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CONIC FRAMEWORKS INFINITESIMAL RIGIDITY*

COLIN CROS^{†‡}, PIERRE-OLIVIER AMBLARD[†], CHRISTOPHE PRIEUR[†], AND
JEAN-FRANÇOIS DA ROCHA[‡]

Abstract. This paper introduces new structures called conic frameworks and their rigidity. They are composed by agents and a set of directed constraints between pairs of agents. When the structure cannot be flexed while preserving the constraints, it is said to be rigid. If only smooth deformations are considered a sufficient condition for rigidity is called infinitesimal rigidity. In conic frameworks, each agent u has a spatial position x_u and a clock offset represented by a bias β_u . If the constraint from Agent u to Agent w is in the framework, the pseudo-range from u to w , defined as $\|x_u - x_w\| + \beta_w - \beta_u$, is set. Pseudo-ranges appear when measuring inter-agent distances using a Time-of-Arrival method. This paper completely characterizes infinitesimal rigidity of conic frameworks whose agents are in general position. Two characterizations are introduced: one for unidimensional frameworks, the other for multidimensional frameworks. They both rely on the graph of constraints and use a decoupling between space and bias variables. In multidimensional cases, this new conic paradigm sharply reduces the minimal number of constraints required to maintain a formation with respect to classical Two-Way Ranging methods.

Key words. Rigidity theory, rigid graphs, pseudo-range

AMS subject classifications. 52C25, 70B15

1. Introduction. Consider a group of n agents whose geometric formation has to be maintained, a fleet of Unmanned Aerial Vehicles (UAV) or autonomous cars for example. Maintaining the formation means preserving the relative positions between the agents as the group might move as a rigid entity from one point to another. A natural way to achieve this result is (i) to select some distances between pairs of agents and, (ii) to force the agents to preserve these distances using some command [3, 13]. The choice of the distances in Phase (i) must ensure that the formation cannot flex. For example, to maintain a square formation between four agents in the plane, constraining the sides and one diagonal of the square is sufficient. Another solution could be to impose directly the positions of the agents, but this would need an external positioning system, such as the satellite systems GPS or Galileo. The first option is considered here. It is free from external systems but supposes that distances between pairs of agents can be constrained and therefore measured. An efficient way to measure the distance between two agents is to send a signal from one to the other, measure the delay between the emission and the reception, and multiply that delay by the speed of the signal. If the agents' clocks are synchronized, this procedure provides the distance. In general however, this procedure only gives a *pseudo-range* which is the sum of the actual distance between the agents and a bias reflecting the lack of synchronization. This bias is the difference between the clock offsets of the agents (to some absolute reference) multiplied by the speed of the signal. Concretely, consider a pair of agents (e, r) whose positions are (x_e, x_r) and clock offsets are (τ_e, τ_r) . Denote $\beta_z = c\tau_z$ for $z \in \{e, r\}$ the clock offset premultiplied by the speed of signal c . The pseudo-range from e to r is then defined by the asymmetrical map:

$$(1.1) \quad r \left(\begin{pmatrix} x_e \\ \beta_e \end{pmatrix}, \begin{pmatrix} x_r \\ \beta_r \end{pmatrix} \right) = \|x_e - x_r\| + \beta_r - \beta_e$$

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Note in particular that there is no absolute value on the bias: it is positive if the emitter's clock is ahead of the receiver's and negative otherwise. Pseudo-ranges were first introduced in the context of hyperbolic positioning systems, see *e.g.*, [22, Chap. 2] for an introduction with satellite systems.

To retrieve the distance between x_e and x_r , the pseudo-ranges in both directions are usually summed to compensate for the biases. This technique is called symmetrical double-sided two-way ranging (SDS TWR) [1] and is quite used with Ultra WideBand (UWB) technology [28]. To maintain a formation using SDS TWR, each constrained distance is measured using two pseudo-range measurements. This paper introduces a more efficient alternative method by constraining pseudo-ranges instead of distances. In this new scheme, to maintain the formation, in Phase (i) pseudo-ranges are chosen instead of distances and the control of Phase (ii) is adapted to preserve these pseudo-ranges. This paper only focuses on the selection of the pseudo-ranges in Phase (i) and does not deal with the control.

The underlying questions are how many and which pseudo-ranges should be constrained to maintain the formation? These problems are closely associated with the notion of *rigidity* and more precisely of *infinitesimal rigidity*. This notion has been well-studied when considering distances instead of pseudo-ranges [14, 8, 17].

In particular, the minimal number of distances required to preserve a formation of n agents in \mathbb{R}^d is known and denoted as $S_e(n, d)$ [4]. Consequently, preserving the same formation using SDS TWR requires at least $2S_e(n, d)$ pseudo-ranges. In contrast, the new scheme requires only $S_e(n, d) + n - 1$ pseudo-ranges. As $S_e(n, d)$ is greater than $n - 1$, the new scheme always reduces the minimal number of pseudo-ranges. Furthermore, $S_e(n, d) \sim_n dn$, therefore, when n is large the minimal number of pseudo-range constraints is reduced by a fourth in \mathbb{R}^2 and a third in \mathbb{R}^3 compared to the SDS TWR method. This new scheme exploits the fact that constraining two symmetrical pseudo-ranges sets both the distance and the bias difference between the agents: the distance is the half of their sum and the bias difference the half of their difference. Therefore, with $2(n - 1)$ well-chosen pseudo-range constraints, $n - 1$ distances and the bias differences between any pair of agents can be set. Indeed, to set the $n(n - 1)/2$ bias differences, it is sufficient to set $n - 1$ among them, *e.g.*, setting $\beta_i - \beta_1$ for $i = 2, \dots, n$ is a solution. Then, $S_e(n, d) - (n - 1)$ more distances must be set to preserve the formation. With the bias differences set, pseudo-range constraints are equivalent to distance constraints. Thus, in total $S_e(n, d) + n - 1$ pseudo-range constraints are needed. In addition, we prove that the pseudo-ranges can also be chosen without any symmetrical pair. Our main results state that when the agents are in general positions, *infinitesimal rigidity* of the formation depends only on the underlying graph of pseudo-range constraints denoted Γ (and on the ordering of the points in the unidimensional case). When constraining distances instead of pseudo-ranges, infinitesimal rigidity depends also only on the underlying graph of distance constraints [4] denoted G . In both cases, the graph is said to be rigid if it generates infinitesimally rigid formations. We prove that a pseudo-range graph Γ is rigid if and only if it is the *union* of two *independent* graphs H and G where H is a connected graph and G is a distance rigid graph. Intuitively, H sets the bias differences while G sets the distances between the agents. Furthermore, we prove that in multidimensional cases, the directions of the pseudo-ranges do not matter as long as both graphs do not constrain the same pseudo-range.

To prove these results, we have designed a new class of structures representing the geometry of the agents and the pseudo-range constraints. We call them *conic frameworks*. They are an extension of the well-studied bar-and-join structures, that

we call in the following *Euclidean frameworks* to avoid any confusion. A *flexing* of a framework (conic or Euclidean) is a *motion* distorting the framework while preserving its constraints [4]. A framework with no flexing is said to be *rigid*: it cannot be distorted. This paper focuses on a weaker notion: *infinitesimal rigidity*, it ensures that no smooth flexing exists. All these notions are properly introduced in [section 2](#). Then, [section 3](#) characterizes infinitesimal rigidity of unidimensional conic frameworks and [section 4](#) infinitesimal rigidity of multidimensional conic frameworks. Finally, [section 5](#) discusses these two characterizations.

2. Preliminaries.

2.1. Background. *Rigidity* was first introduced as a mechanical notion to study the stability of bar-and-join structures [26, 8]. These structures are composed by nodes linked by incompressible and inextendible bars. A bar-and-join structure is said to be flexible if it can be continuously bent. It is said to be rigid otherwise. For example, in the plane, a square is flexible as it can be turned continuously into a rhombus, whereas a triangle is rigid since the three bars impose the relative positions of the agents. Rigidity has been well-studied for decades [4]. In modern literature, a bar-and-join structure is usually represented through the compact form of a Euclidean framework: the combination (G, p) of a simple undirected graph $G = (V, E)$, whose vertex set is V and edge set is E , and a configuration p . For a general background on graph definitions and properties (incidence matrix, connectivity, cycles, etc.), we refer to [7]. The vertex set V of G is associated with the agents while the edge set E represents the distance constraints. The configuration p is a map from V to \mathbb{R}^d associating to each agent u its coordinates x_u . When the dn coordinates of the agents are not root of any non-null polynomial with integer coefficients, the configuration and the framework are said to be generic. It is known that rigidity of an Euclidean framework is NP-Hard to prove in general [2, Chap. 5]. However, for generic Euclidean frameworks, rigidity is more tractable: it is known that it only depends on the graph [5]. For a given graph G and a given dimension d , either every generic d -dimensional framework (G, p) is rigid or none is. Furthermore, complete characterizations of generic rigidity based on the graph exist but only for dimensions 1 and 2 [26]; until now the question remains open for higher dimensions.

We define conic frameworks as an extension of Euclidean frameworks adapted for the pseudo-range context. The main difference is that the associated graph is directed since it represents pseudo-ranges which are asymmetrical. Furthermore, each agent is now characterized by its position $x \in \mathbb{R}^d$ and its bias $\beta \in \mathbb{R}$ (expressed in units of length for convenience).

DEFINITION 2.1. *A d -dimensional conic framework (Γ, p) is the combination of a simple directed graph $\Gamma = (V, E)$ and a configuration p from V to \mathbb{R}^{d+1} assigning to each vertex $u \in V$, $p(u) = (x_u^\top \quad \beta_u)^\top$ a point of \mathbb{R}^{d+1} .*

To avoid confusion, simple undirected graphs are denoted with the Latin letters G or H and their links are referred as edges while simple directed graph are denoted with the Greek letter Γ and their links are referred as arcs. Furthermore, even if the range of p belongs to \mathbb{R}^{d+1} , we still talk about d -dimensional frameworks, d being the spatial dimension. To simplify some expressions, the point $p(u)$ may be denoted in the following as p_u .

Euclidean frameworks can be considered as particular conic frameworks whose graph Γ is symmetric, *i.e.*, $uw \in E \Leftrightarrow wu \in E$. In this case, Γ can therefore be viewed as an undirected graph. Indeed, constraining the two symmetrical pseudo-

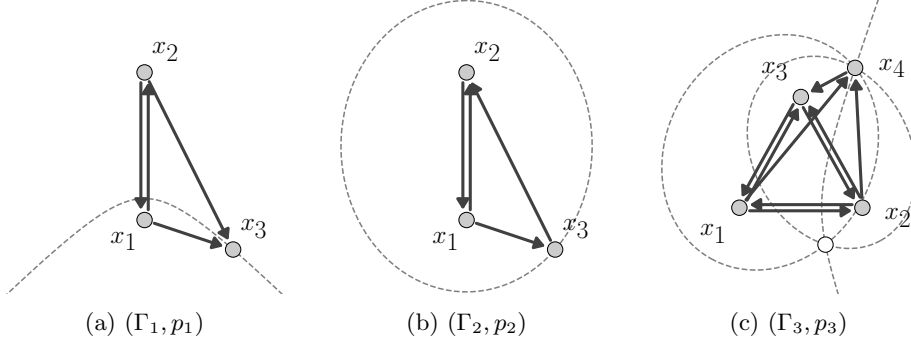


Fig. 1: Examples of 2-dimensional frameworks. The bias axis is not represented.

ranges between a pair of agents is equivalent to constraining their distance and bias difference, that is the idea applied in the SDS TWR method. Several other extensions have already been proposed. For example, [29, 9, 20] introduce linear constraints imposing the movement of some agents into some subspace but still in a Euclidean context. Other authors study the consequences of changing the Euclidean distance by other ones induced by norms: [25] focuses on p -norms (with $p \neq 2$), [23] on polyhedral norms, [24] on unitary invariant matrix norms and [10] more generally on non-Euclidean norms (in the plane). Another interesting extension was introduced in [16]. The authors study frameworks in complex and hyperbolic spaces. In the latter, the constraint between a pair of agents (when arranged to match our notations) is:

$$(2.1) \quad \|x_u - x_w\|^2 - (\beta_u - \beta_w)^2$$

This expression is very similar to (1.1) but creates a completely different problem. Notably, in (2.1) the constraint is symmetrical in u and w . Conic frameworks are, as far as we know, a new concept motivated by the use of pseudo-ranges. Some other works deal with asymmetrical rigidity but, to the best of our knowledge, only in the Euclidean case. One notable application is the persistence of flight formations [18] in which each constraint is preserved by only one agent. This concept is perfectly suited for conic frameworks since pseudo-ranges naturally have a direction.

