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# Blow up for nonlinear parabolic equations with time degeneracy under dynamical boundary conditions

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**Abstract**

We study the blow up behaviour of nonlinear parabolic equations including a time degeneracy, under dynamical boundary conditions. For some exponential and polynomial degeneracies, we develop some energy methods and some spectral comparison techniques and derive upper bounds for the blow up times.


*MSC:* 35B40; 35K55; 35K65; 35R45

*Keywords:* Nonlinear parabolic equations; Time degeneracy; Dynamical boundary conditions; Blow up

**Contents**



## 1. Introduction

In the present paper we study the occurrence of blow up phenomena for several nonlinear parabolic problems including a time degeneracy:

$$\begin{cases} (\exp(ku))_t = \Delta u + \exp(pu) & \text{in } \overline{\Omega} \text{ for } t > 0, \\ B_\sigma(u) := \sigma \partial_t u + \partial_\nu u = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ u(\cdot, 0) = \varphi \in C(\overline{\Omega}) \end{cases} \tag{1}$$

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with $1 \leq k < p \in \mathbb{R}$,

$$\begin{cases} (u^{2k+1})_t = \Delta u + u|u|^{p-1} & \text{in } \overline{\Omega} \text{ for } t > 0, \\ B_\sigma(u) := \sigma \partial_t u + \partial_\nu u = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ u(\cdot, 0) = \varphi \in C(\overline{\Omega}) \end{cases} \tag{2}$$

with $k \in \mathbb{N}$ and $2k + 1 < p \in \mathbb{R}$,

$$\begin{cases} (u^{2k+1})_t = \operatorname{div}(u^m \nabla u) + |u|^p & \text{in } \overline{\Omega} \text{ for } t > 0, \\ B_\sigma(u) := \sigma \partial_t u + \partial_\nu u = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ u(\cdot, 0) = \varphi \in C(\overline{\Omega}) \end{cases} \tag{3}$$

with $k \in \mathbb{N}$, $m \in \mathbb{N}$ and $2k + 1 < p \in \mathbb{R}$.

We assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain, whose boundary $\partial\Omega$ is of class $C^2$. On the time lateral boundary, we impose some dynamical conditions, involving the outer normal derivative and the time derivative. The outer normal unit vector field and the outer normal derivative are respectively denoted by $\nu : \partial\Omega \to \mathbb{R}^n$ and $\partial_\nu$. We suppose the dissipativity condition

$$\sigma \geq 0 \quad \text{on } \partial\Omega \times (0, \infty), \tag{4}$$

and, dealing with classical solutions, we assume that

$$\sigma \in C^1(\partial\Omega \times (0, \infty)). \tag{5}$$

As for the initial data, we require

$$\varphi \in C(\overline{\Omega}), \varphi \geq 0, \varphi \neq 0. \tag{6}$$

As for local existence of solutions for parabolic equations with time degeneracy, we refer to [1] and [5]. As for qualitative theory, we refer to [8], to [3] and [4] for nonlinear parabolic equations under dynamical boundary conditions, and to [2] for equations with time degeneracy. Blow up results for equations including a principal part degeneracy are mentioned in [6,7] and in [4] and [10] for dynamical boundary conditions. As for blow up results for equations containing some gradient term and for Problem (3) in the particular case where $m = 0$, we refer to the author's thesis. As for applications and modellings involving dynamical boundary conditions and time degeneracy, we refer to [9] and [11]. Since the exponential time degeneracy $b(u) := \exp(ku)$ in Problem (1) is a strictly increasing function, we can apply some methods developed in [4] for reaction–diffusion equations. Especially, we can compare solutions for different boundary conditions and establish some lower and upper bounds for the blow up times. For Problem (2) and (3) including a polynomial time degeneracy, the approach is quite different since degeneracies occur when the l.h.s. vanishes. As proved in [2], a comparison principle does not hold for such equations in general, and thereby no uniqueness and flow existence can be deduced in general. We investigate the occurrence of blow up under homogeneous Dirichlet and Neumann boundary conditions by deriving upper bounds for the blow up times. Furthermore, for Problem (3), we study the blow up behaviour in the dynamical case, in particular for $p = m + 1$.

