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# A Gaussian correlation inequality for plurisubharmonic functions

F. Barthe and D. Cordero-Erausquin

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## Abstract

A positive correlation inequality is established for circular-invariant plurisubharmonic functions, with respect to complex Gaussian measures. The main ingredients of the proofs are the Ornstein-Uhlenbeck semigroup, and another natural semigroup associated to the Gaussian  $\bar{\partial}$ -Laplacian.

## 1 Introduction

The motivation of the present work comes from a Gaussian moment inequality in  $\mathbb{C}^n$  due to Arias de Reyna [1]. We will show that his result is in fact a very particular case of a new correlation inequality, which can be seen as the complex analogue of the following correlation inequality for convex function in  $\mathbb{R}^n$  due to Hu [6]: if  $\mu$  is a centered Gaussian measure on  $\mathbb{R}^n$  and if  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$  are *convex* functions in  $L^2(\mu)$  and  $f$  is *even*, then

$$\int f g d\mu \geq \int f d\mu \int g d\mu.$$

We will say that a function on  $\mathbb{C}^n$  is *circular-symmetric* if it is invariant under the action of  $S^1$  (i.e. multiplication by complex numbers of modulus one) ; in other words a function  $f$  defined on  $\mathbb{C}^n$  is circular-symmetric if

$$f(e^{i\theta}w) = f(w) \quad \forall \theta \in \mathbb{R}, \quad \forall w \in \mathbb{C}^n.$$

A function  $u : \mathbb{C}^n \rightarrow [-\infty, +\infty)$  is plurisubharmonic (psh) if it is upper semi-continuous and for all  $a, b \in \mathbb{C}^n$  the function  $z \in \mathbb{C} \mapsto u(a + zb)$  is subharmonic. Classically, a twice continuously differentiable function  $u : \mathbb{C}^n \rightarrow \mathbb{R}$  is psh if and only if for all  $w, z \in \mathbb{C}^n$

$$\sum_{j,k} \partial_{z_j \bar{z}_k}^2 u(z) w_j \bar{w}_k \geq 0,$$

where  $\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$ ,  $\partial_{\bar{z}_j} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$ ,  $z = x + iy$  with  $x, y \in \mathbb{R}^n$ . The later condition means that the complex Hessian  $D_{\mathbb{C}}^2 u$  is pointwise Hermitian semi-definite positive. We refer e.g. to the textbook [5] for more details.

We consider the standard complex Gaussian measure  $\gamma$  on  $\mathbb{C}^n$ ,

$$d\gamma(w) = d\gamma_n(w) = \pi^{-n} e^{-w \cdot \bar{w}} d\ell(w) = \pi^{-n} e^{-|w|^2} d\ell(w),$$

where  $\ell$  denotes the Lebesgue measure on  $\mathbb{C}^n \simeq \mathbb{R}^{2n}$  and  $w \cdot w' = \sum w_j w'_j$  for  $w, w' \in \mathbb{C}^n$ . For convenience, let us introduce the following class of  $L^2(\gamma)$  functions with controlled growth at infinity:

$$\mathcal{G} := \left\{ f : \mathbb{C}^n \rightarrow \mathbb{C}; f \in L^2_{loc}(\lambda) \text{ and } \exists \epsilon, c, C > 0 \text{ such that } |f(w)| \leq e^{c|w|^{2-\epsilon}}, \forall |w| \geq C \right\}.$$

In particular any function (locally  $L^2$ ) dominated by a polynomial function on  $\mathbb{R}^{2n}$  belongs to  $\mathcal{G}$ . Our main result reads as follows:

**Theorem 1** (Correlation for psh functions). *Let  $f, g : \mathbb{C}^n \rightarrow [-\infty, \infty)$  be two plurisubharmonic functions belonging to  $\mathcal{G}$ . If  $f$  is circular-symmetric, then*

$$\int f g d\gamma \geq \int f d\gamma \int g d\gamma.$$

One can extend the result by approximation to more general psh functions in  $L^2(\gamma)$ . The inequality also extends to arbitrary centered complex Gaussian measure, which are images of  $\gamma$  by  $\mathbb{C}$ -linear maps. Indeed composing a psh function with a  $\mathbb{C}$ -linear map gives another psh function.

Let us give some direct consequences of this theorem. First, we see that when  $F, G$  are holomorphic functions, or simply complex polynomial functions, belonging to  $L^2(\gamma)$  and  $F$  is homogeneous, then for any  $\alpha, \beta \geq 0$  we have

$$\int |F|^\alpha |G|^\beta d\gamma \geq \int |F|^\alpha d\gamma \int |G|^\beta d\gamma.$$

Indeed, if  $F$  is holomorphic  $f = |F|^\beta$  is psh, and if  $F$  is homogeneous, then  $f$  is also circular-symmetric. This argument can also be used for products of the form

$$f := |F_1|^{\alpha_1} \dots |F_k|^{\alpha_k}$$

where the  $F_j$  are holomorphic and the  $\alpha_j$ 's are nonnegative real numbers, so that  $f$  is log-psh, in the sense that

$$\log f(w) = \sum_{\ell=1}^k \alpha_\ell \log |F_\ell(w)|$$

is psh. This implies that  $f$  is also psh, and if the holomorphic functions  $F_j$  are homogeneous then  $f$  is also circular-symmetric.

