

# Determinants and Limit Systems in some Idempotent and Non-Associative Algebraic Structure

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June, 2022

## Abstract

This paper considers an idempotent and symmetrical algebraic structure as well as some closely related concepts. A special notion of determinant is introduced and a Cramer formula is derived for a class of limit systems derived from the Hadamard matrix product. Thereby, some standard results arising for Max-Times systems with nonnegative entries appear as a special case. The case of two sided systems is also analyzed. In addition, a notion of eigenvalue in limit is considered. It is shown that one can construct a special semi-continuous regularized polynomial whose zeros are related to the eigenvalues of a matrix with nonnegative entries.

**AMS:** 15A80, 06D50, 06F25

**Keywords:** Idempotent algebraic structure, semilattices, determinant, Cramer's rule, generalized power-mean, Max-Times systems of equations.

## 1 Introduction

Exotic or tropical semirings such as the Max-Plus semiring, have been developed since the late fifties. They have many applications to various fields: performance evaluation of manufacturing systems; graph theory and Markov decision processes; Hamilton-Jacobi theory. However, it is well known that there is no nontrivial algebraic structures satisfying both idempotence, symmetry and having a neutral element. Despite this, there exist methods for symmetrizing an idempotent semiring imitating the familiar construction of  $\mathbb{Z}$  from  $\mathbb{N}$ , for an arbitrary semiring. Symmetrization of idempotent Semirings plays a crucial role to develop an approach in term of determinant in Max-Plus Algebra. Gaubert [19] introduced a balance relation to preserve transitivity. Familiar identities valid in rings admit analogues, replacing equalities by balances. The balance relation yields to relations similar to those arising for ordinary determinants making a lexical change. This symmetrization was invented independently by G. Hegedüs [21] and M. Plus [28]. It follows that solving linear equations in the Max-Plus semi-ring requires to solve systems of linear balances. Results concerning Cramer solutions can be found in [5].

In this paper we have taken a different point of view. We consider an idempotent algebraic structure having the symmetry property and 0 as a neutral

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We are especially grateful to an anonymous referee for very detailed comments and many helpful suggestions that greatly improved the paper.

element. The price to pay is that associativity no longer holds true. More precisely, we focus on a Max-Times algebraic structure which is derived as a limit case of the generalized power-mean that involves an homeomorphic transformation of the real field. The binary operation defined from this algebraic structure was mentioned in [20] as an exercise due to Gerard Duchamp.<sup>1</sup> Though it is not associative it admits an  $n$ -ary extension and satisfies some interesting properties. In particular, one can construct a scalar product which will play an important role in the paper. It has been shown in [8, 9] that such an algebraic structure is useful to extend a Max-Times idempotent convex structure from  $\mathbb{R}_+^n$  to the whole Euclidean vector space. The problem arising with such a cancellative algebraic structure is that it involves a natural  $n$ -ary operation which is not continuous nor associative. One can circumvent this difficulty by considering a special class of semi-continuous (upper and lower) regularized inner products analysed in [9].

The paper focusses on the asymptotic Cramer solutions of a special sequence of generalized power-linear systems. These systems are constructed from an homeomorphic transformation of the usual matrix product involving the Hadamard power for vectors and matrices.<sup>2</sup> The formula of the determinant and Cramer's rule are then derived with respect to the non-associative algebraic structure considered in [8]. Along this line, we give the algebraic form of a sequence of hyperplanes passing through a finite number of points. More importantly, a general class of limit systems is defined over  $\mathbb{R}^n$ . These limit systems involve several inequations that are derived from the semi-continuous (upper and lower) regularizations of the non-associative inner product. They include as a special case all the Max-Times systems defined from a matrix with positive entries. The Kaykobad's conditions established in [22] can then be applied to warrant the asymptotic existence of a positive solution. This algebraic structure does not require any balance relation and one can give an explicit form to some solutions of a two-sided Max-Times system. In addition, it is shown that one can construct a special polynomial to find the eigenvalues of a matrix with nonnegative entries. To do that the limit of the Perron-Frobenius eigenvalue is considered. A parallel viewpoint was adopted in [2] in a Max-Plus context. Asymptotic descriptions of Max-Times algebra eigenvalues via classical eigenvalues of Hadamard powers of nonnegative matrices have also been proposed in [6], [17], [18] and [25].

The paper unfolds as follows. We lay down the groundwork in section 2. In section 3, a suitable notion of determinant is defined with respect to this non-associative algebraic structure. Section 4 considers a class of semi-continuous regularized operators. Hence an explicit algebraic form of the limit of a sequence of generalized hyperplanes is provided. In section 5 a class of limit systems of equations is analyzed for which an explicit Cramer formula is established including the case of Max-Times systems with nonnegative entries. In addition, we provide a solution for a class of two-sided systems and we compare the balance relations and the non-associative algebraic structure used in the paper. Finally, a notion of eigenvalues in limit is analyzed and connected to the algebraic structure proposed in the paper.

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<sup>1</sup>Exercice 41, p. 25.

<sup>2</sup>A similar approach was considered in [2] modulo a logarithmic change in the variables related to the Max-Plus algebraic structure.

## 2 Preliminary Properties

### 2.1 An Idempotent and Non-Associative Algebraic Structure

For all  $p \in \mathbb{N}$ , let us consider a bijection  $\varphi_p : \mathbb{R} \rightarrow \mathbb{R}$  defined by:

$$\varphi_p : x \rightarrow x^{2p+1} \quad (2.1)$$

and  $\phi_p(x_1, \dots, x_n) = (\varphi_p(x_1), \dots, \varphi_p(x_n))$ ; this is closely related to the approach proposed by Ben-Tal [7] and Avriel [4]. One can induce a field structure on  $\mathbb{R}$  for which  $\varphi_p$  becomes a field isomorphism. Given this change of notation via  $\varphi_p$  and  $\phi_p$  we can define a  $\mathbb{R}$ -vector space structure on  $\mathbb{R}^n$  by:  $\lambda \overset{\varphi_p}{+} x = \phi_p^{-1}(\varphi_p(\lambda) \cdot \phi_p(x)) = \lambda \cdot x$  and  $x \overset{\varphi_p}{+} y = \phi_p^{-1}(\phi_p(x) + \phi_p(y))$ ; we call these two operations the indexed scalar product and the indexed sum (indexed by  $\varphi_p$ ).

The  $\varphi_p$ -sum denoted  $\sum^{\varphi_p}$  of  $(x_1, \dots, x_m) \in \mathbb{R}^{n \times m}$  is defined by<sup>3</sup>

$$\sum_{i \in [m]}^{\varphi_p} x_j = \phi_p^{-1} \left( \sum_{j \in [m]} \phi_p(x_j) \right). \quad (2.2)$$

For simplicity, throughout the paper we denote for all  $x, y \in \mathbb{R}^n$ :

$$x \overset{p}{+} y = x \overset{\varphi_p}{+} y. \quad (2.3)$$

Recall that Kuratowski-Painlevé lower limit of the sequence of sets  $\{A_n\}_{n \in \mathbb{N}}$ , denoted  $Li_{n \rightarrow \infty} A_n$ , is the set of points  $x$  for which there exists a sequence  $\{x^{(n)}\}_{n \in \mathbb{N}}$  of points such that  $x^{(n)} \in A_n$  for all  $n$  and  $x = \lim_{n \rightarrow \infty} x^{(n)}$ . The Kuratowski-Painlevé upper limit of the sequence of sets  $\{A_n\}_{n \in \mathbb{N}}$ , denoted  $Ls_{n \rightarrow \infty} A_n$ , is the set of points  $x$  for which there exists a subsequence  $\{x_{n_k}\}_{k \in \mathbb{N}}$  of points such that  $x^{(n_k)} \in A_{n_k}$  for all  $k$  and  $x = \lim_{k \rightarrow \infty} x^{(n_k)}$ . A sequence  $\{A_n\}_{n \in \mathbb{N}}$  of subsets of  $\mathbb{R}^n$  is said to converge, in the Kuratowski-Painlevé sense, to a set  $A$  if  $Ls_{n \rightarrow \infty} A_n = A = Li_{n \rightarrow \infty} A_n$ , in which case we write  $A = Lim_{n \rightarrow \infty} A_n$ .

### 2.2 A Limit Algebraic Structure

In [8] it was shown that for all  $x, y \in \mathbb{R}$  we have:

$$\lim_{p \rightarrow +\infty} x \overset{p}{+} y = \begin{cases} x & \text{if } |x| > |y| \\ \frac{1}{2}(x+y) & \text{if } |x| = |y| \\ y & \text{if } |x| < |y|. \end{cases}$$

Along this line one can introduce the binary operation  $\boxplus$  defined for all  $x, y \in \mathbb{R}$  by:

$$x \boxplus y = \lim_{p \rightarrow +\infty} x \overset{p}{+} y. \quad (2.4)$$

Though the operation  $\boxplus$  does not satisfy associativity, it can be extended by constructing a non-associative algebraic structure which returns to a given  $n$ -tuple a real value. For all  $x \in \mathbb{R}^n$  and all subsets  $I$  of  $[n]$ , let us consider the map  $\xi_I[x] : \mathbb{R} \rightarrow \mathbb{Z}$  defined for all  $\alpha \in \mathbb{R}$  by

$$\xi_I[x](\alpha) = \text{Card}\{i \in I : x_i = \alpha\} - \text{Card}\{i \in I : x_i = -\alpha\}. \quad (2.5)$$

<sup>3</sup>For all positive natural numbers  $n$ ,  $[n] = \{1, \dots, n\}$ .

This map measures the symmetry of the occurrences of a given value  $\alpha$  in the coordinates of a vector  $x$ .

For all  $x \in \mathbb{R}^n$  let  $\mathcal{J}_I(x)$  be a subset of  $I$  defined by

$$\mathcal{J}_I(x) = \left\{ j \in I : \xi_I[x](x_j) \neq 0 \right\} = I \setminus \{ i \in I : \xi_I[x](x_i) = 0 \}. \quad (2.6)$$

$\mathcal{J}_I(x)$  is called **the residual index set** of  $x$ . It is obtained by dropping from  $I$  all the  $i$ 's such that  $\text{Card}\{j \in I : x_j = x_i\} = \text{Card}\{j \in I : x_j = -x_i\}$ .

For all positive natural numbers  $n$  and for all subsets  $I$  of  $[n]$ , let  $F_I : \mathbb{R}^n \rightarrow \mathbb{R}$  be the map defined for all  $x \in \mathbb{R}^n$  by

$$F_I(x) = \begin{cases} \max_{i \in \mathcal{J}_I(x)} x_i & \text{if } \xi_I[x](\max_{i \in \mathcal{J}_I(x)} |x_i|) > 0 \\ \min_{i \in \mathcal{J}_I(x)} x_i & \text{if } \xi_I[x](\max_{i \in \mathcal{J}_I(x)} |x_i|) < 0 \\ 0 & \text{if } \xi_I[x](\max_{i \in \mathcal{J}_I(x)} |x_i|) = 0. \end{cases} \quad (2.7)$$

where  $\xi_I[x]$  is the map defined in (2.5) and  $\mathcal{J}_I(x)$  is the residual index set of  $x$ . The operation that takes an  $n$ -tuple  $(x_1, \dots, x_n)$  of  $\mathbb{R}^n$  and returns a single real element  $F_I(x_1, \dots, x_n)$  is called a  $n$ -ary extension of the binary operation  $\boxplus$  for all natural numbers  $n \geq 1$  and all  $x \in \mathbb{R}^n$ , if  $I$  is a nonempty subset of  $[n]$ . From [8], for all  $n$ -tuple  $x = (x_1, \dots, x_n)$ , one can define the operation:

$$\boxplus_{i \in I} x_i = \lim_{p \rightarrow \infty} \sum_{i \in I}^{\varphi_p} x_i = F_I(x). \quad (2.8)$$

Clearly, this operation encompasses as a special case the binary operation mentioned in [20] and defined in equation (2.2). For all  $(x_1, x_2) \in \mathbb{R}^2$ , we have:

$$\boxplus_{i \in \{1,2\}} x_i = x_1 \boxplus x_2.$$

For example, if  $x = (-3, -2, 3, 3, 1, -3)$ , we have  $F_{[6]}(-3, -2, 3, 3, 1, -3) = F_{[2]}(-2, 1) = -2 = \boxplus_{i \in [6]} x_i$ . There are some basic properties that can be inherited from the above algebraic structure:

(i) If all the elements of the family  $\{x_i\}_{i \in I}$  are mutually non symmetric, then:

$$\boxplus_{i \in I} x_i = \begin{cases} \max_{i \in I} |x_i| & \text{if } \max_{i \in I} x_i > -\min_{i \in I} x_i \\ -\max_{i \in I} |x_i| & \text{if } \max_{i \in I} x_i < -\min_{i \in I} x_i. \end{cases}$$

(ii) For all  $\alpha \in \mathbb{R}$ , one has:  $\alpha \left( \boxplus_{i \in I} x_i \right) = \boxplus_{i \in I} (\alpha x_i)$ ;

(iii) Suppose that  $x \in \epsilon \mathbb{R}_+^n$  where  $\epsilon$  is  $+1$  or  $-1$ . Then  $\boxplus_{i \in I} x_i = \epsilon \max_{i \in I} \{\epsilon x_i\}$ ;

(iv) We have  $|\boxplus_{i \in I} x_i| \leq \boxplus_{i \in I} |x_i|$ ;

(v) For all  $x \in \mathbb{R}^n$ :

$$\left[ x_i \boxplus \left( \boxplus_{j \in I \setminus \{i\}} x_j \right) \right] \in \left\{ 0, \boxplus_{j \in I} x_j \right\} \quad \text{and} \quad \boxplus_{i \in I} x_i = \boxplus_{i \in I} \left[ x_i \boxplus \left( \boxplus_{j \in I \setminus \{i\}} x_j \right) \right].$$

The algebraic structure  $(\mathbb{R}, \boxplus, \cdot)$  can be extended to  $\mathbb{R}^n$ . Suppose that  $x, y \in \mathbb{R}^n$ , and let us denote  $x \boxplus y = (x_1 \boxplus y_1, \dots, x_n \boxplus y_n)$ . Moreover, let us consider  $m$  vectors  $x_1, \dots, x_m \in \mathbb{R}^n$ , and define

$$\boxplus_{j \in [m]} x_j = \left( \boxplus_{j \in [m]} x_{j,1}, \dots, \boxplus_{j \in [m]} x_{j,n} \right). \quad (2.9)$$

The  $n$ -ary operation  $(x_1, \dots, x_n) \rightarrow \bigsqcup_{i \in [n]} x_i$  is not associative. To simplify the notations of the paper, for all  $z \in \{z_{i_1, \dots, i_m} : i_k \in I_k, k \in [m]\}$ , where  $I_1, \dots, I_m$  are  $m$  index subsets of  $\mathbb{N}$ , we use the notation:

$$\bigsqcup_{\substack{i_k \in I_k \\ k \in [m]}} z_{i_1, \dots, i_m} = \bigsqcup_{(i_1, \dots, i_m) \in \prod_{k \in [m]} I_k} z_{i_1, \dots, i_m}. \quad (2.10)$$

Notice that for all  $x \in \mathbb{R}^n$  and all  $y \in \mathbb{R}^m$ :

$$\left( \bigsqcup_{i \in [n]} x_i \right) \left( \bigsqcup_{j \in [m]} y_j \right) = \bigsqcup_{\substack{i \in [n] \\ j \in [m]}} x_i y_j. \quad (2.11)$$

This relation immediately comes from the fact that for all natural numbers  $p$ , we have:

$$\left( \sum_{i \in [n]}^{\varphi_p} x_i \right) \left( \sum_{j \in [m]}^{\varphi_p} y_j \right) = \sum_{i \in [n]}^{\varphi_p} \sum_{j \in [m]}^{\varphi_p} x_i y_j = \sum_{\substack{i \in [n] \\ j \in [m]}}^{\varphi_p} x_i y_j. \quad (2.12)$$

Taking the limit on both sides yields equation (2.11). In the remainder, we will adopt the following notational convention. For all  $x \in \mathbb{R}^n$ :

$$\bigsqcup_{i \in [n]} x_i = x_1 \boxplus \dots \boxplus x_n = F_{[n]}(x). \quad (2.13)$$

### 2.3 Scalar Product

This section presents the algebraic properties induced by the isomorphism of scalar field  $\varphi_p$  on the scalar product. Most of the results have been pointed in details by Avriel [4] and Ben Tal [7]. A norm  $\|\cdot\|$  yields another norm induced by the algebraic operations  $\boxplus$  and  $\cdot$ . The map  $\|\cdot\|_{\varphi_p} : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by  $\|x\|_{\varphi_p} = \varphi_p^{-1}(\|\phi_p(x)\|)$  is a norm over  $\mathbb{R}^n$  endowed with the operations  $\boxplus$  and  $\cdot$ . Since  $\varphi_p$  is continuous over  $\mathbb{R}$ , the topological structure is the same. Along this line it is natural to define a scalar product. If  $\langle \cdot, \cdot \rangle$  is an inner product over  $\mathbb{R}^n$ , then there exists a symmetric bilinear form  $\langle \cdot, \cdot \rangle_{\varphi_p} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  defined by:

$$\langle x, y \rangle_{\varphi_p} = \varphi_p^{-1}(\langle \phi_p(x), \phi_p(y) \rangle) = \left( \sum_{i \in [n]} x_i^{2p+1} y_i^{2p+1} \right)^{\frac{1}{2p+1}}. \quad (2.14)$$

Now, let us denote  $[\langle y, \cdot \rangle_{\varphi_p} \leq \lambda] = \{x \in \mathbb{R}^n : \langle y, x \rangle_{\varphi_p} \leq \lambda\}$  and let  $\langle \cdot, \cdot \rangle_p$  stands for this scalar product.

In the following we introduce the operation  $\langle \cdot, \cdot \rangle_{\infty} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  defined for all  $x, y \in \mathbb{R}^n$  by  $\langle x, y \rangle_{\infty} = \bigsqcup_{i \in [n]} x_i y_i$ . Let  $\|\cdot\|_{\infty}$  be the Tchebychev norm defined by  $\|x\|_{\infty} = \max_{i \in [n]} |x_i|$ . It is established in [8] that for all  $x, y \in \mathbb{R}^n$ , we have: (i)  $\sqrt{\langle x, x \rangle_{\infty}} = \|x\|_{\infty}$ ; (ii)  $|\langle x, y \rangle_{\infty}| \leq \|x\|_{\infty} \|y\|_{\infty}$ ; (iii) For all  $\alpha \in \mathbb{R}$ ,  $\alpha \langle x, y \rangle_{\infty} = \langle \alpha x, y \rangle_{\infty} = \langle x, \alpha y \rangle_{\infty}$ . By definition, we have for all  $x, y \in \mathbb{R}^n$ :

$$\langle x, y \rangle_{\infty} = \lim_{p \rightarrow \infty} \langle x, y \rangle_p \quad (2.15)$$

## 3 Limit of Linear Operators and Determinant

This section is devoted to study the matrix representation of a linear operator defined on the scalar field  $(\mathbb{R}, \boxplus, \cdot)$ . Along this line some limit properties are derived to establish several results in closed algebraic form when  $p \rightarrow \infty$ .

### 3.1 $\varphi_p$ -linear Endomorphisms

Let  $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$  denotes the set of all the linear morphisms defined from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ .  $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$  is then the set of all the linear endomorphisms defined over  $\mathbb{R}^n$ . In the following, we say that a map  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is  $\varphi_p$ -linear if for all  $\lambda \in \mathbb{R}$ ,  $f(\lambda x + y) = \lambda f(x) + f(y)$ . Moreover, for all natural numbers  $p$ , let  $\mathcal{L}^{(p)}(\mathbb{R}^n, \mathbb{R}^n)$  denotes the set of all the  $\varphi_p$ -linear endomorphisms.

