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ABC(T)-GRAPHS: AN AXIOMATIC CHARACTERIZATION OF THE MEDIAN PROCEDURE IN GRAPHS WITH CONNECTED AND G^2 -CONNECTED MEDIANS

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Abstract. The median function is a location/consensus function that maps any profile π (a finite multiset of vertices) to the set of vertices that minimize the distance sum to vertices from π . The median function satisfies several simple axioms: Anonymity (A), Betweenness (B), and Consistency (C). McMorris, Mulder, Novick and Powers (2015) defined the ABC-problem for consensus functions on graphs as the problem of characterizing the graphs (called, ABC-graphs) for which the unique consensus function satisfying the axioms (A), (B), and (C) is the median function. In this paper, we show that modular graphs with G^2 -connected medians (in particular, bipartite Helly graphs) are ABC-graphs. On the other hand, the addition of some simple local axioms satisfied by the median function in all graphs (axioms (T), and (T₂)) enables us to show that all graphs with connected median (comprising Helly graphs, median graphs, basis graphs of matroids and even Δ -matroids) are ABCT-graphs and that benzenoid graphs are ABCT₂-graphs. McMorris et al (2015) proved that the graphs satisfying the pairing property (called the intersecting-interval property in their paper) are ABC-graphs. We prove that graphs with the pairing property constitute a proper subclass of bipartite Helly graphs and we discuss the complexity status of the recognition problem of such graphs.

1. INTRODUCTION

The median problem is one of the oldest optimization problems in Euclidean geometry [24]. Although it can be formulated for all metric spaces, we will consider it for graphs. Given a (not necessarily finite) connected graph $G = (V, E)$, a *profile* is any finite sequence $\pi = (x_1, \dots, x_n)$ of vertices of G . The *total distance* of a vertex v of G is defined by $F_\pi(v) = \sum_{i=1}^n d(v, x_i)$. A vertex v minimizing $F_\pi(v)$ is called a *median vertex* of G with respect to π and the set of all medians vertices is called the *median set* $\text{Med}_G(\pi)$. Finally, the mapping that associate to any profile π of G its median set $\text{Med}_G(\pi)$ is called the *median function*. Consequently, the median function $\text{Med} = \text{Med}_G$ can be viewed as a particular consensus function. In the consensus problem in social group choice, given individual preferences of candidates one has to compute a consensual group decision that best reflects those preferences. It is usually required that the consensus respects some simple axioms ensuring that it remains reasonable and rational. However, by the classical Arrow's [3] impossibility theorem, there is no consensus function satisfying natural "fairness" axioms. In this respect, the Kemeny median [21, 22] is an important consensus function satisfying most of fairness axioms. It corresponds to the median problem in graphs of permutahedra.

Following Arrow's axiomatic approach to consensus functions, one may wish to characterize axiomatically a consensus function M of a given type among all consensus functions. Then the goal is either to characterize M by a set of axioms or to characterize the instances for which M is the unique consensus function satisfying a set of natural axioms. Holzman [20] was the first to study a location function axiomatically as a consensus problem. Namely, he characterized the barycenter function of trees, namely the function B mapping each profile $\pi = (x_1, \dots, x_n)$ to the set of vertices minimizing the mean $F_\pi(v) = \sum_{i=1}^n d^2(v, x_i)$. Notice also that the axiomatic characterization of the barycentric function on the line or in \mathbb{R}^d is a classical problem [1], having its origins in the paper [23] by Kolmogorov. Foster and Vohra [17] axiomatically characterized the median function Med on continuous trees (where the location can be on edges), and McMorris, Mulder and Roberts [28] characterized Med in the case of combinatorial trees. They also showed that three basic axioms (Anonymity (A), Betweenness (B) and Consistency (C)) are sufficient to characterize the median function of cube-free median graphs. Median graphs constitute the most important class of graphs in metric graph theory [8]. Those are the graphs in which for each triplet of vertices there exists exactly one vertex lying on shortest paths between any pair of vertices of the triplet. The nice result of Mulder and Novick [29] shows that in fact the axioms (A), (B), and (C) characterize Med in all median graphs. They called the consensus functions that respects the axioms (A), (B), and (C) *ABC-functions*. Recently, McMorris, Mulder, Novick, and Powers [27] introduced the

ABC-problem as the characterization of graphs (which we call *ABC-graphs*) with unique ABC-functions (which then necessarily coincide with Med). Additionally to median graphs, they proved that the graphs satisfying the intersecting-intervals property (which we call *pairing property*) are ABC-graphs. On the other hand, they showed that the complete graph K_n admits at least two ABC-functions for any $n \geq 3$. Finally, they asked to find additional axioms that hold for the median function on all graphs and which added to (A), (B),(C) characterize Med.

Bandelt and Chepoi [7] characterized the *graphs with connected medians*, i.e., the graphs in which all median sets induce connected subgraphs (equivalently, for all profiles π , the function F_π is unimodal). They also established that several important classes of graphs (Helly graphs, basis graphs of matroids, and weakly median graphs) are graphs with connected medians. Recently, in [10] we generalized these results and characterized the graphs with G^p -connected medians, i.e., the graphs in which all median sets induce connected subgraphs in the p th power G^p of the graph G . We also established that some important classes of graphs have G^2 -connected medians (chordal graphs, bridged graphs, graphs with convex balls, bipartite Helly graphs, benzenoids).

In this paper, we continue the research on the ABC-problem in several directions. First, we prove that modular graphs with G^2 -connected medians (and in particular, bipartite Helly graphs) are ABC-graphs. Second, we prove that graphs with connected medians are ABCT-graphs, i.e., graphs in which the median function is the unique consensus function satisfying (A),(B),(C), and a new axiom (T), requiring that the consensus function returns all the elements of the profile when the profile is a triplet of three pairwise adjacent vertices. We extend (T) to a similar axiom (T₂) on triplets of vertices at pairwise distance 2 and prove that benzenoids are ABCT₂-graphs (but we show that the 6-cycle, which is the simplest benzenoid, has at least two ABC-functions). Finally, we prove that the graphs with the pairing or double-pairing property constitute a proper subclass of bipartite Helly graphs. We characterize the graphs with the pairing/double-pairing property as bipartite Helly graphs satisfying a local condition on balls of radius 2. We show that the problem of deciding if a bipartite Helly graph satisfies the double pairing property is in co-NP. However we do not know if one can decide in polynomial time (or even in non-deterministic polynomial time) if a bipartite Helly graph satisfies the double-pairing or the pairing property.

2. PRELIMINARIES

2.1. Graphs. All graphs $G = (V, E)$ in this paper are undirected, simple, and connected; V is the vertex-set and E is the edge-set of G . We write $u \sim v$ if $u, v \in V$ are adjacent. Given a graph $G = (V, E)$, the *distance* $d_G(u, v)$ between two vertices u and v is the length of a shortest (u, v) -path. If there is no ambiguity, we denote $d(u, v) = d_G(u, v)$. For a vertex v and a set $A \subset V$, we denote by $d(v, A) = \min\{d(v, x) : x \in A\}$ the distance from v to the set A . The *interval* $I(u, v)$ between two vertices u and v is the set of all vertices on shortest (u, v) -paths, i.e. $I(u, v) = \{w : d(u, w) + d(w, v) = d(u, v)\}$. We denote by $I^\circ(u, v) = I(u, v) \setminus \{u, v\}$ the “interior” of the interval $I(u, v)$. We say that a pair of vertices (u, v) of G is a *2-pair* if $d(u, v) = 2$. An induced subgraph H (or the corresponding vertex-set of H) of a graph G is *gated* if for every vertex x outside H there exists a vertex x' in H (the *gate* of x) such that $x' \in I(x, y)$ for any y of H . The gate x' of x is necessarily unique. A *ball* of radius r and center v is the set $B_r(v)$ of all vertices x of G such that $d(v, x) \leq r$. A *half-ball* of a bipartite graph $G = (X \cup Y, E)$ is the intersection of a ball of G with one of the two color classes X or Y . By $N(v)$ and $N(A)$ we denote the neighborhood of a vertex v or a set $A \subset V$. Let H be a subgraph of a graph G . We call H an *isometric subgraph* of G if $d_H(u, v) = d_G(u, v)$ for each pair of vertices u, v of H .

Vertices v_1, v_2, v_3 of a graph G form a *metric triangle* $v_1v_2v_3$ if the intervals $I(v_1, v_2), I(v_2, v_3)$, and $I(v_3, v_1)$ pairwise intersect only in the common end-vertices, i.e., $I(v_i, v_j) \cap I(v_i, v_k) = \{v_i\}$ for any $1 \leq i, j, k \leq 3$. If $d(v_1, v_2) = d(v_2, v_3) = d(v_3, v_1) = k$, then this metric triangle is called *equilateral of size k* . An equilateral metric triangle uvw of size k is called *strongly equilateral* if $d(v_3, x) = k$ for any vertex $x \in I(v_1, v_2)$. A metric triangle $v_1v_2v_3$ of G is a *quasi-median* of the triplet x, y, z if the following metric equalities are satisfied:

$$\begin{aligned} d(x, y) &= d(x, v_1) + d(v_1, v_2) + d(v_2, y), \\ d(y, z) &= d(y, v_2) + d(v_2, v_3) + d(v_3, z), \\ d(z, x) &= d(z, v_3) + d(v_3, v_1) + d(v_1, x). \end{aligned}$$

If the size of $v_1v_2v_3$ is zero, which means that v_1, v_2 , and v_3 are the same vertex v , then this vertex v is called a *median* of x, y, z . A median may not exist and may not be unique. On the other hand, a quasi-median of every triplet x, y, z always exists.

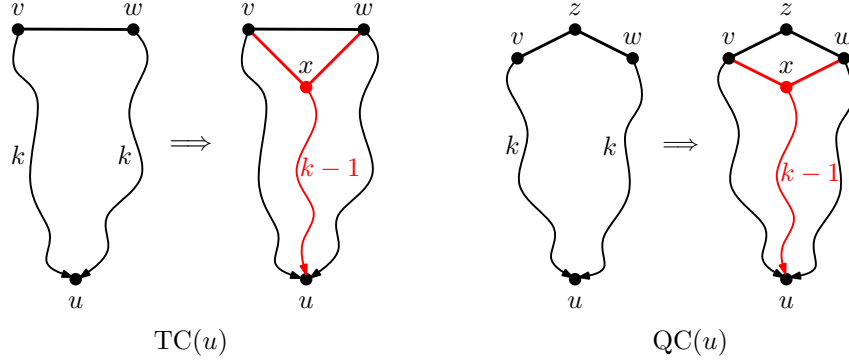


FIGURE 1. Triangle and quadrangle conditions

2.2. Classes of graphs. In this subsection, we recall the definitions of some classes of graphs that will be investigated in the paper. A graph G is *weakly modular* [6, 13] if for any vertex u its distance function d satisfies the following triangle and quadrangle conditions (see Figure 1):

- *Triangle condition* $\text{TC}(u)$: for any two vertices v, w with $1 = d(v, w) < d(u, v) = d(u, w)$ there exists a common neighbor x of v and w such that $d(u, x) = d(u, v) - 1$.
- *Quadrangle condition* $\text{QC}(u)$: for any three vertices v, w, z with $d(v, z) = d(w, z) = 1$ and $2 = d(v, w) \leq d(u, v) = d(u, w) = d(u, z) - 1$, there exists a common neighbor x of v and w such that $d(u, x) = d(u, v) - 1$.

All metric triangles of weakly modular graphs are equilateral. In fact, a graph G is weakly modular if and only if any metric triangle $v_1v_2v_3$ of G is strongly equilateral [13], i.e., if all quasi-medians of triplets are strongly equilateral metric triangles. If all quasi-medians are medians, i.e., metric triangles of size 0, then such weakly modular graphs are called *modular*. Modular graphs in which every triplet of vertices has a unique quasi-median are called *median graphs*. Median graphs constitutes the most important main class of graphs in metric graph theory [8]. Analogously to median graphs, *weakly median graphs* are the weakly modular graphs in which all triplets have unique quasi-medians.

Another important subclass of weakly modular graphs is the class of Helly graphs. A graph G is a *Helly graph* if the family of balls of G has the Helly property, that is, every finite collection of pairwise intersecting balls of G has a nonempty intersection. Analogously, a *bipartite Helly graph* is a graph such that the collection of half-balls of G has the Helly property. Bipartite Helly graphs are modular.

A graph $G = (V, E)$ is called *meshed* [7] if for any three vertices u, v, w with $d(v, w) = 2$, there exists a common neighbor x of v and w such that $2d(u, x) \leq d(u, v) + d(u, w)$. Metric triangles of meshed graphs are equilateral [7]. It is well-known that meshed graphs satisfy the triangle condition (see [12, Lemma 2.23] for example) but they do not necessarily satisfy the quadrangle condition. All weakly modular graphs and all graphs with connected medians are meshed [7]. Another important subclasses of meshed graphs with connected medians are basis graphs of matroids [26] and of even Δ -matroids [15].

A *hexagonal grid* is a grid formed by a tessellation of the plane \mathbb{R}^2 into regular hexagons. Let Z be a cycle of the hexagonal grid. A *benzenoid system* is a subgraph of the hexagonal grid induced by the vertices located in the closed region of the plane bounded by Z . Equivalently, a benzenoid system is a finite connected plane graph in which every interior face is a regular hexagon of side length 1. Benzenoid systems play an important role in chemical graph theory [19].

2.3. Axioms. Given a graph $G = (V, E)$, we call a *profile* any sequence $\pi = (x_1, \dots, x_n)$ of vertices of G (notice that the same vertex may occur several times in π or not at all). The set $\{1, \dots, n\}$ can be interpreted as a set of agents and for each agent $i \in [n]$, x_i is called the *location* of i in G . Often the profile $\pi = (x_1, \dots, x_n)$ is referred to as the *location profile*. Denote by V^* the set of all profiles of finite length. For two profiles $\pi, \pi' \in V^*$, we write $\pi' \subseteq \pi$ if π' is a subprofile of π . A *consensus function* on a graph $G = (V, E)$ is any function $L : V^* \rightarrow 2^V \setminus \emptyset$. For two profiles π and ρ , denote by $\pi\rho$ the concatenation of π and ρ . For a positive integer k and a profile π , denote by π^k the k concatenations of the profile π . When $k = 2$, we say that the profile π^2 is a *double-profile*. For a vertex v , the *weight* $\pi(v)$ of v is the number of occurrences of v in π . We call a vertex v a *majority vertex* of π if $\pi(v) \geq \frac{1}{2}|\pi|$. If π has a majority vertex then we say that π is a *majority profile*. We say that a profile is *tie* if it has two majority vertices u and v , i.e., $\pi(v) = \pi(u) = \frac{1}{2}|\pi|$. Denote by $\text{Maj}(\pi)$ the majority vertices of a profile π and by $\text{supp}(\pi)$ the *support* of a profile π , i.e., the vertices v that appear in π .

Given a profile $\pi = (x_1, \dots, x_n)$ and a vertex v of G , let

$$F_\pi(v) = \sum_{i=1}^n d(v, x_i) = \sum_{x \in V} \pi(x) d(v, x).$$

A vertex v of G minimizing $F_\pi(v)$ is called a *median vertex* of G with respect to π and the set of all medians vertices is called the *median set* $\text{Med}(\pi)$. Finally, the mapping Med that associate to any profile π of G its median set $\text{Med}(\pi)$ is called the *median function*. Clearly, Med is a consensus function.

Recall now the following simple axioms (A), (B), (C) for consensus functions on graphs, introduced in [27, 28, 29]:

- (A) *Anonymity*: for any profile $\pi = (x_1, x_2, \dots, x_n)$ and any permutation σ of $\{1, 2, \dots, n\}$, $L(\pi) = L(\pi^\sigma)$ where $\pi^\sigma = (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$.
- (B) *Betweenness*: $L(u, v) = I(u, v)$.
- (C) *Consistency*: for any profiles π and ρ , if $L(\pi) \cap L(\rho) \neq \emptyset$, then $L(\pi\rho) = L(\pi) \cap L(\rho)$.

We also consider the following triangle axioms, that are meaningful only for graphs with triangles:

- (T) *Triangle*: for any three pairwise adjacent vertices u, v, w of G , $L(u, v, w) = \{u, v, w\}$.
- (T⁻) *weak Triangle*: for any three pairwise adjacent vertices u, v, w of G , if $u \in L(u, v, w)$, then $\{u, v, w\} \subseteq L(u, v, w)$.