**Theorem 2.** *Let  $F_1, \dots, F_N$  be a family of homogeneous polynomial functions on  $\mathbb{C}^n$ . Then for any  $\alpha, \dots, \alpha_N \geq 0$  and  $k \leq N - 1$  we have*

$$\int \prod_{j=1}^N |F_j|^{\alpha_j} d\gamma \geq \left( \int \prod_{j=1}^k |F_j|^{\alpha_j} d\gamma \right) \left( \int \prod_{j=k+1}^N |F_j|^{\alpha_j} d\gamma \right) \geq \prod_{j=1}^N \int |F_j|^{\alpha_j} d\gamma.$$

A standard complex Gaussian vector in  $\mathbb{C}^n$  is a random vector taking values in  $\mathbb{C}^n$  according to the distribution  $\gamma = \gamma_n$ . A random vector  $X = (X_1, \dots, X_N) \in \mathbb{C}^N$  is a centered complex Gaussian vector if there is an  $n$ , a standard complex Gaussian vector  $G$  in  $\mathbb{C}^n$  and a  $\mathbb{C}$ -linear map  $A : \mathbb{C}^n \rightarrow \mathbb{C}^N$  such that  $X = AG$ . It turns out that the law for  $X$  is then characterized by its complex covariance matrix  $(\mathbb{E}(X_k \overline{X_\ell}))_{1 \leq k, \ell \leq N}$ . Denoting by  $a_1, \dots, a_N \in \mathbb{C}^n$  the rows the matrix of  $A$  in the canonical basis,  $X_j = G \cdot a_j$ . Applying the later theorem to the complex linear forms  $F_j(w) = w \cdot a_j$  yields the following result.

**Theorem 3.** *Let  $(X_1, \dots, X_N) \in \mathbb{C}^N$  be a centered complex Gaussian vector, and let  $\alpha_1, \dots, \alpha_N \in \mathbb{R}^+$ . Then, for any  $k \leq N - 1$*

$$\mathbb{E} \prod_{j=1}^N |X_j|^{\alpha_j} \geq \left( \mathbb{E} \prod_{j=1}^k |X_j|^{\alpha_j} \right) \left( \mathbb{E} \prod_{j=k+1}^N |X_j|^{\alpha_j} \right) \quad (1)$$

and in particular

$$\mathbb{E} \prod_{j=1}^N |X_j|^{\alpha_j} \geq \prod_{j=1}^N \mathbb{E} |X_j|^{\alpha_j}. \quad (2)$$

In other words, among centered complex Gaussian vectors  $(X_1, \dots, X_N) \in \mathbb{C}^N$  with fixed diagonal covariance (i.e.  $(\mathbb{E}|X_j|^2)_{j \leq N}$  fixed) the expectation of  $\prod_{j=1}^N |X_j|^{\alpha_j}$  is minimal when the variables are independent.

Inequality (2) is an extension of an inequality of Arias de Reyna [1], who established the particular case where all the  $\alpha_j = 2p_j$  are even integer by rewriting the left hand side in terms of a permanent of a  $2m$  matrix ( $m = \sum p_j$ ) and using an inequality for permanents due to Lieb. Actually, Inequality (1) in the case where the  $\alpha_j$  are even integers is equivalent to Lieb's permanent inequality, so in particular we are giving a new proof of this inequality.

In the next section we will introduce the tools that will be used in the proof, that is two semi-groups: the usual Ornstein-Uhlenbeck semi-group and another natural semi-group associated to the  $\bar{\partial}$  operator (the generator of which could be called, depending from the context, Landau or magnetic Laplacian). In the last section we give the proof of our correlation inequality.

## 2 Semi-groups

To get the result, we will let the circular-symmetric psh function evolve along the Ornstein-Uhlenbeck semi-group on  $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n \simeq \mathbb{R}^{2n}$  associated with the measure  $\gamma$  and the scalar product :  $\langle w, w' \rangle = \Re(w \cdot \overline{w'})$ . We recall that its generator is given, for smooth functions  $f$ , writing  $w = x + iy$ ,  $x, y \in \mathbb{R}^n$ , by

$$\begin{aligned} L^{\text{ou}} f(w) &:= \frac{1}{4} \Delta f(w) - \frac{1}{2} \langle w, \nabla \rangle f(w) \\ &= \sum_{j=1}^n \frac{1}{4} \left( \partial_{x_j x_j}^2 f(w) + \partial_{y_j y_j}^2 f(w) \right) - \frac{1}{2} (x_j \partial_{x_j} f(w) + y_j \partial_{y_j} f(w)). \end{aligned}$$

Note that the normalization differs slightly from the usual one on  $\mathbb{R}^{2n}$  because our Gaussian measure has complex covariance  $\text{Id}_n$  but real covariance equal to  $\frac{1}{2} \text{Id}_{2n}$ . Accordingly, the spectrum of  $-L^{\text{ou}}$  on  $L^2(\gamma)$  is here  $\frac{1}{2} \mathbb{N} = \{0, \frac{1}{2}, 1, \dots\}$ . The Ornstein-Uhlenbeck semi-group  $P_t^{\text{ou}} = e^{tL^{\text{ou}}}$  admits the representation, for suitable functions  $f : \mathbb{C}^n \rightarrow \mathbb{C}$ ,

$$P_t^{\text{ou}} f(z) = \int f(e^{-t/2} z + \sqrt{1 - e^{-t}} w) d\gamma(w) \quad (3)$$

$$= \pi^{-n} (1 - e^{-t})^{-n} \int f(w) e^{-\frac{1}{1-e^{-t}} |w - e^{-t/2} z|^2} d\ell(w) \quad (4)$$

As usual in semi-group methods, it is convenient to work with a nice stable space of functions. Here, we can for instance consider

$$\mathcal{G}^\infty := \{f \in C^\infty(\mathbb{C}^n); f \text{ and all its derivatives belong to } \mathcal{G}\}.$$

Note that for  $f \in \mathcal{G}$ , we can define  $P_t^{\text{ou}} f$  by (3) and then we have that  $(t, z) \mapsto P_t^{\text{ou}} f(z)$  is smooth on  $(0, \infty) \times \mathbb{C}^n$ , with  $P_t^{\text{ou}} f \in \mathcal{G}^\infty$  and  $\partial_t P_t^{\text{ou}} f = L^{\text{ou}} P_t^{\text{ou}} f$  for every  $t > 0$ , and  $P_t^{\text{ou}} f \rightarrow f$  in  $L^2(\gamma)$  as  $t \rightarrow 0$ . We refer to [3] for details. Let us also mention here that with formula (3) it is readily checked that properties like convexity, subharmonicity, pluri-subharmonicity are preserved along  $P_t^{\text{ou}}$ .