## 2. Exponential time degeneracy

In this section we are interested in Problem (1). Note that for $\tau > 0$, its parabolic partial differential equation can be written in $\overline{\Omega} \times (0, \tau]$ in the form

$$\partial_t u = \frac{1}{k} \exp(-ku) \Delta u + \frac{1}{k} \exp((p-k)u) =: F[u],$$

with $F \in C^1(\overline{\Omega} \times (0, \tau] \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2})$. Then, some of the qualitative results developed in [3] and [4] hold. In particular, the comparison principle Theorem 7.1 of [4] and its corollaries imply the unicity and the non-negativity in $\overline{\Omega} \times [0, \tau]$ of the solution of Problem (1). Therefore, throughout this section, we note

$$u_\sigma \in C\left(\overline{\Omega} \times [0, T(\sigma, \varphi))\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, T(\sigma, \varphi))\right)$$

the maximal solution of Problem (1), where $T(\sigma, \varphi)$ denotes the blow up time of $u_\sigma$, i.e. its maximal existence time in $C(\overline{\Omega})$, defined as

$$T(\sigma, \varphi) = \inf \left\{ s > 0 \middle| \lim_{t \nearrow s} \sup\{|u_\sigma(x,t)| \mid x \in \overline{\Omega}\} = \infty \right\}.$$

By the comparison principle, we can prove the decreasing character of $T(\sigma, \varphi)$ with respect to the initial value $\varphi$:

**Theorem 1.** *Suppose $\varphi_1, \varphi_2 \in C(\overline{\Omega})$. Then*

$$0 \le \varphi_1 \le \varphi_2 \implies T(\sigma, \varphi_1) \ge T(\sigma, \varphi_2).$$

As in [4], the additional assumptions

$$\varphi \in C^2(\overline{\Omega}), \quad \frac{1}{k}\exp(-k\varphi)\Delta\varphi + \frac{1}{k}\exp((p-k)\varphi) \ge 0 \quad \text{in } \overline{\Omega}, \tag{7}$$

and

$$\sigma \in C^1(\partial\Omega) \tag{8}$$

imply

**Lemma 2.** *Let $u \in C^{2,1}(\overline{\Omega} \times [0, \tau])$ be a solution of Problem (1). Then u satisfies $\partial_t u \ge 0$ in $\overline{\Omega} \times [0, \tau]$.*

Then by the comparison principle, $T(\sigma, \varphi)$ is increasing with respect to the coefficient $\sigma$:

**Theorem 3.** *Under Condition (7), suppose that the functions $0 \le \sigma_1 \le \sigma_2$ satisfy (8). Then $u_{\sigma_1} \ge u_{\sigma_2}$ in $\overline{\Omega} \times [0, T(\sigma_1, \varphi))$ and*

$$T(\sigma_1, \varphi) \le T(\sigma_2, \varphi).$$

Moreover, following the comparative techniques of [4] we can establish a relation between the blow up times of the classical solutions of Problem (1) fulfilling different boundary conditions. Indeed, recall that the Neumann boundary condition $\partial_\nu u = 0$ on $\partial\Omega$ is the dynamical boundary condition $B_\sigma(u) = 0$ with $\sigma = 0$ and that the Dirichlet condition $u = \varphi$ on $\partial\Omega$ corresponds to the case $\sigma \equiv \infty$. Introduce

$$v \in C\left(\overline{\Omega} \times [0, T(\infty, \varphi))\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, T(\infty, \varphi))\right)$$

as the maximal solution of

$$\begin{cases} (\exp(kv))_t = \Delta v + \exp(pv) & \text{in } \overline{\Omega} \times (0, T(\infty, \varphi)), \\ v(\cdot, 0) = \varphi & \text{in } \overline{\Omega}, \\ v = \varphi & \text{on } \partial\Omega \times (0, T(\infty, \varphi)), \end{cases} \tag{9}$$

and

$$w \in C\left(\overline{\Omega} \times [0, T_0(\infty, \tilde{\varphi}))\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, T_0(\infty, \tilde{\varphi}))\right)$$

as the maximal solution of

$$\begin{cases} (\exp(kw))_t = \Delta w + \exp(pw) & \text{in } \overline{\Omega} \times (0, T_0(\infty, \tilde{\varphi})), \\ w(\cdot, 0) = \tilde{\varphi} & \text{in } \overline{\Omega}, \\ w = 0 & \text{on } \partial\Omega \times (0, T_0(\infty, \tilde{\varphi})), \end{cases} \tag{10}$$

with

$$\tilde{\varphi} \in C(\overline{\Omega}), \qquad \tilde{\varphi} = 0 \quad \text{on } \partial\Omega, \qquad \tilde{\varphi} \le \varphi. \tag{11}$$