Similarly to a Euclidean framework, a conic framework is said to be flexible if it can be bent while preserving the constraints, *i.e.*, the pseudo-ranges on its arcs; it is said to be rigid otherwise. The key notion of *infinitesimal rigidity* is introduced after the following basic examples of conic frameworks.

2.2. Examples of conic frameworks. Figure 1 presents three 2-dimensional conic frameworks, this paragraph investigates their rigidity or flexibility. In the figures, agents are represented by circles, pseudo-range constraints by arrows and, for the sake of clarity, the bias axes are not represented.

First, consider the conic framework in Figure 1a. This framework has 4 arcs corresponding to the pseudo-ranges from 1 to 2, from 2 to 1, from 1 to 3 and from 2 to 3; we denote $\rho_{i,j}$ the pseudo-range from i to j . They constrain the following

equations:

$$(2.2) \quad \|x_1 - x_2\| + \beta_2 - \beta_1 = \rho_{1,2}$$

$$(2.3) \quad \|x_1 - x_2\| + \beta_1 - \beta_2 = \rho_{2,1}$$

$$(2.4) \quad \|x_1 - x_3\| + \beta_3 - \beta_1 = \rho_{1,3}$$

$$(2.5) \quad \|x_2 - x_3\| + \beta_3 - \beta_2 = \rho_{2,3}$$

Since both pseudo-ranges between the Agents 1 and 2 are constrained, the first two equations impose that both the distance and the bias difference between agents 1 and 2 are constrained: their relative positions and biases are set. Moreover, since the pseudo-ranges $\rho_{1,3}$ and $\rho_{2,3}$ are also constrained, subtracting (2.4) from (2.5) gives:

$$(2.6) \quad \|x_1 - x_3\| - \|x_2 - x_3\| = \rho_{1,3} - \rho_{2,3} + \beta_1 - \beta_2 = \text{constant}$$

Therefore, the position of Agent 3 lies on a branch of hyperbola whose foci are x_1 and x_2 . This branch is represented by the dashed line in the figure. Note that the bias β_3 is obtained by reinjecting the distance into either (2.4) or (2.5) and is not constant along this line: it decreases as the distances $\|x_1 - x_3\|$ increases. Moving Agent 3 along this 3-dimensional curve of position preserves the four pseudo-range constraints but not the formation. Therefore, this conic framework is flexible.

The second conic framework in Figure 1b is very similar. The only difference lies in the direction of the arc between 2 and 3. In Figure 1a, $\rho_{2,3}$ was constrained whereas it is now $\rho_{3,2}$. This transforms (2.5) to:

$$(2.7) \quad \|x_2 - x_3\| + \beta_2 - \beta_3 = \rho_{3,2}$$

This time, summing (2.4) and (2.7) gives:

$$(2.8) \quad \|x_1 - x_3\| + \|x_2 - x_3\| = \rho_{1,3} + \rho_{3,2} + \beta_1 - \beta_2 = \text{constant}$$

Consequently, x_3 lies on an ellipse, also represented by a dashed line in the figure. Similarly, moving Agent 3 on this curve of position preserves the pseudo-range constraints therefore, this conic framework is also flexible. These two first examples underline how important the directions of the arcs are: flipping an arc changes the flexing of the framework.

The third conic framework in Figure 1c is more complex. The three agents 1, 2 and 3 are fully connected, therefore, all the distances and bias differences between them are constrained: their relative positions and biases are set. Agent 4 is connected to each of them by one unique arc. Each pair of arcs constrains the position x_4 to lie on a curve which is either a branch of hyperbola or an ellipse depending on whether both arcs point in the same direction or not. These three curves are still represented by dashed lines, they intersect twice: at x_4 of course and at a second point represented by a white dot. Those two points are suitable positions for Agent 4: placing it in one of these loci (with the corresponding bias) satisfies all the constraints. However, Agent 4 cannot move so the conic framework is rigid. Note that at this two loci, the associated bias are different since, for example, the distances to x_2 are different.

2.3. Infinitesimal rigidity of conic frameworks. This paragraph introduces the notions of *rigidity* and *infinitesimal rigidity* for conic frameworks. They require the following definitions that are adaptations of concepts from Euclidean rigidity theory, see e.g., [14, 17].

DEFINITION 2.2. Two conic frameworks (Γ, p) and (Γ, p') are said to be congruent if the pseudo-ranges between any pair of agents are equal in both configurations, i.e., $\forall (u, w) \in V^2, r(p_u, p_w) = r(p'_u, p'_w)$.

Definition 2.2 is equivalent to say that the distances and bias differences between any pair of agents are equal in both configurations. Indeed, the distance $\|x_u - x_w\|$ and the bias difference $\beta_u - \beta_w$ are linked with the pseudo-ranges $r(p_u, p_w)$ and $r(p_w, p_u)$ by the following invertible system:

$$(2.9) \quad \|x_u - x_w\| = \frac{r(p_u, p_w) + r(p_w, p_u)}{2} \quad \beta_u - \beta_w = \frac{r(p_w, p_u) - r(p_u, p_w)}{2}$$

DEFINITION 2.3. A motion from a conic framework (Γ, p) to another conic framework (Γ, p') is an application: $P : [0, 1] \times V \rightarrow \mathbb{R}^{d+1}$ satisfying:

1. $\forall u \in V, P(0, u) = p(u)$ and $P(1, u) = p'(u)$;
2. $\forall u, t \mapsto P(t, u)$ is continuous;
3. $\forall u, w \in E, \forall t \in [0, 1], r(P(t, u), P(t, w)) = r(p(u), p(w))$.

Note that motions must only preserve the pseudo-ranges along the arcs of the framework. They may create congruent or non-congruent frameworks. In the latter case, they are called *flexing*.

DEFINITION 2.4. A flexing of a conic framework (Γ, p) is a motion starting at (Γ, p) such that for any $t > 0$, the frameworks $(\Gamma, P(t))$ and (Γ, p) are not congruent.

For example, the frameworks depicted in Figure 1a and Figure 1b have a flexing generated by the motion of Agent 3 along the curve of position. Of course not all motions are flexing, rigid motions (combinations of rotations and translations of the spatial position) and translations of all the biases are not. These particular motions preserve the formation and are said to be trivial. We can now define rigidity of conic frameworks.

DEFINITION 2.5. A conic framework with no flexing is said to be rigid.

Infinitesimal rigidity is a strengthening of rigidity at the first order. Consider that the agents are moving smoothly. The position of Agent u at time t is denoted $p_u(t) = (x_u(t)^\top \quad \beta_u(t)^\top)^\top$ and its instantaneous velocity is denoted $q_u(t) = (v_u(t)^\top \quad \alpha_u(t)^\top)^\top$ where:

$$(2.10) \quad v_u(t) = \frac{dx_u(t)}{dt} \quad \alpha_u(t) = \frac{d\beta_u(t)}{dt}$$

If the agents always keep distinct positions, i.e., $\forall t, u \neq w \Rightarrow x_u(t) \neq x_w(t)$, the distances are also differentiable. Then the preservation of a pseudo-range $\rho_{u,w}$ and its derivative give that $\forall t \in [0, 1]$:

$$(2.11) \quad \|x_u(t) - x_w(t)\| + \beta_w(t) - \beta_u(t) = \rho_{u,w}$$

$$(2.12) \quad \frac{(x_u(t) - x_w(t))^\top \cdot (v_u(t) - v_w(t))}{\|x_u(t) - x_w(t)\|} + \alpha_w(t) - \alpha_u(t) = 0$$

Applying (2.12) at $t = 0$ gives:

$$(2.13) \quad \frac{(x_u - x_w)^\top \cdot (v_u - v_w)}{\|x_u - x_w\|} + \alpha_w - \alpha_u = 0$$

where $p_u = p_u(0)$ and $q_u = q_u(0)$ denote the initial position and velocity of Agent u for the sake of clarity.

The vector $q_u = (v_u^\top \ \alpha_u)^\top \in \mathbb{R}^{d+1}$ denotes the instantaneous velocity of Agent u . All these velocities are stacked into a vector $q = (v_1^\top \ \dots \ v_n^\top \ \alpha_1 \ \dots \ \alpha_n)^\top \in \mathbb{R}^{(d+1)n}$ called an instantaneous velocity vector (where n denotes the number of agents). An instantaneous velocity vector is said to be admissible for the conic framework (Γ, p) if (2.13) is satisfied for every arc or equivalently:

$$(2.14) \quad \forall uw \in E, \quad (x_u - x_w)^\top \cdot (v_u - v_w) + \|x_u - x_w\| (\alpha_w - \alpha_u) = 0$$

Stacking the equations of (2.14) gives that $M(\Gamma, p)q = 0$ where $M(\Gamma, p)$ is a matrix called the (conic) rigidity matrix of the framework. This rigidity matrix has $(d+1)n$ columns, one per variable, and $|E|$ rows (where $|\cdot|$ denotes the cardinal), one per arc. Therefore, the set of admissible instantaneous velocity vectors is a vector space.

The conic rigidity matrix is separated in two blocks. The first dn columns correspond to the spatial variables and form the classical Euclidean rigidity matrix $M_e(\Gamma, p)$, see *e.g.*, [17]. The last n columns correspond to the bias variables and form a matrix $B(\Gamma, p)$ defined as:

$$(2.15) \quad B(\Gamma, p) = D(\Gamma, p)B(\Gamma)$$

where $D(\Gamma, p) = \text{diag}(\dots, \|x_u - x_w\|, \dots)$ is the diagonal matrix whose i -th entry is the distance between the points connected by the i -th arc and $B(\Gamma)$ is the transpose of the incidence matrix of the graph, see *e.g.*, [7]. Note that the direction of the arcs only appear in $B(\Gamma)$ and $B(\Gamma, p)$.