Let  $\mathcal{M}_n(\mathbb{R})$  denotes the set of all the  $n \times n$  matrices defined over  $\mathbb{R}$ . Let  $\Phi_p : \mathcal{M}_n(\mathbb{R}) \rightarrow \mathcal{M}_n(\mathbb{R})$  be the map defined for any matrix  $A = (a_{i,j})_{\substack{i=1\dots n \\ j=1\dots n}} \in \mathcal{M}_n(\mathbb{R})$  as:

$$\Phi_p(A) = (\varphi_p(a_{i,j}))_{\substack{i=1\dots n \\ j=1\dots n}} = (a_{i,j}^{2p+1})_{\substack{i=1\dots n \\ j=1\dots n}}. \quad (3.1)$$

Its reciprocal is the map  $\Phi_p^{-1} : \mathcal{M}_n(\mathbb{R}) \rightarrow \mathcal{M}_n(\mathbb{R})$  defined by:

$$\Phi_p^{-1}(A) = (\varphi_p^{-1}(a_{i,j}))_{\substack{i=1\dots n \\ j=1\dots n}} = (a_{i,j}^{\frac{1}{2p+1}})_{\substack{i=1\dots n \\ j=1\dots n}}. \quad (3.2)$$

$\Phi_p$  is a natural extension of the map  $\phi_p$  from  $\mathbb{R}^n$  to  $\mathcal{M}_n(\mathbb{R})$ .  $\Phi_p(A)$  is the  $2p+1$  Hadamard power of matrix  $A$ . In the following we introduce the matrix product:

$$A \cdot^p x = \sum_{j \in [n]}^{\varphi_p} x_j \cdot a^j, \quad (3.3)$$

where  $a^j$  stands for the  $j$ -th column of  $A$ . It is straightforward to show that this formulation is equivalent to the following:

$$A \cdot^p x = \phi_p^{-1}(\Phi_p(A) \cdot \phi_p(x)). \quad (3.4)$$

Another equivalent formulation involves the inner product  $\langle \cdot, \cdot \rangle_p$ :

$$A \cdot^p x = \sum_{i \in [n]}^{\varphi_p} \langle a_i, x \rangle_p e_i, \quad (3.5)$$

where  $a_i$  is the  $i$ -th line of matrix  $A$  and  $\{e_i\}_{i \in [n]}$  is the canonical basis of  $\mathbb{R}^n$ .

It is easy to see that the map  $x \mapsto A \cdot^p x$  is  $\varphi_p$ -linear. Conversely, if  $g$  is a  $\varphi_p$ -linear map then it can be represented by a matrix  $A$  such that  $g(x) = A \cdot^p x$  for all  $x \in \mathbb{R}^n$ . If  $A, B \in \mathcal{M}_n(\mathbb{R})$ , the product  $B \cdot^p A$  is the matrix representation of the map:

$$x \mapsto B \cdot^p A \cdot^p x = \phi_p^{-1}(\Phi_p(B) \Phi_p(A) \phi_p(x)). \quad (3.6)$$

Notice that the identity matrix  $I$  is invariant with respect to  $\Phi_p$ .

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear endomorphism and let  $A$  be its matrix representation in the canonical basis. The map  $T^{(p)} : \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathcal{L}^{(p)}(\mathbb{R}^n, \mathbb{R}^n)$  defined for all  $x \in \mathbb{R}^n$  by:

$$T^{(p)}(f)(x) = f^{(p)}(x) := \phi_p^{-1}(\Phi_p(A) \phi_p(x))$$

is called **the  $\varphi_p$ -linear transformation of  $f$** .

A  $\varphi_p$ -linear endomorphism  $g$  is invertible if and only if  $\Phi_p(A)$  is invertible. For any  $n \times n$  matrix  $A$ , let  $|A|$  denotes its determinant. Let us introduce the following definition of a  $\varphi_p$ -**determinant**

$$|A|_p = \varphi_p^{-1} |\Phi_p(A)|. \quad (3.7)$$

Let  $S_n$  be the set of all the permutations defined on  $[n]$ . The Leibnitz formula yields

$$|A|_p = \left( \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i \in [n]} a_{i, \sigma(i)}^{2p+1} \right)^{\frac{1}{2p+1}}. \quad (3.8)$$

In the remainder, if  $A$  is the matrix of a linear endomorphism  $f$ , then we define the  $\varphi_p$ -determinant of  $f$  as  $|f|_p = |A|_p$ . If  $f^{(p)}$  is invertible, then we have the equivalences:

$$y = f^{(p)}(x) \iff y = \phi_p^{-1}(\Phi_p(A)\phi_p(x)) \iff \phi_p^{-1}(\Phi_p(A)^{-1}\phi_p(y)) = x. \quad (3.9)$$

Along this line, the  $\varphi_p$ -cofactor matrix  $A^{*,p}$  is defined as:

$$A^{*,p} = (a_{i,j}^{*,p})_{\substack{i \in [n] \\ j \in [n]}} = \left( (-1)^{i+j} |A_{i,j}|_p \right)_{\substack{i \in [n] \\ j \in [n]}}, \quad (3.10)$$

where  $A_{i,j}$  is obtained from matrix  $A$  by dropping line  $i$  and column  $j$ . The  $\phi_p$ -inverse matrix of a  $\varphi_p$ -invertible matrix  $A$  (such that  $|A|_p \neq 0$ ) is then defined as:

$$A^{-1,p} = \frac{1}{|A|_p} {}^t A^{*,p}, \quad (3.11)$$

where  ${}^t A$  denotes the transpose of  $A$ . Suppose that  $f$  is a linear endomorphism having a matrix representation  $A$  in the canonical basis and let  $b \in \mathbb{R}^n$ . Given a system of  $\varphi_p$ -linear equations of the form:

$$f^{(p)}(x) = b \iff A \cdot^p x = b, \quad (3.12)$$

if  $|A|_p \neq 0$ , then the solution is  $x^* = A^{-1,p} \cdot^p b$ .

## 3.2 Limit Properties

**Proposition 3.2.1** *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear endomorphism having a matrix representation  $A$  in the canonical basis of  $\mathbb{R}^n$ . For all  $p \in \mathbb{N}$ , let  $f^{(p)}$  be its  $\varphi_p$ -linear transformation. Then:*

$$\lim_{p \rightarrow \infty} f^{(p)}(x) = \bigsqcup_{j \in [n]} x_j a^j = \bigsqcup_{i \in [n]} \langle a_i, x \rangle_{\infty} e_i,$$

where  $a_i$  and  $a^j$  respectively mean the  $i$ -th row and the  $j$ -th column of  $A$ .

**Proof:** For all  $i \in [n]$ , we have from [8]:

$$\lim_{p \rightarrow \infty} \sum_{j \in [n]}^{\varphi_p} x_j a_{i,j} = \lim_{p \rightarrow \infty} \left( \sum_{j \in [n]} x_j^{2p+1} a_{i,j}^{2p+1} \right)^{\frac{1}{2p+1}} = \bigsqcup_{j \in [n]} x_j a_{i,j} = \langle a_i, x \rangle_{\infty}.$$

Therefore

$$\lim_{p \rightarrow \infty} \sum_{j \in [n]}^{\varphi_p} x_j a^j = \bigsqcup_{j \in [n]} x_j a^j.$$

The last equality immediately follows.  $\square$

For any squared matrix  $A$ ,

$$|A|_{\infty} = \bigsqcup_{\sigma \in S_n} (\text{sgn}(\sigma) \prod_{i \in [n]} a_{i, \sigma(i)}) \quad (3.13)$$

is called **the determinant in limit of  $A$** . Moreover, for any linear endomorphism  $f$  whose matrix is  $A$ , the determinant in limit of  $f$  is defined as  $|f|_{\infty} = |A|_{\infty}$ .

**Proposition 3.2.2** For all  $A \in \mathcal{M}_n(\mathbb{R})$ , we have:

$$\lim_{p \rightarrow \infty} |A|_p := |A|_\infty = \bigsqcup_{\sigma \in S_n} \left( \text{sgn}(\sigma) \prod_{i \in [n]} a_{i, \sigma(i)} \right).$$

**Proof:** From [8] we have:

$$\lim_{p \rightarrow \infty} \left( \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i \in [n]} a_{i, \sigma(i)}^{2p+1} \right)^{\frac{1}{2p+1}} = \bigsqcup_{\sigma \in S_n} \left( \text{sgn}(\sigma) \prod_{i \in [n]} a_{i, \sigma(i)} \right). \quad \square$$

In Max-Plus algebra the term determinant of a matrix  $A$  is often used for a max-permanent (see [13], page. 30). To that end, we consider a notion of Max-Times permanent which is a multiplicative version of the notion of permanent used in a Max-Plus context. For all matrices  $A \in \mathcal{M}_n(\mathbb{R}_+)$ , the Max-Times permanent of  $A$  is defined as:

$$\text{per}(A) = \bigsqcup_{\sigma \in S_n} \prod_{i \in [n]} a_{i, \sigma(i)} = \max_{\sigma \in S_n} \prod_{i \in [n]} a_{i, \sigma(i)}. \quad (3.14)$$

The main difference, compared to the definition of limit determinants, is that the terms of the equation are not alternated. This is a multiplicative version of the notion of permanent arising in Max-Plus algebra that can be derived from an exponential transformation. This version of the notion of permanent is particularly useful for examining the geometric and algebraic eigenvalues of a non-negative matrix. This question is analysed in Section 6.

**Proposition 3.2.3** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear endomorphism having a matrix representation  $A$ . For all  $p \in \mathbb{N}$ , let  $f^{(p)}$  be its  $\varphi_p$ -linear transformation. If  $|A|_\infty \neq 0$ , then there is some  $p_0 \in \mathbb{N}$  such that for all  $p \geq p_0$ ,  $f^{(p)}$  is  $\varphi_p$ -invertible and for all  $b \in \mathbb{R}^n$ , there exists a solution  $x^{(p)}$  to the system  $A \cdot^p x = b$  with:

$$x_i^{(p)} = \frac{|A^{(i)}|_p}{|A|_p} = \frac{|\Phi_p(A^{(i)})|_{\frac{1}{2p+1}}}{|\Phi_p(A)|_{\frac{1}{2p+1}}},$$

where  $A^{(i)}$  is obtained from  $A$  by dropping column  $i$  and replacing it with  $b$ . Moreover, we have:

$$\lim_{p \rightarrow \infty} x^{(p)} = x^*,$$

and for all  $i \in [n]$

$$x_i^* = \frac{|A^{(i)}|_\infty}{|A|_\infty}.$$

**Proof:** Since  $|A|_\infty \neq 0$  and  $\lim_{p \rightarrow \infty} |A|_p = |A|_\infty$ , there is some  $p_0$  such that for all  $p \geq p_0$ ,  $|A|_p \neq 0$ , which implies that  $A$  is  $\varphi_p$ -invertible. In such a case, there exists a unique solution to the system  $A \cdot^p x = b$ , that is  $x^{(p)} = A^{-1 \cdot p} b$ . Moreover, we have:

$$A \cdot^p x = b \iff \phi_p^{-1}(\Phi_p(A)\phi_p(x)) = b \iff \Phi_p(A)\phi_p(x) = \phi_p(b).$$

Since  $f^{(p)}$  is  $\varphi_p$ -invertible, it follows that  $\Phi_p(A)$  is invertible. Set  $u = \phi_p(x)$ . The system  $\Phi_p(A)u = \phi_p(b)$  has a solution for all  $p \geq p_0$ . Applying the Cramer's rule the solution is the vector  $u^p$  satisfying the relation:

$$u^{(p)} = \frac{|\Phi_p(A)^{(i)}|}{|\Phi_p(A)|} = \frac{|\Phi_p(A^{(i)})|}{|\Phi_p(A)|}.$$



Setting  $x^{(p)} = \phi_p^{-1}(u^{(p)})$ , we obtain the result. From Proposition 3.2.2,  $\lim_{p \rightarrow \infty} |A^{(i)}|_p = |A^{(i)}|_\infty$  and  $\lim_{p \rightarrow \infty} |A|_p = |A|_\infty$ , which ends the proof.  $\square$

The next properties are useful. We first establish the following Lemma.

**Lemma 3.2.4** *Suppose that there is some  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  such that  $\bigoplus_{i \in [n]} x_i = 0$ . Then for all  $p \in \mathbb{N}$ ,  $\sum_{i \in [n]}^{\varphi_p} x_i = 0$ . Moreover, for all matrices  $A \in \mathcal{M}_n(\mathbb{R})$ , if  $|A|_\infty = 0$  then  $|A|_p = 0$  for all  $p \in \mathbb{N}$ .*

**Proof:** Let  $\Lambda[x] = \{\alpha \in \mathbb{R}_+ : |x_i| = \alpha, i \in [n]\}$ . Since  $\bigoplus_{i \in [n]} x_i = 0$ , we have for all  $\alpha$ ,  $\xi[x](\alpha) = \text{Card}\{i : x_i = \alpha\} - \text{Card}\{i : x_i = -\alpha\} = 0$ . Hence, for all  $\alpha \in \Lambda[x]$ ,  $\sum_{|x_i|=\alpha}^{\varphi_p} x_i = 0$ . Thus

$$\sum_{i \in [n]}^{\varphi_p} x_i = \sum_{\alpha \in \Lambda[x]}^{\varphi_p} \sum_{|x_i|=\alpha}^{\varphi_p} x_i = 0.$$

The second part of the statement is an immediate consequence of the Leibniz formula.  $\square$

For all  $p \in \mathbb{N} \cup \{\infty\}$  and for all matrices  $A \in \mathcal{M}_n(\mathbb{R})$ , let us denote  $|a^1, \dots, a^n|_p = |A|_p$  where the  $a^j$ 's are the column vectors of  $A$ .

**Proposition 3.2.5** *For all  $A \in \mathcal{M}_n(\mathbb{R})$ , we have the following properties.*

- (a) For all  $\alpha \in \mathbb{R}$ ,  $|a^1, \dots, \alpha a^j, \dots, a^n|_\infty = \alpha |A|_\infty$ ;
- (b) For all permutations  $\sigma$  of  $S_n$ ,  $|a^{\sigma(1)}, \dots, a^{\sigma(n)}|_\infty = \text{sgn}(\sigma) |A|_\infty$ ;
- (c) If there exists  $\alpha \in \mathbb{R}^n \setminus \{0\}$  such that  $\bigoplus_{j \in [n]} \alpha_j a^j = 0$  then  $|A|_\infty = 0$ .
- (d) If  $|A|_\infty = 0$  then there exists a sequence  $\{\alpha^{(p)}\}_{p \in \mathbb{N}} \subset \mathbb{R}^n \setminus \{0\}$  such that  $\sum_{j \in [n]}^p \alpha_j^{(p)} a^j = 0$  for all  $p \in \mathbb{N}$ .

**Proof:** (a) Since  $|A|_p = \varphi_p^{-1}(|\Phi_p(A)|)$ , we deduce that for all natural numbers  $p$ ,  $|a^1, \dots, \alpha a^j, \dots, a^n|_p = \alpha |a^1, \dots, a^n|_p$ . Taking the limit yields the result. (b) Similarly, for all permutations  $\sigma \in S_n$ ,  $|a^{\sigma(1)}, \dots, a^{\sigma(n)}|_p = \text{sgn}(\sigma) |A|_p$ , which yields (b), by taking the limit. (c) If there exists  $\alpha \in \mathbb{R}^n \setminus \{0\}$  such that  $\bigoplus_{j \in [n]} \alpha_j a^j = 0$ , from Lemma 3.2.4, then we deduce that for all natural numbers  $p$ ,  $\sum_{j \in [n]}^p \alpha_j^{(p)} a^j = 0$ . However, this implies that  $|A|_p = 0$  for all  $p$ . Hence  $|A|_\infty = \lim_{p \rightarrow \infty} |A|_p = 0$ . (d) If  $|A|_\infty = 0$ , then  $\bigoplus_{\sigma \in S_n} (\text{sgn}(\sigma) \prod_{i \in [n]} a_{i, \sigma(i)}) = 0$ . Thus, from Lemma 3.2.4

$$\sum_{\sigma \in S_n}^{\varphi_p} (\text{sgn}(\sigma) \prod_{i \in [n]} a_{i, \sigma(i)}) = 0$$

for all natural numbers  $p$ . Hence, for all  $p$ , there is  $\alpha^{(p)} \in \mathbb{R}^n \setminus \{0\}$  such that  $\sum_{j \in [n]}^p \alpha_j^{(p)} a^j = 0$ , which ends the proof.  $\square$

Determinants are intimately linked to the exterior product of vectors that is an algebraic construction used to study areas, volumes, and their higher-dimensional analogues. Paralleling the earlier definitions, a map  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is called a  $\varphi_p$ -**multilinear form** if it is  $\varphi_p$ -linear in each argument. A  $\varphi_p$ -multilinear form is alternating if for each permutation  $\sigma \in S_n$  we have

$f(x_1, \dots, x_n) = \text{sgn}(\sigma)f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ . For all natural numbers  $r$  an alternating  $\varphi_p$ -linear  $r$ -form is a map defined for all  $x_1, x_2, \dots, x_r \in \mathbb{R}^n$  as:

$$(f_1 \overset{p}{\wedge} f_2 \overset{p}{\wedge} \dots \overset{p}{\wedge} f_r)(x_1, \dots, x_r) = \sum_{\sigma \in S_r}^{\varphi_p} \text{sign}(\sigma) f_1^{(p)}(x_{\sigma(1)}) \dots f_r^{(p)}(x_{\sigma(r)}), \quad (3.15)$$

where for any  $i$ ,  $f_i$  is a linear form and, and  $f_i^{(p)}$  is the corresponding  $\varphi_p$ -transformation.  $\overset{p}{\wedge}$  is called the  $\varphi_p$ -exterior product of the linear forms  $f_1, \dots, f_r$ . Let  $\{e_1^*, \dots, e_n^*\}$  be the canonical basis of the dual space  $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$ . Suppose that  $r = n$  and let  $f = \sum_{i \in [n]} f_i(e_i) e_i^*$  be the linear endomorphism constructed from  $f_1, \dots, f_n$ .