We are led to

**Theorem 4.** *Under Conditions* (7) and (8)

$$u_0 \geq u_\sigma \geq v \geq w$$

*in the domain of definition of* $u_0$, $u_\sigma$ *and* $v$ *respectively, and*

$$T(0,\varphi) \leq T(\sigma,\varphi) \leq T(\infty,\varphi) \leq T_0(\infty,\tilde{\varphi}).$$

Furthermore, we can establish a lower bound for the blow up time. Consider the maximal solution $z \in C^1([0,t_1))$ of the EDO

$$\begin{cases} \dot{z} = (1/k)\exp((p-k)z) & \text{in } [0,t_1), \\ z(0) = \|\varphi\|_\infty. \end{cases} \tag{12}$$

By Theorems 1 and 3, we obtain

**Theorem 5.**

$$T(\sigma,\varphi) \geq T(0,\varphi) \geq T(0,\|\varphi\|_\infty) = \frac{k}{p-k}\exp((k-p)\|\varphi\|_\infty) = t_1.$$

Now, we derive upper bounds for the blow up time $T(\sigma,\varphi)$. First, we establish an optimal upper bound under Neumann boundary condition, which corresponds to $t_1$ for a constant initial data $\varphi$.

**Theorem 6.** *Suppose* $\sigma = 0$. *Then the classical maximal solution* $u_0$ *of Problem* (1) *blows up in finite time* $T(0,\varphi)$ *satisfying*

$$T(0,\varphi) \leq \frac{k}{p-k}|\Omega|^{(p-k)/k}\left(\int_\Omega \exp(k\varphi)\,\mathrm{d}x\right)^{(k-p)/k} =: t_2.$$

**Proof.** The Neumann boundary condition implies $\int_\Omega \Delta u\,\mathrm{d}x = \int_{\partial\Omega} \partial_\nu u\,\mathrm{d}s = 0$. Set $M(t) = \int_\Omega \exp(ku(\cdot,t))\,\mathrm{d}x$, then $\dot{M} = \int_\Omega \exp(pu)\,\mathrm{d}x$, and by Hölder's inequality,

$$\dot{M} \geq |\Omega|^{(k-p)/k} M^{p/k}.$$

Integration between 0 and $t > 0$ leads to

$$\int_{M(0)}^{M(t)} \eta^{-p/k}\,\mathrm{d}\eta \geq |\Omega|^{(k-p)/k} t$$

and

$$M(t) \geq \left(M(0)^{(k-p)/k} - \frac{p-k}{k}|\Omega|^{(k-p)/k} t\right)^{k/(k-p)}.$$

The right hand side of the inequality becomes infinite iff

$$t = \frac{k}{p-k} M(0)^{(k-p)/k}|\Omega|^{(p-k)/k} = t_2$$

then $T(0,\varphi) \leq t_2$. □

Now, dealing with spectral methods described in [4], we derive an upper bound for the blow up time for any boundary condition already mentioned. Throughout, we note $\lambda = \lambda(\Omega)$ the minimal eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ and

$$\psi \in H_0^1(\Omega) \tag{13}$$

an eigenfunction belonging to $\lambda$ fulfilling

$$0 < \psi \leq 1 \quad \text{in } \Omega. \tag{14}$$

Clearly $\psi$ satisfies

$$\partial_\nu \psi \leq 0 \quad \text{on } \partial\Omega. \tag{15}$$

The following holds:

**Theorem 7.** *Let $c > 0$, $T > 0$ and $1 \leq k < p$ denote real constants and let*

$$u \in C\left(\overline{\Omega} \times [0, T)\right) \cap C^{2,1}(\Omega \times (0, T))$$

