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A Gaussian correlation inequality for plurisubharmonic functions

F. Barthe and D. Cordero-Erausquin

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Abstract

A positive correlation inequality is established for circular-invariant plurisubharmonic functions, with respect to complex Gaussian measures. The main ingredients of the proofs are the Ornstein-Uhlenbeck semigroup, and another natural semigroup associated to the Gaussian $\bar{\partial}$ -Laplacian.

1 Introduction

The motivation of the present work comes from a Gaussian moment inequality in \mathbb{C}^n due to Arias de Reyna [1]. We will show that his result is in fact a very particular case of a new correlation inequality, which can be seen as the complex analogue of the following correlation inequality for convex function in \mathbb{R}^n due to Hu [6]: if μ is a centered Gaussian measure on \mathbb{R}^n and if $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ are *convex* functions in $L^2(\mu)$ and f is *even*, then

$$\int f g d\mu \geq \int f d\mu \int g d\mu.$$

We will say that a function on \mathbb{C}^n is *circular-symmetric* if it is invariant under the action of S^1 (i.e. multiplication by complex numbers of modulus one) ; in other words a function f defined on \mathbb{C}^n is circular-symmetric if

$$f(e^{i\theta}w) = f(w) \quad \forall \theta \in \mathbb{R}, \quad \forall w \in \mathbb{C}^n.$$

A function $u : \mathbb{C}^n \rightarrow [-\infty, +\infty)$ is plurisubharmonic (psh) if it is upper semi-continuous and for all $a, b \in \mathbb{C}^n$ the function $z \in \mathbb{C} \mapsto u(a + zb)$ is subharmonic. Classically, a twice continuously differentiable function $u : \mathbb{C}^n \rightarrow \mathbb{R}$ is psh if and only if for all $w, z \in \mathbb{C}^n$

$$\sum_{j,k} \partial_{z_j \bar{z}_k}^2 u(z) w_j \bar{w}_k \geq 0,$$

where $\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$, $\partial_{\bar{z}_j} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$, $z = x + iy$ with $x, y \in \mathbb{R}^n$. The later condition means that the complex Hessian $D_{\mathbb{C}}^2 u$ is pointwise Hermitian semi-definite positive. We refer e.g. to the textbook [5] for more details.

We consider the standard complex Gaussian measure γ on \mathbb{C}^n ,

$$d\gamma(w) = d\gamma_n(w) = \pi^{-n} e^{-w \cdot \bar{w}} d\ell(w) = \pi^{-n} e^{-|w|^2} d\ell(w),$$

where ℓ denotes the Lebesgue measure on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ and $w \cdot w' = \sum w_j w'_j$ for $w, w' \in \mathbb{C}^n$. For convenience, let us introduce the following class of $L^2(\gamma)$ functions with controlled growth at infinity:

$$\mathcal{G} := \left\{ f : \mathbb{C}^n \rightarrow \mathbb{C}; f \in L^2_{loc}(\lambda) \text{ and } \exists \epsilon, c, C > 0 \text{ such that } |f(w)| \leq e^{c|w|^{2-\epsilon}}, \forall |w| \geq C \right\}.$$

In particular any function (locally L^2) dominated by a polynomial function on \mathbb{R}^{2n} belongs to \mathcal{G} . Our main result reads as follows:

Theorem 1 (Correlation for psh functions). *Let $f, g : \mathbb{C}^n \rightarrow [-\infty, \infty)$ be two plurisubharmonic functions belonging to \mathcal{G} . If f is circular-symmetric, then*

$$\int f g d\gamma \geq \int f d\gamma \int g d\gamma.$$

One can extend the result by approximation to more general psh functions in $L^2(\gamma)$. The inequality also extends to arbitrary centered complex Gaussian measure, which are images of γ by \mathbb{C} -linear maps. Indeed composing a psh function with a \mathbb{C} -linear map gives another psh function.

Let us give some direct consequences of this theorem. First, we see that when F, G are holomorphic functions, or simply complex polynomial functions, belonging to $L^2(\gamma)$ and F is homogeneous, then for any $\alpha, \beta \geq 0$ we have

$$\int |F|^\alpha |G|^\beta d\gamma \geq \int |F|^\alpha d\gamma \int |G|^\beta d\gamma.$$

Indeed, if F is holomorphic $f = |F|^\beta$ is psh, and if F is homogeneous, then f is also circular-symmetric. This argument can also be used for products of the form

$$f := |F_1|^{\alpha_1} \dots |F_k|^{\alpha_k}$$

where the F_j are holomorphic and the α_j 's are nonnegative real numbers, so that f is log-psh, in the sense that

$$\log f(w) = \sum_{\ell=1}^k \alpha_\ell \log |F_\ell(w)|$$

is psh. This implies that f is also psh, and if the holomorphic functions F_j are homogeneous then f is also circular-symmetric.