Another natural operator will be used. Indeed, since pluri-subharmonicity is characterized through a “ $\partial_{\bar{z}\bar{z}}$  operation”, we shall also use the following differential operator on smooth functions  $f$  on  $\mathbb{C}^n$ :

$$Lf = \sum_{j=1}^n \left( \partial_{z_j \bar{z}_j}^2 f - \bar{z}_j \partial_{\bar{z}_j} f \right) = \sum_{j=1}^n e^{|z|^2} \partial_{z_j} (e^{-|z|^2} \partial_{\bar{z}_j} f)$$

Note that  $Lf = 0$  if (and only if, see below)  $f$  is holomorphic. Formally  $L = -\bar{\partial}^* \bar{\partial}$  on  $L^2(\gamma)$  equipped with the Hermitian structure  $(f, g) = \int f \bar{g} d\gamma$ . More precisely, denoting for a differentiable function

$$\partial_{\bar{z}} f = (\partial_{\bar{z}_1} f, \dots, \partial_{\bar{z}_n} f) \quad \text{and} \quad \partial_z f := (\partial_{z_1} f, \dots, \partial_{z_n} f)$$

we have the following standard fact.

**Fact 4** (Integration by parts). *For regular enough functions  $f, g : \mathbb{C}^n \rightarrow \mathbb{C}$ , for instance for functions in  $\mathcal{G}^\infty$ , we have*

$$\int (Lf) \bar{g} d\gamma = - \int \partial_{\bar{z}} f \cdot \overline{\partial_{\bar{z}} g} d\gamma \quad (5)$$

and in particular  $\int (Lf) \bar{f} d\gamma = - \int |\partial_{\bar{z}} f|^2 d\gamma \leq 0$ . We can also write

$$\int (Lf) g d\gamma = - \int \partial_{\bar{z}} f \cdot \partial_z g d\gamma = \int f(\bar{L}g) d\gamma \quad (6)$$

where  $\bar{L}f := \sum_{j=1}^n \left( \partial_{z_j \bar{z}_j}^2 f - z_j \partial_{z_j} f \right)$ .

Indeed, it suffices to sum over  $j$  the equations

$$\int [e^{|z|^2} \partial_{z_j} (e^{-|z|^2} \partial_{\bar{z}_j} f)] \bar{g} d\gamma = \pi^{-n} \int \partial_{z_j} (e^{-|z|^2} \partial_{\bar{z}_j} f) \bar{g} d\lambda(z) = - \int \partial_{\bar{z}_j} f \partial_{z_j} \bar{g} d\gamma = - \int \partial_{\bar{z}_j} f \overline{\partial_{z_j} g} d\gamma.$$

The assumption that  $f, g \in \mathcal{G}^\infty$  guarantees that the boundary terms (at infinity) in the integration by parts vanish.

As a consequence, we see that the Gaussian measure  $\gamma$  is invariant for  $L$ , and actually that  $L$  is a symmetric nonpositive operator on  $L^2(\gamma)$  with the above-mentioned Hermitian structure. The kernel of  $L$  is the Bargmann space  $\mathcal{H}_0$  formed by the holomorphic functions on  $\mathbb{C}^n$  that belong to  $L^2(\gamma)$ .

We want to work with the semi-group  $P_t = e^{tL}$  which is also Hermitian (formally):

$$\int (P_t f) \bar{g} d\gamma = \int f \overline{P_t g} d\gamma \quad (7)$$

Although we will not explicitly use it, let us discuss a bit the (well known) spectral analysis of  $L$  on the complex Hilbert space  $L^2(\gamma)$ . This analysis is indeed fairly standard using the ideas introduced by Landau. Following for instance the presentation given in [4, Section 4], consider the ‘‘annihilation’’ operators  $a_j = \partial_{\bar{z}_j}$  and their adjoints, the ‘‘creation’’ operators  $b_j := a_j^* = \bar{z}_j - \partial_{z_j}$ . Then  $L = - \sum_{j \leq n} b_j \circ a_j$ , with  $[a_j, b_j] = 1$ , and all these operators commute for distinct indices  $j$ . Plainly, if a function  $f$  and a scalar  $\lambda \in \mathbb{C}$  satisfy  $-Lf = \lambda f$ , then  $-L(a_j f) = (\lambda - 1)a_j f$  and  $-L(b_j f) = (\lambda + 1)b_j f$ . This implies that the spectrum of  $-L$  is  $\mathbb{N}$  and that the eigenspace associated to the eigenvalue  $k \in \mathbb{N}$  is given by the sum of the spaces  $b^m \mathcal{H}_0$  with  $m = (m_1, \dots, m_n) \in \mathbb{N}^n$ ,  $|m| := \sum_{j \leq n} m_j = k$  and the convention  $b^m := b_1^{m_1} \circ \dots \circ b_n^{m_n}$ . Moreover, if we introduce the classical projection  $\Pi_0 : L^2(\gamma) \rightarrow \mathcal{H}_0$  onto holomorphic functions

