mathcal{B}$.

D'après Paul Germain [Germain-1973] : LEMME FONDAMENTAL Soient $\phi(M)$ une fonction définie et continue dans le domaine $\mathcal{B}$ et $\mathcal{F}$ une famille dense dans $\mathcal{B}$. Si pour tout $\mathcal{D}$ de $\mathcal{F}$ l'intégrale de ϕ dans $\mathcal{D}$ est nulle, alors la fonction ϕ est identiquement nulle dans $\mathcal{B}$.

Ainsi, l'équation 4.69 nous conduit à

$$\overset{\bullet}{e}(\vec{x}) + p_{\text{int}}(\vec{x}) - r(\vec{x}) + \frac{1}{\rho_t}\text{div}\vec{q}(\vec{x}) = 0 \tag{4.70}$$

$$\iff \overset{\bullet}{e}(\vec{x}) = -p_{\text{int}}(\vec{x}) + r(\vec{x}) - \frac{1}{\rho_t}\text{div}\vec{q}(\vec{x}) \tag{4.71}$$

$$\iff \overset{\bullet}{e}(\vec{x}) = \frac{1}{\rho_t}.\text{Tr}\left(\widetilde{\boldsymbol{T}}(\vec{x}) \boldsymbol{.} \widetilde{\boldsymbol{D}}(\vec{x})\right) + r(\vec{x}) - \frac{1}{\rho_t}\text{div}\vec{q}(\vec{x}) \tag{4.72}$$

pour tout point matériel M du corps $\mathcal{B}$. Ecrit succinctement, en tout point M du corps, selon

$$\overset{\bullet}{e} = -p_{\text{int}} + r - \frac{1}{\rho_t}\text{div}\vec{q} \tag{4.73}$$

$$= \frac{1}{\rho_t}.\text{Tr}\left(\widetilde{\boldsymbol{T}} \boldsymbol{.} \widetilde{\boldsymbol{D}}\right) + r - \frac{1}{\rho_t}\text{div}\vec{q} \tag{4.74}$$

$$= \text{Tr}\left(\widetilde{\boldsymbol{\Sigma}} \boldsymbol{.} \widetilde{\boldsymbol{D}}\right) + r - \frac{1}{\rho_t}\text{div}\vec{q} \tag{4.75}$$

Si nous reprenons le même raisonnement partant de l'équation 4.67, mais cette fois çi par une intégrale dans la configuration de référence Ω_0, nous obtenons que :

$$\overset{\bullet}{e} = -p_{\text{int}} + r - \frac{1}{\rho_0}\text{DIV}\vec{Q} \tag{4.76}$$

$$= \frac{1}{\rho_0}.\text{Tr}\left(\widetilde{\boldsymbol{S}} \boldsymbol{.} \overset{\bullet}{\widetilde{\boldsymbol{E}}}\right) + r - \frac{1}{\rho_0}\text{DIV}\vec{Q} \tag{4.77}$$

4.11 Entropy and second principle

We propose the second principle as follows. We assume that a quantity S, so-called *entropy*, exists for each material particle M of mass dm, and is the product of a function s of the particle state and mass dm :

$$S[M, dm] = s(M) \ . \ dm = s(\theta, \widetilde{\boldsymbol{\varepsilon}}, V_i) \ . \ dm \tag{4.78}$$

such that, for all transformations, its evolution from time t to $t + dt$ satisfies :

$$dS[M, dm] = \frac{\delta Q_{\text{ext}}[M, dm]}{\theta} + \delta S_{\text{irrev}}[M, dm] \tag{4.79}$$

where $\delta Q_{\text{ext}}[M, dm]$ is the heat received by the particle from its surroundings, and if irreversible phenomena occur inside the particle, from t to $t + dt$,

$$\delta S_{\text{irrev}}[M, dm] > 0 \tag{4.80}$$

else

$$\delta S_{\text{irrev}}[M, dm] = 0 \ . \tag{4.81}$$

The term $s\,(M)$ is the *specific entropy* of material point M.

Assumption – *We assume that the specific entropy is invariant.*

We define the *specific intrinsic dissipation* φ_1 of a material point as

$$\varphi_1\,(M) = \frac{\theta.\delta S_{\text{irrev}}[M, dm]}{dt\ .\ dm} \ . \tag{4.82}$$

The specific intrinsic dissipation φ_1 is invariant.

According to equation 4.22,

$$\delta Q_{\text{ext}}[M, dm] = \left(r - \frac{1}{\rho_t}\text{div}\vec{q}\right) dm\ .\ dt \tag{4.83}$$

So,

$$\overset{\bullet}{S}[M, dm] = \frac{\left(r - \frac{1}{\rho_t}\text{div}\vec{q}\right) dm}{\theta} + \frac{\varphi_1}{\theta} dm \tag{4.84}$$

and

$$\overset{\bullet}{s} = \frac{\left(r - \frac{1}{\rho_t}\text{div}\vec{q}\right)}{\theta} + \frac{\varphi_1}{\theta} \tag{4.85}$$

According to relation 4.74, we obtain also :

$$\begin{aligned}
\overset{\bullet}{e} &= \frac{1}{\rho_t}.\text{Tr}\left(\widetilde{\boldsymbol{T}} \boldsymbol{.} \widetilde{\boldsymbol{D}}\right) + \theta.\overset{\bullet}{s} - \varphi_1 && (4.86)\\
&= \frac{1}{\rho_0}.\text{Tr}\left(\widetilde{\boldsymbol{S}} \boldsymbol{.} \overset{\bullet}{\widetilde{\boldsymbol{E}}}\right) + \theta.\overset{\bullet}{s} - \varphi_1 && (4.87)
\end{aligned}$$

As the kinetic energy, the received heat rate and the internal energy, the entropy of part $\mathcal{D}$ is

$$\begin{aligned}
S[\mathcal{D}] &= \iiint_{\mathcal{D}} s\,(M)\ .\ dm && (4.88)\\
&= \iiint_{\mathcal{D}_t} s\,(\vec{x})\ .\ \rho_t\,(\vec{x})\,.\text{dv} && (4.89)\\
&= \iiint_{\mathcal{D}_0} s\left(\vec{X}\right)\ .\ \rho_0\left(\vec{X}\right).\text{dV} && (4.90)
\end{aligned}$$

and

$$\overset{\bullet}{S}[\mathcal{D}] = \iiint_{\mathcal{D}_0} \overset{\bullet}{s}\left(\vec{X}\right) . \rho_0\left(\vec{X}\right) .\mathrm{dV} \tag{4.91}$$
$$= \iiint_{\mathcal{D}_t} \overset{\bullet}{s}\left(\vec{x}\right) . \rho_t\left(\vec{x}\right) .\mathrm{dv} \tag{4.92}$$
$$= \iiint_{\mathcal{D}} \overset{\bullet}{s}\left(M\right) . dm \tag{4.93}$$

According to relation 4.85 and this one :

$$\iiint_{\mathcal{D}_t} \frac{\mathrm{div}\vec{q}}{\theta}.\mathrm{dv} = \iiint_{\mathcal{D}_t} \frac{\overrightarrow{\mathrm{grad}\theta} \centerdot \vec{q}}{\theta^2}.\mathrm{dv} + \oint_{\mathcal{S}_t} \frac{\vec{q} \centerdot \vec{ds}}{\theta} \tag{4.94}$$

we obtain

$$\overset{\bullet}{S}[\mathcal{D}] = \underbrace{\iiint_{\mathcal{D}} \frac{r}{\theta} dm - \oint_{\mathcal{S}_t} \frac{\vec{q} \centerdot \vec{ds}}{\theta}}_{\overset{\bullet}{S}_{\mathrm{ext}}[\mathcal{D}]} + \underbrace{\iiint_{\mathcal{D}} \left[\overbrace{\frac{-\overrightarrow{\mathrm{grad}\theta} \centerdot \vec{q}}{\rho_t.\theta}}^{\varphi_2} + \varphi_1 \right] . \frac{dm}{\theta}}_{\overset{\bullet}{S}_{\mathrm{int}}[\mathcal{D}]} \tag{4.95}$$

We define the *specific thermal dissipation* φ_2 of a material point as

$$\varphi_2\left(M\right) = \frac{-\overrightarrow{\mathrm{grad}\theta} \centerdot \vec{q}}{\rho_t.\theta} . \tag{4.96}$$

The specific thermal dissipation φ_2 is invariant.

Assumption – *We assume that the specific thermal dissipation is never negative :*

$$\varphi_2\left(M\right) \geqslant 0 . \tag{4.97}$$

4.12 Helmholtz free energy

We define the free energy F of material particle as

$$F[M, dm] = E[M, dm] - \theta . S[M, dm] \tag{4.98}$$

and the specific free energy ψ as

$$F[M, dm] = \psi\left(M\right)\ .\ dm = \psi(\theta, \widetilde{\boldsymbol{\varepsilon}}, V_i)\ .\ dm \tag{4.99}$$

So,

$$\psi = e - \theta.s \tag{4.100}$$

and

$$\begin{aligned} F[\mathcal{D}] &= \iiint_{\mathcal{D}} \psi\left(M\right)\ .\ dm && (4.101) \\ &= \iiint_{\mathcal{D}_t} \psi\left(\vec{x}\right)\ .\ \rho_t\left(\vec{x}\right).d\mathrm{v} && (4.102) \\ &= \iiint_{\mathcal{D}_0} \psi\left(\vec{X}\right)\ .\ \rho_0\left(\vec{X}\right).d\mathrm{V} && (4.103) \end{aligned}$$

and

$$\begin{aligned} \overset{\bullet}{F}[\mathcal{D}] &= \iiint_{\mathcal{D}_0} \overset{\bullet}{\psi}\left(\vec{X}\right)\ .\ \rho_0\left(\vec{X}\right).d\mathrm{V} && (4.104) \\ &= \iiint_{\mathcal{D}_t} \overset{\bullet}{\psi}\left(\vec{x}\right)\ .\ \rho_t\left(\vec{x}\right).d\mathrm{v} && (4.105) \\ &= \iiint_{\mathcal{D}} \overset{\bullet}{\psi}\left(M\right)\ .\ dm && (4.106) \end{aligned}$$

Let us calculate the rate of specific free energy :

$$\begin{aligned} \overset{\bullet}{\psi} &= \overset{\bullet}{e} - \overset{\bullet}{\theta}.s - \theta.\overset{\bullet}{s} && (4.107) \\ &= \left[\frac{1}{\rho_t}.\mathrm{Tr}\left(\widetilde{\boldsymbol{T}}\boldsymbol{.}\widetilde{\boldsymbol{D}}\right) + r - \frac{1}{\rho_t}\mathrm{div}\vec{q}\right] \\ &\quad -\overset{\bullet}{\theta}.s - \left[r - \frac{1}{\rho_t}\mathrm{div}\vec{q} + \varphi_1\right] && (4.108) \end{aligned}$$

So,

$$\begin{aligned} \overset{\bullet}{\psi} &= \frac{1}{\rho_t}.\mathrm{Tr}\left(\widetilde{\boldsymbol{T}}\boldsymbol{.}\widetilde{\boldsymbol{D}}\right) - s.\overset{\bullet}{\theta} - \varphi_1 && (4.109) \\ &= \frac{1}{\rho_0}.\mathrm{Tr}\left(\widetilde{\boldsymbol{S}}\boldsymbol{.}\overset{\bullet}{\widetilde{\boldsymbol{E}}}\right) - s.\overset{\bullet}{\theta} - \varphi_1 && (4.110) \end{aligned}$$

The specific free energy is invariant.

Chapitre 5

Isotropic hyperelastic laws

5.1 Definition

A hyperelastic law may be defined as follows :

1. $(\theta, \widetilde{\boldsymbol{E}})$ is a complete set of thermodynamic state variables, where θ is the temperature and $\widetilde{\boldsymbol{E}}$ is the right Green-Lagrange strain tensor,
2. the thermodynamic properties are described by a potential function $W(\theta, \widetilde{\boldsymbol{E}})$, equal to the free energy per unit of reference volume,
3. the specific intrinsic dissipation is always null.

5.2 Free energy rate

By definition

$$\rho_0.\psi = W(\theta, \widetilde{\boldsymbol{E}}) + \rho_0.\psi_0 \tag{5.1}$$

where ψ_0 is an unknown constant. So

$$\rho_0.\overset{\bullet}{\psi} = \left(\frac{\partial W}{\partial \theta}\right)_{/\widetilde{\boldsymbol{E}}}(\theta, \widetilde{\boldsymbol{E}})\ .\overset{\bullet}{\theta} + \widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\boldsymbol{\theta}}(\theta, \widetilde{\boldsymbol{E}})\ : \overset{\bullet}{\widetilde{\boldsymbol{E}}} \tag{5.2}$$

$$= \left(\frac{\partial W}{\partial \theta}\right)_{/\widetilde{\boldsymbol{E}}}(\theta, \widetilde{\boldsymbol{E}})\ .\overset{\bullet}{\theta} + \mathrm{Tr}\left[\widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\boldsymbol{\theta}}(\theta, \widetilde{\boldsymbol{E}})\ \boldsymbol{.}\ \overset{\bullet}{\widetilde{\boldsymbol{E}}}^{\mathbf{T}}\right] \tag{5.3}$$

This equation is true whatever the state and whatever its evolution rate :

$$\forall(\theta, \widetilde{\boldsymbol{E}}) \quad , \quad \forall(\overset{\bullet}{\theta}, \overset{\bullet}{\widetilde{\boldsymbol{E}}}) \tag{5.4}$$

Note 1. *Equation 5.2 defines tensor*

$$\widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)_{/\boldsymbol{\theta}}}(\theta, \widetilde{\boldsymbol{E}}) \tag{5.5}$$

In this definition, this tensor is in scalar product with a symmetric tensor. Then if the skew-symmetric part of this tensor is not null, this part will never have any effect on the result of this scalar product. Indeed, the scalar product between a skew-symmetric tensor and a symmetric tensor is null. Hence, we can assume, without any loss in generality, that this tensor is always symmetric.

Since equation 4.109,

$$\begin{aligned}
\rho_0.\overset{\bullet}{\psi} &= \left(\frac{\partial W}{\partial \theta}\right)_{/\widetilde{\boldsymbol{E}}}(\theta, \widetilde{\boldsymbol{E}})\ .\overset{\bullet}{\theta} + \widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)_{/\boldsymbol{\theta}}}(\theta, \widetilde{\boldsymbol{E}})\ :\overset{\bullet}{\widetilde{\boldsymbol{E}}} && (5.6)\\
&= J.\mathrm{Tr}\left(\widetilde{\boldsymbol{T}}\boldsymbol{.}\widetilde{\boldsymbol{D}}\right) - \rho_0.s.\overset{\bullet}{\theta} - \rho_0.\varphi_1 && (5.7)\\
&= J.\widetilde{\boldsymbol{T}} : \widetilde{\boldsymbol{D}} - \rho_0.s.\overset{\bullet}{\theta} - \rho_0.\varphi_1 && (5.8)
\end{aligned}$$

$$\forall(\theta, \widetilde{\boldsymbol{E}}) \quad , \quad \forall(\overset{\bullet}{\theta}, \overset{\bullet}{\widetilde{\boldsymbol{E}}})$$

Since item 3 of the hyperelasticity definition,

$$\begin{aligned}
\rho_0.\overset{\bullet}{\psi} &= \left(\frac{\partial W}{\partial \theta}\right)_{/\widetilde{\boldsymbol{E}}}(\theta, \widetilde{\boldsymbol{E}})\ .\overset{\bullet}{\theta} + \widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)_{/\boldsymbol{\theta}}}(\theta, \widetilde{\boldsymbol{E}})\ :\overset{\bullet}{\widetilde{\boldsymbol{E}}} && (5.9)\\
&= -\rho_0.s.\overset{\bullet}{\theta} + J.\widetilde{\boldsymbol{T}} : \widetilde{\boldsymbol{D}} && (5.10)
\end{aligned}$$

$$\forall(\theta, \widetilde{\boldsymbol{E}}) \quad , \quad \forall(\overset{\bullet}{\theta}, \overset{\bullet}{\widetilde{\boldsymbol{E}}})$$

5.3 Stress-strain equation

Let us limit this equation to the cases where $\overset{\bullet}{\widetilde{\boldsymbol{E}}} = \widetilde{\boldsymbol{0}}$ (and also $\widetilde{\boldsymbol{D}} = \widetilde{\boldsymbol{0}}$) :

$$\begin{aligned}
\rho_0.\overset{\bullet}{\psi} &= \left(\frac{\partial W}{\partial \theta}\right)_{/\widetilde{\boldsymbol{E}}}(\theta, \widetilde{\boldsymbol{E}})\ .\overset{\bullet}{\theta} && (5.11)\\
&= -\rho_0.s.\overset{\bullet}{\theta} && (5.12)
\end{aligned}$$

$$\forall(\theta, \widetilde{\boldsymbol{E}}) \quad , \quad \forall\overset{\bullet}{\theta}$$

Or

$$\left[\left(\frac{\partial W}{\partial \theta}\right)_{/\widetilde{\boldsymbol{E}}}(\theta, \widetilde{\boldsymbol{E}}) + \rho_0.s(\theta, \widetilde{\boldsymbol{E}})\right].\overset{\bullet}{\theta} \quad = \quad 0 \tag{5.13}$$

$$\forall(\theta, \widetilde{\boldsymbol{E}}) \qquad , \qquad \forall\overset{\bullet}{\theta}$$

This leads to

$$\left(\frac{\partial W}{\partial \theta}\right)_{/\widetilde{\boldsymbol{E}}}(\theta, \widetilde{\boldsymbol{E}}) + \rho_0.s(\theta, \widetilde{\boldsymbol{E}}) \quad = \quad 0 \tag{5.14}$$

$$\forall(\theta, \widetilde{\boldsymbol{E}})$$

Or

$$\forall(\theta, \widetilde{\boldsymbol{E}}) \qquad \rho_0.s(\theta, \widetilde{\boldsymbol{E}}) = -\left(\frac{\partial W}{\partial \theta}\right)_{/\widetilde{\boldsymbol{E}}}(\theta, \widetilde{\boldsymbol{E}}) \tag{5.15}$$

Let us go back to the general case, unlimited, but taking into account this previous equation :

$$\widetilde{\left(\frac{\boldsymbol{\partial W}}{\boldsymbol{\partial \widetilde{E}}}\right)}_{/\boldsymbol{\theta}}(\theta, \widetilde{\boldsymbol{E}}) \; : \overset{\bullet}{\widetilde{\boldsymbol{E}}} \quad = \quad J.\widetilde{\boldsymbol{T}} : \widetilde{\boldsymbol{D}} \quad = \quad -\rho_0.p_{\text{int}} \tag{5.16}$$

$$\forall(\theta, \widetilde{\boldsymbol{E}}) \qquad , \qquad \forall\overset{\bullet}{\widetilde{\boldsymbol{E}}}$$

Or

$$J.\widetilde{\boldsymbol{T}} : \widetilde{\boldsymbol{D}} - \widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\!/\theta}(\theta, \widetilde{\boldsymbol{E}}) : \dot{\widetilde{\boldsymbol{E}}} = 0 \tag{5.17}$$

$$J.\widetilde{\boldsymbol{T}} : \left(\widetilde{\boldsymbol{F}}^{-\mathrm{T}} \cdot \dot{\widetilde{\boldsymbol{E}}} \cdot \widetilde{\boldsymbol{F}}^{-1}\right) - \widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\!/\theta}(\theta, \widetilde{\boldsymbol{E}}) : \dot{\widetilde{\boldsymbol{E}}} = 0 \tag{5.18}$$

$$J.\mathrm{Tr}\left[\widetilde{\boldsymbol{T}} \cdot \left(\widetilde{\boldsymbol{F}}^{-\mathrm{T}} \cdot \dot{\widetilde{\boldsymbol{E}}} \cdot \widetilde{\boldsymbol{F}}^{-1}\right)^{\mathrm{T}}\right] - \widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\!/\theta}(\theta, \widetilde{\boldsymbol{E}}) : \dot{\widetilde{\boldsymbol{E}}} = 0 \tag{5.19}$$

$$J.\mathrm{Tr}\left[\widetilde{\boldsymbol{T}} \cdot \widetilde{\boldsymbol{F}}^{-\mathrm{T}} \cdot \dot{\widetilde{\boldsymbol{E}}} \cdot \widetilde{\boldsymbol{F}}^{-1}\right] - \widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\!/\theta}(\theta, \widetilde{\boldsymbol{E}}) : \dot{\widetilde{\boldsymbol{E}}} = 0 \tag{5.20}$$

$$J.\mathrm{Tr}\left[\widetilde{\boldsymbol{F}}^{-1} \cdot \widetilde{\boldsymbol{T}} \cdot \widetilde{\boldsymbol{F}}^{-\mathrm{T}} \cdot \dot{\widetilde{\boldsymbol{E}}}\right] - \widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\!/\theta}(\theta, \widetilde{\boldsymbol{E}}) : \dot{\widetilde{\boldsymbol{E}}} = 0 \tag{5.21}$$

$$\left(J\widetilde{\boldsymbol{F}}^{-1} \cdot \widetilde{\boldsymbol{T}} \cdot \widetilde{\boldsymbol{F}}^{-\mathrm{T}}\right) : \dot{\widetilde{\boldsymbol{E}}} - \widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\!/\theta}(\theta, \widetilde{\boldsymbol{E}}) : \dot{\widetilde{\boldsymbol{E}}} = 0 \tag{5.22}$$

$$\left[\left(J\widetilde{\boldsymbol{F}}^{-1} \cdot \widetilde{\boldsymbol{T}} \cdot \widetilde{\boldsymbol{F}}^{-\mathrm{T}}\right) - \widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\!/\theta}(\theta, \widetilde{\boldsymbol{E}})\right] : \dot{\widetilde{\boldsymbol{E}}} = 0 \tag{5.23}$$

$$\forall(\theta, \widetilde{\boldsymbol{E}}) \quad , \quad \forall \dot{\widetilde{\boldsymbol{E}}}$$

Assumption – *We assume that the stress tensor $\widetilde{\boldsymbol{T}}$ does not depend on the strain rate $\dot{\widetilde{\boldsymbol{E}}}$.*

Then, and because $\dot{\widetilde{\boldsymbol{E}}}$ may reach any point of the space of second-order symmetric tensors, we obtain :

$$\forall(\theta, \widetilde{\boldsymbol{E}}) \qquad \left[\left(J\widetilde{\boldsymbol{F}}^{-1} \cdot \widetilde{\boldsymbol{T}} \cdot \widetilde{\boldsymbol{F}}^{-\mathrm{T}}\right) - \widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\!/\theta}(\theta, \widetilde{\boldsymbol{E}})\right] \text{ is skew-symmetric} \tag{5.24}$$

Or

$$\forall(\theta, \widetilde{\boldsymbol{E}}) \qquad \left[\left(J\widetilde{\boldsymbol{F}}^{-1} \cdot \widetilde{\boldsymbol{T}} \cdot \widetilde{\boldsymbol{F}}^{-\mathrm{T}}\right) - \widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\!/\theta}(\theta, \widetilde{\boldsymbol{E}})\right] + \left[\left(J\widetilde{\boldsymbol{F}}^{-1} \cdot \widetilde{\boldsymbol{T}} \cdot \widetilde{\boldsymbol{F}}^{-\mathrm{T}}\right) - \widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\!/\theta}(\theta, \widetilde{\boldsymbol{E}})\right]^{\mathrm{T}} = \widetilde{\boldsymbol{0}} \tag{5.25}$$

$\Longleftrightarrow$

$$\forall(\theta, \widetilde{\boldsymbol{E}}) \qquad 2\left(J\widetilde{\boldsymbol{F}}^{-1}\boldsymbol{.}\widetilde{\boldsymbol{T}}\boldsymbol{.}\widetilde{\boldsymbol{F}}^{-\mathrm{T}}\right) = \widetilde{\left(\frac{\partial W}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\theta}(\theta, \widetilde{\boldsymbol{E}}) + \widetilde{\left(\frac{\partial W}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\theta}^{\mathbf{T}}(\theta, \widetilde{\boldsymbol{E}}) \quad (5.26)$$

Because of note 1, we may assume that

$$\widetilde{\left(\frac{\partial W}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\theta}(\theta, \widetilde{\boldsymbol{E}}) \quad (5.27)$$

is symmetric.