With these notations, $M(\Gamma, p) = [M_e(\Gamma, p) \ B(\Gamma, p)]$. For example, the rigidity matrix of the conic framework in Figure 1a is:

$$(2.16) \quad M(\Gamma, p) = \begin{bmatrix} x_1^\top - x_2^\top & x_2^\top - x_1^\top & 0_d^\top & -d_{1,2} & d_{1,2} & 0 \\ x_1^\top - x_2^\top & x_2^\top - x_1^\top & 0_d^\top & d_{1,2} & -d_{1,2} & 0 \\ x_1^\top - x_3^\top & 0_d^\top & x_3^\top - x_1^\top & -d_{1,3} & 0 & d_{1,3} \\ 0_d^\top & x_2^\top - x_3^\top & x_3^\top - x_2^\top & 0 & -d_{2,3} & d_{2,3} \end{bmatrix}$$

where $d_{u,w} = \|x_u - x_w\|$ and 0_d denotes the null vector of \mathbb{R}^d . This decomposition provides a bound on the rank of the conic rigidity matrix:

$$(2.17) \quad \text{rank } M(\Gamma, p) \leq S(n, d) = S_e(n, d) + n - 1$$

In (2.17), $n - 1$ represents the maximal rank of the incidence matrix (the vector filled with ones always annihilates the incidence matrix). The term S_e is the maximal rank of the Euclidean rigidity matrix [17]:

$$(2.18) \quad S_e(n, d) = \begin{cases} dn - \binom{d+1}{2} & \text{if } n \geq d + 1 \\ \binom{n}{2} = S_e(n, n - 1) & \text{if } n \leq d \end{cases}$$

We are now in position to define infinitesimal rigidity in the context of conic frameworks. The following definition generalizes the definition of infinitesimal rigidity in the Euclidean context [5].

DEFINITION 2.6. *A d -dimensional conic framework (Γ, p) with n vertices is infinitesimally rigid if and only if $\text{rank } M(\Gamma, p) = S(n, d)$.*

The bound $S(n, d)$ on the rank comes from the fact that trivial motions induce instantaneous velocity vectors that are always admissible and are therefore said to be trivial. These trivial vectors form a vector space of dimension at most $\binom{d+1}{2} + 1$ depending

on the number of points and their geometry. Indeed, there are $\binom{d}{2}$ spatial rotations, d spatial translations and 1 bias translation. An equivalent definition for infinitesimal rigidity is the following.

DEFINITION 2.7. *A conic framework (Γ, p) is infinitesimally rigid if and only if every admissible instantaneous velocity vector is trivial.*

If a conic framework is infinitesimally rigid, the only possible smooth motions lead to congruent frameworks, therefore it cannot be flexed. These definitions are extensions of infinitesimal rigidity of Euclidean frameworks [17]:

DEFINITION 2.8. *A d -dimensional Euclidean framework (G, p) with n vertices is infinitesimally rigid if and only if $\text{rank } M_e(\Gamma, p) = S_e(n, d)$.*

Finally, the definition of generic configuration is also extended to conic frameworks.

DEFINITION 2.9. *A configuration is said to be generic if the set of the dn coordinates of x_1, \dots, x_n are not root of any non-trivial polynomial with integer coefficients.*

A conic framework (Γ, p) is said to be generic if p is generic.

Note that the definition is independent from the bias.

The remainder of this paper characterizes infinitesimal rigidity of generic conic frameworks. [Definition 2.6](#) is based on the rank of the rigidity matrix. Computing a rank may be numerically imprecise, *e.g.*, if the agents are almost aligned, the rigidity matrix may have small eigenvalues that may cause an underestimation of the rank. The characterizations introduced in the following depend only on the graph of the framework. Since graphs are discrete mathematical objects (they can be encoded by arrays of integers), the characterizations do not suffer from the rounding issues that may appeared in a rank computation.

3. Infinitesimal rigidity of unidimensional frameworks. This section focuses on infinitesimal rigidity of unidimensional conic frameworks, *i.e.*, $d = 1$. The main result is [Theorem 3.1](#) that provides a necessary and sufficient condition for a conic framework (generic or not) to be infinitesimally rigid in this particular case.

The specificity of the unidimensional case is the natural ordering of \mathbb{R} which orients the arcs. An arc uw is said to be increasing if $x_u < x_w$, decreasing if $x_u > x_w$ and null if $x_u = x_w$. This orientation depends on the configuration but not on the biases. The set of increasing and decreasing arcs are denoted E_+ and E_- :

$$E_+ = \{uw \in E \mid x_u < x_w\} \quad E_- = \{uw \in E \mid x_u > x_w\}$$

Note that a symmetrical pair of (non-null) arcs is distributed in both E_+ and E_- . The undirected graphs induced by these two edge sets are denoted $G_+ = (V, E_+)$ and $G_- = (V, E_-)$. They are deduced from (Γ, p) and depend on both Γ and p as illustrated in [Figure 2](#). The connectivities of G_+ and G_- provide the main result of this section:

THEOREM 3.1. *Let (Γ, p) be a unidimensional conic framework. (Γ, p) is infinitesimally rigid if and only if both G_+ and G_- are connected.*

Proof. First, assume G_+ and G_- are both connected and let q be an admissible instantaneous velocity, *i.e.*, $M(\Gamma, p)q = 0$. Let us prove that q is trivial. In this particular unidimensional case, it means that the vectors q_u are equal as the only trivial motions are translations. Furthermore, in the context, q is admissible for a

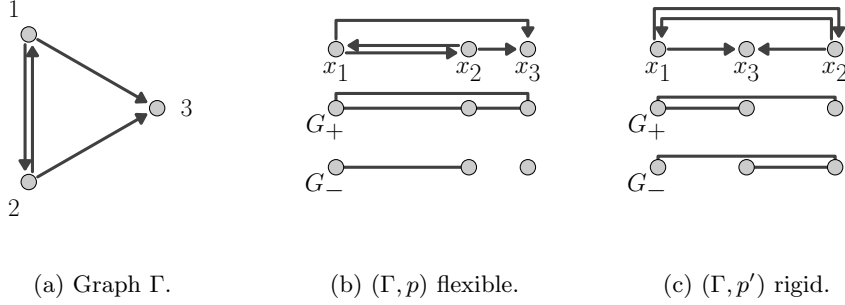


Fig. 2: Example of two unidimensional conic frameworks having the same graph Γ (Figure 2a) with their corresponding graphs G_+ and G_- . Theorem 3.1 states that the conic framework in Figure 2b is not infinitesimally rigid as G_- is disconnected while the conic framework in Figure 2c is infinitesimally rigid.

framework if it satisfies:

$$(3.1) \quad \forall uw \in E, \quad (x_u - x_w) [v_u - v_w - \text{sign}(x_u - x_w)(\alpha_u - \alpha_w)] = 0$$

where sign is the sign function. This constraint is the simplification of (2.14) in the unidimensional case. Consequently, an increasing arc $uw \in E$, imposes $v_u + \alpha_u = v_w + \alpha_w$, and a decreasing arc $uw \in E$ imposes $v_u - \alpha_u = v_w - \alpha_w$. Since both G_+ and G_- are connected, both equalities hold for any pair of vertices $(u, w) \in V^2$. Consequently, for any pair $(u, w) \in V^2$, $v_u = v_w$ and $\alpha_u = \alpha_w$, i.e., q is trivial.

Conversely, suppose without loss of generality, that G_- is not connected and let us prove that there exists a non-trivial instantaneous velocity vector. Since G_- is not connected, there is a non-trivial subset of vertices $U \in \mathcal{P}(V) \setminus \{\emptyset, V\}$ (where $\mathcal{P}(V)$ denotes the power set of V) such that U and $U^c = V \setminus U$ are not connected in G_- . Let q be the instantaneous velocity vector defined as: $q_u = (1 \ -1)^\top$ if $u \in U$ and $q_u = 0$ if $u \in U^c$. As $|U| \geq 1, |U^c| \geq 1$, q is not trivial. Let us prove that q is admissible for (Γ, p) , i.e., $M(\Gamma, p)q = 0$. Clearly, q satisfies all the arc constraints between any pair of vertices of U and any pair of vertices of U^c . Let us then consider, without loss of generality, an arc $uw \in E$ with $u \in U$ and $w \in U^c$, the case $w \in U$ and $u \in U^c$ would be symmetrical. The arc uw can not be decreasing since U and U^c are not connected in G_- . If uw is null, q satisfies the constraint. Then, suppose that uw is increasing. Equation (3.1) becomes:

$$(3.2) \quad \|x_u - x_w\| (v_u - v_w + \alpha_u - \alpha_w) = \|x_u - x_w\| (1 - 1) = 0$$

Thus, q is admissible and the framework is not infinitesimally rigid. \square

This theorem gives a simple and efficient way to numerically check whether a unidimensional conic framework is infinitesimally rigid. To verify if a conic framework is infinitesimally rigid, (i) decompose E into E_+ and E_- and (ii) test the connectivity of G_+ and G_- . Both phases can be computed with $O(|E|)$ operations, see e.g., [11].

Furthermore, as G_+ and G_- depend on the configuration p , infinitesimal rigidity of unidimensional conic frameworks is not a generic property of their graph. For example, the conic frameworks in Figure 2 have the same graph but only one is

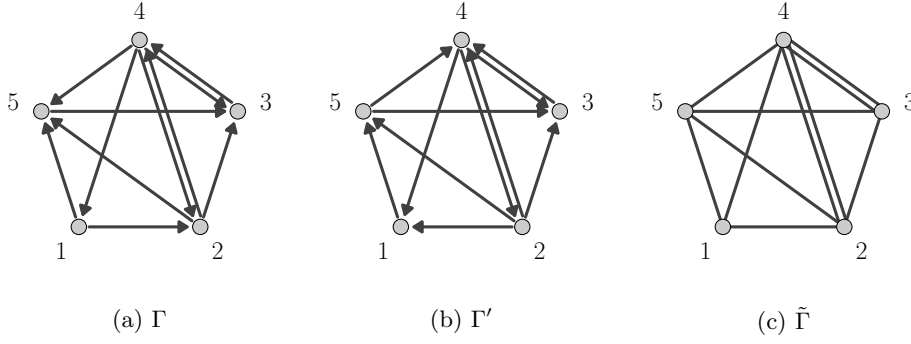


Fig. 3: Example of two equivalent directed graphs Γ and Γ' with the conic graph $\tilde{\Gamma}$ associated with their equivalence class. Only the arcs from 1 to 2 and from 4 to 5 have been reversed from Γ to Γ' . The set E_D of double edges is represented by double lines.

infinitesimally rigid. This is a major difference compared to Euclidean frameworks [17] or multidimensional conic frameworks as explained in [section 4](#).

4. Rigidity of multidimensional frameworks. This section focuses on infinitesimal rigidity of multidimensional conic frameworks, *i.e.*, $d \geq 2$. It contains two main results: [Theorem 4.1](#) and [Theorem 4.2](#).

4.1. Statement of the main results. To clearly state the two theorems, some additional definitions are required.

Two simple directed graphs with the same vertex set are *equivalent* if they have the same underlying undirected multi-graph. This multi-graph may have simple and double edges. We call *conic graph* an undirected multi-graph $\tilde{\Gamma} = (V, E_S, E_D)$ associated with such an equivalence class. E_S is the set of simple edges and E_D the set of double edges. [Figure 3](#) presents an example of two equivalent directed graphs with the associated conic graph. For a conic framework (Γ, p) , if the equivalence class of Γ is $\tilde{\Gamma}$, we say that $\tilde{\Gamma}$ is the conic graph of the framework. We are now in position to state the first theorem.