$$\Pi_0 f(z) := \int f(w) e^{z \cdot \bar{w}} d\gamma(w) = \int f(z + w) e^{-\bar{z} \cdot w} d\gamma(w),$$

then the projector  $\Pi_k$  on the  $k$ -eigenspace can be expressed in terms of  $\Pi_0$  and the creation and annihilation operators. This allows to compute the reproducing kernel of  $\Pi_k$ , in terms of classical families of orthogonal polynomials. Next, one can sum over  $k$  and obtain the kernel  $K_t(z, w)$  for  $e^{tL} = \sum_k e^{-kt} \Pi_k$ . Only the formula for  $K_t$  will be useful in the sequel and we shall actually *check* below that this suggested formula is indeed the kernel of  $e^{tL}$ .

An explicit formula for  $K_t$  can be found in [2]: setting

$$K_t(z, w) := \frac{1}{\pi^n (1 - e^{-t})^n} \exp \left( z \cdot \bar{w} - \frac{e^{-t} |z - w|^2}{1 - e^{-t}} - |w|^2 \right), \quad (8)$$

then

$$\begin{aligned} P_t f(z) &= \int f(w) K_t(z, w) d\ell(w) \\ &= (1 - e^{-t})^{-n} \int f(w) e^{z \cdot \bar{w} - \frac{e^{-t} |z - w|^2}{1 - e^{-t}}} d\gamma(w) \end{aligned} \quad (9)$$

Next, let us note that by performing the change of variable  $w = z + \sqrt{1 - e^{-t}} \xi$  for fixed  $z$  we find

$$P_t f(z) = \int f(z + \sqrt{1 - e^{-t}} \xi) e^{-\sqrt{1 - e^{-t}} \bar{z} \cdot \xi} d\gamma(\xi). \quad (10)$$

On this formula, we see immediately that  $P_0 = \text{Id}$  and  $P_\infty = \Pi_0$ .

To avoid discussions regarding unbounded operators and existence of semi-groups, we will proceed in the opposite direction and use the previous formula to *define*  $P_t f$ . Actually, to be fair, we should mention that later, in the proof of our result, we only need to work with smooth functions  $f \in \mathcal{G}^\infty$ ; these functions provide nice initial data, ensuring existence and uniqueness of strong solutions for the semi-group equation. Nevertheless, we feel it is of independent interest to start from the integral formula (9) or (10) and derive from it the semi-group properties; we will first check that (9) solves (11) indeed. The drawback is that some properties that are obvious (formally) for  $e^{tL} f$  need to be checked thoroughly when using this kernel representation, in particular because the kernel (8) is not Markovian.

Formulas (9)-(10) make sense pointwise, for  $z \in \mathbb{C}^n$  fixed, as long as  $f \in L^2(\gamma)$ ; actually we have the pointwise estimate  $|P_t f(z)| \leq C_t(z) \|f\|_{L^2(\gamma)}$  for some constant  $C_t(z) > 0$  depending on  $t$  and  $z$ . In order to derive some properties of  $P_t f$ , a stronger integrability condition (as  $f \in \mathcal{G}$ ) will be assumed.

**Lemma 5.** *Given  $f \in \mathcal{G}$ , let us define  $P_t f$  using the formula (9) or (10). Then one has that  $(t, z) \rightarrow P_t f(z)$  is smooth on  $(0, \infty) \times \mathbb{C}^n$ , with for any  $t > 0$ ,  $P_t f \in \mathcal{G}_\infty$  and*

$$\partial_t P_t f = L P_t f. \quad (11)$$

Moreover we have

$$\|P_t f\|_{L^2(\gamma)} \leq \|f\|_{L^2(\gamma)}$$

and  $P_t f \rightarrow f$  in  $L^2(\gamma)$  as  $t \rightarrow 0$ .

*Proof.* It is readily checked that, for any fixed  $w \in \mathbb{C}^n$  one has

$$\partial_t K_t(\cdot, w) = L K_t(\cdot, w).$$

Moreover, for  $T, R, k > 0$  fixed, there exists constants  $c = c(T, R, k), C = C(T, R, k)$  such that for  $F = \partial K_t$ , or  $F = K_t$  or else  $F$  being any of the first  $k$ th partial derivatives of  $K_t$ , it holds that  $|F(z, w)| \leq C e^{-c|w|^2}$  for all  $w \in \mathbb{C}^n$ , all  $t \in (0, T)$  and all  $|z| \leq R$ . From this and the definition of  $\mathcal{G}$ , we can call upon dominated convergence to conclude to the smoothness of  $(t, z) \rightarrow P_t f(z)$  and to the fact that  $P_t f \in \mathcal{G}^\infty$  with  $\partial_t P_t f = L P_t f$ .

Regarding the contraction property, we want to avoid direct computations or spectral arguments, and so we make a detour and use some obvious but important properties of  $P_t$ .

