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A Gaussian correlation inequality for plurisubharmonic functions

F. Barthe and D. Cordero-Erausquin

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Abstract

A positive correlation inequality is established for circular-invariant plurisubharmonic functions, with respect to complex Gaussian measures. The main ingredients of the proofs are the Ornstein-Uhlenbeck semigroup, and another natural semigroup associated to the Gaussian $\overline{\partial}$-Laplacian.

1 Introduction

The motivation of the present work comes from a Gaussian moment inequality in $\mathbb{C}^n$ due to Arias de Reyna [1]. We will show that his result is in fact a very particular case of a new correlation inequality, which can be seen as the complex analogue of the following correlation inequality for convex function in $\mathbb{R}^n$ due to Hu [6]: if μ is a centered Gaussian measure on $\mathbb{R}^n$ and if $f, g : \mathbb{R}^n \to \mathbb{R}$ are *convex* functions in $L^2(\mu)$ and f is *even*, then

$$\int f\, g\, d\mu \geq \int f\, d\mu \int g\, d\mu.$$

We will say that a function on $\mathbb{C}^n$ is *circular-symmetric* if it is invariant under the action of S^1 (i.e. multiplication by complex numbers of modulus one) ; in other words a function f defined on $\mathbb{C}^n$ is circular-symmetric if

$$f(e^{i\theta} w) = f(w) \qquad \forall \theta \in \mathbb{R}, \quad \forall w \in \mathbb{C}^n.$$

A function $u : \mathbb{C}^n \to [-\infty, +\infty)$ is plurisubharmonic (psh) if it is upper semi-continuous and for all $a, b \in \mathbb{C}^n$ the function $z \in \mathbb{C} \mapsto u(a + zb)$ is subharmonic. Classically, a twice continuously differentiable function $u : \mathbb{C}^n \to \mathbb{R}$ is psh if and only if for all $w, z \in \mathbb{C}^n$

$$\sum_{j,k} \partial^2_{z_j \overline{z_k}} u(z)\, w_j \overline{w_k} \geq 0,$$

where $\partial_{z_j} = \frac{1}{2}\big(\partial_{x_j} - i\partial_{y_j}\big)$, $\partial_{\overline{z_j}} = \frac{1}{2}\big(\partial_{x_j} + i\partial_{y_j}\big)$, $z = x + iy$ with $x, y \in \mathbb{R}^n$. The later condition means that the complex Hessian $D^2_{\mathbb{C}} u$ is pointwize Hermitian semi-definite positive. We refer e.g. to the textbook [5] for more details.

We consider the standard complex Gaussian measure γ on $\mathbb{C}^n$,

$$d\gamma(w) = d\gamma_n(w) = \pi^{-n} e^{-w \cdot \overline{w}} d\ell(w) = \pi^{-n} e^{-|w|^2} d\ell(w),$$

where ℓ denotes the Lebesgue measure on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ and $w \cdot w' = \sum w_j w'_j$ for $w, w' \in \mathbb{C}^n$. For convenience, let us introduce the following class of $L^2(\gamma)$ functions with controlled growth at infinity:

$$\mathcal{G} := \Big\{ f : \mathbb{C}^n \to \mathbb{C}\,;\, f \in L^2_{loc}(\lambda) \,\text{and}\, \exists \epsilon, c, C > 0 \,\text{such that}\, |f(w)| \leq e^{c|w|^{2-\epsilon}},\, \forall |w| \geq C \Big\}.$$

In particular any function (locally L^2) dominated by a polynomial function on $\mathbb{R}^{2n}$ belongs to $\mathcal{G}$. Our main result reads as follows:

Theorem 1 (Correlation for psh functions)**.** *Let $f, g : \mathbb{C}^n \to [-\infty, \infty)$ be two plurisubharmonic functions belonging to $\mathcal{G}$. If f is* circular-symmetric*, then*

$$\int f\, g\, d\gamma \geq \int f\, d\gamma \int g\, d\gamma.$$

One can extend the result by approximation to more general psh functions in $L^2(\gamma)$. The inequality also extends to arbitrary centered complex Gaussian measure, which are images of γ by $\mathbb{C}$-linear maps. Indeed composing a psh function with a $\mathbb{C}$-linear map gives another psh function.

Let us give some direct consequences of this theorem. First, we see that when F, G are holomorphic functions, or simply complex polynomial functions, belonging to $L^2(\gamma)$ and F is homogeneous, then for any $\alpha, \beta \geq 0$ we have

$$\int |F|^\alpha\, |G|^\beta\, d\gamma \geq \int |F|^\alpha\, d\gamma \int |G|^\beta\, d\gamma.$$

Indeed, if F is holomorphic $f = |F|^\beta$ is psh, and if F is homogeneous, then f is also circular-symmetric. This argument can also be used for products of the form

$$f := |F_1|^{\alpha_1} \dots |F_k|^{\alpha_k}$$

where the F_j are holomorphic and the α_j's are nonnegative real numbers, so that f is log-psh, in the sense that

$$\log f(w) = \sum_{\ell=1}^{k} \alpha_\ell\, \log |F_\ell(w)|$$

is psh. This implies that f is also psh, and if the holomorphic functions F_j are homogeneous then f is also circular-symmetric.