THEOREM 4.1. *Let $d \geq 2$ and $\tilde{\Gamma}$ be a conic graph. Either every generic d -dimensional conic framework whose underlying conic graph is $\tilde{\Gamma}$ is infinitesimally rigid or none of them is. In this former case, $\tilde{\Gamma}$ is said to be rigid in \mathbb{R}^d .*

[Theorem 4.1](#) implies that two generic conic frameworks having equivalent graphs have the same infinitesimal rigidity. For example, in [Figure 1](#) of [section 2](#), the graphs Γ_1 and Γ_2 are equivalent and neither the conic frameworks (Γ_1, p_1) nor (Γ_2, p_2) is infinitesimal rigid.

The second main result characterizes rigidity of conic graphs. It involves a *decomposition* of the conic graph into two special *Euclidean graphs*. We call a Euclidean graph a simple undirected graph—such graphs are associated with Euclidean frameworks. Given two Euclidean graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with the same set of vertices, we define their *union* as the conic graph $\tilde{\Gamma} = G_1 \cup G_2 = (V, E_S, E_D)$ obtained by the union of their edge sets: $E_S = E_1 \triangle E_2$ (symmetric difference) and $E_D = E_1 \cap E_2$. The pair (G_1, G_2) is called a decomposition of $\tilde{\Gamma}$ and is generally

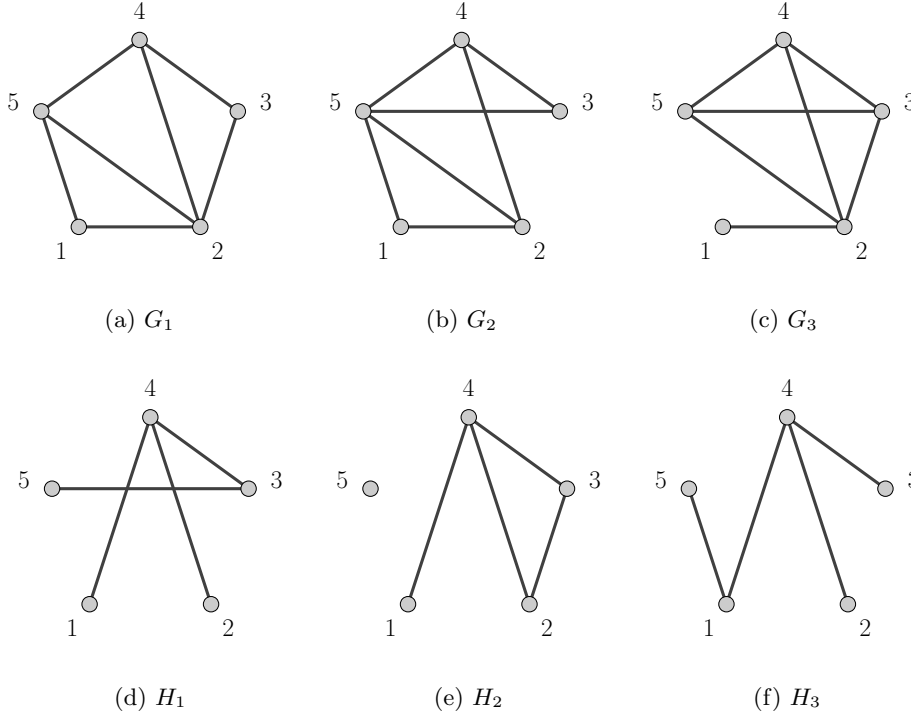


Fig. 4: Examples of decompositions of the conic graph $\tilde{\Gamma}$ of Figure 3c. Each column presents a decomposition of $\tilde{\Gamma}$ as $G \cup H$.

not unique. For example, Figure 4 provides three different decompositions of the conic graph of Figure 3c. Similarly to conic graphs, a Euclidean graph G is said to be rigid in \mathbb{R}^d if the d -dimensional generic Euclidean frameworks whose graph is G are infinitesimally rigid [17]. Indeed, infinitesimal rigidity of Euclidean frameworks also depends only on the graphs of the frameworks [4]. We can now state the second theorem.

THEOREM 4.2. *Let $d \geq 2$ and $\tilde{\Gamma}$ be a conic graph. $\tilde{\Gamma}$ is rigid in \mathbb{R}^d if and only if there exists (G, H) a decomposition of $\tilde{\Gamma}$ such that G is rigid in \mathbb{R}^d and H is connected.*

If a conic graph is rigid, Theorem 4.2 solely implies that at least one decomposition satisfies the two hypotheses. It does not imply that any decomposition of a rigid conic graph does. For example, the conic graph $\tilde{\Gamma}$ represented in Figure 3c is rigid in \mathbb{R}^2 since the decomposition (G_1, H_1) in Figure 4 satisfies the hypotheses: G_1 is rigid in \mathbb{R}^2 and H_1 is connected. However, the other decompositions (G_2, H_2) and (G_3, H_3) do not satisfy one of the properties: in the former H_2 is not connected and the latter G_3 is not rigid in \mathbb{R}^2 . Note that finding a decomposition satisfying the two hypotheses is not trivial. Even if $\tilde{\Gamma}$ is rigid and G is *minimally rigid* (i.e., becomes flexible after removing any edge), the resulting graph H may be unconnected, e.g., consider the decomposition (G_2, H_2) in Figure 4. Similarly, if $\tilde{\Gamma}$ is rigid and H is minimally connected (i.e., is a spanning tree), G may be flexible, e.g., consider the decomposition (G_3, H_3) in Figure 4.

The proofs of [Theorem 4.1](#) and [Theorem 4.2](#) are presented in the following subsections.

4.2. Proof of [Theorem 4.1](#). [Theorem 4.1](#) states that infinitesimal rigidity of generic conic frameworks depends only on their conic graph. This section proves a slightly stronger result: the rank of the rigidity matrix of a generic conic framework depends only on its conic graph. The approach is similar to the one used for Euclidean frameworks in *e.g.*, [17]. First, remind that the rank may be defined as the order of a highest order non-vanishing minor. For conic rigidity matrices, the minors can be viewed as functions of the coordinates of the points. Contrary to the minors of Euclidean rigidity matrices, these minor functions are not polynomial in the coordinates but belong to some larger space we call \mathbb{L} . The sketch of the proof is as follows. First, [Lemma 4.3](#) characterizes the space \mathbb{L} . Then, [Lemma 4.4](#) proves that if a generic configuration annihilates a minor function, then the minor function is the null function. As a consequence, [Lemma 4.5](#) proves that the generic configurations maximize the rank of the rigidity matrix over all the possible configurations. Finally, by exploiting the structure of \mathbb{L} , [Lemma 4.6](#) proves similarly that this maximum generic rank is a property not only of the graph of the framework but of its conic graph. This section ends with the detailed proof of [Theorem 4.1](#).

The entries of the rigidity matrix of a conic framework with n agents are function of the dn coordinates of the points: $\{x_u^{(i)} \mid u \in V \text{ and } i \in \{1, \dots, d\}\}$ where $x_u^{(i)}$ denotes the i -th coordinate of Agent u . These entries are either a linear function or a distance function, see (2.16) for an example of a conic rigidity matrix. Therefore the minor functions of the conic rigidity matrix are functions of these dn coordinates.

More precisely, for a conic framework (Γ, p) whose underlying conic graph is $\tilde{\Gamma} = (V, E_S, E_D)$, the minor functions belong to the space $\mathbb{L} = \mathbb{L}(E_S \cup E_D)$ defined as follows. For any set of edges $E \subset \{uw \mid 1 \leq u < w \leq n\}$, the space $\mathbb{L}(E)$ is defined as:

$$(4.1) \quad \mathbb{L}(E) = \left\{ \sum_{F \in \mathcal{P}(E)} P_F \prod_{uw \in F} D_{u,w} \mid \forall F, P_F \in \mathbb{K} \right\}$$

where $\mathbb{K} = \mathbb{Q}(X_1^{(1)}, \dots, X_n^{(d)})$ is the field of rational functions with integer coefficients in dn variables, $\mathcal{P}(E)$ denotes the power set of E and, the distance $D_{u,w}$ is the function in dn variables:

$$(4.2) \quad D_{u,w} : \begin{cases} \mathbb{R}^{dn} & \rightarrow \mathbb{R} \\ x_1^{(1)}, \dots, x_n^{(d)} & \mapsto \sqrt{\sum_{i=1}^d (x_u^{(i)} - x_w^{(i)})^2} \end{cases}$$

The definitions of $\mathbb{L}(E)$ and \mathbb{K} may appear excessively complex. They have been both chosen to provide a field structure to $\mathbb{L}(E)$ as explained in the following lemma.

LEMMA 4.3. *Let $d \geq 2$, $E \subset \{uw \mid 1 \leq u < w \leq n\}$ be a set of edges and $m = |E|$. Then, $\mathbb{L}(E)/\mathbb{K}$ is a field extension of degree 2^m . Furthermore, the family $\{\prod_{uw \in F} D_{u,w} \mid F \in \mathcal{P}(E)\}$ is a basis of $\mathbb{L}(E)$ viewed as a \mathbb{K} -vector space. We call this basis the natural basis of $\mathbb{L}(E)$.*

[Lemma 4.3](#) involves several elements from field theory. They will not be discussed as they come from a branch of mathematics, namely algebra, different than and far from the rest of the concepts used in this paper. Nonetheless, these concepts can be

found *e.g.*, in [27] and the proof of Lemma 4.3 is provided in Appendix A.1 for the sake of completeness. Only the implications of this lemma are explained here. The first important point is that \mathbb{L} is a field. Therefore, every nonzero element has a multiplicative inverse. Second, \mathbb{L} is a \mathbb{K} -vector space of dimension 2^m and one natural basis is known. For example, if $E = \{ab, bc, ac\}$, the natural basis of \mathbb{L} has eight elements: the constant function equals to 1, the three distance functions $D_{a,b}, D_{b,c}, D_{a,c}$, the three products of two distance functions $D_{a,b}D_{b,c}, D_{b,c}D_{a,c}, D_{a,c}D_{a,b}$ and the product of the three distance functions $D_{a,b}D_{b,c}D_{a,c}$. Finally, Lemma 4.3 also implies that the polynomials P_F involved in (4.1) are unique as they are the coordinates on the natural basis.

Remind that a vector x is said to be *generic* if its coordinates are not root of any non-null polynomial with integer coefficients. By definition, the dn coordinates of a generic configuration form a generic vector of \mathbb{R}^{dn} . The structure of \mathbb{L} allows to extend this property to \mathbb{L} as explained in the following lemma.

LEMMA 4.4. *Let $d \geq 2$, $E \subset \{uw \mid 1 \leq u < w \leq n\}$ be a set of edges and $f \in \mathbb{L}(E)$. If there exists $x \in \mathbb{R}^{dn}$ is a generic vector such that $f(x) = 0$, then $f = 0$.*

Proof. Let x be a generic vector and $m = |E|$. For every $E' \subset E$, by Lemma 4.3, $\mathbb{L}(E')$ is a field and a \mathbb{K} -vector space of dimension $2^{|E'|}$ whose natural basis is composed of the products between the distance functions.

Let us prove the lemma by induction on the number of distance functions appearing in the expression of f . Let us prove that: $\forall k \in \{0, \dots, m\}$, if $f \in \mathbb{L}(E')$ with $|E'| = k$ and if $f(x) = 0$, then $f = 0$.