**Proposition 3.2.6** *Let us consider  $n$  linear forms  $f_1, \dots, f_n$ . Then for all  $x_1, x_2, \dots, x_n \in \mathbb{R}^n$ , we have*

$$(f_1 \overset{p}{\wedge} f_2 \overset{p}{\wedge} \dots \overset{p}{\wedge} f_n)(x_1, \dots, x_n) = |f|_p (e_1^* \overset{p}{\wedge} e_2^* \overset{p}{\wedge} \dots \overset{p}{\wedge} e_n^*)(x_1, \dots, x_n).$$

Moreover, we have:

$$\lim_{p \rightarrow \infty} (f_1 \overset{p}{\wedge} f_2 \overset{p}{\wedge} \dots \overset{p}{\wedge} f_n)(x_1, \dots, x_n) = |f|_\infty (e_1^* \overset{\infty}{\wedge} e_2^* \overset{\infty}{\wedge} \dots \overset{\infty}{\wedge} e_n^*)(x_1, \dots, x_n),$$

where

$$(e_1^* \overset{\infty}{\wedge} e_2^* \overset{\infty}{\wedge} \dots \overset{\infty}{\wedge} e_n^*)(x_1, \dots, x_n) = \lim_{p \rightarrow \infty} (e_1^* \overset{p}{\wedge} e_2^* \overset{p}{\wedge} \dots \overset{p}{\wedge} e_n^*)(x_1, \dots, x_n) = |x_1, \dots, x_n|_\infty.$$

**Proof:** Suppose that for  $i = 1, \dots, n$  there is a vector  $a_i \in \mathbb{R}^n$  such that  $f_i(x) = \langle a_i, x \rangle$ . Then  $f_i^{(p)}(x) = \varphi_p^{-1}(\langle \phi_p(a_i), \phi_p(x) \rangle) = (\sum_{i \in [n]} a_i^{2p+1} x_i^{2p+1})^{\frac{1}{2p+1}}$ . It follows that:

$$\begin{aligned} (f_1 \overset{p}{\wedge} f_2 \overset{p}{\wedge} \dots \overset{p}{\wedge} f_n)(x_1, \dots, x_n) &= \left( \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i \in [n]} \langle \phi_p(a_i), \phi_p(x_{\sigma(i)}) \rangle \right)^{\frac{1}{2p+1}} \\ &= \left( \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i \in [n]} a_i^{2p+1} x_{\sigma(i)}^{2p+1} \right)^{\frac{1}{2p+1}}. \end{aligned}$$

For each  $i$ , let  $g_i^{(p)} : \mathbb{R}^n \rightarrow \mathbb{R}$  be the linear form defined by  $g_i^{(p)}(z) = \langle \phi_p(a_i), z \rangle$ . It follows that:

$$(f_1 \overset{p}{\wedge} f_2 \overset{p}{\wedge} \dots \overset{p}{\wedge} f_n)(x_1, \dots, x_n) = \varphi_p^{-1} \left( (g_1^{(p)} \wedge g_2^{(p)} \wedge \dots \wedge g_n^{(p)})(\phi_p(x_1), \dots, \phi_p(x_n)) \right).$$

Let  $\{e_1^*, \dots, e_n^*\}$  be the canonical basis of the dual space  $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$ . From the usual properties of an alternating  $n$ -form we deduce that:

$$(g_1^{(p)} \wedge g_2^{(p)} \wedge \dots \wedge g_n^{(p)})(\phi_p(x_1), \dots, \phi_p(x_n)) = |\Phi_p(A)| (e_1^* \wedge e_2^* \wedge \dots \wedge e_n^*)(\phi_p(x_1), \dots, \phi_p(x_n)),$$

where  $\Phi_p(A)$  is the matrix whose line  $i$  is the vector  $\phi_p(a_i)$ . Since for all  $i$  and all  $x \in \mathbb{R}^n$  we have  $\langle e_i, x \rangle_p = x_i$ , this canonical basis is also, independently of  $p$ , the canonical basis of  $\mathcal{L}_p(\mathbb{R}^n, \mathbb{R})$ . Since  $|A|_p = \varphi_p^{-1}(|\Phi_p(A)|)$ , it follows that:

$$\begin{aligned} (f_1^{(p)} \overset{p}{\wedge} f_2^{(p)} \overset{p}{\wedge} \dots \overset{p}{\wedge} f_n^{(p)})(x_1, \dots, x_n) &= |A|_p \left( \sum_{i \in [n]} \langle e_i, \phi_p(x) \rangle \right)^{\frac{1}{2p+1}} \\ &= |A|_p (e_1^* \overset{p}{\wedge} e_2^* \overset{p}{\wedge} \dots \overset{p}{\wedge} e_n^*)(x_1, \dots, x_n). \end{aligned}$$

However, we have

$$(e_1^* \overset{p}{\wedge} e_2^* \overset{p}{\wedge} \cdots \overset{p}{\wedge} e_n^*)(x_1, \dots, x_n) = |x_1, \dots, x_n|_p.$$

We then obtain the final result taking the limit.  $\square$

For all  $(f_1, f_2, \dots, f_n) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})^n$ , let  $f_1 \overset{\infty}{\wedge} f_2 \overset{\infty}{\wedge} \cdots \overset{\infty}{\wedge} f_n$  denote the pointwise limit of the sequence  $\{f_1 \overset{p}{\wedge} f_2 \overset{p}{\wedge} \cdots \overset{p}{\wedge} f_n\}_{n \in \mathbb{N}}$ . From Proposition 3.2.6, we have:

$$f_1 \overset{\infty}{\wedge} f_2 \overset{\infty}{\wedge} \cdots \overset{\infty}{\wedge} f_n = |A|_{\infty} (e_1^* \overset{\infty}{\wedge} e_2^* \overset{\infty}{\wedge} \cdots \overset{\infty}{\wedge} e_n^*). \quad (3.16)$$

Consequently, since primal and dual spaces are isomorphic, one can define for all  $v_1, v_2, \dots, v_n \in \mathbb{R}^n$  the exterior product:

$$(v_1 \overset{\infty}{\wedge} v_2 \overset{\infty}{\wedge} \cdots \overset{\infty}{\wedge} v_n) = |v_1, v_2, \dots, v_n|_{\infty} (e_1 \overset{\infty}{\wedge} e_2 \overset{\infty}{\wedge} \cdots \overset{\infty}{\wedge} e_n). \quad (3.17)$$

Notice however, that although this definition extends as a limit case the usual definition of exterior product, it does not satisfy the additivity property in each arguments with respect to the operation  $\boxplus$ .

## 4 Semi-continuous Regularization and Limit of Hyperplanes

### 4.1 Semi-continuous Regularizations

In the following, we say that a map  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a **B-form** if there exists some  $a \in \mathbb{R}^n$  such that:

$$f(x) = \boxplus_{i \in [n]} a_i x_i = \langle a, x \rangle_{\infty}. \quad (4.1)$$

The function above is depicted in Figure 2.2.2.

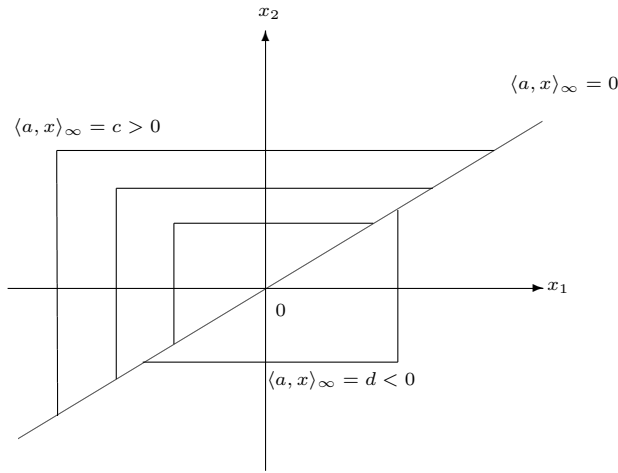


Figure 4.1.1.1: The level lines of the form  $\langle a, \cdot \rangle_{\infty}$

These functions were used in [8, 9] to establish a separation theorem for  $\mathbb{B}$ -convex sets [11].<sup>4</sup> All the points such that  $\langle a, x \rangle_{\infty} = 0$  are represented by the diagonal

<sup>4</sup>A relaxed definition of  $\mathbb{B}$ -convexity was proposed in [8]: a subset  $C$  of  $\mathbb{R}^n$  is  $\mathbb{B}^{\sharp}$ -convex if for all  $x, y \in C$  and all  $t \in [0, 1]$ ,  $x \boxplus ty \in C$ .

line. In the following, for all subsets  $E$  of  $\mathbb{R}^n$   $\text{cl}(E)$  and  $\text{int}(E)$  respectively stand for the closure and the interior of  $E$ .

For all maps  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  and all real numbers  $c$ , the notation  $[f \leq c]$  stands for the set  $f^{-1}([-\infty, c])$ . Similarly,  $[f < c]$  stands for  $f^{-1}([-\infty, c])$  and  $[f \geq c] = [-f \leq -c]$ .

For all  $u, v \in \mathbb{R}$ , let us define the binary operation

$$u \ominus v = \begin{cases} u & \text{if } |u| > |v| \\ \min\{u, v\} & \text{if } |u| = |v| \\ v & \text{if } |u| < |v|. \end{cases}$$

An elementary calculus shows that  $u \boxplus v = \frac{1}{2}[u \ominus v - [(-u) \ominus (-v)]]$ .

Similarly one can introduce a symmetrical binary operation defined for all  $u, v \in \mathbb{R}$  defined as:

$$u \oplus v = \begin{cases} u & \text{if } |u| > |v| \\ \max\{u, v\} & \text{if } |u| = |v| \\ v & \text{if } |u| < |v|. \end{cases}$$

Equivalently, one has:  $u \oplus v = -[(-u) \ominus (-v)]$ . This means that  $u \boxplus v = \frac{1}{2}[(u \ominus v) + (u \oplus v)]$ . Notice that the operations  $\ominus$  and  $\oplus$  are associative. Given  $m$  elements  $u_1, \dots, u_m$  of  $\mathbb{R}$ , not all of which are 0, let  $I_+$ , respectively  $I_-$ , be the set of indices for which  $0 < u_i$ , respectively  $u_i < 0$ . We can then write  $u_1 \ominus \dots \ominus u_m = (\ominus_{i \in I_+} u_i) \ominus (\ominus_{i \in I_-} u_i) = (\max_{i \in I_+} u_i) \ominus (\min_{i \in I_-} u_i)$  from which we have:

$$u_1 \ominus \dots \ominus u_m = \begin{cases} \max_{i \in I_+} u_i & \text{if } I_- = \emptyset \text{ or } \max_{i \in I_-} |u_i| < \max_{i \in I_+} u_i \\ \min_{i \in I_-} u_i & \text{if } I_+ = \emptyset \text{ or } \max_{i \in I_+} u_i < \max_{i \in I_-} |u_i| \\ \min_{i \in I_-} u_i & \text{if } \max_{i \in I_-} |u_i| = \max_{i \in I_+} u_i. \end{cases} \quad (4.2)$$

We define a **lower  $\mathbb{B}$ -form** on  $\mathbb{R}_+^n$  as a map  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  such that for all  $(x_1, \dots, x_n) \in \mathbb{R}_+^n$ ,

$$g(x_1, \dots, x_n) = \langle a, x \rangle_\infty^- = a_1 x_1 \ominus \dots \ominus a_n x_n. \quad (4.3)$$

It was established in [8] that for all  $c \in \mathbb{R}$ ,  $g^{-1}([-\infty, c]) = \{x \in \mathbb{R}^n : g(x) \leq c\}$  is closed. It follows that a  $\mathbb{B}$ -form is lower semi-continuous. It was established in [12] that  $g^{-1}([-\infty, c]) \cap \mathbb{R}_+^n$  is a  $\mathbb{B}$ -halfspace, that is a  $\mathbb{B}$ -convex subset of  $\mathbb{R}_+^n$  whose complement in  $\mathbb{R}_+^n$  is also  $\mathbb{B}$ -convex.

Similarly, one can define an **upper  $\mathbb{B}$ -form** as a map  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  such that, for all  $(x_1, \dots, x_n) \in \mathbb{R}^n$ ,

$$h(x_1, \dots, x_n) = \langle a, x \rangle_\infty^+ = a_1 x_1 \oplus \dots \oplus a_n x_n. \quad (4.4)$$

For all  $x \in \mathbb{R}^n$ , we clearly, have the following identities

$$\langle a, x \rangle_\infty^+ = -\langle a, -x \rangle_\infty^- \text{ and } \langle a, x \rangle_\infty^- = -\langle a, -x \rangle_\infty^+. \quad (4.5)$$

The largest (smallest) lower (upper) semi-continuous minorant (majorant) of a map  $f$  is said to be the lower (upper) semi-continuous regularization of  $f$ . In the next statements it is shown that the lower (upper)  $\mathbb{B}$ -forms are the lower (upper) semi-continuous regularizations of the  $\mathbb{B}$ -forms.

**Proposition 4.1.1** [8] *Let  $g$  be a lower  $\mathbb{B}$ -form defined by  $g(x_1, \dots, x_n) = a_1 x_1 \ominus \dots \ominus a_n x_n$ , for some  $a \in \mathbb{R}^n$ . Then  $g$  is the lower semi-continuous regularization of the map  $x \mapsto \langle a, x \rangle_\infty = \boxplus_{i \in [n]} a_i x_i$ .*

The following corollary is then immediate.

**Corollary 4.1.2** [8] *Let  $h$  be an upper  $\mathbb{B}$ -form defined by  $h(x_1, \dots, x_n) = a_1 x_1 \overset{+}{\smile} \dots \overset{+}{\smile} a_n x_n$ , for some  $a \in \mathbb{R}^n$ . Then  $h$  is the upper semi-continuous regularization of the map  $x \mapsto \langle a, x \rangle_\infty = \bigsqcup_{i \in [n]} a_i x_i$ .*

For the sake of simplicity let us denote for all  $x \in \mathbb{R}^n$ :

$$\overset{-}{\smile}_{i \in [n]} x_i = x_1 \overset{-}{\smile} \dots \overset{-}{\smile} x_n \quad \text{and} \quad \overset{+}{\smile}_{i \in [n]} x_i = x_1 \overset{+}{\smile} \dots \overset{+}{\smile} x_n. \quad (4.6)$$

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a  $\mathbb{B}$ -form. Let  $f^-$  and  $f^+$  be respectively the lower and upper semi-continuous regularizations of  $f$  over  $\mathbb{R}^n$ . It is shown in [9] that for all  $c \in \mathbb{R}$ ,

$$\text{cl} [f \leq c] = [f^- \leq c] \quad \text{and} \quad \text{cl} [f \geq c] = [f^+ \geq c]. \quad (4.7)$$

The following lemma is useful. A map  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is called a dual  $\mathbb{B}$ -form if there exists some  $a \in \mathbb{R}^n$  such that, for all  $x \in \mathbb{R}^n$ ,  $f(x) = \langle a, x \rangle_\infty$ .

**Lemma 4.1.3** *For all dual  $\mathbb{B}$ -forms  $f$  we have*

$$[f^- \leq 0] \cap [f^+ \geq 0] = [f^- + f^+ = 0].$$

**Proof:** Suppose that  $x \in [f^- \leq 0] \cap [f^+ \geq 0]$ . If  $f^-(x) = f^+(x) = 0$ , the inclusion is trivial. Suppose now that  $f^-(x) < 0$  and  $f^+(x) > 0$ . There exists  $a \in \mathbb{R}^n$  such that  $f^-(x) = a_1 x_1 \overset{-}{\smile} \dots \overset{-}{\smile} a_n x_n$  and  $f^+(x) = a_1 x_1 \overset{+}{\smile} \dots \overset{+}{\smile} a_n x_n$ . Hence there is some  $i_- \in [n]$  such that  $f^-(x) = a_{i_-} x_{i_-} < 0$  and some  $i_+ \in [n]$  such that  $f^+(x) = a_{i_+} x_{i_+} > 0$ . Moreover

$$f^+(x) = \max_{i \in [n]} a_i x_i = \max_{i \in [n]} |a_i x_i| = a_{i_+} x_{i_+}$$

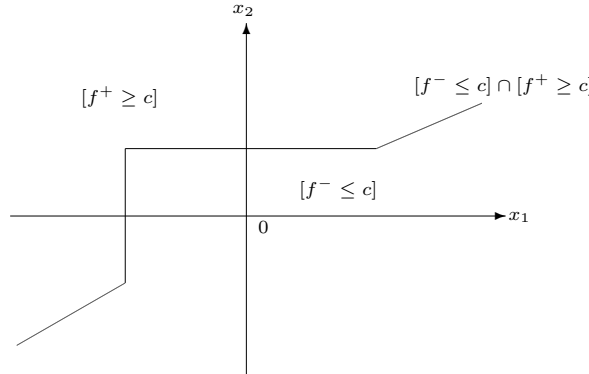
and

$$f^-(x) = \min_{i \in [n]} a_i x_i = -\max_{i \in [n]} |a_i x_i| = a_{i_-} x_{i_-}$$

Consequently,  $|a_{i_-} x_{i_-}| = |a_{i_+} x_{i_+}|$  and  $a_{i_-} x_{i_-} = -a_{i_+} x_{i_+}$ . Thus  $f^-(x) + f^+(x) = 0$ . Hence  $[f^- \leq 0] \cap [f^+ \geq 0] \subset [f^- + f^+ = 0]$ . Conversely if  $x \in [f^- + f^+ = 0]$ , we have  $f^-(x) f^+(x) \leq 0$ , which implies the converse inclusion and ends the proof.  $\square$

In the remainder, it will be useful to consider the lower and upper semi-continuous determinant defined as:

$$|A|_\infty^- = \overset{-}{\smile}_{\sigma \in S_n} (\text{sgn}(\sigma) \cdot \prod_{i \in [n]} a_{i, \sigma(i)}) \quad \text{and} \quad |A|_\infty^+ = \overset{+}{\smile}_{\sigma \in S_n} (\text{sgn}(\sigma) \cdot \prod_{i \in [n]} a_{i, \sigma(i)}). \quad (4.8)$$



**Figure 4.1.1.1** Lower and Upper halfspaces.

## 4.2 Kuratowski-Painlevé Limit of Hyperplanes

This section is devoted to analyze the Kuratowski-Painlevé limit of a sequence of half-spaces defined on the scalar field  $(\mathbb{R}, \frac{p}{\cdot}, \cdot)$ . The next result was established in [9]. These half-spaces are called  $\varphi_p$ -halfspaces.

**Proposition 4.2.1** *Let  $f$  be a  $\mathbb{B}$ -form defined by  $f(x) = \langle a, x \rangle_\infty$  for some  $a \in \mathbb{R}^n \setminus \{0\}$ . For any natural number  $p$  let  $f_p : \mathbb{R}^n \rightarrow \mathbb{R}$  be a map defined by  $f_p(x) = \langle a^{(p)}, x \rangle_p$  where  $\{a^{(p)}\}_{p \in \mathbb{N}}$  is a sequence of  $\mathbb{R}^n \setminus \{0\}$ . If there exists a sequence  $\{c_p\}_{p \in \mathbb{N}} \subset \mathbb{R}$  such that  $\lim_{p \rightarrow \infty} (a^{(p)}, c_p) = (a, c)$ , then:*

$$\text{Lim}_{p \rightarrow \infty} [f_p \leq c_p] = \text{cl} [f \leq c] = [f^- \leq c]$$

and

$$\text{Lim}_{p \rightarrow \infty} [f_p \geq c_p] = \text{cl} [f \geq c] = [f^+ \geq c].$$

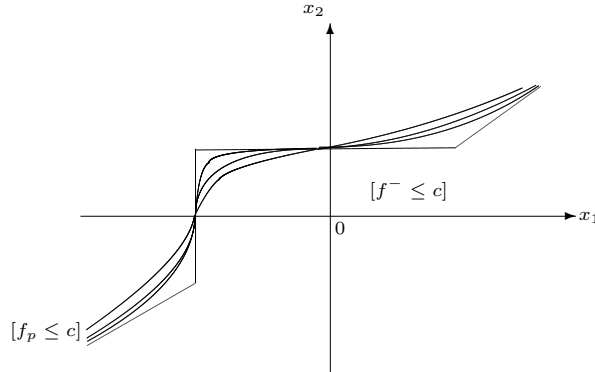


Figure 4.2 Limit of a sequence of  $\varphi_p$ -halfspaces.

In the following, one can go a bit further by showing that a sequence of  $\phi_p$ -hyperplanes defined for all  $p \in \mathbb{N}$  as  $[\langle a^{(p)}, \cdot \rangle_p = c_p]$  has a Painlevé-Kuratowski limit.