*be a maximal solution of*

$$\begin{cases} (\exp(ku))_t \geq \Delta u + c \exp(pu) & \text{in } \Omega \times (0, T), \\ u \geq 0 & \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = \varphi \geq 0, \neq 0. \end{cases} \tag{16}$$

*If*

$$|\Omega| \left(\frac{\lambda}{c}\right)^{k/(p-k)} < \int_\Omega \exp(k\varphi)\psi \, \mathrm{d}x, \tag{17}$$

*then*

$$T \leq \frac{k}{\lambda(p-k)} \ln\left(\frac{c}{c - \lambda|\Omega|^{(p-k)/k} \left(\int_\Omega \exp(k\varphi)\, \psi \, \mathrm{d}x\right)^{(k-p)/p}}\right) =: t_3.$$

**Proof.** Define $M(t) = \int_\Omega \exp(ku(\cdot, t))\, \psi \, \mathrm{d}x$. Clearly, (13) and (15) imply

$$\int_{\partial\Omega} \partial_\nu u \, \psi \, \mathrm{d}s = 0, \qquad \int_{\partial\Omega} u \, \partial_\nu \psi \, \mathrm{d}s \leq 0. \tag{18}$$

Thus Green's Formula leads to

$$\int_\Omega \Delta u \, \psi \, \mathrm{d}x = \int_\Omega u \, \Delta\psi \, \mathrm{d}x + \int_{\partial\Omega} (\partial_\nu u \, \psi - u \, \partial_\nu \psi) \, \mathrm{d}s \geq -\lambda \int_\Omega u \, \psi \, \mathrm{d}x.$$

Since $u \geq 0$, by the assumptions on $k$ and $p$, we have $\exp(ku) \geq u$ and $\psi^{p/k} \leq \psi$. Then

$$\dot{M} \geq -\lambda \int_\Omega \exp(ku)\psi \, \mathrm{d}x + c \int_\Omega \exp(pu)\psi^{p/k} \, \mathrm{d}x.$$

Hölder's inequality implies

$$\int_\Omega \exp(pu)\psi^{p/k} \, \mathrm{d}x \geq |\Omega|^{(k-p)/k} M^{p/k}.$$

Thus

$$\dot{M} \geq -\lambda M + \alpha M^\beta = \alpha M(M^{\beta-1} - \delta)$$

with

$$\alpha = c|\Omega|^{(k-p)/k}, \qquad \beta = \frac{p}{k} \quad \text{and} \quad \delta = \frac{\lambda}{\alpha}.$$

Integration between 0 and $t > 0$ yields

$$\alpha t \leq \int_{M(0)}^{M(t)} \frac{1}{\eta(\eta^{\beta-1} - \delta)} \, \mathrm{d}\eta = \frac{1}{\delta} \left\{ \int_{M(0)}^{M(t)} \frac{\eta^{\beta-2}}{\eta^{\beta-1} - \delta} \, \mathrm{d}\eta - \int_{M(0)}^{M(t)} \frac{\mathrm{d}\eta}{\eta} \right\}$$

and

$$\frac{1}{\beta - 1} \int_{M(0)^{\beta-1}}^{M(t)^{\beta-1}} \frac{\mathrm{d}\xi}{\xi - \delta} - \int_{M(0)}^{M(t)} \frac{\mathrm{d}\eta}{\eta} \geq \lambda t. \tag{19}$$

$M$ is increasing with respect to $t$ since $\dot{M} = k\int_\Omega \exp(ku)\psi\,\partial_t u\,\mathrm{d}x \geq 0$ by Lemma 2. Then (17) implies $M(t) > \delta^{1/(\beta-1)}$ for $t \geq 0$ and (19) yields

$$\ln\frac{M(t)^{\beta-1}-\delta}{M(0)^{\beta-1}-\delta} - \ln\frac{M(t)^{\beta-1}}{M(0)^{\beta-1}} \geq (\beta-1)\lambda t.$$

We obtain

$$1 > \frac{M(t)^{\beta-1}-\delta}{M(t)^{\beta-1}} \geq (1-\delta M(0)^{1-\beta})\exp((\beta-1)\lambda t) > 0.$$