Base case: If $k = 0$, f is a rational function with integer coefficients, *i.e.*, $f = P/Q$ with P and Q two polynomials with integer coefficients. By definition, since x is generic and $P(x) = 0$, P is the null function and therefore $f = 0$.

Inductive step: Let E' have a cardinality of $k + 1$ with $k \geq 0$, $f \in \mathbb{L}(E')$ with $f(x) = 0$ and $uw \in E'$. Any function $h \in \mathbb{L}(E')$ can be uniquely decomposed, by separating the natural basis of $\mathbb{L}(E')$, as $h_1 + D_{u,w}h_2$ with $h_1, h_2 \in \mathbb{L}(E' \setminus \{uw\})$. Let $f = f_1 + D_{u,w}f_2$ be this decomposition applied to f . Furthermore, let $\tilde{f} = f_1 - D_{u,w}f_2$ and $g = f\tilde{f} = f_1^2 - D_{u,w}^2f_2^2$. As $D_{u,w}^2$ is a polynomial, $g \in \mathbb{L}(E' \setminus \{uw\})$ whose cardinal is k . Since f vanishes at x , g also vanishes at x and by the induction hypothesis, $g = 0$. Therefore, since $\mathbb{L}(E')$ is a field, either $f = 0$ or $\tilde{f} = 0$. By definition, f and \tilde{f} have the same coordinates up to a sign in the natural basis, thus $f = 0$. \square

Lemma 4.4 extends the definition of generic point. By definition, if a generic point annihilates a function $f \in \mathbb{K}$, then f is the null function. This result remains true if f belongs to \mathbb{L} . Since \mathbb{L} is the field, the minor function of the rigidity matrix belong to \mathbb{L} . Lemma 4.4 implies the two following main lemmas.

LEMMA 4.5. *Let $d \geq 2$ and (Γ, p) be a d -dimensional conic framework. If p is generic then:*

$$(4.3) \quad \text{rank } M(\Gamma, p) = \max_{p'} \text{rank } M(\Gamma, p')$$

where the maximum is taken over all the configurations p' of Γ in \mathbb{R}^{d+1} .

Proof. Let (Γ, p) be a generic conic framework, p' be any other configuration and $r = \text{rank } M(\Gamma, p)$. Let us prove that $\text{rank } M(\Gamma, p') \leq r$. To do so, let us prove that any minor of order $(r+1)$ of $M(\Gamma, p')$ is null. Let N be a square sub-matrix of order $(r+1)$ of $M(\Gamma, p)$. Consider f_N the function from \mathbb{R}^{dn} to \mathbb{R} associating to the coordinates of the points the minor corresponding to N . As $r = \text{rank } M(\Gamma, p)$, by definition of the

rank, $f_N(p) = 0$. As p is generic and $f_N \in \mathbb{L}$, by [Lemma 4.4](#), $f_N = 0$. In particular, $f_N(p') = 0$, and thus $\text{rank } M(\Gamma, p') \leq r$. \square

The rank of the conic rigidity matrix is therefore a generic property of the graph. The following lemma extends this property to the conic graph.

LEMMA 4.6. *Let $d \geq 2$, (Γ, p) be a d -dimensional conic framework and Γ' be a graph equivalent to Γ . If p is generic then:*

$$(4.4) \quad \text{rank } M(\Gamma, p) = \text{rank } M(\Gamma', p)$$

Proof. Let Γ and Γ' be two equivalent graphs and p be a generic configuration. Let us prove that $\text{rank } M(\Gamma', p) \leq \text{rank } M(\Gamma, p)$. Then by symmetry, the equality will hold. Similarly to the proof of [Lemma 4.5](#), denote $r = \text{rank } M(\Gamma, p)$, consider N a square sub-matrix of order $(r + 1)$ of $M(\Gamma, p)$ and denote f_N the function associated to the corresponding minor. As p is generic and $f_N \in \mathbb{L}$, by [Lemma 4.4](#), $f_N = 0$. Furthermore, let f'_N be the minor function of $M(\Gamma', p)$ associated to the same columns and to the rows associated with the equivalent arcs in Γ' . Flipping an arc uw is equivalent to replace the entries of the rigidity matrix in $D_{u,w}$ by their opposites. Consequently, in the natural basis of \mathbb{L} , f_N and f'_N have the same coordinates up to the sign. Therefore, $f'_N = 0$ too and in particular, $f'_N(p) = 0$. Thus $\text{rank } M(\Gamma', p) \leq \text{rank } M(\Gamma, p)$. \square

Combining this two lemmas gives the proof of [Theorem 4.1](#).

Proof of Theorem 4.1. Let (Γ, p) and (Γ', p') be two generic conic frameworks with Γ and Γ' equivalent. Since both p and p' are generic, by [Lemma 4.5](#):

$$(4.5) \quad \text{rank } M(\Gamma, p) = \max_{p''} \text{rank } M(\Gamma, p'') = \text{rank } M(\Gamma, p')$$

Then, since Γ and Γ' are equivalent, by [Lemma 4.6](#): $\text{rank } M(\Gamma, p') = \text{rank } M(\Gamma', p')$. Therefore, $\text{rank } M(\Gamma, p) = \text{rank } M(\Gamma', p')$, thus (Γ, p) and (Γ', p') are either both infinitesimally rigid or none of them are. \square

4.3. Proof of Theorem 4.2. Since [Theorem 4.2](#) is an equivalence, the two following implications need to be proved.

I.1 If G is a Euclidean rigid graph in \mathbb{R}^d and H is a Euclidean connected graph then, their union $\tilde{\Gamma} = G \cup H$ is rigid in \mathbb{R}^d .

I.2 If $\tilde{\Gamma}$ is rigid in \mathbb{R}^d then, there exists (G, H) a decomposition of $\tilde{\Gamma}$ such that G is rigid in \mathbb{R}^d and H is connected.

[Implication I.1](#) is proved first. It involves the following additional definitions on Euclidean and conic graphs.

DEFINITION 4.7. *Let $d \geq 2$.*

A Euclidean graph G is independent in \mathbb{R}^d if for any d -dimensional generic framework (G, p) , the rows of its Euclidean rigidity matrix are independent.

A conic graph $\tilde{\Gamma}$ is independent in \mathbb{R}^d if for any d -dimensional generic framework (Γ, p) whose underlying conic graph is $\tilde{\Gamma}$, the rows of its conic rigidity matrix are independent.

The definitions are correct since the rank of the Euclidean rigidity matrix does not depend on the generic configuration [5] and similarly, the rank of the conic rigidity matrix does depend neither on the generic configuration nor on the graph but only on the equivalence class of the graph ([Theorem 4.1](#)).

DEFINITION 4.8. *Let $d \geq 2$.*

A Euclidean graph G is minimally rigid in \mathbb{R}^d if it is independent in \mathbb{R}^d and has $S_e(n, d)$ edges.

A conic graph $\tilde{\Gamma}$ is minimally rigid in \mathbb{R}^d if it is independent in \mathbb{R}^d and has $S(n, d)$ edges.

Note that the number of edges of a conic graph $\tilde{\Gamma} = (V, E_S, E_D)$ is $|E_S| + 2|E_D|$. As the name suggests, minimally rigid graphs are rigid. Indeed, the generic rank of the rigidity matrix of an independent graph is equal to its numbers of edges. Minimally rigid graphs achieve therefore the maximum rank. Furthermore, as the name also suggests, minimally rigid graphs are the rigid graphs with the minimal number of edges.

Implication I.1 is a consequence of Lemma 4.10. To prove this lemma, the following result from algebra is required, its proof is provided in Appendix A.2.

LEMMA 4.9. *Let $d \geq 2$, E_H and E_G be two sets of edges, and A be a square matrix of order $|E_H|$ whose entries are functions from \mathbb{R}^{dn} to \mathbb{R} .*

Suppose $E_H \cap E_G = \emptyset$ and the entries of A belong to $\mathbb{L}(E_G)$. Then, for any generic configuration p , the matrix $B = D(H, p) + A(p)$ is invertible.

LEMMA 4.10. *Let $d \geq 2$, $G = (V, E_G)$ and $H = (V, E_H)$ be two Euclidean graphs.*

If G is independent in \mathbb{R}^d and if H is a spanning forest then, their union $\tilde{\Gamma} = G \cup H$ is independent in \mathbb{R}^d .

Proof. Let $G = (V, E_G)$ be an independent Euclidean graph and $H = (V, E_H)$ be a spanning forest. Furthermore, let $\tilde{\Gamma} = G \cup H$ and (Γ, p) be a generic conic framework whose underlying conic graph is $\tilde{\Gamma}$. Let us prove that the rows of the conic rigidity matrix $M(\Gamma, p)$ are independent. Therefore, consider $\omega \in \text{Ker } M(\Gamma, p)^\top$ and let us prove that $\omega = 0$.

By sorting the arcs of Γ starting with those associated with the edges of G then those associated with the edges H , the rigidity matrix becomes:

$$(4.6) \quad M(\Gamma, p) = \begin{bmatrix} M_e(G, p) & B(G, p) \\ M_e(H, p) & B(H, p) \end{bmatrix}$$

By a slight abuse of notation, $B(G, p)$ and $B(H, p)$ denote the bias blocks of the conic rigidity matrix of G and H considered as directed graphs (whose arcs are directed according to Γ).

By also decomposing $\omega = (\omega_G^\top \quad \omega_H^\top)^\top$, the assumption $M(\Gamma, p)^\top \omega = 0$ gives:

$$(4.7) \quad M_e(G, p)^\top \omega_G + M_e(H, p)^\top \omega_H = 0$$

$$(4.8) \quad B(G, p)^\top \omega_G + B(H, p)^\top \omega_H = 0$$

Introducing the incidence matrices of G and H , see (2.15), equation (4.8) becomes:

$$(4.9) \quad B(G)^\top D(G, p) \omega_G + B(H)^\top D(H, p) \omega_H = 0$$

Then, since G is independent, the rows of $M_e(G, p)$ are independent and therefore $M_e(G, p)M_e(G, p)^\top$ is invertible. Equation (4.7) implies that:

$$(4.10) \quad \omega_G + A \omega_H = 0$$

with $A = [M_e(G, p)M_e(G, p)^\top]^{-1} M_e(G, p)M_e(H, p)^\top$.

Similarly, since H is a forest, the columns of its incidence matrix, *i.e.*, the rows of $B(H)$, are also independent [7]. Equation (4.9) implies that:

$$(4.11) \quad C\omega_G + \omega_H = 0$$

with $C = D(H, p)^{-1}\tilde{C}D(G, p)$ and $\tilde{C} = [B(H)B(H)^\top]^{-1}B(H)B(G)^\top$.

Combining (4.10) and (4.11), ω is in the null space of the matrix P defined as:

$$(4.12) \quad P = \begin{bmatrix} I_{|E_G|} & A \\ C & I_{|E_H|} \end{bmatrix}$$

Then, proving that P is invertible is sufficient to prove that $\omega = 0$. Introducing the Schur complement of the first block, P is invertible if the matrix $S = I_{|E_H|} - CA$ is invertible.

To prove that S is invertible, Lemma 4.9 will be used but to apply it, the double edges must be isolated. Let us first introduce the following edge sets.

$$(4.13) \quad E_D = E_G \cap E_H \quad E_{G'} = E_G \setminus E_D \quad E_{H'} = E_H \setminus E_D$$

The set E_D is the set of double edges of $\tilde{\Gamma}$, $E_{G'}$ and $E_{H'}$ form a decomposition of the set of simple edges E_S of $\tilde{\Gamma}$: $E_{G'} \cup E_{H'} = E_S$ and $E_{G'} \cap E_{H'} = \emptyset$. We order both E_G and E_H starting with the common edges E_D .