Theorem 2. *Let $F_1, \dots, F_N$ be a family of homogeneous polymomial functions on $\mathbb{C}^n$. Then for any $\alpha, \dots, \alpha_N \geq 0$ and $k \leq N-1$ we have*

$$\int \prod_{j=1}^{N} |F_j|^{\alpha_j}\, d\gamma \geq \Big(\int \prod_{j=1}^{k} |F_j|^{\alpha_j}\, d\gamma \Big) \Big(\int \prod_{j=k+1}^{N} |F_j|^{\alpha_j}\, d\gamma \Big) \geq \prod_{j=1}^{N} \int |F_j|^{\alpha_j}\, d\gamma.$$

A standard complex Gaussian vector in $\mathbb{C}^n$ is a random vector taking values in $\mathbb{C}^n$ according to the distribution $\gamma = \gamma_n$. A random vector $X = (X_1, \dots, X_N) \in \mathbb{C}^N$ is a centered complex Gaussian vector if there is an n, a standard complex Gaussian vector G in $\mathbb{C}^n$ and a $\mathbb{C}$-linear map $A : \mathbb{C}^n \to \mathbb{C}^N$ such that $X = AG$. It turns out that the law for X is then characterized by its complex covariance matrix $\big(\mathbb{E}(X_k \overline{X_\ell})\big)_{1 \leq k, \ell \leq N}$. Denoting by $a_1, \dots a_N \in \mathbb{C}^n$ the rows the matrix of A in the canonical basis, $X_j = G \cdot a_j$. Applying the later theorem to the complex linear forms $F_j(w) = w \cdot a_j$ yields the following result.

Theorem 3. *Let $(X_1, \dots, X_N) \in \mathbb{C}^N$ be a centered complex Gaussian vector, and let $\alpha_1, \dots, \alpha_N \in \mathbb{R}^+$. Then, for any $k \leq N-1$*

$$\mathbb{E} \prod_{j=1}^{N} |X_j|^{\alpha_j} \geq \Big(\mathbb{E} \prod_{j=1}^{k} |X_j|^{\alpha_j} \Big) \Big(\mathbb{E} \prod_{j=k+1}^{N} |X_j|^{\alpha_j} \Big) \tag{1}$$

and in particular

$$\mathbb{E} \prod_{j=1}^{N} |X_j|^{\alpha_j} \geq \prod_{j=1}^{N} \mathbb{E}\, |X_j|^{\alpha_j}. \tag{2}$$

In other words, among centered complex Gaussian vectors $(X_1, \dots, X_N) \in \mathbb{C}^N$ with fixed diagonal covariance (i.e. $(\mathbb{E}|X_j|^2)_{j \leq N}$ fixed) the expectation of $\prod_{j=1}^{N} |X_j|^{\alpha_j}$ is minimal when the variables are independent.

Inequality (2) is an extension of an inequality of Arias de Reyna [1], who established the particular case where all the $\alpha_j = 2p_j$ are even integer by rewriting the left hand side in terms of a permanent of a $2m$ matrix ($m = \sum p_j$) and using an inequality for permanents due to Lieb. Actually, Inequality (1) in the case where the α_j are even integers is equivalent to Lieb's permanent inequality, so in particular we are giving a new proof of this inequality.

In the next section we will introduce the tools that will be used in the proof, that is two semi-groups: the usual Ornstein-Uhlenbeck semi-group and another natural semi-group associated to the $\overline{\partial}$ operator (the generator of which could be called, depending from the context, Landau or magnetic Laplacian). In the last section we give the proof of our correlation inequality.

2 Semi-groups

To get the result, we will let the circular-symmetric psh function evolve along the Ornstein-Uhlenbeck semi-group on $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n \simeq \mathbb{R}^{2n}$ associated with the measure γ and the scalar product : $\langle w, w' \rangle = \Re(w \cdot \overline{w'})$. We recall that its generator is given, for smooth functions f, writing $w = x + iy$, $x, y \in \mathbb{R}^n$, by

$$\begin{aligned} L^{\mathrm{ou}} f(w) &:= \frac{1}{4}\Delta f(w) - \frac{1}{2}\langle w, \nabla \rangle f(w) \\ &= \sum_{j=1}^{n} \frac{1}{4}\left(\partial^2_{x_j x_j} f(w) + \partial^2_{y_j y_j} f(w)\right) - \frac{1}{2}\left(x_j \partial_{x_j} f(w) + y_j \partial_{y_j} f(w)\right). \end{aligned}$$

Note that the normalization differs slightly from the usual one on $\mathbb{R}^{2n}$ because our Gaussian measure has complex covariance Id_n but real covariance equal to $\frac{1}{2}\mathrm{Id}_{2n}$. Accordingly, the spectrum of $-L^{\mathrm{ou}}$ on $L^2(\gamma)$ is here $\frac{1}{2}\mathbb{N} = \{0, \frac{1}{2}, 1, \ldots\}$. The Ornstein-Uhlenbeck semi-group $P_t^{\mathrm{ou}} = e^{tL^{\mathrm{ou}}}$ admits the representation, for suitable functions $f : \mathbb{C}^n \to \mathbb{C}$,

$$P_t^{\mathrm{ou}} f(z) = \int f(e^{-t/2} z + \sqrt{1 - e^{-t}}\, w)\, d\gamma(w) \tag{3}$$

$$= \pi^{-n}(1 - e^{-t})^{-n} \int f(w)\, e^{-\frac{1}{1-e^{-t}}|w - e^{-t/2} z|^2}\, d\ell(w) \tag{4}$$