So,

$$\forall(\theta, \widetilde{\boldsymbol{E}}) \qquad J\widetilde{\boldsymbol{F}}^{-1}\boldsymbol{.}\widetilde{\boldsymbol{T}}\boldsymbol{.}\widetilde{\boldsymbol{F}}^{-\mathrm{T}} = \widetilde{\left(\frac{\partial W}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\theta}(\theta, \widetilde{\boldsymbol{E}}) \quad (5.28)$$

5.4 Second Piola-Kirchhoff stress $\widetilde{\boldsymbol{S}}$

This tensor is sometimes called stress tensor of Piola-Lagrange, and also «pk2».

Definition – *The second stress tensor of Piola-Kirchhoff is defined as*

$$\widetilde{\boldsymbol{S}} = (\det\widetilde{\boldsymbol{F}}).\left(\widetilde{\boldsymbol{F}}^{-1}\boldsymbol{.}\widetilde{\boldsymbol{T}}\boldsymbol{.}\widetilde{\boldsymbol{F}}^{-\mathrm{T}}\right) \ . \quad (5.29)$$

By using this stress tensor, the stress-strain equation of the hyperelastic law may be the following :

$$\forall(\theta, \widetilde{\boldsymbol{E}}) \qquad \widetilde{\boldsymbol{S}} = \widetilde{\left(\frac{\partial W}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\theta}(\theta, \widetilde{\boldsymbol{E}}) \quad (5.30)$$

Then, the rate of free energy is :

$$\rho_0.\overset{\bullet}{\psi} \quad = \quad -\rho_0.s\ .\overset{\bullet}{\theta} + \widetilde{\boldsymbol{S}} : \overset{\bullet}{\widetilde{\boldsymbol{E}}} \quad (5.31)$$

And, second Piola-Kirchhoff stress $\widetilde{\boldsymbol{S}}$ is the **work-conjugate** of the right Green-Lagrange strain $\widetilde{\boldsymbol{E}}$ (if the energy is considered per unit of reference volume).

This stress tensor is symmetric.

5.5 Kirchhoff stress $\widetilde{\boldsymbol{\tau}}$

Then, the opposite of the power of internal forces per unit of reference volume is :

$$-\rho_0.p_{\text{int}} = \widetilde{\boldsymbol{S}} : \dot{\widetilde{\boldsymbol{E}}} = J.\widetilde{\boldsymbol{T}} : \widetilde{\boldsymbol{D}} \tag{5.32}$$

Definition – *The Kirchhoff stress tensor is defined as*

$$\widetilde{\boldsymbol{\tau}} = J.\widetilde{\boldsymbol{T}} \;= (\det\widetilde{\boldsymbol{F}}).\widetilde{\boldsymbol{T}} \;= \frac{\rho_0}{\rho_t}.\widetilde{\boldsymbol{T}}\ . \tag{5.33}$$

Then,

$$-\rho_0.p_{\text{int}} = \widetilde{\boldsymbol{\tau}} : \widetilde{\boldsymbol{D}} \tag{5.34}$$

Kirchhoff stress $\widetilde{\boldsymbol{\tau}}$ is the *pseudo-work-conjugate* of stretching tensor $\widetilde{\boldsymbol{D}}$ (if the energy is considered per unit of reference volume). We write here *pseudo-work-conjugate*, because the stretching tensor is not a strain measure.

This stress tensor is symmetric.

5.6 First Piola-Kirchhoff stress $\widetilde{\Pi}$

Let us derive the opposite of internal power as follows :

$$\begin{aligned}
-\rho_0.p_{\text{int}} &= J.\widetilde{\boldsymbol{T}} : \widetilde{\boldsymbol{D}} && (5.35)\\
&= J\mathrm{Tr}\left[\widetilde{\boldsymbol{T}} \boldsymbol{.} \widetilde{\boldsymbol{D}}^{\mathbf{T}}\right] && (5.36)\\
&= J\mathrm{Tr}\left[\widetilde{\boldsymbol{T}} \boldsymbol{.} (\widetilde{\boldsymbol{D}} + \widetilde{\boldsymbol{W}})^{\mathbf{T}}\right] && (5.37)\\
&= J\mathrm{Tr}\left[\widetilde{\boldsymbol{T}} \boldsymbol{.} \widetilde{\boldsymbol{L}}^{\mathbf{T}}\right] && (5.38)\\
&= J\mathrm{Tr}\left[\widetilde{\boldsymbol{T}} \boldsymbol{.} \left(\dot{\widetilde{\boldsymbol{F}}} \boldsymbol{.} \widetilde{\boldsymbol{F}}^{-1}\right)^{\mathbf{T}}\right] && (5.39)\\
&= J\mathrm{Tr}\left[\widetilde{\boldsymbol{T}} \boldsymbol{.} \widetilde{\boldsymbol{F}}^{-\mathbf{T}} \boldsymbol{.} \left(\dot{\widetilde{\boldsymbol{F}}}\right)^{\mathbf{T}}\right] && (5.40)\\
&= \left(J\widetilde{\boldsymbol{T}} \boldsymbol{.} \widetilde{\boldsymbol{F}}^{-\mathbf{T}}\right) : \dot{\widetilde{\boldsymbol{F}}} && (5.41)
\end{aligned}$$

Definition – *The first Piola-Kirchhoff stress tensor, also called Boussinesq stress tensor, is defined as*

$$\widetilde{\boldsymbol{\Pi}} = J\widetilde{\boldsymbol{T}} \cdot \widetilde{\boldsymbol{F}}^{-\mathrm{T}} = \widetilde{\boldsymbol{F}} \cdot \widetilde{\boldsymbol{S}} \; . \tag{5.42}$$

Then,

$$-\rho_0.p_{\text{int}} = \widetilde{\boldsymbol{\Pi}} : \dot{\widetilde{\boldsymbol{F}}} \tag{5.43}$$

The first Piola-Kirchhoff stress $\widetilde{\boldsymbol{\Pi}}$ is the *work-conjugate* of the deformation gradient $\widetilde{\boldsymbol{F}}$ (if the energy is considered per unit of reference volume).

This stress tensor is NOT symmetric.

5.7 What if the material is incompressible ?

If the material is always incompressible

$$J = 1 \tag{5.44}$$

and whatever the thermomechanical state

$$\dot{J} = 0 \tag{5.45}$$

This means that

$$\det(\widetilde{\boldsymbol{F}}) = 1 = \det(\widetilde{\boldsymbol{I}} + 2\widetilde{\boldsymbol{E}}) \tag{5.46}$$

and whatever the thermomechanical state

$$\mathrm{Tr}\widetilde{\boldsymbol{D}} = 0 \tag{5.47}$$

So the stretching tensor is always deviatoric.

Let us apply again the method of the previous sections :

$$\begin{aligned} -\rho_0.p_{\text{int}} = \widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\theta}(\theta, \widetilde{\boldsymbol{E}}) \; : \dot{\widetilde{\boldsymbol{E}}} \; &= \; J.\widetilde{\boldsymbol{T}} : \widetilde{\boldsymbol{D}} \\ \forall(\theta, \widetilde{\boldsymbol{E}}) \text{ such that } \det(\widetilde{\boldsymbol{I}} &+ 2\widetilde{\boldsymbol{E}}) = 1 \\ &\text{and } \forall\widetilde{\boldsymbol{D}} \text{ symmetric and deviatoric} \end{aligned} \tag{5.48}$$

Or

$$J.\widetilde{\boldsymbol{T}} : \widetilde{\boldsymbol{D}} - \widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\!/\boldsymbol{\theta}}(\theta, \widetilde{\boldsymbol{E}}) : \overset{\bullet}{\widetilde{\boldsymbol{E}}} = 0 \tag{5.49}$$

$$J.\widetilde{\boldsymbol{T}} : \widetilde{\boldsymbol{D}} - \widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\!/\boldsymbol{\theta}}(\theta, \widetilde{\boldsymbol{E}}) : \left(\widetilde{\boldsymbol{F}}^{\mathbf{T}} \boldsymbol{.} \widetilde{\boldsymbol{D}} \boldsymbol{.} \widetilde{\boldsymbol{F}}\right) = 0 \tag{5.50}$$

$$J.\widetilde{\boldsymbol{T}} : \widetilde{\boldsymbol{D}} - \mathrm{Tr}\left[\widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\!/\boldsymbol{\theta}}(\theta, \widetilde{\boldsymbol{E}}) \boldsymbol{.} \left(\widetilde{\boldsymbol{F}}^{\mathbf{T}} \boldsymbol{.} \widetilde{\boldsymbol{D}} \boldsymbol{.} \widetilde{\boldsymbol{F}}\right)^{\mathbf{T}}\right] = 0 \tag{5.51}$$

$$J.\widetilde{\boldsymbol{T}} : \widetilde{\boldsymbol{D}} - \mathrm{Tr}\left[\widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\!/\boldsymbol{\theta}}(\theta, \widetilde{\boldsymbol{E}}) \boldsymbol{.} \widetilde{\boldsymbol{F}}^{\mathbf{T}} \boldsymbol{.} \widetilde{\boldsymbol{D}} \boldsymbol{.} \widetilde{\boldsymbol{F}}\right] = 0 \tag{5.52}$$

$$J.\widetilde{\boldsymbol{T}} : \widetilde{\boldsymbol{D}} - \mathrm{Tr}\left[\widetilde{\boldsymbol{F}} \boldsymbol{.} \widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\!/\boldsymbol{\theta}}(\theta, \widetilde{\boldsymbol{E}}) \boldsymbol{.} \widetilde{\boldsymbol{F}}^{\mathbf{T}} \boldsymbol{.} \widetilde{\boldsymbol{D}}\right] = 0 \tag{5.53}$$

$$J.\widetilde{\boldsymbol{T}} : \widetilde{\boldsymbol{D}} - \left[\widetilde{\boldsymbol{F}} \boldsymbol{.} \widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\!/\boldsymbol{\theta}}(\theta, \widetilde{\boldsymbol{E}}) \boldsymbol{.} \widetilde{\boldsymbol{F}}^{\mathbf{T}}\right] : \widetilde{\boldsymbol{D}} = 0 \tag{5.54}$$

$$\left[J.\widetilde{\boldsymbol{T}} - \widetilde{\boldsymbol{F}} \boldsymbol{.} \widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\!/\boldsymbol{\theta}}(\theta, \widetilde{\boldsymbol{E}}) \boldsymbol{.} \widetilde{\boldsymbol{F}}^{\mathbf{T}}\right] : \widetilde{\boldsymbol{D}} = 0 \tag{5.55}$$

$$\forall(\theta, \widetilde{\boldsymbol{E}}) \text{ such that } \det(\widetilde{\boldsymbol{I}} + 2\widetilde{\boldsymbol{E}}) = 1 \tag{5.56}$$
$$\text{and } \forall\widetilde{\boldsymbol{D}} \text{ symmetric and deviatoric}$$

Then, the symmetric part of

$$J.\widetilde{\boldsymbol{T}} - \widetilde{\boldsymbol{F}} \boldsymbol{.} \widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\!/\boldsymbol{\theta}}(\theta, \widetilde{\boldsymbol{E}}) \boldsymbol{.} \widetilde{\boldsymbol{F}}^{\mathbf{T}} \tag{5.57}$$

is isotropic, that is to say that it exists a real number k such that

$$\left[J.\widetilde{\boldsymbol{T}} - \widetilde{\boldsymbol{F}} \boldsymbol{.} \widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\!/\boldsymbol{\theta}}(\theta, \widetilde{\boldsymbol{E}}) \boldsymbol{.} \widetilde{\boldsymbol{F}}^{\mathbf{T}}\right] + \left[J.\widetilde{\boldsymbol{T}} - \widetilde{\boldsymbol{F}} \boldsymbol{.} \widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\!/\boldsymbol{\theta}}(\theta, \widetilde{\boldsymbol{E}}) \boldsymbol{.} \widetilde{\boldsymbol{F}}^{\mathbf{T}}\right]^{\mathbf{T}} = 2k.\widetilde{\boldsymbol{I}} \tag{5.58}$$

Since, this tensor

$$J.\widetilde{\boldsymbol{T}} - \widetilde{\boldsymbol{F}} \boldsymbol{.} \widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\!/\boldsymbol{\theta}}(\theta, \widetilde{\boldsymbol{E}}) \boldsymbol{.} \widetilde{\boldsymbol{F}}^{\mathbf{T}} \tag{5.59}$$

is symmetric, we obtain that

$$J.\widetilde{\boldsymbol{T}} - \widetilde{\boldsymbol{F}} \boldsymbol{.} \widetilde{\left(\frac{\partial \boldsymbol{W}}{\partial \widetilde{\boldsymbol{E}}}\right)}_{/\!/\boldsymbol{\theta}}(\theta, \widetilde{\boldsymbol{E}}) \boldsymbol{.} \widetilde{\boldsymbol{F}}^{\mathbf{T}} = k.\widetilde{\boldsymbol{I}} \tag{5.60}$$

This leads to

$$J.\widetilde{\boldsymbol{F}}^{-\mathbf{1}} \boldsymbol{.} \widetilde{\boldsymbol{T}} \boldsymbol{.} \widetilde{\boldsymbol{F}}^{-\mathbf{T}} - \widetilde{\left(\frac{\boldsymbol{\partial W}}{\boldsymbol{\partial \widetilde{E}}}\right)}_{/\!/\boldsymbol{\theta}}(\theta, \widetilde{\boldsymbol{E}}) \quad = \quad k.\widetilde{\boldsymbol{F}}^{-\mathbf{1}} \boldsymbol{.} \widetilde{\boldsymbol{F}}^{-\mathbf{T}} \tag{5.61}$$

$$\widetilde{\boldsymbol{S}} - \widetilde{\left(\frac{\boldsymbol{\partial W}}{\boldsymbol{\partial \widetilde{E}}}\right)}_{/\!/\boldsymbol{\theta}}(\theta, \widetilde{\boldsymbol{E}}) \quad = \quad k.\widetilde{\boldsymbol{C}}^{-\mathbf{1}} \tag{5.62}$$

Or, it exists a real number k such that

$$\widetilde{\boldsymbol{S}} = \widetilde{\left(\frac{\boldsymbol{\partial W}}{\boldsymbol{\partial \widetilde{E}}}\right)}_{/\!/\boldsymbol{\theta}}(\theta, \widetilde{\boldsymbol{E}}) + k.\widetilde{\boldsymbol{C}}^{-\mathbf{1}}$$

$$J.\widetilde{\boldsymbol{T}} = \widetilde{\boldsymbol{F}} \boldsymbol{.} \widetilde{\left(\frac{\boldsymbol{\partial W}}{\boldsymbol{\partial \widetilde{E}}}\right)}_{/\!/\boldsymbol{\theta}}(\theta, \widetilde{\boldsymbol{E}}) \boldsymbol{.} \widetilde{\boldsymbol{F}}^{\mathbf{T}} + k.\widetilde{\boldsymbol{I}}$$

$$\forall(\theta, \widetilde{\boldsymbol{E}}) \text{ such that } \det(\widetilde{\boldsymbol{I}} + 2\widetilde{\boldsymbol{E}}) = 1 \tag{5.63}$$

5.8 Isotropic hyperelasticity

Now we limit the hyperelasticity to isotropic behavior, useful for rubber materials, for instance.

That case may be described by a potential as a function of three free invariants of the right Green-Lagrange strain tensor $\widetilde{\boldsymbol{E}}$:

$$W(\theta, \widetilde{\boldsymbol{E}}) = \omega\left(\theta, I_1(\widetilde{\boldsymbol{E}}), I_2(\widetilde{\boldsymbol{E}}), I_3(\widetilde{\boldsymbol{E}})\right) \tag{5.64}$$

where for all second-order symmetric tensor $\widetilde{\boldsymbol{X}}$,

$$I_1(\widetilde{\boldsymbol{X}}) \quad = \quad \mathrm{Tr}\widetilde{\boldsymbol{X}} = \widetilde{\boldsymbol{I}} : \widetilde{\boldsymbol{X}} \tag{5.65}$$

$$I_2(\widetilde{\boldsymbol{X}}) \quad = \quad \mathrm{Tr}(\widetilde{\boldsymbol{X}} \boldsymbol{.} \widetilde{\boldsymbol{X}}) = \widetilde{\boldsymbol{X}} : \widetilde{\boldsymbol{X}} \tag{5.66}$$

$$I_3(\widetilde{\boldsymbol{X}}) \quad = \quad \mathrm{Tr}(\widetilde{\boldsymbol{X}} \boldsymbol{.} \widetilde{\boldsymbol{X}} \boldsymbol{.} \widetilde{\boldsymbol{X}}) = (\widetilde{\boldsymbol{X}} \boldsymbol{.} \widetilde{\boldsymbol{X}}) : \widetilde{\boldsymbol{X}} \tag{5.67}$$

The choice of these three free invariants is not unique. *Free* means here, that one invariant can not be a function of the two others.

It exists a bijective function between these three invariants of $\widetilde{\boldsymbol{E}}$ and the three principal stretches $(\lambda_1, \lambda_2, \lambda_3)$, the eigenvalues of strain tensors $\widetilde{\boldsymbol{U}}$ and $\widetilde{\boldsymbol{V}}$. What is true for $\widetilde{\boldsymbol{E}}$ is also true for all strain measure tensors, left and right. This property leads to a lot (infinitely ?) of ways to define the potential function 5.64. For example, we may propose :

$$W(\theta, \widetilde{\boldsymbol{E}}) = \omega_a \Big(\theta, I_1(\widetilde{\boldsymbol{B}}), I_2(\widetilde{\boldsymbol{B}}), I_3(\widetilde{\boldsymbol{B}})\Big) \tag{5.68}$$

$$W(\theta, \widetilde{\boldsymbol{E}}) = \omega_b \Big(\theta, I_1(\widetilde{\boldsymbol{LnV}}), I_2(\widetilde{\boldsymbol{LnV}}), I_3(\widetilde{\boldsymbol{LnV}})\Big) \tag{5.69}$$

$$W(\theta, \widetilde{\boldsymbol{E}}) = \omega_c \Big(\theta, \mathrm{Tr}(\widetilde{\boldsymbol{LnU}}), \sec(\widetilde{\boldsymbol{LnU}}), \det(\widetilde{\boldsymbol{LnU}})\Big) \tag{5.70}$$

And so on ...