For a double edge, the corresponding rows in the Euclidean rigidity matrices are equal and the corresponding columns in the incidence matrices are opposed (since the arcs are opposed). Therefore, $\forall i \in \{1, \dots, |E_D|\}$:

$$(4.14) \quad M(G, p)^\top e_i = M(H, p)^\top e_i \quad B(G)^\top e_i = -B(H)^\top e_i$$

Using once again the independence of the rows of $M_e(G, p)$ and $B(H)$, (4.14) gives:

$$(4.15) \quad Ae_i = e_i \quad Ce_i = -e_i \quad \tilde{C}e_i = -e_i$$

Then, the matrices A , C and \tilde{C} can be decomposed as:

$$(4.16) \quad A = \begin{bmatrix} I_{|E_D|} & A_1 \\ 0 & A_2 \end{bmatrix} \quad C = \begin{bmatrix} -I_{|E_D|} & C_1 \\ 0 & C_2 \end{bmatrix} \quad \tilde{C} = \begin{bmatrix} -I_{|E_D|} & \tilde{C}_1 \\ 0 & \tilde{C}_2 \end{bmatrix}$$

where the blocks A_1 , A_2 , C_1 , C_2 , \tilde{C}_1 and \tilde{C}_2 do not have any particular structure.

Therefore, the matrix S is re-expressed as:

$$(4.17) \quad S = \begin{bmatrix} 2I_{|E_D|} & A_1 - C_1A_2 \\ 0 & S_2 \end{bmatrix}$$

with $S_2 = I_{|E_{H'}|} - C_2A_2$.

Consequently, P and S are invertible if S_2 is invertible.

Note that by construction, $C_2 = D(H', p)^{-1}\tilde{C}_2D(G', p)$ where H' and G' denote the graph induced by the edge sets $E_{H'}$ and $E_{G'}$. Then, S_2 is invertible if the matrix $\tilde{S}_2 = D(H', p) - \tilde{C}_2D(G', p)A_2$ is invertible.

The sets $E_{H'}$ and $E_{G'}$ are disjoint. Furthermore, by considering every entry as a function of the coordinates, the entries of \tilde{C} and A belong to \mathbb{K} . Then, the entries of $\tilde{C}_2D(G', p)A_2$ belong to $\mathbb{L}(E_{G'})$. Therefore by Lemma 4.9, \tilde{S}_2 is invertible. Thus S_2 , S , P are invertible and $\omega = 0$. \square

With this lemma, [Implication I.1](#) can be proved.

Proof of [Implication I.1](#). Let G be a Euclidean rigid graph in \mathbb{R}^d and H be a Euclidean connected graph H . Let $\tilde{\Gamma} = G \cup H$ be their union. Let us prove that $\tilde{\Gamma}$ is rigid in \mathbb{R}^d .

Since G is rigid in \mathbb{R}^d , $\exists G'$ a sub-graph of G minimally rigid in \mathbb{R}^d . Since H is connected, $\exists H'$ a sub-graph of H which is a spanning tree. Applying [Lemma 4.10](#), their union $\tilde{\Gamma}' = G' \cup H'$ is independent in \mathbb{R}^d . Since minimally rigid Euclidean graphs have $S_e(n, d)$ edges and spanning trees have $n-1$ edges, $\tilde{\Gamma}'$ has $S_e(n, d) + n - 1 = S(n, d)$ edges. Since $\tilde{\Gamma}'$ is independent in \mathbb{R}^d and has $S(n, d)$ edges, it is minimally rigid in \mathbb{R}^d . Thus, $\tilde{\Gamma}$ is rigid in \mathbb{R}^d . \square

To prove [Implication I.2](#), a decomposition of a rigid conic graph must be found. As already mentioned, a decomposition (G, H) may not satisfy the conditions of [Theorem 4.2](#). The existence of the particular decomposition is proved by induction without loss of generality for a minimally rigid conic graph. At each steps, a decomposition (G, H) is proposed having G minimally rigid and H with fewer connected component. The induction will stop when H is connected.

The proof relies on two lemmas: [Lemma 4.11](#) ensures that a decomposition (G, H) having G minimally rigid exists and [Lemma 4.13](#) proves that if a decomposition (G, H) exists with G minimally rigid and H not connected, then another decomposition (G', H') exists with G' minimally rigid and H' having fewer connected components than H .

LEMMA 4.11. *Let $d \geq 2$. Let $\tilde{\Gamma} = (V, E_S, E_D)$ be a conic graph.*

If $\tilde{\Gamma}$ is minimally rigid in \mathbb{R}^d then, there exists (G, H) a decomposition of $\tilde{\Gamma}$ such that G is minimally rigid in \mathbb{R}^d .

Proof. Proving that there exists a set of edges E_G containing the double edges $\tilde{\Gamma}$ such that the Euclidean graph $G = (V, E_G)$ is minimally rigid is sufficient. The graph H would then be deduced by letting its edge set be $E_H = E_S \triangle E_G$ (symmetric difference).

Let $G_D = (V, E_D)$ and $G_S = (V, E_S)$ be the Euclidean graphs induced by the double edges and the simple edges of $\tilde{\Gamma}$. Consider a generic conic framework (Γ, p) , with the correct ordering of the arcs, the conic rigidity matrix of (Γ, p) is:

$$(4.18) \quad M(\Gamma, p) = \begin{bmatrix} M_e(G_D, p) & B(G_D, p) \\ M_e(G_D, p) & -B(G_D, p) \\ M_e(G_S, p) & B(G_S, p) \end{bmatrix}$$

First, let us prove that the Euclidean graph G_D is independent. If G_D was dependent, there would exist a non-null vector $\omega_D \in \text{Ker } M_e(G_D)^\top$. Therefore the vector $\omega = \begin{pmatrix} \omega_D^\top & \omega_D^\top & 0_{|E_S|}^\top \end{pmatrix}^\top$ would be a non-null vector of $\text{Ker } M(\Gamma, p)^\top$. This would contradict the fact that $\tilde{\Gamma}$ is independent, thus G_D is independent.

Now, let us prove that E_D can be extended with edges from E_S to create E_G an independent set of $S_e(n, d)$ edges. Since $\tilde{\Gamma}$ is rigid, the Euclidean graph $G_1 = (V, E_D \cup E_S)$ is rigid. Indeed, if G_1 was not rigid, there would exist a non-trivial Euclidean velocity vector $v \in \text{Ker } M_e(G_1, p)$ [17]. Then, $q = (v^\top \ 0_n^\top)^\top$ would be a non-trivial velocity vector admissible for (Γ, p) which contradicts that $\tilde{\Gamma}$ is rigid. Since G_1 is rigid, the rank of the Euclidean rigidity matrix $M_e(G_1, p)$ is $S_e(n, d)$. Consequently, the independent set E_D can be completed with edges from E_S to create a set E_G of $S_e(n, d)$ independent edges. The resulting graph $G = (V, E_G)$ is minimally rigid and generates the decomposition. \square

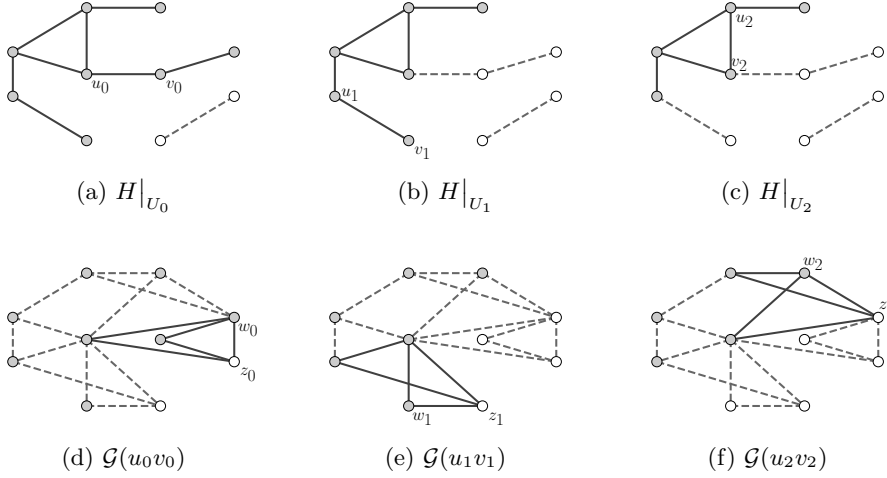


Fig. 5: Illustrations of [Lemma 4.12](#) in \mathbb{R}^2 . Each column presents an example with a different set U_i represented by the gray vertices (the others are white). The first row represents the graph H with the edge $u_i v_i$ labeled and the second row represents the graph G with the edge $w_i z_i$ labeled. The restriction of H to U_i is represented in solid lines while the rest of the graph is on dashed lines. For the graph G , the edge set $\mathcal{G}(u_i v_i)$ (introduced in the proof of [Lemma 4.12](#)) is represented in solid lines while the other edges are on dashed lines.

To prove the second key lemma, the following result is required. It is a consequence of [Lemma 4.10](#).

LEMMA 4.12. *Let $d \geq 2$, $G = (V, E_G)$ and $H = (V, E_H)$ be two Euclidean graphs, $\tilde{\Gamma} = G \cup H$ be their union and $U \subset V$ be a subset of vertices. Furthermore, assume that G and $\tilde{\Gamma}$ are both minimally rigid in \mathbb{R}^d .*

If $H|_U$, the restriction of H to U , is connected and has a cycle, then there exist $uv \in E_H$ and $wz \in E_G$ with $u, v, w \in U$ and $z \notin U$ such that the graphs G' obtained from G by replacing the edge wz by the edge uv is also minimally rigid in \mathbb{R}^d .

For example, consider the decomposition (G_2, H_2) introduced in [Figure 4](#) and the set $U = \{1, 2, 3, 4\}$. Both G_2 and the union $G_2 \cup H_2$ are rigid in \mathbb{R}^2 . Furthermore, the sub-graph $H_2|_U$ is connected and has a cycle $\mathcal{C} = \{2, 3, 4\}$. Therefore, [Lemma 4.12](#) applies. With $u = 2$, $v = w = 3$ and $z = 5$: the graph G' obtained is the graph G_1 of [Figure 4](#) which is indeed minimally rigid. [Figure 5](#) present more complex examples sharing the same decomposition (G, H) but with three different sets U .

Proof of Lemma 4.12. Let us consider a generic configuration p . In this proof, the edges are associated with their corresponding rows in the Euclidean rigidity matrix viewed as vectors, *i.e.*, the edge uw is associated with the vector $M_e(\{uw\}, p)^\top \in \mathbb{R}^{dn}$.

Since G is minimally rigid, any edge can be uniquely written as a linear combination of E_G . The generation comes from rigidity (E_G has the maximal rank) and the uniqueness from the independence of E_G . In that sense, E_G is a basis of the space of edges. For any edge uv , an edge wz from the basis E_G is said to generate uv if the coordinate associated with wz in the decomposition of uv on E_G is not-null. For an

edge uv , we denote $\mathcal{G}(uv)$ the set of its generating edges. Graphically, $\mathcal{G}(uv)$ is the smallest subset F of E_G such that the family $F \cup \{uv\}$ is dependent. For example, in [Figure 5](#) the three sets $\mathcal{G}(u_i v_i)$ are highlighted in the graph G . For any $uv \notin E_G$, replacing any edge of $\mathcal{G}(uv)$ by uv creates a new basis and therefore a new minimally rigid graph. Therefore, it is sufficient to prove that there exists an edge $uv \in E_H$ with $u, v \in U$ such that $\mathcal{G}(uv)$ contains an edge wz with $w \in U$ and $z \notin U$.