As usual in semi-group methods, it is convenient to work with a nice stable space of functions. Here, we can for instance consider

$$\mathcal{G}^\infty := \left\{ f \in C^\infty(\mathbb{C}^n)\,;\, f \text{ and all its derivatives belong to } \mathcal{G} \right\}.$$

Note that for $f \in \mathcal{G}$, we can *define* $P_t^{\mathrm{ou}} f$ by (3) and then we have that $(t, z) \mapsto P_t^{\mathrm{ou}} f(z)$ is smooth on $(0, \infty) \times \mathbb{C}^n$, with $P_t^{\mathrm{ou}} f \in \mathcal{G}^\infty$ and $\partial_t P_t^{\mathrm{ou}} f = L^{\mathrm{ou}} P_t^{\mathrm{ou}} f$ for every $t > 0$, and $P_t^{\mathrm{ou}} f \to f$ in $L^2(\gamma)$ as $t \to 0$. We refer to [3] for details. Let us also mention here that with formula (3) it is readily checked that properties like convexity, subharmonicity, pluri-subharmonicity are preserved along P_t^{ou}.

Another natural operator will be used. Indeed, since pluri-subharmonicity is characterized through a "$\partial^2_{z\overline{z}}$ operation", we shall also use the following differential operator on smooth functions f on $\mathbb{C}^n$:

$$Lf = \sum_{j=1}^{n} \left(\partial^2_{z_j \overline{z_j}} f - \overline{z_j}\, \partial_{\overline{z_j}} f\right) = \sum_{j=1}^{n} e^{|z|^2} \partial_{z_j}\left(e^{-|z|^2} \partial_{\overline{z_j}} f\right)$$

Note that $Lf = 0$ if (and only if, see below) f is holomorphic. Formally $L = -\overline{\partial}^* \overline{\partial}$ on $L^2(\gamma)$ equipped with the Hermitian structure $(f, g) = \int f \overline{g}\, d\gamma$. More precisely, denoting for a differentiable function

$$\partial_{\overline{z}} f = (\partial_{\overline{z_1}} f, \ldots, \partial_{\overline{z_n}} f) \quad \text{and} \quad \partial_z f := (\partial_{z_1} f, \ldots, \partial_{z_n} f)$$

we have the following standard fact.

Fact 4 (Integration by parts)**.** *For regular enough functions $f, g : \mathbb{C}^n \to \mathbb{C}$, for instance for functions in $\mathcal{G}^\infty$, we have*

$$\int (Lf)\, \overline{g}\, d\gamma = -\int \partial_{\overline{z}} f \cdot \overline{\partial_{\overline{z}} g}\, d\gamma \tag{5}$$

and in particular $\int (Lf)\, \overline{f}\, d\gamma = -\int |\partial_{\overline{z}} f|^2\, d\gamma \leq 0$. We can also write

$$\int (Lf)\, g\, d\gamma = -\int \partial_{\overline{z}} f \cdot \partial_z g\, d\gamma = \int f(\overline{L} g)\, d\gamma \tag{6}$$

where $\overline{L} f := \sum_{j=1}^n \left(\partial^2_{z_j \overline{z_j}} f - z_j\, \partial_{z_j} f \right)$.

Indeed, it suffices to sum over j the equations

$$\int \left[e^{|z|^2} \partial_{z_j} \left(e^{-|z|^2} \partial_{\overline{z_j}} f \right) \right] \overline{g}\, d\gamma = \pi^{-n} \int \partial_{z_j} \left(e^{-|z|^2} \partial_{\overline{z_j}} f \right) \overline{g}\, d\lambda(z) = -\int \partial_{\overline{z_j}} f\, \partial_{z_j} \overline{g}\, d\gamma = -\int \partial_{\overline{z_j}} f\, \overline{\partial_{\overline{z_j}} g}\, d\gamma.$$

The assumption that $f, g \in \mathcal{G}^\infty$ guarantees that the boundary terms (at infinity) in the integration by parts vanish.

As a consequence, we see that the Gaussian measure γ is invariant for L, and actually that L is a symmetric nonpositive operator on $L^2(\gamma)$ with the above-mentioned Hermitian structure. The kernel of L is the Bargmann space $\mathcal{H}_0$ formed by the holomorphic functions on $\mathbb{C}^n$ that belong to $L^2(\gamma)$.

We want to work with the semi-group $P_t = e^{tL}$ which is also Hermitian (formally):

$$\int (P_t f)\, \overline{g}\, d\gamma = \int f \overline{P_t g}\, d\gamma \tag{7}$$

Although we will not explicitly use it, let us discuss a bit the (well known) spectral analysis of L on the complex Hilbert space $L^2(\gamma)$. This analysis is indeed fairly standard using the ideas introduced by Landau. Following for instance the presentation given in [4, Section 4], consider the "annihilation" operators $a_j = \partial_{\overline{z_j}}$ and their adjoints, the "creation" operators $b_j := a_j^* = \overline{z_j} - \partial_{z_j}$. Then $L = -\sum_{j \leq n} b_j \circ a_j$, with $[a_j, b_j] = 1$, and all these operators commute for distinct indices j. Plainly, if a function f and a scalar $\lambda \in \mathbb{C}$ satisfy $-Lf = \lambda f$, then $-L(a_j f) = (\lambda - 1) a_j f$ and $-L(b_j f) = (\lambda + 1) b_j f$. This implies that the spectrum of $-L$ is $\mathbb{N}$ and that the eigenspace associated to the eigenvalue $k \in \mathbb{N}$ is given by the sum of the spaces $b^m \mathcal{H}_0$ with $m = (m_1, \ldots, m_n) \in \mathbb{N}^n$, $|m| := \sum_{j \leq n} m_j = k$ and the convention $b^m := b_1^{m_1} \circ \ldots \circ b_n^{m_n}$. Moreover, if we introduce the classical projection $\Pi_0 : L^2(\gamma) \to \mathcal{H}_0$ onto holomorphic functions