5.9 Calculus method for the partial derivatives of the potential

In the case of the isotropic hyperelasticity of equation 5.64 :

$$\begin{aligned}
\left(\widetilde{\frac{\boldsymbol{\partial W}}{\boldsymbol{\partial \widetilde{E}}}}\right)_{/\boldsymbol{\theta}}(\theta, \widetilde{\boldsymbol{E}}) \;=\; & \left(\frac{\partial \omega}{\partial I_1}\right)_{/\theta, I_2, I_3}\Big(\theta, I_1(\widetilde{\boldsymbol{E}}), I_2(\widetilde{\boldsymbol{E}}), I_3(\widetilde{\boldsymbol{E}})\Big) \;.\; \widetilde{\frac{\boldsymbol{\partial I_1(\widetilde{E})}}{\boldsymbol{\partial \widetilde{E}}}} \\
+ & \left(\frac{\partial \omega}{\partial I_2}\right)_{/\theta, I_1, I_3}\Big(\theta, I_1(\widetilde{\boldsymbol{E}}), I_2(\widetilde{\boldsymbol{E}}), I_3(\widetilde{\boldsymbol{E}})\Big) \;.\; \widetilde{\frac{\boldsymbol{\partial I_2(\widetilde{E})}}{\boldsymbol{\partial \widetilde{E}}}} \\
+ & \left(\frac{\partial \omega}{\partial I_3}\right)_{/\theta, I_1, I_2}\Big(\theta, I_1(\widetilde{\boldsymbol{E}}), I_2(\widetilde{\boldsymbol{E}}), I_3(\widetilde{\boldsymbol{E}})\Big) \;.\; \widetilde{\frac{\boldsymbol{\partial I_3(\widetilde{E})}}{\boldsymbol{\partial \widetilde{E}}}}
\end{aligned} \tag{5.71}$$

Now, how calculate the terms

$$\widetilde{\frac{\boldsymbol{\partial I_i(\widetilde{E})}}{\boldsymbol{\partial \widetilde{E}}}} \quad ? \tag{5.72}$$

We may broaden this question to all scalar functions of a second-order tensor $f(\widetilde{\boldsymbol{X}})$:

$$\widetilde{\frac{\boldsymbol{\partial f(\widetilde{X})}}{\boldsymbol{\partial \widetilde{X}}}} \quad ? \tag{5.73}$$

We propose here a relatively simple method. The idea is to calculate the time derivative of the result of function $f(\widetilde{\boldsymbol{X}})$. By using the calculus properties of the simple derivative from 2.56 to 2.66, we have to show a relation between the time derivative of this result and the time derivative of tensor $\widetilde{\boldsymbol{X}}$, according to the following shape :

$$\overline{f(\widetilde{\boldsymbol{X}})}^{\bullet} = \widetilde{\boldsymbol{H}} : \dot{\widetilde{\boldsymbol{X}}} \tag{5.74}$$

Then, we have calculated the derivative, since

$$\widetilde{\frac{\partial f(\widetilde{\boldsymbol{X}})}{\partial \widetilde{\boldsymbol{X}}}} = \widetilde{\boldsymbol{H}} \tag{5.75}$$

Here, some examples.

$$\begin{aligned} \overline{I_1(\widetilde{\boldsymbol{X}})}^{\bullet} &= \mathrm{Tr}\dot{\widetilde{\boldsymbol{X}}} = \mathrm{Tr}\left[\dot{\widetilde{\boldsymbol{X}}}\right] = \mathrm{Tr}\left[\dot{\widetilde{\boldsymbol{X}}}^{\mathbf{T}}\right] && (5.76)\\ &= \mathrm{Tr}\left[\widetilde{\boldsymbol{I}} \,.\, \dot{\widetilde{\boldsymbol{X}}}^{\mathbf{T}}\right] = \widetilde{\boldsymbol{I}} : \dot{\widetilde{\boldsymbol{X}}} && (5.77)\end{aligned}$$

So,

$$\widetilde{\frac{\partial I_1(\widetilde{\boldsymbol{X}})}{\partial \widetilde{\boldsymbol{X}}}} = \widetilde{\boldsymbol{I}}\ . \tag{5.78}$$

A second example :

$$\begin{aligned} \overline{I_2(\widetilde{\boldsymbol{X}})}^{\bullet} &= \overline{\mathrm{Tr}(\widetilde{\boldsymbol{X}} \,.\, \widetilde{\boldsymbol{X}})}^{\bullet} = \mathrm{Tr}\left[\overline{\widetilde{\boldsymbol{X}} \,.\, \widetilde{\boldsymbol{X}}}^{\bullet}\right] = \mathrm{Tr}\left[\dot{\widetilde{\boldsymbol{X}}} \,.\, \widetilde{\boldsymbol{X}} + \widetilde{\boldsymbol{X}} \,.\, \dot{\widetilde{\boldsymbol{X}}}\right] && (5.79)\\ &= \mathrm{Tr}\left[2\widetilde{\boldsymbol{X}} \,.\, \dot{\widetilde{\boldsymbol{X}}}\right] = \mathrm{Tr}\left[\dot{\widetilde{\boldsymbol{X}}}^{\mathbf{T}} \,.\, 2\widetilde{\boldsymbol{X}}^{\mathbf{T}}\right] = \mathrm{Tr}\left[2\widetilde{\boldsymbol{X}}^{\mathbf{T}} \,.\, \dot{\widetilde{\boldsymbol{X}}}^{\mathbf{T}}\right] && (5.80)\\ &= 2\widetilde{\boldsymbol{X}}^{\mathbf{T}} : \dot{\widetilde{\boldsymbol{X}}} && (5.81)\end{aligned}$$

So,

$$\widetilde{\frac{\partial I_2(\widetilde{\boldsymbol{X}})}{\partial \widetilde{\boldsymbol{X}}}} = 2\widetilde{\boldsymbol{X}}^{\mathbf{T}}\ . \tag{5.82}$$

An other example :

$$\begin{aligned} \overline{I_3(\widetilde{\boldsymbol{X}})}^{\bullet} &= \overline{\mathrm{Tr}(\widetilde{\boldsymbol{X}} \,.\, \widetilde{\boldsymbol{X}} \,.\, \widetilde{\boldsymbol{X}})}^{\bullet} = \mathrm{Tr}\left[\overline{\widetilde{\boldsymbol{X}} \,.\, \widetilde{\boldsymbol{X}} \,.\, \widetilde{\boldsymbol{X}}}^{\bullet}\right] && (5.83)\\ &= \mathrm{Tr}\left[\dot{\widetilde{\boldsymbol{X}}} \,.\, \widetilde{\boldsymbol{X}} \,.\, \widetilde{\boldsymbol{X}} + \widetilde{\boldsymbol{X}} \,.\, \dot{\widetilde{\boldsymbol{X}}} \,.\, \widetilde{\boldsymbol{X}} + \widetilde{\boldsymbol{X}} \,.\, \widetilde{\boldsymbol{X}} \,.\, \dot{\widetilde{\boldsymbol{X}}}\right] && (5.84)\\ &= \mathrm{Tr}\left[3\widetilde{\boldsymbol{X}} \,.\, \widetilde{\boldsymbol{X}} \,.\, \dot{\widetilde{\boldsymbol{X}}}\right] = (3\widetilde{\boldsymbol{X}}^{\mathbf{T}} \,.\, \widetilde{\boldsymbol{X}}^{\mathbf{T}}) : \dot{\widetilde{\boldsymbol{X}}} && (5.85)\end{aligned}$$

So,

$$\frac{\widetilde{\boldsymbol{\partial I_3(\widetilde{X})}}}{\boldsymbol{\partial \widetilde{X}}} = 3\widetilde{\boldsymbol{X}}^{\mathbf{T}} \boldsymbol{.} \widetilde{\boldsymbol{X}}^{\mathbf{T}} . \tag{5.86}$$

Some more examples. Relation 1.146[p22] gives :

$$\frac{\widetilde{\boldsymbol{\partial \mathrm{sec}(\widetilde{X})}}}{\boldsymbol{\partial \widetilde{X}}} = \mathrm{Tr}\widetilde{\boldsymbol{X}} \ . \ \widetilde{\boldsymbol{I}} - \widetilde{\boldsymbol{X}}^{\mathbf{T}} \tag{5.87}$$

Relations 1.147[p22] and 1.133[p21] give :

$$\frac{\widetilde{\boldsymbol{\partial \mathrm{det}(\widetilde{X})}}}{\boldsymbol{\partial \widetilde{X}}} = \left[(\mathrm{sec}\widetilde{\boldsymbol{X}})\widetilde{\boldsymbol{I}} - (\mathrm{Tr}\widetilde{\boldsymbol{X}})\widetilde{\boldsymbol{X}} + \widetilde{\boldsymbol{X}} \boldsymbol{.} \widetilde{\boldsymbol{X}}\right]^{\mathbf{T}} \tag{5.88}$$

$$= (\mathrm{det}\widetilde{\boldsymbol{X}}) \ . \ \widetilde{\boldsymbol{X}}^{-\mathbf{T}} \tag{5.89}$$

This last equation is true only if tensor $\widetilde{\boldsymbol{X}}$ is invertible.

In the particular case of potential definition 5.68, we need to calculate

$$\frac{\widetilde{\boldsymbol{\partial I_i(\widetilde{B})}}}{\boldsymbol{\partial \widetilde{E}}} . \tag{5.90}$$

The first term could be calculated as follows :

$$\overset{\bullet}{\overline{I_1(\widetilde{\boldsymbol{B}})}} = \mathrm{Tr}\overset{\bullet}{\widetilde{\boldsymbol{B}}} = \mathrm{Tr}\left[\overset{\bullet}{\widetilde{\boldsymbol{B}}}\right] = \mathrm{Tr}\left[\overset{\bullet}{\widetilde{\boldsymbol{F}}} \boldsymbol{.} \widetilde{\boldsymbol{F}}^{\mathbf{T}} + \widetilde{\boldsymbol{F}} \boldsymbol{.} \overset{\bullet}{\widetilde{\boldsymbol{F}}}^{\mathbf{T}}\right] \tag{5.91}$$

$$= \mathrm{Tr}\left[\overset{\bullet}{\widetilde{\boldsymbol{F}}} \boldsymbol{.} \widetilde{\boldsymbol{F}}^{\mathbf{T}}\right] + \mathrm{Tr}\left[\widetilde{\boldsymbol{F}} \boldsymbol{.} \overset{\bullet}{\widetilde{\boldsymbol{F}}}^{\mathbf{T}}\right] \tag{5.92}$$

$$= \mathrm{Tr}\left[\widetilde{\boldsymbol{F}}^{\mathbf{T}} \boldsymbol{.} \overset{\bullet}{\widetilde{\boldsymbol{F}}}\right] + \mathrm{Tr}\left[\overset{\bullet}{\widetilde{\boldsymbol{F}}}^{\mathbf{T}} \boldsymbol{.} \widetilde{\boldsymbol{F}}\right] \tag{5.93}$$

$$= \mathrm{Tr}\left[\widetilde{\boldsymbol{F}}^{\mathbf{T}} \boldsymbol{.} \overset{\bullet}{\widetilde{\boldsymbol{F}}} + \overset{\bullet}{\widetilde{\boldsymbol{F}}}^{\mathbf{T}} \boldsymbol{.} \widetilde{\boldsymbol{F}}\right] = \mathrm{Tr}\left[\overset{\bullet}{\overline{\widetilde{\boldsymbol{F}}^{\mathbf{T}} \boldsymbol{.} \widetilde{\boldsymbol{F}}}}\right] \tag{5.94}$$

$$= \mathrm{Tr}\left[\overset{\bullet}{\widetilde{\boldsymbol{C}}}\right] = \mathrm{Tr}\left[2\overset{\bullet}{\widetilde{\boldsymbol{E}}}\right] = 2\widetilde{\boldsymbol{I}} : \overset{\bullet}{\widetilde{\boldsymbol{E}}} \tag{5.95}$$

So,

$$\frac{\widetilde{\boldsymbol{\partial I_1(\widetilde{B})}}}{\boldsymbol{\partial \widetilde{E}}} = 2\widetilde{\boldsymbol{I}} \ . \tag{5.96}$$

Another way to prove the same, is as follows, by using the calculus properties of the objective rates, equations 7.32, 7.33 :

$$\begin{aligned} \overset{\bullet}{\overline{I_1(\widetilde{\boldsymbol{B}})}} &= \mathrm{Tr}\overset{\bullet}{\widetilde{\boldsymbol{B}}} = \mathrm{Tr}\left[\overset{\circ j}{\widetilde{\boldsymbol{B}}}\right] = \mathrm{Tr}\left[\widetilde{\boldsymbol{B}}\,\boldsymbol{.}\,\widetilde{\boldsymbol{D}} + \widetilde{\boldsymbol{D}}\,\boldsymbol{.}\,\widetilde{\boldsymbol{B}}\right] && (5.97)\\ &= \mathrm{Tr}\left[2\widetilde{\boldsymbol{B}}\,\boldsymbol{.}\,\widetilde{\boldsymbol{D}}\right] && (5.98)\end{aligned}$$

Since

$$\overset{\bullet}{\widetilde{\boldsymbol{E}}} = \widetilde{\boldsymbol{F}}^{\mathbf{T}}\boldsymbol{.}\,\widetilde{\boldsymbol{D}}\,\boldsymbol{.}\,\widetilde{\boldsymbol{F}} \iff \widetilde{\boldsymbol{D}} = \widetilde{\boldsymbol{F}}^{-\mathbf{T}}\boldsymbol{.}\,\overset{\bullet}{\widetilde{\boldsymbol{E}}}\,\boldsymbol{.}\,\widetilde{\boldsymbol{F}}^{-\mathbf{1}}\,, \tag{5.99}$$

$$\begin{aligned} \overset{\bullet}{\overline{I_1(\widetilde{\boldsymbol{B}})}} &= \mathrm{Tr}\left[2\widetilde{\boldsymbol{B}}\,\boldsymbol{.}\,\widetilde{\boldsymbol{F}}^{-\mathbf{T}}\boldsymbol{.}\,\overset{\bullet}{\widetilde{\boldsymbol{E}}}\,\boldsymbol{.}\,\widetilde{\boldsymbol{F}}^{-\mathbf{1}}\right] = \mathrm{Tr}\left[2\widetilde{\boldsymbol{F}}^{-\mathbf{1}}\boldsymbol{.}\,\widetilde{\boldsymbol{B}}\,\boldsymbol{.}\,\widetilde{\boldsymbol{F}}^{-\mathbf{T}}\boldsymbol{.}\,\overset{\bullet}{\widetilde{\boldsymbol{E}}}\right] && (5.100)\\ &= \mathrm{Tr}\left[2\widetilde{\boldsymbol{F}}^{-\mathbf{1}}\boldsymbol{.}\,\widetilde{\boldsymbol{F}}\,\boldsymbol{.}\,\widetilde{\boldsymbol{F}}^{\mathbf{T}}\boldsymbol{.}\,\widetilde{\boldsymbol{F}}^{-\mathbf{T}}\boldsymbol{.}\,\overset{\bullet}{\widetilde{\boldsymbol{E}}}\right] = \mathrm{Tr}\left[2\overset{\bullet}{\widetilde{\boldsymbol{E}}}\right] && (5.101)\\ &= 2\widetilde{\boldsymbol{I}} : \overset{\bullet}{\widetilde{\boldsymbol{E}}} && (5.102)\end{aligned}$$

So,

$$\frac{\widetilde{\boldsymbol{\partial I_1(\widetilde{B})}}}{\boldsymbol{\partial \widetilde{E}}} = 2\widetilde{\boldsymbol{I}} \ . \tag{5.103}$$

5.10 State variables θ and $\widetilde{\boldsymbol{LnU}}$

In this chapter, the definition of the hyperelastic laws was made by choosing $(\theta, \widetilde{\boldsymbol{E}})$ as state variables. In the present section, we investigate an other choice of state variables :

$$(\theta, \widetilde{\boldsymbol{LnU}}) \tag{5.104}$$

By following the same method, we obtain

$$\begin{aligned} -\rho_0.p_{\mathrm{int}} &= \widetilde{\boldsymbol{S}} : \overset{\bullet}{\widetilde{\boldsymbol{E}}} = J.\widetilde{\boldsymbol{T}} : \widetilde{\boldsymbol{D}} && (5.105)\\ &= \left(\frac{\widetilde{\boldsymbol{\partial W}}}{\boldsymbol{\partial \widetilde{LnU}}}\right)_{\!/\theta}(\theta, \widetilde{\boldsymbol{LnU}}) \ : \overset{\bullet}{\widetilde{\boldsymbol{LnU}}} && (5.106)\end{aligned}$$

$$\forall(\theta, \widetilde{\boldsymbol{LnU}}) \qquad , \qquad \forall \overset{\bullet}{\widetilde{\boldsymbol{E}}} \text{ or } \overset{\bullet}{\widetilde{\boldsymbol{LnU}}} \text{ or } \widetilde{\boldsymbol{D}}$$

According to relation 2.84[p46],

$$
\begin{aligned}
\widetilde{\boldsymbol{S}} : \dot{\widetilde{\boldsymbol{E}}} &= \left(\frac{\partial \widetilde{\boldsymbol{W}}}{\partial \widetilde{\boldsymbol{LnU}}}\right)_{/\theta}(\theta, \widetilde{\boldsymbol{LnU}}) \ : \left[\widetilde{\overset{4}{\gamma}}_{(\widetilde{\boldsymbol{U}})} : \left(\widetilde{\boldsymbol{U}}^{-1} \boldsymbol{.}\, \dot{\widetilde{\boldsymbol{E}}} \,\boldsymbol{.} \widetilde{\boldsymbol{U}}^{-1}\right)\right] \\
&= \left(\widetilde{\boldsymbol{U}}^{-1} \boldsymbol{.}\, \dot{\widetilde{\boldsymbol{E}}} \,\boldsymbol{.} \widetilde{\boldsymbol{U}}^{-1}\right) \ : \left[\widetilde{\overset{4}{\gamma}}_{(\widetilde{\boldsymbol{U}})} : \left(\frac{\partial \widetilde{\boldsymbol{W}}}{\partial \widetilde{\boldsymbol{LnU}}}\right)_{/\theta}(\theta, \widetilde{\boldsymbol{LnU}})\right] \\
&= \mathrm{Tr}\left(\widetilde{\boldsymbol{U}}^{-1} \boldsymbol{.}\, \dot{\widetilde{\boldsymbol{E}}} \,\boldsymbol{.} \widetilde{\boldsymbol{U}}^{-1} \boldsymbol{.} \left[\widetilde{\overset{4}{\gamma}}_{(\widetilde{\boldsymbol{U}})} : \left(\frac{\partial \widetilde{\boldsymbol{W}}}{\partial \widetilde{\boldsymbol{LnU}}}\right)_{/\theta}(\theta, \widetilde{\boldsymbol{LnU}})\right]\right) \\
&= \mathrm{Tr}\left(\widetilde{\boldsymbol{U}}^{-1} \boldsymbol{.} \left[\widetilde{\overset{4}{\gamma}}_{(\widetilde{\boldsymbol{U}})} : \left(\frac{\partial \widetilde{\boldsymbol{W}}}{\partial \widetilde{\boldsymbol{LnU}}}\right)_{/\theta}(\theta, \widetilde{\boldsymbol{LnU}})\right] \boldsymbol{.}\, \widetilde{\boldsymbol{U}}^{-1} \boldsymbol{.}\, \dot{\widetilde{\boldsymbol{E}}}\right) \\
&= \left(\widetilde{\boldsymbol{U}}^{-1} \boldsymbol{.} \left[\widetilde{\overset{4}{\gamma}}_{(\widetilde{\boldsymbol{U}})} : \left(\frac{\partial \widetilde{\boldsymbol{W}}}{\partial \widetilde{\boldsymbol{LnU}}}\right)_{/\theta}(\theta, \widetilde{\boldsymbol{LnU}})\right] \boldsymbol{.}\, \widetilde{\boldsymbol{U}}^{-1}\right) : \dot{\widetilde{\boldsymbol{E}}} \\
&\qquad \forall (\theta, \widetilde{\boldsymbol{LnU}}) \quad , \quad \forall \dot{\widetilde{\boldsymbol{E}}}
\end{aligned}
\tag{5.107}
$$

Then,

$$
\forall (\theta, \widetilde{\boldsymbol{LnU}}) \quad , \quad \widetilde{\boldsymbol{S}} \ = \ \widetilde{\boldsymbol{U}}^{-1} \boldsymbol{.} \left[\widetilde{\overset{4}{\gamma}}_{(\widetilde{\boldsymbol{U}})} : \left(\frac{\partial \widetilde{\boldsymbol{W}}}{\partial \widetilde{\boldsymbol{LnU}}}\right)_{/\theta}(\theta, \widetilde{\boldsymbol{LnU}})\right] \boldsymbol{.}\, \widetilde{\boldsymbol{U}}^{-1} \tag{5.108}
$$

and,

$$
\forall (\theta, \widetilde{\boldsymbol{LnU}}) \quad , \quad J.\widetilde{\boldsymbol{T}} \ = \ \widetilde{\overset{4}{\gamma}}_{(\widetilde{\boldsymbol{V}})} : \left[\widetilde{\boldsymbol{R}} \boldsymbol{.} \left(\frac{\partial \widetilde{\boldsymbol{W}}}{\partial \widetilde{\boldsymbol{LnU}}}\right)_{/\theta}(\theta, \widetilde{\boldsymbol{LnU}}) \boldsymbol{.}\, \widetilde{\boldsymbol{R}}^{\mathbf{T}}\right] \tag{5.109}
$$

So,

$$
\begin{aligned}
\forall (\theta, \widetilde{\boldsymbol{LnU}}) \quad , \quad \left[\widetilde{\overset{4}{\gamma}}_{(\widetilde{\boldsymbol{U}})}\right]^{-1} : (\widetilde{\boldsymbol{U}} \boldsymbol{.} \widetilde{\boldsymbol{S}} \boldsymbol{.} \widetilde{\boldsymbol{U}}) \ &= \ \widetilde{\boldsymbol{R}}^{\mathbf{T}} \boldsymbol{.} \left\{\left[\widetilde{\overset{4}{\gamma}}_{(\widetilde{\boldsymbol{V}})}\right]^{-1} : (J.\widetilde{\boldsymbol{T}})\right\} \boldsymbol{.}\, \widetilde{\boldsymbol{R}} \\
&= \left(\frac{\partial \widetilde{\boldsymbol{W}}}{\partial \widetilde{\boldsymbol{LnU}}}\right)_{/\theta}(\theta, \widetilde{\boldsymbol{LnU}})
\end{aligned}
\tag{5.110}
$$

Then, the stress tensor $\widetilde{\boldsymbol{S_{lnU}}}$, defined as

$$
\widetilde{\boldsymbol{S_{lnU}}} \ = \ \left[\widetilde{\overset{4}{\gamma}}_{(\widetilde{\boldsymbol{U}})}\right]^{-1} : (\widetilde{\boldsymbol{U}} \boldsymbol{.} \widetilde{\boldsymbol{S}} \boldsymbol{.} \widetilde{\boldsymbol{U}}) \ = \ \widetilde{\boldsymbol{R}}^{\mathbf{T}} \boldsymbol{.} \left\{\left[\widetilde{\overset{4}{\gamma}}_{(\widetilde{\boldsymbol{V}})}\right]^{-1} : (J.\widetilde{\boldsymbol{T}})\right\} \boldsymbol{.}\, \widetilde{\boldsymbol{R}} \ , \tag{5.111}
$$

is the **work conjugate** of the right logarithmic strain $\widetilde{\boldsymbol{LnU}}$ (if the energy is considered per unit of reference volume).