First, we will use the semi-group property: for  $t, s > 0$  that  $P_t \circ P_s = P_{t+s}$ . This can be seen in two ways. One can invoke that for a *smooth* function  $F := P_{t_0} f \in \mathcal{G}^\infty$  then there is existence and uniqueness for the equation  $\partial_t F_t = L F_t$  with initial condition  $F_0 = F$  and we have seen that  $(P_{t_0+t} f)$  is a solution; from there one can conclude. Or else, in a more pedestrian way, one can check that

$$\int K_t(z, w) K_s(\xi, z) d\ell(z) = K_{t+s}(\xi, w).$$

For this, one may use that given  $z, \xi \in \mathbb{C}^n$  and  $c \in \mathbb{C}$  with  $\Re(c) > 0$ ,

$$\int e^{z \cdot w + \xi \cdot \bar{w}} e^{-c|w|^2} d\ell(w) = \pi^n c^{-n} e^{z \cdot \xi / c}. \quad (12)$$

Next, recall that  $P_t$  is Hermitian, in the sense (7), on  $\mathcal{G} \subset L^2(\gamma)$ ; this can be seen directly from the integral formula since  $K_t(z, w) = \overline{K_t(w, z)}$ . Finally, we will use that  $\|P_t f\|_{L^2(\gamma)}$  decreases for  $t \in (0, \infty)$ . This is immediate from the non-positivity of  $L$  since for  $t > 0$  we have  $P_t f \in \mathcal{G}^\infty$  and

$$\frac{d}{dt} \int |P_t f|^2 d\gamma = 2 \int (L P_t f) \overline{P_t f} d\gamma = -2 \int |\partial_{\bar{z}} P_t f|^2 d\gamma \leq 0.$$

So for our  $f \in \mathcal{G}$  and  $t > 0$  we have

$$\|P_t f\|_{L^2(\gamma)}^2 = \int (P_{2t} f) \bar{f} d\gamma \leq \|P_{2t} f\|_{L^2(\gamma)} \|f\|_{L^2(\gamma)} \leq \|P_t f\|_{L^2(\gamma)} \|f\|_{L^2(\gamma)}$$

which implies the contraction property in  $L^2(\gamma)$ .

To prove the continuity at  $t = 0$  in  $L^2(\gamma)$  we first assume that  $f$  is smooth and compactly supported. Using (10) we see that  $P_t f$  converge point-wise to  $f$  and that for  $t \in (0, 1)$  we have  $|P_t f(z)| \leq C e^{c|z|}$  for some constant  $c, C > 0$ ; so we can conclude by dominated convergence. For  $f \in \mathcal{G}$  and  $\epsilon > 0$ , introduce  $g$  smooth compactly supported such that  $\|f - g\|_{L^2(\gamma)} \leq \epsilon$  and let  $\delta > 0$  be such that  $t \leq \delta$  ensures that  $\|P_t g - g\|_{L^2(\gamma)} \leq \epsilon$  holds. For  $t \leq \delta$ ,

$$\|P_t f - f\|_{L^2(\gamma)} \leq \|P_t f - P_t g\|_{L^2(\gamma)} + \|P_t g - g\|_{L^2(\gamma)} + \|g - f\|_{L^2(\gamma)} \leq 2\|f - g\|_{L^2(\gamma)} + \epsilon \leq 3\epsilon.$$

This establishes the desired continuity.  $\square$

**Remark 6** (Contraction property).

1. We observe that some results, which can be deduced from the spectral decomposition and Hilbertian analysis, may be derived in a soft way thanks to flexible semigroup techniques. We have proved that starting from formula (9) we have  $\|P_t f\|_{L^2(\gamma)} \leq \|f\|_{L^2(\gamma)}$  on the dense subspace  $\mathcal{G}$ , which together with the pointwise estimate given just before the previous Lemma implies by density that

$$\|P_t\|_{L^2(\gamma) \rightarrow L^2(\gamma)} \leq 1.$$

Formally, by letting  $t \rightarrow \infty$  in  $\|P_t\|_{L^2(\gamma) \rightarrow L^2(\gamma)} \leq 1$  we recover that

$$\|\Pi_0\|_{L^2(\gamma) \rightarrow L^2(\gamma)} \leq 1.$$

Actually, the convergence of  $P_t$  towards  $\Pi_0$  can be quantified rigorously through Hörmander's inequality,

$$\|\varphi - \Pi_0 \varphi\|_{L^2(\gamma)}^2 \leq \|\partial_{\bar{z}} \varphi\|_{L^2(\gamma)}^2 = \int (-L\varphi) \varphi d\gamma$$

valid for any suitable  $\varphi$ , for instance for  $\varphi \in \mathcal{G}^\infty$ . Note that from formula (10),  $P_t$  acts as the identity on holomorphic functions, so  $P_t \Pi_0 = \Pi_0 P_t = \Pi_0$ . A classical Grönwall type argument (using the previous Lemma to justify the computation of  $\frac{d}{dt} \int |P_t(f - \Pi_0 f)|^2 d\gamma$ ) ensures that for  $f \in \mathcal{G}$  and  $t \geq 0$

$$\|P_t f - \Pi_0 f\|_{L^2(\gamma)}^2 \leq e^{-2t} \|f - \Pi_0 f\|_{L^2(\gamma)}^2.$$

2. In analogy with the Markovian case  $P_t^{\text{ou}}$  we may wonder if  $P_t$  is also a contraction on some  $L^p(\gamma)$ . However, for any  $p \neq 2$  we have

$$\|P_t\|_{L^p(\gamma) \rightarrow L^p(\gamma)} = +\infty,$$

as it can be seen by taking, in dimension  $n = 1$ , for  $a \in \mathbb{R}$ ,

$$f_a(w) := e^{aw + \bar{w}}, \quad w \in \mathbb{C}.$$

Indeed, repeated applications of (12) with  $c = 1$  show, setting  $s_t := \sqrt{1 - e^{-t}}$  and using (10), that  $P_t f_a(z) = e^{s_t^2 a} e^{az + (1 - s_t^2) \bar{z}}$  and that

$$\frac{\|P_t f_a\|_{L^p(\gamma)}^p}{\|f_a\|_{L^p(\gamma)}^p} = C(t, p) e^{a s_t^2 (p - p^2/2)}.$$

The next result describes how derivatives and  $P_t$  commute, an important issue in semi-group methods.