One can easily see that the median function Med satisfies the axioms (A), (B), (C), (T), and (T⁻). According to [27], the consensus functions that satisfies the axioms (A), (B) and (C) are called *ABC-functions*. We say that a graph G is an *ABC-graph* if there exists a unique ABC-function on G , i.e., if the median function is the only ABC-function. The *ABCT*- and *ABCT⁻*-functions and the *ABCT*-, *ABCT⁻*-graphs are defined in a similar way. Characterizing the ABC-graphs (and the ABCT-graphs) is an open problem. The first problem is referred in [27] as to the *ABC-problem* and we will refer to the other as to the *ABCT-problem*.

2.4. Graphs with connected and G^p -connected medians. In this subsection we recall some definitions and results from [7] and [10]. The p th power of a graph $G = (V, E)$ is a graph G^p having the same vertex-set V as G and two vertices u and v are adjacent in G^p if and only if $d_G(u, v) \leq p$. Let f be a real-valued function defined on the vertex set V of G . We say that f is *unimodal* on G^p if any local minimum of f in G^p is a global minimum of f , i.e., if for a vertex u we have $f(u) \leq f(v)$ for any $v \in V$ such that $d_G(u, v) \leq p$, then $f(u) \leq f(v)$ for any $v \in V$. Analogously, we say that the function f has *G^p -connected* (respectively, *G^p -isometric*) minima, if the set of minima of f induce a connected (respectively, isometric) subgraph of G^p . Finally, we say that G is a graph with *G^p -connected* (respectively, *G^p -isometric*) medians if for any profile π , the function F_π is a function with *G^p -connected* (respectively, *G^p -isometric*) minima. If $p = 1$, we say that G has connected/isometric medians.

A p -geodesic between two vertices u, v is a finite sequence of vertices $P = (u = w_0, w_1, \dots, w_n = v)$ included in a (u, v) -geodesic of G such that $d_G(w_i, w_{i+1}) \leq p$ for any $i = 0, \dots, n-1$. A function f defined on the vertex-set V of a graph G is called *p -weakly peakless* if any pair of vertices u, v of G can be connected by a p -geodesic $P = (u = w_0, w_1, \dots, w_{n-1}, w_n = v)$ along which f is peakless, i.e., if $0 \leq i < j < k \leq p$ implies $f(w_j) \leq \max\{f(w_i), f(w_k)\}$ and equality holds only if $f(w_i) = f(w_j) = f(w_k)$. For $p = 1$, the p -weakly peakless functions are called *pseudopeakless* [7]. Generalizing a result for pseudopeakless functions of [7], it was shown in [10] that a function f is p -weakly peakless if and only if it is locally- p -weakly peakless, i.e., for any u, v such that $p+1 \leq d(u, v) \leq 2p$ there exists $w \in I^\circ(u, v)$ such that $f(w) \leq \max\{f(u), f(v)\}$, and equality holds only if $f(u) = f(w) = f(v)$.

Generalizing a result from [7] for $p = 1$, in [10] we characterized graphs with G^p -connected medians in the following way:

Theorem 1 ([10]). *For a graph G and an integer $p \geq 1$, the following conditions are equivalent:*

- (1) F_π is unimodal on G^p for any profile π ;
- (2) F_π is p -weakly peakless for any profile π ;
- (3) G is a graph with G^p -isometric medians;
- (4) G is a graph with G^p -connected medians.

In [7], it was established that the graphs with connected medians are meshed. Consequently, they satisfy the triangle condition TC and their metric triangles are equilateral.

It was established in [7] and [10] that many important classes of graphs are graphs with connected or G^2 -connected medians. In particular, Helly graphs, weakly median and median graphs, basis graphs of

matroids and of even Δ -matroids have connected medians [7], and bipartite Helly graphs and benzenoids have G^2 -connected medians [10]. The results of our paper concern all graphs with connected medians (including the classes mentioned above), all modular graphs with G^2 -connected medians (in particular bipartite Helly graphs), and benzenoids.

3. ABC(T)-GRAPHS

In this section, we prove two results: (1) that graphs with connected medians are ABCT-graphs and (2) that modular graphs with G^2 -connected medians are ABC-graphs.

3.1. Properties of ABC-functions. We continue with some useful properties of ABC-functions.

Lemma 2. *Let L be an ABC-function on a graph G . Then for any profile π , any vertex $x \in V$, and any $x' \in L(\pi, x)$, we have $x' \in L(\pi, x') \subseteq L(\pi, x)$.*

Proof. Pick any $x'' \in L(\pi, x')$ and consider the profile $\pi'' = \pi, x, x', x''$. Since $x' \in L(\pi, x)$ and $x' \in L(x', x'') = I(x', x'')$, we have $x' \in L(\pi'') = L(\pi, x) \cap L(x', x'')$. Since $x'' \in L(\pi, x')$ and $x'' \in L(x, x'') = I(x, x'')$, we have $x'' \in L(\pi'') = L(\pi, x') \cap L(x, x'')$. Consequently, $x' \in L(\pi, x')$ and $x'' \in L(\pi, x)$, establishing $x' \in L(\pi, x') \subseteq L(\pi, x)$. \square

Lemma 2 implies that if $x \in L(u, v, w)$, then $x \in L(u, v, x)$ and $L(u, v, x) \subseteq L(u, v, w)$.

Lemma 3. *Let L be an ABC-function on a graph G . Then for any profile π with $|\pi| \geq 2$ and any vertex $x \in V$ such that $x \in L(\pi, x)$, we have $\bigcap_{y \in \pi} I(x, y) = \{x\}$.*

Proof. Pick $z \in \bigcap_{y \in \pi} I(x, y)$, let $k = |\pi|$, and consider the profile $\pi' = \pi, x^k, z^{k-1}$. Since $x \in L(\pi, x)$ and $x \in L(x^{k-1}, z^{k-1}) = L(x, z) = I(x, z)$, $x \in L(\pi') = L(\pi, x) \cap I(x, z)$. Since $z \in \bigcap_{y \in \pi} L(y, x) = \bigcap_{y \in \pi} I(y, x)$ and $z \in L(z^k) = L(z) = \{z\}$, $L(\pi') = \bigcap_{y \in \pi} I(y, x) \cap \{z\} = \{z\}$ and consequently $x = z$. \square

Lemma 3 implies that if $\{u, v, w\} \subseteq L(u, v, w)$, then u, v, w form a metric triangle, i.e., the intervals $I(u, v)$, $I(u, w)$ and $I(v, w)$ pairwise intersect only on the common end vertices.

3.2. Graphs with connected medians are ABCT-graphs. In this section, we show that the graphs in which the median set of any profile is connected satisfy the ABCT-property. This generalizes the result of [29] for median graphs that are precisely the bipartite graphs in which the median set of any profile is connected. The following lemma establishes that for the ABC-functions in graphs with connected medians, (T^-) is equivalent to (T) :

Lemma 4. *If G is a graph satisfying the triangle condition TC and L is an $ABCT^-$ -function, then L also satisfies axiom (T) .*

Proof. Consider three pairwise adjacent vertices u, v, w and assume that there exists $x \in L(u, v, w) \setminus \{u, v, w\}$. Since $x \in L(u, v, w)$, by Lemma 2, we have $x \in L(u, v, x)$ and consequently, by Lemma 3, $I(u, x) \cap I(v, x) = \{x\}$. Since G satisfies the triangle condition, necessarily we get $x \sim u, v$. Since $x \in L(u, v, x)$, by (T^-) , we get that $u, v, x \in L(u, v, x)$. Consequently, $u \in L(u, v, x) \cap L(u, w) = L(u, v, w, x, u)$ and since $L(u, w) = \{u, w\}$, we get that $L(u, v, w, x, u) \subseteq \{u, w\}$. But we have $x \in L(u, v, w) \cap L(u, x) = L(u, v, w, x, u)$, leading to a contradiction. Consequently, $L(u, v, w) \subseteq \{u, v, w\}$ and by (T^-) , we have $L(u, v, w) = \{u, v, w\}$. \square

In the following, we consider graphs satisfying the triangle condition TC and ABC-functions on these graphs. We generalize some ideas of Mulder and Nowick [29] to establish that if G has connected medians, then L is the median function on G . Since median graphs have connected medians and do not contain triangles, this provides a new proof of the main result of [29] which does not use specific properties of median graphs other than the fact that in median graphs, the median set of any profile is connected.

For an edge uv of $G = (V, E)$, let $W_{uv} = \{x \in V : d(u, x) < d(v, x)\}$, $W_{vu} = \{x \in V : d(v, x) < d(u, x)\}$, and $W_{uv}^- = \{x \in V : d(u, x) = d(v, x)\}$. Note that if G is bipartite, then $W_{uv}^- = \emptyset$. Given a profile π , we denote by π_{uv} (respectively, π_{vu}, π_{uv}^-) the restriction of π to W_{uv} (respectively, E_{vu}, E_{uv}^-). The following observation is immediate:

Lemma 5. *For any edge uv of a graph G and any $x \in W_{uv}$ and $y \in W_{vu}$, we have $u, v \in L(v, x) = I(v, x)$ and $u, v \in L(u, y) = I(u, y)$.*

Lemma 6. *Let G be a graph satisfying the triangle condition TC. For any edge uv of G and any $z \in W_{uv}^-$, we have $u, v \in L(u, v, z)$.*

Proof. By TC(z), there exists $w \sim u, v$ such that $d(w, z) = d(u, z) - 1 = d(v, z) - 1$.

Claim 7. *Either $u, v, w \in L(u, v, z)$ or $u, v, w \notin L(u, v, z)$.*

Proof. By symmetry, it suffices to show that $u \in L(u, v, z)$ if and only if $w \in L(u, v, z)$. Suppose that $\{u, w\} \cap L(u, v, z) \neq \emptyset$ and consider the profile (u, v, z, u, w) . Since $L(u, w) = \{u, w\}$ and $\{u, w\} \cap L(u, v, z) \neq \emptyset$, we have $L(u, v, z, u, w) = \{u, w\} \cap L(u, v, z)$. By (T), $L(u, v, w) = \{u, v, w\}$ and since $L(u, z) = I(u, z)$, we have $u, w \in L(u, v, w) \cap L(u, z) = L(u, v, z, u, w) = \{u, w\} \cap L(u, v, z)$. Consequently, $u \in L(u, v, z)$ iff $w \in L(u, v, z)$. \square

If $u \in L(u, v, z)$ or $v \in L(u, v, z)$, we are done by Claim 7. Suppose now that $u, v \notin L(u, v, z)$ and let $z' \in L(u, v, z)$. By Lemma 2, $z' \in L(u, v, z') \subseteq L(u, v, z)$ and by Lemma 3, $I(u, z') \cap I(v, z') = \{z'\}$. Since G satisfies TC(z'), necessarily, $z' \sim u, v$. Therefore, by (T), $L(u, v, z') = \{u, v, z'\}$ and consequently, $u, v \in L(u, v, z') \subseteq L(u, v, z)$. \square

Given a graph $G = (V, E)$ and a profile $\pi \in V^*$, let π_{uv} be the restriction of π to the set W_{uv} , π_{vu} be the restriction of π to W_{vu} , and π_{uv}^- be the restriction of π to W_{uv}^- .

Lemma 8. *Let $G = (V, E)$ be a graph satisfying the triangle condition TC and $\pi \in V^*$ be any profile. For any edge uv of G , we have $u, v \in L(\pi')$ where $\pi' = \pi, u^{l+p}, v^{k+p}$ with $l = |\pi_{vu}|$, $k = |\pi_{uv}|$, and $p = |\pi_{uv}^-|$.*

Proof. Let $\pi_{uv} = (x_1, \dots, x_k)$, $\pi_{vu} = (y_1, \dots, y_l)$, and $\pi_{uv}^- = (z_1, \dots, z_p)$. Consider the profile $\pi' = \pi, u^{l+p}, v^{k+p}$. Note that $u, v \in \bigcap_{i=1}^k L(v, x_i) = \bigcap_{i=1}^k I(v, x_i)$ and that $u, v \in \bigcap_{i=1}^l L(u, y_i) = \bigcap_{i=1}^l I(u, y_i)$ by Lemma 5. Note also that $u, v \in \bigcap_{i=1}^p L(u, v, z_i)$ by Lemma 6. Consequently, $u, v \in L(v, x_1, \dots, v, x_k, u, y_1, \dots, u, y_l, u, v, z_1, \dots, u, v, z_p) = L(x_1, \dots, x_k, y_1, \dots, y_l, z_1, \dots, z_p, u^{l+p}, v^{k+p}) = L(\pi')$. \square

Lemma 9. *Let $G = (V, E)$ be a graph satisfying the triangle condition TC and $\pi \in V^*$ be any profile. For any edge uv of G the following holds:*

- (1) *If $F_{G,\pi}(u) = F_{G,\pi}(v)$, then $u \in L(\pi)$ iff $v \in L(\pi)$.*
- (2) *If $F_{G,\pi}(u) > F_{G,\pi}(v)$, then $u \notin L(\pi)$.*

Proof. Let $l = |\pi_{vu}|$, $k = |\pi_{uv}|$, and $p = |\pi_{uv}^-|$. Assume that $F_{G,\pi}(u) \geq F_{G,\pi}(v)$. Observe that $F_{G,\pi}(u) - F_{G,\pi}(v) = |\pi_{vu}| - |\pi_{uv}| = l - k \geq 0$. As in Lemma 8, we consider the profile $\pi' = \pi, u^{l+p}, v^{k+p}$. Note that $\pi' = \pi, u^{l-k}, u^{k+p}, v^{k+p}$.

Suppose first that $l = k$ (i.e., $F_{G,\pi}(u) = F_{G,\pi}(v)$) and that $u \in L(\pi)$. Since $u \in L(u^{k+p}, v^{k+p}) = L(u, v) = \{u, v\}$, we have $L(\pi') = L(\pi) \cap \{u, v\}$. By Lemma 8, $u, v \in L(\pi')$ and thus $u, v \in L(\pi)$.

Suppose now that $l > k$ (i.e., $F_{G,\pi}(u) > F_{G,\pi}(v)$) and that $u \in L(\pi)$. Since $u \in L(u^{k+p}, v^{k+p}) = \{u, v\}$ and $u \in L(u^{l-k}) = L(u) = \{u\}$, we have $L(\pi') = L(\pi) \cap \{u, v\} \cap \{u\} = \{u\}$, contradicting Lemma 8. \square

Theorem 10. *Any graph G with connected medians is an ABCT-graph and an ABCT⁻-graph.*

Proof. Since graphs with connected medians satisfy the triangle condition, Lemma 9 holds for G . Consider an ABCT-function L on V and a profile π . We first show that $L(\pi) \subseteq \text{Med}_G(\pi)$. Pick $u \in L(\pi)$. By Theorem 1(1) for $p = 1$, if $u \notin \text{Med}_G(\pi)$, then there exists $v \in N(u)$ such that $F_{G,\pi}(v) < F_{G,\pi}(u)$. Since By Lemma 9(2), $u \notin L(\pi)$, a contradiction.

We now show that $\text{Med}_G(\pi) \subseteq L(\pi)$. By Theorem 1(4) for $p = 1$, $\text{Med}_G(\pi)$ is connected in G . Consequently, if $L(\pi) \subsetneq \text{Med}_G(\pi)$, there exists $u \in L(\pi)$ and $v \in \text{Med}_G(\pi) \setminus L(\pi)$ such that $u \sim v$. Since $u, v \in \text{Med}_G(\pi)$, $F_{G,\pi}(u) = F_{G,\pi}(v)$ and we obtain a contradiction with Lemma 9(1). Thus, the graphs with connected medians verifies the ABCT-property. Since they satisfy the triangle condition, by Lemma 4, they also satisfy the ABCT⁻-property. \square

Remark. Since the complete graph K_3 has two ABC-functions [27] and by Theorem 10 only one ABCT-function (or ABCT⁻-function), axioms (T) and (T⁻) are independent from the axioms (A), (B), and (C).

3.3. Modular graphs with G^2 -connected medians are ABC-graphs. In this subsection we show that all modular graphs G with G^2 -connected medians are ABC-graphs. This result generalizes the results of [29] and [27] stating that median graphs and graphs satisfying the pairing property are ABC-graphs. Indeed, in Section 4, we establish that graphs with the pairing property are bipartite Helly graphs, and thus they are modular with G^2 -connected medians.