In the following particular case of potential

$$W(\theta, \widetilde{\boldsymbol{LnU}}) = \frac{\lambda(\theta)}{2} \left(\mathrm{Tr}\widetilde{\boldsymbol{LnU}}\right)^2 + \mu(\theta)\,\mathrm{Tr}(\widetilde{\boldsymbol{LnU}} \boldsymbol{.} \widetilde{\boldsymbol{LnU}}) \tag{5.112}$$

its partial derivative

$$\left(\frac{\widetilde{\boldsymbol{\partial W}}}{\boldsymbol{\partial}\widetilde{\boldsymbol{LnU}}}\right)_{/\boldsymbol{\theta}}(\theta, \widetilde{\boldsymbol{LnU}}) = \lambda(\theta) \left(\mathrm{Tr}\widetilde{\boldsymbol{LnU}}\right) \widetilde{\boldsymbol{I}} + 2\mu(\theta)\,\widetilde{\boldsymbol{LnU}} \tag{5.113}$$

Then,

$$\begin{aligned}
\forall(\theta, \widetilde{\boldsymbol{LnU}}) \quad , \quad \widetilde{\boldsymbol{S}} = &\\
\widetilde{\boldsymbol{U}}^{-\mathbf{1}} \boldsymbol{.} &\left[\widetilde{\gamma}^{\mathbf{4}}_{(\widetilde{\boldsymbol{U}})} : \left\{\lambda(\theta) \left(\mathrm{Tr}\widetilde{\boldsymbol{LnU}}\right) \widetilde{\boldsymbol{I}} + 2\mu(\theta)\,\widetilde{\boldsymbol{LnU}}\right\}\right] \boldsymbol{.} \widetilde{\boldsymbol{U}}^{-\mathbf{1}} \\
&= \widetilde{\boldsymbol{U}}^{-\mathbf{1}} \boldsymbol{.} \left[\lambda(\theta) \left(\mathrm{Tr}\widetilde{\boldsymbol{LnU}}\right) \widetilde{\boldsymbol{I}} + 2\mu(\theta)\,\widetilde{\boldsymbol{LnU}}\right] \boldsymbol{.} \widetilde{\boldsymbol{U}}^{-\mathbf{1}}
\end{aligned} \tag{5.114}$$

and,

$$\begin{aligned}
\forall(\theta, \widetilde{\boldsymbol{LnU}}) \quad , \quad J.\widetilde{\boldsymbol{T}} = &\\
\widetilde{\gamma}^{\mathbf{4}}_{(\widetilde{\boldsymbol{V}})} : &\left[\widetilde{\boldsymbol{R}} \boldsymbol{.} \left\{\lambda(\theta) \left(\mathrm{Tr}\widetilde{\boldsymbol{LnU}}\right) \widetilde{\boldsymbol{I}} + 2\mu(\theta)\,\widetilde{\boldsymbol{LnU}}\right\} \boldsymbol{.} \widetilde{\boldsymbol{R}}^{\mathbf{T}}\right] \\
&= \widetilde{\gamma}^{\mathbf{4}}_{(\widetilde{\boldsymbol{V}})} : \left[\lambda(\theta) \left(\mathrm{Tr}\widetilde{\boldsymbol{LnV}}\right) \widetilde{\boldsymbol{I}} + 2\mu(\theta)\,\widetilde{\boldsymbol{LnV}}\right] \\
&= \lambda(\theta) \left(\mathrm{Tr}\widetilde{\boldsymbol{LnV}}\right) \widetilde{\boldsymbol{I}} + 2\mu(\theta)\,\widetilde{\boldsymbol{LnV}}
\end{aligned} \tag{5.115}$$

Chapitre 6

Reference frames

Let us condiser the following straightforward way for modelling mechanical behaviors. We observe the matter, its movements and the forces applied on it ; the result is the time evolutions of the strain and the stress. Based on these observations, and possibly on mechanical phenomena at the microscopic scale, we propose a constitutive law in agreement with these.

When we observe the matter, actually a choice is made implicitly ; the choice of the reference frame. So, let us describe again this process, by mentioning this choice explicitly :

1. we choose a reference frame,
2. we observe the matter, i.e. its strain and stress,
3. we propose a constitutive law, which is a relation between the time evolutions of the strain and the stress, in agreement with the experimental observations.

Then, this natural process involves the concept of reference frame, and so requires to define it.

6.1 Sensitization to and familiar reference frame

For sailing, beginners learn what are the true and apparent winds. The true wind is felt when we are onshore or fixed with the seabed. The apparent wind is felt by all people on board of a moving sailboat. Indeed, the boat speed produces a wind that is added to the true wind, whose result is the apparent wind. For the crew member who controls a sail, only the apparent wind is sufficient to know. That is to say that the relevant reference frame is the one fixed with the sail hull, and not the one of the shore or seabed ground. This concept of reference frame is exactly the same as the one required in finite transformations. Moreover, in proposing constitutive laws in finite transformation, some reference frames are more relevant that others.

The reference frames, already defined by the academic lectures about the mechanics of material points or of rigid bodies, are exactly from the same concept as the one of the mechanics of deformable bodies in finite transformation. Let us recall, that, in this field, the material point velocity is always defined from a specified reference frame :

$$\overrightarrow{V(M)_{/R_0}} \ , \tag{6.1}$$

is the velocity of point M from reference frame R_0.

The point velocity changes when the reference frame is changed from R_0 to R_1 ;

$$\overrightarrow{V(M)_{/R_0}} = \overrightarrow{V(M)_{/R_1}} + \overrightarrow{V(O_1)_{/R_0}} + \overrightarrow{\Omega_{R_1/R_0}} \wedge \overrightarrow{O_1M} \ , \tag{6.2}$$

where O_1 is a fixed point of reference frame R_1. Quantities $\overrightarrow{V(O_1)_{/R_0}}$ and $\overrightarrow{\Omega_{R_1/R_0}}$ describe respectively the rigid translation movement and the rigid rotation of frame R_1 from reference frame R_0.

6.2 Definition of the reference frames

Section 2 of [DelPiero-2018] proposes a definition of the reference frames based on an observer who can measure the distances between two any material points of any body (fig. 6.1, 6.2).

Let us consider a close system, i.e. that does not exchange any matter with its surroundings.

This system is called *body $\mathcal{B}$*. Body $\mathcal{B}$ is not a set of three-dimensionnal positions of material points. The body is a set of material points or particles M. This set has a very low or simple mathematical structure ; notably, it is neither a vector space nor affine space. But connexions between the material points exists into this body, whatever their positions into a three-dimensional vector space. This is like the element definition (node list of each element) of a finite-element model mesh ; this element definition is done whatever the positions of the nodes.

The choice of the *observer* that measure distances between material points is the fundamental basis, the foundation of the reference frame and later the strain measure and all the mechanics.

2 Geometry of the Classical Continuum

A *body* is a set $\mathcal{B}$ whose elements are the *material points*. It is assumed that an *observer* can measure the distances $\mathcal{D}(X, X_\emptyset)$ between pairs $(X, X_\emptyset)$ of material points. The distances are communicated to a *placer*, whose task is to place the material points on a three-dimensional Euclidean point space $\mathcal{E}$, in such a way that the Euclidean distances of their images $\chi(X)$, $\chi(X_\emptyset)$ coincide with the distances measured on the body

$$\big|\chi(X) - \chi(X_\emptyset)\big| = \mathcal{D}(X, X_\emptyset). \tag{1}$$

A map $\chi : \mathcal{B} \to \mathcal{E}$ which satisfies this condition is a *placement* of the body in $\mathcal{E}$ corresponding to the distance $\mathcal{D}$.

For two placements χ and χ', the map $f = \chi' \circ \chi^{-1}$ which with every point $x = \chi(X)$ of $\chi(\mathcal{B})$ associates the point $x' = \chi'(X)$ of $\chi'(\mathcal{B})$ is the *deformation* from χ to χ'. A deformation is supposed to be continuous and differentiable, so that the expansion

$$f(x) = f(x_\emptyset) + \nabla f(x_\emptyset)[x - x_\emptyset] + o\big(|x - x_\emptyset|\big) \quad \forall x \in \chi(\mathcal{B}) \tag{2}$$

holds. A placement is far from being determined by the distance function $\mathcal{D}$. Indeed, for any deformation f from $\chi(\mathcal{B})$ to $\mathcal{E}$ such that

$$\big|f(x) - f(x_\emptyset)\big| = |x - x_\emptyset| \quad \forall x, x_\emptyset \in \chi(\mathcal{B}), \tag{3}$$

FIGURE 6.1 – First part of section 2 of [DelPiero-2018].

the distances in $\chi(\mathcal{B})$ and those in $f(\chi(\mathcal{B}))$ are both equal to $\mathcal{D}$. The equivalence class of all placements corresponding to the same distance $\mathcal{D}$ is a *configuration* of the body. Then two placements belong to the same configuration if they differ by a distance-preserving deformation. By consequence, *configuration* can be considered as a synonym of *distance function*.[4]

From (2) and (3) for a distance-preserving deformation f we have

$$\big|\nabla f(x_\emptyset)[x - x_\emptyset]\big| = |x - x_\emptyset| + o\big(|x - x_\emptyset|\big) \quad \forall x, x_\emptyset \in \chi(\mathcal{B}). \tag{4}$$

From elementary tensor analysis it follows that this relation holds if and only if there is an orthogonal tensor Q, independent of x, such that

$$f(x) - f(x_\emptyset) = Q\,[x - x_\emptyset] \quad \forall x, x_\emptyset \in \chi(\mathcal{B}). \tag{5}$$

Then every deformation f between placements belonging to the same configuration is determined by a *rigid translation* $f(x_\emptyset)$ and a *rigid rotation* Q.

A way for selecting a representative in each equivalence class is to take a *reference frame*, that is, a set of points, not necessarily belonging to $\mathcal{B}$, such that the position in $\mathcal{E}$ of every point of $\mathcal{B}$ is uniquely determined by its distances from the points of the reference frame.[5]

A distance function and a reference frame determine a *reference placement* χ_R, which can be considered as the representative of the body on $\mathcal{E}$. For every pair X, $X_\emptyset$ of material points, the vector $e = \chi_R(X) - \chi_R(X_\emptyset)$ is the image of an array of material points, called a *material direction*. For any deformation f from χ_R the image $\nabla f[e]$ of e is the image of this material direction under the deformation f.

FIGURE 6.2 – Second part of section 2 of [DelPiero-2018].

How to choose this *measuring observer* in such a way that he is the relevant foundation for describing the mechanical behavior of parts and structures ?

One solution is to choose a piece of material that do not undergoes any thermomechanical event. No force is applied on it, and its temperature is homogeneous and constant along the time, in order to avoid any thermal expansion. In other words, this piece of material is a *prototype meter* (*mètre-étalon*) as the one of the *Metre Convention*, an international treaty that was signed in Paris on 20 May 1875.

If we assume, that this prototype meter could be translated and rotated in order to measure the distance between any pair of material points *instantaneously*, without any thermomechanical event inside it, the *measuring observer* may be carried out by this way.

The term *instantaneously* is necessary, because in the following we consider that this distance measure is available between all material points at one time.

Based on that, we will define, at a chosen time or date, a reference frame, for instance a, in a slight different way, as a function from the material point set of a body to an euclidian vectorial space of their positions :

$$\mathcal{B} \xrightarrow{a} \mathcal{P} \tag{6.3}$$
$$M \longrightarrow \vec{x}^{\backslash a} \tag{6.4}$$

Euclidian means that a scalar product is chosen and may give the lengths and angles between these three-dimensionnal positions.

A reference frame is not unique. Another reference frame may be chosen :

$$\mathcal{B} \xrightarrow{r} \mathcal{P} \tag{6.5}$$
$$M \longrightarrow \vec{x}^{\backslash r} \tag{6.6}$$

But between two reference frames, the following relations exits, at a chosen time :

$$\forall M \in \mathcal{B} \qquad \vec{x}^{\backslash r}(M) = \left(\widetilde{\boldsymbol{R}^{ar}}\right)^{\mathbf{T}} \cdot \vec{x}^{\backslash a}(M) - t^{\vec{ar}} \tag{6.7}$$

Between these body positions from two reference frames, we observe a translation, described by $-t^{\vec{ar}}$, and a rotation $\left(\widetilde{\boldsymbol{R}^{ar}}\right)^{\mathbf{T}}$.

The rotation $\widetilde{\boldsymbol{R}^{ar}}$ will be called the rotation of frame r from frame a, and $t^{\vec{ar}}$ will be called the translation of frame r from frame a.

Properties –

$$\widetilde{\boldsymbol{R}^{aa}} = \widetilde{\boldsymbol{I}} \qquad t^{\vec{aa}} = \vec{0} \tag{6.8}$$

$$\widetilde{\boldsymbol{R}^{ra}} = \left(\widetilde{\boldsymbol{R}^{ar}}\right)^{\mathbf{T}} \qquad \vec{t^{ra}} = -\widetilde{\boldsymbol{R}^{ar}} \cdot \vec{t^{ar}} \tag{6.9}$$

$$\widetilde{\boldsymbol{R}^{ar}} = \widetilde{\boldsymbol{R}^{ab}} \cdot \widetilde{\boldsymbol{R}^{br}} \qquad \vec{t^{ar}} = \widetilde{\boldsymbol{R}^{rb}} \cdot \vec{t^{ab}} + \vec{t^{br}} \tag{6.10}$$

Then, the velocity vector is impacted as follows, by the reference frame change :

$$\begin{aligned}
\forall M \in \mathcal{B} \qquad \vec{v}^{\backslash r}(M) &= \left(\widetilde{\boldsymbol{R}^{ar}}\right)^{\mathbf{T}} \cdot \vec{v}^{\backslash a}(M) \\
&\quad + \left(\dot{\widetilde{\boldsymbol{R}^{ar}}}\right)^{\mathbf{T}} \cdot \vec{x}^{\backslash a}(M) - \dot{\vec{t^{ar}}} && (6.11) \\
&= \widetilde{\boldsymbol{R}^{ra}} \cdot \vec{v}^{\backslash a}(M) \\
&\quad + \widetilde{\boldsymbol{\Omega}^{ra}} \cdot \left(\vec{x}^{\backslash r}(M) + \vec{t^{ar}}\right) - \dot{\vec{t^{ar}}} && (6.12)
\end{aligned}$$

where $\widetilde{\boldsymbol{\Omega}^{ra}}$ is the rotation velocity of frame a from reference frame r (see eq. 6.55 page 104). Moreover, the acceleration vector is impacted as follows, by the reference frame change :

$$\begin{aligned}
\forall M \in \mathcal{B} \qquad \vec{a}^{\backslash r}(M) &= \widetilde{\boldsymbol{R}^{ra}} \cdot \vec{a}^{\backslash a}(M) \\
&\quad + \left(\dot{\widetilde{\boldsymbol{\Omega}^{ra}}} - \widetilde{\boldsymbol{\Omega}^{ra}} \cdot \widetilde{\boldsymbol{\Omega}^{ra}}\right) \cdot \left(\vec{x}^{\backslash r}(M) + \vec{t^{ar}}\right) \\
&\quad - \ddot{\vec{t^{ar}}} + 2\widetilde{\boldsymbol{\Omega}^{ra}} \cdot \left(\vec{v}^{\backslash r}(M) + \dot{\vec{t^{ar}}}\right) && (6.13)
\end{aligned}$$

Let us choose two material points, M and M', infinitely near, one to the other :

$$\begin{aligned}
\forall MM' \in \mathcal{B} \qquad d\vec{x}^{\backslash r}(MM') &= \left(\widetilde{\boldsymbol{R}^{ar}}\right)^{\mathbf{T}} \cdot d\vec{x}^{\backslash a}(MM') && (6.14) \\
&= \widetilde{\boldsymbol{R}^{ra}} \cdot d\vec{x}^{\backslash a}(MM') && (6.15)
\end{aligned}$$

The reference frame defines the material positions into space $\mathcal{P}$, but do not impact the masses of the body. Although this is not necessary, only to be clear, we write :

$$m[\mathcal{D}] = m^{\backslash r}[\mathcal{D}] = m^{\backslash a}[\mathcal{D}] \tag{6.16}$$

for all part $\mathcal{D}$ of body $\mathcal{B}$. And for a material particle at material point M of the body $\mathcal{B}$, the infinitesimal mass of this particle is the same from all reference frames :

$$dm = dm^{\backslash r} = dm^{\backslash a} \tag{6.17}$$

Because relation 6.7 is an isometric transformation, the lengths, the surfaces and the volumes remain the same :

$$s^{\backslash r}[\mathcal{D}] = s^{\backslash a}[\mathcal{D}] \tag{6.18}$$

$$\mathrm{v}^{\backslash r}[\mathcal{D}] = \mathrm{v}^{\backslash a}[\mathcal{D}] \tag{6.19}$$

for all part $\mathcal{D}$ of body $\mathcal{B}$. Notably, this is true for the infinitesimal volume of a material particle :

$$d\mathrm{v}^{\backslash r} = d\mathrm{v}^{\backslash a} \tag{6.20}$$

And then

$$d\mathrm{v}^{\backslash r} = d\mathrm{v}^{\backslash a} \tag{6.21}$$

$$\vec{ds}^{\backslash r} \,\boldsymbol{.}\, d\vec{x}^{\backslash r} = \vec{ds}^{\backslash a} \,\boldsymbol{.}\, d\vec{x}^{\backslash a} \tag{6.22}$$

$$\vec{ds}^{\backslash r} \,\boldsymbol{.}\, \left(\widetilde{\boldsymbol{R^{ra}}} \,\boldsymbol{.}\, d\vec{x}^{\backslash a}\right) = \vec{ds}^{\backslash a} \,\boldsymbol{.}\, d\vec{x}^{\backslash a} \tag{6.23}$$

$$d\vec{x}^{\backslash a} \,\boldsymbol{.}\, \left(\widetilde{\boldsymbol{R^{ar}}} \,\boldsymbol{.}\, \vec{ds}^{\backslash r}\right) = \vec{ds}^{\backslash a} \,\boldsymbol{.}\, d\vec{x}^{\backslash a} \tag{6.24}$$

Because this is true for all $d\vec{x}\,(MM')$, we obtain :

$$\widetilde{\boldsymbol{R^{ar}}} \,\boldsymbol{.}\, \vec{ds}^{\backslash r} = \vec{ds}^{\backslash a} \tag{6.25}$$

Or

$$\vec{ds}^{\backslash r} = \widetilde{\boldsymbol{R^{ra}}} \,\boldsymbol{.}\, \vec{ds}^{\backslash a} \tag{6.26}$$

Assumption – *We will assume that the forces satisfy :*

$$\vec{F}^{\backslash r} = \widetilde{\boldsymbol{R^{ra}}} \,\boldsymbol{.}\, \vec{F}^{\backslash a} \tag{6.27}$$

Assumption – *We will assume that the temperatures satisfy :*

$$\theta^{\backslash r} = \theta^{\backslash a} \tag{6.28}$$

According to the present definition of the reference frames, the position of a body may be defined from several different reference frames at one same time. This means that the time is the same from all reference frames :

$$t = t^{\backslash r} = t^{\backslash a} \tag{6.29}$$

This means that we adopted the framework of the newtonian mechanics and not the one of the relativistic mechanics.

6.3 Objective or invariant vector, tensor

During the present lecture, we have defined physical quantities of different mathematical types ; scalar, vector or tensor. These quantities were defined by equation or function of other already defined quantities. For example, the stretching tensor $\widetilde{\boldsymbol{D}}$.

A lot of quantities, defined before this chapter about reference frames, required the choice of a reference frame. But we have not explicitly mentioned this choice. Actually, we have chosen this reference frame arbitrarily.

This arbitrarily chosen frame was called reference frame a.

What would happen if we had chosen an other different reference frame arbitrarily ?

Let us call this other possibly-chosen reference frame b.

What are the differences, or possibly the relations, between the quantities defined from the choices a and b of the reference frame ?

The movement of frame b from reference frame a can be described by rotation

$$\widetilde{\boldsymbol{R}^{\boldsymbol{ab}}}_{(t)} \ , \tag{6.30}$$

that is a function of time.

For the sake of simplicity and without any loss in generality range, we may assume, that at initial time $t = 0$

$$\widetilde{\boldsymbol{R}^{\boldsymbol{ab}}}_{(0)} = \widetilde{\boldsymbol{I}} \tag{6.31}$$

We write $\vec{a}^{\backslash a}$ and $\vec{a}^{\backslash b}$ the vectors defined on the basis of the reference frame choices a and b, respectively. So do we for the tensors ; $\widetilde{\boldsymbol{A}}^{\backslash a}$ and $\widetilde{\boldsymbol{A}}^{\backslash b}$. And for the scalar quantities ; $\alpha^{\backslash a}$ and $\alpha^{\backslash b}$.

Let us define what are **objective vector and tensor**, for any choices of two reference frames a and b, as follows :

$$\forall t \quad \vec{a}^{\backslash b}_{(t)} = \widetilde{\boldsymbol{R}^{\boldsymbol{ba}}}_{(t)} \cdot \vec{a}^{\backslash a}_{(t)} \tag{6.32}$$

$$\forall t \quad \widetilde{\boldsymbol{A}}^{\backslash b}_{(t)} = \widetilde{\boldsymbol{R}^{\boldsymbol{ba}}}_{(t)} \cdot \widetilde{\boldsymbol{A}}^{\backslash a}_{(t)} \cdot \widetilde{\boldsymbol{R}^{\boldsymbol{ab}}}_{(t)} \tag{6.33}$$

We also define **invariant scalar, vector and tensor**, for any choices of two reference frames a and b, as follows :

$$\begin{aligned}
\forall t \quad \alpha^{\backslash b}_{(t)} &= \alpha^{\backslash a}_{(t)} && (6.34)\\
\forall t \quad \vec{a}^{\backslash b}_{(t)} &= \widetilde{\boldsymbol{R^{ba}}}_{(0)} \cdot \vec{a}^{\backslash a}_{(t)} = \vec{a}^{\backslash a}_{(t)} && (6.35)\\
\forall t \quad \widetilde{\boldsymbol{A}}^{\backslash b}_{(t)} &= \widetilde{\boldsymbol{R^{ba}}}_{(0)} \cdot \widetilde{\boldsymbol{A}}^{\backslash a}_{(t)} \cdot \widetilde{\boldsymbol{R^{ab}}}_{(0)} = \widetilde{\boldsymbol{A}}^{\backslash a}_{(t)} && (6.36)\\
& && (6.37)
\end{aligned}$$

Exercise 6.1. *Prove that*

$$\forall t \quad \widetilde{\boldsymbol{F}}^{\backslash b}_{(t)} = \widetilde{\boldsymbol{R^{ba}}}_{(t)} \cdot \widetilde{\boldsymbol{F}}^{\backslash a}_{(t)} \cdot \widetilde{\boldsymbol{R^{ab}}}_{(0)} = \widetilde{\boldsymbol{R^{ba}}}_{(t)} \cdot \widetilde{\boldsymbol{F}}^{\backslash a}_{(t)} \tag{6.38}$$

Exercise 6.2. *Is the transformation gradient $\widetilde{\boldsymbol{F}}$ objective, invariant ?*

Exercise 6.3. *Prove that the volume change J is invariant.*

Exercise 6.4. *Prove that the right Cauchy-Green strain tensor $\widetilde{\boldsymbol{C}}$ is invariant.*

Exercise 6.5. *Prove that the left Cauchy-Green strain tensor $\widetilde{\boldsymbol{B}}$ is objective.*

Exercise 6.6. *Prove that the right pure strain tensor $\widetilde{\boldsymbol{U}}$ is invariant.*

Exercise 6.7. *Prove that the left pure strain tensor $\widetilde{\boldsymbol{V}}$ is objective.*

Exercise 6.8. *Prove that the Cauchy stress tensor $\widetilde{\boldsymbol{T}}$ is objective.*

Exercise 6.9. *Prove that the Piola-Kirchhoff 2 stress tensor $\widetilde{\boldsymbol{S}}$ is invariant.*

Exercise 6.10. *Prove that the stretching tensor $\widetilde{\boldsymbol{D}}$ is objective.*

Exercise 6.11. *Prove that the norm of an* ***invariant or objective*** *vector, defined as*

$$||\vec{a}|| = \sqrt{\vec{a} \cdot \vec{a}}\ , \tag{6.39}$$

is invariant.