Theorem 2. *Let F_1, \dots, F_N be a family of homogeneous polynomial functions on \mathbb{C}^n . Then for any $\alpha, \dots, \alpha_N \geq 0$ and $k \leq N - 1$ we have*

$$\int \prod_{j=1}^N |F_j|^{\alpha_j} d\gamma \geq \left(\int \prod_{j=1}^k |F_j|^{\alpha_j} d\gamma \right) \left(\int \prod_{j=k+1}^N |F_j|^{\alpha_j} d\gamma \right) \geq \prod_{j=1}^N \int |F_j|^{\alpha_j} d\gamma.$$

A standard complex Gaussian vector in \mathbb{C}^n is a random vector taking values in \mathbb{C}^n according to the distribution $\gamma = \gamma_n$. A random vector $X = (X_1, \dots, X_N) \in \mathbb{C}^N$ is a centered complex Gaussian vector if there is an n , a standard complex Gaussian vector G in \mathbb{C}^n and a \mathbb{C} -linear map $A : \mathbb{C}^n \rightarrow \mathbb{C}^N$ such that $X = AG$. It turns out that the law for X is then characterized by its complex covariance matrix $(\mathbb{E}(X_k \overline{X_\ell}))_{1 \leq k, \ell \leq N}$. Denoting by $a_1, \dots, a_N \in \mathbb{C}^n$ the rows the matrix of A in the canonical basis, $X_j = G \cdot a_j$. Applying the later theorem to the complex linear forms $F_j(w) = w \cdot a_j$ yields the following result.

Theorem 3. *Let $(X_1, \dots, X_N) \in \mathbb{C}^N$ be a centered complex Gaussian vector, and let $\alpha_1, \dots, \alpha_N \in \mathbb{R}^+$. Then, for any $k \leq N - 1$*

$$\mathbb{E} \prod_{j=1}^N |X_j|^{\alpha_j} \geq \left(\mathbb{E} \prod_{j=1}^k |X_j|^{\alpha_j} \right) \left(\mathbb{E} \prod_{j=k+1}^N |X_j|^{\alpha_j} \right) \quad (1)$$

and in particular

$$\mathbb{E} \prod_{j=1}^N |X_j|^{\alpha_j} \geq \prod_{j=1}^N \mathbb{E} |X_j|^{\alpha_j}. \quad (2)$$

In other words, among centered complex Gaussian vectors $(X_1, \dots, X_N) \in \mathbb{C}^N$ with fixed diagonal covariance (i.e. $(\mathbb{E}|X_j|^2)_{j \leq N}$ fixed) the expectation of $\prod_{j=1}^N |X_j|^{\alpha_j}$ is minimal when the variables are independent.

Inequality (2) is an extension of an inequality of Arias de Reyna [1], who established the particular case where all the $\alpha_j = 2p_j$ are even integer by rewriting the left hand side in terms of a permanent of a $2m$ matrix ($m = \sum p_j$) and using an inequality for permanents due to Lieb. Actually, Inequality (1) in the case where the α_j are even integers is equivalent to Lieb's permanent inequality, so in particular we are giving a new proof of this inequality.

In the next section we will introduce the tools that will be used in the proof, that is two semi-groups: the usual Ornstein-Uhlenbeck semi-group and another natural semi-group associated to the $\bar{\partial}$ operator (the generator of which could be called, depending from the context, Landau or magnetic Laplacian). In the last section we give the proof of our correlation inequality.

2 Semi-groups

To get the result, we will let the circular-symmetric psh function evolve along the Ornstein-Uhlenbeck semi-group on $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n \simeq \mathbb{R}^{2n}$ associated with the measure γ and the scalar product : $\langle w, w' \rangle = \Re(w \cdot \overline{w'})$. We recall that its generator is given, for smooth functions f , writing $w = x + iy$, $x, y \in \mathbb{R}^n$, by

$$\begin{aligned} L^{\text{ou}} f(w) &:= \frac{1}{4} \Delta f(w) - \frac{1}{2} \langle w, \nabla \rangle f(w) \\ &= \sum_{j=1}^n \frac{1}{4} \left(\partial_{x_j x_j}^2 f(w) + \partial_{y_j y_j}^2 f(w) \right) - \frac{1}{2} (x_j \partial_{x_j} f(w) + y_j \partial_{y_j} f(w)). \end{aligned}$$

Note that the normalization differs slightly from the usual one on \mathbb{R}^{2n} because our Gaussian measure has complex covariance Id_n but real covariance equal to $\frac{1}{2} \text{Id}_{2n}$. Accordingly, the spectrum of $-L^{\text{ou}}$ on $L^2(\gamma)$ is here $\frac{1}{2} \mathbb{N} = \{0, \frac{1}{2}, 1, \dots\}$. The Ornstein-Uhlenbeck semi-group $P_t^{\text{ou}} = e^{tL^{\text{ou}}}$ admits the representation, for suitable functions $f : \mathbb{C}^n \rightarrow \mathbb{C}$,

$$P_t^{\text{ou}} f(z) = \int f(e^{-t/2} z + \sqrt{1 - e^{-t}} w) d\gamma(w) \quad (3)$$

$$= \pi^{-n} (1 - e^{-t})^{-n} \int f(w) e^{-\frac{1}{1-e^{-t}} |w - e^{-t/2} z|^2} d\ell(w) \quad (4)$$

As usual in semi-group methods, it is convenient to work with a nice stable space of functions. Here, we can for instance consider

$$\mathcal{G}^\infty := \{f \in C^\infty(\mathbb{C}^n); f \text{ and all its derivatives belong to } \mathcal{G}\}.$$

Note that for $f \in \mathcal{G}$, we can define $P_t^{\text{ou}} f$ by (3) and then we have that $(t, z) \mapsto P_t^{\text{ou}} f(z)$ is smooth on $(0, \infty) \times \mathbb{C}^n$, with $P_t^{\text{ou}} f \in \mathcal{G}^\infty$ and $\partial_t P_t^{\text{ou}} f = L^{\text{ou}} P_t^{\text{ou}} f$ for every $t > 0$, and $P_t^{\text{ou}} f \rightarrow f$ in $L^2(\gamma)$ as $t \rightarrow 0$. We refer to [3] for details. Let us also mention here that with formula (3) it is readily checked that properties like convexity, subharmonicity, pluri-subharmonicity are preserved along P_t^{ou} .