For a bipartite graph $G = (V, E)$ and a 2-pair (u, v) of G , let $X_{uv} = \{x \in V : d(u, x) < d(v, x)\}$, $X_{vu} = \{x \in V : d(v, x) < d(u, x)\}$, and $X_{uv}^- = \{x \in V : d(u, x) = d(v, x)\}$. Since G is bipartite, for any $x \in X_{uv}$, $d(v, x) = d(u, x) + 2$ and $u \in I(v, x)$. The following observation is trivial:

Lemma 11. *For any bipartite graph G , any 2-pair (u, v) of G , any $x \in X_{uv}$ and $y \in X_{vu}$, for any ABC-function L , we have $u, v \in L(v, x) = I(v, x)$ and $u, v \in L(u, y) = I(u, y)$.*

The following lemma is similar to Theorem 5 of [27]. The difference is that here we consider a subgraph $K_{2,n}$ in an arbitrary bipartite graph G .

Lemma 12. *For any bipartite graph G , any 2-pair (u, v) and any distinct vertices $w_1, w_2, \dots, w_n \in I^\circ(u, v)$, with $n \geq 2$, for any ABC-function L , we have $u, v \in L(w_1, w_2, \dots, w_n)$ and $u, v \in L(u, v, w_1, w_2, \dots, w_n)$.*

Proof. Let $\pi = (w_1, w_2, \dots, w_n)$. For any $1 \leq i \leq n$, we have $u, v \in L(w_i, w_{i+1}) = I(w_i, w_{i+1})$ (with the convention that $w_{n+1} = w_1$). Consequently, $u, v \in L(w_1, w_2) \cap L(w_2, w_3) \cap \dots \cap L(w_{n-1}, w_n) \cap L(w_n, w_1) = L(w_1, w_2, w_2, w_3, \dots, w_{n-1}, w_n, w_n, w_1) = L(\pi\pi) = L(\pi)$. Since $u, v \in L(u, v) = I(u, v)$, $u, v \in L(u, v) \cap L(\pi) = L(u, v, w_1, w_2, \dots, w_n)$. \square

Lemma 13. *For a modular graph G and a 2-pair (u, v) , if there exists a profile π such that F_π is not pseudopeakless on (u, v) , then $I^\circ(u, v)$ contains at least three vertices w_1, w_2, w_3 .*

Proof. Suppose that $I^\circ(u, v)$ contains at most two vertices w_1, w_2 (where w_1 and w_2 may coincide). Since G is modular, for any vertex x of π either w_1 or w_2 belongs to $I(x, u) \cap I(x, v) \cap I(u, v)$, yielding $d(x, w_1) + d(x, w_2) \leq d(x, u) + d(x, v)$. Therefore, $F_\pi(w_1) + F_\pi(w_2) \leq F_\pi(u) + F_\pi(v)$, contrary to the assumption that F_π is not pseudopeakless on (u, v) . \square

Therefore, we investigate now the properties of an ABC-function L on 2-pairs (u, v) of G for which $I^\circ(u, v)$ contains at least three vertices.

Lemma 14. *Let G be a bipartite graph, (u, v) be a 2-pair with $w_1, w_2, w_3 \in I^\circ(u, v)$, and $z \in X_{uv}^-$. Then $u, v \in L(u, v, w_1, w_2, w_3, z)$.*

Proof. Let $k = d(u, z) = d(v, z)$. We first consider the case where u, v belong to one of the intervals $I(w_1, z), I(w_2, z), I(w_3, z)$. Note that this covers the case when $z \in \{w_1, w_2, w_3\}$.

Claim 15. *If $\max\{d(z, w_1), d(z, w_2), d(z, w_3)\} = k + 1$, then $u, v \in L(u, v, w_1, w_2, w_3, z)$.*

Proof. Let $d(z, w_1) = k + 1$. Since $u, v \in L(w_1, z) = I(w_1, z)$, $u, v \in L(u, v) = I(u, v)$, and $u, v \in L(w_2, w_3) = I(w_2, w_3)$, necessarily $u, v \in L(w_1, z) \cap L(u, v) \cap L(w_2, w_3) = L(u, v, w_1, w_2, w_3, z)$. \square

By Claim 15 and since G is bipartite, we can assume that $d(w_1, z) = d(w_2, z) = d(w_3, z) = k - 1$.

Claim 16. *Either $u, v, w_1, w_2, w_3 \in L(u, v, w_1, w_2, w_3, z)$ or $u, v, w_1, w_2, w_3 \notin L(u, v, w_1, w_2, w_3, z)$.*

Proof. By symmetry, it suffices to show that $u \in L(u, v, w_1, w_2, w_3, z)$ iff $w_1 \in L(u, v, w_1, w_2, w_3, z)$. Suppose that $L(u, v, w_1, w_2, w_3, z) \cap \{u, w_1\} \neq \emptyset$ and consider the profile $\pi' = (u, v, w_1, w_2, w_3, z, u, w_1)$. Note that $L(u, w_1) = I(u, w_1) = \{u, w_1\}$ and thus $L(\pi') = L(u, v, w_1, w_2, w_3, z) \cap L(u, w_1) = L(u, v, w_1, w_2, w_3, z) \cap \{u, w_1\}$. Note also that $u, w_1 \in L(u, z) = I(u, z)$, $u, w_1 \in L(u, v) = I(u, v)$, $u, w_1 \in L(w_1, w_2) = I(w_1, w_2)$, and $u, w_1 \in L(w_1, w_3) = I(w_1, w_3)$. Consequently, $u, w_1 \in L(u, z) \cap L(u, v) \cap L(w_1, w_2) \cap L(w_1, w_3) = L(u, z, u, v, w_1, w_2, w_1, w_3) = L(\pi') = L(u, v, w_1, w_2, w_3, z) \cap \{u, w_1\}$. Therefore, $u, w_1 \in L(u, v, w_1, w_2, w_3, z)$ and we are done. \square

Now, we prove the assertion of the lemma. Let $z' \in L(u, v, w_1, w_2, w_3, z)$. By Lemma 2, $z' \in L(u, v, w_1, w_2, w_3, z')$ and $L(u, v, w_1, w_2, w_3, z') \subseteq L(u, v, w_1, w_2, w_3, z)$. Suppose first that $d(u, z') \neq d(v, z')$ and assume without loss of generality that $d(u, z') < d(v, z')$. In this case, $d(v, z') = d(w_1, z') + 1 = d(w_2, z') + 1 = d(w_2, z') + 1 = d(u, z') + 2$. Then $u \in L(z', v) = I(v, z')$, $u \in L(w_2, w_3) = I(w_2, w_3)$, and $u \in L(u, w_1) = \{u, w_1\}$. Consequently, $L(u, v, w_1, w_2, w_3, z') = I(v, z') \cap I(w_2, w_3) \cap \{u, w_1\} = \{u\}$. Therefore $u \in L(u, v, w_1, w_2, w_3, z') \subseteq L(u, v, w_1, w_2, w_3, z)$ and we are done by Claim 16.

Suppose now that $d(u, z') = d(v, z')$ and let $k' = d(u, z')$. If $\max\{d(z', w_1), d(z', w_2), d(z', w_3)\} = k' + 1$, say $d(w_1, z') = k' + 1$, by Claim 15 applied to the profile $(u, v, w_1, w_2, w_3, z')$, we have $u, v \in L(u, v, w_1, w_2, w_3, z') \subseteq L(u, v, w_1, w_2, w_3, z)$ and we are done. Thus, we can assume that $d(w_1, z') = d(w_2, z') = d(w_3, z') = k' - 1$. Consider the profile $\pi'' = (u, v, w_1, w_2, w_3, z', w_1, z')$. Since $z' \in L(u, v, w_1, w_2, w_3, z')$ and $z' \in L(w_1, z') = I(w_1, z')$, we have $L(\pi'') = L(u, v, w_1, w_2, w_3, z') \cap L(w_1, z')$. Observe that $w_1 \in L(u, z') = I(u, z')$, $w_1 \in L(v, z') = I(v, z')$, $w_1 \in L(w_1, w_2) = I(w_1, w_2)$, and $w_1 \in L(w_1, w_3) = I(w_1, w_3)$. Consequently, $w_1 \in L(u, z') \cap L(v, z') \cap L(w_1, w_2) \cap L(w_1, w_3) = L(u, z', v, z', w_1, w_2, w_1, w_3) = L(\pi'') = L(u, v, w_1, w_2, w_3, z') \cap L(w_1, z')$. Therefore, $w_1 \in L(u, v, w_1, w_2, w_3, z') \subseteq L(u, v, w_1, w_2, w_3, z)$ and we are done by Claim 16. \square

For any profile π on a bipartite graph G and any 2-pair (u, v) , let π_{uv} and π_{vu} be the restrictions of π on the sets X_{uv} and X_{vu} , respectively, and π_{uv}^- be the restriction of π on X_{uv}^- .

Lemma 17. *Let G be a bipartite graph, (u, v) be a 2-pair with $w_1, w_2, w_3 \in I^\circ(u, v)$. Then for any profile π , we have $u, v \in L(\pi')$ where $\pi' = \pi, u^{\ell+p}, v^{k+p}, w_1^p, w_2^p, w_3^p$ with $\ell = |\pi_{vu}|$, $k = |\pi_{uv}|$, and $p = |\pi_{uv}^-|$.*

Proof. Set $\pi_{uv} = (x_1, \dots, x_k)$, $\pi_{vu} = (y_1, \dots, y_\ell)$, and $\pi_{uv}^- = (z_1, \dots, z_p)$. Consider the profile $\pi' = \pi, u^{\ell+p}, v^{k+p}, w_1^p, w_2^p, w_3^p$. By Lemma 11, $u, v \in \bigcap_{i=1}^k L(v, x_i) = \bigcap_{i=1}^k I(v, x_i)$ and $u, v \in \bigcap_{i=1}^\ell L(u, y_i) = \bigcap_{i=1}^\ell I(u, y_i)$. Note also that $u, v \in \bigcap_{i=1}^p L(u, v, w_1, w_2, w_3, z_i)$ by Lemma 14. Consequently, $u, v \in L(v^k, x_1, \dots, x_k, u^\ell, y_1, \dots, y_\ell, u^p, v^p, w_1^p, w_2^p, w_3^p, z_1, \dots, z_p) = L(x_1, \dots, x_k, y_1, \dots, y_\ell, z_1, \dots, z_p, u^{\ell+p}, v^{k+p}, w_1^p, w_2^p, w_3^p) = L(\pi')$. \square

Lemma 18. *Let G be a bipartite graph and (u, v) be a 2-pair with $w_1, w_2, w_3 \in I^\circ(u, v)$. For any profile π on G the following holds:*

- (1) *If $F_{G, \pi}(u) = F_{G, \pi}(v)$, then $u \in L(\pi)$ iff $v \in L(\pi)$.*
- (2) *If $F_{G, \pi}(u) > F_{G, \pi}(v)$, then $u \notin L(\pi)$.*

Proof. Let $\ell = |\pi_{vu}|$, $k = |\pi_{uv}|$, and $p = |\pi_{uv}^-|$. Suppose without loss of generality that $F_{G, \pi}(u) \geq F_{G, \pi}(v)$. Then $F_{G, \pi}(u) - F_{G, \pi}(v) = |\pi_{vu}| - |\pi_{uv}| = \ell - k \geq 0$. As in Lemma 17, we consider the profile $\pi' = \pi, u^{\ell+p}, v^{k+p}, w_1^p, w_2^p, w_3^p$. Note that $\pi' = \pi, u^{\ell-k}, v^k, u^p, v^p, w_1^p, w_2^p, w_3^p$.

Suppose first that $F_{G, \pi}(u) = F_{G, \pi}(v)$ (i.e., $\ell = k$) and that $u \in L(\pi)$. Since $u \in L(u^k, v^k) = L(u, v) = I(u, v)$ and $u \in L(u^p, v^p, w_1^p, w_2^p, w_3^p) = L(u, v, w_1, w_2, w_3)$ by Lemma 12, we have $L(\pi') = L(\pi) \cap L(u, v, w_1, w_2, w_3) \cap I(u, v)$. By Lemma 17, $u, v \in L(\pi')$ and thus $u, v \in L(\pi)$.

Suppose now that $F_{G, \pi}(u) > F_{G, \pi}(v)$ (i.e., $\ell > k$) and that $u \in L(\pi)$. Since $u \in L(u^k, v^k) \cap L(u^p, v^p, w_1^p, w_2^p, w_3^p) = I(u, v) \cap L(u, v, w_1, w_2, w_3)$ and $u \in L(u^{\ell-k}) = L(u) = \{u\}$, we have $L(\pi') = L(\pi) \cap I(u, v) \cap L(u, v, w_1, w_2, w_3) \cap \{u\} = \{u\}$, contradicting Lemma 17. \square

Theorem 19. *Any modular graph $G = (V, E)$ with G^2 -connected medians is an ABC-graph. In particular, any bipartite Helly graph is an ABC-graph.*

Proof. Consider an ABC-function L on V and a profile $\pi \in V^*$. We first show that $L(\pi) \subseteq \text{Med}_G(\pi)$. Pick any $u \in L(\pi)$. If $u \notin \text{Med}_G(\pi)$, then by Theorem 1(1) for $p = 2$ there exists a vertex v such that $1 \leq d(u, v) \leq 2$ and $F_{G, \pi}(v) < F_{G, \pi}(u)$. Since G is bipartite, G satisfies the triangle condition. Therefore, if $d(u, v) = 1$ we obtain a contradiction with Lemma 9(2). Thus, we can suppose that $d(u, v) = 2$ and that $F_{G, \pi}(w) \geq F_{G, \pi}(u)$ for any neighbor w of u . By Lemma 13 applied to the 2-pair (u, v) , there exists three distinct vertices $w_1, w_2, w_3 \in I^\circ(u, v)$. Consequently, by Lemma 18(2), $u \notin L(\pi)$, a contradiction. This shows that $L(\pi) \subseteq \text{Med}_G(\pi)$.

Now we prove the converse inclusion $\text{Med}_G(\pi) \subseteq L(\pi)$. Suppose by way of contradiction that $L(\pi)$ is a proper subset of $\text{Med}_G(\pi)$. Pick two vertices $u \in L(\pi)$ and $v \in \text{Med}_G(\pi) \setminus L(\pi)$ minimizing $d(u, v)$. Since $\text{Med}_G(\pi)$ is G^2 -connected, $d(u, v) \leq 2$. Since $u, v \in \text{Med}_G(\pi)$, $F_{G, \pi}(u) = F_{G, \pi}(v)$. If $d(u, v) = 1$, we obtain a contradiction with Lemma 9(1). If $d(u, v) = 2$, by our choice of u and v , we must have $F_{G, \pi}(w) > F_{G, \pi}(u)$ for any $w \in I^\circ(u, v)$. Thus, by Lemma 13 there exists three distinct vertices $w_1, w_2, w_3 \in I^\circ(u, v)$ and we obtain a contradiction with Lemma 18.

The second assertion is a consequence of [10, Proposition 62] establishing that bipartite Helly graphs have G^2 connected medians since bipartite Helly graphs are modular. \square

4. GRAPHS WITH THE PAIRING PROPERTY

In this section, we investigate the graphs with the pairing and double-pairing properties. MacMorris et al. [27] established that the graphs with the pairing property are ABC-graphs. We show that this also holds for graphs with the double-pairing property. We also prove that both those classes of graphs are

proper subclasses of bipartite Helly graphs, showing that Theorem 19 is a strict generalization of [27, Theorem 4]. We characterize the graphs with the pairing property in a local-to-global way as bipartite Helly graphs where each ball of radius 2 satisfies the pairing property. Finally, we show that the problem of deciding if a graph satisfies the double-pairing property is in co-NP.