Exercise 6.12. *Prove that the trace of an* ***invariant or objective*** *second-order tensor is invariant.*

Exercise 6.13. *Let $\widetilde{\boldsymbol{A}}$ be an* ***invariant or objective*** *second-order tensor. Prove that the following scalars*

$$\mathrm{Tr}\left(\widetilde{\boldsymbol{A}} \cdot \widetilde{\boldsymbol{A}}\right), \qquad \mathrm{Tr}\left(\widetilde{\boldsymbol{A}} \cdot \widetilde{\boldsymbol{A}} \cdot \widetilde{\boldsymbol{A}}\right), \qquad \mathrm{Tr}\left(\widetilde{\boldsymbol{A}}^{\mathbf{T}} \cdot \widetilde{\boldsymbol{A}}\right) \tag{6.40}$$

are invariant.

Exercise 6.14. *Let* $\widetilde{\boldsymbol{A}}$ *be an* ***invariant or objective*** *second-order tensor. Prove that the following scalars*

$$\sec\widetilde{\boldsymbol{A}} \qquad \det\widetilde{\boldsymbol{A}} \tag{6.41}$$

are invariant.

Exercise 6.15. *Let be* $\widetilde{\boldsymbol{A}}$ *and* $\widetilde{\boldsymbol{B}}$ *two* ***both invariant*** *second-order tensors, or two* ***both objective*** *second-order tensors. Prove that the following scalar*

$$\mathrm{Tr}\left(\widetilde{\boldsymbol{A}}\,\boldsymbol{.}\,\widetilde{\boldsymbol{B}}\right), \tag{6.42}$$

is invariant.

Exercise 6.16. *Let us propose the following elastic constitutive law*

$$\widetilde{\boldsymbol{T}}^{\backslash a} = \lambda.\mathrm{Tr}\left(\widetilde{\boldsymbol{J}}^{\backslash a}\right).\widetilde{\boldsymbol{I}} + 2\mu.\widetilde{\boldsymbol{J}}^{\backslash a} \ . \tag{6.43}$$

Write this law from an other different reference frame r*, i.e. an equation involving quantities from frame* r *only* $(-^{\backslash r})$*.*

6.4 Reference frame of the proper rotation

Definition – *We define the reference frame of the proper rotation «rp», as*

$$\widetilde{\boldsymbol{R}^{arp}} = \widetilde{\boldsymbol{R}}^{\backslash a} \ , \tag{6.44}$$

where $\widetilde{\boldsymbol{R}}$ *is the proper rotation tensor obtained from the polar decomposition of the transformation gradient* $\widetilde{\boldsymbol{F}}$ *(chap. 2).*

Property –

$$\widetilde{\boldsymbol{U}} = \widetilde{\boldsymbol{U}}^{\backslash a} = \widetilde{\boldsymbol{V}}^{\backslash rp} \ ; \tag{6.45}$$

the right pure strain $\widetilde{\boldsymbol{U}}$ *from the arbitrarily-chosen base reference frame* a *is the same tensor as the left pure strain tensor* $\widetilde{\boldsymbol{V}}$ *from the reference frame of the proper rotation.*

Indeed,

$$\widetilde{\boldsymbol{V}}^{\backslash rp} = \widetilde{\boldsymbol{R}^{arp}}^{\mathbf{T}}\boldsymbol{.}\,\widetilde{\boldsymbol{V}}^{\backslash a}\boldsymbol{.}\,\widetilde{\boldsymbol{R}^{arp}} \tag{6.46}$$

$$= \widetilde{\boldsymbol{R}}^{\backslash a\,\mathbf{T}}\boldsymbol{.}\,\widetilde{\boldsymbol{V}}^{\backslash a}\boldsymbol{.}\,\widetilde{\boldsymbol{R}}^{\backslash a} \tag{6.47}$$

since $\widetilde{\boldsymbol{F}}^{\backslash a} = \widetilde{\boldsymbol{V}}^{\backslash a} \boldsymbol{.} \widetilde{\boldsymbol{R}}^{\backslash a} = \widetilde{\boldsymbol{R}}^{\backslash a} \boldsymbol{.} \widetilde{\boldsymbol{U}}^{\backslash a}$,

$$\widetilde{\boldsymbol{V}}^{\backslash rp} = \widetilde{\boldsymbol{R}}^{\backslash a\mathbf{T}} \boldsymbol{.} \left(\widetilde{\boldsymbol{R}}^{\backslash a} \boldsymbol{.} \widetilde{\boldsymbol{U}}^{\backslash a}\right) \tag{6.48}$$

$$= \widetilde{\boldsymbol{I}} \boldsymbol{.} \widetilde{\boldsymbol{U}}^{\backslash a} \tag{6.49}$$

$$= \widetilde{\boldsymbol{U}}^{\backslash a} . \tag{6.50}$$

This property is also true for other strain tensors ;

properties –

$$\widetilde{\boldsymbol{P}} = \widetilde{\boldsymbol{P}}^{\backslash a} = \widetilde{\boldsymbol{J}}^{\backslash rp} , \tag{6.51}$$

$$\widetilde{\boldsymbol{C}} = \widetilde{\boldsymbol{C}}^{\backslash a} = \widetilde{\boldsymbol{B}}^{\backslash rp} , \tag{6.52}$$

$$\widetilde{\boldsymbol{E_n}} = \widetilde{\boldsymbol{E_n}}^{\backslash a} = \widetilde{\boldsymbol{e_n}}^{\backslash rp} ; \tag{6.53}$$

These right strain tensors from the base reference frame are identical to their left counterpart tensors from the frame of the proper rotation.

Exercise 6.17. *Prove the preceding equalities.*

6.5 Rotation velocity of a reference frame

Let us recall that the rotation of a frame r from another reference frame a is written as

$$\widetilde{\boldsymbol{R}^{ar}} . \tag{6.54}$$

We define the rotation velocity of r from a, as follows :

definition – *the rotation velocity of the reference frame r from frame a is defined by the following skew-symmetric tensor*

$$\widetilde{\boldsymbol{\Omega}^{ar}} = \dot{\widetilde{\boldsymbol{R}^{ar}}} \boldsymbol{.} \widetilde{\boldsymbol{R}^{ar}}^{\mathbf{T}} = -\widetilde{\boldsymbol{R}^{ar}} \boldsymbol{.} \dot{\widetilde{\boldsymbol{R}^{ar}}}^{\mathbf{T}} , \tag{6.55}$$

where $\dot{\widetilde{\boldsymbol{A}}}$ is the time derivative of tensor $\widetilde{\boldsymbol{A}}$;

$$\dot{\widetilde{\boldsymbol{A}}} = \lim_{dt \to 0} \left[\frac{\widetilde{\boldsymbol{A}}_{(t+dt)} - \widetilde{\boldsymbol{A}}_{(t)}}{dt} \right] . \tag{6.56}$$

Exercise 6.18. *Prove that tensor $\widetilde{\boldsymbol{\Omega}^{ar}}$ is skew-symmetric, i.e.*

$$\widetilde{\boldsymbol{\Omega}^{ar}}^{\mathbf{T}} = -\widetilde{\boldsymbol{\Omega}^{ar}} . \tag{6.57}$$

Properties –

$$\widetilde{\boldsymbol{\Omega}^{ar}} = -\widetilde{\boldsymbol{R}^{ar}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{\Omega}^{ra}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{R}^{ar}}^{\mathbf{T}} \tag{6.58}$$

$$\widetilde{\boldsymbol{\Omega}^{ar}} = \widetilde{\boldsymbol{\Omega}^{ab}} + \widetilde{\boldsymbol{R}^{ab}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{\Omega}^{br}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{R}^{ab}}^{\mathbf{T}} \tag{6.59}$$

6.6 The corotational reference frame j

Definition – *The corotational reference frame j is the frame such that its rotation velocity is*

$$\widetilde{\boldsymbol{\Omega}^{aj}} = \widetilde{\boldsymbol{W}}^{\backslash a} , \tag{6.60}$$

that is to say, such that :

$$\widetilde{\boldsymbol{R}^{aj}}_{(t_0)} = \widetilde{\boldsymbol{I}} \tag{6.61}$$

and

$$\forall t, \qquad \overset{\bullet}{\widetilde{\boldsymbol{R}^{aj}}}_{(t)} = \widetilde{\boldsymbol{W}}^{\backslash a}_{(t)} \,\boldsymbol{.}\, \widetilde{\boldsymbol{R}^{aj}}_{(t)} \tag{6.62}$$

Property – *In the cases of uniaxial compression-tension test and of bi-axial tension test, if the base reference frame a is fixed to the test machine rigid frame, we have*

$$\forall t, \qquad \widetilde{\boldsymbol{R}^{arp}}_{(t)} = \widetilde{\boldsymbol{R}^{aj}}_{(t)} = \widetilde{\boldsymbol{I}} \ , \tag{6.63}$$

that is to say, that the three reference frame a, rp and j are the same frame.

Exercise 6.19. *Calculate the rotation of the corotational reference frame j during a simple shear test, until ultra-high shear strains.*

Prove that the rotation angle in radian unit, is equal to the classical shear strain $\frac{\gamma}{2}$, regardless of its sign.

6.7 Transformation en cinématique irrotationnelle

Définition – *Nous définissons les transformations* $\widetilde{\boldsymbol{F}}$ ***en cinématique irrotationnelle****, comme celles pour lesquelles les directions principales de* $\widetilde{\boldsymbol{U}}$ *sont fixes (dans le référentiel de base) :*

$$\overset{\bullet}{\vec{u}_i} = \vec{0}, \qquad i = 1, 2, 3. \tag{6.64}$$

Propriété – *En cinématique irrotationnelle,*

$$\widetilde{\boldsymbol{\Omega}^{arp}} = \widetilde{\boldsymbol{\Omega}^{aj}} \ , \tag{6.65}$$

les vitesses de rotation des deux référentiels rp et j sont identiques.

En effet,

$$\widetilde{\boldsymbol{F}} = \widetilde{\boldsymbol{R}} \,.\, \widetilde{\boldsymbol{U}} \tag{6.66}$$

$$\Longleftrightarrow \qquad \overset{\bullet}{\widetilde{\boldsymbol{F}}} = \overset{\bullet}{\widetilde{\boldsymbol{R}}} \,.\widetilde{\boldsymbol{U}} + \widetilde{\boldsymbol{R}}.\overset{\bullet}{\widetilde{\boldsymbol{U}}} \tag{6.67}$$

$$\widetilde{\boldsymbol{L}} \,.\, \widetilde{\boldsymbol{F}} = \tag{6.68}$$

$$\widetilde{\boldsymbol{L}} \,.\, \widetilde{\boldsymbol{R}} \,.\, \widetilde{\boldsymbol{U}} = \tag{6.69}$$

$$\Longleftrightarrow \qquad \widetilde{\boldsymbol{L}} = \overset{\bullet}{\widetilde{\boldsymbol{R}}} \,.\widetilde{\boldsymbol{R}}^{\mathbf{T}} + \widetilde{\boldsymbol{R}}.\overset{\bullet}{\widetilde{\boldsymbol{U}}} \,.\widetilde{\boldsymbol{U}}^{-1} \,.\, \widetilde{\boldsymbol{R}}^{\mathbf{T}} \tag{6.70}$$

Calculons le taux de rotation

$$\widetilde{\boldsymbol{W}} = \frac{1}{2}\left(\widetilde{\boldsymbol{L}} - \widetilde{\boldsymbol{L}}^{\mathbf{T}}\right) \tag{6.71}$$

$$= \overset{\bullet}{\widetilde{\boldsymbol{R}}} \,.\widetilde{\boldsymbol{R}}^{\mathbf{T}} + \frac{1}{2}\widetilde{\boldsymbol{R}} \,.\left(\overset{\bullet}{\widetilde{\boldsymbol{U}}} \,.\widetilde{\boldsymbol{U}}^{-1} - \widetilde{\boldsymbol{U}}^{-1} \,.\, \overset{\bullet}{\widetilde{\boldsymbol{U}}}\right) .\, \widetilde{\boldsymbol{R}}^{\mathbf{T}} \tag{6.72}$$

D'autre part, la matrice du tenseur $\widetilde{\boldsymbol{U}}$ dans sa base de vecteurs propres est

$$Mat_{\vec{u_i}}\left(\widetilde{\boldsymbol{U}}\right) = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} . \tag{6.73}$$

En cinématique irrotationnelle, la base $(\vec{u_1}, \vec{u_2}, \vec{u_3})$ est fixe donc on peut dériver les composantes de la matrice précédente pour avoir la matrice de la dérivée du tenseur ;

$$Mat_{\vec{u_i}}\left(\overset{\bullet}{\widetilde{\boldsymbol{U}}}\right) = \left[Mat_{\vec{u_i}}\left(\widetilde{\boldsymbol{U}}\right)\right]^{\cdot} = \begin{bmatrix} \dot{\lambda}_1 & 0 & 0 \\ 0 & \dot{\lambda}_2 & 0 \\ 0 & 0 & \dot{\lambda}_3 \end{bmatrix} . \tag{6.74}$$

Ainsi,

$$Mat_{\vec{u_i}}\left(\overset{\bullet}{\widetilde{\boldsymbol{U}}}\,.\widetilde{\boldsymbol{U}}\right) = \begin{bmatrix} \dot{\lambda_1} & 0 & 0 \\ 0 & \dot{\lambda_2} & 0 \\ 0 & 0 & \dot{\lambda_3} \end{bmatrix} \times \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{6.75}$$

$$= \begin{bmatrix} \dot{\lambda_1}.\lambda_1 & 0 & 0 \\ 0 & \dot{\lambda_2}.\lambda_2 & 0 \\ 0 & 0 & \dot{\lambda_3}.\lambda_3 \end{bmatrix} \tag{6.76}$$

$$= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \times \begin{bmatrix} \dot{\lambda_1} & 0 & 0 \\ 0 & \dot{\lambda_2} & 0 \\ 0 & 0 & \dot{\lambda_3} \end{bmatrix} \tag{6.77}$$

$$= Mat_{\vec{u_i}}\left(\widetilde{\boldsymbol{U}}.\,\overset{\bullet}{\widetilde{\boldsymbol{U}}}\right) . \tag{6.78}$$

Donc, en cinématique irrotationnelle,

$$\overset{\bullet}{\widetilde{\boldsymbol{U}}}\,.\widetilde{\boldsymbol{U}} = \widetilde{\boldsymbol{U}}.\,\overset{\bullet}{\widetilde{\boldsymbol{U}}} \tag{6.79}$$

$$\Longleftrightarrow \qquad \widetilde{\boldsymbol{U}}^{-1}.\,\overset{\bullet}{\widetilde{\boldsymbol{U}}} = \overset{\bullet}{\widetilde{\boldsymbol{U}}}\,.\widetilde{\boldsymbol{U}}^{-1} . \tag{6.80}$$

Pour conclure, la relation 6.72 et la dernière égalité conduisent à

$$\widetilde{\boldsymbol{W}} = \overset{\bullet}{\widetilde{\boldsymbol{R}}}\,.\widetilde{\boldsymbol{R}}^{\mathbf{T}} \tag{6.81}$$

$$\Longleftrightarrow \qquad \widetilde{\boldsymbol{\Omega^{aj}}} = \widetilde{\boldsymbol{\Omega^{arp}}} . \tag{6.82}$$

6.8 Elastic law and reference frames

Let us consider the following elastic law

$$\widetilde{\boldsymbol{S}}^{\backslash r} = 2\mu.\widetilde{\boldsymbol{J}}^{\backslash r} \tag{6.83}$$

whatever the choice of reference frame r.

According to its definition, this law has this expression whatever the reference frame, so a user, so-called ua, may choose a reference frame a and another user, so-called ub, may choose a different reference frame b. So for user ua :

$$\widetilde{\boldsymbol{S}}^{\backslash a}_{ua} = 2\mu.\widetilde{\boldsymbol{J}}^{\backslash a}_{ua} \tag{6.84}$$

and for user ub :

$$\begin{aligned}
\widetilde{\boldsymbol{S}}_{ub}^{\backslash b} &= 2\mu.\widetilde{\boldsymbol{J}}_{ub}^{\backslash b} && (6.85)\\
\widetilde{\boldsymbol{S}}_{ub}^{\backslash a} &= 2\mu.\widetilde{\boldsymbol{R^{ba}}} \cdot \widetilde{\boldsymbol{J}}_{ub}^{\backslash a} \cdot \widetilde{\boldsymbol{R^{ab}}} && (6.86)\\
\widetilde{\boldsymbol{S}}_{ub}^{\backslash a} &= \widetilde{\boldsymbol{R^{ba}}} \cdot \left(2\mu.\widetilde{\boldsymbol{J}}_{ub}^{\backslash a}\right) \cdot \widetilde{\boldsymbol{R^{ab}}} && (6.87)
\end{aligned}$$

since $\widetilde{\boldsymbol{S}}$ is invariant and $\widetilde{\boldsymbol{J}}$ is objective.

Then, let us give to these two users the same deformation of a body along the time, and we ask them to calculate the stress according to this elastic law. So, their strain measure tensors will satisfy :

$$\widetilde{\boldsymbol{J}}_{ua}^{\backslash a} = \widetilde{\boldsymbol{J}}_{ub}^{\backslash a} = \widetilde{\boldsymbol{R^{ab}}} \cdot \widetilde{\boldsymbol{J}}_{ub}^{\backslash b} \cdot \widetilde{\boldsymbol{R^{ba}}} = \widetilde{\boldsymbol{J}}^{\backslash a} \tag{6.88}$$

And, user ua will calculate the following stress :

$$\widetilde{\boldsymbol{S}}_{ua}^{\backslash a} = 2\mu.\widetilde{\boldsymbol{J}}^{\backslash a} \tag{6.89}$$

while user ub will calculate the following one :

$$\widetilde{\boldsymbol{S}}_{ub}^{\backslash a} = \widetilde{\boldsymbol{R^{ba}}} \cdot \left(2\mu.\widetilde{\boldsymbol{J}}^{\backslash a}\right) \cdot \widetilde{\boldsymbol{R^{ab}}} \tag{6.90}$$

So,

$$\widetilde{\boldsymbol{S}}_{ua}^{\backslash a} \neq \widetilde{\boldsymbol{S}}_{ub}^{\backslash a} \tag{6.91}$$

We can conclude that the choice of the reference frame has an impact on or changes the behavior according to this law. The choice of the reference frame is not specified, so this choice is arbitrary. It means that the behavior of this law will be arbitrary or impacted by an hasardous parameter, that is the reference frame.

So, this elastic law is **problematic**.

This problem will not arise with the following elastic law :

$$\widetilde{\boldsymbol{S}}^{\backslash r} = 2\mu.\widetilde{\boldsymbol{P}}^{\backslash r} \tag{6.92}$$

whatever the choice of reference frame r, since $\widetilde{\boldsymbol{S}}$ and $\widetilde{\boldsymbol{P}}$ are both invariants,

or this one :

$$\widetilde{\boldsymbol{T}}^{\backslash r} = 2\mu.\widetilde{\boldsymbol{J}}^{\backslash r} \tag{6.93}$$

whatever the choice of reference frame r, since $\widetilde{\boldsymbol{T}}$ and $\widetilde{\boldsymbol{J}}$ are both objectives and since the function is isotropic.

In conclusion, when one defines an elastic law whatever the choice of the reference frame, it is mandatory to check that the reference frame choice has no impact on the behavior of the law.

Another way to define an elastic law is to specify a choice of reference frame, for example like this :

$$\widetilde{\boldsymbol{S}}^{\backslash rp} = 2\mu.\widetilde{\boldsymbol{J}}^{\backslash rp} \tag{6.94}$$

where the reference frame rp is the one in proper rotation.

This law is the same as

$$\widetilde{\boldsymbol{S}}^{\backslash a} = 2\mu.\widetilde{\boldsymbol{P}}^{\backslash a} \tag{6.95}$$

since

$$\widetilde{\boldsymbol{S}}^{\backslash rp} = \widetilde{\boldsymbol{S}}^{\backslash a} \qquad \widetilde{\boldsymbol{J}}^{\backslash rp} = \widetilde{\boldsymbol{P}}^{\backslash a} \tag{6.96}$$

Chapitre 7

Objective rates

Some materials require constitutive laws involving strain and stress rates.

For example, the polymer materials include generally a great amount of viscosity. The Maxwell law may modelise viscoelastic behaviors. In one dimension, it is the combination of an purely elastic element ($\sigma = E.\varepsilon$) and a viscous element (**FIG.O1**).

The viscous element describes a purely viscous behavior ; $\sigma = \eta.\dot{\varepsilon}$, where η is the viscosity parameter and $\dot{\varepsilon}$ is the strain rate or time derivative. In three dimensions, a Maxwell law may be ;

$$\overset{\circ}{\widetilde{\boldsymbol{T}}} + \frac{2\mu}{\eta}.\widetilde{\boldsymbol{T}} = \lambda.\mathrm{Tr}\left(\widetilde{\overset{\circ}{\boldsymbol{LnV}}}\right).\widetilde{\boldsymbol{I}} + 2\mu.\widetilde{\overset{\circ}{\boldsymbol{LnV}}}\ , \tag{7.1}$$

where (λ, μ) are the Lamé coefficients of elasticity and η is the viscosity parameter. Or the other one :

$$\overset{\bullet}{\widetilde{\boldsymbol{S}}} + \frac{2\mu}{\eta}.\widetilde{\boldsymbol{S}} = \lambda.\mathrm{Tr}\left(\overset{\bullet}{\widetilde{\boldsymbol{E}}}\right).\widetilde{\boldsymbol{I}} + 2\mu.\overset{\bullet}{\widetilde{\boldsymbol{E}}}\ , \tag{7.2}$$

with the stress-strain couple $(\widetilde{\boldsymbol{S}}, \widetilde{\boldsymbol{E}})$.

An other example of constitutive law involving strain and stress rates are some elastoplastic laws.

Rigorously, these behaviors are time-independent and so their constitutive laws should require neither strain nor stress rate. But, practically, these laws are expressed by means of strain and stress rates.

An example is the elasto-plastic laws of hypo-elastic type : one of them is

$$\overset{\circ}{\widetilde{\boldsymbol{\tau}}} = \lambda.\mathrm{Tr}\widetilde{\boldsymbol{D}}.\widetilde{\boldsymbol{I}} + 2\mu.\widetilde{\boldsymbol{D}} - \frac{2\mu}{Q_0^2}.\mathrm{Tr}\left(\underline{\widetilde{\boldsymbol{\tau}}} \cdot \underline{\widetilde{\boldsymbol{D}}}\right).\underline{\widetilde{\boldsymbol{\tau}}}\ . \tag{7.3}$$

In the preceding law definitions, the strain and stress rates are written $\overset{\circ}{(.)}$, or $\overset{\bullet}{(.)}$. The first one is an objective time derivative required for objective vectors and tensors, while the second is the simple time derivative, that we apply to the invariant vectors and tensors.