Another natural operator will be used. Indeed, since pluri-subharmonicity is characterized through a “ $\partial_{\bar{z}\bar{z}}$ operation”, we shall also use the following differential operator on smooth functions f on \mathbb{C}^n :

$$Lf = \sum_{j=1}^n \left(\partial_{z_j \bar{z}_j}^2 f - \bar{z}_j \partial_{\bar{z}_j} f \right) = \sum_{j=1}^n e^{|z|^2} \partial_{z_j} (e^{-|z|^2} \partial_{\bar{z}_j} f)$$

Note that $Lf = 0$ if (and only if, see below) f is holomorphic. Formally $L = -\bar{\partial}^* \bar{\partial}$ on $L^2(\gamma)$ equipped with the Hermitian structure $(f, g) = \int f \bar{g} d\gamma$. More precisely, denoting for a differentiable function

$$\partial_{\bar{z}} f = (\partial_{\bar{z}_1} f, \dots, \partial_{\bar{z}_n} f) \quad \text{and} \quad \partial_z f := (\partial_{z_1} f, \dots, \partial_{z_n} f)$$

we have the following standard fact.

Fact 4 (Integration by parts). *For regular enough functions $f, g : \mathbb{C}^n \rightarrow \mathbb{C}$, for instance for functions in \mathcal{G}^∞ , we have*

$$\int (Lf) \bar{g} d\gamma = - \int \partial_{\bar{z}} f \cdot \overline{\partial_{\bar{z}} g} d\gamma \quad (5)$$

and in particular $\int (Lf) \bar{f} d\gamma = - \int |\partial_{\bar{z}} f|^2 d\gamma \leq 0$. We can also write

$$\int (Lf) g d\gamma = - \int \partial_{\bar{z}} f \cdot \partial_{\bar{z}} g d\gamma = \int f(\bar{L}g) d\gamma \quad (6)$$

where $\bar{L}f := \sum_{j=1}^n \left(\partial_{z_j \bar{z}_j}^2 f - z_j \partial_{z_j} f \right)$.

Indeed, it suffices to sum over j the equations

$$\int [e^{|z|^2} \partial_{z_j} (e^{-|z|^2} \partial_{\bar{z}_j} f)] \bar{g} d\gamma = \pi^{-n} \int \partial_{z_j} (e^{-|z|^2} \partial_{\bar{z}_j} f) \bar{g} d\lambda(z) = - \int \partial_{\bar{z}_j} f \partial_{z_j} \bar{g} d\gamma = - \int \partial_{\bar{z}_j} f \overline{\partial_{z_j} g} d\gamma.$$

The assumption that $f, g \in \mathcal{G}^\infty$ guarantees that the boundary terms (at infinity) in the integration by parts vanish.

As a consequence, we see that the Gaussian measure γ is invariant for L , and actually that L is a symmetric nonpositive operator on $L^2(\gamma)$ with the above-mentioned Hermitian structure. The kernel of L is the Bargmann space \mathcal{H}_0 formed by the holomorphic functions on \mathbb{C}^n that belong to $L^2(\gamma)$.

We want to work with the semi-group $P_t = e^{tL}$ which is also Hermitian (formally):

$$\int (P_t f) \bar{g} d\gamma = \int f \overline{P_t g} d\gamma \quad (7)$$

Although we will not explicitly use it, let us discuss a bit the (well known) spectral analysis of L on the complex Hilbert space $L^2(\gamma)$. This analysis is indeed fairly standard using the ideas introduced by Landau. Following for instance the presentation given in [4, Section 4], consider the ‘‘annihilation’’ operators $a_j = \partial_{\bar{z}_j}$ and their adjoints, the ‘‘creation’’ operators $b_j := a_j^* = \bar{z}_j - \partial_{z_j}$. Then $L = - \sum_{j \leq n} b_j \circ a_j$, with $[a_j, b_j] = 1$, and all these operators commute for distinct indices j . Plainly, if a function f and a scalar $\lambda \in \mathbb{C}$ satisfy $-Lf = \lambda f$, then $-L(a_j f) = (\lambda - 1)a_j f$ and $-L(b_j f) = (\lambda + 1)b_j f$. This implies that the spectrum of $-L$ is \mathbb{N} and that the eigenspace associated to the eigenvalue $k \in \mathbb{N}$ is given by the sum of the spaces $b^m \mathcal{H}_0$ with $m = (m_1, \dots, m_n) \in \mathbb{N}^n$, $|m| := \sum_{j \leq n} m_j = k$ and the convention $b^m := b_1^{m_1} \circ \dots \circ b_n^{m_n}$. Moreover, if we introduce the classical projection $\Pi_0 : L^2(\gamma) \rightarrow \mathcal{H}_0$ onto holomorphic functions

$$\Pi_0 f(z) := \int f(w) e^{z \cdot \bar{w}} d\gamma(w) = \int f(z + w) e^{-\bar{z} \cdot w} d\gamma(w),$$

then the projector Π_k on the k -eigenspace can be expressed in terms of Π_0 and the creation and annihilation operators. This allows to compute the reproducing kernel of Π_k , in terms of classical families of orthogonal polynomials. Next, one can sum over k and obtain the kernel $K_t(z, w)$ for $e^{tL} = \sum_k e^{-kt} \Pi_k$. Only the formula for K_t will be useful in the sequel and we shall actually *check* below that this suggested formula is indeed the kernel of e^{tL} .