**Proposition 4.2.2** *Let  $f$  be a  $\mathbb{B}$ -form defined by  $f(x) = \langle a, x \rangle_\infty$  for some  $a \in \mathbb{R}^n \setminus \{0\}$ . For any natural number  $p$  let  $f_p : \mathbb{R}^n \rightarrow \mathbb{R}$  be a map defined by  $f_p(x) = \langle a^{(p)}, x \rangle_p$  where  $\{a^{(p)}\}_{p \in \mathbb{N}}$  is a sequence of  $\mathbb{R}^n \setminus \{0\}$ . If there exists a sequence  $\{c_p\}_{p \in \mathbb{N}} \subset \mathbb{R}$  such that  $\lim_{p \rightarrow \infty} (a^{(p)}, c_p) = (a, c)$ , then:*

$$\text{Lim}_{p \rightarrow \infty} [f_p = c_p] = [f^- \leq c] \cap [f^+ \geq c].$$

**Proof:** By definition, for all  $p$ , we have  $[f_p = c_p] = [f_p \leq c_p] \cap [f_p \geq c_p]$ . Hence, we have the inclusion:

$$\begin{aligned} \text{Ls}_{p \rightarrow \infty} [f_p = c_p] &= \text{Ls}_{p \rightarrow \infty} ([f_p \leq c_p] \cap [f_p \geq c_p]) \\ &\subset \left( \text{Ls}_{p \rightarrow \infty} [f_p \leq c_p] \right) \cap \left( \text{Ls}_{p \rightarrow \infty} [f_p \geq c_p] \right) \\ &= [f^- \leq c] \cap [f^+ \geq c]. \end{aligned}$$

In the following, we show that  $[f^- \leq c] \cap [f^+ \geq c] \subset \text{Li}_{p \rightarrow \infty} [f_p = c_p]$ . From Proposition 4.2.1, we have  $\text{Li}_{p \rightarrow \infty} [f_p \leq c_p] = [f^- \leq c]$  and  $\text{Li}_{p \rightarrow \infty} [f_p \geq c_p] = [f^+ \geq c]$ . Suppose that  $x \in [f^- \leq c] \cap [f^+ \geq c]$ . This implies that there exist two sequences  $\{y^{(p)}\}_{p \in \mathbb{N}}$  and  $\{z^{(p)}\}_{p \in \mathbb{N}}$  respectively such that for any  $p$ ,  $y^{(p)} \in [f_p \leq c_p]$  and  $z^{(p)} \in [f_p \geq c_p]$  with  $x = \lim_{p \rightarrow \infty} y^{(p)} = \lim_{p \rightarrow \infty} z^{(p)}$ .

For all  $p$ , the map  $f_p$  is continuous. Therefore, for all natural numbers  $p$ , there exists some  $\alpha_p \in [0, 1]$  such that  $f_p(\alpha_p y^{(p)} + (1 - \alpha_p)z^{(p)}) = c_p$ . Set  $w^{(p)} = \alpha_p y^{(p)} + (1 - \alpha_p)z^{(p)}$ . We have for all natural numbers  $p$

$$\begin{aligned} \|x - w^{(p)}\| &= \|\alpha_p(x - y^{(p)}) + (1 - \alpha_p)(x - z^{(p)})\| \\ &\leq \alpha_p \|x - y^{(p)}\| + (1 - \alpha_p) \|x - z^{(p)}\| \\ &\leq \|x - y^{(p)}\| + \|x - z^{(p)}\|. \end{aligned}$$

By hypothesis  $\lim_{p \rightarrow \infty} \|x - y^{(p)}\| = \lim_{p \rightarrow \infty} \|x - z^{(p)}\| = 0$ . Thus  $\lim_{p \rightarrow \infty} \|x - w^{(p)}\| = 0$ . Since  $w_p \in [f_p = c_p]$  for all  $p$ , we deduce that  $x \in \text{Li}_{p \rightarrow \infty}[f_p = c_p]$ . Consequently,  $[f^- \leq c] \cap [f^+ \geq c] \subset \text{Li}_{p \rightarrow \infty}[f_p = c_p]$ . Since we have the sequence of inclusions

$$Ls_{p \rightarrow \infty}[f_p = c_p] \subset [f^- \leq c] \cap [f^+ \geq c] \subset \text{Li}_{p \rightarrow \infty}[f_p = c_p],$$

we deduce that

$$\text{Lim}_{p \rightarrow \infty}[f_p = c_p] = [f^- \leq c] \cap [f^+ \geq c]. \quad \square$$

### 4.3 Limit Hyperplane Passing Through $n$ Points

In this subsection we give the equation of a limit hyperplane passing through  $n$  points. Given  $n$  points  $v_1, \dots, v_n$  in  $\mathbb{R}^n$ , let  $V$  be the  $n \times n$  matrix whose each column is a vector  $v_i$ . If  $|v_1, \dots, v_n|_p \neq 0$  then let  $H_p(V)$  denote the  $\varphi_p$ -hyperplane passing through  $v_1, \dots, v_n$ . In the following, the symbol  $\overset{p}{-}$  means for all  $\alpha, \beta \in \mathbb{R}$ ,  $\alpha \overset{p}{-} \beta = \alpha \overset{p}{+} (-\beta)$ . Moreover, the symbol  $\boxplus$  means  $\alpha \boxplus \beta = \alpha \boxplus (-\beta)$ .

**Proposition 4.3.1** *Let  $V$  be the  $n \times n$  matrix with  $v_i$  as  $i$ -th column for each  $i$ . Let  $V_{(i)}$  be the matrix obtained from  $V$  by replacing line  $i$  with the transpose of the unit vector  $\mathbb{1}_n$ . Suppose that  $|V|_\infty \neq 0$ . Then*

$$\text{Lim}_{p \rightarrow \infty} H_p(V) = \left\{ x \in \mathbb{R}^n : \overset{-}{\underset{i \in [n]}{|V_{(i)}|_\infty x_i}} \leq |V|_\infty \leq \overset{+}{\underset{i \in [n]}{|V_{(i)}|_\infty x_i}} \right\}.$$

**Proof:** First note that since  $|V|_\infty \neq 0$ , there exists  $p_0 \in \mathbb{N}$  such that for all  $p \geq p_0$ ,  $|V|_p \neq 0$ . Therefore for all  $p \geq p_0$ , there exists a hyperplane  $H_p(V)$  which contains  $v_1, \dots, v_p$ . Therefore, there exists some  $a^{(p)} \in \mathbb{R}^n$  and some  $c_p \in \mathbb{R}$  such that

$$H_p(V) = [\langle a^{(p)}, \cdot \rangle_p = c].$$

Suppose that  $x \in H_p(V)$ . For all  $i \in [n]$ :

$$\langle a^{(p)}, v_i \overset{p}{-} x \rangle_p = c \overset{p}{-} c = 0.$$

Let us denote  $F_p(V) = [\langle a^{(p)}, \cdot \rangle_p = 0]$ . Since  $F_p(V)$  is a  $n - 1$ -dimensional  $\varphi_p$ -subspace of  $\mathbb{R}^n$ :

$$|v_1 - x, v_2 - x, \dots, v_n - x|_p = 0.$$

Let  $V_{i,j}$  be the matrix obtained suppressing line  $i$  and column  $j$ . It follows

that:

$$\begin{aligned}
& |v_1 - x, v_2 - x, \dots, v_n - x|_p \\
&= |V|_p \binom{p}{p} |x, v_2, \dots, v_n|_p \binom{p}{p} |v_1, x, v_3, \dots, v_n|_p \binom{p}{p} |v_1, v_2, \dots, v_{n-1}, x|_p \\
&= |V|_p \binom{p}{p} \sum_{j \in [n]} \binom{\varphi_p}{i \in [n]} (-1)^{i+j} |V_{i,j}|_p x_i = |V|_p \binom{p}{p} \sum_{i \in [n]} \binom{\varphi_p}{j \in [n]} (-1)^{i+j} |V_{i,j}|_p x_i \\
&= |V|_p \binom{p}{p} \sum_{i \in [n]} |V_{(i)}|_p x_i = 0.
\end{aligned}$$

Therefore, we have:

$$H_p(V) = \left\{ x \in \mathbb{R}^n : \sum_{i \in [n]} |V_{(i)}|_p x_i = |V|_p \right\}.$$

Since  $\lim_{p \rightarrow \infty} |V_{(i)}|_p = |V_{(i)}|_\infty$  and  $\lim_{p \rightarrow \infty} |V|_p = |V|_\infty$ , we deduce the result from Proposition 4.2.2.  $\square$

A simple intuition is given in the case  $n = 2$  with two points. The hyperplane passing from two points  $u$  and  $v$  is a line. For all  $p \in \mathbb{N}$ , let us denote  $D_p(u, v)$  the  $\varphi_p$ -line spanned by  $u$  and  $v$  in  $\mathbb{R}^2$ . If  $p = 0$ , all points  $x = (x_1, x_2) \in D_0(u, v)$  satisfy the relation:

$$|u - x, v - x| = \begin{vmatrix} u_1 - x_1 & v_1 - x_1 \\ u_2 - x_2 & v_2 - x_2 \end{vmatrix} = 0. \quad (4.9)$$

Equivalently, we have

$$(v_2 - u_2)x_1 + (u_1 - v_1)x_2 = |u, v|. \quad (4.10)$$

For all points  $z \in D_p(u, v)$  we have the relation:

$$|u \binom{p}{p} x, v \binom{p}{p} x|_p = \begin{vmatrix} u_1 \binom{p}{p} x_1 & v_1 \binom{p}{p} x_1 \\ u_2 \binom{p}{p} x_2 & v_2 \binom{p}{p} x_2 \end{vmatrix} = 0 \iff (v_2 \binom{p}{p} u_2)x_1 + (u_1 \binom{p}{p} v_1)x_2 = |u, v|_p.$$

We obtain that

$$\begin{aligned}
& \text{Lim}_{p \rightarrow \infty} D_p(u, v) \\
&= \left\{ x \in \mathbb{R}^2 : \begin{vmatrix} 1 & 1 \\ u_2 & v_2 \end{vmatrix}_\infty x_1 \ominus \begin{vmatrix} u_1 & v_1 \\ 1 & 1 \end{vmatrix}_\infty x_2 \leq |u, v|_\infty \leq \begin{vmatrix} 1 & 1 \\ u_2 & v_2 \end{vmatrix}_\infty x_1 \oplus \begin{vmatrix} u_1 & v_1 \\ 1 & 1 \end{vmatrix}_\infty x_2 \right\}.
\end{aligned} \quad (4.11)$$

Hence

$$\begin{aligned}
& \text{Lim}_{p \rightarrow \infty} D_p(u, v) \\
&= \left\{ x \in \mathbb{R}^2 : (v_2 \boxminus u_2)x_1 \ominus (u_1 \boxminus v_1)x_2 \leq |u, v|_\infty \leq (v_2 \boxminus u_2)x_1 \oplus (u_1 \boxminus v_1)x_2 \right\}.
\end{aligned} \quad (4.12)$$

**Example 4.3.2** Suppose that  $n = 3$  and that  $v_1 = (1, 0, -3)$ ,  $v_2 = (2, -1, 1)$ ,  $v_3 = (4, 1, 2)$ . We have

$$V = \begin{pmatrix} 1 & 2 & 4 \\ 0 & -1 & 1 \\ -3 & 1 & 2 \end{pmatrix}.$$



Thus

$$V_{(1)} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ -3 & 1 & 2 \end{pmatrix}, V_{(2)} = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 1 & 1 \\ -3 & 1 & 2 \end{pmatrix} \text{ and } V_{(3)} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Hence  $|V|_\infty = 1 \cdot (-1) \cdot 2 \boxplus 2 \cdot 1 \cdot (-3) \boxplus 0 \cdot 1 \cdot 4 \boxplus (-4) \cdot (-1) \cdot (-3) \boxplus (-1) \cdot 1 \cdot 1 \boxplus 0 \cdot 2 \cdot 2 = -12$ ;  $|V_{(1)}|_\infty = 1 \cdot (-1) \cdot 2 \boxplus 1 \cdot 1 \cdot (-3) \boxplus 0 \cdot 1 \cdot 1 \boxplus (-1) \cdot (-1) \cdot (-3) \boxplus (-1) \cdot 1 \cdot 1 \boxplus 0 \cdot 1 \cdot 2 = -3$ ;  $|V_{(2)}|_\infty = 1 \cdot 1 \cdot 2 \boxplus 2 \cdot 1 \cdot (-3) \boxplus 1 \cdot 1 \cdot 4 \boxplus (-4) \cdot 1 \cdot (-3) \boxplus (-1) \cdot 1 \cdot 1 \boxplus (-1) \cdot 2 \cdot 2 = 12$ ;  $|V_{(3)}|_\infty = 1 \cdot (-1) \cdot 1 \boxplus 2 \cdot 1 \cdot 1 \boxplus 0 \cdot 1 \cdot 4 \boxplus (-4) \cdot (-1) \cdot 1 \boxplus (-1) \cdot 1 \cdot 1 \boxplus 0 \cdot 2 \cdot 1 = 4$ .

$$H_\infty(V) = \{(x_1, x_2, x_3) : (-3)x_1 \ominus 12x_2 \ominus 4x_3 \leq -12 \leq (-3)x_1 \oplus 12x_2 \oplus 4x_3\}.$$

It is easy to check that  $v_1, v_2, v_3 \in H_\infty(V)$ .

## 5 Limit Systems of Equations

### 5.1 Limit Solutions of a Sequence of Systems of Equations

For all  $\varphi_p$ -linear endomorphisms  $f : x \mapsto A \cdot^p x$ , where  $A \in \mathcal{M}_n(\mathbb{R})$ , we consider a sequence of  $\varphi_p$ -linear systems of the form  $A \cdot^p x = b$ . For all  $x \in \mathbb{R}^d$ , let  $B_\infty(x, d) = \{z \in \mathbb{R}^n : \|x - z\|_\infty \leq d\}$  denote the ball of center  $x$  and radius  $d$ . Moreover, let  $C_\infty(x, d) = \{z \in \mathbb{R}^n : \|x - z\|_\infty = d\}$  denote the circle of center  $x$  and radius  $d$ .

**Proposition 5.1.1** *Suppose that  $A \in \mathcal{M}_n(\mathbb{R})$  is a square matrix and let  $b \in \mathbb{R}^n$ . Then, there exists a unique  $x^* \in Ls_{p \rightarrow \infty} \{x \in \mathbb{R}^n : A \cdot^p x = b\}$  if and only if  $|A|_\infty \neq 0$ . Moreover, if  $|A|_\infty \neq 0$  then  $x^* = \sum_{i \in [n]} \frac{|A^{(i)}|_\infty}{|A|_\infty} e_i$  and*

$$\{x^*\} = \text{Lim}_{p \rightarrow \infty} \{x \in \mathbb{R}^n : A \cdot^p x = b\}.$$

**Proof:** If  $|A|_\infty \neq 0$  then there is some  $p_0$  such that for all  $p \geq p_0$ ,  $|A|_p \neq 0$ . Thus for all  $p \geq p_0$ ,  $x^{(p)} = \sum_{i \in [n]} \frac{|A^{(i)}|_p}{|A|_p} e_i$  is solution of the system  $A \cdot^p x = b$  and therefore  $x^{(p)} \in \{x \in \mathbb{R}^n : A \cdot^p x = b\}$ . However,  $x^* = \sum_{i \in [n]} \frac{|A^{(i)}|_\infty}{|A|_\infty} e_i = \lim_{p \rightarrow \infty} x^{(p)}$ . Thus  $x^* \in Li_{p \rightarrow \infty} \{x \in \mathbb{R}^n : A \cdot^p x = b\}$ . Moreover, for all  $p \geq p_0$ , since  $|A|_p \neq 0$  we have  $\{x^{(p)}\} = \{x \in \mathbb{R}^n : A \cdot^p x = b\}$ . Consequently  $x^*$  is the unique solution. This implies that  $x^* \in Ls_{p \rightarrow \infty} \{x \in \mathbb{R}^n : A \cdot^p x = b\}$ . Moreover, for each increasing sequence of natural numbers  $\{p_k\}_{k \in \mathbb{N}}$ ,  $x^{(p_k)} = \sum_{i \in [n]} \frac{|A^{(i)}|_{p_k}}{|A|_{p_k}} e_i$  is the unique solution of the system of the form  $A \cdot^{p_k} x = b$ . Hence  $Li_{p \rightarrow \infty} \{x \in \mathbb{R}^n : A \cdot^p x = b\} = Ls_{p \rightarrow \infty} \{x \in \mathbb{R}^n : A \cdot^p x = b\} = \{x^*\}$  which ends the first part of the statement.

To complete the proof, suppose that  $\{x^*\} = Ls_{p \rightarrow \infty} \{x \in \mathbb{R}^n : A \cdot^p x = b\}$  with  $|A|_\infty = 0$  and let us show a contradiction. This implies that for any  $p \in \mathbb{N}$  we have  $|A|_p = 0$  from Lemma 3.2.4. Thus, for any  $p$ , the system  $\{x \in \mathbb{R}^n : A \cdot^p x = b\}$  has either an infinity of solutions or is an empty set. If, for all  $p \in \mathbb{N}$ , it is an empty set then the upper limit of the sequence of solution sets is empty. By assumption this is not the case. Suppose that  $x^* \in Ls_{p \rightarrow \infty} \{x \in \mathbb{R}^n : A \cdot^p x = b\}$ . In such case there exists a subsequence  $\{p_k\}_{k \in \mathbb{N}}$  such that  $x^* = \lim_{k \rightarrow \infty} x^{(p_k)}$  where for all  $k$ ,  $x^{(p_k)} \in \{x \in \mathbb{R}^n : A \cdot^{p_k} x = b\}$  that is a  $\varphi_{p_k}$ -affine subspace that contains an infinity of points. For any  $k$  let us consider

the ball  $B_\infty(x^{(p_k)}, 1]$  of center  $x^{(p_k)}$  and of radius 1. Since  $\{x \in \mathbb{R}^n : A^{p_k} x = b\}$  is a  $\varphi_{p_k}$  affine subspace of  $\mathbb{R}^n$ , for all  $k$  there exists a vector  $v^{(p_k)} \neq 0$  such that  $A^{p_k} v^{(p_k)} = 0$ . This implies that:

$$\{x^{(p_k)} \overset{p_k}{+} \delta v^{(p_k)} : \delta \in \mathbb{R}\} \subset \{x \in \mathbb{R}^n : A^{p_k} x = b\}.$$

Let  $\delta_{p_k} = \sup\{\delta : x^{(p_k)} \overset{p_k}{+} \delta v^{(p_k)} \in B_\infty(x^{(p_k)}, 1]\}$ . Since the map  $\delta \mapsto x^{(p_k)} \overset{p_k}{+} \delta v^{(p_k)}$  is a continuous vector valued function,  $y^{(p_k)} = x^{(p_k)} \overset{p_k}{+} \delta_{p_k} v^{(p_k)} \in C_\infty(x^{(p_k)}, 1)$  which implies that  $d_\infty(x^{(p_k)}, y^{(p_k)}) = 1$ . Now since  $\{x^{(p_k)}\}_{k \in \mathbb{N}}$  converges to  $x^*$ , There exists some  $d > 0$  and  $k_d \in \mathbb{N}$  such that for all  $k \geq k_d$ ,  $x^{(p_k)}, y^{(p_k)} \in B_\infty(x^*, d]$ . Since  $B_\infty(x^*, d]$  is a compact subset of  $\mathbb{R}^n$  one can extract a sequence  $\{y^{(p_{k_r})}\}_{r \in \mathbb{N}}$  which converges to some  $y^* \in Ls_{p \rightarrow \infty}\{x \in \mathbb{R}^n : A^p x = b\}$ . However, for all  $r \in \mathbb{N}$ ,  $d_\infty(x^{(p_{k_r})}, y^{(p_{k_r})}) = 1$ , and we deduce that  $d_\infty(x^*, y^*) = 1$ . This implies that  $x^* \neq y^*$  which contradicts the unicity. Consequently, if the upper limit of the sequence of solution sets has a unique element, then  $|A|_\infty \neq 0$ .  $\square$

In the following, for all matrices  $A \in \mathcal{M}_{n,l}(\mathbb{R})$  and  $B \in \mathcal{M}_{l,m}(\mathbb{R})$  let us define the product:

$$A \boxtimes B = \left( \bigoplus_{\substack{k \in [l] \\ j \in [m]}} a_{i,k} b_{k,j} \right)_{\substack{i \in [n] \\ j \in [m]}}. \quad (5.1)$$

The lower and upper semi-continuous regularized products are respectively defined as:

$$A \bar{\boxtimes} B = \left( \bar{\bigoplus}_{\substack{k \in [l] \\ j \in [m]}} a_{i,k} b_{k,j} \right)_{\substack{i \in [n] \\ j \in [m]}} \quad \text{and} \quad A \overset{+}{\boxtimes} B = \left( \overset{+}{\bigoplus}_{\substack{k \in [l] \\ j \in [m]}} a_{i,k} b_{k,j} \right)_{\substack{i \in [n] \\ j \in [m]}}. \quad (5.2)$$

By construction, it follows that for all vectors  $x \in \mathbb{R}^n$ , the matrix-vector products derived from  $\bar{\boxtimes}$  and  $\overset{+}{\boxtimes}$  are defined by:

$$A \bar{\boxtimes} x = \begin{pmatrix} \langle a_1, x \rangle_\infty^- \\ \vdots \\ \langle a_n, x \rangle_\infty^- \end{pmatrix} \quad \text{and} \quad A \overset{+}{\boxtimes} x = \begin{pmatrix} \langle a_1, x \rangle_\infty^+ \\ \vdots \\ \langle a_n, x \rangle_\infty^+ \end{pmatrix}. \quad (5.3)$$

The next result is an immediate consequence.