First, let us prove that there exists an edge $uv \in E_H$ with $u, v \in U$ such that $\mathcal{G}(uv)$ contains an edge wz with $z \notin U$ (and without any constraint on w). By contradiction, let us assume that for any $uv \in E_H$ with $u, v \in U$, every edge $wz \in \mathcal{G}(uv)$ has $w, z \in U$; this means that the edges of $H|_U$ are generated by edges of $G|_U$. Let Γ be a directed graph whose underlying conic graph is $\tilde{\Gamma}$ and let $\Gamma|_U$ its restriction to U . By assumption, the graph $H|_U$ is connected and has a cycle, let us assume without loss of generality that $H|_U = \hat{H}|_U \cup H'|_U$ with $\hat{H}|_U$ being a spanning tree on U and $H'|_U$ forming the cycles. By assumption, the rows of the Euclidean rigidity matrix of $H|_U$ are linear combinations of those of $G|_U$. Therefore, there exist two matrices A and A' such that:

$$(4.19) \quad M_e(\hat{H}|_U, p) = A M_e(G|_U, p) \quad M_e(H'|_U, p) = A' M_e(G|_U, p)$$

Furthermore, since $\hat{H}|_U$ is a spanning tree, the rows of $B(\hat{H}|_U, p)$ generates any rows in a bias matrix associated with an edge between two vertices of U . This is a consequence of a well-known result in graph theory: in an incidence matrix the columns associated with a spanning tree form a basis of the columns, see *e.g.*, [\[7\]](#). Therefore, there also exist two matrices C and C' such that:

$$(4.20) \quad B(G|_U, p) = C B(\hat{H}|_U, p) \quad B(H'|_U, p) = C' B(\hat{H}|_U, p)$$

With these notations and the correct ordering of the arcs, the conic rigidity matrix of $\Gamma|_U$ becomes:

$$(4.21) \quad M(\Gamma|_U, p) = \begin{bmatrix} M_e(G|_U, p) & B(G|_U, p) \\ M_e(\hat{H}|_U, p) & B(\hat{H}|_U, p) \\ M_e(H'|_U, p) & B(H'|_U, p) \end{bmatrix} = \begin{bmatrix} M_e(G|_U, p) & C B(\hat{H}|_U, p) \\ A M_e(G|_U, p) & B(\hat{H}|_U, p) \\ A' M_e(G|_U, p) & C' B(\hat{H}|_U, p) \end{bmatrix}$$

In term of rank, this conic rigidity matrix is equivalent to the matrix:

$$(4.22) \quad \begin{bmatrix} (I - CA)M_e(G|_U, p) & 0 \\ A M_e(G|_U, p) & B(\hat{H}|_U, p) \\ A' M_e(G|_U, p) & C' B(\hat{H}|_U, p) \end{bmatrix}$$

Since $G|_U$ is independent and $\hat{H}|_U$ is a spanning tree, as corollary of [Lemma 4.10](#), the matrix $I - CA$ is invertible (see the proof of [Lemma 4.10](#)). Therefore the rows corresponding to H' are linear combinations of the others. This is a contradiction since $\tilde{\Gamma}$ is independent. Thus, there exists an edge $uv \in E_H$ with $u, v \in U$ such that there exists an edge $wz \in \mathcal{G}(uv)$ with $z \notin U$.

We conclude by noting that the edges in $\mathcal{G}(uv)$ must be connected and that $\mathcal{G}(uv)$ must contain edges incident to u and v . Consequently, there is an edge $wz \in \mathcal{G}(uv)$ with $w \in U$ and $z \notin U$. \square

LEMMA 4.13. *Let $\tilde{\Gamma}$ be a conic graph and (G, H) be a decomposition of $\tilde{\Gamma}$.*

If $\tilde{\Gamma}$ and G are both minimally rigid in \mathbb{R}^d and if H has $k \geq 2$ connected components, then there exists (G', H') another decomposition of $\tilde{\Gamma}$ such that G' is minimally rigid in \mathbb{R}^d and H' has $k - 1$ connected components.

Proof. As $\tilde{\Gamma}$ is minimally rigid, it has $S(n, d)$ edges. Similarly, as G is minimally rigid, it has $S_e(n, d)$ edges. Therefore, the graph H has $n - 1$ edges. By a well-known result on graphs, see *e.g.*, [7], either H is a tree or it contains a cycle. Since H has more than one component, it is not a tree and therefore it admits at least one cycle. Let \mathcal{C} be a cycle of H .

Let us construct a new decomposition (G', H') from (G, H) in which G' is minimally rigid and the connected component of \mathcal{C} in H is connected with another connected component in H' . This new decomposition is obtained by exchanging some edges between E_G and E_H selected using Lemma 4.12 (possibly several times). Note that applying directly Lemma 4.12 with $U = \mathcal{C}$ might not work since the exchange could create another cycle in H : that is the case for the cycle presented in Figure 5. In that example, the only exchangeable edge in $H|_{\mathcal{C}}$ is u_2v_2 since the other two edges are already in G (and therefore cannot be exchanged). Unfortunately, $\mathcal{G}(u_2v_2)$ contains only edges between vertices belonging to U_0 , therefore any exchange would create a new cycle in H : for example exchanging u_2v_2 with w_2z_2 would give the decomposition (G', H') in Figure 6.

To avoid, this issue, let us first select several pairs of edges by the following induction. Start by letting U_0 be the connected component of \mathcal{C} in H and H_0 be the restriction of H to U_0 . By Lemma 4.12, there exist two edges $u_0v_0 \in E_H$ and $w_0z_0 \in E_G$ preserving the rigidity of G with $u_0, v_0, w_0 \in U_0$ and $z_0 \notin U_0$. Then, while $u_i v_i$ is not in a cycle of H (not necessary \mathcal{C}), let U_{i+1} be the connected component of \mathcal{C} in $H_i \setminus \{u_i v_i\}$ (the graph H_i without the edge $u_i v_i$) and H_{i+1} be the restriction of H to U_{i+1} . By Lemma 4.12, there exist two edges $u_{i+1}v_{i+1} \in E_H$ and $w_{i+1}z_{i+1} \in E_G$ preserving the rigidity of G with $u_{i+1}, v_{i+1}, w_{i+1} \in U_{i+1}$ and $z_{i+1} \notin U_{i+1}$. This procedure is illustrated on Figure 5: the sets U_i have been chosen according to the induction.

The induction finishes since the cardinal of U_i decreases at each iteration and U_i always contains at least the cycle \mathcal{C} . Let K denote the number of iterations. At the end of the induction, K pairs of edges $(u_i v_i, w_i z_i) \in E_H \times E_G$ with $i \in \{0, \dots, K - 1\}$ have been selected. In the example of Figure 5, $K = 3$.

To construct the new decomposition, some pairs of edges among the K selected are exchanged. They are also chosen by induction. The idea is to exchange at each step a pair $(u_i v_i, w_i z_i)$ whose edge $u_i v_i$ belongs to a cycle in H . If so, removing $u_i v_i$ from H preserves its connected components and then, adding $w_i z_i$ to H either connects two connected components or creates a new cycle but not in H_i : in some H_j with $j < i$. Concretely, let $\sigma(i)$ denote the index of the pair chosen at step i . The first pair is the last selected pair: $\sigma(0) = K - 1$. Then, while $z_{\sigma(i)} \in U_0$, the $(i + 1)$ -th index is $\sigma(i + 1) = \max \{j \mid z_{\sigma(i)} \in U_j\}$. If $z_{\sigma(i)} \notin U_0$, the induction stops as U_0 has been connected with its complementary. If $z_{\sigma(i)} \in U_0$, then exchanging $u_{\sigma(i)} v_{\sigma(i)}$ with $w_{\sigma(i)} z_{\sigma(i)}$ creates a new cycle passing through $u_{\sigma(i+1)} v_{\sigma(i+1)}$. For example, consider again the decomposition introduced in Figure 5, two exchanges are performed. The first exchange between $u_2 v_2$ and $w_2 z_2$ creates a cycle passing through $u_0 v_0$ since

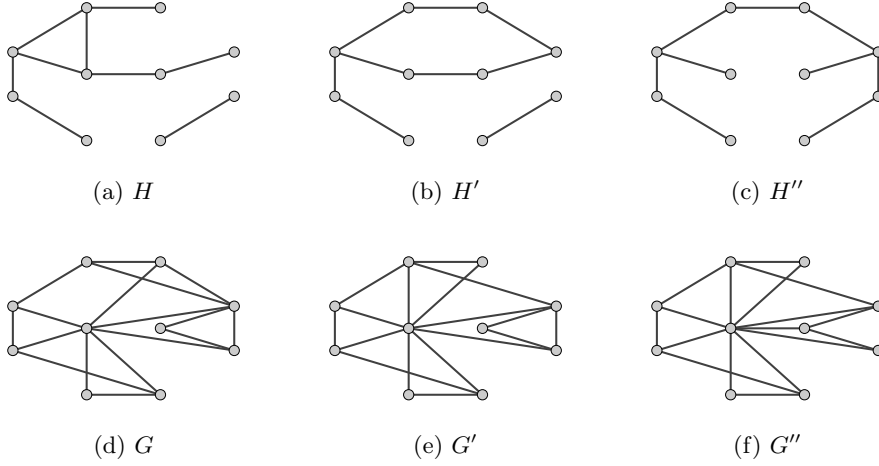


Fig. 6: Illustration of Lemma 4.13.

$z_2 \in U_0$ but $z_2 \notin U_1$. The second exchange between u_0v_0 and w_0z_0 connects the two components of H and consequently stops the induction. The initial decomposition (G, H) and the decompositions after each exchange are illustrated in Figure 6: (G', H') is the decomposition after the first exchange and (G'', H'') the decomposition after the second. By construction, each exchange preserves the minimal rigidity of G . \square

Implication I.2 can now be proved.

Proof of Implication I.2. Let $\tilde{\Gamma}$ be, without loss of generality, a minimally rigid conic graph (the redundant edges can be placed in any Euclidean graph).

By Lemma 4.11, a decomposition (G, H) of $\tilde{\Gamma}$ with G minimally rigid exists. If the graph H is connected then the proof is over. Otherwise, H has $k \geq 2$ components, then using Lemma 4.13, by a trivial induction, there exists a decomposition (G', H') of $\tilde{\Gamma}$ with G minimally rigid and H connected. \square

5. Discussions. Two characterizations of generic infinitesimal rigidity have been introduced: the first for unidimensional conic frameworks and the second for multidimensional conic frameworks. Although different, they are based on a similar decoupling of the space and the bias variables: the positions are constrained by a Euclidean rigid graph while the biases are simply constrained by a connected graph. To reach the maximum rank and to ensure infinitesimal rigidity, those two graphs must generate complementary constraints, *i.e.*, their constraints must be independent. In the unidimensional case, this comes from the distinction between increasing and decreasing arcs while in the multidimensional case, it is a consequence of the linear independence between distances and coordinates.