**Lemma 7** (Commutation relations). *For any suitable  $f$ , say  $f \in \mathcal{G}^\infty$ , and  $t > 0$  we have for every  $1 \leq j \leq n$  and  $z \in \mathbb{C}^n$ ,*

$$\partial_{z_j}(P_t f(z)) = P_t(\partial_{z_j} f)(z) \quad \text{and} \quad \partial_{\bar{z}_j}(P_t f(z)) = e^{-t} P_t(\partial_{\bar{z}_j} f)(z).$$

*Proof.* We use (10). The first equality is obvious. For the second one, setting  $s_t = \sqrt{1 - e^{-t}}$ , we have

$$\partial_{\bar{z}_j}(P_t f)(z) = P_t(\partial_{\bar{z}_j} f)(z) - s_t \int f(z + s_t \xi) e^{-s_t \bar{z}_j \xi} \xi_j d\gamma(\xi),$$

and

$$\begin{aligned} \pi^{-n} \int f(z + s_t \xi) e^{-s_t \bar{z}_j \xi} \xi_j e^{-\xi \cdot \bar{\xi}} d\ell(\xi) &= -\pi^{-n} \int f(z + s_t \xi) e^{-s_t \bar{z}_j \xi} \partial_{\xi_j}(e^{-\xi \cdot \bar{\xi}}) d\ell(\xi) \\ &= s_t \int (\partial_{\bar{z}_j} f)(z + s_t \xi) e^{-s_t \bar{z}_j \xi} d\gamma(\xi) = s_t P_t(\partial_{\bar{z}_j} f)(z). \end{aligned}$$

□

Now, and for the rest of this section, we focus on the case of circular-symmetric functions. Given  $\theta \in \mathbb{R}$  and a function  $f$  we denote  $f_\theta$  the function  $f_\theta(w) = f(e^{i\theta} w)$ . Note that

$$P_t(f_\theta) = (P_t f)_\theta. \quad (13)$$

Recall that a function  $f$  is said to be circular-symmetric if  $f_\theta = f$  for every  $\theta$ . It is worth noting that a holomorphic function on  $\mathbb{C}^n$  is necessarily constant when circular-symmetric. Indeed if  $h : \mathbb{C} \rightarrow \mathbb{C}$  has both properties then invariance and the Cauchy formula give  $h(1) = \int_0^{2\pi} h(e^{i\theta}) d\theta / (2\pi) = h(0)$ ; next if  $f : \mathbb{C}^n \rightarrow \mathbb{C}$  is holomorphic and circular symmetric, then for any  $(z_1, \dots, z_n) \in \mathbb{C}^n$ , the function  $h : z \in \mathbb{C} \mapsto f(z z_1, \dots, z z_n)$  is also holomorphic and circular-symmetric, hence  $f(z_1, \dots, z_n) = h(1) = h(0) = f(0)$ . Accordingly if  $f \in L^2(\gamma)$  is circular-symmetric then  $\Pi_0 f \equiv \int f d\gamma$  since the gaussian density is also circular-symmetric. Actually, much more can be said, as we shall see below.

Let us first investigate the relation between  $L$  and  $L^{\text{ou}}$ . Note that one can write

$$L f = L^{\text{ou}} f + \frac{i}{2} \sum_{j=1}^n (y_j \partial_{x_j} f - x_j \partial_{y_j} f). \quad (14)$$

So we have  $L^{\text{ou}} = \Re(L) = \frac{L + \bar{L}}{2}$  where  $\bar{L} f = \sum_{j=1}^n \left( \partial_{z_j}^2 \bar{z}_j f - z_j \partial_{z_j} f \right)$  has a kernel formed by the anti-holomorphic functions. The operators  $L$  and  $\bar{L}$  are Hermitian symmetric, whereas  $L^{\text{ou}}$  is symmetric for the real and the Hermitian product, and preserves the subspace of real valued functions. As we said, its spectrum is  $-\frac{1}{2}\mathbb{N}$ , as can be seen also from the formula  $L^{\text{ou}} = \frac{L + \bar{L}}{2}$ . Let us illustrate this on two examples, obtained by applying the creation operator  $b_1 = \bar{z}_1 - \partial_{z_1}$  to the holomorphic functions  $z \mapsto 1$ , and  $z \mapsto z_1$ . The function  $z \mapsto \bar{z}_1$  is an eigenfunction for  $L$  with eigenvalue  $-1$ , for  $\bar{L}$  with eigenvalue  $0$ , and for  $L^{\text{ou}}$  with eigenvalue  $-1/2$ . The function  $z \mapsto |z_1|^2 - 1$  is an eigenfunction for  $L$  with eigenvalue  $-1$ , for  $\bar{L}$  with eigenvalue  $-1$ , and for  $L^{\text{ou}}$  with eigenvalue  $-1$ .

The special role played by circular-symmetric functions is due to the fact that these operators, and the associated semi-groups, coincide for them.

**Lemma 8** (Action of  $L$  and  $P_t$  on circular-symmetric functions). *If  $f$  is a smooth circular-symmetric function, then we have*

$$L f = \bar{L} f = L^{\text{ou}} f.$$

*In particular we have, when  $f, g \in \mathcal{G}^\infty$  and  $f$  is circular-symmetric,*

$$\int (L f) g d\gamma = \int f L g d\gamma = - \int \partial_z f \cdot \partial_{\bar{z}} g d\gamma.$$



Also, if  $f \in L^2(\gamma)$  is circular-symmetric then we have

$$P_t f = P_t^{\text{ou}} f$$

for every  $t \geq 0$ .