We will define these time derivatives in the present chapter.

7.1 Why using objective rates ?

Let us consider two viscous laws :

$$\widetilde{\boldsymbol{S}}^{\backslash a} = \eta \ . \ \overset{\bullet}{\widetilde{\boldsymbol{LnU}}}^{\backslash a} \tag{7.4}$$

whatever the choice of a, and

$$\widetilde{\boldsymbol{T}}^{\backslash a} = \eta \ . \ \overset{\bullet}{\widetilde{\boldsymbol{LnV}}}^{\backslash a} \tag{7.5}$$

whatever the choice of a.

The constant parameter η is the viscosity.

According to its definition, law 7.4 has this expression whatever the reference frame, so a user, so-called ub, may choose a reference frame a and another user, so-called uc, may choose a different reference frame b. So for user ub :

$$\widetilde{\boldsymbol{S}}_{ub}^{\backslash a} = \eta \ . \ \overset{\bullet}{\widetilde{\boldsymbol{LnU}}}_{ub}^{\backslash a} \tag{7.6}$$

and for user uc :

$$\widetilde{\boldsymbol{S}}_{uc}^{\backslash b} = \eta \ . \ \overset{\bullet}{\widetilde{\boldsymbol{LnU}}}_{uc}^{\backslash b} \tag{7.7}$$

$$\widetilde{\boldsymbol{S}}_{uc}^{\backslash a} = \eta \ . \ \overset{\bullet}{\widetilde{\boldsymbol{LnU}}}_{uc}^{\backslash a} \tag{7.8}$$

since $\widetilde{\boldsymbol{S}}$ and $\widetilde{\boldsymbol{LnU}}$ are invariant.

So, the choice of the reference frame does not impact the behavior of law 7.4. This law is well defined.

Let us check this for law 7.5. For user ub :

$$\widetilde{\boldsymbol{T}}_{ub}^{\backslash a} = \eta \ . \ \overset{\bullet}{\widetilde{\boldsymbol{LnV}}}_{ub}^{\backslash a} \tag{7.9}$$

and for user uc :

$$\widetilde{\boldsymbol{T}}_{uc}^{\backslash b} = \eta \, . \, \overset{\bullet}{\widetilde{\boldsymbol{LnV}}}{}_{uc}^{\backslash b} \tag{7.10}$$

$$\widetilde{\boldsymbol{R}^{ba}} \centerdot \widetilde{\boldsymbol{T}}_{uc}^{\backslash a} \centerdot \widetilde{\boldsymbol{R}^{ab}} = \eta \, . \, \overset{\bullet}{\overline{\left[\widetilde{\boldsymbol{R}^{ba}} \centerdot \widetilde{\boldsymbol{LnV}}{}_{uc}^{\backslash a} \centerdot \widetilde{\boldsymbol{R}^{ab}}\right]}} \tag{7.11}$$

since $\widetilde{\boldsymbol{T}}$ and $\widetilde{\boldsymbol{LnV}}$ are objective. After some calculus, we obtain that :

$$\widetilde{\boldsymbol{T}}_{uc}^{\backslash a} = \eta \, . \, \overset{\bullet}{\widetilde{\boldsymbol{LnV}}}{}_{uc}^{\backslash a} + \eta \, . \, \left(\widetilde{\boldsymbol{LnV}}{}_{uc}^{\backslash a} \centerdot \widetilde{\boldsymbol{\Omega}^{ab}} - \widetilde{\boldsymbol{\Omega}^{ab}} \centerdot \widetilde{\boldsymbol{LnV}}{}_{uc}^{\backslash a} \right) \tag{7.12}$$

So, the choice of the reference frame impacts or changes the behavior of law 7.5. This law is **problematic**.

In order to obtain a well-defined law 7.5, we may replace the simple rate $\overset{\bullet}{\widetilde{\boldsymbol{LnV}}}$ by a particular rate $\overset{\circ}{\widetilde{\boldsymbol{LnV}}}$ that is **objective** :

$$\overset{\circ}{\widetilde{\boldsymbol{LnV}}}{}^{\backslash b} = \widetilde{\boldsymbol{R}^{ba}} \centerdot \overset{\circ}{\widetilde{\boldsymbol{LnV}}}{}^{\backslash a} \centerdot \widetilde{\boldsymbol{R}^{ab}} \tag{7.13}$$

whatever the choices of a and b.

But how to do that ?

In order to do that, let us consider the following property :

Property – *If tensor $\widetilde{\boldsymbol{A}}$ is objective, then tensor $\widetilde{\boldsymbol{B}}$ defined, whatever the choice of a, as*

$$\widetilde{\boldsymbol{B}}^{\backslash a} = \widetilde{\boldsymbol{R}^{ja}} \centerdot \widetilde{\boldsymbol{A}}^{\backslash a} \centerdot \widetilde{\boldsymbol{R}^{aj}} = \widetilde{\boldsymbol{A}}^{\backslash j} \tag{7.14}$$

is invariant.

Based on this property, we may change the problematic law 7.5 such that :

$$\widetilde{\boldsymbol{T}}^{\backslash j} = \eta \, . \, \overset{\bullet}{\widetilde{\boldsymbol{LnV}}}{}^{\backslash j} \tag{7.15}$$

Now, tensors $\widetilde{\boldsymbol{T}}^{\backslash j}$ and $\widetilde{\boldsymbol{LnV}}^{\backslash j}$ are invariant, so this law is of the same type of the well-defined law 7.4 and **is also well defined**.

Let us write this law 7.15 into an other expression :

$$\widetilde{\boldsymbol{T}}^{\backslash j} = \eta \;.\; \overset{\bullet}{\widetilde{\boldsymbol{LnV}}}^{\backslash j} \tag{7.16}$$

$$\widetilde{\boldsymbol{R}^{ja}} \;\boldsymbol{.}\; \widetilde{\boldsymbol{T}}^{\backslash a} \;\boldsymbol{.}\; \widetilde{\boldsymbol{R}^{aj}} = \tag{7.17}$$

$$\Longleftrightarrow \tag{7.18}$$

$$\widetilde{\boldsymbol{T}}^{\backslash a} = \eta \;.\; \left\{ \widetilde{\boldsymbol{R}^{aj}} \;\boldsymbol{.}\; \left[\overset{\bullet}{\widetilde{\boldsymbol{LnV}}}^{\backslash j} \right] \;\boldsymbol{.}\; \widetilde{\boldsymbol{R}^{ja}} \right\} \tag{7.19}$$

So, this well-defined law has a new expression :

$$\widetilde{\boldsymbol{T}}^{\backslash a} = \eta \;.\; \left(\overset{\circ j}{\widetilde{\boldsymbol{LnV}}} \right)^{\backslash a} \tag{7.20}$$

whatever the choice of reference frame a, and where

$$\left(\overset{\circ j}{\widetilde{\boldsymbol{LnV}}} \right)^{\backslash a} = \widetilde{\boldsymbol{R}^{aj}} \;\boldsymbol{.}\; \left[\overset{\bullet}{\widetilde{\boldsymbol{LnV}}}^{\backslash j} \right] \;\boldsymbol{.}\; \widetilde{\boldsymbol{R}^{ja}} \tag{7.21}$$

7.2 Objective rate into a reference frame

Definition – *The objective rate into a reference frame r is defined, for an* ***objective*** *second-order tensor, as*

$$\left(\overset{\circ r}{\widetilde{\boldsymbol{A}}} \right)^{\backslash a} = \widetilde{\boldsymbol{R}^{ar}} \boldsymbol{.} \overset{\bullet}{\overline{\left[\widetilde{\boldsymbol{A}}^{\backslash r} \right]}} \boldsymbol{.} \widetilde{\boldsymbol{R}^{ar}}^{\mathbf{T}} \tag{7.22}$$

$$= \widetilde{\boldsymbol{R}^{ar}} \boldsymbol{.} \overset{\bullet}{\overline{\left[\widetilde{\boldsymbol{R}^{ar}}^{\mathbf{T}} \boldsymbol{.} \widetilde{\boldsymbol{A}}^{\backslash a} \boldsymbol{.} \widetilde{\boldsymbol{R}^{ar}} \right]}} \boldsymbol{.} \widetilde{\boldsymbol{R}^{ar}}^{\mathbf{T}} \tag{7.23}$$

and for an ***objective*** *vector, as*

$$\left(\overset{\circ r}{\vec{a}} \right)^{\backslash a} = \widetilde{\boldsymbol{R}^{ar}} \boldsymbol{.} \overset{\bullet}{\overline{\left[\vec{a}^{\backslash r} \right]}} \tag{7.24}$$

$$= \widetilde{\boldsymbol{R}^{ar}} \boldsymbol{.} \overset{\bullet}{\overline{\left[\widetilde{\boldsymbol{R}^{ar}}^{\mathbf{T}} \boldsymbol{.} \vec{a}^{\backslash a} \right]}} \tag{7.25}$$

This time derivative shoud be applied only to **objective** vector and tensor, and never to invariant vector nor tensor.

For that reason, the following expressions have no mechanical meaning :

$$\overset{or}{\widetilde{\boldsymbol{U}}} \qquad \overset{or}{\widetilde{\boldsymbol{S}}} \qquad \overset{or}{\widetilde{\boldsymbol{\Pi}}} \qquad \left[\overset{or}{\widetilde{\boldsymbol{A}}}^{\backslash j}\right] \, . \tag{7.26}$$

This objective rate is the vector or tensor rate observed from reference frame r,

$$\left[\overset{\bullet}{\widetilde{\boldsymbol{A}}}^{\backslash r}\right] \qquad \left[\overset{\bullet}{\vec{a}}^{\backslash r}\right] \, , \tag{7.27}$$

but this rate is observed from base reference frame a.

Properties – *Relations between these objective rates and the simple time derivative or simple rate are :*

$$\overset{or}{\widetilde{\boldsymbol{A}}} = \overset{\bullet}{\widetilde{\boldsymbol{A}}} + \widetilde{\boldsymbol{A}} \cdot \widetilde{\boldsymbol{\Omega}^{ar}} - \widetilde{\boldsymbol{\Omega}^{ar}} \cdot \widetilde{\boldsymbol{A}} \tag{7.28}$$

$$\overset{or}{\vec{a}} = \overset{\bullet}{\vec{a}} - \widetilde{\boldsymbol{\Omega}^{ar}} \cdot \vec{a} \tag{7.29}$$

Exercise 7.1. *Prove the preceding properties.*

This objective rate is identical to the concept of the point velocity from a reference frame, within the mechanics of the material points

$$\overrightarrow{V(M)_{/R_1}} \, , \tag{7.30}$$

which is the velocity of point M from frame R_1, but this velocity is observed from base reference frame R_0. Indeed, according to the material point mechanics theory, we have the following velocity composition relation :

$$\overrightarrow{V(\vec{a})_{/R_1}} = \overrightarrow{V(\vec{a})_{/R_0}} - \overrightarrow{\Omega_{R_1/R_0}} \wedge \vec{a} \, , \tag{7.31}$$

which is exactly relation 7.29.

Properties –

$$\overset{or}{\widetilde{\boldsymbol{I}}} = \widetilde{\boldsymbol{0}} \tag{7.32}$$

$$\left[\overset{or}{\widetilde{\boldsymbol{A}}}\right]^{\mathbf{T}} = \left[\overset{or}{\widetilde{\boldsymbol{A}}^{\mathbf{T}}}\right] \qquad \mathrm{Tr}\left[\overset{or}{\widetilde{\boldsymbol{A}}}\right] = \left[\overset{\cdot}{\mathrm{Tr}\,\widetilde{\boldsymbol{A}}}\right]$$

$$\left[\overset{or}{\lambda.\widetilde{\boldsymbol{A}}}\right] = \dot{\lambda}.\widetilde{\boldsymbol{A}} + \lambda.\overset{or}{\widetilde{\boldsymbol{A}}} \qquad \left[\overset{or}{\underline{\widetilde{\boldsymbol{A}}}}\right] = \underline{\left[\overset{or}{\widetilde{\boldsymbol{A}}}\right]}$$

$$\left[\overset{or}{\widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{B}}}\right] = \overset{or}{\widetilde{\boldsymbol{A}}} + \overset{or}{\widetilde{\boldsymbol{B}}} \qquad \left[\overset{or}{\widetilde{\boldsymbol{A}} \boldsymbol{.} \widetilde{\boldsymbol{B}}}\right] = \overset{or}{\widetilde{\boldsymbol{A}}} \boldsymbol{.} \widetilde{\boldsymbol{B}} + \widetilde{\boldsymbol{A}} \boldsymbol{.} \overset{or}{\widetilde{\boldsymbol{B}}} \tag{7.33}$$

$$\left[\overset{or}{\widetilde{\boldsymbol{A}} \boldsymbol{.} \vec{x}}\right] = \overset{or}{\widetilde{\boldsymbol{A}}} \boldsymbol{.} \vec{x} + \widetilde{\boldsymbol{A}} \boldsymbol{.} \overset{or}{\vec{x}}$$

$$\left[\overset{or}{\vec{x} + \vec{y}}\right] = \overset{or}{\vec{x}} + \overset{or}{\vec{y}} \qquad \left[\vec{x} \overset{\cdot}{\boldsymbol{.}} \vec{y}\right] = \overset{or}{\vec{x}} \boldsymbol{.} \vec{y} + \vec{x} \boldsymbol{.} \overset{or}{\vec{y}}$$

Warning –

$$\left[\overset{or}{\widetilde{\boldsymbol{A}}}\right]^{-\mathbf{1}} \neq \left[\overset{or}{\widetilde{\boldsymbol{A}}^{-\mathbf{1}}}\right] = -\widetilde{\boldsymbol{A}}^{-\mathbf{1}} \boldsymbol{.} \overset{or}{\widetilde{\boldsymbol{A}}} \boldsymbol{.} \widetilde{\boldsymbol{A}}^{-\mathbf{1}} \tag{7.34}$$

7.3 Green-Naghdi objective rate

Definition – *The objective rate of Green-Naghdi or Green-Naghdi-McInnis is the objective rate into the* ***reference frame of the proper rotation*** *rp.*

Properties – *Relations between the Green-Naghdi objective rate and the simple time derivative or simple rate are :*

$$\overset{orp}{\widetilde{\boldsymbol{A}}} = \overset{\bullet}{\widetilde{\boldsymbol{A}}} + \widetilde{\boldsymbol{A}} \boldsymbol{.} \widetilde{\boldsymbol{\Omega}^{arp}} - \widetilde{\boldsymbol{\Omega}^{arp}} \boldsymbol{.} \widetilde{\boldsymbol{A}} \tag{7.35}$$

$$\overset{orp}{\vec{a}} = \overset{\bullet}{\vec{a}} - \widetilde{\boldsymbol{\Omega}^{arp}} \boldsymbol{.} \vec{a}\ , \tag{7.36}$$

where $\widetilde{\boldsymbol{\Omega}^{arp}} = \overset{\bullet}{\widetilde{\boldsymbol{R}}} \boldsymbol{.} \widetilde{\boldsymbol{R}}^{\mathbf{T}}$.

7.4 Jaumann objective rate

Definition – *The objective rate of Jaumann is the objective rate into the* ***corotational reference frame*** *j.*

Properties – *Relations between the Jaumann objective rate and the simple time derivative or simple rate are :*

$$\overset{\circ j}{\widetilde{\boldsymbol{A}}} = \overset{\bullet}{\widetilde{\boldsymbol{A}}} + \widetilde{\boldsymbol{A}} \cdot \widetilde{\boldsymbol{W}} - \widetilde{\boldsymbol{W}} \cdot \widetilde{\boldsymbol{A}} \tag{7.37}$$

$$\overset{\circ j}{\vec{a}} = \overset{\bullet}{\vec{a}} - \widetilde{\boldsymbol{W}} \cdot \vec{a} \tag{7.38}$$

7.5 Dérivées objectives convectives

Définitions – *La* ***dérivée convective deux fois contravariante****, appelée aussi* ***dérivée d'Oldroyd*** *ou encore* ***dérivée surconvectée****, est définie pour un tenseur d'ordre deux ainsi*

$$L^{\cdot\cdot}(\widetilde{\boldsymbol{A}}) = \widetilde{\boldsymbol{F}} \cdot \overset{\bullet}{\overline{\left[\widetilde{\boldsymbol{F}}^{-1} \cdot \widetilde{\boldsymbol{A}} \cdot \widetilde{\boldsymbol{F}}^{-\mathrm{T}}\right]}} \cdot \widetilde{\boldsymbol{F}}^{\mathrm{T}} \,. \tag{7.39}$$

La ***dérivée convective deux fois covariante****, appelée aussi* ***dérivée de Cotter-Rivlin*** *ou encore* ***dérivée sousconvectée****, est définie pour un tenseur d'ordre deux ainsi*

$$L_{\cdot\cdot}(\widetilde{\boldsymbol{A}}) = \widetilde{\boldsymbol{F}}^{-\mathrm{T}} \cdot \overset{\bullet}{\overline{\left[\widetilde{\boldsymbol{F}}^{\mathrm{T}} \cdot \widetilde{\boldsymbol{A}} \cdot \widetilde{\boldsymbol{F}}\right]}} \cdot \widetilde{\boldsymbol{F}}^{-1} \,. \tag{7.40}$$

La ***dérivée convective contra-covariante*** *est définie pour un tenseur d'ordre deux ainsi*

$$L^{\cdot}{}_{\cdot}(\widetilde{\boldsymbol{A}}) = \widetilde{\boldsymbol{F}} \cdot \overset{\bullet}{\overline{\left[\widetilde{\boldsymbol{F}}^{-1} \cdot \widetilde{\boldsymbol{A}} \cdot \widetilde{\boldsymbol{F}}\right]}} \cdot \widetilde{\boldsymbol{F}}^{-1} \,. \tag{7.41}$$

La ***dérivée convective co-contravariante*** *est définie pour un tenseur d'ordre deux ainsi*

$$L_{\cdot}{}^{\cdot}(\widetilde{\boldsymbol{A}}) = \widetilde{\boldsymbol{F}}^{-\mathrm{T}} \cdot \overset{\bullet}{\overline{\left[\widetilde{\boldsymbol{F}}^{\mathrm{T}} \cdot \widetilde{\boldsymbol{A}} \cdot \widetilde{\boldsymbol{F}}^{-\mathrm{T}}\right]}} \cdot \widetilde{\boldsymbol{F}}^{\mathrm{T}} \,. \tag{7.42}$$

La ***dérivée convective contravariante*** *est définie pour un vecteur ainsi*

$$L^{\cdot}(\vec{a}) = \widetilde{\boldsymbol{F}} \cdot \overset{\bullet}{\left(\widetilde{\boldsymbol{F}}^{-1} \cdot \vec{a}\right)} \,. \tag{7.43}$$

La **dérivée convective covariante** est définie pour un vecteur ainsi

$$L_{\cdot}(\vec{a}) = \widetilde{\boldsymbol{F}}^{-\mathbf{T}} \boldsymbol{\cdot} \left(\dot{\widetilde{\boldsymbol{F}}}^{\mathbf{T}} \boldsymbol{\cdot} \vec{a} \right) \quad . \tag{7.44}$$

Properties – *Les relations entre ces dérivées convectives et la dérivée simple sont les suivantes :*

$$L^{\cdot\cdot}(\widetilde{\boldsymbol{A}}) = \dot{\widetilde{\boldsymbol{A}}} - \widetilde{\boldsymbol{A}} \boldsymbol{\cdot} \widetilde{\boldsymbol{L}}^{\mathbf{T}} - \widetilde{\boldsymbol{L}} \boldsymbol{\cdot} \widetilde{\boldsymbol{A}} \tag{7.45}$$

$$L_{\cdot\cdot}(\widetilde{\boldsymbol{A}}) = \dot{\widetilde{\boldsymbol{A}}} + \widetilde{\boldsymbol{A}} \boldsymbol{\cdot} \widetilde{\boldsymbol{L}} + \widetilde{\boldsymbol{L}}^{\mathbf{T}} \boldsymbol{\cdot} \widetilde{\boldsymbol{A}} \tag{7.46}$$

$${L^{\cdot}}_{\cdot}(\widetilde{\boldsymbol{A}}) = \dot{\widetilde{\boldsymbol{A}}} + \widetilde{\boldsymbol{A}} \boldsymbol{\cdot} \widetilde{\boldsymbol{L}} - \widetilde{\boldsymbol{L}} \boldsymbol{\cdot} \widetilde{\boldsymbol{A}} \tag{7.47}$$

$${L_{\cdot}}^{\cdot}(\widetilde{\boldsymbol{A}}) = \dot{\widetilde{\boldsymbol{A}}} - \widetilde{\boldsymbol{A}} \boldsymbol{\cdot} \widetilde{\boldsymbol{L}}^{\mathbf{T}} + \widetilde{\boldsymbol{L}}^{\mathbf{T}} \boldsymbol{\cdot} \widetilde{\boldsymbol{A}} \tag{7.48}$$

$$L^{\cdot}(\vec{a}) = \dot{\vec{a}} - \widetilde{\boldsymbol{L}} \boldsymbol{\cdot} \vec{a} \tag{7.49}$$

$$L_{\cdot}(\vec{a}) = \dot{\vec{a}} + \widetilde{\boldsymbol{L}}^{\mathbf{T}} \boldsymbol{\cdot} \vec{a} \tag{7.50}$$

Properties – *Les relations entre ces dérivées convectives et la dérivée de Jaumann sont les suivantes :*

$$L^{\cdot\cdot}(\widetilde{\boldsymbol{A}}) = \overset{\circ j}{\widetilde{\boldsymbol{A}}} - \widetilde{\boldsymbol{A}} \boldsymbol{\cdot} \widetilde{\boldsymbol{D}} - \widetilde{\boldsymbol{D}} \boldsymbol{\cdot} \widetilde{\boldsymbol{A}} \tag{7.51}$$

$$L_{\cdot\cdot}(\widetilde{\boldsymbol{A}}) = \overset{\circ j}{\widetilde{\boldsymbol{A}}} + \widetilde{\boldsymbol{A}} \boldsymbol{\cdot} \widetilde{\boldsymbol{D}} + \widetilde{\boldsymbol{D}} \boldsymbol{\cdot} \widetilde{\boldsymbol{A}} \tag{7.52}$$

$${L^{\cdot}}_{\cdot}(\widetilde{\boldsymbol{A}}) = \overset{\circ j}{\widetilde{\boldsymbol{A}}} + \widetilde{\boldsymbol{A}} \boldsymbol{\cdot} \widetilde{\boldsymbol{D}} - \widetilde{\boldsymbol{D}} \boldsymbol{\cdot} \widetilde{\boldsymbol{A}} \tag{7.53}$$

$${L_{\cdot}}^{\cdot}(\widetilde{\boldsymbol{A}}) = \overset{\circ j}{\widetilde{\boldsymbol{A}}} - \widetilde{\boldsymbol{A}} \boldsymbol{\cdot} \widetilde{\boldsymbol{D}} + \widetilde{\boldsymbol{D}} \boldsymbol{\cdot} \widetilde{\boldsymbol{A}} \tag{7.54}$$

$$L^{\cdot}(\vec{a}) = \overset{\circ j}{\vec{a}} - \widetilde{\boldsymbol{D}} \boldsymbol{\cdot} \vec{a} \tag{7.55}$$

$$L_{\cdot}(\vec{a}) = \overset{\circ j}{\vec{a}} + \widetilde{\boldsymbol{D}} \boldsymbol{\cdot} \vec{a} \tag{7.56}$$