$$\Pi_0 f(z) := \int f(w)\, e^{z \cdot \overline{w}} d\gamma(w) = \int f(z + w)\, e^{-\overline{z} \cdot w} d\gamma(w),$$

then the projector Π_k on the k-eigenspace can be expressed in terms of Π_0 and the creation and annihilation operators. This allows to compute the reproducing kernel of Π_k, in terms of classical families of orthogonal polynomials. Next, one can sum over k and obtain the kernel $K_t(z, w)$ for $e^{tL} = \sum_k e^{-kt} \Pi_k$. Only the formula for K_t will be useful in the sequel and we shall actually *check* below that this suggested formula is indeed the kernel of e^{tL}.

An explicit formula for K_t can be found in [2]: setting

$$K_t(z, w) := \frac{1}{\pi^n (1 - e^{-t})^n} \exp\left(z \cdot \overline{w} - \frac{e^{-t} |z - w|^2}{1 - e^{-t}} - |w|^2 \right), \tag{8}$$

then

$$\begin{aligned} P_t f(z) &= \int f(w)\, K_t(z, w) d\ell(w) \\ &= (1 - e^{-t})^{-n} \int f(w)\, e^{z \cdot \overline{w} - \frac{e^{-t} |z - w|^2}{1 - e^{-t}}}\, d\gamma(w) \end{aligned} \tag{9}$$

Next, let us note that by performing the change of variable $w = z + \sqrt{1-e^{-t}}\,\xi$ for fixed z we find

$$P_t f(z) = \int f(z + \sqrt{1-e^{-t}}\,\xi)\, e^{-\sqrt{1-e^{-t}}\,\overline{z}\cdot\xi}\, d\gamma(\xi). \tag{10}$$

On this formula, we see immediately that $P_0 = \mathrm{Id}$ and $P_\infty = \Pi_0$.

To avoid discussions regarding unbounded operators and existence of semi-groups, we will proceed in the opposite direction and use the previous formula to *define* $P_t f$. Actually, to be fair, we should mention that later, in the proof of our result, we only need to work with smooth functions $f \in \mathcal{G}^\infty$; these functions provide nice initial data, ensuring existence and uniqueness of strong solutions for the semi-group equation. Nevertheless, we feel it is of independent interest to start from the integral formula (9) or (10) and derive from it the semi-group properties ; we will first check that (9) solves (11) indeed. The drawback is that some properties that are obvious (formally) for $e^{tL}f$ need to be checked thoroughly when using this kernel representation, in particular because the kernel (8) is not Markovian.

Formulas (9)-(10) make sense pointwise, for $z \in \mathbb{C}^n$ fixed, as long as $f \in L^2(\gamma)$; actually we have the pointwise estimate $|P_t f(z)| \le C_t(z)\|f\|_{L^2(\gamma)}$ for some constant $C_t(z) > 0$ depending on t and z. In order to derive some properties of $P_t f$, a stronger integrability condition (as $f \in \mathcal{G}$) will be assumed.

Lemma 5. *Given $f \in \mathcal{G}$, let us* define *$P_t f$ using the formula (9) or (10). Then one has that $(t,z) \to P_t f(z)$ is smooth on $(0,\infty)\times\mathbb{C}^n$, with for any $t > 0$, $P_t f \in \mathcal{G}_\infty$ and*

$$\partial_t P_t f = L P_t f. \tag{11}$$

Moreover we have

$$\|P_t f\|_{L^2(\gamma)} \le \|f\|_{L^2(\gamma)}$$

and $P_t f \to f$ in $L^2(\gamma)$ as $t \to 0$.

Proof. It is readily checked that, for any fixed $w \in \mathbb{C}^n$ one has

$$\partial_t K_t(\cdot, w) = L\, K_t(\cdot, w).$$

Moreover, for $T, R, k > 0$ fixed, there exists constants $c = c(T,R,k), C = C(T,R,k)$ such that for $F = \partial K_t$, or $F = K_t$ or else F being any of the first kth partial derivatives of K_t, it holds that $|F(z,w)| \le Ce^{-c|w|^2}$ for all $w \in \mathbb{C}^n$, all $t \in (0,T)$ and all $|z| \le R$. From this and the definition of $\mathcal{G}$, we can call upon dominated convergence to conclude to the smoothness of $(t,z) \to P_t f(z)$ and to the fact that $P_t f \in \mathcal{G}^\infty$ with $\partial_t P_t f = L P_t f$.

Regarding the contraction property, we want to avoid direct computations or spectral arguments, and so we make a detour and use some obvious but important properties of P_t.