4.1. Pairing and double-pairing properties. For an even profile π of length $k = 2n$ of G , a *pairing* P of π is a partition of π into n disjoint pairs. For a pairing P , define $D_\pi(P) = \sum_{\{a,b\} \in P} d(a,b)$. A pairing P of π maximizing the function D_π is called a *maximum pairing*. The notion of pairing was defined by Gerstel and Zaks [18]; they also proved the following weak duality between the functions F_π and D_π :

Lemma 20. *For any even profile π of length $k = 2n$ of G , for any pairing P of π and any vertex v of G , $D_\pi(P) \leq F_\pi(v)$ and the equality holds if and only if $v \in \bigcap_{\{a,b\} \in P} I(a,b)$.*

Proof. Let $P = \{\{a_i, b_i\} : i = 1, \dots, n\}$. By triangle inequality, $D_\pi(P) = \sum_{i=1}^n d(a_i, b_i) \leq \sum_{i=1}^n (d(a_i, v) + d(v, b_i)) = \sum_{x \in \pi} d(v, x) = F_\pi(v)$. Observe that $D_\pi(P) = F_\pi(v)$ if and only if $d(a_i, b_i) = d(a_i, v) + d(v, b_i)$ for all $1 \leq i \leq n$, i.e., if and only if $v \in I(a_i, b_i)$ for all $1 \leq i \leq n$. \square

We say that a graph G satisfies the *pairing property* if for any even profile π there exists a pairing P of π and a vertex v of G such that $D_\pi(P) = F_\pi(v)$, i.e., the functions F_π and D_π satisfy the strong duality. Such a pairing is called a *perfect pairing*. We say that a graph G satisfies the *double-pairing property* if for any profile π , the profile π^2 admits a perfect pairing. Clearly a graph satisfying the pairing property also satisfies the double-pairing property.

By Lemma 20, the pairing property of [18] coincides with the *intersecting-intervals property* of [27]. By [27, Theorem 4] the graphs satisfying the pairing property are ABC-graphs. It was shown in [18] that trees satisfy the pairing property. More generally, it was shown in [28] and independently in [16] that cube-free median graphs also satisfy the pairing property. It was proven in [27] that the complete bipartite graph $K_{2,n}$ satisfies the pairing property. As observed in [27], the 3-cube is a simple example of a graph not satisfying the pairing property. The investigation of the structure of graphs with the pairing property was formulated as an open problem in [18].

We use the following lemma, whose first assertion corresponds to [27, Theorem 4]. This Lemma 21 establishes that any graph with the pairing or the double-pairing property is an ABC-graph.

Lemma 21. *Let L be an ABC-function on a graph G . Then for any even profile π in G that admits a perfect pairing, we have $L(\pi) = \text{Med}(\pi)$. Furthermore, for any profile π such that π^{2k} has a perfect pairing for some $k \geq 1$, we have $L(\pi) = \text{Med}(\pi)$.*

Proof. Consider a perfect pairing P of an even profile π and let $v \in V$ such that $D_\pi(P) = F_\pi(v)$. By Lemma 20, v is a median of π and $v \in \bigcap_{\{a,b\} \in P} I(a,b)$. Consequently, $\text{Med}(\pi) = \bigcap_{\{a,b\} \in P} I(a,b)$. By (B) and (C), $L(\pi) = \bigcap_{\{a,b\} \in P} L(a,b) = \bigcap_{\{a,b\} \in P} I(a,b)$, and thus $L(\pi) = \text{Med}(\pi)$.

Consider now a profile π and a perfect pairing P of π^{2k} . By the first assertion, we have $L(\pi^{2k}) = \text{Med}(\pi^{2k})$. By (C), we have $L(\pi^{2k}) = \bigcap_{i=1}^{2k} L(\pi) = L(\pi)$ and $\text{Med}(\pi^{2k}) = \text{Med}(\pi)$. \square

Given a profile π on a graph $G = (V, E)$, deciding whether π (or π^2) admits a perfect pairing can be reduced to a perfect π -matching problem as follows. Consider a vertex $u \in \text{Med}(\pi)$ and the auxiliary graph $A_u = (V, E_u)$ where $vw \in E_u$ if and only if $u \in I(v, w)$ in G , i.e., $d(v, w) = d(v, u) + d(u, w)$. Then π admits a perfect pairing if and only if the graph A_u has a perfect π -matching, i.e., there exists a multiset P of edges of E_u such that each vertex $v \in V$ belongs to exactly $\pi(v)$ edges of P .

In fact, the previous remark holds for any weight function $b : V \rightarrow \mathbb{R}^+$ (note that profiles are the weight functions with integer values). Analogously to the definitions of F_π and Med_π , for a weight function b , let $F_b : v \in V \mapsto \sum_{z \in V} b(z)d(v, z)$ and let $\text{Med}(b)$ be the set of vertices minimizing F_b . Recall that given a weight function $b : V \rightarrow \mathbb{R}^+$, a fractional perfect b -matching of $A_u = (V, E_u)$ is a weight function $x : E_u \rightarrow \mathbb{R}^+$ on the edges of A_u such that for each $v \in V$, $\sum_{e \in E_u : v \in e} x(e) = b(v)$.

Lemma 22. *For any weight function $b : V \rightarrow \mathbb{R}^+$ such that A_u admits a fractional perfect b -matching, we have $u \in \text{Med}(b)$.*

Proof. Consider a fractional perfect b -matching $x : E_u \rightarrow \mathbb{R}^+$ of A_u and observe by triangle inequality that for any $v \in V$,

$$F_b(v) = \sum_{z \in V} b(z)d(v, z) = \sum_{z \in V} \left(\sum_{e : z \in e} x(e) \right) d(v, z) = \sum_{e = zz' \in E_u} x(e) (d(v, z) + d(v, z')) \geq \sum_{e = zz' \in E_u} x(e) d(z, z').$$

The last inequality is an equality if and only if $v \in I(z, z')$ for all edges $e = zz'$ such that $x(e) > 0$. By the definition of the edges of A_u , we thus conclude that $F_b(u) = \sum_{e=zz' \in E_u} x(e)d(z, z')$ and consequently, that $u \in \text{Med}(b)$. \square

Since deciding if a graph G admits a fractional perfect b -matching can be done in (strongly) polynomial time [2, 25], deciding if a profile π on a graph G has a perfect pairing can be done in polynomial time.

For a given weight function b , the set of all fractional perfect b -matchings of A_u is described by the following polytope $M(b, u)$ (see [30]):

$$M(b, u) \quad \begin{cases} \sum_{e:v \in e} x(e) = b(v) & \text{for all } v \in V \\ x(e) \geq 0 & \text{for all } e \in E_u \end{cases}$$

If b is a profile π , i.e., $b(v) \in \mathbb{N}$ for all $v \in V$, then the polytope $M(b, u)$ is either empty (and A_u does not have any fractional perfect b -matching) or each vertex $\mathbf{x} = (x(e))_{e \in E_u}$ of $M(b, u)$ is half-integral [4] (see also [30, Theorem 30.2]), i.e., for each $e \in E_u$, we have $x(e) \in \{0, \frac{1}{2}, 1\}$. Consequently, we have the following result.

Lemma 23. *Given a graph G , an integral weight function $b : V \rightarrow \mathbb{N}$ such that $b(v)$ is even for all $v \in V$, and a vertex $u \in V$, then either the auxiliary graph A_u has an integral perfect b -matching, or A_u has no fractional perfect b -matching.*

Lemma 21 suggests that graphs with the $2k$ -pairing property (i.e., graphs such that for any profile π , the profile π^{2k} has a perfect pairing) can be interesting generalizations of graphs with the pairing or double-pairing properties. However, the following result shows that they coincide with the graphs with the double-pairing property.

Lemma 24. *Given a graph $G = (V, E)$, a vertex $u \in V$, and a profile π on G , if $M(\pi, u)$ is non-empty, then π^2 admits a perfect pairing. Consequently, if G has a perfect π^{2k} -pairing for some integer $k \geq 1$, then G has a perfect π^2 -pairing.*

Proof. Observe that if $M(\pi, u)$ is non-empty, then $M(\pi^2, u)$ is also non-empty. By Lemma 23, A_u admits an integral perfect π^2 -matching. Consequently, π^2 admits a perfect pairing.

To prove the second assertion, consider a profile π on $G = (V, E)$ such that π^{2k} admits a perfect pairing and pick any $u \in \text{Med}(\pi^{2k}) = \text{Med}(\pi)$. Observe that if π^{2k} has a perfect pairing, then A_u has a fractional perfect π -matching (where $x(e)$ is a multiple of $\frac{1}{2k}$ for each $e \in E_u$). Consequently, the polytope $M(\pi, u)$ is non-empty and thus π^2 admits a perfect pairing. \square

4.2. (Double-)pairing property implies bipartite Hellyness. We now establish that graphs with the double-pairing property (and thus with the pairing property) are bipartite Helly graphs. Recall that bipartite Helly graphs are graphs in which the collection of half-balls of G has the Helly property. The bipartite Helly graphs have been characterized in several nice ways by Bandelt, Dählmann, and Schütte [9] (the papers [5, 9] also characterize the bipartite Helly graphs via medians, Condorcet vertices, and plurality vertices):

Theorem 25 ([9]). *For a bipartite graph G , the following conditions are equivalent:*

- (1) G is a bipartite Helly graph;
- (2) G is a modular graph of breadth at most two;
- (3) G is a modular graph such that every induced B_n ($n \geq 4$) extends to \widehat{B}_n in G ;
- (4) G satisfies the following interval condition: for any vertices u and v with $d(u, v) \geq 3$, the neighbors of v in $I(u, v)$ have a second common neighbor x in $I(u, v)$.

The graph B_n is the bipartite complete graph $K_{n,n}$ minus a perfect matching, i.e., the bipartition of B_n is defined by a_1, \dots, a_n and b_1, \dots, b_n and a_i is adjacent to b_j if and only if $i \neq j$. The graph \widehat{B}_n is obtained from B_n by adding two adjacent new vertices a and b such that a is adjacent to all vertices b_1, \dots, b_n and b is adjacent to all vertices a_1, \dots, a_n . Recall that a graph G is of breadth at most two [9] if for any vertex u and any set of vertices W of G , the following implication holds: $\bigcap_{v \in W} I(u, v) = \{u\}$ implies that there exists two vertices $w', w'' \in W$ such that $I(u, w') \cap I(u, w'') = \{u\}$.

Proposition 26. *If a graph G satisfies the double-pairing property, then G is a bipartite Helly graph.*

Proof. First we prove that G is bipartite. Suppose the contrary and let C be an odd cycle of minimum length of G . Then C is an isometric cycle of G . Let uv be an edge of C and w be the vertex of C opposite to uv . Let $d(u, w) = d(v, w) = k \geq 1$. Consider the profile $\tau = (u, v, w)$ and let $\pi = \tau^2 = (u, u, v, v, w, w)$.

Then $F_\pi(u) = F_\pi(v) = 2k + 2$. For any other vertex x of G , let $k_1 = d(x, u) \geq 1$, $k_2 = d(x, v) \geq 1$, and $k_3 = d(x, w)$, where $k_1 + k_3 \geq k$ and $k_2 + k_3 \geq k$. Notice that $F_\pi(x) = 2k_1 + 2k_2 + 2k_3 \geq 2k + 2k_2 \geq 2k + 2$. Thus $u, v \in \text{Med}(\pi)$. Up to symmetry, π has three non-trivial pairings: $P_1 = \{\{u, v\}, \{v, w\}, \{w, u\}\}$, $P_2 = \{\{u, w\}, \{u, w\}, \{v, v\}\}$, $P_3 = \{\{u, v\}, \{u, v\}, \{w, w\}\}$. Notice that $D_\pi(P_1) = 2k + 1$, $D_\pi(P_2) = 2k$, and $D_\pi(P_3) = 2$, i.e., $D_\pi(P_i) < F_\pi(u)$ for $i = 1, 2, 3$, and thus $\pi = \tau^2$ does not admit a perfect pairing.

By Theorem 25, G is a bipartite Helly graph if and only if G is a modular graph and every induced B_n ($n \geq 4$) extends to \widehat{B}_n . To prove that G is modular, since G is bipartite, it suffices to show that G satisfies the quadrangle condition. Suppose by way of contradiction that G contains four vertices u, v, w, y such that $y \sim u, v$ and $d(u, w) = d(v, w) = d(y, w) - 1 = k \geq 2$ and $I(u, w) \cap I(v, w) = \{w\}$. Consider the profile $\tau = (u, v, w)$ and let $\pi = \tau^2 = (u, u, v, v, w, w)$. Then $F_\pi(u) = F_\pi(v) = 2k + 4$. Pick any other vertex x of G and let $d(x, u) = k_1 \geq 1$, $d(x, v) = k_2 \geq 1$, and $d(x, w) = k_3 \geq 0$. Then $k_1 + k_3 \geq k$ and $k_2 + k_3 \geq k$. Notice that $F_\pi(x) = 2k_1 + 2k_2 + 2k_3$. Suppose that $F_\pi(x) < F_\pi(u)$, i.e., $2k_1 + 2k_2 + 2k_3 < 2k + 4$. Since $k_1 + k_3 \geq k$ (resp. $k_2 + k_3 \geq k$) this is possible only if $k_2 = 1$ (resp. $k_1 = 1$). Thus x is a common neighbor of u and v . Since $I(u, w) \cap I(v, w) = \{w\}$ and G is bipartite, we must have $k_3 = k + 1$, i.e., $F_\pi(x) = 2k + 6 > 2k + 4$, a contradiction. This shows that $u, v \in \text{Med}(\pi)$. Again, up to symmetry, π has three non-trivial pairings: $P_1 = \{\{u, v\}, \{v, w\}, \{w, u\}\}$, $P_2 = \{\{u, w\}, \{u, w\}, \{v, v\}\}$, and $P_3 = \{\{u, v\}, \{u, v\}, \{w, w\}\}$. Note that $D_\pi(P_1) = 2k + 2$, $D_\pi(P_2) = 2k$, and $D_\pi(P_3) = 4$, i.e., $D_\pi(P_i) < F_\pi(u)$ for $i = 1, 2, 3$, and thus $\pi = \tau^2$ does not admit a perfect pairing. Consequently, any graph G satisfying the double-pairing property is modular.

It remains to prove that every induced B_n ($n \geq 4$) extends to \widehat{B}_n . Suppose by way of contradiction that G contains induced $B_n, n \geq 3$ which are not included in \widehat{B}_n and pick such a B_n with a maximum number of vertices (this maximum exists since G is finite). Suppose that the vertex set of B_n is defined by the bipartition $\{a_1, \dots, a_n\} \cup \{b_1, \dots, b_n\}$. Consider the profile $\tau = \{a_1, \dots, a_n\}$ and let $\pi = \tau^2 = \{a_1, a_1, \dots, a_n, a_n\}$. One can easily see that if P is a maximum pairing of π , then $D_\pi(P) = 2n$. Notice that $F_\pi(b_i) = 2(n - 1) + 6 = 2n + 4$ and $F_\pi(a_i) = 4n - 4$. Therefore, if π has a perfect pairing, then there exists a vertex $x \neq a_1, \dots, a_n, b_1, \dots, b_n$ such that $F_\pi(x) = D_\pi(P) = 2n$. If in the unique bipartition of the vertices of G , x belongs to the same part as a_1, \dots, a_n , then $d(x, a_i) \geq 2$ and thus $F_\pi(x) \geq 4n$. Therefore, x belongs to the same part as b_1, \dots, b_n and to have $F_\pi(x) = D_\pi(P) = 2n$ necessarily x must be adjacent to all vertices a_1, \dots, a_n . Similarly, there exists a vertex $y \neq a_1, \dots, a_n$ adjacent to all vertices b_1, \dots, b_n . If the vertices x and y are not adjacent in G , then x, y together with $a_1, \dots, a_n, b_1, \dots, b_n$ define a B_{n+1} , contrary to the maximality choice of B_n . Thus, any graph G satisfying the pairing property is a bipartite Helly graph. \square

4.3. Bipartite Helly graphs without the pairing property. Unfortunately, there exist bipartite Helly graphs that do not satisfy the double-pairing property, and thus that do not satisfy the pairing property. To construct them, we relate the bipartite Helly graphs with the Helly hypergraphs. A hypergraph $\mathcal{H} = (X, \mathcal{E})$ consists of a set X and a family \mathcal{E} of subsets of X . A hypergraph \mathcal{H} is called a *Helly hypergraph* [11] if for any subfamily \mathcal{E}' of \mathcal{E} of pairwise intersecting sets the intersection $\bigcap \mathcal{E}'$ is nonempty. For any vertex u of G we can define a hypergraph \mathcal{H}_u in the following way: the ground-set of \mathcal{H}_u is the set $N(u)$ of neighbors of the vertex u and for each vertex v of G , the set $R_v := N(u) \cap I(u, v)$ is a hyperedge of \mathcal{H}_u . Then the condition 2 of Theorem 25 can be rephrased in the following way:

Lemma 27. *A graph G is a bipartite Helly graph if and only if G is modular and for any vertex u , \mathcal{H}_u is a Helly hypergraph.*

The next lemma allows us to construct bipartite Helly graphs from Helly hypergraphs. With any hypergraph $\mathcal{H} = (X, \mathcal{E})$ we associate the following graph $R(\mathcal{H})$, which we call the *incidence graph* of \mathcal{H} . The vertex-set of $R(\mathcal{H})$ consists of the set X , a vertex u which is adjacent to all vertices of X , and a vertex v_H for each hyperedge $H \in \mathcal{E}$ which is adjacent exactly to all vertices of X which belong to H . We call the vertices $x \in X$ *v-vertices* and the vertices $v_H, H \in \mathcal{H}$ *h-vertices*. Notice that $R(\mathcal{H})$ is a bipartite graph of diameter at most 4.