$M$ becomes infinite iff the ratio above approaches unity, i.e., iff

$$t = \frac{1}{(\beta-1)\lambda}\ln\frac{1}{1-\delta M(0)^{1-\beta}} = t_3.$$

Then $u$ blows up at the latest for $t = t_3$ and $T \leq t_3$. □

## 3. Polynomial time degeneracy

This section is devoted to the blow up behaviour of solutions of Problem (2) and Problem (3), including a polynomial time degeneracy. In the case where $p \leq 2k+1$, we can establish an a priori bound for these problems thanks to Theorem 1 from [2] and show global existence. Indeed, the solutions grow at most exponentially because for any $\tau > 0$

$$\max_{\overline{\Omega}\times[0,\tau]} |u| \leq \mathrm{e}^{2/(2k+1)\tau}\max\left\{\max_{\overline{\Omega}}\varphi, 1\right\}.$$

First, we consider Problem (2). For $p > 2k+1$, we derive upper bounds for the blow up times under homogeneous Dirichlet boundary condition and Neumann boundary condition, by using an energy type method. Introduce the energy functional $E : H^1(\Omega)\cap L^{p+1}(\Omega) \to \mathbb{R}$, defined by

$$E(u) = \frac{1}{2}\int_\Omega \|\nabla u\|_2^2\,\mathrm{d}x - \frac{1}{p+1}\int_\Omega |u|^{p+1}\,\mathrm{d}x.$$

We have the following

**Lemma 8.** *The function $t \mapsto E(u(\cdot,t))$ is decreasing if $u \in C(\overline{\Omega}\times[0,\tau])\cap C^{2,1}(\overline{\Omega}\times(0,\tau])$ is a solution of Problem (2) under Neumann boundary condition or Dirichlet boundary condition.*

**Proof.** First, note that under the Dirichlet boundary condition $u = \varphi$ on $\partial\Omega$ for $t > 0$ and under Neumann boundary condition, we have

$$\int_{\partial\Omega} \partial_t u\,\partial_\nu u\,\mathrm{d}s = 0.$$

Then, thanks to Green's formula

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(u(t)) &= \int_\Omega (\partial_t\nabla u, \nabla u)\,\mathrm{d}x - \int_\Omega u|u|^{p-1}\,\partial_t u\,\mathrm{d}x\\ &= -\int_\Omega \partial_t u\,\Delta u\,\mathrm{d}x + \int_{\partial\Omega}\partial_t u\,\partial_\nu u\,\mathrm{d}s - \int_\Omega u|u|^{p-1}\partial_t u\,\mathrm{d}x\end{aligned}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}E(u(t)) \leq -(2k+1)\int_\Omega u^{2k}(\partial_t u)^2\,\mathrm{d}x \leq 0.$$ □

**Theorem 9.** *Suppose $\varphi \in H_0^1(\Omega)\cap C(\overline{\Omega})$ and $E(\varphi) \leq 0$. Let*

$$u \in C(\overline{\Omega}\times[0,T))\cap C^{2,1}(\overline{\Omega}\times(0,T))$$

*be a maximal solution of*

$$\begin{cases} (u^{2k+1})_t = \Delta u + u|u|^{p-1} & \text{in } \overline{\Omega} \times (0,T), \\ u(\cdot,0) = \varphi & \text{in } \overline{\Omega}, \\ u = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{20}$$

*Then u blows up in finite time T satisfying*

$$T \le \frac{(p+1)(2k+1)}{(p-1)(p-2k-1)} |\Omega|^{(p-2k-1)/(2k+2)} \left( \int_\Omega \varphi^{2k+2}(x)\,\mathrm{d}x \right)^{(2k+1-p)/(2k+2)} =: t_4.$$

**Proof.** Introduce

$$N(t) = |u^{k+1}|_{2,\Omega}^2 = \int_\Omega u^{2k+2}(x,t)\,\mathrm{d}x.$$

Then

$$\begin{aligned} \dot{N} &= \frac{2k+2}{2k+1} \int_\Omega u (u^{2k+1})_t \,\mathrm{d}x \\ &= \frac{2k+2}{2k+1} \left( \int_\Omega u \Delta u \,\mathrm{d}x + \int_\Omega |u|^{p+1} \,\mathrm{d}x \right). \end{aligned}$$