Property – *Une propriété assez connue est la suivante. La dérivée de Jaumann peut être calculée avec les dérivées convectives ainsi :*

$$\overset{\circ j}{\widetilde{\boldsymbol{A}}} = \frac{1}{2}\left(L^{\cdot\cdot}(\widetilde{\boldsymbol{A}}) + L_{\cdot\cdot}(\widetilde{\boldsymbol{A}}) \right) = \frac{1}{2}\left({L^{\cdot}}_{\cdot}(\widetilde{\boldsymbol{A}}) + {L_{\cdot}}^{\cdot}(\widetilde{\boldsymbol{A}}) \right) \quad . \tag{7.57}$$

Properties de la dérivée deux fois contravariante –

$$L^{\cdot\cdot}(\widetilde{\boldsymbol{I}}) = -\widetilde{\boldsymbol{L}} - \widetilde{\boldsymbol{L}}^{\mathbf{T}} = -2\widetilde{\boldsymbol{D}} \tag{7.58}$$

$$\begin{aligned}
&\left[L^{\cdot\cdot}(\widetilde{\boldsymbol{A}})\right]^{\mathbf{T}} = L^{\cdot\cdot}(\widetilde{\boldsymbol{A}}^{\mathbf{T}})\\
&\mathrm{Tr}\left[L^{\cdot\cdot}(\widetilde{\boldsymbol{A}})\right] = \left[\overset{\cdot}{\mathrm{Tr}\,\widetilde{\boldsymbol{A}}}\right] - 2\mathrm{Tr}\left(\widetilde{\boldsymbol{A}} \boldsymbol{\cdot} \widetilde{\boldsymbol{D}}\right)\\
&L^{\cdot\cdot}(\widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{B}}) = L^{\cdot\cdot}(\widetilde{\boldsymbol{A}}) + L^{\cdot\cdot}(\widetilde{\boldsymbol{B}})\\
&L^{\cdot\cdot}(\widetilde{\boldsymbol{A}} \boldsymbol{\cdot} \widetilde{\boldsymbol{B}}) = L^{\cdot\cdot}(\widetilde{\boldsymbol{A}}) \boldsymbol{\cdot} \widetilde{\boldsymbol{B}} + \widetilde{\boldsymbol{A}} \boldsymbol{\cdot} L^{\cdot\cdot}(\widetilde{\boldsymbol{B}}) + 2\widetilde{\boldsymbol{A}} \boldsymbol{\cdot} \widetilde{\boldsymbol{D}} \boldsymbol{\cdot} \widetilde{\boldsymbol{B}}
\end{aligned} \tag{7.59}$$

Properties de la dérivée deux fois covariante –

$$L_{\cdot\cdot}(\widetilde{\boldsymbol{I}}) = \widetilde{\boldsymbol{L}} + \widetilde{\boldsymbol{L}}^{\mathbf{T}} = 2\widetilde{\boldsymbol{D}} \tag{7.60}$$

$$\begin{aligned}
&\left[L_{\cdot\cdot}(\widetilde{\boldsymbol{A}})\right]^{\mathbf{T}} = L_{\cdot\cdot}(\widetilde{\boldsymbol{A}}^{\mathbf{T}})\\
&\mathrm{Tr}\left[L_{\cdot\cdot}(\widetilde{\boldsymbol{A}})\right] = \left[\overset{\cdot}{\mathrm{Tr}\,\widetilde{\boldsymbol{A}}}\right] + 2\mathrm{Tr}\left(\widetilde{\boldsymbol{A}} \boldsymbol{\cdot} \widetilde{\boldsymbol{D}}\right)\\
&L_{\cdot\cdot}(\widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{B}}) = L_{\cdot\cdot}(\widetilde{\boldsymbol{A}}) + L_{\cdot\cdot}(\widetilde{\boldsymbol{B}})\\
&L_{\cdot\cdot}(\widetilde{\boldsymbol{A}} \boldsymbol{\cdot} \widetilde{\boldsymbol{B}}) = L_{\cdot\cdot}(\widetilde{\boldsymbol{A}}) \boldsymbol{\cdot} \widetilde{\boldsymbol{B}} + \widetilde{\boldsymbol{A}} \boldsymbol{\cdot} L_{\cdot\cdot}(\widetilde{\boldsymbol{B}}) - 2\widetilde{\boldsymbol{A}} \boldsymbol{\cdot} \widetilde{\boldsymbol{D}} \boldsymbol{\cdot} \widetilde{\boldsymbol{B}}
\end{aligned} \tag{7.61}$$

Properties de la dérivée contra-covariante –

$$L^{\cdot}{}_{\cdot}(\widetilde{\boldsymbol{I}}) = \widetilde{\mathbf{0}} \tag{7.62}$$

$$\begin{aligned}
&\left[L^{\cdot}{}_{\cdot}(\widetilde{\boldsymbol{A}})\right]^{\mathbf{T}} = L^{\cdot}{}_{\cdot}(\widetilde{\boldsymbol{A}}^{\mathbf{T}}) + \left(\widetilde{\boldsymbol{L}} + \widetilde{\boldsymbol{L}}^{\mathbf{T}}\right) \boldsymbol{\cdot} \widetilde{\boldsymbol{A}}^{\mathbf{T}} - \widetilde{\boldsymbol{A}}^{\mathbf{T}} \boldsymbol{\cdot} \left(\widetilde{\boldsymbol{L}} + \widetilde{\boldsymbol{L}}^{\mathbf{T}}\right)\\
&\mathrm{Tr}\left[L^{\cdot}{}_{\cdot}(\widetilde{\boldsymbol{A}})\right] = \left[\overset{\cdot}{\mathrm{Tr}\,\widetilde{\boldsymbol{A}}}\right]\\
&L^{\cdot}{}_{\cdot}(\widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{B}}) = L^{\cdot}{}_{\cdot}(\widetilde{\boldsymbol{A}}) + L^{\cdot}{}_{\cdot}(\widetilde{\boldsymbol{B}})\\
&L^{\cdot}{}_{\cdot}(\widetilde{\boldsymbol{A}} \boldsymbol{\cdot} \widetilde{\boldsymbol{B}}) = L^{\cdot}{}_{\cdot}(\widetilde{\boldsymbol{A}}) \boldsymbol{\cdot} \widetilde{\boldsymbol{B}} + \widetilde{\boldsymbol{A}} \boldsymbol{\cdot} L^{\cdot}{}_{\cdot}(\widetilde{\boldsymbol{B}})
\end{aligned} \tag{7.63}$$

Properties de la dérivée co-contravariante –

$$L_{\cdot}{}^{\cdot}(\widetilde{\boldsymbol{I}}) = \widetilde{\mathbf{0}} \tag{7.64}$$

$$\begin{aligned}
&\left[L_{\cdot}{}^{\cdot}(\widetilde{\boldsymbol{A}})\right]^{\mathbf{T}} = L_{\cdot}{}^{\cdot}(\widetilde{\boldsymbol{A}}^{\mathbf{T}}) - \left(\widetilde{\boldsymbol{L}} + \widetilde{\boldsymbol{L}}^{\mathbf{T}}\right) \boldsymbol{\cdot} \widetilde{\boldsymbol{A}}^{\mathbf{T}} + \widetilde{\boldsymbol{A}}^{\mathbf{T}} \boldsymbol{\cdot} \left(\widetilde{\boldsymbol{L}} + \widetilde{\boldsymbol{L}}^{\mathbf{T}}\right)\\
&\mathrm{Tr}\left[L_{\cdot}{}^{\cdot}(\widetilde{\boldsymbol{A}})\right] = \left[\overset{\cdot}{\mathrm{Tr}\,\widetilde{\boldsymbol{A}}}\right]\\
&L_{\cdot}{}^{\cdot}(\widetilde{\boldsymbol{A}} + \widetilde{\boldsymbol{B}}) = L_{\cdot}{}^{\cdot}(\widetilde{\boldsymbol{A}}) + L_{\cdot}{}^{\cdot}(\widetilde{\boldsymbol{B}})\\
&L_{\cdot}{}^{\cdot}(\widetilde{\boldsymbol{A}} \boldsymbol{\cdot} \widetilde{\boldsymbol{B}}) = L_{\cdot}{}^{\cdot}(\widetilde{\boldsymbol{A}}) \boldsymbol{\cdot} \widetilde{\boldsymbol{B}} + \widetilde{\boldsymbol{A}} \boldsymbol{\cdot} L_{\cdot}{}^{\cdot}(\widetilde{\boldsymbol{B}})
\end{aligned} \tag{7.65}$$

Properties de la dérivée contravariante d'un vecteur –

$$
\begin{aligned}
&L^{\cdot}(\vec{x}+\vec{y}) = L^{\cdot}(\vec{x}) + L^{\cdot}(\vec{y}) \\
&\overset{\cdot}{[\vec{x} \bullet \vec{y}]} = L^{\cdot}(\vec{x}) \bullet \vec{y} + \vec{x} \bullet L^{\cdot}(\vec{y}) + 2\vec{x} \bullet \left[\widetilde{\boldsymbol{D}} \bullet \vec{y}\right] \\
&L^{\cdot}(\widetilde{\boldsymbol{A}} \bullet \vec{x}) = L^{\cdot\cdot}(\widetilde{\boldsymbol{A}}) \bullet \vec{x} + \widetilde{\boldsymbol{A}} \bullet L^{\cdot}(\vec{x}) + 2\widetilde{\boldsymbol{A}} \bullet \widetilde{\boldsymbol{D}} \bullet \vec{x} \\
&L^{\cdot}(\widetilde{\boldsymbol{A}} \bullet \vec{x}) = L_{\cdot\cdot}(\widetilde{\boldsymbol{A}}) \bullet \vec{x} + \widetilde{\boldsymbol{A}} \bullet L^{\cdot}(\vec{x}) - 2\widetilde{\boldsymbol{D}} \bullet \widetilde{\boldsymbol{A}} \bullet \vec{x} \\
&L^{\cdot}(\widetilde{\boldsymbol{A}} \bullet \vec{x}) = L^{\cdot}{}_{\cdot}(\widetilde{\boldsymbol{A}}) \bullet \vec{x} + \widetilde{\boldsymbol{A}} \bullet L^{\cdot}(\vec{x})
\end{aligned}
\tag{7.66}
$$

Properties de la dérivée covariante d'un vecteur –

$$
\begin{aligned}
&L_{\cdot}(\vec{x}+\vec{y}) = L_{\cdot}(\vec{x}) + L_{\cdot}(\vec{y}) \\
&\overset{\cdot}{[\vec{x} \bullet \vec{y}]} = L_{\cdot}(\vec{x}) \bullet \vec{y} + \vec{x} \bullet L_{\cdot}(\vec{y}) - 2\vec{x} \bullet \left[\widetilde{\boldsymbol{D}} \bullet \vec{y}\right] \\
&L_{\cdot}(\widetilde{\boldsymbol{A}} \bullet \vec{x}) = L^{\cdot\cdot}(\widetilde{\boldsymbol{A}}) \bullet \vec{x} + \widetilde{\boldsymbol{A}} \bullet L_{\cdot}(\vec{x}) + 2\widetilde{\boldsymbol{D}} \bullet \widetilde{\boldsymbol{A}} \bullet \vec{x} \\
&L_{\cdot}(\widetilde{\boldsymbol{A}} \bullet \vec{x}) = L_{\cdot\cdot}(\widetilde{\boldsymbol{A}}) \bullet \vec{x} + \widetilde{\boldsymbol{A}} \bullet L_{\cdot}(\vec{x}) - 2\widetilde{\boldsymbol{A}} \bullet \widetilde{\boldsymbol{D}} \bullet \vec{x} \\
&L_{\cdot}(\widetilde{\boldsymbol{A}} \bullet \vec{x}) = L_{\cdot}{}^{\cdot}(\widetilde{\boldsymbol{A}}) \bullet \vec{x} + \widetilde{\boldsymbol{A}} \bullet L_{\cdot}(\vec{x})
\end{aligned}
\tag{7.67}
$$

7.6 Dérivée objective de Truesdell

Définition – *Une **dérivée de Truesdell** pour un tenseur d'ordre deux est définie ainsi*

$$\overset{\circ T}{\widetilde{\boldsymbol{A}}} = L^{\cdot\cdot}(\widetilde{\boldsymbol{A}}) + (\mathrm{Tr}\widetilde{\boldsymbol{D}}).\widetilde{\boldsymbol{A}} \ . \tag{7.68}$$

Property – *La dérivée de Truesdell de la contrainte de Cauchy présente une relation simple avec la dérivée simple de la seconde contrainte de Piola-Kirchhoff :*

$$\overset{\bullet}{\widetilde{\boldsymbol{S}}} = (\det\widetilde{\boldsymbol{F}}).\widetilde{\boldsymbol{F}}^{-1} \bullet \overset{\circ T}{\widetilde{\boldsymbol{T}}} \bullet \widetilde{\boldsymbol{F}}^{-\mathbf{T}} \ . \tag{7.69}$$

Une relation identique relie la contrainte de Cauchy à la seconde contrainte de Piola-Kirchhoff ; $\widetilde{\boldsymbol{S}} = (\det\widetilde{\boldsymbol{F}}).\widetilde{\boldsymbol{F}}^{-\mathbf{1}} \bullet \widetilde{\boldsymbol{T}} \bullet \widetilde{\boldsymbol{F}}^{-\mathbf{T}}$.

Property – *La dérivée de Truesdell de la contrainte de Cauchy présente une relation simple avec la dérivée d'Oldroyd de la contrainte de Kirchhoff :*

$$L^{\cdot\cdot}(\widetilde{\boldsymbol{\tau}}) = (\det\widetilde{\boldsymbol{F}}).\overset{\circ T}{\widetilde{\boldsymbol{T}}} \ . \tag{7.70}$$

Exercise 7.2. *Propose a viscoelastic law of Maxwell type for the 3-dimension case, well defined for the finite transformation.*

In the 1-dimension case, a Maxwell law is :

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{G} + \frac{\sigma}{\eta} \tag{7.71}$$

where G and η are respectively the elasticity modulus and the viscosity parameter.

7.7 Strain rate $\widetilde{\boldsymbol{D}}$ and objective rate of strain tensors

According its definition 2.75[p45],, the stretching tensor $\widetilde{\boldsymbol{D}}$ is not a time derivative of a strain tensor. However, this tensor $\widetilde{\boldsymbol{D}}$ is related to the objective rates of all objective strain tensors, by notably linear maps. We will investigate these linear relations in this section.

Property – *The relation between the Jaumann objective rate of the left Cauchy-Green strain and $\widetilde{\boldsymbol{D}}$ is :*

$$\overset{\circ j}{\widetilde{\boldsymbol{B}}} = \widetilde{\boldsymbol{D}} \boldsymbol{.} \widetilde{\boldsymbol{B}} + \widetilde{\boldsymbol{B}} \boldsymbol{.} \widetilde{\boldsymbol{D}} = \overset{\sim 4}{\boldsymbol{N}}_{(\widetilde{\boldsymbol{B}})} : \widetilde{\boldsymbol{D}} \,. \tag{7.72}$$

The definition of fourth-order tensor $\overset{\sim 4}{\boldsymbol{N}}$ is given by 1.157[p24].

Indeed,

$$\begin{aligned}
\overset{\circ j}{\widetilde{\boldsymbol{B}}} &= \dot{\widetilde{\boldsymbol{B}}} + \frac{1}{2}\widetilde{\boldsymbol{B}} \boldsymbol{.} \left(\widetilde{\boldsymbol{L}} - \widetilde{\boldsymbol{L}}^{\mathbf{T}}\right) - \frac{1}{2}\left(\widetilde{\boldsymbol{L}} - \widetilde{\boldsymbol{L}}^{\mathbf{T}}\right) \boldsymbol{.} \widetilde{\boldsymbol{B}} && (7.73)\\
&= \left(\widetilde{\boldsymbol{F}} \boldsymbol{.} \widetilde{\boldsymbol{F}}^{\mathbf{T}}\right)^{\bullet} + \frac{1}{2}\widetilde{\boldsymbol{B}} \boldsymbol{.} \widetilde{\boldsymbol{L}} - \frac{1}{2}\widetilde{\boldsymbol{B}} \boldsymbol{.} \widetilde{\boldsymbol{L}}^{\mathbf{T}} - \frac{1}{2}\widetilde{\boldsymbol{L}} \boldsymbol{.} \widetilde{\boldsymbol{B}} + \frac{1}{2}\widetilde{\boldsymbol{L}}^{\mathbf{T}} \boldsymbol{.} \widetilde{\boldsymbol{B}} && (7.74)\\
&= \dot{\widetilde{\boldsymbol{F}}} \boldsymbol{.} \widetilde{\boldsymbol{F}}^{\mathbf{T}} + \widetilde{\boldsymbol{F}} \boldsymbol{.} \dot{\widetilde{\boldsymbol{F}}}^{\mathbf{T}} \\
&\quad + \frac{1}{2}\widetilde{\boldsymbol{B}} \boldsymbol{.} \widetilde{\boldsymbol{L}} - \frac{1}{2}\widetilde{\boldsymbol{B}} \boldsymbol{.} \widetilde{\boldsymbol{L}}^{\mathbf{T}} - \frac{1}{2}\widetilde{\boldsymbol{L}} \boldsymbol{.} \widetilde{\boldsymbol{B}} + \frac{1}{2}\widetilde{\boldsymbol{L}}^{\mathbf{T}} \boldsymbol{.} \widetilde{\boldsymbol{B}} && (7.75)
\end{aligned}$$

Since $\overset{\bullet}{\widetilde{\boldsymbol{F}}} = \widetilde{\boldsymbol{L}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{F}}$,

$$\overset{\circ j}{\widetilde{\boldsymbol{B}}} = \widetilde{\boldsymbol{L}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{F}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{F}}^{\mathbf{T}} + \widetilde{\boldsymbol{F}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{F}}^{\mathbf{T}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{L}}^{\mathbf{T}} + \frac{1}{2}\widetilde{\boldsymbol{B}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{L}} - \frac{1}{2}\widetilde{\boldsymbol{B}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{L}}^{\mathbf{T}} - \frac{1}{2}\widetilde{\boldsymbol{L}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{B}} + \frac{1}{2}\widetilde{\boldsymbol{L}}^{\mathbf{T}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{B}} \tag{7.76}$$

$$= \widetilde{\boldsymbol{L}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{B}} + \widetilde{\boldsymbol{B}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{L}}^{\mathbf{T}} + \frac{1}{2}\widetilde{\boldsymbol{B}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{L}} - \frac{1}{2}\widetilde{\boldsymbol{B}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{L}}^{\mathbf{T}} - \frac{1}{2}\widetilde{\boldsymbol{L}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{B}} + \frac{1}{2}\widetilde{\boldsymbol{L}}^{\mathbf{T}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{B}} \tag{7.77}$$

$$= \frac{1}{2}\left(\widetilde{\boldsymbol{L}} + \widetilde{\boldsymbol{L}}^{\mathbf{T}}\right) \,\boldsymbol{.}\, \widetilde{\boldsymbol{B}} + \widetilde{\boldsymbol{B}} \,\boldsymbol{.}\, \frac{1}{2}\left(\widetilde{\boldsymbol{L}} + \widetilde{\boldsymbol{L}}^{\mathbf{T}}\right) \tag{7.78}$$

$$= \widetilde{\boldsymbol{D}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{B}} + \widetilde{\boldsymbol{B}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{D}} \tag{7.79}$$

Property – *The relation between the Jaumann objective rate of the inverse of the left Cauchy-Green strain and* $\widetilde{\boldsymbol{D}}$ *is :*

$$\left(\overset{\circ j}{\widetilde{\boldsymbol{B}}^{-1}}\right) = -\widetilde{\boldsymbol{D}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{B}}^{-1} - \widetilde{\boldsymbol{B}}^{-1} \,\boldsymbol{.}\, \widetilde{\boldsymbol{D}} = -\overset{\sim 4}{\boldsymbol{N}}_{(\widetilde{\boldsymbol{B}}^{-1})} : \widetilde{\boldsymbol{D}}\ . \tag{7.80}$$

Property – *The relation between the Jaumann objective rate of the Almansi-Euler strain and* $\widetilde{\boldsymbol{D}}$ *is :*

$$\overset{\circ j}{\widetilde{\boldsymbol{A}}} = \widetilde{\boldsymbol{D}} - \widetilde{\boldsymbol{D}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{A}} - \widetilde{\boldsymbol{A}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{D}} \tag{7.81}$$

$$= \frac{1}{2}\left(\widetilde{\boldsymbol{D}} \,\boldsymbol{.}\, \widetilde{\boldsymbol{B}}^{-1} + \widetilde{\boldsymbol{B}}^{-1} \,\boldsymbol{.}\, \widetilde{\boldsymbol{D}}\right) = \frac{1}{2}\overset{\sim 4}{\boldsymbol{N}}_{(\widetilde{\boldsymbol{B}}^{-1})} : \widetilde{\boldsymbol{D}}\ . \tag{7.82}$$

Exercise 7.3. *Prove relation 7.81.*

Property – *The preceding property 7.81 and relation 7.52 lead to :*

$$L_{..}(\widetilde{\boldsymbol{A}}) = \widetilde{\boldsymbol{D}}\ . \tag{7.83}$$