An explicit formula for K_t can be found in [2]: setting

$$K_t(z, w) := \frac{1}{\pi^n (1 - e^{-t})^n} \exp \left(z \cdot \bar{w} - \frac{e^{-t} |z - w|^2}{1 - e^{-t}} - |w|^2 \right), \quad (8)$$

then

$$\begin{aligned} P_t f(z) &= \int f(w) K_t(z, w) d\ell(w) \\ &= (1 - e^{-t})^{-n} \int f(w) e^{z \cdot \bar{w} - \frac{e^{-t} |z - w|^2}{1 - e^{-t}}} d\gamma(w) \end{aligned} \quad (9)$$

Next, let us note that by performing the change of variable $w = z + \sqrt{1 - e^{-t}} \xi$ for fixed z we find

$$P_t f(z) = \int f(z + \sqrt{1 - e^{-t}} \xi) e^{-\sqrt{1 - e^{-t}} \bar{z} \cdot \xi} d\gamma(\xi). \quad (10)$$

On this formula, we see immediately that $P_0 = \text{Id}$ and $P_\infty = \Pi_0$.

To avoid discussions regarding unbounded operators and existence of semi-groups, we will proceed in the opposite direction and use the previous formula to *define* $P_t f$. Actually, to be fair, we should mention that later, in the proof of our result, we only need to work with smooth functions $f \in \mathcal{G}^\infty$; these functions provide nice initial data, ensuring existence and uniqueness of strong solutions for the semi-group equation. Nevertheless, we feel it is of independent interest to start from the integral formula (9) or (10) and derive from it the semi-group properties; we will first check that (9) solves (11) indeed. The drawback is that some properties that are obvious (formally) for $e^{tL} f$ need to be checked thoroughly when using this kernel representation, in particular because the kernel (8) is not Markovian.

Formulas (9)-(10) make sense pointwise, for $z \in \mathbb{C}^n$ fixed, as long as $f \in L^2(\gamma)$; actually we have the pointwise estimate $|P_t f(z)| \leq C_t(z) \|f\|_{L^2(\gamma)}$ for some constant $C_t(z) > 0$ depending on t and z . In order to derive some properties of $P_t f$, a stronger integrability condition (as $f \in \mathcal{G}$) will be assumed.

Lemma 5. *Given $f \in \mathcal{G}$, let us define $P_t f$ using the formula (9) or (10). Then one has that $(t, z) \rightarrow P_t f(z)$ is smooth on $(0, \infty) \times \mathbb{C}^n$, with for any $t > 0$, $P_t f \in \mathcal{G}_\infty$ and*

$$\partial_t P_t f = L P_t f. \quad (11)$$

Moreover we have

$$\|P_t f\|_{L^2(\gamma)} \leq \|f\|_{L^2(\gamma)}$$

and $P_t f \rightarrow f$ in $L^2(\gamma)$ as $t \rightarrow 0$.

Proof. It is readily checked that, for any fixed $w \in \mathbb{C}^n$ one has

$$\partial_t K_t(\cdot, w) = L K_t(\cdot, w).$$

Moreover, for $T, R, k > 0$ fixed, there exists constants $c = c(T, R, k), C = C(T, R, k)$ such that for $F = \partial K_t$, or $F = K_t$ or else F being any of the first k th partial derivatives of K_t , it holds that $|F(z, w)| \leq C e^{-c|w|^2}$ for all $w \in \mathbb{C}^n$, all $t \in (0, T)$ and all $|z| \leq R$. From this and the definition of \mathcal{G} , we can call upon dominated convergence to conclude to the smoothness of $(t, z) \rightarrow P_t f(z)$ and to the fact that $P_t f \in \mathcal{G}^\infty$ with $\partial_t P_t f = L P_t f$.

Regarding the contraction property, we want to avoid direct computations or spectral arguments, and so we make a detour and use some obvious but important properties of P_t .