Lemma 28. *A hypergraph $\mathcal{H} = (X, \mathcal{E})$ is Helly if and only if its incidence graph $R(\mathcal{H})$ is a bipartite Helly graph.*

Proof. One direction follows from Lemma 27. Conversely, suppose that \mathcal{H} is Helly. Notice that for any two hyperedges H, H' , $d(v_H, v_{H'}) = 4$ if $H \cap H' = \emptyset$ and $d(v_H, v_{H'}) = 2$ if $H \cap H' \neq \emptyset$. Notice also that $d(x, x') = 2$ for any $x, x' \in X$ and $d(u, v_H) = 2, d(x, v_H) \leq 3$ for any $H \in \mathcal{E}$ and $x \in X$. We assert that $R(\mathcal{H})$ satisfies the condition 4 of Theorem 25. We have to consider two cases. First, consider two vertices $v_H, v_{H'}$ with $d(v_H, v_{H'}) = 4$. Then H and H' are disjoint and the neighbors of v_H in $I(v_H, v_{H'})$ are

exactly the vertices of H . They have u as the second common neighbor in $I(v_H, v_{H'})$. Second, consider two vertices $x \in X$ and v_H with $d(x, v_H) = 3$. This implies that $x \notin H$. Then again the neighbors of v_H in $I(v_H, x)$ have u as the second common neighbor in $I(v_H, x)$. On the other hand, the neighbors of x in $I(x, v_H)$ are the vertex u and all the vertices $v_{H'}$ such that $H' \cap H \neq \emptyset$. Since $x \in H' \cap H''$ for any two such vertices $v_{H'}, v_{H''}$, applying the Helly property to H and all hyperedges H' such that $v_{H'} \in I(x, v_H)$ we find a common point $y \in X$. In $R(\mathcal{H})$, y is adjacent to v_H , all $v_{H'} \in I(x, v_H)$, and to u . This shows that $R(\mathcal{H})$ is a bipartite Helly graph. \square

There is a simple way to construct Helly hypergraphs from graphs via hypergraph duality. The *dual* of a hypergraph $\mathcal{H} = (X, \mathcal{E})$ is the hypergraph $\mathcal{H}^* = (X^*, \mathcal{E}^*)$ whose vertex-set X^* is in bijection with the edge-set \mathcal{E} of \mathcal{H} and whose edge-set \mathcal{E}^* is in bijection with the vertex-set X , namely \mathcal{E}^* consists of all $S_x = \{H_j \in \mathcal{E} : x \in H_j\}$, $x \in X$. By definition, $(\mathcal{H}^*)^* = \mathcal{H}$. The *clique hypergraph* of a graph $B = (V, E)$ is the hypergraph $\mathcal{C}(B)$ whose vertices are the vertices of B and whose hyperedges are the maximal cliques of B . The following is a standard fact from hypergraph theory:

Lemma 29 ([11]). *The dual $(\mathcal{C}(B))^*$ of the clique hypergraph of any graph B is a Helly hypergraph.*

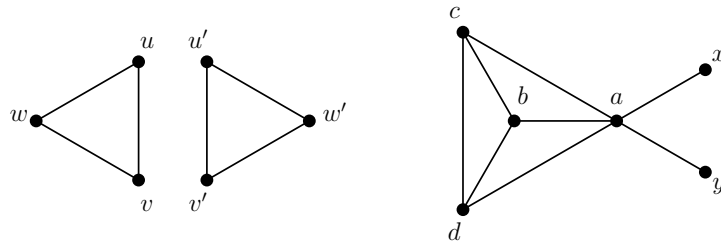


FIGURE 2. The graphs from the proof of Proposition 30

Proposition 30. *There exist bipartite Helly graphs which do not satisfy the pairing property (respectively, the double-pairing property).*

Proof. We construct a bipartite Helly graph not satisfying the pairing property that has the form $R := R((\mathcal{C}(B))^*)$ for a specially chosen graph B . That R is bipartite Helly follows from Lemmas 28 and 29. By definition of the dual hypergraph, the vertices of $(\mathcal{C}(B))^*$ are the maximal cliques of B and the hyperedges of $(\mathcal{C}(B))^*$ are in bijection with the vertices of B . For each vertex b of B we denote by $H(b)$ the corresponding hyperedge of $(\mathcal{C}(B))^*$; $H(b)$ consists of all maximal cliques of B containing b . Therefore the h -vertices of R are in bijection with the vertices of B : $v_{H(b)} \leftrightarrow b$. Notice also that for any distinct $b, b' \in V(B)$, $d(v_{H(b)}, v_{H(b')}) = 2$ if and only if $b \sim b'$.

As B we consider a graph with an even number $2m$ of vertices, we consider the even profile on B of size $2m$ $\pi = (v_{H(b)} : b \in V(B))$, and we consider the double-profile $\tau = \pi^2$. We want to ensure that the central vertex u of R is in $\text{Med}(\pi) = \text{Med}(\tau)$. Observe that $F_\pi(u) = 4m$. Moreover, for any v -vertex x in R , x corresponds to a maximum clique K in B and the set of neighbors of x in R is precisely $K \cup \{u\}$. Consequently, $F_\pi(x) = |K| + 3(2m - |K|)$ and thus $F_\pi(u) \leq F_\pi(x)$ if and only if $|K| \leq m$. Thus we require that all maximal cliques of B have size at most m ; equivalently, all maximal stable sets of \overline{B} must have size at most m . For any h -vertex $v_{H(b)}$, $F_\pi(v_{H(b)}) = 2k + 4(2m - 1 - k) = 8m - 2k - 4$ where $k = |N_B(b)| \leq 2m - 1$. Since we want to have $4m = F_\pi(u) \leq F_\pi(v_{H(b)}) = 8m - 2k - 4$, i.e., $k \leq 2m - 2$, we require that every vertex of B is not adjacent to at least one vertex of B ; equivalently, the minimum degree of \overline{B} must be at least 1.

If the profile π (respectively, τ) of R would satisfy the pairing property, then we can find a pairing P of π (respectively, of τ) such that for any pair $\{v_{H(b)}, v_{H(b')}\} \in P$ we must have $u \in I(v_{H(b)}, v_{H(b')})$. As we noticed above, this is equivalent to require that $H(b) \cap H(b') = \emptyset$, which is obviously equivalent to the requirement that the vertices b and b' are not adjacent in B . Therefore, the profile π (respectively, τ) of R admits a perfect pairing if and only if the complement \overline{B} of B has a perfect matching (respectively, a τ -perfect matching). Since $\tau = \pi^2$, the weight of each node of B in τ is 2, and thus by Lemma 23 B admits a τ -perfect matching if and only if B admits a fractional perfect matching.

Therefore, we have to construct a graph C with $2m$ vertices with no isolated vertices in which the maximum stable set has size at most m and that does not have a perfect matching (respectively, a fractional perfect matching). Then, we can take $B = \overline{C}$ and $R((\mathcal{C}(B))^*)$ will be a bipartite Helly graph

that does not satisfy the pairing property (respectively, the double-pairing property). For the pairing property, we can take the disjoint union of two triangles as C (see Figure 2, left). In this graph, there is no isolated vertex, any maximum stable set is of size 2, and clearly it does not contain a perfect matching. For the double-pairing property, we can take as C the graph on 6 vertices obtained by considering a complete graph on four vertices a, b, c, d in which we added two nodes x, y that are only adjacent to a (see Figure 2, right). In this graph, there is no isolated vertex, any stable set is of size at most 3, and it does not contain a fractional perfect matching. Indeed, the stable set $S = \{x, y\}$ is of size 2, but $N(S) = \{a\}$ is of size 1, preventing C from having a fractional perfect matching. \square

4.4. A local-to-global characterization of graphs with the pairing property. In this subsection, we provide a local-to-global characterization of bipartite Helly graphs with the pairing property. Note that balls in bipartite Helly graphs induce bipartite Helly graphs:

Lemma 31. *If G is a bipartite Helly graph, then for every vertex u and every integer k , the subgraph of G induced by the ball $B_k(u)$ is an isometric subgraph of G , which is a bipartite Helly graph.*

Proof. Let $H = G[B_k(u)]$. First notice that H is an isometric subgraph of G . Indeed, pick any $x, y \in B_k(u)$. Since G is modular, the triplet x, y, u has a median vertex z . Since $x, y \in B_k(u)$ and $z \in I(u, x) \cap I(u, y)$, the vertex z and the intervals $I(x, z)$ and $I(z, y)$ belong to the ball $B_k(u)$. Since $z \in I(x, y)$ and $I(x, z) \cup I(z, y) \subset B_k(u)$, x and y can be connected in H by a shortest path of G , thus H is an isometric subgraph of G .

Now, we prove that H is a bipartite Helly graph. Let $\frac{1}{2}B'_{r_i}(x_i), i = 1, \dots, n$ be a collection of pairwise intersecting half-balls of H . Denote by $\frac{1}{2}B_{r_i}(x_i), i = 1, \dots, n$ the respective half-balls of G . Suppose without loss of generality that $\frac{1}{2}B_{r_i}(x_i) = B_{r_i}(x_i) \cap X$, where X and Y are the color classes of G . Let $\frac{1}{2}B_r(u) = B_r(u) \cap X$. We assert that for any $i = 1, \dots, n$, we have $\frac{1}{2}B_{r_i}(x_i) \cap \frac{1}{2}B_r(u) \neq \emptyset$. Indeed, if x_i belongs to X , then $x_i \in \frac{1}{2}B_{r_i}(x_i)$ and since $x_i \in B_r(x)$ we also have $x_i \in \frac{1}{2}B_r(u)$. Otherwise, if $x_i \in Y$, then $r_i \geq 1$ and any neighbor z of x_i in $I(x_i, u)$ belongs to $\frac{1}{2}B_{r_i}(x_i)$ and to $B_r(u)$ and thus to $\frac{1}{2}B_r(u)$. Consequently, $\frac{1}{2}B_r(u), \frac{1}{2}B_{r_1}(x_1), \dots, \frac{1}{2}B_{r_n}(x_n)$ is a collection of pairwise intersecting half-balls of G . By the Helly property, these half-balls contain a common vertex y . Since $y \in \frac{1}{2}B_r(u) \subset B_r(u)$, y is a common vertex of the half-balls $\frac{1}{2}B'_{r_1}(x_1), \dots, \frac{1}{2}B'_{r_n}(x_n)$ of H . \square

For a vertex u of a bipartite Helly graph we define a graph B_u in the following way: the vertices of B_u is the set $B_2(u)$ of all vertices at distance at most 2 from u and two such vertices v, v' are adjacent in B_u if and only if $u \in I(v, v')$. We call B_u the *local graph* of the vertex u .

We say that a graph $G = (V, E)$ satisfies the *matching-stable-set property* (respectively, *double-matching-stable-set property*) if for any profile π (respectively, any double-profile $\pi = \tau^2$) on G ,

- (1) either there exists a vertex z of G such that $\pi(z) > \pi(N_G(z))$,
- (2) or there exists a maximal stable set S such that $\pi(S) > \pi(N_G(S))$,
- (3) or there exists a perfect π -matching in G .

Observe that for a maximal stable set S , we have $S \cup N_G(S) = V$, thus condition (2) can be restated by requiring that there exists a (maximal) stable set S such that $\pi(S) > \frac{1}{2}\pi(V)$.

Given a graph $G = (V, E)$ and a double-profile π on V , by the fractional Hall condition [4, 30], the graph G admits a perfect π -matching if and only if for any stable set S of G , we have $\pi(N_G(S)) \geq \pi(S)$. By this condition, for any double-profile π such that G does not admit a perfect π -matching, there exists a stable set S such that $\pi(N_G(S)) < \pi(S)$. Such a set is called a *disabling stable set* for π . A family \mathcal{D} of stable sets of G is called *disabling* for G if for any double-profile π , either G admits a perfect π -matching, or there exists $S \in \mathcal{D}$ disabling π . Then, a graph G satisfies the double-matching-stable-set property if and only if the family of all maximal and minimal stable sets of G is a disabling family.

Proposition 32. *A bipartite Helly graph G satisfies the pairing property (respectively, the double-pairing property) if and only if all local graphs $B_u, u \in V$ satisfy the matching-stable-set property (respectively, the double-matching-stable-set property).*

Proof. First, let $G = (V, E)$ be a bipartite Helly graph satisfying the pairing property (respectively, the double-pairing property). Consider a vertex $u \in V$ and the corresponding local graph $B_u = (V_u, E_u)$ as well as an even profile (respectively, a double profile) π on $V_u \subseteq V$. Note that V_u is the set of vertices at distance at most 2 from u in G . Let $X = N_G(u)$ and $Z = \{z \in V_u : d_G(u, z) = 2\}$.

Suppose first that there exists $z \in Z$ such that $F_\pi(z) < F_\pi(u)$. Let $X_1 = X \cap N_G(z)$ and $X_2 = X \setminus X_1$. Let $Z_1 = \{z' \in Z : d_G(z, z') \leq 2\}$ and $Z_2 = Z \setminus Z_1$. Observe that since G is bipartite, for any $x \in X$,

$xz \in E_u$ if and only if $x \in X_2$ and for any $z' \in Z$, $zz' \in E_u$ if and only if $z' \in Z_2$. Note also that $F_\pi(z) - F_\pi(u) = 2\pi(u) + 2\pi(X_2) + 2\pi(Z_2) - 2\pi(z)$. Since $F_\pi(z) < F_\pi(u)$ and since $N_{B_u}(z) = \{u\} \cup X_2 \cup Z_2$, we have $\pi(N_{B_u}(z)) < \pi(z)$ and thus condition (1) holds.

Suppose now that there exists $x \in X$ such that $F_\pi(x) < F_\pi(u)$. Let $Z_1 = Z \cap N_G(x)$ and $Z_2 = Z \setminus Z_1$. Observe that in B_u , $S = Z_1 \cup \{x\}$ is a stable set. Note that $F_\pi(x) - F_\pi(u) = \pi(u) - \pi(x) + \sum_{x' \in X \setminus \{x\}} \pi(x') - \sum_{z \in Z_1} \pi(z) + \sum_{z \in Z_2} \pi(z) = \pi(\{u\} \cup (X \setminus \{x\}) \cup Z_2) - \pi(\{x\} \cup Z_1)$. Since $F_\pi(x) < F_\pi(u)$, we have $\pi(S) > \pi(V_u \setminus S) = \pi(N_{B_u}(S))$ and thus condition (2) holds. Thus, we can assume that u is a local median of π in G^2 . Since bipartite Helly graphs have G^2 -connected medians [10, Proposition 62], $u \in \text{Med}_G(\pi)$. Since G satisfies the pairing property (respectively, the double-pairing property), there exists a pairing P of π such that $\text{Med}_G(\pi) = \bigcap_{\{a,b\} \in P} I(a,b)$. Hence for any $\{a,b\} \in P$, we have $u \in I(a,b)$ and thus $ab \in E_u$. Consequently, in this case, the pairing P defines a π -matching in B_u and condition (3) holds.