Property – *The relation between the Jaumann objective rate of the left quasilogarithmic strain and* $\widetilde{\boldsymbol{D}}$ *is :*

$$\overset{\circ j}{\widetilde{\boldsymbol{QLnV}}} = \frac{1}{4}\left[\widetilde{\boldsymbol{D}} \,\boldsymbol{.}\, \left(\widetilde{\boldsymbol{B}} + \widetilde{\boldsymbol{B}}^{-1}\right) + \left(\widetilde{\boldsymbol{B}} + \widetilde{\boldsymbol{B}}^{-1}\right) \,\boldsymbol{.}\, \widetilde{\boldsymbol{D}}\right] \tag{7.84}$$

$$= \frac{1}{4}\overset{\sim 4}{\boldsymbol{N}}_{(\widetilde{\boldsymbol{B}}+\widetilde{\boldsymbol{B}}^{-1})} : \widetilde{\boldsymbol{D}}\ . \tag{7.85}$$

Property – *The relation between the Green-Naghdi objective rate of the left pure strain and $\widetilde{\boldsymbol{D}}$ is :*

$$\overset{orp}{\widetilde{\boldsymbol{V}}} \,.\, \widetilde{\boldsymbol{V}}^{-1} + \widetilde{\boldsymbol{V}}^{-1} \,.\, \overset{orp}{\widetilde{\boldsymbol{V}}} \;=\; 2\widetilde{\boldsymbol{D}} \;, \tag{7.86}$$

that is equivalent to

$$\widetilde{\boldsymbol{D}} \;=\; \frac{1}{2}\, \overset{\sim 4}{\boldsymbol{N}}_{(\widetilde{\boldsymbol{V}}^{-1})} : \overset{orp}{\widetilde{\boldsymbol{V}}} \;\;. \tag{7.87}$$

Indeed, according to the polar decomposition $\widetilde{\boldsymbol{F}} = \widetilde{\boldsymbol{R}} \,.\, \widetilde{\boldsymbol{U}}$, so

$$\overset{\bullet}{\widetilde{\boldsymbol{F}}} \;=\; \overset{\bullet}{\widetilde{\boldsymbol{R}}} \,.\, \widetilde{\boldsymbol{U}} + \widetilde{\boldsymbol{R}} \,.\, \overset{\bullet}{\widetilde{\boldsymbol{U}}} \tag{7.88}$$

$$\widetilde{\boldsymbol{L}} \,.\, \widetilde{\boldsymbol{R}} \,.\, \widetilde{\boldsymbol{U}} \;=\; \overset{\bullet}{\widetilde{\boldsymbol{R}}} \,.\, \widetilde{\boldsymbol{U}} + \widetilde{\boldsymbol{R}} \,.\, \overset{\bullet}{\widetilde{\boldsymbol{U}}} \tag{7.89}$$

Then,

$$\widetilde{\boldsymbol{L}} = \overset{\bullet}{\widetilde{\boldsymbol{R}}} \,.\, \widetilde{\boldsymbol{R}}^{\mathbf{T}} + \widetilde{\boldsymbol{R}} \,.\, \overset{\bullet}{\widetilde{\boldsymbol{U}}} \,.\, \widetilde{\boldsymbol{U}}^{-1} \,.\, \widetilde{\boldsymbol{R}}^{\mathbf{T}} \,, \tag{7.90}$$

By transposing this relation,

$$\widetilde{\boldsymbol{L}}^{\mathbf{T}} = -\, \overset{\bullet}{\widetilde{\boldsymbol{R}}} \,.\, \widetilde{\boldsymbol{R}}^{\mathbf{T}} + \widetilde{\boldsymbol{R}} \,.\, \widetilde{\boldsymbol{U}}^{-1} \,.\, \overset{\bullet}{\widetilde{\boldsymbol{U}}} \,.\, \widetilde{\boldsymbol{R}}^{\mathbf{T}} \,. \tag{7.91}$$

Now, we add the two preceding relations ;

$$2\widetilde{\boldsymbol{D}} \;=\; \widetilde{0} + \widetilde{\boldsymbol{R}} \,.\, \left(\overset{\bullet}{\widetilde{\boldsymbol{U}}} \,.\, \widetilde{\boldsymbol{U}}^{-1} + \widetilde{\boldsymbol{U}}^{-1} \,.\, \overset{\bullet}{\widetilde{\boldsymbol{U}}} \right) \,.\, \widetilde{\boldsymbol{R}}^{\mathbf{T}} \tag{7.92}$$

$$\begin{aligned} &= \left(\widetilde{\boldsymbol{R}} \,.\, \overset{\bullet}{\widetilde{\boldsymbol{U}}} \,.\, \widetilde{\boldsymbol{R}}^{\mathbf{T}} \right) . \left(\widetilde{\boldsymbol{R}} \,.\, \widetilde{\boldsymbol{U}}^{-1} \widetilde{\boldsymbol{R}}^{\mathbf{T}} \right) \\ &\quad + \left(\widetilde{\boldsymbol{R}} \,.\, \widetilde{\boldsymbol{U}}^{-1} \widetilde{\boldsymbol{R}}^{\mathbf{T}} \right) . \left(\widetilde{\boldsymbol{R}} \,.\, \overset{\bullet}{\widetilde{\boldsymbol{U}}} \,.\, \widetilde{\boldsymbol{R}}^{\mathbf{T}} \right) \;. \end{aligned} \tag{7.93}$$

Since, first, $\widetilde{\boldsymbol{V}} \,.\, \widetilde{\boldsymbol{R}} = \widetilde{\boldsymbol{R}} \,.\, \widetilde{\boldsymbol{U}}$, so their inverses are also equal, $\widetilde{\boldsymbol{R}}^{\mathbf{T}} \,.\, \widetilde{\boldsymbol{V}}^{-1} = \widetilde{\boldsymbol{U}}^{-1} \,.\, \widetilde{\boldsymbol{R}}^{\mathbf{T}}$, which leads to

$$\widetilde{\boldsymbol{V}}^{-1} = \widetilde{\boldsymbol{R}} \,.\, \widetilde{\boldsymbol{U}}^{-1} \,.\, \widetilde{\boldsymbol{R}}^{\mathbf{T}} \,, \tag{7.94}$$

and second, $\widetilde{\boldsymbol{R}} \,.\, \overset{\bullet}{\widetilde{\boldsymbol{U}}} \,.\, \widetilde{\boldsymbol{R}}^{\mathbf{T}} = \widetilde{\boldsymbol{R}} \,.\, \left(\overset{\bullet}{\widetilde{\boldsymbol{V}}}^{\backslash rp} \right) \,.\, \widetilde{\boldsymbol{R}}^{\mathbf{T}} = \overset{orp}{\widetilde{\boldsymbol{V}}}$.

Consequently,

$$2\widetilde{\boldsymbol{D}} = \overset{orp}{\widetilde{\boldsymbol{V}}} \,.\, \widetilde{\boldsymbol{V}}^{-1} + \widetilde{\boldsymbol{V}}^{-1} \,.\, \overset{orp}{\widetilde{\boldsymbol{V}}} \;\;. \tag{7.95}$$

Property – *An other relation between* $\overset{\circ rp}{\widetilde{\boldsymbol{V}}}$ and $\widetilde{\boldsymbol{D}}$ *was proven by [Hoger-1986] :*

$$\begin{aligned}\overset{\circ rp}{\widetilde{\boldsymbol{V}}} = \frac{1}{I_1.I_2 - I_3}\Big\{ & \widetilde{\boldsymbol{V}}^2 \boldsymbol{.} \widetilde{\boldsymbol{D}} \boldsymbol{.} \widetilde{\boldsymbol{V}}^2 - I_1 \left(\widetilde{\boldsymbol{V}}^2 \boldsymbol{.} \widetilde{\boldsymbol{D}} \boldsymbol{.} \widetilde{\boldsymbol{V}} + \widetilde{\boldsymbol{V}} \boldsymbol{.} \widetilde{\boldsymbol{D}} \boldsymbol{.} \widetilde{\boldsymbol{V}}^2\right) \\ & + (I_1^2 + I_2).\widetilde{\boldsymbol{V}} \boldsymbol{.} \widetilde{\boldsymbol{D}} \boldsymbol{.} \widetilde{\boldsymbol{V}} \; - \; I_3 \left(\widetilde{\boldsymbol{V}} \boldsymbol{.} \widetilde{\boldsymbol{D}} + \widetilde{\boldsymbol{D}} \boldsymbol{.} \widetilde{\boldsymbol{V}}\right) \\ & + I_1.I_3.\widetilde{\boldsymbol{D}} \Big\} \ , \end{aligned} \tag{7.96}$$

where the invariants are :
- $I_1 = \lambda_1 + \lambda_2 + \lambda_3$
- $I_2 = \lambda_1.\lambda_2 + \lambda_2.\lambda_3 + \lambda_3.\lambda_1$
- $I_3 = \lambda_1.\lambda_2.\lambda_3$

Property – *If the eigenvalues of* $\widetilde{\boldsymbol{V}}$ *(principal elongations* λ_i*) are distinct, the Green-Naghdi objective rate of the left logarithmic strain is related to the stretching tensor as follows :*

$$\overset{\circ rp}{\widetilde{\boldsymbol{LnV}}} = \widetilde{\boldsymbol{D}} + \widetilde{\boldsymbol{\Delta\Omega}} \boldsymbol{.} \widetilde{\boldsymbol{LnV}} - \widetilde{\boldsymbol{LnV}} \boldsymbol{.} \widetilde{\boldsymbol{\Delta\Omega}} + \frac{1}{2}\left(\widetilde{\boldsymbol{V}} \boldsymbol{.} \widetilde{\boldsymbol{\Delta\Omega}} \boldsymbol{.} \widetilde{\boldsymbol{V}}^{-1} - \widetilde{\boldsymbol{V}}^{-1} \boldsymbol{.} \widetilde{\boldsymbol{\Delta\Omega}} \boldsymbol{.} \widetilde{\boldsymbol{V}}\right) \tag{7.97}$$

where $\widetilde{\boldsymbol{\Delta\Omega}} = \left(\widetilde{\boldsymbol{\Omega_{dpV}}} - \widetilde{\boldsymbol{\Omega^{arp}}}\right)$ and $\widetilde{\boldsymbol{\Omega_{dpV}}}$ *is the rotation velocity of the eigendirections of* $\widetilde{\boldsymbol{V}}$ *from the reference frame a. This tensor* $\widetilde{\boldsymbol{\Delta\Omega}}$ *is skew-symmetric.*

Property – *According to [Hoger-1986], a relation between the Green-Naghdi objective rate of the left logarithmic strain* $\widetilde{\boldsymbol{LnV}}$ *and the stretching tensor* $\widetilde{\boldsymbol{D}}$ *is :*

$$\overset{\circ rp}{\widetilde{\boldsymbol{LnV}}} = \overset{\sim 4}{\boldsymbol{\gamma}}_{(\widetilde{\boldsymbol{V}})} : \widetilde{\boldsymbol{D}} \ , \tag{7.98}$$

The definition of fourth-order tensor $\overset{\sim 4}{\boldsymbol{\gamma}}_{(\widetilde{\boldsymbol{V}})}$ *is given by 1.160[p24].*

[Hoger-1986] gives also an other relation between $\overset{\circ rp}{\widetilde{\boldsymbol{LnV}}}$ and $\widetilde{\boldsymbol{D}}$. This relation is complicated but could be applied into a numerical calculus.

Property – *According to [Hoger-1986], a relation between the Jaumann objective rate of the left logarithmic strain* $\widetilde{\boldsymbol{LnV}}$ *and the stretching tensor* $\widetilde{\boldsymbol{D}}$ *is :*

$$\overset{\circ j}{\widetilde{\boldsymbol{LnV}}} = \overset{\sim 4}{\boldsymbol{\delta}}_{(\widetilde{\boldsymbol{V}})} : \widetilde{\boldsymbol{D}} \ . \tag{7.99}$$

The definition of fourth-order tensor $\overset{\sim 4}{\boldsymbol{\delta}}_{(\widetilde{\boldsymbol{V}})}$ is given by 1.173[p26].

Into the article [Scheidler-1991], relations are given, between the stretching tensor $\widetilde{\boldsymbol{D}}$ and the Green-Naghdi and Jaumann objective rates of all strain tensors.

Property – *A relation between the rotation velocity of the corotational Jaumann and the Green-Naghdi reference frames is :*

$$\widetilde{\widetilde{\boldsymbol{W}}} \;=\; \widetilde{\widetilde{\boldsymbol{\Omega}^{arp}}} + \frac{1}{2}\,\overset{\sim 4}{\boldsymbol{R}}_{(\widetilde{\boldsymbol{V}}^{-1})}:\overset{\circ rp}{\widetilde{\boldsymbol{V}}} \;. \tag{7.100}$$

The definition of fourth-order tensor $\overset{\sim 4}{\boldsymbol{R}}$ is given by 1.159[p24].

Property – *A relation between the Jaumann and the Green-Naghdi objective rates of a second-order tensor is :*

$$\overset{\circ rp}{\widetilde{\boldsymbol{A}}} = \overset{\circ j}{\widetilde{\boldsymbol{A}}} + \left[\overset{\sim 4}{\boldsymbol{R}}_{(\widetilde{\boldsymbol{A}})}:\overset{\sim 4}{\boldsymbol{R}}_{(\widetilde{\boldsymbol{V}}^{-1})}:\left(\overset{\sim 4}{\boldsymbol{N}}_{(\widetilde{\boldsymbol{V}}^{-1})}\right)^{-1}\right]:\widetilde{\boldsymbol{D}}\;. \tag{7.101}$$

The definitions of the fourth-order tensors $\overset{\sim 4}{\boldsymbol{N}}$ and $\overset{\sim 4}{\boldsymbol{R}}$ is given by 1.157[p24] and 1.159[p24].

Chapitre 8

Some approaches for proposing constitutive laws

In this chapter, we propose some ways in order to write constitutive laws for finite transformation.

Subsequently, $\mathcal{F}$ is a function of several tensor variables and whose result is also tensorial.

8.1 Approach 1 – with invariant tensors

$$\mathcal{F}\left(\overset{\bullet}{\widetilde{\boldsymbol{S}}},\ \overset{\bullet}{\widetilde{\boldsymbol{\varepsilon_U}}},\ \widetilde{\boldsymbol{S}},\ \widetilde{\boldsymbol{\varepsilon_U}} \right) \ = \ \widetilde{\boldsymbol{0}}\ , \tag{8.1}$$

where $\widetilde{\boldsymbol{\varepsilon_U}}$ is a right strain tensor.

For example :

$$\widetilde{\boldsymbol{S}} = \lambda.(\mathrm{Tr}\widetilde{\boldsymbol{LnU}}).\widetilde{\boldsymbol{I}} + 2\mu.\widetilde{\boldsymbol{LnU}}\ , \tag{8.2}$$

$$\overset{\bullet}{\widetilde{\boldsymbol{S}}} = \lambda.\mathrm{Tr}\left(\overset{\bullet}{\widetilde{\boldsymbol{E}}}\right).\widetilde{\boldsymbol{I}} + 2\mu.\ \overset{\bullet}{\widetilde{\boldsymbol{E}}}\ . \tag{8.3}$$

8.2 Approach 2 – with objective tensors and from a material reference frame

We propose that a *material reference frame* is a frame *which rotates with the deformation of the material particle*. Let be m a material reference frame. It is material, if and only if :

$$\widetilde{\boldsymbol{R}^{am}} \;=\; \mathcal{R}(\widetilde{\boldsymbol{F}}) \;, \tag{8.4}$$

where $\mathcal{R}$ is a tensor-variable function, whose result is a rotation tensor, such that :

$$\forall \text{ rotation tensor } \widetilde{\boldsymbol{Q}}, \quad \forall \text{ second-order tensor } \widetilde{\boldsymbol{A}} \;/\; \det\widetilde{\boldsymbol{A}} > 0$$
$$\mathcal{R}(\widetilde{\boldsymbol{Q}} \boldsymbol{.} \widetilde{\boldsymbol{A}}) = \widetilde{\boldsymbol{Q}} \boldsymbol{.} \mathcal{R}(\widetilde{\boldsymbol{A}}) \;. \tag{8.5}$$

In the present lecture, we have presented two material reference frame ; the corotational frame j and the proper rotation frame rp. But other material reference frames may be proposed.

A constitutive law may be defined as :

1. Definition of a material reference frame m ; for example by defining the function $\mathcal{R}(\widetilde{\boldsymbol{F}})$.
2. $$\mathcal{F}\left(\overset{\bullet}{\widetilde{\boldsymbol{T}}}{}^{\backslash m},\; \overset{\bullet}{\widetilde{\varepsilon_{\boldsymbol{V}}}}{}^{\backslash m},\; \widetilde{\boldsymbol{T}}^{\backslash m},\; \widetilde{\varepsilon_{\boldsymbol{V}}}^{\backslash m} \right) \;=\; \widetilde{\boldsymbol{0}} \;, \tag{8.6}$$

   where $\widetilde{\varepsilon_{\boldsymbol{V}}}$ is a left strain tensor. An other objective stress tensor may replace the Cauchy stress tensor $\widetilde{\boldsymbol{T}}$.

8.3 Approach 3 – with objective tensors and an objective rate

A constitutive law may be defined as :

1. Definition of an objective rate $\overset{\circ}{\widetilde{\boldsymbol{A}}}$; for example the Jaumann or the Green-Naghdi rates, but other objective rates into a material reference frame may be proposed.
2. $$\mathcal{F}\left(\overset{\circ}{\widetilde{\boldsymbol{T}}},\; \overset{\circ}{\widetilde{\varepsilon_{\boldsymbol{V}}}},\; \widetilde{\boldsymbol{T}},\; \widetilde{\varepsilon_{\boldsymbol{V}}} \right) \;=\; \widetilde{\boldsymbol{0}} \;, \tag{8.7}$$

   where $\widetilde{\varepsilon_{\boldsymbol{V}}}$ is a left strain tensor. An other objective stress tensor may replace the Cauchy stress tensor $\widetilde{\boldsymbol{T}}$. Be careful, function $\mathcal{F}$ has to be **hemitropic** :

$$\begin{gathered}\forall \text{ rotation tensor } \widetilde{\boldsymbol{Q}}, \quad \forall \text{ second-order tensors } \widetilde{\boldsymbol{A}}, \widetilde{\boldsymbol{B}}, \widetilde{\boldsymbol{C}}, \widetilde{\boldsymbol{D}} \\ \mathcal{F}\left(\widetilde{\boldsymbol{Q}} \boldsymbol{\cdot} \widetilde{\boldsymbol{A}} \boldsymbol{\cdot} \widetilde{\boldsymbol{Q}}^{\mathbf{T}}, \widetilde{\boldsymbol{Q}} \boldsymbol{\cdot} \widetilde{\boldsymbol{B}} \boldsymbol{\cdot} \widetilde{\boldsymbol{Q}}^{\mathbf{T}}, \widetilde{\boldsymbol{Q}} \boldsymbol{\cdot} \widetilde{\boldsymbol{C}} \boldsymbol{\cdot} \widetilde{\boldsymbol{Q}}^{\mathbf{T}}, \widetilde{\boldsymbol{Q}} \boldsymbol{\cdot} \widetilde{\boldsymbol{D}} \boldsymbol{\cdot} \widetilde{\boldsymbol{Q}}^{\mathbf{T}}\right) = \\ \widetilde{\boldsymbol{Q}} \boldsymbol{\cdot} \mathcal{F}(\widetilde{\boldsymbol{A}}, \widetilde{\boldsymbol{B}}, \widetilde{\boldsymbol{C}}, \widetilde{\boldsymbol{D}}) \boldsymbol{\cdot} \widetilde{\boldsymbol{Q}}^{\mathbf{T}}\end{gathered} \tag{8.8}$$

For example :

$$\overset{\circ j}{\widetilde{\boldsymbol{T}}} = \lambda.\mathrm{Tr}\left(\overset{\circ j}{\widetilde{\boldsymbol{A}}}\right).\widetilde{\boldsymbol{I}} + 2\mu.\overset{\circ j}{\widetilde{\boldsymbol{A}}} , \tag{8.9}$$

$$\overset{\circ rp}{\widetilde{\boldsymbol{\tau}}} = \lambda.\mathrm{Tr}\left(\overset{\circ rp}{\widetilde{\boldsymbol{J}}}\right).\widetilde{\boldsymbol{I}} + 2\mu.\overset{\circ rp}{\widetilde{\boldsymbol{J}}} - \frac{2\mu}{Q_0^2}.\mathrm{Tr}\left(\underline{\widetilde{\boldsymbol{\tau}}} \boldsymbol{\cdot} \underline{\overset{\circ rp}{\widetilde{\boldsymbol{J}}}}\right).\underline{\widetilde{\boldsymbol{\tau}}} . \tag{8.10}$$

8.4 Approach 4 – with an objective stress tensor, an objective rate and the stretching tensor $\widetilde{\boldsymbol{D}}$

Many elastoplastic and elasto-viscoplastic constitutive laws are proposed according to this way :

1. Definition of an objective rate $\overset{\circ}{\widetilde{\boldsymbol{A}}}$.
2. $$\mathcal{F}\left(\overset{\circ}{\widetilde{\boldsymbol{T}}},\ \widetilde{\boldsymbol{D}},\ \widetilde{\boldsymbol{T}}\right) = \widetilde{\mathbf{0}} , \tag{8.11}$$

   where the function $\mathcal{F}$ has to be hemitropic (rel. 8.8). An other objective stress tensor may replace the Cauchy stress tensor $\widetilde{\boldsymbol{T}}$.

For example :

$$\overset{\circ j}{\widetilde{\boldsymbol{T}}} = \lambda.(\mathrm{Tr}\widetilde{\boldsymbol{D}}).\widetilde{\boldsymbol{I}} + 2\mu.\widetilde{\boldsymbol{D}} , \tag{8.12}$$

$$\overset{\circ j}{\widetilde{\boldsymbol{T}}} = \lambda.(\mathrm{Tr}\widetilde{\boldsymbol{D}}).\widetilde{\boldsymbol{I}} + 2\mu.\widetilde{\boldsymbol{D}} - \frac{2\mu}{Q_0^2}.\mathrm{Tr}\left(\underline{\widetilde{\boldsymbol{T}}} \boldsymbol{\cdot} \underline{\widetilde{\boldsymbol{D}}}\right).\underline{\widetilde{\boldsymbol{T}}} . \tag{8.13}$$

Exercise 8.1. *Let us propose the following constitutive elastic law ;*

$$\widetilde{\boldsymbol{S}} = \lambda.(\mathrm{Tr}\widetilde{\boldsymbol{E}}).\widetilde{\boldsymbol{I}} + 2\mu.\widetilde{\boldsymbol{E}} \ , \tag{8.14}$$

This is the Hooke's law with the stress-strain couple $(\widetilde{\boldsymbol{S}}, \widetilde{\boldsymbol{E}})$*. Parameters* (λ, μ) *are the Lamé's elasticity coefficients and are constant :*

1. *which approach is used here ?*
2. *write the same law with approach 3.*
3. *write the same law with approach 1 but between only stress and strain rates.*

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