**Proposition 5.1.2** *Suppose that  $A \in \mathcal{M}_n(\mathbb{R})$  is a square matrix and let  $b \in \mathbb{R}^n$ . If  $\bar{x} \in Ls_{p \rightarrow \infty}\{x \in \mathbb{R}^n : A^p x = b\}$ , then  $\bar{x}$  is a solution of the system:*

$$\begin{cases} A \bar{\boxtimes} x \leq b \\ A \overset{+}{\boxtimes} x \geq b, \quad x \in \mathbb{R}^n. \end{cases} \quad (5.4)$$

Moreover, if  $|A|_\infty \neq 0$  then  $x^* = \sum_{i \in [n]} \frac{|A^{(i)}|_\infty}{|A|_\infty} e_i$  is a solution of the system (5.4).

**Proof:** From Proposition 4.2.2, for all  $i \in [n]$ :

$$\begin{aligned} Ls_{p \rightarrow \infty}\{x : \langle a_i, x \rangle_p = b_i\} &= Lim_{p \rightarrow \infty}\{x : \langle a_i, x \rangle_p = b_i\} \\ &= [x : \langle a_i, x \rangle_\infty^- \leq b_i] \cap [x : \langle a_i, x \rangle_\infty^+ \geq b_i]. \end{aligned}$$

From (5.15), we deduce that:

$$\begin{aligned}
Ls_{p \rightarrow \infty} \{x \in \mathbb{R}^n : A^p x = b\} &= Ls_{p \rightarrow \infty} \bigcap_{i \in [n]} [\langle a_i, \cdot \rangle_p = b_i] \\
&\subset \bigcap_{i \in [n]} Ls_{p \rightarrow \infty} [\langle a_i, \cdot \rangle_p = b_i] \\
&\subset \bigcap_{i \in [n]} \left( [\langle a_i, \cdot \rangle_\infty^- \leq b_i] \cap [\langle a_i, \cdot \rangle_\infty^+ \geq b_i] \right),
\end{aligned}$$

which implies that if  $\bar{x} \in Ls_{p \rightarrow \infty} \{x \in \mathbb{R}^n : A^p x = b\}$  then it satisfies the system (5.4). If  $|A|_\infty \neq 0$ , from Proposition 5.1.1  $\{x^*\} = Lim_{p \rightarrow \infty} \{x \in \mathbb{R}^n : A^p x = b\}$  and this implies that  $x^*$  satisfies system (5.4).  $\square$

Since it contains any element of the upper limit set  $Ls_{p \rightarrow \infty} \{x \in \mathbb{R}^n : A^p x = b\}$ , the system (5.4) is called a **limit system**.

**Example 5.1.3** Let us consider the matrix

$$A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

with  $a_1 = (-1, 1)$ ,  $a_2 = (1, 1)$  and suppose that  $b_1 = 2$ ,  $b_2 = 3$ . Now, let us consider the matrices:

$$A^{(1)} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} \quad A^{(2)} = \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}.$$

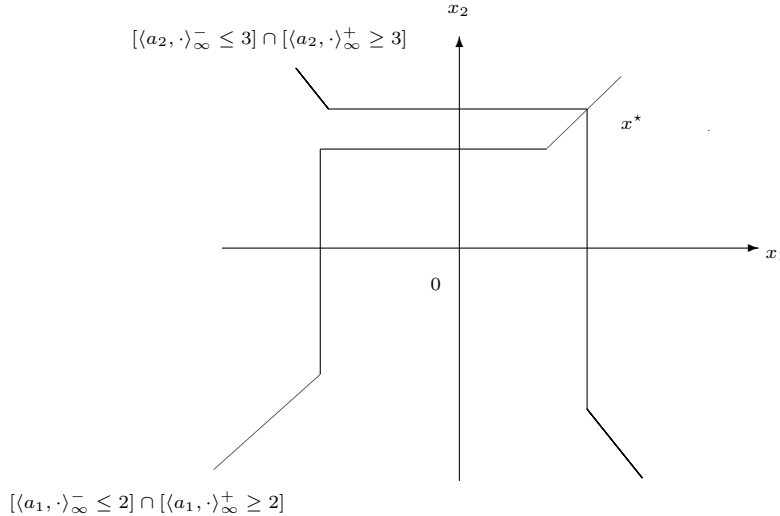
We have  $|A|_\infty = ((-1) \cdot 1) \boxplus ((-1) \cdot 1) = -1$ ;  $|A^{(1)}|_\infty = (2 \cdot 1 \boxplus (-3) \cdot 1) = -3$ ;  $|A^{(2)}|_\infty = ((-1) \cdot 3 \boxplus (-2) \cdot 1) = -3$ . We obtain the solutions:  $x_1^* = \frac{-3}{-1} = 3$   $x_2^* = \frac{-3}{-1} = 3$ . One can then check that:

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \boxtimes \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} (-3) \ominus 3 \\ 3 \ominus 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix} \leq \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

and

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \boxplus \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} (-3) \oplus 3 \\ 3 \oplus 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \geq \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Therefore  $x^* = (3, 3)$  is a solution of the limit system. This example is depicted in Figure 5.2.



**Figure 5.2** Example of a two dimensional Limit System.

**Example 5.1.4** Let us consider the matrix:

$$A = \begin{pmatrix} 3 & -1 & 3 \\ 2 & -4 & 1 \\ -4 & 5 & 3 \end{pmatrix}$$

with  $a_1 = (3, 1, -3)$ ,  $a_2 = (2, -4, 1)$ ,  $a_3 = (-4, 5, 3)$ ,  $b_1 = 6$ ,  $b_2 = 8$ ,  $b_3 = 4$ .

The limit system is:

$$\begin{cases} \begin{pmatrix} 3 & -1 & 3 \\ 2 & -4 & 1 \\ -4 & 5 & 3 \end{pmatrix} \boxtimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq \begin{pmatrix} 6 \\ 8 \\ 4 \end{pmatrix} \\ \begin{pmatrix} 3 & -1 & 3 \\ 2 & -4 & 1 \\ -4 & 5 & 3 \end{pmatrix} \boxplus \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \geq \begin{pmatrix} 6 \\ 8 \\ 4 \end{pmatrix} \end{cases} \quad x \in \mathbb{R}^3. \quad (5.5)$$

Now, let us consider the matrices:

$$A^{(1)} = \begin{pmatrix} 6 & -1 & 3 \\ 8 & -4 & 1 \\ 4 & 5 & 3 \end{pmatrix}; \quad A^{(2)} = \begin{pmatrix} 3 & 6 & 3 \\ 2 & 8 & 1 \\ -4 & 4 & 3 \end{pmatrix}; \quad A^{(3)} = \begin{pmatrix} 3 & -1 & 6 \\ 2 & -4 & 8 \\ -4 & 5 & 4 \end{pmatrix}.$$

We have  $|A|_\infty = (-36) \boxplus 4 \boxplus 30 \boxplus (-48) \boxplus (-15) \boxplus 6 = -48$ ;  $|A^{(1)}|_\infty = (-72) \boxplus (-4) \boxplus 120 \boxplus 48 \boxplus 24 \boxplus (-30) = 120$ ;  $|A^{(2)}|_\infty = 72 \boxplus (-24) \boxplus 24 \boxplus 96 \boxplus (-12) \boxplus (-36) = 96$ ;  $|A^{(3)}|_\infty = (-48) \boxplus 32 \boxplus +60 \boxplus (-96) \boxplus 8 \boxplus (-120) = -120$ . We obtain that

$$x_1^* = \frac{120}{-48} = -\frac{5}{2}, \quad x_2^* = \frac{96}{-48} = -2 \quad \text{and} \quad x_3^* = \frac{5}{2}.$$

Let us check that  $x^* = (-\frac{5}{2}, -2, -\frac{5}{2})$  satisfies the system of equations (5.5). We have:

$$\begin{cases} \begin{pmatrix} 3 & -1 & 3 \\ 2 & -4 & 1 \\ -4 & 5 & 3 \end{pmatrix} \boxtimes \begin{pmatrix} -\frac{5}{2} \\ -2 \\ \frac{5}{2} \end{pmatrix} = \begin{pmatrix} -\frac{15}{2} \\ 8 \\ -10 \end{pmatrix} \leq \begin{pmatrix} 6 \\ 8 \\ 4 \end{pmatrix} \\ \begin{pmatrix} 3 & -1 & 3 \\ 2 & -4 & 1 \\ -4 & 5 & 3 \end{pmatrix} \boxplus \begin{pmatrix} -\frac{5}{2} \\ -2 \\ \frac{5}{2} \end{pmatrix} = \begin{pmatrix} \frac{15}{2} \\ 8 \\ 10 \end{pmatrix} \geq \begin{pmatrix} 6 \\ 8 \\ 4 \end{pmatrix} \end{cases} \quad x \in \mathbb{R}^3.$$

Thus  $x^* = (-\frac{5}{2}, -2, -\frac{5}{2})$  satisfies the system (5.5).

In the following, we say that a solution  $x^*$  of the limit system is **regular** if for all  $i \in [m]$ ,  $\langle a_i, x^* \rangle_\infty = \langle a_i, x^* \rangle_\infty^- = \langle a_i, x^* \rangle_\infty^+$ . This implies that  $x^*$  is also solution of the equation

$$A \boxtimes x = b. \quad (5.6)$$

Equivalently, this means that:

$$\begin{cases} \boxplus_{j \in [n]} a_{1,j} x_j = b_1 \\ \vdots \\ \boxplus_{j \in [n]} a_{m,j} x_j = b_m. \end{cases} \quad (5.7)$$

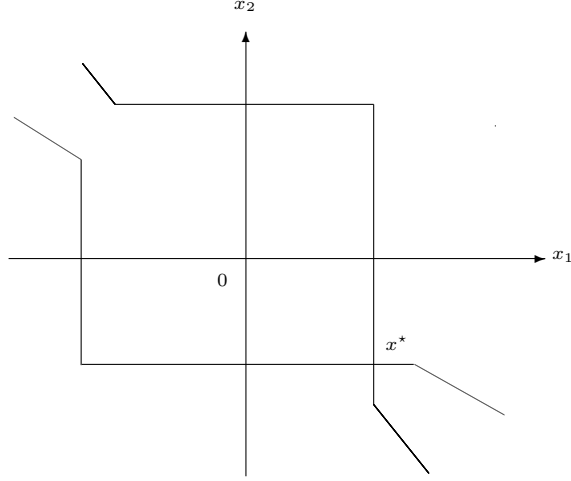


Figure 5.3 Regular Solutions of a Limit System.

## 5.2 Positive Solutions of Positive Systems of Maximum Equations

In the following, we consider a theorem established by Kaykobad [22] that gives a necessary condition for the existence of a positive solution to a positive invertible linear system.

**Theorem 5.2.1** *Suppose that  $A = (a_{i,j})_{i,j \in [n]} \in \mathcal{M}_n(\mathbb{R})$  is a square matrix such that for all  $i, j$   $a_{i,j} \geq 0$  and  $a_{i,i} > 0$  for all  $i \in [n]$ . Suppose moreover that  $b \in \mathbb{R}_{++}^n$ . If for all  $i \in [n]$*

$$b_i > \sum_{j \in [n] \setminus \{i\}} a_{i,j} \frac{b_j}{a_{j,j}}$$

*then  $A$  is invertible and  $A^{-1}b \in \mathbb{R}_{++}^n$ .*

In the following this result is extended to a  $\varphi_p$ -endomorphism.

**Lemma 5.2.2** *Suppose that  $A = (a_{i,j})_{i,j \in [n]} \in \mathcal{M}_n(\mathbb{R})$  is a square matrix such that for all  $i, j$   $a_{i,j} \geq 0$ . Suppose that there exists a permutation  $\sigma : [n] \rightarrow [n]$  such that  $a_{i,\sigma(i)} > 0$  for all  $i \in [n]$ . Suppose moreover that  $b \in \mathbb{R}_{++}^n$ . If for all  $i \in [n]$*

$$b_i > \left( \sum_{j \in [n] \setminus \{i\}} (a_{i,\sigma(j)})^{2p+1} \frac{(b_j)^{2p+1}}{(a_{j,\sigma(j)})^{2p+1}} \right)^{\frac{1}{2p+1}}$$

*then  $A$  is  $\varphi_p$ -invertible and there is a solution  $x^{(p)} \in \mathbb{R}_{++}^n$  to the equation  $A \stackrel{p}{\cdot} x = b$ .*

**Proof:** Let  $\bar{A} = (\bar{a}_{i,j})_{i,j \in [n]}$  be the  $n \times n$  matrix defined by  $\bar{a}_{i,j} = a_{i,\sigma(j)}$ . The system  $\bar{A} \stackrel{p}{\cdot} x = b$  is equivalent to  $\Phi_p(\bar{A})u = \phi_p(b)$  setting  $u = \phi_p(x)$ . Since  $a_{i,\sigma(i)} > 0$  for all  $i$ , we deduce that for all  $i$ ,  $\bar{a}_{i,i} > 0$ . Since by definition  $\Phi_p(\bar{A}) = (\bar{a}_{i,j}^{2p+1})_{i,j \in [n]}$  and  $\phi_p(b) = (b_1^{2p+1}, \dots, b_n^{2p+1})$ , it follows from Theorem 5.2.1 that this system has a positive solution if:

$$(b_i)^{2p+1} > \sum_{j \in [n] \setminus \{i\}} (\bar{a}_{i,j})^{2p+1} \frac{(b_j)^{2p+1}}{(\bar{a}_{j,j})^{2p+1}}.$$

Equivalently, we deduce that the system  $A^p x = b$  has a solution if

$$(b_i)^{2p+1} > \sum_{j \in [n] \setminus \{i\}} (a_{i,\sigma(j)})^{2p+1} \frac{(b_j)^{2p+1}}{(a_{j,\sigma(j)})^{2p+1}}$$

which ends the proof.  $\square$

First, we consider systems of max-equations, that is, systems of the form

$$\begin{cases} \max\{a_{1,1}x_1, \dots, a_{1,n}x_n\} & = b_1 \\ \vdots & \vdots \\ \max\{a_{m,1}x_1, \dots, a_{m,n}x_n\} & = b_m \end{cases} \quad (5.8)$$

where  $a_i = (a_{i,1}, \dots, a_{i,n}) \in \mathbb{R}_+^n$ ,  $i = 1, \dots, m$ ,  $b = (b_1, \dots, b_m) \in \mathbb{R}_+^m$  and the solution  $(x_1, \dots, x_n)$  is to be found in  $\mathbb{R}_+^n$ . Notice that if  $b_i = 0$  then we have to take  $x_j = 0$  for each  $j$  such that  $a_{i,j} > 0$ , and, as far as equation  $i$  is concerned, the other values  $x_l$  are irrelevant; equation  $i$  can therefore be removed from the system and the number of variables decreases. In other words, we can assume that  $b_i > 0$  for all  $i$ . In the remainder these types of systems will be called a system of maximum-equations. We can assume that for all  $j$  there is at least one index  $i$  such that  $a_{i,j} > 0$ ; let  $\eta(j) = \{i : a_{i,j} > 0\}$  and

$$\bar{x} = \sum_{i \in [n]} \left( \min_{i \in \eta(j)} \frac{b_i}{a_{i,j}} \right) e_i. \quad (5.9)$$

It is known (see e.g. [13]) that if the system of maximum equations (5.8) has some solution, then  $\bar{x}$  is a solution and, for any solution  $x$  one has  $x \leq \bar{x}$ .  $\bar{x}$  is called the principal solution of the system (5.8). This condition is equivalent to the following.

**Lemma 5.2.3** *Let  $A \in \mathcal{M}_n(\mathbb{R})$  be a square matrix such that  $a_{i,j} \geq 0$  for all  $i, j \in [n]$ . For all  $j \in [n]$ , let us denote  $\eta(j) = \{i : a_{i,j} > 0\}$  and assume that  $\eta(j)$  is nonempty. Suppose moreover that  $b \in \mathbb{R}_{++}^n$ . The system of maximum equations (5.8) has a solution in  $\mathbb{R}_+^n$  if and only if there exists a permutation  $\sigma : [n] \rightarrow [n]$  such that for all  $i \in [n]$  we have  $a_{i,\sigma(i)} > 0$  and*

$$b_i \geq \max_{j \in [n] \setminus \{i\}} a_{i,\sigma(j)} \frac{b_j}{a_{j,\sigma(j)}}.$$

Moreover, this solution is unique if and only if for all  $i \in [n]$

$$b_i > \max_{j \in [n] \setminus \{i\}} a_{i,\sigma(j)} \frac{b_j}{a_{j,\sigma(j)}}.$$

**Proof:** The system (5.8) has a solution if and only if the point

$$\bar{x} = \sum_{j \in [n]} \left( \min_{i \in \eta(j)} \frac{b_i}{a_{i,j}} \right) e_j$$

is a solution. Suppose that  $\bar{x}$  is a solution. This implies that for all  $k$ , we have  $\bar{x}_k = \min_{i \in \eta(k)} \left( \frac{b_i}{a_{i,k}} \right)$ . Moreover, for all  $j$

$$\max_{k \in [n]} (a_{j,k} \bar{x}_k) = \max_{k \in [n]} \left\{ a_{j,k} \left( \min_{i \in \eta(k)} \frac{b_i}{a_{i,k}} \right) \right\} = b_j.$$

Therefore, there is some  $\sigma(j) \in [n]$  such that:

$$a_{j,\sigma(j)}\bar{x}_{\sigma(j)} = a_{j,\sigma(j)} \min_{i \in \eta(\sigma(j))} \left( \frac{b_i}{a_{i,\sigma(j)}} \right) = b_j.$$

Note that, since  $b_j > 0$ , this condition implies that  $a_{j,\sigma(j)} > 0$ . Hence, we deduce the equality  $\min_{i \in \eta(\sigma(j))} \frac{b_i}{a_{i,\sigma(j)}} = \frac{b_j}{a_{j,\sigma(j)}}$ . Therefore, for all  $i$  and all  $i \in \eta(\sigma(j))$ , we have  $\frac{b_i}{a_{i,\sigma(j)}} \geq \frac{b_j}{a_{j,\sigma(j)}}$ , and equivalently:

$$b_i \geq a_{i,\sigma(j)} \frac{b_j}{a_{j,\sigma(j)}}.$$

Moreover, if  $i \notin \eta(\sigma(j))$  then  $a_{i,\sigma(j)} = 0$ . Consequently, for all  $i, j \in [n]$ ,  $b_i \geq a_{i,\sigma(j)} \frac{b_j}{a_{j,\sigma(j)}}$ . Therefore, we deduce that

$$b_i \geq \max_{j \in [n] \setminus \{i\}} a_{i,\sigma(j)} \frac{b_j}{a_{j,\sigma(j)}}. \quad (1)$$

Let us prove that for each  $j$ ,  $\sigma(j)$  can be chosen such that  $\sigma$  defines a permutation on  $[n]$ . If this is not the case then there is no surjective map  $j \mapsto \sigma(j)$  such that, for all  $j$ ,  $a_{j,\sigma(j)}\bar{x}_{\sigma(j)} = \max_{k \in [n]} a_{j,k}\bar{x}_k = b_j$ . Consequently, there is some  $k_0 \in [n]$  such that for all  $i \in \eta(k_0)$ ,  $a_{i,k_0}\bar{x}_{k_0} < b_i$ . This is equivalent to  $\bar{x}_{k_0} < \frac{b_i}{a_{i,k_0}}$  for all  $i \in \eta(k_0)$  which means that  $\bar{x}_{k_0} < \min_{i \in \eta(k_0)} \frac{b_i}{a_{i,k_0}} = \bar{x}_{k_0}$  that is a contradiction. Hence, the first implication is established.