Assume now that for any $u \in V$, the local graph $B_u = (V_u, E_u)$ satisfies the matching-stable-set property (respectively, the double-matching-stable-set property), and consider an even profile (respectively, a double-profile) π on G . Consider a vertex $u \in \text{Med}_G(\pi)$. For each vertex $v \in \pi$ such that $d_G(u, v) \geq 2$, by Theorem 25(4), there exists a vertex $z_v \in I(u, v)$ such that $d_G(u, z_v) = 2$ and $z_v \sim I(u, v) \cap N_G(u)$. We construct a profile π' on $V_u = B_2(u, G)$ by replacing each occurrence of v in π with $d_G(u, v) \geq 2$ by such a z_v (we keep in π' each occurrence of u in π and each occurrence of $x \in N_G(u)$ in π). Observe that π' is an even profile and that we can assume that π' is a double-profile when π is a double profile. Note that if $u \in I(v, v')$, then $u \in I(z_v, z_{v'})$. Conversely, if $u \in I(z_v, z_{v'})$, then $u \in I(v, v')$. Indeed, since G is bipartite, either $d_G(v, v') = d_G(v, u) + d_G(u, v')$, or $d_G(v, v') \leq d_G(v, u) + d_G(u, v') - 2$. In the second case, consider the balls $B_1(u)$ of G and the balls of radius $d_G(u, v) - 1$ and $d_G(u, v') - 1$ respectively centered at v and v' . Since these three balls pairwise intersect, there exists a neighbor x of u such that $x \in I(u, v) \cap I(u, v')$. This implies that $x \sim z_v, z_{v'}$ by the definition of z_v and $z_{v'}$, and thus $u \notin I(z_v, z_{v'})$.

Since u is a median of π in G , u is also a median of π' in G . If $\pi'(u) \geq \pi'(V_u \setminus \{u\})$, then $\pi(u) \geq \pi(V \setminus \{u\})$ and in this case, there exists a pairing P of π such that for each $\{a, b\} \in P$, $a = u$ or $b = u$. In this case, $u \in I(a, b)$ for each $\{a, b\} \in P$, and thus $\text{Med}_G(\pi) = \bigcap_{\{a,b\} \in P} I(a, b)$. Assume now that $\pi'(u) < \pi'(V \setminus \{u\})$.

Suppose first that there exists a stable set S of B_u such that $\pi'(S) > \pi'(V_u \setminus S)$. Since u is a universal vertex in B_u and since $\pi'(u) < \pi'(V \setminus \{u\})$, necessarily $u \notin S$. Let $X = N_G(u)$ and $Z = \{z : d_G(u, z) = 2\}$. Since for all distinct $x, x' \in X$, we have $xx' \in E_u$, we have $|X \cap S| \leq 1$. If $X \cap S = \{x\}$ and $\pi'(x) > \pi'(V_u \setminus \{x\})$, then $\pi(x) > \pi(V \setminus \{x\})$, and in this case, $\text{Med}_G(\pi) = \{x\}$, contradicting the choice of u . Consequently, if there exists $x \in X \cap S$, then $S \setminus \{x\} \neq \emptyset$ and $S \setminus \{x\} \subseteq Z$. Moreover, if there exists $x \in X \cap S$, for each $z \in Z \cap S$, $d_G(x, z) = 1$ and thus $Z \cap X \subseteq N_G(x)$. Since S is a stable set in B_u , for all distinct $z, z' \in S \cap Z$, we have $d_G(z, z') = 2$. Consequently, by considering all balls of radius 1 centered at the vertices of Z , we get that there exists $x \in X$ such that $S \subseteq N_G(x)$. Therefore, in any case, there exists x such that $S \subseteq \{x\} \cup N_G(x)$ and since $\{x\} \cup N_G(x)$ is a stable set of B_u , we can assume that $S = \{x\} \cup N_G(x)$. In this case, $F_{\pi'}(x) - F_{\pi'}(u) = \pi'(V \setminus S) - \pi(S) < 0$, contradicting the fact that $u \in \text{Med}_G(\pi')$.

Suppose now that there exists $z \in V_u$ such that $\pi'(z) > \pi'(N_{B_u}(z))$. If $z \in X$, let $S = \{z\} \cup (N_G(z) \cap Z)$. Observe that S is a stable set of B_u and that $N_{B_u}(z) = V_u \setminus S$. Consequently, there exists a stable set S of B_u such that $\pi'(S) > \pi'(V_u \setminus S)$, but we already know that this is impossible. We can thus assume that $z \in Z$. Observe that for each $y \in V_u \setminus \{z\}$, either $d_G(u, y) = d_G(z, y)$, or $y \in N_{B_u}(z)$ and $d_G(z, y) = d_G(u, y) + 2$. Consequently, $F_{\pi'}(z) - F_{\pi'}(u) = 2\pi'(N_{B_u}(z)) - 2\pi'(z) < 0$, contradicting the fact that $u \in \text{Med}_G(\pi')$.

Consequently, we can assume that there exists a π -matching M in B_u . From such a matching M , one can obtain a pairing P of π such that if $\{v, v'\} \in P$, then $z_v z_{v'} \in M$. Consequently, for each $\{v, v'\} \in P$, we have $u \in I(z_v, z_{v'})$ and $u \in I(v, v')$. Therefore we have $u \in \bigcap_{\{v,v'\} \in P} I(v, v') = \text{Med}_G(\pi)$. This shows that G satisfies the pairing property (respectively, the double-pairing property). \square

The following result follows from Lemma 31 and the proof of Proposition 32:

Corollary 33. *A bipartite Helly graph G satisfies the pairing property (respectively, the double-pairing property) if and only if all bipartite Helly graphs induced by the balls of radius 2 of G satisfy the pairing property (respectively, the double-pairing property).*

4.5. Recognizing graphs with the double-pairing property. We were not able to settle the complexity of deciding if a graph has the double-pairing property. However, we can show that this problem

is in co-NP. As explained above, if we are given a profile π , one can check whether π admits a perfect pairing in time that is polynomial in the size of G and the size of the profile π . In order to prove that recognizing graphs satisfying the pairing property (respectively, the double-pairing property) is in co-NP, it would then be sufficient to show that when a graph G does not satisfy the pairing property (respectively, the double-pairing property), there exists an even profile π (respectively, a double-profile π) that does not admit a perfect pairing and whose size is polynomial in the size of G . Unfortunately, we were not able to prove that these profiles of polynomial size always exist. For the double-pairing property, we establish the result by reformulating the problem as a problem of inclusion of polytopes.

For a graph $G = (V, E)$ and a vertex $u \in V$, we define two polytopes $\text{Ma}(u)$ and $\text{Me}(u)$ as follows. The polytope $\text{Me}(u)$ consists of all weight functions $b : V \rightarrow \mathbb{R}^+$ such that u belongs to the median set $\text{Med}(b)$. In particular, for each profile π on G such that $u \in \text{Med}(\pi)$, the weight function defined by π belongs to $\text{Me}(u)$.

$$\text{Me}(u) \quad \begin{cases} \sum_{w \in V} b(w)(d(v, w) - d(u, w)) \geq 0 & \text{for all } v \in V, \\ b(v) \geq 0 & \text{for all } v \in V. \end{cases}$$

Note that $\text{Me}(u)$ is a non-empty polytope defined by a linear number of inequalities.

The second polytope $\text{Ma}(u)$ consists of all weight functions $b : V \rightarrow \mathbb{R}^+$ such that the auxiliary graph A_u admits a fractional perfect b -matching. The description of the polytope is defined using the inequalities given by the fractional Hall condition.

$$\text{Ma}(u) \quad \begin{cases} \sum_{v \in N(S)} b(v) - \sum_{v \in S} b(v) \geq 0 & \text{for all stable sets } S \text{ of } A_u = (V, E_u), \\ b(v) \geq 0 & \text{for all } v \in V. \end{cases}$$

The number of inequalities defining $\text{Ma}(u)$ is potentially exponential in the size of G . Note however that the separation problem on $\text{Ma}(u)$ can be solved in polynomial time by solving a fractional b -matching problem. Such an algorithm either provides a b -matching and b is a point of $\text{Ma}(u)$, or it enables to compute a set S such that b violates the constraint corresponding to S .

Observe that both polytopes $\text{Me}(u)$ and $\text{Ma}(u)$ contain the origin $\mathbf{0}$ and that they are stable by multiplication by a positive scalar.

Lemma 34. *Given a graph $G = (V, E)$, for any vertex $u \in V$, $\text{Ma}(u) \subseteq \text{Me}(u)$.*

Proof. For any weight function $b : V \rightarrow \mathbb{R}^+$, if b is a point of $\text{Ma}(u)$, then by the fractional Hall Condition, there exists a perfect b -matching in A_u and thus, by Lemma 22, $u \in \text{Med}(b)$. This establishes that $b \in \text{Me}(u)$. \square

Proposition 35. *A graph $G = (V, E)$ satisfies the double-pairing property if and only if $\text{Ma}(u) = \text{Me}(u)$ for all $u \in V$.*

Proof. First suppose that G satisfies the double-pairing property. Since both polytopes are defined by constraints with integral coefficients, their extremal points and rays are defined by rational coordinates. Since both polytopes are stable by multiplication by a positive scalar, it suffices to prove that any point of $\text{Me}(u)$ corresponding to a double-profile π belongs to $\text{Ma}(u)$. Since G satisfies the double-pairing property, for any such double profile π , there exists a perfect π -matching in A_u , and thus $\pi \in \text{Ma}(u)$.

Conversely, assume that $\text{Me}(u) \subseteq \text{Ma}(u)$ for all $u \in V$ and consider a double-profile $\tau = \pi^2$ on G . Pick any $u \in \text{Med}(\tau) = \text{Med}(\pi)$. Then $\pi \in \text{Me}(u) \subseteq \text{Ma}(u)$, and thus, $\text{M}(\pi, u)$ is non-empty. By Lemma 24, $\tau = \pi^2$ admits a perfect pairing. This establishes that G satisfies the double-pairing property. \square

We now explain how to decide if a given graph G does not satisfy the double-pairing property in non-deterministic polynomial time. To do so, we first guess a vertex u such that $\text{Ma}(u) \subsetneq \text{Me}(u)$. We construct the auxiliary graph A_u and we guess a stable set S of A_u such that the corresponding constraint of $\text{Ma}(u)$ separates a point of $\text{Me}(u)$ from $\text{Ma}(u)$. In order to check that this constraint indeed separates a point of $\text{Me}(u)$ from $\text{Ma}(u)$, we minimize the function $(b(v))_{v \in V} \mapsto \sum_{v \in N(S)} b(v) - \sum_{v \in S} b(v)$ on the polytope $\text{Me}(u)$. This can be done in polynomial time since this is a linear program with a linear number of constraints. If the minimum is negative, then $\text{Me}(u)$ is not contained in $\text{Ma}(u)$ and thus G does not satisfy the double-pairing property by Lemma 35. We thus have the following result.

Proposition 36. *Recognizing graphs satisfying the double-pairing property belongs to co-NP.*

Remark. We do not know if we can extend this method for the recognition of graphs with the pairing property. This is due to the fact that the set of profiles that admit a perfect pairing is not stable by

convex combinations. Indeed, consider the graph on the left Figure 2 that is made of two disjoint triangles u, v, w and u', v', w' , and denote it by C . Let B be the complement of C and consider the bipartite Helly graph $R((\mathcal{C}(B))^*)$ defined in Subsection 4.3. Then, since C does not admit a perfect matching, the even profile $\pi = (u, v, w, u', v', w')$ does not admit a perfect pairing in $R((\mathcal{C}(B))^*)$. However, both the empty profile and the double profile π^2 admit a perfect pairing in $R((\mathcal{C}(B))^*)$.

Question. Can we recognize graphs satisfying the double-pairing property (respectively, the pairing property) in polynomial time? Is the recognition of graphs satisfying the pairing property in co-NP?

5. BENZENOID GRAPHS

The main tools in our paper are the axioms (A), (B), (C), and (T), satisfied by the median function Med in all graphs. Nevertheless, these axioms are difficult to handle and do not allow to establish even simple structural properties of ABC- or ABCT-graphs. In this section, we formulate two other axioms (T_2) and (E_k) satisfied by Med . We show that the 6-cycle C_6 has two ABC-functions. Finally, we prove that benzenoids (which are obtained from 6-cycles by amalgamations) are ABCT₂- and ABCE₂-graphs.

5.1. Axioms for equilateral metric triangles. As for axiom (T), the axioms we consider deal with triplets of vertices. First, we start with a generalizations of axioms (T) and (T^-) :

(T_2) for any equilateral metric triangle uvw of size 2, $\{u, v, w\} \subseteq L(u, v, w)$.

Unfortunately, there is no straightforward way to generalize (T) and (T_2) to triplets of vertices at distance k . However, we can define the axiom (E_k) , which coincides with (T^-) when $k = 1$:

(E_k) *Equilateral*: for any equilateral metric triangle uvw of size k , if $u \in L(u, v, w)$, then $\{u, v, w\} \subseteq L(u, v, w)$.

Lemma 37. *The median function Med satisfies the axioms (T_2) and (E_k) .*

Proof. To prove (T_2) , pick any equilateral metric triangle uvw of size 2. Then $F_\pi(u) = F_\pi(v) = F_\pi(w) = 4$. Any vertex z of G is not adjacent to at least one of the vertices u, v, w , thus $F_\pi(z) \geq 4$, proving that $\{u, v, w\} \subseteq \text{Med}(u, v, w)$. To prove (E_k) , pick any equilateral metric triangle uvw of size k and suppose that $u \in \text{Med}(u, v, w)$. Since $F_\pi(u) = 2k = F_\pi(v) = F_\pi(w)$, we conclude that $v, w \in \text{Med}(u, v, w)$. \square

5.2. C_6 has two ABC-functions. In this subsection, we define an ABC-function L_6 on C_6 different from Med . Denote by $V = \{v_0, v_1, v_2, v_3, v_4, v_5\}$ the ordered set of vertices of the graph C_6 such that v_i is a neighbor of v_{i-1} and v_{i+1} (all additions are done modulo 6). For a profile π on C_6 , let $\pi_i = \pi(v_i)$ denote the number of occurrences of v_i in π .

Since $I(v_i, v_{i+3}) = V$, for each ABC-function L on C_6 , we have $L(v_i, v_{i+3}) = I(v_i, v_{i+3}) = V$ and $L(\pi, v_i, v_{i+3}) = L(\pi)$. For each profile π , we denote by π° the profile such that $\pi_i^\circ = \pi_i - \min\{\pi_i, \pi_{i+3}\}$ for each $i \in \{0, \dots, 5\}$. Notice that either π_i° or π_{i+3}° is equal to 0 and that $L(\pi^\circ) = L(\pi)$. If $\pi_i^\circ, \pi_{i+2}^\circ, \pi_{i+4}^\circ$ are not equal to 0, then we call π° an *alternate* profile.