First, we will use the semi-group property: for $t, s > 0$ that $P_t \circ P_s = P_{t+s}$. This can be seen in two ways. One can invoke that for a *smooth* function $F := P_{t_0} f \in \mathcal{G}^\infty$ then there is existence and uniqueness for the equation $\partial_t F_t = L F_t$ with initial condition $F_0 = F$ and we have seen that $(P_{t_0+t} f)$ is a solution; from there one can conclude. Or else, in a more pedestrian way, one can check that

$$\int K_t(z, w) K_s(\xi, z) d\ell(z) = K_{t+s}(\xi, w).$$

For this, one may use that given $z, \xi \in \mathbb{C}^n$ and $c \in \mathbb{C}$ with $\Re(c) > 0$,

$$\int e^{z \cdot w + \xi \cdot \bar{w}} e^{-c|w|^2} d\ell(w) = \pi^n c^{-n} e^{z \cdot \xi / c}. \quad (12)$$

Next, recall that P_t is Hermitian, in the sense (7), on $\mathcal{G} \subset L^2(\gamma)$; this can be seen directly from the integral formula since $K_t(z, w) = \overline{K_t(w, z)}$. Finally, we will use that $\|P_t f\|_{L^2(\gamma)}$ decreases for $t \in (0, \infty)$. This is immediate from the non-positivity of L since for $t > 0$ we have $P_t f \in \mathcal{G}^\infty$ and

$$\frac{d}{dt} \int |P_t f|^2 d\gamma = 2 \int (L P_t f) \overline{P_t f} d\gamma = -2 \int |\partial_{\bar{z}} P_t f|^2 d\gamma \leq 0.$$

So for our $f \in \mathcal{G}$ and $t > 0$ we have

$$\|P_t f\|_{L^2(\gamma)}^2 = \int (P_{2t} f) \bar{f} d\gamma \leq \|P_{2t} f\|_{L^2(\gamma)} \|f\|_{L^2(\gamma)} \leq \|P_t f\|_{L^2(\gamma)} \|f\|_{L^2(\gamma)}$$

which implies the contraction property in $L^2(\gamma)$.

To prove the continuity at $t = 0$ in $L^2(\gamma)$ we first assume that f is smooth and compactly supported. Using (10) we see that $P_t f$ converge point-wise to f and that for $t \in (0, 1)$ we have $|P_t f(z)| \leq C e^{c|z|}$ for some constant $c, C > 0$; so we can conclude by dominated convergence. For $f \in \mathcal{G}$ and $\epsilon > 0$, introduce g smooth compactly supported such that $\|f - g\|_{L^2(\gamma)} \leq \epsilon$ and let $\delta > 0$ be such that $t \leq \delta$ ensures that $\|P_t g - g\|_{L^2(\gamma)} \leq \epsilon$ holds. For $t \leq \delta$,

$$\|P_t f - f\|_{L^2(\gamma)} \leq \|P_t f - P_t g\|_{L^2(\gamma)} + \|P_t g - g\|_{L^2(\gamma)} + \|g - f\|_{L^2(\gamma)} \leq 2\|f - g\|_{L^2(\gamma)} + \epsilon \leq 3\epsilon.$$

This establishes the desired continuity. \square

Remark 6 (Contraction property).

1. We observe that some results, which can be deduced from the spectral decomposition and Hilbertian analysis, may be derived in a soft way thanks to flexible semigroup techniques. We have proved that starting from formula (9) we have $\|P_t f\|_{L^2(\gamma)} \leq \|f\|_{L^2(\gamma)}$ on the dense subspace \mathcal{G} , which together with the pointwise estimate given just before the previous Lemma implies by density that

$$\|P_t\|_{L^2(\gamma) \rightarrow L^2(\gamma)} \leq 1.$$

Formally, by letting $t \rightarrow \infty$ in $\|P_t\|_{L^2(\gamma) \rightarrow L^2(\gamma)} \leq 1$ we recover that

$$\|\Pi_0\|_{L^2(\gamma) \rightarrow L^2(\gamma)} \leq 1.$$

Actually, the convergence of P_t towards Π_0 can be quantified rigorously through Hörmander's inequality,

$$\|\varphi - \Pi_0 \varphi\|_{L^2(\gamma)}^2 \leq \|\partial_{\bar{z}} \varphi\|_{L^2(\gamma)}^2 = \int (-L\varphi) \varphi d\gamma$$

valid for any suitable φ , for instance for $\varphi \in \mathcal{G}^\infty$. Note that from formula (10), P_t acts as the identity on holomorphic functions, so $P_t \Pi_0 = \Pi_0 P_t = \Pi_0$. A classical Grönwall type argument (using the previous Lemma to justify the computation of $\frac{d}{dt} \int |P_t(f - \Pi_0 f)|^2 d\gamma$) ensures that for $f \in \mathcal{G}$ and $t \geq 0$

$$\|P_t f - \Pi_0 f\|_{L^2(\gamma)}^2 \leq e^{-2t} \|f - \Pi_0 f\|_{L^2(\gamma)}^2.$$

2. In analogy with the Markovian case P_t^{ou} we may wonder if P_t is also a contraction on some $L^p(\gamma)$. However, for any $p \neq 2$ we have

$$\|P_t\|_{L^p(\gamma) \rightarrow L^p(\gamma)} = +\infty,$$

as it can be seen by taking, in dimension $n = 1$, for $a \in \mathbb{R}$,

$$f_a(w) := e^{aw + \bar{w}}, \quad w \in \mathbb{C}.$$

Indeed, repeated applications of (12) with $c = 1$ show, setting $s_t := \sqrt{1 - e^{-t}}$ and using (10), that $P_t f_a(z) = e^{s_t^2 a} e^{az + (1 - s_t^2) \bar{z}}$ and that

$$\frac{\|P_t f_a\|_{L^p(\gamma)}^p}{\|f_a\|_{L^p(\gamma)}^p} = C(t, p) e^{a s_t^2 (p - p^2/2)}.$$

The next result describes how derivatives and P_t commute, an important issue in semi-group methods.