Green's formula and homogeneous Dirichlet boundary condition imply

$$\begin{aligned} \int_\Omega u \Delta u \,\mathrm{d}x &= -\int_\Omega \|\nabla u\|^2 \,\mathrm{d}x + \underbrace{\int_{\partial\Omega} u \partial_\nu u \,\mathrm{d}s}_{=0} \\ &= -2E(u) - \frac{2}{p+1} \int_\Omega |u|^{p+1} \,\mathrm{d}x. \end{aligned} \tag{21}$$

Thus, since by Lemma 8, $E(u) \le E(\varphi) \le 0$, we are led to

$$\dot{N} \ge \underbrace{\frac{(2k+2)(p-1)}{(2k+1)(p+1)}}_{:=\alpha_0} \int_\Omega |u|^{p+1} \,\mathrm{d}x > 0.$$

Since $p > 2k+1$, Hölder's inequality implies

$$\int_\Omega |u|^{p+1} \,\mathrm{d}x \ge |\Omega|^{(2k+1-p)/(2k+2)} \left( \int_\Omega u^{2k+2} \right)^{(p+1)/(2k+2)},$$

thus

$$\dot{N} \ge \alpha_1 N^\beta \tag{22}$$

with $\alpha_1 = \alpha_0 |\Omega|^{(2k+1-p)/(2k+2)}$ and $\beta = (p+1)/(2k+2)$. Integration between 0 and $t > 0$ yields

$$N(t) \ge (N(0)^{1-\beta} - \alpha_1(\beta-1)t)^{1/(1-\beta)}.$$

The right hand side of the inequality becomes infinite iff $t = t_4$, where $t_4$ is defined as in the assertion. Then $u$ blows up at the latest for $t = t_4$ and $T \le t_4$. □

Since (21) remains valid for Neumann boundary condition instead of homogeneous Dirichlet boundary condition, we obtain

**Theorem 10.** *Suppose $\sigma = 0$, $\varphi \in H^1(\Omega) \cap C(\overline{\Omega})$ and $E(\varphi) \le 0$. Let u be a maximal classical solution of Problem (2), then u blows up in finite time T satisfying*

$$T \le t_4,$$

*with $t_4$ as defined in Theorem 9.*

Now, we are interested in Problem (3) including a possible degenerate principal part. Even if a comparison principle does not hold because of the nonuniqueness of solutions, Theorems 2 and 4 from [2] imply the following qualitative result

**Theorem 11.** *Let $\tau > 0$ be a real constant and $u \in C(\overline{\Omega} \times [0, \tau]) \cap C^{2,1}(\overline{\Omega} \times (0, \tau])$ a solution of*

$$\begin{cases} (u^{2k+1})_t = \operatorname{div}(u^m \nabla u) + |u|^p & \text{in } \overline{\Omega} \times (0, \tau], \\ B_\sigma(u) := \sigma \partial_t u + \partial_\nu u = 0 & \text{on } \partial\Omega \times (0, \tau], \\ u(\cdot, 0) = \varphi \in C(\overline{\Omega}) \end{cases} \tag{23}$$

*then $u$ satisfies*

$$u \geq 0 \quad \text{in } \overline{\Omega} \times [0, \tau] \quad \text{and} \quad \min_{\overline{\Omega} \times [0,\tau]} u = \min_{\overline{\Omega}} \varphi. \tag{24}$$

We estimate the blow up times for different boundary conditions from above by spectral comparison. We use the notations $\lambda$ and $\psi$ already introduced in Section 2. First, we study the special case $p = m + 1$ and we establish an upper bound for the blow up times under the general boundary condition $u \geq 0$.