Consider the following consensus function L_6 :

$$L_6(\pi) = \begin{cases} \{v_i\} & \text{if } \pi_i^\circ, \pi_{i+2}^\circ, \pi_{i+4}^\circ > 0 \text{ and } i = \min\{j : \pi_j^\circ = \max\{\pi_i^\circ, \pi_{i+2}^\circ, \pi_{i+4}^\circ\}\} \\ \text{Med}(\pi) & \text{otherwise.} \end{cases}$$

Note that by definition of L , if π° is not an alternate profile, then $L_6(\pi) = \text{Med}(\pi) = \text{Med}(\pi^\circ) = L_6(\pi^\circ)$. Obviously, $L_6 \neq \text{Med}$ since for the profile $\pi = (v_0, v_2, v_4)$ we have $\text{Med}(\pi) = \{v_0, v_2, v_4\}$ whereas $L_6(\pi) = \{v_0\}$. To prove that L_6 is an ABC-function, we use the following lemmas:

Lemma 38. *For any $\pi, \rho \in V^*$, if $\sigma = \pi\rho$ and $\tau = \pi^\circ\rho^\circ$, then $\sigma^\circ = \tau^\circ$.*

Proof. For any $i \in \{0, \dots, 5\}$, we have:

$$\begin{aligned} \tau_i^\circ &= \tau_i - \min\{\tau_i, \tau_{i+3}\} = \pi_i^\circ + \rho_i^\circ - \min\{\pi_i^\circ + \rho_i^\circ, \pi_{i+3}^\circ + \rho_{i+3}^\circ\} \\ &= \pi_i^\circ + \rho_i^\circ - (\min\{\pi_i + \rho_i, \pi_{i+3} + \rho_{i+3}\} - \min\{\pi_i, \pi_{i+3}\} - \min\{\rho_i, \rho_{i+3}\}) \\ &= \pi_i + \rho_i - \min\{\pi_i + \rho_i, \pi_{i+3} + \rho_{i+3}\} = \sigma_i - \min\{\sigma_i, \sigma_{i+3}\} = \sigma_i^\circ, \end{aligned}$$

concluding the proof. \square

Lemma 39. *Let L be an ABC-function on C_6 and π a profile such that π° is nonempty and non-alternate. A vertex v_i belongs to $L(\pi) = L(\pi^\circ)$ if and only if up to symmetry we are in the following cases of π° and of $L(\pi)$:*

$$\begin{aligned}
(i) \quad & \text{if } \pi^\circ = (v_i^{\pi_i^\circ}) & L(\pi) &= \{v_i\} \\
(ii) \quad & \text{if } \pi^\circ = (v_i^{\pi_i^\circ}, v_{i+1}^{\pi_{i+1}^\circ}) \quad \text{and } \pi_i^\circ \geq \pi_{i+1}^\circ, & L(\pi) &= \begin{cases} I(v_i, v_{i+1}) & \text{if } \pi_i^\circ = \pi_{i+1}^\circ, \\ \{v_i\} & \text{otherwise} \end{cases} \\
(iii) \quad & \text{if } \pi^\circ = (v_i^{\pi_i^\circ}, v_{i+2}^{\pi_{i+2}^\circ}) \quad \text{and } \pi_i^\circ \geq \pi_{i+2}^\circ, & L(\pi) &= \begin{cases} I(v_i, v_{i+2}) & \text{if } \pi_i^\circ = \pi_{i+2}^\circ, \\ \{v_i\} & \text{otherwise} \end{cases} \\
(iv) \quad & \text{if } \pi^\circ = (v_{i-1}^{\pi_{i-1}^\circ}, v_{i+1}^{\pi_{i+1}^\circ}) \quad \text{and } \pi_{i+1}^\circ = \pi_{i-1}^\circ, & L(\pi) &= I(v_{i+1}, v_{i-1}) \\
(v) \quad & \text{if } \pi^\circ = (v_i^{\pi_i^\circ}, v_{i+1}^{\pi_{i+1}^\circ}, v_{i+2}^{\pi_{i+2}^\circ}) \quad \text{and } \pi_i^\circ \geq \pi_{i+1}^\circ + \pi_{i+2}^\circ, & L(\pi) &= \begin{cases} I(v_i, v_{i+1}) & \text{if } \pi_i^\circ = \pi_{i+1}^\circ + \pi_{i+2}^\circ, \\ \{v_i\} & \text{otherwise} \end{cases} \\
(vi) \quad & \text{if } \pi^\circ = (v_{i-1}^{\pi_{i-1}^\circ}, v_i^{\pi_i^\circ}, v_{i+1}^{\pi_{i+1}^\circ}) \quad \text{and } \pi_i^\circ \geq |\pi_{i+1}^\circ - \pi_{i-1}^\circ|, & L(\pi) &= \begin{cases} I(v_i, v_{i+1}) & \text{if } \pi_i^\circ + \pi_{i-1}^\circ = \pi_{i+1}^\circ, \\ I(v_i, v_{i-1}) & \text{if } \pi_i^\circ + \pi_{i+1}^\circ = \pi_{i-1}^\circ, \\ \{v_i\} & \text{otherwise.} \end{cases}
\end{aligned}$$

Proof. Since π° is non-alternate, we can assume without loss of generality that $\pi_3^\circ = \pi_4^\circ = \pi_5^\circ = 0$. Let $a = \pi_0^\circ$, $b = \pi_1^\circ$, $c = \pi_2^\circ$ and up to symmetry, we can assume that $a \geq c$. Note that since π° is not empty, $a + b + c \geq 1$. If $a = c$ and $b = 0$ (in this case, $a = c \geq 1$), then $L(\pi) = L(\pi^\circ) = L(v_0, v_2) = I(v_0, v_2)$, and for v_0, v_2 , we are in Case (iii) while for v_1 , we are in Case (iv). If $a = c$ and $b > 0$, then since $b \in L(v_0^\circ, v_2^\circ) = I(v_0, v_2)$, either $L(\pi^\circ) = L(v_1^b) = \{v_1\}$ if $a = c = 0$, or $L(\pi^\circ) = L(v_0^a, v_2^c) \cap L(v_1^b) = \{v_1\}$ if $a = c > 0$. In the first case, we are in case (i), while in the second case, we are in case (vi). Suppose now that $a > c$ and let $a' = a - c = |a - c|$. If $c = 0$, then $L(\pi^\circ) = L(v_0^{a'}, v_1^b)$. If $c > 0$, then $L(\pi^\circ) = L(v_0^c, v_2^c) \cap L(v_0^{a'}, v_1^b) = L(v_0^{a'}, v_1^b)$ since $L(v_0^c, v_2^c) = I(v_0, v_2)$ and since $L(v_0^{a'}, v_1^b) \subseteq \{v_0, v_1\} \subseteq I(v_0, v_2)$. If $a' = b$ (i.e., if $a = b + c$), then $L(\pi^\circ) = L(v_0^{a'}, v_1^b) = \{v_0, v_1\}$. In this case, if $c = 0$, then we are in Case (ii) for v_0 and v_1 , and if $c > 0$, then we are in Case (v) for v_0 and in Case (vi) for v_1 . If $a' < b$ (i.e., if $b > |a - c|$), then $L(\pi^\circ) = L(v_0^{a'}, v_1^b) = \{v_1\}$ and we are in Case (ii) if $c = 0$ and in Case (vi) if $c > 0$. If $a' > b$ (i.e., if $a > c$), then $L(\pi^\circ) = L(v_0^{a'}) = \{v_0\}$. Then, we are in Case (i) if $b = c = 0$, in Case (ii) if $b > c = 0$, in Case (iii) if $c > b = 0$, and in Case (v) if $b > 0$ and $c > 0$. \square

The next corollary follows from previous lemma and will be used next:

Corollary 40. *If π° is a non-alternate profile, then for any ABC-function L , $L(\pi) = \text{Med}(\pi)$.*

We now establish the main result of this subsection:

Proposition 41. *L_6 is an ABC-function on C_6 .*

Proof. Obviously, L_6 satisfies the axioms (A) and (B). It remains to show that L_6 also satisfies (C). Let $\pi = \rho\sigma$ for $\rho, \sigma \in V^*$. We assert that if $L_6(\rho) \cap L_6(\sigma) \neq \emptyset$, then $L_6(\rho\sigma) = L_6(\rho) \cap L_6(\sigma)$.

Suppose that $v_i \in L_6(\rho) \cap L_6(\sigma) = L_6(\rho^\circ) \cap L_6(\sigma^\circ)$. By the definition of L_6 and Lemma 39, this implies that $\sigma_{i+3}^\circ = \rho_{i+3}^\circ = 0$. Consequently, either σ° (respectively, ρ°) is not an alternate profile, or $\sigma_i^\circ, \sigma_{i+2}^\circ, \sigma_{i+4}^\circ > 0$ (respectively, $\rho_i^\circ, \rho_{i+2}^\circ, \rho_{i+4}^\circ > 0$). By Lemma 38, we have $\pi_{i+3}^\circ = 0$ and consequently, either π° is not an alternate profile, or $\pi_i^\circ, \pi_{i+2}^\circ, \pi_{i+4}^\circ > 0$.

If ρ° and σ° are both alternate profiles, then π° is an alternate profile and for $j \in \{i+2, i+4\}$, we have $\pi_i^\circ = \rho_i^\circ + \sigma_i^\circ \geq \rho_j^\circ + \sigma_j^\circ$ where the equality holds only if $\rho_i^\circ = \rho_j^\circ$ and $\sigma_i^\circ = \sigma_j^\circ$. Note that if the equality holds, then $0 \leq i < j \leq 5$, since $L_6(\rho) = \{v_i\}$. Consequently, either $\pi_i^\circ > \pi_j^\circ$ or $\pi_i^\circ = \pi_j^\circ$ and $0 \leq i < j \leq 5$. Therefore we have $L_6(\pi) = \{v_i\} = L_6(\rho) \cap L_6(\sigma)$ and we are done.

If neither ρ° nor σ° are alternate profiles, then $L_6(\rho) = L_6(\rho^\circ) = \text{Med}(\rho)$ and $L_6(\sigma) = L_6(\sigma^\circ) = \text{Med}(\sigma)$. In this case, $L_6(\rho) \cap L_6(\sigma) = \text{Med}(\rho) \cap \text{Med}(\sigma) = \text{Med}(\rho\sigma) = \text{Med}(\pi)$. If π° is not an alternate profile, then $L_6(\pi) = L_6(\pi^\circ) = \text{Med}(\pi^\circ) = \text{Med}(\pi)$ and we are done. If π° is an alternate profile, then $\pi_i^\circ, \pi_{i+2}^\circ, \pi_{i+4}^\circ > 0$. Since ρ° is not an alternate profile and since $v_i \in L_6(\rho^\circ)$, by Lemma 38, either $\rho_{i+2}^\circ = 0$ or $\rho_{i+4}^\circ = 0$. Similarly, either $\sigma_{i+2}^\circ = 0$ or $\sigma_{i+4}^\circ = 0$. Since $\rho_{i+2}^\circ + \sigma_{i+2}^\circ \geq \pi_{i+2}^\circ > 0$ and $\rho_{i+4}^\circ + \sigma_{i+4}^\circ \geq \pi_{i+4}^\circ > 0$, we can assume without loss of generality that $\rho_{i+2}^\circ > 0$, $\rho_{i+4}^\circ = 0$, $\sigma_{i+2}^\circ = 0$, and $\sigma_{i+4}^\circ > 0$. By Lemma 39, we can thus assume that $\rho^\circ = (v_i^a, v_{i+1}^b, v_{i+2}^c)$ where $a > 0, c > 0, b \geq 0$, and $a \geq b + c$, and that $\sigma^\circ = (v_{i+4}^{d'}, v_{i+5}^{e'}, v_i^{a'})$ where $a' > 0, d' > 0, e' \geq 0$, and $a' \geq d' + e'$. Consequently,

since π° is an alternate profile, $\pi^\circ = (v_i^{a+a'}, v_{i+2}^{c-e'}, v_{i+4}^{d'-b})$. Since $a \geq c > 0$ and $a' \geq d' > 0$, we have $a + a' > c - e'$ and $a + a' > d' - b$, and thus $L_6(\pi) = L_6(\pi^\circ) = \{v_i\} = \text{Med}(\pi)$ and we are done.

Suppose now that ρ° is an alternate profile and that σ° is not an alternate profile. Then $\rho^\circ = (v_i^a, v_{i+2}^c, v_{i+4}^d)$ with $a \geq c > 0$ and $a \geq d > 0$. Moreover if $a = c$ (respectively, $a = d$), then $0 \leq i < i + 2 \leq 5$ (respectively, $0 \leq i < i + 4 \leq 5$). For σ° , we consider the different cases given by Lemma 39. In all cases, we show that $L_6(\pi) = L_6(\pi^\circ) = \{v_i\} = L_6(\rho^\circ) \cap L_6(\sigma^\circ)$.

In Case (i) or (iii), $\sigma^\circ = (v_i^{a'}, v_{i+2}^{c'})$ with $a' > 0$ and $a' \geq c' \geq 0$, and $\pi^\circ = (v_i^{a+a'}, v_{i+2}^{c+c'}, v_{i+4}^d)$ is an alternate profile. Note that $a + a' > d$, that $a + a' \geq c + c'$ and that when $a + a' = c + c'$, we have $a = c$ and thus $0 \leq i < i + 2 \leq 5$ since $L_6(\rho^\circ) = \{v_i\}$. In any case, we have $L_6(\pi^\circ) = \{v_i\}$.

In Case (ii) or (v), $\sigma^\circ = (v_i^{a'}, v_{i+1}^{b'}, v_{i+2}^{c'})$ with $b' > 0, c' \geq 0$, and $a' \geq b' + c'$. In this case, $\rho^\circ \sigma^\circ = (v_i^{a+a'}, v_{i+1}^{b'+c'}, v_{i+2}^{c+c'}, v_{i+4}^d)$. If $b' \geq d$, then $\pi^\circ = (v_i^{a+a'}, v_{i+1}^{b'-d}, v_{i+2}^{c+c'})$ and $L_6(\pi^\circ) = \{v_i\}$ since $a + a' \geq c + b' + c' > b' - d + c + c'$. If $b' < d$, then $\pi^\circ = (v_i^{a+a'}, v_{i+2}^{c+c'}, v_{i+4}^{d-b'})$ is an alternate profile. Since $a + a' \geq c + c' + b' > c + c'$ and $a + a' > d > d - b'$, we also have $L_6(\pi^\circ) = \{v_i\}$ in this case.

In Case (iv) or (vi), $\sigma^\circ = (v_{i-1}^{e'}, v_i^{a'}, v_{i+1}^{b'})$ with $a' \geq 0, b' > 0, e' > 0, a' \geq e' - b'$, and $a' \geq b' - e'$. In this case, $\rho^\circ \sigma^\circ = (v_{i-1}^{e'}, v_i^{a+a'}, v_{i+1}^{b'+e'}, v_{i+2}^{c'}, v_{i+4}^d)$. If $d > b'$ and $c > e'$, then $\pi^\circ = (v_i^{a+a'}, v_{i+2}^{c-e'}, v_{i+4}^{d-b'})$ is an alternate profile and $L_6(\pi^\circ) = \{v_i\}$ since $a + a' > c - e'$ and $a + a' > d - b'$. If $d \leq b'$ and $c > e'$, then $\pi^\circ = (v_i^{a+a'}, v_{i+1}^{b'-d}, v_{i+2}^{c-e'})$ and $L_6(\pi) = \{v_i\}$ since $a + a' \geq c + b' - e' > b' - d + c - e'$. For similar reasons, we have $L_6(\pi) = \{v_i\}$ if $d > b'$ and $c \leq e'$. If $d \leq b'$ and $c \leq e'$, then $\pi^\circ = (v_{i-1}^{e'-c}, v_i^{a+a'}, v_{i+1}^{b'-d})$. Without loss of generality, assume that $e' - c \leq b' - d$. Let $\tau_1 = (v_{i-1}^{e'-c}, v_{i+1}^{e'-c})$ and $\tau_2 = (v_i^{a+a'}, v_{i+1}^{b'-d-e'+c})$ and observe that $\pi^\circ = \tau_1 \tau_2$ if $e' > c$ and $\pi^\circ = \tau_2$ if $e' = c$. If $e' > c$, then $L_6(\tau_1) = I(v_{i-1}, v_{i+1})$. Since $a + a' \geq c + b' - e' > c + b' - e' - d$, we have $L_6(\tau_2) = \{v_i\}$. Thus, in any case, we have $L_6(\pi^\circ) = L_6(\tau_2) = \{v_i\}$. \square

5.3. Benzenoid graphs are ABCT₂- and ABCE₂-graphs. By [10, Proposition 70], benzenoids have G^2 -connected medians. However they are not modular graphs (triplets of vertices at distance 2 in each 6-cycle do not have medians). Since, by Proposition 41 C_6 is not an ABC-graph, benzenoids are not ABC-graphs (as a graph family). In this subsection, we show that they are ABCT₂- and ABCE₂-graphs.

It was shown in [14] that any benzenoid graph $G = (V, E)$ can be isometrically embedded into the Cartesian product $T_1 \square T_2 \square T_3$ of three trees T_1, T_2, T_3 . Namely, let E_1, E_2 , and E_3 be the edges of G on the three directions of the hexagonal grid and let $G_i = (V, E \setminus E_i)$ be the graph obtained from G by removing the edges of $E_i, i = 1, 2, 3$. The tree T_i has the connected components of G_i as the set of vertices and two such connected components P and P' are adjacent in T_i if and only if there is an edge $uv \in E_i$ with one end in P and another end in P' . The isometric embedding $\varphi : V \rightarrow T_1 \square T_2 \square T_3$ maps any vertex v of G to a triplet (v_1, v_2, v_3) , where v_i is the connected component of G_i containing $v, i = 1, 2, 3$ [14]. The inner faces of a benzenoid G are called *hexagons*. They correspond to the hexagons of the underlying hexagonal grid. A path P of G is an *incomplete hexagon* if P is a path of length 3 that contains an edge from each class E_1, E_2, E_3 and P is not included in a hexagon of G .