Conic frameworks are truly a new concept. To first state the obvious, a d -dimensional conic framework is not simply a Euclidean framework of dimension $d+1$. If so, since rigid conic graphs are characterized by a condition on their underlying rigid Euclidean graphs, there would be an induction construction of rigid Euclidean graphs in any dimension. Unfortunately, there is no known characterization of Euclidean rigidity in dimension $d > 2$. The main difference between a d -dimensional conic

graph and a $(d+1)$ -dimensional Euclidean graph comes from the additional entries in rigidity matrices. For conic graphs, these entries are distances which are generically linearly independent. In contrast, for $(d+1)$ -dimensional Euclidean graph the additional entries are dependent: *e.g.*, if the additional spatial variable is θ , the additional entries are in the form of $\theta_u - \theta_w$ and for example: $\theta_1 - \theta_2$, $\theta_2 - \theta_3$ and $\theta_3 - \theta_1$ are clearly dependent. Furthermore, conic frameworks are also different from hyperbolic frameworks described in [16]: the absence of absolute value around the bias difference in the pseudo-range equation makes the constraint asymmetrical. This also allows the graph to have pair of vertices connected by two arcs. Interestingly enough, in multidimensional cases, this asymmetry has no impact on infinitesimal rigidity: two frameworks with equivalent graphs have the same infinitesimal rigidity. However, if they are flexible, their flexing may be different as illustrated by the conic frameworks of Figure 1.

One of the greatest interests of the conic paradigm is the preservation of flight formation. First as mentioned in introduction, it reduces the minimal number of pseudo-range constraints required with respect to a classical SDS TWR method. Remind that SDS TWR requires at least $2S_e(n, d) \sim_n 2dn$ pseudo-range constraints whereas the conic method only requires $S(n, d) \sim_n (d+1)n$. Consequently, when used in the plane, SDS TWR method performs at least about 25% of redundant measurements and about 33% in 3D space. Furthermore, from an implementation point of view, the pseudo-range approach has another advantage: some agents can be only broadcasting. In the context of formation persistence [18], every constraint is maintained by only one agent, called the *follower*. If the graph has some good properties the whole formation is preserved. This technique greatly simplifies the control. With the SDS TWR approach, an agent having several followers has to interact with every one of them to compute the distances. When the number of followers increases, the update rate necessarily decreases, which may induce a loss of precision. With the conic method in contrast, a agent having several followers may have no interaction with them: he could simply broadcast its position and bias, then, each follower could compute the pseudo-range without any feedback. This approach allows significant scale-up in the system as the number of followers would not be limited by the channel capacity.

From a computational point of view, testing infinitesimal rigidity is simple for unidimensional frameworks. It requires to divide the arcs according to their orientation and then apply twice a connectivity test algorithm to both G_+ and G_- . These three steps may be realized with a complexity of $O(|E|)$, with a bread search first algorithm for example. In the multidimensional case, infinitesimal rigidity is a property of the conic graph. Rigidity of conic graphs relies on their underlying rigid Euclidean graphs and underlying connected graphs. When $d = 2$, several efficient algorithms have been developed to test rigidity of Euclidean graphs with a complexity of $O(n^2)$ [19, 17, 12, 21]. Unfortunately, the number of spanning trees in a graph may increase exponentially and therefore testing every possible decomposition would be inefficient. To bypass this issue, the construction of the decomposition of the conic graph presented could be implemented. The construction of the initial Euclidean rigid graph may be realized using the pebble-game algorithm of [21] and the construction of the sets $\mathcal{G}(uv)$ using the algorithm introduced in [6] to find the redundantly rigid components of a graph. This will be the focus of further works. Randomized algorithm may also be considered to test rigidity of conic graphs. This approach was already proposed for Euclidean frameworks [17]. The idea is to randomly generate a configuration and computing the rank of its rigidity matrix. For example the algorithm

introduced in [15] may be adapted to test conic infinitesimal rigidity.

Finally, the limits of infinitesimal rigidity must be underlined. Infinitesimal rigidity considers only instantaneous velocities of conic frameworks. As a consequence, it ensures that no smooth deformation of the framework exists. However, it is weaker than rigidity which considers every flexing (smooth or not). For Euclidean frameworks, two stronger results exist. First, infinitesimal rigidity implies rigidity and second, those two concepts are equivalent for generic Euclidean frameworks, see *e.g.*, [4]. However, the proofs of these two results use elements from differential geometry that are beyond the scope of this paper. Therefore, they are, for now, only conjectured for conic frameworks.

CONJECTURE 5.1. *Let (Γ, p) be a conic framework.*

If (Γ, p) is minimally rigid, then (Γ, p) is rigid.

CONJECTURE 5.2. *Let (Γ, p) be a generic conic framework.*

(Γ, p) is minimally rigid if and only if (Γ, p) is rigid.

Appendix A. Proof of algebraic results.

This appendix provides the proofs of the algebraic results. It has been isolated because it uses materials very different than those introduced in the body of the paper. For general concepts of field theory (field extension, extension order, etc.) please refer to *e.g.*, [27].

A.1. Proof of Lemma 4.3. Lemma 4.3 is a particular application of the following result.

LEMMA A.1. *Let $\mathbb{K} = \mathbb{Q}(X_1, \dots, X_N)$ be the field of fractions in N variables with coefficients in \mathbb{Q} and (R_1, \dots, R_m) be a family of functions such that:*

H.1 $\forall i \in \{1, \dots, m\}, R_i^2 \in \mathbb{K};$

H.2 $\forall i \in \{1, \dots, m\}, R_i^2 \notin \mathbb{K}^{(2)},$ with $\mathbb{K}^{(2)} = \{P^2 \mid P \in \mathbb{K}\};$

H.3 $\forall I \in \mathcal{P}(\{1, \dots, m\}) \setminus \{\emptyset\}, R_I = \prod_{i \in I} R_i \notin \mathbb{K}.$

Then $\mathbb{L} = \mathbb{K}[R_1, \dots, R_m]$ is a field and \mathbb{L}/\mathbb{K} is a field extension of order 2^m .

Proof. The proof is realized by induction over m . The property to prove is $\mathbf{P}(k)$: “For any R_1, \dots, R_k satisfying the three hypotheses, $\mathbb{L} = \mathbb{K}[R_1, \dots, R_k]$ is a field and \mathbb{L}/\mathbb{K} is a field extension of order 2^k .”

Initialization. For $k = 1$, let R satisfy the three hypotheses. To prove that $\mathbb{K}[R]$ is a field, proving that every non-null element has an inverse is sufficient. Let $P \in \mathbb{K}[R]$, $P \neq 0$. Since $R^2 \in \mathbb{K}$, there exists $(A, B) \in \mathbb{K}^2$ with $(A, B) \neq (0, 0)$ such that $P = A + BR$. If $B = 0$, $P = A \in \mathbb{K}$ therefore P is invertible. If $B \neq 0$, using Hypothesis H.2, $R^2 \neq A^2/B^2$, therefore $A^2 - B^2R^2 \neq 0$. Then, $(A - BR)/(A^2 - B^2R^2) \in \mathbb{K}[R]$ is the inverse of P . The extension is of order 2 by Hypothesis H.1 and Hypothesis H.3. Thus, $\mathbf{P}(1)$ is true.

Induction step. Let assume $\mathbf{P}(k)$ for $k \geq 1$ and prove $\mathbf{P}(k+1)$. Let R_1, \dots, R_{k+1} be $k+1$ functions satisfying the three hypotheses. We denote $\mathbb{L}_k = \mathbb{K}[R_1, \dots, R_k]$. First, let us prove that \mathbb{L}_{k+1} is a field. Proving that $R_{k+1}^2 \notin \mathbb{L}_k^{(2)}$ is sufficient since then, with the same arguments as for the initialization every non-null element of \mathbb{L}_{k+1} would have an inverse.

Let us assume by contradiction that $R_{k+1}^2 \in \mathbb{L}_k^{(2)}$. By induction hypothesis, $\mathbb{L}_k = \mathbb{K}_{k-1}[R_k]$. Therefore, there exist $A, B \in \mathbb{K}_{k-1}$ such that:

$$(A.1) \quad R_{k+1}^2 = (A + BR_k)^2 = A^2 + B^2R_k^2 + 2ABR_k$$

If $AB \neq 0$, then $R_k \in \mathbb{L}_{k-1}$ which contradicts the induction hypothesis. Then, nec-

essarily A or B is null. If $B = 0$, then $R_{k+1}^2 = A^2 \in \mathbb{L}_{k-1}^{(2)}$. This also contradicts the induction hypothesis when considering the k functions $R_1, \dots, R_{k-1}, R_{k+1}$. Therefore $A = 0$. If $A = 0$ then, $R_{k+1}^2 = B^2 R_k^2$ and $(R_{k+1} R_k)^2 = (B R_k^2)^2 \in \mathbb{L}_{k-1}^{(2)}$. Similarly, this also contradicts the induction hypothesis when considering the k functions $R_1, \dots, R_{k-1}, R_k R_{k+1}$ (which satisfies the three hypotheses). These contradictions give that $R_{k+1}^2 \notin \mathbb{L}_k^{(2)}$ and thus, \mathbb{L}_{k+1} is a field.

To prove the order, let us use the induction hypothesis:

$$(A.2) \quad [\mathbb{L}_{k+1} : \mathbb{K}] = [\mathbb{L}_{k+1} : \mathbb{L}_k][\mathbb{L}_k : \mathbb{K}] = 2^k [\mathbb{L}_{k+1} : \mathbb{L}_k]$$

Since $R_{k+1} \notin \mathbb{L}_k$ and $R_{k+1}^2 \in \mathbb{L}_k$, $[\mathbb{L}_{k+1} : \mathbb{L}_k] = 2$ and $[\mathbb{L}_{k+1} : \mathbb{K}] = 2^{k+1}$. $\mathbf{P}(k+1)$ is true. \square

Proof of Lemma 4.3. Let $E \subset \{uw \mid 1 \leq u < w \leq n\}$ be a set of edges and $m = |E|$. The set of m distance functions $D_{u,w}$ do satisfy the three conditions of Lemma A.1 when $d \geq 2$.

Note however that when $d = 1$, the distance functions do not satisfy Hypothesis H.2 of Lemma A.1. \square

A.2. Proof of Lemma 4.9.

Proof. Let E_H and E_G be two disjoint sets of edges, A be a square matrix of order $|E_H|$ whose entries are functions in $\mathbb{L}(E_G)$, and p be a generic configuration.

Let us denote $E = E_G \cup E_H$ and $B : p \mapsto D(H, p) + A(p)$. The entries of B belong to $\mathbb{L}(E)$. Therefore, its determinant also belongs to $\mathbb{L}(E)$. The decomposition of $\det B$ on the natural basis of $\mathbb{L}(E)$ gives:

$$(A.3) \quad \det B = \sum_{F \in \mathcal{P}(E)} \alpha_F \prod_{uw \in F} D_{u,w}$$

where the α_F are the coordinates of $\det B$ on the natural basis of $\mathbb{L}(E)$.

Furthermore, since E_H and E_G are disjoint and since the entries of A are in $\mathbb{L}(E_G)$, by an easy induction on the cardinal of E_H , the coordinates associated to E_H is 1.

Therefore $\det B \neq 0$. Then, by Lemma 4.4, $\det B(p)$ is not null and $B(p)$ is invertible. \square

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