**Theorem 12.** *Suppose $m \geq \max\{2k, 1\}$ and let*

$$u \in C\left(\overline{\Omega} \times [0, T)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, T)\right)$$

*be a maximal solution of*

$$\begin{cases} (u^{2k+1})_t = \operatorname{div}(u^m \nabla u) + |u|^{m+1} & \text{in } \overline{\Omega} \times (0, T), \\ u \geq 0 & \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = \varphi & \text{in } \overline{\Omega}. \end{cases} \tag{25}$$

*If*

$$\lambda < m + 1,$$

*then*

$$T \leq \frac{(m+1)(2k+1)|\Omega|^{(m-2k)/(2k+1)}}{(m-2k)(m+1-\lambda)} \left( \int_\Omega \varphi^{2k+1} \psi \, \mathrm{d}x \right)^{(2k-m)/(2k+1)} =: t_5.$$

**Proof.** By (13) and (15), (18) remains valid for $u^{m+1}$ instead of $u$. Thus,

$$\int_\Omega \operatorname{div}(u^m \nabla u) \, \psi \, \mathrm{d}x = \frac{1}{m+1} \int_\Omega \Delta(u^{m+1}) \, \psi \, \mathrm{d}x \geq -\frac{\lambda}{m+1} \int_\Omega u^{m+1} \psi \, \mathrm{d}x.$$

Introduce

$$M(t) = \int_\Omega u^{2k+1}(\cdot, t) \, \psi \, \mathrm{d}x,$$

then by (24) we obtain the differential inequality

$$\dot{M} \geq \frac{(m+1-\lambda)}{m+1} \int_\Omega u^{m+1} \psi \, \mathrm{d}x.$$

Moreover, Hölder's inequality

$$\int_\Omega u^{2k+1} \psi \, \mathrm{d}x \leq |\Omega|^{(m-2k)/(m+1)} \left( \int_\Omega u^{m+1} \psi^{(m+1)/(2k+1)} \, \mathrm{d}x \right)^{(2k+1)/(m+1)} \tag{26}$$

and (14) imply

$$\dot{M} \geq \underbrace{\frac{(m+1-\lambda)}{m+1} |\Omega|^{(2k-m)/(2k+1)}}_{:=C} M^{(m+1)/(2k+1)}.$$

Integration leads to

$$M(t) \geq \left( M(0)^{(2k-m)/(2k+1)} - \frac{m-2k}{2k+1} Ct \right)^{(2k+1)/(2k-m)}.$$

The term in brackets vanishes iff $t = t_5$, with $t_5$ as defined in the assertion. Then $T \leq t_5$ since $M(t) \leq |\Omega| \times \|u(\cdot,t)\|_\infty^{2k+1}$. □

Note that Theorem 12 is valid in particular under Neumann, Dirichlet or dynamical boundary conditions. Furthermore, in the general case where $p > 2k+1$ and $m \in \mathbb{N}$, we can derive an upper bound for the blow up times for Neumann and homogeneous Dirichlet boundary conditions, thanks to the estimate of the $L^{2k+1}$-norm.

**Theorem 13.** *Suppose $\sigma = 0$. Let $k, m \in \mathbb{N}$ and $2k+1 < p \in \mathbb{R}$. Let $u$ be a maximal classical solution of Problem (3). Then $u$ blows up in finite time $T$ satisfying*

$$T \leq \frac{(2k+1)|\Omega|^{(p-2k-1)/(2k+1)}}{(p-2k-1)} \left( \int_\Omega \varphi^{2k+1} \, \mathrm{d}x \right)^{(2k+1-p)/(2k+1)} =: t_6.$$

**Proof.** Note that Neumann boundary condition implies

$$\int_\Omega \Delta(u^{m+1}) \, \mathrm{d}x = (m+1) \int_{\partial\Omega} u^m \partial_\nu u \, \mathrm{d}s = 0. \tag{27}$$

Define $M$ by $M(t) = \int_\Omega u(\cdot,t)^{2k+1} \, \mathrm{d}x$. Eqs. (24) and (27) and Hölder's inequality yield

$$\dot{M} = \int_\Omega u^p \, \mathrm{d}x \geq |\Omega|^{(2k+1-p)(2k+1)} M^{p/(2k+1)}.$$

Integration between 0 and $t > 0$ implies

$$M(t) \geq \left( M(0)^{(2k+1-p)/(2k+1)} - \frac{p-2k-1}{2k+1} |\Omega|^{(2k+1-p)/(2k+1)} t \right)^{(2k+1)/(2k+1-p)}.$$

Then $u$ becomes infinite iff $T \leq t_6$, where $t_6$ is defined as in the assertion. □