Lemma 7 (Commutation relations). *For any suitable f , say $f \in \mathcal{G}^\infty$, and $t > 0$ we have for every $1 \leq j \leq n$ and $z \in \mathbb{C}^n$,*

$$\partial_{z_j}(P_t f(z)) = P_t(\partial_{z_j} f)(z) \quad \text{and} \quad \partial_{\bar{z}_j}(P_t f(z)) = e^{-t} P_t(\partial_{\bar{z}_j} f)(z).$$

Proof. We use (10). The first equality is obvious. For the second one, setting $s_t = \sqrt{1 - e^{-t}}$, we have

$$\partial_{\bar{z}_j}(P_t f)(z) = P_t(\partial_{\bar{z}_j} f)(z) - s_t \int f(z + s_t \xi) e^{-s_t \bar{z}_j \xi} \xi_j d\gamma(\xi),$$

and

$$\begin{aligned} \pi^{-n} \int f(z + s_t \xi) e^{-s_t \bar{z}_j \xi} \xi_j e^{-\xi \cdot \bar{\xi}} d\ell(\xi) &= -\pi^{-n} \int f(z + s_t \xi) e^{-s_t \bar{z}_j \xi} \partial_{\xi_j}(e^{-\xi \cdot \bar{\xi}}) d\ell(\xi) \\ &= s_t \int (\partial_{\bar{z}_j} f)(z + s_t \xi) e^{-s_t \bar{z}_j \xi} d\gamma(\xi) = s_t P_t(\partial_{\bar{z}_j} f)(z). \end{aligned}$$

□

Now, and for the rest of this section, we focus on the case of circular-symmetric functions. Given $\theta \in \mathbb{R}$ and a function f we denote f_θ the function $f_\theta(w) = f(e^{i\theta} w)$. Note that

$$P_t(f_\theta) = (P_t f)_\theta. \quad (13)$$

Recall that a function f is said to be circular-symmetric if $f_\theta = f$ for every θ . It is worth noting that a holomorphic function on \mathbb{C}^n is necessarily constant when circular-symmetric. Indeed if $h : \mathbb{C} \rightarrow \mathbb{C}$ has both properties then invariance and the Cauchy formula give $h(1) = \int_0^{2\pi} h(e^{i\theta}) d\theta / (2\pi) = h(0)$; next if $f : \mathbb{C}^n \rightarrow \mathbb{C}$ is holomorphic and circular symmetric, then for any $(z_1, \dots, z_n) \in \mathbb{C}^n$, the function $h : z \in \mathbb{C} \mapsto f(z z_1, \dots, z z_n)$ is also holomorphic and circular-symmetric, hence $f(z_1, \dots, z_n) = h(1) = h(0) = f(0)$. Accordingly if $f \in L^2(\gamma)$ is circular-symmetric then $\Pi_0 f \equiv \int f d\gamma$ since the gaussian density is also circular-symmetric. Actually, much more can be said, as we shall see below.

Let us first investigate the relation between L and L^{ou} . Note that one can write

$$L f = L^{\text{ou}} f + \frac{i}{2} \sum_{j=1}^n (y_j \partial_{x_j} f - x_j \partial_{y_j} f). \quad (14)$$

So we have $L^{\text{ou}} = \Re(L) = \frac{L + \bar{L}}{2}$ where $\bar{L} f = \sum_{j=1}^n \left(\partial_{z_j}^2 \bar{z}_j f - z_j \partial_{z_j} f \right)$ has a kernel formed by the anti-holomorphic functions. The operators L and \bar{L} are Hermitian symmetric, whereas L^{ou} is symmetric for the real and the Hermitian product, and preserves the subspace of real valued functions. As we said, its spectrum is $-\frac{1}{2}\mathbb{N}$, as can be seen also from the formula $L^{\text{ou}} = \frac{L + \bar{L}}{2}$. Let us illustrate this on two examples, obtained by applying the creation operator $b_1 = \bar{z}_1 - \partial_{z_1}$ to the holomorphic functions $z \mapsto 1$, and $z \mapsto z_1$. The function $z \mapsto \bar{z}_1$ is an eigenfunction for L with eigenvalue -1 , for \bar{L} with eigenvalue 0 , and for L^{ou} with eigenvalue $-1/2$. The function $z \mapsto |z_1|^2 - 1$ is an eigenfunction for L with eigenvalue -1 , for \bar{L} with eigenvalue -1 , and for L^{ou} with eigenvalue -1 .

The special role played by circular-symmetric functions is due to the fact that these operators, and the associated semi-groups, coincide for them.

Lemma 8 (Action of L and P_t on circular-symmetric functions). *If f is a smooth circular-symmetric function, then we have*

$$L f = \bar{L} f = L^{\text{ou}} f.$$

In particular we have, when $f, g \in \mathcal{G}^\infty$ and f is circular-symmetric,

$$\int (L f) g d\gamma = \int f L g d\gamma = - \int \partial_z f \cdot \partial_{\bar{z}} g d\gamma.$$

Also, if $f \in L^2(\gamma)$ is circular-symmetric then we have

$$P_t f = P_t^{\text{ou}} f$$

for every $t \geq 0$.