Fist, we will use the semi-group property: for $t, s > 0$ that $P_t \circ P_s = P_{t+s}$. This can be seen in two ways. One can invoke that for a *smooth* function $F := P_{t_0} f \in \mathcal{G}^\infty$ then there is existence and uniqueness for the equation $\partial_t F_t = L F_t$ with initial condition $F_0 = F$ and we have seen that $(P_{t_0+t} f)$ is a solution; from there one can conclude. Or else, in a more pedestrian way, one can check that

$$\int K_t(z,w) K_s(\xi, z)\, d\ell(z) = K_{t+s}(\xi, w).$$

For this, one may use that given $z, \xi \in \mathbb{C}^n$ and $c \in \mathbb{C}$ with $\Re(c) > 0$,

$$\int e^{z\cdot w + \xi\cdot\overline{w}} e^{-c|w|^2}\, d\ell(w) = \pi^n c^{-n}\, e^{z\cdot\xi\,/c}. \tag{12}$$

Next, recall that P_t is Hermitian, in the sense (7), on $\mathcal{G} \subset L^2(\gamma)$; this can be seen directly from the integral formula since $K_t(z,w) = \overline{K_t(w,z)}$. Finally, we will use that $\|P_t f\|_{L^2(\gamma)}$ decreases for $t \in (0,\infty)$. This is immediate from the non-positivity of L since for $t > 0$ we have $P_t f \in \mathcal{G}^\infty$ and

$$\frac{d}{dt}\int |P_t f|^2 d\gamma = 2\int (L P_t f)\overline{P_t f}\, d\gamma = -2\int |\partial_{\overline{z}} P_t f|^2\, d\gamma \le 0.$$

So for our $f \in \mathcal{G}$ and $t > 0$ we have

$$\|P_t f\|^2_{L^2(\gamma)} = \int (P_{2t} f)\, \overline{f}\, d\gamma \leq \|P_{2t} f\|_{L^2(\gamma)}\, \|f\|_{L^2(\gamma)} \leq \|P_t f\|_{L^2(\gamma)}\, \|f\|_{L^2(\gamma)}$$

which implies the contraction property in $L^2(\gamma)$.

To prove the continuity at $t = 0$ in $L^2(\gamma)$ we first assume that f is smooth and compactly supported. Using (10) we see that $P_t f$ converge point-wise to f and that for $t \in (0, 1)$ we have $|P_t f(z)| \leq C\, e^{c|z|}$ for some constant $c, C > 0$; so we can conclude by dominated convergence. For $f \in \mathcal{G}$ and $\epsilon > 0$, introduce g smooth compactly supported such that $\|f - g\|_{L^2(\gamma)} \leq \epsilon$ and let $\delta > 0$ be such that $t \leq \delta$ ensures that $\|P_t g - g\|_{L^2(\gamma)} \leq \epsilon$ holds. For $t \leq \delta$,

$$\|P_t f - f\|_{L^2(\gamma)} \leq \|P_t f - P_t g\|_{L^2(\gamma)} + \|P_t g - g\|_{L^2(\gamma)} + \|g - f\|_{L^2(\gamma)} \leq 2\|f - g\|_{L^2(\gamma)} + \epsilon \leq 3\epsilon.$$

This establishes the desired continuity. □

Remark 6 (Contraction property)**.**

1. *We observe that some results, which can be deduced from the spectral decomposition and Hilbertian analysis, may be derived in a soft way thanks to flexible semigroup techniques. We have proved that starting from formula (9) we have $\|P_t f\|_{L^2(\gamma)} \leq \|f\|_{L^2(\gamma)}$ on the dense subspace $\mathcal{G}$, which together with the pointwise estimate given just before the previous Lemma implies by density that*

$$\|P_t\|_{L^2(\gamma) \to L^2(\gamma)} \leq 1.$$

Formally, by letting $t \to \infty$ in $\|P_t\|_{L^2(\gamma) \to L^2(\gamma)} \leq 1$ we recover that

$$\|\Pi_0\|_{L^2(\gamma) \to L^2(\gamma)} \leq 1.$$

Actually, the convergence of P_t towards Π_0 can be quantified rigorously through Hörmander's inequality,

$$\|\varphi - \Pi_0 \varphi\|^2_{L^2(\gamma)} \leq \|\partial_{\overline{z}} \varphi\|^2_{L^2(\gamma)} = \int (-L\varphi)\varphi\, d\gamma$$

valid for any suitable φ, for instance for $\varphi \in \mathcal{G}^\infty$. Note that from formula (10), P_t acts as the identity on holomorphic functions, so $P_t \Pi_0 = \Pi_0 P_t = \Pi_0$. A classical Grönwall type argument (using the previous Lemma to justify the computation of $\frac{d}{dt} \int |P_t(f - \Pi_0 f)|^2\, d\gamma$) ensures that for $f \in \mathcal{G}$ and $t \geq 0$

$$\|P_t f - \Pi_0 f\|^2_{L^2(\gamma)} \leq e^{-2t}\, \|f - \Pi_0 f\|^2_{L^2(\gamma)}.$$

2. *In analogy with the Markovian case P_t^{ou} we may wonder if P_t is also a contraction on some $L^p(\gamma)$. However, for any $p \neq 2$ we have*

$$\|P_t\|_{L^p(\gamma) \to L^p(\gamma)} = +\infty,$$

as it can be seen by taking, in dimension $n = 1$, for $a \in \mathbb{R}$,

$$f_a(w) := e^{aw + \overline{w}}, \qquad w \in \mathbb{C}.$$

Indeed,repeated applications of (12) with $c = 1$ show, setting $s_t := \sqrt{1 - e^{-t}}$ and using (10), that $P_t f_a(z) = e^{s_t^2 a} e^{az + (1 - s_t^2)\overline{z}}$ and that

$$\frac{\|P_t f_a\|^p_{L^p(\gamma)}}{\|f_a\|^p_{L^p(\gamma)}} = C(t, p)\, e^{a\, s_t^2 (p - p^2/2)}.$$

The next result describes how derivatives and P_t commute, an important issue in semi-group methods.