To prove the converse note that, condition (1) implies that for all  $i$ , we have  $a_{i,\sigma(i)} \left( \frac{b_i}{a_{i,\sigma(i)}} \right) = b_i = \max_{k \in [n]} \left( a_{i,\sigma(k)} \frac{b_k}{a_{k,\sigma(k)}} \right)$ . Therefore the vector  $\sum_{i \in [n]} \left( \frac{b_i}{a_{i,\sigma(i)}} \right) e_{\sigma(i)}$  is a solution. Since  $a_{i,\sigma(i)} > 0$  for all  $i$ , this is the greatest solution and it follows that

$$\bar{x}_{\sigma(i)} = \frac{b_i}{a_{i,\sigma(i)}}$$

for all  $i \in [n]$ . To complete the proof, let us establish a necessary and sufficient condition for the uniqueness of a solution. From the first part of the statement, the strict inequality

$$b_i > \max_{j \in [n] \setminus \{i\}} a_{i,\sigma(j)} \frac{b_j}{a_{j,\sigma(j)}} \quad (2)$$

is equivalent to

$$\max_{j \in [n]} \left( a_{i,\sigma(j)} \bar{x}_{\sigma(j)} \right) = a_{i,\sigma(i)} \bar{x}_{\sigma(i)} = b_i > \max_{j \in [n] \setminus \{i\}} \left( a_{i,\sigma(j)} \bar{x}_{\sigma(j)} \right)$$

for all  $i \in [n]$ . Suppose now that (2) holds and that there is some  $u \in \mathbb{R}_+^n$  with  $u \leq \bar{x}$  that is a solution. It follows that for all  $i$ , we have  $\max_{j \in [n]} \left( a_{i,\sigma(j)} \bar{x}_{\sigma(j)} \right) = \max_{j \in [n]} \left( a_{i,\sigma(j)} u_{\sigma(j)} \right) = b_i$ . Since  $\bar{x} \geq u$ , we have

$$a_{i,\sigma(i)} \bar{x}_{\sigma(i)} > \max_{j \in [n] \setminus \{i\}} \left( a_{i,\sigma(j)} u_{\sigma(j)} \right).$$

Therefore  $a_{i,\sigma(i)} \bar{x}_{\sigma(i)} = a_{i,\sigma(i)} u_{\sigma(i)} = b_i$ . Since  $a_{i,\sigma(i)} \neq 0$ , we deduce that  $x_{\sigma(i)} = u_{\sigma(i)}$ . Consequently, if  $u$  is non-negative solution then  $u = \bar{x}$  which proves the uniqueness of the solution. Conversely, suppose that there is a unique solution of the system (5.8). Let  $\bar{x}^{(i)}$  be a vector of  $\mathbb{R}_+^n$  such that  $\bar{x}_j^{(i)} = 0$  if  $j = i$

and  $\bar{x}_j^{(i)} = \bar{x}_j$  if  $j \neq i$ . We have  $\bar{x}^{(i)} \leq \bar{x}$  and  $\bar{x}_i^{(i)} = 0 < \frac{b_i}{\sigma^{(i)}} = \bar{x}_i$ . Since there is a unique solution,  $b_i = \max_{j \in [n]} (a_{i,\sigma(j)} \bar{x}_{\sigma(j)}) > \max_{j \in [n]} (a_{i,\sigma(j)} \bar{x}_{\sigma(j)}^{(i)}) = \max_{j \in [n] \setminus \{i\}} (a_{i,\sigma(j)} \bar{x}_{\sigma(j)})$  which proves (2) and ends the proof.  $\square$

The next statement shows that if the limit system (5.4) has a regular solution, then there exists a nonnegative solution to the system of maximum equations (5.8).

**Lemma 5.2.4** *Let  $A \in \mathcal{M}_n(\mathbb{R}_+)$  be a square matrix such that  $a_{i,j} \geq 0$  for all  $i, j \in [n]$ . Suppose moreover that  $b \in \mathbb{R}_+^n$ . Any solution of the limit system (5.4) in  $\mathbb{R}_+^n$  is a solution of the system of maximum equations (5.8). Moreover, if the limit system has a regular solution  $x^* \in \mathbb{R}^n$  then the system of maximum equations (5.8) has a solution in  $\mathbb{R}_+^n$  given by  $\sum_{i \in [n]} |x_i^*| e_i$ .*

**Proof:** First note that if  $x^* \in \mathbb{R}_+^n$  is a solution of the limit system, then we have  $\max_{j \in [n]} a_{i,j} x_j = b_i = \langle a_i, x^* \rangle_\infty^+ = \langle a_i, x^* \rangle_\infty^-$ . Hence  $x^*$  is a solution of the system (5.8). Suppose now that  $x^*$  is a regular solution of the semi-continuous regularized system (5.4). This implies that for all  $i$ ,  $b_i = \langle a_i, x^* \rangle_\infty^+ = \langle a_i, x^* \rangle_\infty^-$ .

Let us prove that  $y^* = \sum_{i \in [n]} |x_i^*| e_i$  is a solution of the system (5.8). Let  $J_o = \{j : x_j^* < 0\}$ . Since  $A \in \mathcal{M}_n(\mathbb{R}_+)$  and  $y^* \in \mathbb{R}_+^n$ , for all  $i \in [n]$  we have  $b_i = \langle a_i, x^* \rangle_\infty^- \leq \langle a_i, y^* \rangle_\infty^-$ . If  $y^*$  is not solution of the system (5.8) then, there is some  $i \in [n]$  and some  $j_o \in J_o$  such that  $\langle a_i, y^* \rangle_\infty^- = a_{i,j_o} |x_{j_o}^*| > b_i$ . However, this implies that  $\langle a_i, x^* \rangle_\infty^- = a_{i,j_o} x_{j_o}^* < 0 \leq b_i$ , which is a contradiction. Consequently  $y^*$  is a solution of the system (5.4).  $\square$

In the following, a condition is given to ensure that the Cramer formula expressed in this idempotent and non-associative algebraic structure yields a solution to a system of maximum equations. This is a limit case of the condition proposed by Kaykobad [22] when  $p \rightarrow \infty$ .

**Proposition 5.2.5** *Let  $A \in \mathcal{M}_n(\mathbb{R})$  be a square matrix such that  $a_{i,j} \geq 0$  for all  $i, j \in [n]$ . Suppose that  $b \in \mathbb{R}_{++}^n$ . If there exists a permutation  $\sigma : [n] \rightarrow [n]$  such that for all  $i$ , we have  $a_{i,\sigma(i)} > 0$  and*

$$b_i > \max_{j \in [n] \setminus \{i\}} a_{i,\sigma(j)} \frac{b_j}{a_{j,\sigma(j)}},$$

then  $|A|_\infty \neq 0$ . Moreover, there exists a unique solution  $x^* = \sum_{i \in [n]} \frac{|A^{(i)}|_\infty}{|A|_\infty} e_i \in \mathbb{R}_+^n$  to the system of maximum equations (5.8).

Conversely, suppose that  $|A|_\infty \neq 0$ . If  $x^* = \sum_{i \in [n]} \frac{|A^{(i)}|_\infty}{|A|_\infty} e_i$  is a unique regular solution of the limit system (5.4) then  $x^*$  is a nonnegative solution of the system of maximum equations (5.8).

**Proof:** We first establish that  $|A|_\infty \neq 0$ . Let us consider the system  $A \cdot^p x = b$ . We have established that if

$$b_i > \left( \sum_{j \in [n] \setminus \{i\}} (a_{i,j})^{2p+1} \frac{(b_j)^{2p+1}}{(a_{j,j})^{2p+1}} \right)^{\frac{1}{2p+1}} \quad (1)$$

for all  $i \in [n]$ , then  $A$  is  $\varphi_p$ -invertible and there is a solution  $x^{(p)} \in \mathbb{R}_{++}^n$  to the system  $A \cdot^p x = b$ .



By hypothesis there exists a permutation  $\sigma : [n] \rightarrow [n]$  such that for all  $i$ , we have  $a_{i,\sigma(i)} > 0$  and

$$b_i > \max_{j \in [n] \setminus \{i\}} a_{i,\sigma(j)} \frac{b_j}{a_{j,\sigma(j)}}.$$

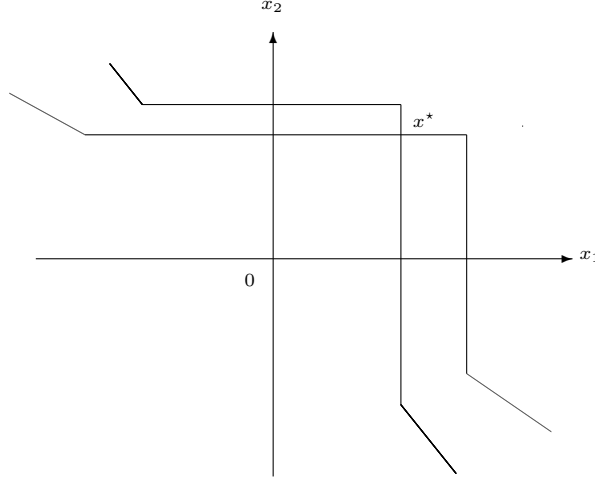
However, we have

$$\max_{j \in [n] \setminus \{i\}} a_{i,\sigma(j)} \frac{b_j}{a_{j,\sigma(j)}} = \lim_{p \rightarrow \infty} \left( \sum_{j \in [n] \setminus \{i\}} (a_{i,\sigma(j)})^{2p+1} \frac{(b_j)^{2p+1}}{(a_{j,\sigma(j)})^{2p+1}} \right)^{\frac{1}{2p+1}}.$$

Hence, there is some  $p_0 \in \mathbb{N}$  such that for all  $p \geq p_0$ , condition (1) is satisfied, which implies that  $A$  is  $\varphi_p$ -invertible. Therefore, for all  $p \geq p_0$  we have  $|A|_p \neq 0$ . From Lemma 3.2.4, the condition  $|A|_\infty = 0$  implies  $|A|_p = 0$  for all  $p$ . Hence, we deduce that  $|A|_\infty \neq 0$ . Moreover, for all  $p \geq p_0$ ,  $x^{(p)} = \sum_{i \in [n]} \frac{|A^{(i)}|_p}{|A|_p} e_i \in \mathbb{R}_{++}^n$  is a solution of the system  $A \cdot^p x = b$ . However  $x^* = \sum_{i \in [n]} \frac{|A^{(i)}|_\infty}{|A|_\infty} e_i = \lim_{p \rightarrow \infty} x^{(p)}$ . It follows that  $x^* \in \mathbb{R}_+^n$ . We only need to prove that for all  $i \in [n]$ ,  $\max_j a_{i,j} x_j^* = b_i$ . We have shown that  $x^* \in \text{Li}_{p \rightarrow \infty} \{x \in \mathbb{R}^n : A \cdot^p x = b\}$ . We also have  $\text{Li}_{p \rightarrow \infty} \{x \in \mathbb{R}^n : A \cdot^p x = b\} \subset \bigcap_{i \in [n]} ( [\langle a_i, \cdot \rangle_\infty^- \leq b_i] \cap [\langle a_i, \cdot \rangle_\infty^+ \geq b_i] )$ . However since  $a_i \geq 0$  for all  $i$ , it follows that for all  $i$ :

$$\langle a_i, x^* \rangle_\infty^- = \langle a_i, x^* \rangle_\infty^+ = \max_j a_{i,j} x_j^* = b_i.$$

Therefore  $x^*$  is a solution of the system of maximum equations. Moreover, from Lemma 5.2.3, this solution is unique. Conversely, if  $|A|_\infty \neq 0$ , then  $x^* \in \text{Li}_{p \rightarrow \infty} \{x \in \mathbb{R}^n : A \cdot^p x = b\}$ . Consequently, if  $x^*$  is regular, we deduce from Lemma 5.2.4 that  $x^*$  is a nonnegative solution system of maximum equations (5.8).  $\square$



**Figure 5.4** Positive Solutions of Limit Systems

Note that if there is a unique positive solution of the system (5.8) then  $x^*$  is identical to the principal solution  $\bar{x}$ . We illustrate these results on simple numerical examples.

**Example 5.2.6** *Let us consider the following system:*

$$\begin{cases} \max\{2x_1, 3x_2\} & = 1 \\ \max\{4x_1, x_2\} & = 1. \end{cases} \quad (5.10)$$

We have  $a_1 = (2, 3)$ ,  $a_2 = (4, 1)$ ,  $b_1 = 1$  and  $b_2 = 1$ , from which we obtain  $\bar{x}_1 = \min\{\frac{1}{2}, \frac{1}{4}\} = \frac{1}{4}$  and  $\bar{x}_2 = \min\{\frac{1}{3}, \frac{1}{1}\} = \frac{1}{3}$ . One can check that  $(\frac{1}{4}, \frac{1}{3})$  is a solution of the system (5.10). Let us consider the matrices:

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \quad A^{(1)} = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \quad A^{(2)} = \begin{pmatrix} 2 & 1 \\ 4 & 1 \end{pmatrix}.$$

We have  $|A|_\infty = (2 \cdot 1 \boxplus (-3 \cdot 4)) = -12$ ;  $|A^{(1)}|_\infty = (1 \cdot 1 \boxplus (-3 \cdot 1)) = -3$ ;  $|A^{(2)}|_\infty = (2 \cdot 1 \boxplus (-1 \cdot 4)) = -4$ . One can then retrieve the principal solution:

$$x_1^* = \frac{1}{4} \quad x_2^* = \frac{-4}{-12} = \frac{1}{3}.$$

In the following a three dimensional example is given.

**Example 5.2.7** Let us consider the following system:

$$\begin{cases} \max\{x_1, 3x_2, 4x_3\} = 1 \\ \max\{2x_1, 5x_2, x_3\} = 1 \\ \max\{4x_1, 2x_2, x_3\} = 1. \end{cases} \quad (5.11)$$

We have  $a_1 = (1, 3, 4)$ ,  $a_2 = (2, 5, 1)$ ,  $a_3 = (4, 2, 1)$ ,  $b_1 = b_2 = b_3 = 1$ , from which we get  $\bar{x}_1 = \min\{1, \frac{1}{2}, \frac{1}{4}\} = \frac{1}{4}$ ,  $\bar{x}_2 = \min\{\frac{1}{2}, \frac{1}{5}, \frac{1}{3}\} = \frac{1}{5}$ , and  $\bar{x}_3 = \min\{\frac{1}{4}, 1, 1\} = \frac{1}{4}$ . One can check that  $(\frac{1}{4}, \frac{1}{5}, \frac{1}{4})$  is a solution of the system (5.11). Let us consider the matrices:

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 5 & 1 \\ 4 & 2 & 1 \end{pmatrix} \quad A^{(1)} = \begin{pmatrix} 1 & 3 & 4 \\ 1 & 5 & 1 \\ 1 & 2 & 1 \end{pmatrix} \quad A^{(2)} = \begin{pmatrix} 1 & 1 & 4 \\ 2 & 1 & 1 \\ 4 & 1 & 1 \end{pmatrix} \quad A^{(3)} = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 4 & 2 & 1 \end{pmatrix}.$$

We have:

$$|A|_\infty = 1 \cdot 5 \cdot 1 \boxplus 3 \cdot 1 \cdot 4 \boxplus 2 \cdot 2 \cdot 4 \boxplus (-4 \cdot 5 \cdot 4) \boxplus (-1 \cdot 2 \cdot 1) \boxplus (-2 \cdot 3 \cdot 1) = -80;$$

$$|A^{(1)}|_\infty = 1 \cdot 5 \cdot 1 \boxplus 3 \cdot 1 \cdot 1 \boxplus 1 \cdot 2 \cdot 4 \boxplus (-4 \cdot 5 \cdot 1) \boxplus (-3 \cdot 1 \cdot 1) \boxplus (-2 \cdot 1 \cdot 1) = -20;$$

$$|A^{(2)}|_\infty = 1 \cdot 1 \cdot 1 \boxplus 2 \cdot 1 \cdot 4 \boxplus 1 \cdot 1 \cdot 4 \boxplus (-4 \cdot 1 \cdot 4) \boxplus (-1 \cdot 2 \cdot 1) \boxplus (-1 \cdot 1 \cdot 1) = -16 \text{ and}$$

$$|A^{(3)}|_\infty = 1 \cdot 5 \cdot 1 \boxplus 2 \cdot 2 \cdot 1 \boxplus 3 \cdot 1 \cdot 4 \boxplus (-4 \cdot 5 \cdot 1) \boxplus (-1 \cdot 2 \cdot 1) \boxplus (-2 \cdot 3 \cdot 1) = -20.$$

One can then retrieve the principal solution:

$$x_1^* = \frac{-20}{-80} = \frac{1}{4} \quad x_2^* = \frac{-16}{-80} = \frac{1}{5} \quad x_3^* = \frac{-20}{-80} = \frac{1}{4}.$$

### 5.3 Limit Two-Sided Systems

Let  $A, C \in \mathcal{M}_n(\mathbb{R})$  and let  $b, d \in \mathbb{R}^n$ . We consider the following system:

$$\begin{cases} (A \boxtimes x) \bar{\circ} d \leq (C \boxtimes x) \bar{\circ} b \\ (A \boxtimes x) \dot{\circ} d \geq (C \boxtimes x) \dot{\circ} b, \quad x \in \mathbb{R}^n \end{cases} \quad (5.12)$$

In the following, we provide a sufficient condition for the existence of a solution. To do that we introduce the matrix:

$$A \boxplus C = (a_{i,j} \boxplus c_{i,j})_{i,j \in [n]}.$$

**Proposition 5.3.1** Let  $A, C \in \mathcal{M}_n(\mathbb{R})$  and let  $b, d \in \mathbb{R}^n$ . If  $|A \boxplus C|_\infty \neq 0$ , then

$$x^* = \sum_{i \in [n]} \frac{|(A \boxplus C)^{(i)}|_\infty}{|A \boxplus C|_\infty} e_i$$

is a solution of the system (5.12), where  $(A \boxplus C)^{(i)}$  is the matrix obtained by replacing the  $i$ -th column with  $b \boxplus d$ . Moreover,  $\{x^*\} = \text{Lim}_{p \rightarrow \infty} \{x \in \mathbb{R}^n : (A \boxplus C)^{\cdot p} x = (b \boxplus d)\}$ . It follows that  $x^*$  is a solution of the limit system:

$$\begin{cases} (A \boxplus C) \bar{\boxtimes} x \leq b \boxplus d \\ (A \boxplus C) \bar{\boxplus} x \geq b \boxplus d, \quad x \in \mathbb{R}^n. \end{cases} \quad (5.13)$$