Proof. Writing $w = x + iy$, $x, y \in \mathbb{R}^n$, the symmetry rewrites as $f((\cos(\theta)x - \sin(\theta)y) + i(\cos(\theta)y + \sin(\theta)x)) = f(x + iy)$. Taking the derivative at $\theta = 0$ we find

$$\sum_{j=1}^n \left(-y_j \partial_{x_j} f(x + iy) + x_j \partial_{y_j} f(x + iy) \right) = 0,$$

and this for every $x, y \in \mathbb{R}^n$. This implies in view of (14) that $Lf = L^{\text{ou}} f = \bar{L}f$. Next, for any smooth function g we have, using that $Lf = \bar{L}f$ and (6),

$$\int (Lf)g \, d\gamma = \int (\bar{L}f)g \, d\gamma = \int fLg \, d\gamma = - \int \partial_z f \cdot \partial_{\bar{z}} g \, d\gamma.$$

Although it is formally trivial that equality of L and L^{ou} on circular-symmetric functions implies equality of the semi-groups P_t and P_t^{ou} , a bit more should be said since we defined the semi-group using the explicit formula (9). And it is anyway instructive to compute the kernels on circular-symmetric functions. Denote by K_t^{ou} the kernel of the Ornstein-Uhlenbeck semi-group that we recalled above: $K_t^{\text{ou}}(z, w) = \pi^{-n} (1 - e^{-t})^{-n} e^{-\frac{1}{1-e^{-t}}|w - e^{-t/2}z|^2}$. So we have, setting $c_t := e^{-t/2}$ and $s_t := \sqrt{1 - e^{-t}}$,

$$K_t^{\text{ou}}(z, w) = \pi^{-n} s_t^{-2n} e^{-s_t^{-2}|w|^2 - s_t^{-2}c_t^2|z|^2} e^{s_t^{-2}c_t(w \cdot \bar{z} + \bar{w} \cdot z)}$$

and

$$K_t(z, w) = \pi^{-n} s_t^{-2n} e^{-s_t^{-2}|w|^2 - s_t^{-2}c_t^2|z|^2} e^{s_t^{-2}(c_t^2 w \cdot \bar{z} + \bar{w} \cdot z)}.$$

Note that only the last exponentials differ in these two formulas. When f is circular-symmetric, in order to check that $P_t f = P_t^{\text{ou}} f$ it suffices to check that for fixed w, z, t one has

$$\frac{1}{2\pi} \int_0^{2\pi} K_t(z, e^{i\theta} w) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} K_t^{\text{ou}}(z, e^{i\theta} w) \, d\theta.$$

Observe that for $a, b \in \mathbb{C}$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} e^{ae^{i\theta} + be^{-i\theta}} \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \sum_{p, q \in \mathbb{N}} \frac{a^p b^q}{p! q!} e^{i(p-q)\theta} \, d\theta = \sum_{n \geq 0} \frac{(ab)^n}{(n!)^2} = B(ab)$$

with $B(x) := \sum_{n \geq 0} \frac{x^n}{(n!)^2}$. Therefore, we have

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} K_t(z, e^{i\theta} w) \, d\theta &= \pi^{-n} s_t^{-2n} e^{-s_t^{-2}|w|^2 - s_t^{-2}c_t^2|z|^2} B(s_t^{-4}c_t^2|w \cdot \bar{z}|^2) \\ &= \frac{1}{2\pi} \int_0^{2\pi} K_t^{\text{ou}}(z, e^{i\theta} w) \, d\theta, \end{aligned}$$

as wanted. \square

3 Proof of Theorem 1

First, let us note that we can assume that g is smooth, and actually that $g \in \mathcal{G}^\infty$. Indeed, if $g \in \mathcal{G}$ then $P_t^{\text{ou}} g \in \mathcal{G}^\infty$ and we mentioned that $P_t^{\text{ou}} g$ converges to g in $L^2(\gamma)$ and therefore also in $L^1(\gamma)$, as $t \rightarrow 0$. Consequently, if we know the conclusion for a function in \mathcal{G}^∞ , then

$$\int f P_t^{\text{ou}} g \, d\gamma \geq \int f \, d\gamma \int P_t^{\text{ou}} g \, d\gamma$$

and by passing to the limit when $t \rightarrow 0$ we know it also for $g \in \mathcal{G}$. For the same reason, we can assume that $f \in \mathcal{G}^\infty$, recalling that $P_t^{\text{ou}}f$ is circular-symmetric when f is.

So in the sequel, we are given two psh functions $f, g \in \mathcal{G}^\infty$, with f circular-symmetric.

As in the proof of the correlation for convex functions [6], we will compute some kind of second derivative in t for integrals involving $P_t^{\text{ou}}f$; recall that $\partial_t P_t^{\text{ou}}f = L^{\text{ou}}P_t^{\text{ou}}f$. The novelty is that, along the way, we will also use $P_t f$ which satisfies $\partial_t P_t f = L P_t f$ (Lemma 5).