Lemma 7 (Commutation relations)**.** *For any suitable f, say $f \in \mathcal{G}^\infty$, and $t > 0$ we have for every $1 \le j \le n$ and $z \in \mathbb{C}^n$,*

$$\partial_{z_j}(P_t f(z)) = P_t(\partial_{z_j} f)(z) \qquad \text{and} \qquad \partial_{\overline{z_j}}(P_t f(z)) = e^{-t} P_t(\partial_{\overline{z_j}} f)(z).$$

Proof. We use (10). The first equality is obvious. For the second one, setting $s_t = \sqrt{1-e^{-t}}$, we have

$$\partial_{\overline{z_j}}(P_t f)(z) = P_t(\partial_{\overline{z_j}} f)(z) - s_t \int f(z + s_t\xi)\, e^{-s_t\, \overline{z}\xi}\, \xi_j\, d\gamma(\xi),$$

and

$$\begin{aligned}\pi^{-n} \int f(z + s_t\xi)\, e^{-s_t\, \overline{z}\xi}\, \xi_j e^{-\xi\cdot\overline{\xi}}\, d\ell(\xi) &= -\pi^{-n} \int f(z + s_t\xi)\, e^{-s_t\, \overline{z}\xi}\, \partial_{\overline{\xi_j}}(e^{-\xi\cdot\overline{\xi}})\, d\ell(\xi) \\ &= s_t \int (\partial_{\overline{z_j}} f)(z + s_t\xi)\, e^{-s_t\, \overline{z}\xi}\, d\gamma(\xi) = s_t\, P_t(\partial_{\overline{z_j}} f)(z).\end{aligned}$$

□

Now, and for the rest of this section, we focus on the case of circular-symmetric functions. Given $\theta \in \mathbb{R}$ and a function f we denote f_θ the function $f_\theta(w) = f(e^{i\theta} w)$. Note that

$$P_t(f_\theta) = (P_t f)_\theta. \tag{13}$$

Recall that a function f is said to be circular-symmetric if $f_\theta = f$ for every θ. It is worth noting that a holomorphic function on $\mathbb{C}^n$ is necessarily constant when circular-symmetric. Indeed if $h : \mathbb{C} \to \mathbb{C}$ has both properties then invariance and the Cauchy formula give $h(1) = \int_0^{2\pi} h(e^{i\theta}) d\theta/(2\pi) = h(0)$; next if $f : \mathbb{C}^n \to \mathbb{C}$ is holomorphic and circular symmetric, then for any $(z_1, \dots, z_n) \in \mathbb{C}^n$, the function $h : z \in \mathbb{C} \mapsto f(zz_1, \dots, zz_n)$ is also holomorphic and circular-symmetric, hence $f(z_1, \dots, z_n) = h(1) = h(0) = f(0)$. Accordingly if $f \in L^2(\gamma)$ is circular-symmetric then $\Pi_0 f \equiv \int f\, d\gamma$ since the gaussian density is also circular-symmetric. Actually, much more can be said, as we shall see below.

Let us first investigate the relation between L and L^{ou}. Note that one can write

$$Lf = L^{\mathrm{ou}} f + \frac{i}{2} \sum_{j=1}^n \big(y_j \partial_{x_j} f - x_j \partial_{y_j} f\big). \tag{14}$$

So we have $L^{\mathrm{ou}} = \Re(L) = \frac{L+\overline{L}}{2}$ where $\overline{L} f = \sum_{j=1}^n \left(\partial^2_{z_j \overline{z_j}} f - z_j\, \partial_{z_j} f\right)$ has a kernel formed by the anti-holomorphic functions. The operators L and $\overline{L}$ are Hermitian symmetric, whereas L^{ou} is symmetric for the real and the Hermitian product, and preserves the subspace of real valued functions. As we said, its spectrum is $-\frac{1}{2}\mathbb{N}$, as can be seen also from the formula $L^{\mathrm{ou}} = \frac{L+\overline{L}}{2}$. Let us illustrate this on two examples, obtained by applying the creation operator $b_1 = \overline{z_1} - \partial_{z_1}$ to the holomorphic functions $z \mapsto 1$, and $z \mapsto z_1$. The function $z \mapsto \overline{z_1}$ is an eigenfunction for L with eigenvalue -1, for $\overline{L}$ with eigenvalue 0, and for L^{ou} with eigenvalue $-1/2$. The function $z \mapsto |z_1|^2 - 1$ is an eigenfunction for L with eigenvalue -1, for $\overline{L}$ with eigenvalue -1, and for L^{ou} with eigenvalue -1.

The special role played by circular-symmetric functions is due to the fact that these operators, and the associated semi-groups, coincide for them.

Lemma 8 (Action of L and P_t on circular-symmetric functions)**.** *If f is a smooth circular-symmetric function, then we have*

$$Lf = \overline{L} f = L^{\mathrm{ou}} f.$$

In particular we have, when $f, g \in \mathcal{G}^\infty$ and f is circular-symmetric,

$$\int (Lf) g\, d\gamma = \int f Lg\, d\gamma = -\int \partial_z f \cdot \partial_{\overline{z}} g\, d\gamma.$$

Also, if $f \in L^2(\gamma)$ is circular-symmetric then we have

$$P_t f = P_t^{\mathrm{ou}} f$$

for every $t \geq 0$.