Since (27) is still valid for homogeneous Dirichlet boundary condition for $m \geq 1$, we have

**Theorem 14.** *Let $T > 0$ denote a real constant, $k \in \mathbb{N}$, $1 \leq m \in \mathbb{N}$ and $2k+1 < p \in \mathbb{R}$. Let*

$$u \in C\left(\overline{\Omega} \times [0,T)\right) \cap C^{2,1}\left(\overline{\Omega} \times (0,T)\right)$$

*be a maximal solution of*

$$\begin{cases} (u^{2k+1})_t = \operatorname{div}(u^m \nabla u) + |u|^p & \text{in } \overline{\Omega} \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = \varphi \in C_0(\overline{\Omega}), & 0 \leq \varphi \not\equiv 0. \end{cases} \tag{28}$$

*Then $T \leq t_6$, with $t_6$ as defined in Theorem 13.*

Of special interest is the occurrence of finite time blow up for Problem (3), in the case where a dynamical boundary condition is imposed with $0 \leq \sigma < \infty$.

**Theorem 15.** *Let $p \in \mathbb{R}$, $m \in \mathbb{N}$ and $k \in \mathbb{N}$ with $p > 2k+1 > m+1$. If there exists $c \in (0,1)$ such that*

$$\lambda \leq (m+1)\, c \left( |\Omega|^{-1} \left( \min_{\overline{\Omega}} \varphi \right)^{2k+1} \int_\Omega \psi \, \mathrm{d}x \right)^{(p-m-1)/(2k+1)} \tag{29}$$

*then the maximal classical solutions of Problem (3) blow up at the latest for*

$$t = \frac{(2k+1)|\Omega|^{(p-2k-1)/(2k+1)}}{(p-2k-1)(1-c)} \left( \int_\Omega \varphi^{2k+1} \, \psi \, \mathrm{d}x \right)^{(2k+1-p)/(2k+1)} =: t_7.$$

**Proof.** Set $M(t) = \int_\Omega u^{2k+1}(\cdot, t)\,\psi\,\mathrm{d}x$ then

$$\dot{M} \geq -\frac{\lambda}{m+1}\int_\Omega u^{m+1}\,\psi\,\mathrm{d}x + \int_\Omega u^p\,\psi\,\mathrm{d}x.$$

Note that for the moment it is not clear that $\dot{M} \geq 0$. Eq. (14) and Hölder's inequalities yield

$$\int_\Omega u^p\,\psi\,\mathrm{d}x \geq |\Omega|^{(2k+1-p)/(2k+1)} M^{p/(2k+1)}$$

and

$$\int_\Omega u^{m+1}\,\psi\,\mathrm{d}x \leq |\Omega|^{(2k-m)/(2k+1)} M^{(m+1)/(2k+1)}.$$

Then

$$\dot{M} \geq -\alpha M^\beta + \delta M^\gamma = \delta M^\gamma\left(1 - \frac{\alpha}{\delta}M^{\beta-\gamma}\right)$$

with

$$\alpha = \frac{\lambda}{m+1}|\Omega|^{(2k-m)/(2k+1)}, \qquad \beta = \frac{m+1}{2k+1}, \qquad \delta = |\Omega|^{(2k+1-p)/(2k+1)} \quad \text{and} \quad \gamma = \frac{p}{2k+1}.$$

By (24) and (29),

$$M^{\gamma-\beta} \geq \frac{\alpha}{\delta\, c}$$

then we have

$$\dot{M} \geq (1-c)\,\delta M^\gamma > 0.$$

By integration,

$$\int_{M(0)}^{M(t)} \frac{\mathrm{d}\eta}{\eta^\gamma} \geq (1-c)\,\delta t$$

and

$$M(t) \geq \left(M(0)^{1-\gamma} - (1-c)(\gamma-1)\,\delta t\right)^{1/(1-\gamma)}.$$

Since $\gamma > 1$, the term in brackets vanishes iff $t = \frac{M(0)^{1-\gamma}}{(1-c)(\gamma-1)\,\delta} =: t_7$. □

Note that the upper bound $t_6$, established under Neumann and homogeneous Dirichlet boundary conditions, satisfies $t_6 < t_7$.

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