**Proof:** Let  $(a \boxplus c)_i = a_i \boxplus c_i$  denote the  $i$ -th line of the matrix  $A \boxplus C$ . Moreover, for all natural numbers  $p$ , let us denote  $a_i \overset{p}{-} c_i$  the  $i$ -th line of matrix  $A \overset{p}{-} C$ . We have  $\lim_{p \rightarrow \infty} a_i \overset{p}{-} c_i = a_i \boxplus c_i$  and  $\lim_{p \rightarrow \infty} b_i \overset{p}{-} d_i = b_i \boxplus d_i$ . It follows from Proposition 4.2.1 that for all  $i$ :

$$\text{Li}_{p \rightarrow \infty} [\langle a_i \overset{p}{-} c_i, \cdot \rangle_p \leq b_i \overset{p}{-} d_i] = [\langle a_i \boxplus c_i, \cdot \rangle_\infty \leq b_i \boxplus d_i].$$

Proposition 4.2.1 also implies  $\text{Li}_{p \rightarrow \infty} [\langle a_i \boxplus c_i, \cdot \rangle_p \leq b_i \boxplus d_i] = [\langle a_i \boxplus c_i, \cdot \rangle_\infty \leq b_i \boxplus d_i]$ . Thus, we deduce that:

$$\text{Li}_{p \rightarrow \infty} [\langle a_i \overset{p}{-} c_i, \cdot \rangle_p \leq b_i \overset{p}{-} d_i] = \text{Li}_{p \rightarrow \infty} [\langle a_i \boxplus c_i, \cdot \rangle_p \leq b_i \boxplus d_i].$$

Moreover, we have:

$$\text{Li}_{p \rightarrow \infty} \bigcap_{i \in [n]} [\langle a_i \boxplus c_i, \cdot \rangle_p = b_i \boxplus d_i] \subset \bigcap_{i \in [n]} \text{Li}_{p \rightarrow \infty} [\langle a_i \boxplus c_i, \cdot \rangle_p = b_i \boxplus d_i].$$

Hence, we deduce that

$$\text{Li}_{p \rightarrow \infty} \bigcap_{i \in [n]} [\langle a_i \boxplus c_i, \cdot \rangle_p = b_i \boxplus d_i] \subset \bigcap_{i \in [n]} \text{Li}_{p \rightarrow \infty} [\langle a_i \overset{p}{-} c_i, \cdot \rangle_p \leq b_i \overset{p}{-} d_i].$$

Moreover, since  $|A \boxplus C|_\infty \neq 0$ , from Proposition 5.1.1

$$x^* = \sum_{i \in [n]} \frac{|(A \boxplus C)^{(i)}|_\infty}{|A \boxplus C|_\infty} e_i \in \text{Lim}_{p \rightarrow \infty} \bigcap_{i \in [n]} [\langle a_i \boxplus c_i, \cdot \rangle_p = b_i \boxplus d_i]. \quad (5.14)$$

Hence, we deduce that

$$x^* \in \bigcap_{i \in [n]} \text{Li}_{p \rightarrow \infty} [\langle a_i \overset{p}{-} c_i, \cdot \rangle_p = b_i \overset{p}{-} d_i]. \quad (5.15)$$

For all natural numbers  $p$ , let us denote:  $E_i^{(p)} = \{z \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^2 : \langle (a_i, -c_i, d_i, -b_i), z \rangle_p \leq 0\}$ ,  $F_1 = \{z \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^2 : z_i = z_{i+n} : i \in [n]\}$  and  $F_2 = \{z \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^2 : z_{2n+1} = z_{2n+2} = 1\}$ . However,

$$\{(x, x, 1, 1) \in \mathbb{R}^{2n+2} : \langle a_i \overset{p}{-} c_i, x \rangle_p \leq b_i \overset{p}{-} d_i\} = E_i^{(p)} \cap F_1 \cap F_2.$$

Therefore

$$\text{Li}_{p \rightarrow \infty} \{(x, x, 1, 1) \in \mathbb{R}^{2n+2} : \langle a_i \overset{p}{-} c_i, x \rangle_p \leq b_i \overset{p}{-} d_i\} \subset \text{Li}_{p \rightarrow \infty} (E_i^{(p)} \cap B_1 \cap B_2).$$

It follows that

$$z^* = (x^*, x^*, 1, 1) \in (Li_{p \rightarrow \infty} E_i^{(p)}) \cap (F_1 \cap F_2).$$

However

$$Li_{p \rightarrow \infty} E_i^{(p)} = [\langle (a_i, -c_i, d_i, -b_i), \cdot \rangle_{\infty}^- \leq 0] \cap [\langle (a_i, -c_i, d_i, -b_i), \cdot \rangle_{\infty}^+ \geq 0].$$

Hence:

$$\begin{aligned} (Li_{p \rightarrow \infty} E_i^{(p)}) \cap (F_1 \cap F_2) = \\ \left\{ (x, x, 1, 1) \in \mathbb{R}^{2n+2} : \left( \bigcirc_{j \in [n]} a_{i,j} x_j \right) \bar{\circ} \left( \bigcirc_{j \in [n]} (-c_{i,j}) x_j \right) \bar{\circ} d_i \bar{\circ} (-b_i) \leq 0, \right. \\ \left. \left( \bigoplus_{j \in [n]} a_{i,j} x_j \right) \oplus \left( \bigoplus_{j \in [n]} (-c_{i,j}) x_j \right) \oplus d_i \oplus (-b_i) \geq 0 \right\} \end{aligned}$$

Now, note that, for all real numbers  $\alpha, \beta$

$$\alpha \leq \beta \iff \alpha \bar{\circ} (-\beta) \leq 0 \iff 0 \leq (-\alpha) \oplus \beta.$$

Starting from System (5.12), we have for all  $i \in [n]$ :

$$\begin{aligned} \left( \bigcirc_{j \in [n]} a_{i,j} x_j \right) \bar{\circ} \left( \bigcirc_{j \in [n]} (-c_{i,j}) x_j \right) \bar{\circ} d_i \bar{\circ} (-b_i) \leq 0 \\ \iff \left( \bigcirc_{j \in [n]} a_{i,j} x_j \right) \bar{\circ} d_i \leq \left( \bigcirc_{j \in [n]} c_{i,j} x_j \right) \bar{\circ} b_i \end{aligned}$$

and

$$\begin{aligned} \left( \bigoplus_{j \in [n]} a_{i,j} x_j \right) \oplus \left( \bigoplus_{j \in [n]} (-c_{i,j}) x_j \right) \oplus d_i \oplus (-b_i) \geq 0 \\ \iff \left( \bigoplus_{j \in [n]} a_{i,j} x_j \right) \oplus d_i \geq \left( \bigoplus_{j \in [n]} c_{i,j} x_j \right) \oplus b_i. \end{aligned}$$

Hence from equation (5.15), and since  $(x^*, x^*, 1, 1) \in \bigcap_{i \in [n]} (Li_{p \rightarrow \infty} E_i^{(p)}) \cap (F_1 \cap F_2)$  we deduce that  $x^*$  satisfies system (5.12). To conclude, since  $|A \boxplus B|_{\infty} \neq 0$ , from Proposition 5.1.1 and equation (5.14) we deduce that:

$$\{x^*\} = Lim_{p \rightarrow \infty} \{x \in \mathbb{R}^n : (A \boxplus C)^P x = b \boxplus d\}.$$

Moreover, from Proposition 5.1.2 we deduce that  $x^*$  is a solution of the system (5.13).  $\square$

If the matrices  $A = (a_{i,j})_{i,j \in [n]}$ ,  $C = (c_{i,j})_{i,j \in [n]}$  and the vectors  $b$  and  $d$  have positive entries, the problem of finding a nonnegative solution to the system (5.12) can be written:

$$\begin{cases} \max\{a_{1,1}x_1, \dots, a_{1,n}x_n, d_1\} & = & \max\{c_{1,1}x_1, \dots, c_{1,n}x_n, b_1\} \\ \vdots & & \vdots \\ \max\{a_{n,1}x_1, \dots, a_{n,n}x_n, d_n\} & = & \max\{c_{n,1}x_1, \dots, c_{n,n}x_n, b_n\}. \end{cases} \quad (5.16)$$

A solution of the system (5.12) is said to be **regular** if for all  $i \in [n]$ :

$$\left( \bigcirc_{j \in [n]} a_{i,j} x_j^* \right) \bar{\circ} d_i = \left( \bigoplus_{j \in [n]} a_{i,j} x_j^* \right) \oplus d_i \quad \text{and} \quad \left( \bigcirc_{j \in [n]} c_{i,j} x_j^* \right) \bar{\circ} b_i = \left( \bigoplus_{j \in [n]} c_{i,j} x_j^* \right) \oplus b_i. \quad (5.17)$$

**Proposition 5.3.2** *Let  $A, C \in \mathcal{M}_n(\mathbb{R}_+)$  and let  $b, d \in \mathbb{R}_+^n$ . If  $x^*$  is a regular solution of the system (5.12) then it is a solution of the system (5.16), moreover  $\sum_{i \in [n]} |x_i^*| e_i$  is a nonnegative solution of (5.16).*

**Proof:** Suppose that  $x^*$  is a regular solution system (5.12). Let us denote  $y^* = \sum_{i \in [n]} |x_i^*| e_i$ . For any equation (i), we consider four cases:

(i)  $\bigwedge_{j \in [n]} a_{i,j} x_j^* \bigwedge d_i = b_i$ . In such a case, since  $b_i \geq 0$ ,  $\bigwedge_{j \in [n]} a_{i,j} y_j^* \bigwedge d_i = b_i$

(ii)  $b_i = \bigwedge_{j \in [n]} c_{i,j} x_j^* \bigwedge b_i$ . Similarly, since  $d_i \geq 0$ ,  $\bigwedge_{j \in [n]} c_{i,j} y_j^* \bigwedge b_i = d_i$

(iii) Suppose that (i) and (ii) do not hold. In such a case:

$$\bigwedge_{j \in [n]} a_{i,j} x_j^* \bigwedge d_i = \bigwedge_{j \in [n]} a_{i,j} x_j^* = \bigwedge_{j \in [n]} c_{i,j} x_j^* = \bigwedge_{j \in [n]} c_{i,j} x_j^* \bigwedge b_i.$$

If  $\bigwedge_{j \in [n]} a_{i,j} x_j^* = \bigwedge_{j \in [n]} c_{i,j} x_j^* < 0$ , then there is some  $j_0, k_0 \in [n]$  such that  $x_{j_0}^* < 0$ ,  $x_{k_0}^* < 0$  and such that

$$a_{i,j_0} x_{j_0} = \bigwedge_{j \in [n]} a_{i,j} x_j^* = \bigwedge_{j \in [n]} c_{i,j} x_j^* = c_{i,k_0} x_{k_0}.$$

It follows that

$$\bigwedge_{j \in [n]} a_{i,j} y_j^* = -a_{i,j_0} x_{j_0}^* = -c_{i,k_0} x_{k_0}^* = \bigwedge_{j \in [n]} c_{i,j} y_j^* > 0,$$

which implies that

$$\bigwedge_{j \in [n]} a_{i,j} y_j^* \bigwedge d_i = \bigwedge_{j \in [n]} c_{i,j} y_j^* \bigwedge b_i.$$

Since these properties hold for all  $i$ , we deduce the result.  $\square$

**Example 5.3.3** *Let us consider the system*

$$\begin{cases} \max\{2x_1, x_2, 3\} = \max\{x_1, x_2, 4\} \\ \max\{x_1, 3x_2, 2\} = \max\{2x_1, 2x_2, 3\}. \end{cases} \quad (5.18)$$

We have  $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ ,  $C = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ ,  $b = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ , and  $d = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$$A \boxplus C = \begin{pmatrix} 2 & 0 \\ -2 & 3 \end{pmatrix} \quad \text{and} \quad b \boxplus d = \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

It follows that:

$$(A \boxplus C)^{(1)} = \begin{pmatrix} 4 & 0 \\ 3 & 3 \end{pmatrix} \quad \text{and} \quad (A \boxplus C)^{(2)} = \begin{pmatrix} 2 & 4 \\ -2 & 3 \end{pmatrix}.$$

We obtain

$$x_1^* = \frac{12}{6} = 2 \quad \text{and} \quad x_2^* = \frac{8}{6} = \frac{4}{3}.$$

## 5.4 Some Remarks on the Symmetrisation of Idempotent Semirings

The above algebraic structure can be viewed as some kind of non-associative symmetrization of the idempotent semi-ring  $(\mathbb{R}_+, \vee, \cdot)$ . However there exists another approach to construct a ring involving a balance relation and symmetrizing  $(\mathbb{R}_+, \vee, \cdot)$  (see [21] and [28] in a Max-Plus context). Following the usual construction of integers from natural numbers, one can introduce the following balance relation defined on  $\mathbb{R}_+^2 \times \mathbb{R}_+^2$  by:

$$(x_+, x_-) \nabla (y_+, y_-) \iff \max\{x_+, y_-\} = \max\{y_+, x_-\}, \quad (5.19)$$

where  $x_+, x_-, y_+, y_- \in \mathbb{R}_+$ . Let us denote  $\mathbf{x} = (x_+, x_-)$  for all  $(x_+, x_-) \in \mathbb{R}_+^2$  and consider the quotient  $\mathbb{S} = \mathbb{R}_+^2 \setminus \nabla$ . Let us define the operations  $\oplus$  and  $\otimes$  on  $\mathbb{S}$  as:

$$\mathbf{x} \oplus \mathbf{y} = (x_+ \oplus y_+, x_- \oplus y_-) = (\max\{x_+, y_+\}, \max\{x_-, y_-\}), \quad (5.20)$$

and

$$\mathbf{t} \otimes \mathbf{x} = (t_+ x_+ \oplus t_- x_-, t_+ x_- \oplus t_- x_+). \quad (5.21)$$

$\mathbb{S}$  can be decomposed in three equivalence classes  $\mathbb{S}_\oplus$ ,  $\mathbb{S}_\ominus$  and  $\mathbb{S}_\circ$  respectively associated to the sets  $\{(x_+, t) : t < x_+\}$  (called positive),  $\{(t, x_-) : t < x_-\}$  (called negative) and  $\{(x_\circ, x_\circ)\}$  called balanced. All the familiar identities valid in rings admit analogues replacing equalities by balances. This means that associativity holds over  $\mathbb{S}$ . It follows that the binary operation  $\oplus$  defined on  $\mathbb{S}$  cannot be identified to the binary operation  $\boxplus$ . However, it can be related to the semi-continuous regularized operators  $\bar{\cdot}$  and  $\hat{\cdot}$ .

Let  $V : \mathbb{S} \rightarrow \mathbb{R}$  be the map defined as  $V(\oplus x_+) = x_+$  for all  $x_+ \in \mathbb{R}_+$ ,  $V(\ominus x_-) = -x_-$  for all  $x_- \in \mathbb{R}_+$ , and  $V(x_\circ, x_\circ) = 0$  for all  $x_\circ \in \mathbb{R}_+$ . Suppose that  $(\mathbf{x}_1, \dots, \mathbf{x}_m) \in \mathbb{S}^m$ . Then

$$V\left(\bigoplus_{i \in [m]} \mathbf{x}_i\right) = V\left(\max_{i \in [n]} x_{i,+}, \max_{i \in [n]} x_{i,-}\right) = \frac{1}{2} \left( \overset{+}{\smile}_{i \in [m]} V(\mathbf{x}_i) + \overset{-}{\smile}_{i \in [m]} V(\mathbf{x}_i) \right). \quad (5.22)$$

Suppose that  $\mathbf{A} = (\mathbf{a}_{i,j})_{\substack{i=1 \dots n \\ j=1 \dots n}} \in \mathcal{M}_n(\mathbb{S})$ . A determinant can be derived from this associative algebraic structure as:

$$|\mathbf{A}|_{\mathbb{S}} = \bigoplus_{\sigma \in S_n} \mathbf{sgn}(\sigma) \bigotimes_{i \in [n]} \mathbf{a}_{i, \sigma(i)}, \quad (5.23)$$

where  $\mathbf{sgn}(\sigma) = \oplus 1$  if  $\sigma$  is even and  $\mathbf{sgn}(\sigma) = \ominus 1$  if  $\sigma$  is odd. Suppose that  $A$  is a  $3 \times 3$ -dimensional real matrix

$$A = \begin{pmatrix} 3 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 1 & 3 \end{pmatrix}.$$

The positive components of  $A$  can be identified to  $\mathbb{S}_\oplus$ . If  $\mathbf{A}$  is the corresponding matrix, then  $|\mathbf{A}|_{\mathbb{S}} = (27, 27) \nabla \mathbf{0}$  and we cannot derive a Cramer solution. However, one can check that  $|A|_{\infty} = 12 \neq 0$ .

The symmetrization process described above is in general used in the context of Maslov's semi module where we replace  $\vee$  with  $\oplus$  and  $+$  with  $\otimes$  [24]. Applications can be found in [23] and [26] for Max-Plus. To be more precise, let  $\mathbb{M} = \mathbb{R} \cup \{-\infty\}$ . For  $x$  and  $y$  in  $\mathbb{M}^n$  let  $d_{\mathbb{M}_+}(x, y) = \| \mathbf{e}^x - \mathbf{e}^y \|_{\infty}$  where  $\mathbf{e}^x = (e^{x_1}, \dots, e^{x_n})$ , with the convention  $e^{-\infty} = 0$ , and, for  $u \in \mathbb{R}_+^n$ ,

$\|u\| = \max_{i \in [n]} x_i$ . The map  $x \mapsto \mathbf{e}^x$  is a homeomorphism from  $\mathbb{M}^n$  with the metric  $d_{\mathbb{M}^n}$  to  $\mathbb{R}_+^n$  endowed with the metric induced by the norm  $\|\cdot\|_\infty$ ; its inverse is the map  $\mathbf{ln}(x) = (\ln(x_1), \dots, \ln(x_n))$  from  $\mathbb{R}_+^n$  to  $\mathbb{M}^n$ , with the convention  $\ln(0) = -\infty$ . For all  $t_1, \dots, t_m \in \mathbb{M}$  and all  $x_1, \dots, x_m \in \mathbb{M}^n$ , let us denote:

$$\bigoplus_{i=1}^m t_i \otimes x_i = \bigvee_{i=1}^m (x_i + t_i \mathbb{1}_n). \quad (5.24)$$

In Chapter 3 of [23] it was shown that Max-Plus algebra can also be viewed as a limit algebraic structure via the dequantization principle. For any natural number  $p$ , let  $\rho_p : \mathbb{M} \rightarrow \mathbb{R}_+$  be the map defined for all  $x \in \mathbb{M}$  as  $\rho_p(x) = e^{(2p+1)x}$ . We obviously have  $\rho_p(-\infty) = 0$ . Consequently,  $\rho_p$  is an homeomorphism from  $\mathbb{M}$  to  $\mathbb{R}_+$ . Let us consider the binary operation  $\oplus_p$  defined for all  $x, y \in \mathbb{M}$  as  $x \oplus_p y = \rho_p^{-1}(\rho_p(x) + \rho_p(y)) = \frac{1}{2p+1} \ln(e^{(2p+1)x} + e^{(2p+1)y})$ . We have here replaced the continuous parameter  $h$  with  $\frac{1}{2p+1}$  to keep notations homogeneous. From [23], it can easily be checked that  $\lim_{p \rightarrow \infty} x \oplus_p y = x \oplus y = \max\{x, y\}$ . Moreover, note that for all  $p$  the product  $\rho_p^{-1}(\rho_p(x) \cdot \rho_p(y)) = x + y = x \otimes y$  is invariant.