Proof. Writing $w = x + iy$, $x, y \in \mathbb{R}^n$, the symmetry rewrites as $f((\cos(\theta)x - \sin(\theta)y) + i(\cos(\theta)y + i\sin(\theta)x)) = f(x+iy)$. Taking the derivative at $\theta = 0$ we find

$$\sum_{j=1}^{n} \Big(- y_j \partial_{x_j} f(x+iy) + x_j \partial_{y_j} f(x+iy) \Big) = 0,$$

and this for every $x, y \in \mathbb{R}^n$. This implies in view of (14) that $Lf = L^{\mathrm{ou}} f = \overline{L} f$. Next, for any smooth function g we have, using that $Lf = \overline{L} f$ and (6),

$$\int (Lf) g \, d\gamma = \int (\overline{L} f) g \, d\gamma = \int f L g \, d\gamma = - \int \partial_z f \cdot \partial_{\overline{z}} g \, d\gamma.$$

Although it is formally trivial that equality of L and L^{ou} on circular-symmetric functions implies equality of the semi-groups P_t and P_t^{ou}, a bit more should be said since we defined the semi-group using the explicit formula (9). And it is anyway instructive to compute the kernels on circular-symmetric functions. Denote by K_t^{ou} the kernel of the Ornstein-Uhlenbeck semi-group that we recalled above: $K_t^{\mathrm{ou}}(z,w) = \pi^{-n}(1-e^{-t})^{-n} \, e^{-\frac{1}{1-e^{-t}}|w - e^{-t/2} z|^2}$. So we have, setting $c_t := e^{-t/2}$ and $s_t := \sqrt{1-e^{-t}}$,

$$K_t^{\mathrm{ou}}(z,w) = \pi^{-n} s_t^{-2n} \, e^{-s_t^{-2}|w|^2 - s_t^{-2} c_t^2 |z|^2} \, e^{s_t^{-2} c_t (w \cdot \overline{z} + \overline{w} \cdot z)}$$

and

$$K_t(z,w) = \pi^{-n} s_t^{-2n} e^{-s_t^{-2}|w|^2 - s_t^{-2} c_t^2 |z|^2} \, e^{s_t^{-2}(c_t^2 w \cdot \overline{z} + \overline{w} \cdot z)}.$$

Note that only the last exponentials differ in these two formulas. When f is circular-symmetric, in order to check that $P_t f = P_t^{\mathrm{ou}} f$ it suffices to check that for fixed w, z, t one has

$$\frac{1}{2\pi} \int_0^{2\pi} K_t(z, e^{i\theta} w) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} K_t^{\mathrm{ou}}(z, e^{i\theta} w) \, d\theta.$$

Observe that for $a, b \in \mathbb{C}$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} e^{a e^{i\theta} + b e^{-i\theta}} \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \sum_{p,q \in \mathbb{N}} \frac{a^p b^q}{p! q!} e^{i(p-q)\theta} d\theta = \sum_{n \geq 0} \frac{(ab)^n}{(n!)^2} = B(ab)$$

with $B(x) := \sum_{n \geq 0} \frac{x^n}{(n!)^2}$. Therefore, we have

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} K_t(z, e^{i\theta} w) \, d\theta &= \pi^{-n} s_t^{-2n} \, e^{-s_t^{-2}|w|^2 - s_t^{-2} c_t^2 |z|^2} B(s_t^{-4} c_t^2 |w \cdot \overline{z}|^2) \\ &= \frac{1}{2\pi} \int_0^{2\pi} K_t^{\mathrm{ou}}(z, e^{i\theta} w) \, d\theta, \end{aligned}$$

as wanted. □

3 Proof of Theorem 1

First, let us note that we can assume that g is smooth, and actually that $g \in \mathcal{G}^\infty$. Indeed, if $g \in \mathcal{G}$ then $P_t^{\mathrm{ou}} g \in \mathcal{G}^\infty$ and we mentionned that $P_t^{\mathrm{ou}} g$ converges to g in $L^2(\gamma)$ and therefore also in $L^1(\gamma)$, as $t \to 0$. Consequently, if we know the conclusion for a function in $\mathcal{G}^\infty$, then

$$\int f P_t^{\mathrm{ou}} g \, d\gamma \geq \int f \, d\gamma \int P_t^{\mathrm{ou}} g \, d\gamma$$

and by passing to the limit when $t \to 0$ we know it also for $g \in \mathcal{G}$. For the same reason, we can assume that $f \in \mathcal{G}^\infty$, recalling that $P_t^{\mathrm{ou}} f$ is circular-symmetric when f is.

So in the sequel, we are given two psh functions $f, g \in \mathcal{G}^\infty$, with f circular-symmetric.

As in the proof of the correlation for convex functions [6], we will compute some kind of second derivative in t for integrals involving $P_t^{\mathrm{ou}} f$; recall that $\partial_t P_t^{\mathrm{ou}} f = L^{\mathrm{ou}} P_t^{\mathrm{ou}} f$. The novelty is that, along the way, we will also use $P_t f$ which satisfies $\partial_t P_t f = L P_t f$ (Lemma 5).