Consider

$$\alpha(t) := \int (P_t^{\text{ou}}f) g d\gamma = \int (P_t f) g d\gamma \in \mathbb{R}.$$

The function α is, by construction, smooth on $(0, \infty)$ and continuous on $[0, \infty)$ (see Lemma 5 for the continuity at zero). Since $P_t^{\text{ou}}f$ tends to the constant $\int f d\gamma$ when $t \rightarrow \infty$, we have

$$\alpha(t) \rightarrow \int f d\gamma \int g d\gamma$$

In order to conclude, it suffices to show that α decreases. Actually we will prove that α is convex; which is enough, since a convex function with a bounded limit at $+\infty$ cannot increase. It holds

$$\alpha'(t) = \int L^{\text{ou}}(P_t^{\text{ou}}f) g d\gamma = \int L P_t f g d\gamma. \quad (15)$$

Since $P_t f$ is also circular-symmetric, we can invoke Lemma 8 and write

$$\alpha'(t) = - \int \partial_z P_t f \cdot \partial_{\bar{z}} g d\gamma.$$

Next, using the first commutation relation from Lemma 7 we get

$$\alpha'(t) = - \sum_{j=1}^n \int P_t(\partial_{z_j} f) \partial_{\bar{z}_j} g d\gamma$$

We stress that $\partial_z f$ is no longer circular-symmetric, so we cannot exchange P_t and P_t^{ou} . The second derivative of α is, using Fact 4,

$$\alpha''(t) = - \sum_{j=1}^n \int L(P_t(\partial_{z_j} f)) \partial_{\bar{z}_j} g d\gamma = \sum_{j=1}^n \int \partial_{\bar{z}}(P_t(\partial_{z_j} f)) \cdot \partial_z(\partial_{\bar{z}_j} g) d\gamma = \sum_{j,k=1}^n \int \partial_{\bar{z}_k} P_t(\partial_{z_j} f) \partial_{z_k \bar{z}_j}^2 g d\gamma.$$

Using the commutation relation from Lemma 7 we can write

$$\alpha''(t) = \sum_{j,k=1}^n \int \partial_{\bar{z}_k z_j}^2 (P_t f) \partial_{z_k \bar{z}_j}^2 g d\gamma = \int \text{Tr} \left((D_{\mathbb{C}}^2 P_t f)(z) (D_{\mathbb{C}}^2 g)(z) \right) d\gamma$$

where for a C^2 function h on \mathbb{C}^n the notation $D_{\mathbb{C}}^2 h(z)$ refers to the Hermitian matrix $\left(\partial_{z_j \bar{z}_k}^2 h(z) \right)_{j,k \leq n}$.

Since $P_t f = P_t^{\text{ou}}f$ and g are psh, the corresponding matrices are nonnegative Hermitian matrices, which means that the trace of their product is still nonnegative. This shows that $\alpha'' \geq 0$ and finishes the proof of the theorem. \square

We would like to conclude with a discussion of the differences between the real case and the complex case. After all, we are computing second derivatives of the same object

$$\alpha(t) = \int (P_t^{\text{ou}}f) g d\gamma$$

along the Ornstein-Uhlenbeck semi-group exactly as in the case of convex functions, so what is going on?

In both cases we prove that α decreases by showing that $\alpha' \leq 0$ using the next derivative somehow, but we compute these derivatives differently. The argument for convex function goes as follows. A direct computation and usual commutation properties show that, if $\int \nabla f d\gamma = 0$, which is the case when f is even, then

$$\alpha'(t) = -e^{-t/2} \int_t^\infty \left(\int \operatorname{Tr} \left((D^2 P_s^{\text{ou}} f)(z) (D^2 g)(z) \right) d\gamma(z) \right) e^{s/2} ds$$

where D^2 refers to the usual (real) Hessian on \mathbb{R}^{2n} ; from this we conclude to the correlation for convex functions. On the other hand, we have proved, when f is circular-symmetric, that

$$\alpha'(t) = - \int_t^\infty \left(\int \operatorname{Tr} \left((D_{\mathbb{C}}^2 P_s^{\text{ou}} f)(z) (D_{\mathbb{C}}^2 g)(z) \right) d\gamma(z) \right) ds.$$

Note that we have used here that $\alpha'(t)$ tends to 0 when $t \rightarrow +\infty$: this follows from the fact that α is convex and has a finite limit at $+\infty$, and can also be seen from (15) since $P_t^{\text{ou}} f$ tends to a constant when $t \rightarrow +\infty$. It is because we wanted to work with complex derivatives that we aimed at inserting L in place of L^{ou} ; recall that $\partial_{z_j} f$ need not be circular-symmetric even when f is, although the second derivatives are again circular-symmetric.

Finally, let us observe that if we consider in dimension 1 the circular-symmetric psh functions $f(w) = |w|^{1/3}$ and $g(w) = |w|^4$ on $\mathbb{C} \simeq \mathbb{R}^2$, then, as we already mentioned,

$$\operatorname{Tr} \left((D_{\mathbb{C}}^2 f) (D_{\mathbb{C}}^2 g) \right) = \Delta f \Delta g \geq 0,$$

but a direct computation shows that

$$\operatorname{Tr} \left((D^2 f)(z) (D^2 g)(z) \right) = -\frac{4}{3} |z|^{1/3} \leq 0 \quad \forall z \in \mathbb{C}.$$

Of course, this discrepancy cannot hold at all times for $P_t^{\text{ou}} f$ in place of f (and moreover f is not smooth at zero, although this is not really an issue). But it suggests that the two formulas above for $\alpha'(t)$ are indeed quite different.

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