Next we show that benzenoids are ABCE₂-graphs. As a warmup, we show that C_6 is an ABCE₂-graph. By Corollary 40, we only have to deal with the profiles π such that π° is an alternate profile. We thus need the following lemma:

Lemma 42. *If L is an ABCE₂-function on C_6 , then $L(v_i, v_{i+2}, v_{i+4}) = \{v_i, v_{i+2}, v_{i+4}\}$.*

Proof. Let $\pi = (v_i, v_{i+2}, v_{i+4})$. First, assume that $L(\pi)$ contains v_{i+1} . Let $\pi' = \pi v_{i+1}$. Then $L(\pi') = L(\pi) \cap \{v_{i+1}\} = \{v_{i+1}\}$. Since $L(v_i, v_{i+2}) \cap L(v_{i+1}, v_{i+4})$ contains v_{i+2} , we have $L(\pi') = L(v_i, v_{i+2}, v_{i+1}, v_{i+4}) = L(v_i, v_{i+2}) \cap L(v_{i+1}, v_{i+4})$ and thus $v_{i+2} \in L(\pi')$, a contradiction. For similar reasons, $v_{i+3}, v_{i+5} \notin L(\pi)$. Consequently, $L(v_i, v_{i+2}, v_{i+4}) \subseteq \{v_i, v_{i+2}, v_{i+4}\}$ and thus by (E₂), we have $L(v_i, v_{i+2}, v_{i+4}) = \{v_i, v_{i+2}, v_{i+4}\}$. \square

By Corollary 40 and Lemma 42, we obtain the following result:

Proposition 43. *The 6-cycle C_6 is an ABCE₂-graph.*

Proof. Let π be a profile on C_6 . By Corollary 40, we have to consider only the case where π° is an alternate profile. We assume without loss of generality that $\pi^\circ = (u^{k_1}, v^{k_2}, w^{k_3})$, where u, v, w are pairwise at distance 2 and $1 \leq k_1 \leq k_2 \leq k_3$. Then $\pi^\circ = (u^{k_1}, v^{k_1}, w^{k_1}, v^{k_2-k_1}, w^{k_2-k_1}, w^{k_3-k_2})$. By Lemma 42, $L(u^{k_1}, v^{k_1}, w^{k_1}) = \{u, v, w\}$. If $k_2 > k_1$, then $L(v^{k_2-k_1}, w^{k_2-k_1}) = I(v, w)$, and if $k_3 > k_2$, then $L(w^{k_3-k_2}) = \{w\}$. Consequently, $L(\pi^\circ) = \{u, v, w\}$ if $k_1 = k_2 = k_3$, $L(\pi^\circ) = \{v, w\}$ if $k_1 < k_2 = k_3$,

and $L(\pi^\circ) = \{w\}$ if $k_1 \leq k_2 < k_3$. Consequently, for any $ABCE_2$ function and any profile π , we have $L(\pi) = L(\pi^\circ) = \text{Med}(\pi^\circ) = \text{Med}(\pi)$ and Med is the unique $ABCE_2$ function on C_6 . \square

To show that benzenoids are $ABCE_2$ -graphs, we will need the following known lemma:

Lemma 44 ([10, Claim 71]). *All hexagons and incomplete hexagons in benzenoids are gated.*

Now we show that it suffices to prove that benzenoids are $ABCT_2$ -graphs:

Lemma 45. *On benzenoids, $ABCE_2$ -functions are $ABCT_2$ -functions.*

Proof. Let u, v, w be a triplet of vertices of G such that $d(u, v) = d(u, w) = d(v, w) = 2$, $I(u, v) \cap I(u, w) = \{u\}$, $I(u, v) \cap I(v, w) = \{v\}$, and $I(v, w) \cap I(w, u) = \{w\}$. Note that u, v , and w are pairwise nonadjacent vertices of some hexagon C . We assert that $L(u, v, w) \subseteq \{u, v, w\}$. Let $x \in L(u, v, w)$ and $\pi' = (x, u, v, w)$. Then $L(u, v, w, x) = L(u, v, w) \cap L(x) = \{x\}$. Since C is gated by Lemma 44, let x' be the gate of x in C . First assume that $x' \neq u, v, w$, say $x \sim u, v$. Then w is the opposite to x' vertex of C and $I(u, v) \subseteq L(x, w)$. Thus, $I(u, v) \subseteq L(u, v) \cap L(x, w) = L(\pi')$, a contradiction. Consequently, $x' \in \{u, v, w\}$, say $x' = u$. Hence u belongs to $L(x, v)$. Since u in $L(u, w)$, we conclude that $u \in L(u, v, w, x) = \{x\}$, thus $u = x \in L(u, v, w)$. By (E_2) , $\{u, v, w\} \subseteq L(u, v, w)$. \square

Lemma 46. *Let (u, v) be a 2-pair of a benzenoid G . If there exists a profile π such that F_π is not pseudopeakless on (u, v) , then u and v belong to the same hexagon of G .*

Proof. Let w be a common neighbor of u and v and let x be a vertex such that $2d(w, x) > d(u, x) + d(v, x)$. Since benzenoids are bipartite, $d(u, x) = d(v, x) = d(w, x) - 1$ and since incident edges of G are in different classes, we can suppose that $wu \in E_1$ and $wv \in E_2$. Let y be a neighbor of u in $I(u, x)$. Then uy cannot belong to E_2 , otherwise in T_2 the vertex w_2 would have two neighbors y_2 and v_2 belonging to $I(w_2, x_2)$, contrary to the fact that T_2 is a tree. Hence $uy \in E_3$. Note that (y, u, w, v) is not gated. By Lemma 44, (y, u, w, v) is not an incomplete hexagon, whence u, v belongs to the same hexagon. \square

The next lemma follows from the fact that every vertex of a hexagon C belongs to the interval between two opposite vertices of C .

Lemma 47. *Let C be a hexagon of a benzenoid G and let \bar{x} be the vertex of C opposite to the gate of a vertex x in C . Then every vertex of C belong to $L(x, \bar{x})$.*

The next lemma follows from Lemma 44:

Lemma 48. *Let C be a hexagon of a benzenoid G and let (u, v) be a 2-pair of C . Then for any vertex z of G having v as the gate in C , we have $u, v \in L(u, z) = I(u, z)$.*

The proof of the next lemma is similar to the proof of Lemma 48:

Lemma 49. *Let C be a hexagon of a benzenoid G and let u, v, w be three vertices of C at pairwise distance 2. Then for any vertex z of G that has w as the gate in C , we have $u, v \in L(u, v, z)$.*

Proof. We start with the following claim:

Claim 50. *Either $u, v, w \in L(u, v, z)$ or $u, v, w \notin L(u, v, z)$.*

Proof. The proof is similar to the proof of Claim 7, using (T_2) instead of (T) . By symmetry it suffices to show that $u \in L(u, v, z)$ if and only if $w \in L(u, v, z)$. Suppose that $\{u, w\} \cap L(u, v, z) \neq \emptyset$ and consider the profile (u, v, z, u, w) . Since $u, w \in I(u, w) = L(u, w)$ and $\{u, w\} \cap L(u, v, z) \neq \emptyset$, we have $L(u, v, z, u, w) = I(u, w) \cap L(u, v, z)$. By (T_2) , $\{u, v, w\} \subseteq L(u, v, w)$ and since $L(u, z) = I(u, z)$, we have $u, w \in L(u, v, w) \cap L(u, z) = L(u, v, z, u, w) = I(u, w) \cap L(u, v, z)$. Consequently, $u \in L(u, v, z)$ iff $w \in L(u, v, z)$. \square

If $u \in L(u, v, z)$ or $v \in L(u, v, z)$, by Claim 50 we are done. If $u, v \notin L(u, v, z)$, then let $z' \in L(u, v, z)$. By Lemma 2, $z' \in L(u, v, z')$ and by Lemma 3, $I(u, z') \cap I(v, z') = \{z'\}$. Since $d(u, v) = 2$ and G is bipartite, this implies that $d(u, z') = d(v, z')$. By Lemma 44, every C_6 is gated, and then either $z' = w$ and we are done by Claim 50, or $z' \sim u, v$. We show that the second case cannot happen by considering the profile (u, v, z, z') . If $z' \sim u, v$, then $z' = \bar{w}$ and $I(u, v) = L(u, v) \subseteq L(z', w) \subseteq L(z', z)$. Consequently, if $z' \sim u, v$, we have $z' \in L(u, v, z) \cap L(z') = \{z'\}$ and $z' \in L(u, v) \cap L(z', z) = L(u, v) = I(u, v)$. This implies that $\{z'\} = L(u, v, z, z') = I(u, v)$, a contradiction. \square

Let $C = (v_0, v_1, v_2, v_3, v_4, v_5)$ be a hexagon of G and π_C^i be the restriction of a profile π to the set of all vertices of G that have v_i as their gate in C . By convenience, we consider every index i of v_i modulo 6.

Lemma 51. *For any profile π on G , consider the profile $\pi' = \pi, v_0^{l_1+l_2+l_3+l_4}, v_2^{l_0+l_1+l_4+l_5}, v_4^{l_1}$ where $l_i = |\pi_C^i|$. Then $v_0, v_2 \in L(\pi')$.*

Proof. Let $\pi_C^i = (x_{i,1}, x_{i,2}, \dots, x_{i,l_i})$ for $i \in \{0, 1, \dots, 5\}$. By Lemma 47, $v_0, v_2 \in \bigcap_{j=1}^{l_1} L(x_{1,j}, v_4)$, $v_0, v_2 \in \bigcap_{j=1}^{l_3} L(x_{3,j}, v_0)$ and $v_0, v_2 \in \bigcap_{j=1}^{l_5} L(x_{5,j}, v_2)$. By Lemma 48, $v_0, v_2 \in \bigcap_{j=1}^{l_0} L(x_{0,j}, v_2)$ and $v_0, v_2 \in \bigcap_{j=1}^{l_2} L(x_{2,j}, v_0)$. By Lemma 49, $v_0, v_2 \in \bigcap_{j=1}^{l_4} L(x_{4,j}, v_2, v_0)$. Observe also that $v_0, v_2 \in L(v_0^{l_1}, v_2^{l_1})$. Consequently, $v_0, v_2 \in L(\pi')$. \square

Lemma 52. *Let (v_0, v_2) be a 2-pair of a hexagon $C = (v_0, v_1, \dots, v_5)$ of a benzenoid G . Then for any profile π on G , the following holds:*

- (1) *if $F_\pi(v_0) = F_\pi(v_2)$, then $v_0 \in L(\pi)$ iff $v_2 \in L(\pi)$.*
- (2) *if $F_\pi(v_0) > F_\pi(v_2)$, then $v_0 \notin L(\pi)$.*

Proof. For any $0 \leq i \leq 5$, let $l_i = |\pi_C^i|$. Assume that $F_\pi(v_0) \geq F_\pi(v_2)$ and observe that $F_\pi(v_0) - F_\pi(v_2) = 2(l_2 + l_3) - 2(l_0 + l_5)$. As in Lemma 51, consider the profile $\pi' = (\pi, v_0^{l_1+l_2+l_3+l_4}, v_2^{l_0+l_1+l_4+l_5}, v_4^{l_1})$. Suppose that $F_\pi(v_0) = F_\pi(v_2)$ (then $l_2 + l_3 + l_4 = l_0 + l_4 + l_5 = p$) and that $v_0 \in L(\pi)$. Since $v_0 \in L(v_0^p, v_2^p) = L(v_0, v_2)$, since $v_0 \in L(v_0^{l_1}, v_2^{l_1}, v_4^{l_1}) = L(v_0, v_2, v_4)$ by (T₂), and since $v_0 \in L(\pi)$, we have $L(\pi') = L(\pi) \cap L(v_0^p, v_2^p) \cap L(v_0^{l_1}, v_2^{l_1}, v_4^{l_1})$. By Lemma 51, $v_2 \in L(\pi')$, and thus $v_2 \in L(\pi)$. Suppose now that $F_\pi(v_0) > F_\pi(v_2)$ (then $q = l_2 + l_3 + l_4 > l_0 + l_4 + l_5 = r$). Since $v_0 \in L(v_0^q, v_2^r) = L(v_0, v_2)$, $v_0 \in L(v_0^{l_1}, v_2^{l_1}, v_4^{l_1}) = L(v_0, v_2, v_4)$ by (T₂) and $v_0 \in L(v_0^{q-r}) = L(v_0) = \{v_0\}$, then $L(\pi') = L(\pi) \cap L(v_0^q, v_2^r) \cap L(v_0^{l_1}, v_2^{l_1}, v_4^{l_1}) \cap L(v_0^{q-r}) = \{v_0\}$ if $v_0 \in L(\pi)$, contradicting Lemma 51. \square

Theorem 53. *Benzenoids are $ABCT_2$ -graphs and $ABCE_2$ -graphs.*

Proof. Let L be an $ABCT_2$ -function on a benzenoid graph G and let $\pi \in V^*$. We first show that $L(\pi) \subseteq \text{Med}_G(\pi)$. Pick any $u \in L(\pi)$. If $u \notin \text{Med}(\pi)$, since benzenoids have G^2 -connected medians, by Theorem 1 there exists a vertex v such that $1 \leq d(u, v) \leq 2$ and $F_\pi(v) < F_\pi(u)$. Since G is bipartite, G satisfies the triangle condition TC. Therefore, if $d(u, v) = 1$, we obtain a contradiction with Lemma 9(2). Thus, we can suppose that $d(u, v) = 2$ and that $F_\pi(w) \geq F_\pi(u)$ for any neighbor w of u . By Lemma 46 applied to the pair (u, v) , u, v belong to the same hexagon and consequently by Lemma 52(2), u cannot belong to $L(\pi)$, a contradiction.

Now we show the converse inclusion $\text{Med}(\pi) \subseteq L(\pi)$. We suppose that there exist $u \in L(\pi)$ and $v \in \text{Med}(\pi) \setminus L(\pi)$ minimizing $d(u, v)$. Since $\text{Med}(\pi)$ is G^2 -connected, $d(u, v) \leq 2$. Since $u, v \in \text{Med}(\pi)$, $F_\pi(u) = F_\pi(v)$. If $d(u, v) = 1$, we obtain a contradiction with Lemma 9(1). If $d(u, v) = 2$, by our choice of u and v , we must have $F_\pi(w) > F_\pi(u)$ for any $w \in I^\circ(u, v)$. Thus, by Lemma 46, u and v belong to the same C_6 and we obtain a contradiction with Lemma 52. Consequently, benzenoids are $ABCT_2$ -graphs and by Lemma 45, they are $ABCE_2$ -graphs. \square

6. PERSPECTIVES

In this paper, we considerably extended the classes of graphs whose median function can be characterized by a set of simple axioms. Namely, we proved that graphs with connected medians are $ABCT$ -graphs, the benzenoids graphs are $ABCT_2$ -graphs, and that modular graphs with G^2 -connected medians are ABC -graphs. One important subclass of modular graphs with G^2 -connected medians are bipartite Helly graphs. We showed that the graphs with the pairing or double-pairing property are proper subclasses of bipartite Helly graphs. Median graphs – another previously known class of ABC -graphs – are also modular and have connected medians.

The problem of characterizing ABC -graphs remains open. In view of our results, it is perhaps plausible (but dangerous) to conjecture that ABC -graphs are exactly the modular graphs with G^2 -connected medians. However, proving that ABC -graphs are bipartite or modular was out of reach for us (due to the weakness of the axioms (A), (B), and (C)). We also failed to prove that G is an ABC -graph or an $ABCT$ -graph if and only if each 2-connected component of G is. On the opposite side, one can ask if the question of deciding if G is an ABC -graph or $ABCT$ -graph is decidable. In [10] we showed that a modular graph has G^2 -connected medians if and only if this holds for each interval $I(u, v)$ with $3 \leq d(u, v) \leq 4$ and any profile π included in $I(u, v)$. Furthermore, we presented sufficient combinatorial conditions on 3- and 4-intervals under which a modular graph G has G^2 -connected medians. However, we do not know

if modular graphs with G^2 -connected medians can be characterized combinatorially by considering only profiles of bounded size included in $I(u, v)$. Bridged (and, in particular, chordal graphs) are graphs with G^2 -connected medians. However, it is open to us if they are ABCT-graphs.

We proved that the problem of deciding if a graph satisfies the double pairing property is in co-NP. However we do not know if one can decide in polynomial time (or even in non-deterministic polynomial time) if a graph satisfies the pairing or the double-pairing property.

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