*Proof.* Writing  $w = x + iy$ ,  $x, y \in \mathbb{R}^n$ , the symmetry rewrites as  $f((\cos(\theta)x - \sin(\theta)y) + i(\cos(\theta)y + \sin(\theta)x)) = f(x + iy)$ . Taking the derivative at  $\theta = 0$  we find

$$\sum_{j=1}^n \left( -y_j \partial_{x_j} f(x + iy) + x_j \partial_{y_j} f(x + iy) \right) = 0,$$

and this for every  $x, y \in \mathbb{R}^n$ . This implies in view of (14) that  $Lf = L^{\text{ou}} f = \overline{L}f$ . Next, for any smooth function  $g$  we have, using that  $Lf = \overline{L}f$  and (6),

$$\int (Lf)g \, d\gamma = \int (\overline{L}f)g \, d\gamma = \int fLg \, d\gamma = - \int \partial_z f \cdot \partial_{\overline{z}} g \, d\gamma.$$

Although it is formally trivial that equality of  $L$  and  $L^{\text{ou}}$  on circular-symmetric functions implies equality of the semi-groups  $P_t$  and  $P_t^{\text{ou}}$ , a bit more should be said since we defined the semi-group using the explicit formula (9). And it is anyway instructive to compute the kernels on circular-symmetric functions. Denote by  $K_t^{\text{ou}}$  the kernel of the Ornstein-Uhlenbeck semi-group that we recalled above:  $K_t^{\text{ou}}(z, w) = \pi^{-n} (1 - e^{-t})^{-n} e^{-\frac{1}{1-e^{-t}}|w - e^{-t/2}z|^2}$ . So we have, setting  $c_t := e^{-t/2}$  and  $s_t := \sqrt{1 - e^{-t}}$ ,

$$K_t^{\text{ou}}(z, w) = \pi^{-n} s_t^{-2n} e^{-s_t^{-2}|w|^2 - s_t^{-2}c_t^2|z|^2} e^{s_t^{-2}c_t(w \cdot \overline{z} + \overline{w} \cdot z)}$$

and

$$K_t(z, w) = \pi^{-n} s_t^{-2n} e^{-s_t^{-2}|w|^2 - s_t^{-2}c_t^2|z|^2} e^{s_t^{-2}(c_t^2 w \cdot \overline{z} + \overline{w} \cdot z)}.$$

Note that only the last exponentials differ in these two formulas. When  $f$  is circular-symmetric, in order to check that  $P_t f = P_t^{\text{ou}} f$  it suffices to check that for fixed  $w, z, t$  one has

$$\frac{1}{2\pi} \int_0^{2\pi} K_t(z, e^{i\theta} w) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} K_t^{\text{ou}}(z, e^{i\theta} w) \, d\theta.$$

Observe that for  $a, b \in \mathbb{C}$ , we have

$$\frac{1}{2\pi} \int_0^{2\pi} e^{ae^{i\theta} + be^{-i\theta}} \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \sum_{p, q \in \mathbb{N}} \frac{a^p b^q}{p! q!} e^{i(p-q)\theta} \, d\theta = \sum_{n \geq 0} \frac{(ab)^n}{(n!)^2} = B(ab)$$

with  $B(x) := \sum_{n \geq 0} \frac{x^n}{(n!)^2}$ . Therefore, we have

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} K_t(z, e^{i\theta} w) \, d\theta &= \pi^{-n} s_t^{-2n} e^{-s_t^{-2}|w|^2 - s_t^{-2}c_t^2|z|^2} B(s_t^{-4}c_t^2|w \cdot \overline{z}|^2) \\ &= \frac{1}{2\pi} \int_0^{2\pi} K_t^{\text{ou}}(z, e^{i\theta} w) \, d\theta, \end{aligned}$$

as wanted.  $\square$

### 3 Proof of Theorem 1

First, let us note that we can assume that  $g$  is smooth, and actually that  $g \in \mathcal{G}^\infty$ . Indeed, if  $g \in \mathcal{G}$  then  $P_t^{\text{ou}} g \in \mathcal{G}^\infty$  and we mentioned that  $P_t^{\text{ou}} g$  converges to  $g$  in  $L^2(\gamma)$  and therefore also in  $L^1(\gamma)$ , as  $t \rightarrow 0$ . Consequently, if we know the conclusion for a function in  $\mathcal{G}^\infty$ , then

$$\int f P_t^{\text{ou}} g \, d\gamma \geq \int f \, d\gamma \int P_t^{\text{ou}} g \, d\gamma$$

and by passing to the limit when  $t \rightarrow 0$  we know it also for  $g \in \mathcal{G}$ . For the same reason, we can assume that  $f \in \mathcal{G}^\infty$ , recalling that  $P_t^{\text{ou}}f$  is circular-symmetric when  $f$  is.

So in the sequel, we are given two psh functions  $f, g \in \mathcal{G}^\infty$ , with  $f$  circular-symmetric.

As in the proof of the correlation for convex functions [6], we will compute some kind of second derivative in  $t$  for integrals involving  $P_t^{\text{ou}}f$ ; recall that  $\partial_t P_t^{\text{ou}}f = L^{\text{ou}}P_t^{\text{ou}}f$ . The novelty is that, along the way, we will also use  $P_t f$  which satisfies  $\partial_t P_t f = L P_t f$  (Lemma 5).