Consider

$$\alpha(t) := \int (P_t^{\mathrm{ou}} f)\, g\, d\gamma = \int (P_t f)\, g\, d\gamma \ \in \mathbb{R}.$$

The function α is, by construction, smooth on $(0, \infty)$ and continuous on $[0, \infty)$ (see Lemma 5 for the continuity at zero). Since $P_t^{\mathrm{ou}} f$ tends to the constant $\int f\, d\gamma$ when $t \to \infty$, we have

$$\alpha(t) \to \int f\, d\gamma \int g\, d\gamma$$

In order to conclude, it suffices to show that α decreases. Actually we will prove that α is convex; which is enough, since a convex function with a bounded limit at $+\infty$ cannot increase. It holds

$$\alpha'(t) = \int L^{\mathrm{ou}}(P_t^{\mathrm{ou}} f)\, g\, d\gamma = \int L P_t f\, g\, d\gamma. \tag{15}$$

Since $P_t f$ is also circular-symmetric, we can invoke Lemma 8 and write

$$\alpha'(t) = -\int \partial_z P_t f \cdot \partial_{\overline{z}} g\, d\gamma.$$

Next, using the first commutation relation from Lemma 7 we get

$$\alpha'(t) = -\sum_{j=1}^n \int P_t(\partial_{z_j} f) \partial_{\overline{z}_j} g\, d\gamma$$

We stress that $\partial_z f$ is no longer circular-symmetric, so we cannot exchange P_t and P_t^{ou}. The second derivative of α is, using Fact 4,

$$\alpha''(t) = -\sum_{j=1}^n \int L(P_t(\partial_{z_j} f)) \partial_{\overline{z}_j} g\, d\gamma = \sum_{j=1}^n \int \partial_{\overline{z}}(P_t(\partial_{z_j} f)) \cdot \partial_z(\partial_{\overline{z}_j} g)\, d\gamma = \sum_{j,k=1}^n \int \partial_{\overline{z_k}} P_t(\partial_{z_j} f)\, \partial^2_{z_k \overline{z_j}} g\, d\gamma.$$

Using the commutation relation from Lemma 7 we can write

$$\alpha''(t) = \sum_{j,k=1}^n \int \partial^2_{\overline{z_k} z_j}(P_t f)\, \partial^2_{z_k \overline{z_j}} g\, d\gamma = \int \mathrm{Tr}\Big((D^2_{\mathbb{C}} P_t f)(z)\ (D^2_{\mathbb{C}} g)(z) \Big)\, d\gamma$$

where for a C^2 function h on $\mathbb{C}^n$ the notation $D^2_{\mathbb{C}} h(z)$ refers to the Hermitian matrix $\Big(\partial^2_{z_j \overline{z_k}} h(z)\Big)_{j,k \le n}$. Since $P_t f = P_t^{\mathrm{ou}} f$ and g are psh, the corresponding matrices are nonnegative Hermitian matrices, which means that the trace of their product is still nonnegative. This shows that $\alpha'' \ge 0$ and finishes the proof of the theorem.

□

We would like to conclude with a discussion of the differences between the real case and the complex case. After all, we are computing second derivatives of the same object

$$\alpha(t) = \int (P_t^{\mathrm{ou}} f)\, g\, d\gamma$$

along the Ornstein-Uhlenbeck semi-group exactly as in the case of convex functions, so what is going on?

In both cases we prove that α decreases by showing that $\alpha' \leq 0$ using the next derivative somehow, but we compute these derivatives differently. The argument for convex function goes as follows. A direct computation and usual commutation properties show that, if $\int \nabla f \, d\gamma = 0$, which is the case when f is even, then

$$\alpha'(t) = -e^{-t/2} \int_t^\infty \Big(\int \mathrm{Tr}\Big((D^2 P_s^{\mathrm{ou}} f)(z) \; (D^2 g)(z) \Big) \, d\gamma(z) \Big) e^{s/2} ds$$

where D^2 refers to the usual (real) Hessian on $\mathbb{R}^{2n}$; from this we conclude to the correlation for convex functions. On the other hand, we have proved, when f is circular-symmetric, that

$$\alpha'(t) = -\int_t^\infty \Big(\int \mathrm{Tr}\Big((D^2_{\mathbb{C}} P_s^{\mathrm{ou}} f)(z) \; (D^2_{\mathbb{C}} g)(z) \Big) \, d\gamma(z) \Big) \, ds.$$

Note that we have used here that $\alpha'(t)$ tends to 0 when $t \to +\infty$: this follows from the fact that α is convex and has a finite limit at $+\infty$, and can also be seen from (15) since $P_t^{\mathrm{ou}} f$ tends to a constant when $t \to +\infty$. It is because we wanted to work with complex derivatives that we aimed at inserting L in place of L^{ou}; recall that $\partial_{z_j} f$ need not be circular-symmetric even when f is, although the second derivatives are again circular-symmetric.

Finally, let us observe that if we consider in dimension 1 the circular-symmetric psh functions $f(w) = |w|^{1/3}$ and $g(w) = |w|^4$ on $\mathbb{C} \simeq \mathbb{R}^2$, then, as we already mentioned,

$$\mathrm{Tr}\Big((D^2_{\mathbb{C}} f) \; (D^2_{\mathbb{C}} g) \Big) = \Delta f \, \Delta g \geq 0,$$

but a direct computation shows that

$$\mathrm{Tr}\Big((D^2 f)(z) \; (D^2 g)(z) \Big) = -\frac{4}{3} |z|^{1/3} \leq 0 \quad \forall z \in \mathbb{C}.$$

Of course, this discrepancy cannot hold at all times for $P_t^{\mathrm{ou}} f$ in place of f (and moreover f is not smooth at zero, although this is not really an issue). But it suggests that the two formulas above for $\alpha'(t)$ are indeed quite different.

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