Consider

$$\alpha(t) := \int (P_t^{\text{ou}}f) g d\gamma = \int (P_t f) g d\gamma \in \mathbb{R}.$$

The function  $\alpha$  is, by construction, smooth on  $(0, \infty)$  and continuous on  $[0, \infty)$  (see Lemma 5 for the continuity at zero). Since  $P_t^{\text{ou}}f$  tends to the constant  $\int f d\gamma$  when  $t \rightarrow \infty$ , we have

$$\alpha(t) \rightarrow \int f d\gamma \int g d\gamma$$

In order to conclude, it suffices to show that  $\alpha$  decreases. Actually we will prove that  $\alpha$  is convex; which is enough, since a convex function with a bounded limit at  $+\infty$  cannot increase. It holds

$$\alpha'(t) = \int L^{\text{ou}}(P_t^{\text{ou}}f) g d\gamma = \int L P_t f g d\gamma. \quad (15)$$

Since  $P_t f$  is also circular-symmetric, we can invoke Lemma 8 and write

$$\alpha'(t) = - \int \partial_z P_t f \cdot \partial_{\bar{z}} g d\gamma.$$

Next, using the first commutation relation from Lemma 7 we get

$$\alpha'(t) = - \sum_{j=1}^n \int P_t(\partial_{z_j} f) \partial_{\bar{z}_j} g d\gamma$$

We stress that  $\partial_z f$  is no longer circular-symmetric, so we cannot exchange  $P_t$  and  $P_t^{\text{ou}}$ . The second derivative of  $\alpha$  is, using Fact 4,

$$\alpha''(t) = - \sum_{j=1}^n \int L(P_t(\partial_{z_j} f)) \partial_{\bar{z}_j} g d\gamma = \sum_{j=1}^n \int \partial_{\bar{z}}(P_t(\partial_{z_j} f)) \cdot \partial_z(\partial_{\bar{z}_j} g) d\gamma = \sum_{j,k=1}^n \int \partial_{\bar{z}_k} P_t(\partial_{z_j} f) \partial_{z_k \bar{z}_j}^2 g d\gamma.$$

Using the commutation relation from Lemma 7 we can write

$$\alpha''(t) = \sum_{j,k=1}^n \int \partial_{\bar{z}_k z_j}^2 (P_t f) \partial_{z_k \bar{z}_j}^2 g d\gamma = \int \text{Tr} \left( (D_{\mathbb{C}}^2 P_t f)(z) (D_{\mathbb{C}}^2 g)(z) \right) d\gamma$$

where for a  $C^2$  function  $h$  on  $\mathbb{C}^n$  the notation  $D_{\mathbb{C}}^2 h(z)$  refers to the Hermitian matrix  $\left( \partial_{z_j \bar{z}_k}^2 h(z) \right)_{j,k \leq n}$ .

Since  $P_t f = P_t^{\text{ou}}f$  and  $g$  are psh, the corresponding matrices are nonnegative Hermitian matrices, which means that the trace of their product is still nonnegative. This shows that  $\alpha'' \geq 0$  and finishes the proof of the theorem.  $\square$

We would like to conclude with a discussion of the differences between the real case and the complex case. After all, we are computing second derivatives of the same object

$$\alpha(t) = \int (P_t^{\text{ou}}f) g d\gamma$$

along the Ornstein-Uhlenbeck semi-group exactly as in the case of convex functions, so what is going on?

In both cases we prove that  $\alpha$  decreases by showing that  $\alpha' \leq 0$  using the next derivative somehow, but we compute these derivatives differently. The argument for convex function goes as follows. A direct computation and usual commutation properties show that, if  $\int \nabla f d\gamma = 0$ , which is the case when  $f$  is even, then

$$\alpha'(t) = -e^{-t/2} \int_t^\infty \left( \int \operatorname{Tr} \left( (D^2 P_s^{\text{ou}} f)(z) (D^2 g)(z) \right) d\gamma(z) \right) e^{s/2} ds$$

where  $D^2$  refers to the usual (real) Hessian on  $\mathbb{R}^{2n}$ ; from this we conclude to the correlation for convex functions. On the other hand, we have proved, when  $f$  is circular-symmetric, that

$$\alpha'(t) = - \int_t^\infty \left( \int \operatorname{Tr} \left( (D_{\mathbb{C}}^2 P_s^{\text{ou}} f)(z) (D_{\mathbb{C}}^2 g)(z) \right) d\gamma(z) \right) ds.$$

Note that we have used here that  $\alpha'(t)$  tends to 0 when  $t \rightarrow +\infty$ : this follows from the fact that  $\alpha$  is convex and has a finite limit at  $+\infty$ , and can also be seen from (15) since  $P_t^{\text{ou}} f$  tends to a constant when  $t \rightarrow +\infty$ . It is because we wanted to work with complex derivatives that we aimed at inserting  $L$  in place of  $L^{\text{ou}}$ ; recall that  $\partial_{z_j} f$  need not be circular-symmetric even when  $f$  is, although the second derivatives are again circular-symmetric.

Finally, let us observe that if we consider in dimension 1 the circular-symmetric psh functions  $f(w) = |w|^{1/3}$  and  $g(w) = |w|^4$  on  $\mathbb{C} \simeq \mathbb{R}^2$ , then, as we already mentioned,

$$\operatorname{Tr} \left( (D_{\mathbb{C}}^2 f) (D_{\mathbb{C}}^2 g) \right) = \Delta f \Delta g \geq 0,$$

but a direct computation shows that

$$\operatorname{Tr} \left( (D^2 f)(z) (D^2 g)(z) \right) = -\frac{4}{3} |z|^{1/3} \leq 0 \quad \forall z \in \mathbb{C}.$$

Of course, this discrepancy cannot hold at all times for  $P_t^{\text{ou}} f$  in place of  $f$  (and moreover  $f$  is not smooth at zero, although this is not really an issue). But it suggests that the two formulas above for  $\alpha'(t)$  are indeed quite different.

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