In the following a non-associative symmetrisation is proposed. Suppose now that  $x \in \mathbb{R}_-$  and let us extend the logarithm function to the whole set of real numbers. This we do by introducing the set

$$\tilde{\mathbb{M}} = \mathbb{M} \cup (\mathbb{R} + i\pi) \quad (5.25)$$

where  $i$  is the complex number such that  $i^2 = -1$  and  $\mathbb{R} + i\pi = \{x + i\pi : x \in \mathbb{R}\}$ . In the following we extend the logarithmic function to  $\tilde{\mathbb{M}}$ . Let  $\psi_{\ln} : \mathbb{R} \rightarrow \tilde{\mathbb{M}}$  be the map defined by:

$$\psi_{\ln}(x) = \begin{cases} \ln(x) & \text{if } x > 0 \\ -\infty & \text{if } x = 0 \\ \ln(-x) + i\pi & \text{if } x < 0. \end{cases} \quad (5.26)$$

The map  $x \mapsto \psi_{\ln}(x)$  is an isomorphism from  $\mathbb{R}$  to  $\tilde{\mathbb{M}}$ . Let  $\psi_{\exp}(x) : \tilde{\mathbb{M}} \rightarrow \mathbb{R}$  be its inverse. Notice that  $\psi_{\ln}(-1) = i\pi$ . The scalar multiplication is extended to the binary operation  $\tilde{\otimes} : \tilde{\mathbb{M}} \times \tilde{\mathbb{M}} \rightarrow \tilde{\mathbb{M}}$  defined by

$$\begin{cases} x \tilde{\otimes} y = x + y \\ x \tilde{\otimes} (y + i\pi) = x + y + i\pi \\ (x + i\pi) \tilde{\otimes} (y + i\pi) = x + y \\ (x + i\pi) \tilde{\otimes} -\infty = -\infty \end{cases} \quad (5.27)$$

For all  $z \in \tilde{\mathbb{M}}$  the symmetrical element is  $\tilde{z} = i\pi \otimes z$ . One can then introduce a corresponding absolute value function  $|\cdot|_{\tilde{\mathbb{M}}} : \tilde{\mathbb{M}} \rightarrow \mathbb{R} \cup \{-\infty\}$  defined by:

$$|z|_{\tilde{\mathbb{M}}} = \begin{cases} z - i\pi & \text{if } z \in \mathbb{R} + i\pi \\ z & \text{if } z \in \mathbb{R} \\ -\infty & \text{if } z = -\infty. \end{cases} \quad (5.28)$$

This absolute value allows us to define the following binary operation on  $\tilde{\mathbb{M}} \times \tilde{\mathbb{M}}$ :

$$z \tilde{\boxplus} u = \begin{cases} z & \text{if } |z|_{\tilde{\mathbb{M}}} > |u|_{\tilde{\mathbb{M}}} \\ z & \text{if } z = u \\ -\infty & \text{if } \tilde{z} = u \\ u & \text{if } |z|_{\tilde{\mathbb{M}}} < |u|_{\tilde{\mathbb{M}}}. \end{cases} \quad (5.29)$$

By definition we have  $z \widetilde{\boxplus} u = \psi_{\ln}(\psi_{\exp}(z) \boxplus \psi_{\exp}(u))$ . Moreover, we have  $z \widetilde{\otimes} u = \psi_{\ln}(\psi_{\exp}(z) \otimes \psi_{\exp}(u))$ . For all  $z \in \widetilde{\mathbb{M}}^n$ , let us denote:

$$\widetilde{\boxplus}_{i \in [n]} z_i = \psi_{\ln} \left( \boxplus_{i \in [n]} \psi_{\exp}(z_i) \right). \quad (5.30)$$

In the remainder, we introduce a sign function  $\widetilde{\text{sgn}}$  defined on  $S_p$  such that  $\widetilde{\text{sgn}}(\sigma) = 0$  if  $\sigma$  is even and  $\widetilde{\text{sgn}}(\sigma) = i\pi$  if  $\sigma$  is odd. Suppose that  $A$  is a square matrix of  $\mathcal{M}_n(\widetilde{\mathbb{M}})$ . The symmetrized determinant defined on  $\widetilde{\mathbb{M}}$  is now:

$$|A|_{\widetilde{\mathbb{M}}, \infty} = \psi_{\ln}(|\psi_{\exp}(A)|_{\infty}) = \widetilde{\boxplus}_{\sigma \in S_n} (\widetilde{\text{sgn}}(\sigma) \widetilde{\otimes}_{i \in [n]} a_{i, \sigma(i)}). \quad (5.31)$$

We end this section by showing that the dequantization principle can also be applied to retrieve as a limit case the operation  $\widetilde{\boxplus}$ .

**Lemma 5.4.1** *For all  $z \in \widetilde{\mathbb{M}}^n$ , we have*

$$\widetilde{\boxplus}_{i \in [n]} z_i = \lim_{p \rightarrow \infty} \frac{1}{2p+1} \psi_{\ln} \left( \sum_{i \in [n]} \psi_{\exp}((2p+1)z_i) \right).$$

**Proof:** Let  $\alpha : \widetilde{\mathbb{M}} \rightarrow \{-\infty, 0, i\pi\}$  be the map defined as:

$$\alpha(z) = \begin{cases} i\pi & \text{if } z \in \mathbb{R} + i\pi \\ 0 & \text{if } z \in \mathbb{R} \\ -\infty & \text{if } z = -\infty. \end{cases}$$

This map is the analogue of a sign function defined on  $\widetilde{\mathbb{M}}$ . By construction, for all  $i$  we have from equation (5.28):

$$z_i = \alpha(z_i) + |z_i|_{\widetilde{\mathbb{M}}}$$

with  $|z_i|_{\widetilde{\mathbb{M}}} \in \mathbb{R} \cup \{-\infty\}$ . Moreover

$$\psi_{\exp}(\alpha(z_i) + |z_i|_{\widetilde{\mathbb{M}}}) = \psi_{\exp}(\alpha(z_i)) \psi_{\exp}(|z_i|_{\widetilde{\mathbb{M}}}).$$

Since  $\psi_{\exp}(i\pi) = -1$ ,  $\psi_{\exp}(0) = 1$  and  $\psi_{\exp}(-\infty) = 0$ , we deduce that

$$(\psi_{\exp}(z_1), \dots, \psi_{\exp}(z_n)) \in \mathbb{R}^n.$$

Therefore:

$$\begin{aligned} \lim_{p \rightarrow \infty} \left( \sum_{i \in [n]} \psi_{\exp}((2p+1)z_i) \right)^{\frac{1}{2p+1}} &= \lim_{p \rightarrow \infty} \left( \sum_{i \in [n]} (\psi_{\exp}(z_i))^{2p+1} \right)^{\frac{1}{2p+1}} \\ &= \widetilde{\boxplus}_{i \in [n]} \psi_{\exp}(z_i). \end{aligned}$$

It follows that

$$\lim_{p \rightarrow \infty} \frac{1}{2p+1} \psi_{\ln} \left( \sum_{i \in [n]} \psi_{\exp}((2p+1)z_i) \right) = \psi_{\ln} \left( \widetilde{\boxplus}_{i \in [n]} \psi_{\exp}(z_i) \right) = \widetilde{\boxplus}_{i \in [n]} z_i,$$

which ends the proof.  $\square$



## 6 Eigenvalues in Limit

In the following, we say that  $\lambda \in \mathbb{R}$  is **an eigenvalue of  $A$  in limit**, if: (1) there exists a sequence  $\{(\lambda_p, v_p)\}_{p \in \mathbb{N}} \subset \mathbb{R} \times \mathbb{R}^n$  such that for all  $p$ ,  $v_p \neq 0$  and  $A^p v_p = \lambda_p v_p$ ; (2) there is an increasing sequence  $\{p_k\}_{k \in \mathbb{N}}$  with  $\lim_{k \rightarrow \infty} (\lambda_{p_k}, v_{p_k}) = (\lambda, v)$  and  $v \neq 0$ .  $v$  is called **an eigenvector in limit of  $A$** .

We start with the following intermediary result which will be useful in the following. We say that for all  $\lambda \in \mathbb{R}$ ,  $P_A^{(p)}(\lambda) = |A - \lambda I_n|_p$  where  $I_n$  is the  $n$ -dimensional identity matrix, is a  $\varphi_p$ -**characteristic polynomial in  $\lambda$** .

**Proposition 6.0.1** *Let  $A \in \mathcal{M}_n(\mathbb{R})$  be a square matrix. For all  $\lambda \in \mathbb{R}$ , the  $\varphi_p$ -characteristic polynomial  $P_A^{(p)}(\lambda)$  in  $\lambda$  is*

$$P_A^{(p)}(\lambda) = \sum_{k \in \{0\} \cup [n]}^{\varphi_p} (-1)^{n-k} \sum_{\substack{I \subset [n] \\ \text{Card}(I)=k}}^{\varphi_p} \sum_{\sigma \in S_I}^{\varphi_p} (\text{sign}(\sigma) \prod_{i \in I} a_{i, \sigma(i)}) \lambda^{n-k},$$

where  $S_I$  denotes the set of all the permutations defined on  $I$ . Moreover for all  $\lambda \in \mathbb{R}$

$$P_A^{(\infty)}(\lambda) = \lim_{p \rightarrow \infty} P_A^{(p)}(\lambda) = \bigsqcup_{\substack{I \subset [n], \text{Card}(I)=k \\ \sigma \in S_I, k \in \{0\} \cup [n]}} (-1)^{n-k} (\text{sign}(\sigma) \prod_{i \in I} a_{i, \sigma(i)}) \lambda^{n-k}.$$

**Proof:** The first part is derived using the method of expanding a characteristic polynomial with respect to its principal minors and making the formal substitution  $+ \mapsto \frac{p}{+}$ . Since the empty product is by convention equal to the multiplicative identity, the coefficient of the highest degree term is equal to 1. Let

$$\Xi_n = \bigcup_{k \in \{0\} \cup [n]} \{\sigma : \sigma \in S_I, I \subset [n], \text{Card}(I) = k\}.$$

If  $\text{Card}(I) = k$ , then  $\text{Card}(S_I) = k!$ . Therefore  $\text{Card}(\Xi_n) = \sum_{k=0}^n k! C_n^k$ . Let us denote  $q_n = \sum_{k=0}^n k! C_n^k$  and let  $\mathcal{B}_{q_n} = \{e_1, \dots, e_{q_n}\}$  be the canonical basis of  $\mathbb{R}^{q_n}$ . Since  $\text{Card}(\Xi_n) = q_n$ , there exists a bijective index function  $j : \Xi_n \mapsto [q_n]$  which associates with each permutation of  $\Xi_n$  an integer in  $[q_n]$ . Therefore, we have  $\mathcal{B}_{q_n} = \{e_{j(\sigma)}\}_{\sigma \in \Xi_n}$ . Let us consider the vector  $\gamma_A \in \mathbb{R}^{q_n}$  defined as:

$$\gamma_A = \sum_{k \in \{0\} \cup [n]} \sum_{\substack{\sigma \in S_I \\ I \subset [n], \text{Card}(I)=k}} (\text{sign}(\sigma) \prod_{i \in I} a_{i, \sigma(i)}) e_{j(\sigma)}. \quad (6.1)$$

Let us introduce the transformation  $\tau_A : \mathbb{R} \rightarrow \mathbb{R}^{q_n}$  defined by

$$\tau_A(\lambda) = \sum_{k \in \{0\} \cup [n]} \sum_{\substack{\sigma \in S_I \\ I \subset [n], \text{Card}(I)=k}} \lambda^{n-k} e_{j(\sigma)}. \quad (6.2)$$

An elementary calculus shows that, for all  $\lambda \in \mathbb{R}$

$$P_A^{(p)}(\lambda) = \langle \gamma_A, \tau_A(\lambda) \rangle_p.$$

For all  $u \in \mathbb{R}^{q_n}$ , we have  $\lim_{p \rightarrow \infty} \langle \gamma_A, u \rangle_p = \langle \gamma_A, u \rangle_\infty$ . Hence,  $P_A^\infty(\lambda) = \lim_{p \rightarrow \infty} \langle \gamma_A, \tau_A(\lambda) \rangle_p = \langle \gamma_A, \tau_A(\lambda) \rangle_\infty$ .  $\square$

$P_A^\infty$  is called the limit characteristic polynomial. Let us introduce now the lower and upper characteristic polynomials, respectively defined by

$$P_{A,-}^{(\infty)}(\lambda) = \underset{I \subset [n], \text{Card}(I)=k}{\overset{\smile}{\sigma \in S_I, k \in \{0\} \cup [n]}} (-1)^{n-k} (\text{sign}(\sigma) \prod_{i \in I} a_{i, \sigma(i)}) \lambda^{n-k} \quad (6.3)$$

and

$$P_{A,+}^{(\infty)}(\lambda) = \underset{I \subset [n], \text{Card}(I)=k}{\overset{\smile}{\sigma \in S_I, k \in \{0\} \cup [n]}} (-1)^{n-k} (\text{sign}(\sigma) \prod_{i \in I} a_{i, \sigma(i)}) \lambda^{n-k}. \quad (6.4)$$

**Proposition 6.0.2** *Let  $A \in \mathcal{M}_n(\mathbb{R})$  be a square matrix. We have:*

$$\text{Lim}_{p \rightarrow \infty} [P_A^{(p)} = 0] = [P_{A,-}^\infty \leq 0] \cap [P_{A,+}^\infty \geq 0] = [P_{A,-}^\infty + P_{A,+}^\infty = 0].$$

Moreover,  $\lambda \in \mathbb{R}$  is an eigenvalue in limit if and only if:

$$\lambda \in [P_{A,-}^\infty + P_{A,+}^\infty = 0].$$

**Proof:** Let us denote  $\gamma_A$  and  $\tau_A(\lambda)$  respectively as in equation (6.1) and (6.2). From Proposition 4.2.2, we have  $\text{Lim}_{p \rightarrow \infty} [\langle \gamma_A, \cdot \rangle_p = 0] = [\langle \gamma_A, \cdot \rangle_\infty^- \leq 0] \cap [\langle \gamma_A, \cdot \rangle_\infty^+ \geq 0]$ . Since that map  $\tau_A$  is continuous,  $\text{Lim}_{p \rightarrow \infty} [P_A^{(p)} = 0] = \text{Lim}_{p \rightarrow \infty} [\langle \gamma_A, \tau_A(\cdot) \rangle_p = 0] = [\langle \gamma_A, \tau_A(\cdot) \rangle_\infty^- \leq 0] \cap [\langle \gamma_A, \tau_A(\cdot) \rangle_\infty^+ \geq 0]$ . However, we have for all  $\lambda$

$$P_{A,-}^{(\infty)}(\lambda) = \langle \gamma_A, \tau_A(\lambda) \rangle_\infty^- \quad \text{and} \quad P_{A,+}^{(\infty)}(\lambda) = \langle \gamma_A, \tau_A(\lambda) \rangle_\infty^+.$$

Hence  $\text{Lim}_{p \rightarrow \infty} [P_A^{(p)} = 0] = [P_{A,-}^\infty \leq 0] \cap [P_{A,+}^\infty \geq 0]$ . The last equality is an immediate consequence of Lemma 4.1.3. To complete the proof, from Proposition 4.2.2,  $\lambda \in [P_{A,-}^\infty \leq 0] \cap [P_{A,+}^\infty \geq 0]$  if and only if there is an increasing sequence  $\{p_q\}_{q \in \mathbb{N}}$  and a sequence of real numbers  $\{\lambda_{p_q}\}_{q \in \mathbb{N}}$  such that  $\lim_{q \rightarrow \infty} \lambda_{p_q} = \lambda$  and  $P_A^{(p_q)}(\lambda_{p_q}) = \langle \gamma_A, \tau_A(\lambda_{p_q}) \rangle_{p_q} = 0$  for all  $q$ . Hence, since  $\tau_A$  is continuous,  $\lambda$  is an eigenvalue in limit, if and only if:

$$\tau_A(\lambda) \in \text{Lim}_{p \rightarrow \infty} [\langle \gamma_A, \cdot \rangle_p = 0] = [\langle \gamma_A, \cdot \rangle_\infty^- \leq 0] \cap [\langle \gamma_A, \cdot \rangle_\infty^+ \geq 0].$$

From Lemma 4.1.3 this condition is equivalent to

$$\tau_A(\lambda) \in [\langle \gamma_A, \cdot \rangle_\infty^- + \langle \gamma_A, \cdot \rangle_\infty^+ = 0].$$

Equivalently, we have:

$$\langle \gamma_A, \tau_A(\lambda) \rangle_\infty^- + \langle \gamma_A, \tau_A(\lambda) \rangle_\infty^+ = 0.$$

Thus  $\lambda \in \mathbb{R}$  is an eigenvalue in limit if and only if  $P_{A,-}^\infty(\lambda) + P_{A,+}^\infty(\lambda) = 0$ , which ends the proof.  $\square$

Given a square matrix with positive entries  $A \in \mathcal{M}_n(\mathbb{R}_{++})$ , the Perron-Frobenius theorem states that there is an eigenvalue called the spectral radius of  $A$  and denoted by  $\rho_A$  such that  $\rho_A \geq |\lambda|$  for all eigenvalues of  $A$ , where  $|\cdot|$  denotes the module of  $\lambda$ .  $\rho_A$  is related to an eigenvector  $v_A \in \mathbb{R}_{++}^n$ , with  $Av_A = \rho_A v_A$ .  $\lambda > 0$  is an eigenvalue in the sense of the matrix product  $\boxtimes$  (a  $\boxtimes$ -eigenvalue) if there is a positive vector  $v \in \mathbb{R}_+^n$  such that  $A \boxtimes v = \lambda v$ . In such a case  $\lambda$  is also called a geometric Max-Times eigenvalue. We say that  $\lambda$  is a

$\varphi_p$ -eigenvalue of  $A$  if  $A \cdot^p v = \lambda v$  for some vector  $v \in \mathbb{R}^n \setminus \{0\}$ . If  $A \in \mathcal{M}_n(\mathbb{R}_{++})$ , then  $\Phi_p(A) \in \mathcal{M}_n(\mathbb{R}_{++})$ . Hence  $\Phi_p(A)$  is endowed with a spectral radius  $\rho_{\Phi_p(A)}$ , and a vector  $u_A^{(p)} \in \mathbb{R}_{++}^n$  such that  $\Phi_p(A)u_A^{(p)} = \rho_{\Phi_p(A)}u_A^{(p)}$ . It follows that setting  $v_A^{(p)} = \phi_p^{-1}(u_A^{(p)})$  and  $\rho_A^{(p)} = \varphi_p^{-1}(\rho_{\Phi_p(A)})$  that

$$A \cdot^p v_A^{(p)} = \rho_A^{(p)} v_A^{(p)}. \quad (6.5)$$

In such a case,  $\rho_A^{(p)}$  is called a  $\varphi_p$ -spectral radius of  $A$ . Notice that in the case where  $A \in \mathcal{M}_n(\mathbb{R}_{++})$  there is only one  $\boxtimes$ -eigenvalue in  $\mathbb{R}_{++}$  (see for instance [13]).

The next result was proven in [6] (Proof 4 on page 7 and Theorem 8) which also mentions relation (6.5). The eigenvalue part of Proposition 6.0.3 can also be found in [17] and [18]. Additional related results can be found in [25] (Theorems 2.3 and 2.6).

**Proposition 6.0.3** *Let  $A \in \mathcal{M}_n(\mathbb{R}_{++})$  be a square matrix. For  $\lambda \in \mathbb{R}_{++}$  and all vectors  $v \in \mathbb{R}_+^n \setminus \{0\}$  such that  $A \boxtimes v = \lambda v$ , there is an increasing subsequence  $\{p_q\}_{q \in \mathbb{N}}$  such that  $\lambda = \lim_{q \rightarrow \infty} \rho_A^{(p_q)}$  and  $v = \lim_{q \rightarrow \infty} v_A^{(p_q)}$  where for all  $p$ ,  $A \cdot^p v_A^{(p)} = \rho_A^{(p)} v_A^{(p)}$ .*

**Proposition 6.0.4** *Let  $A \in \mathcal{M}_n(\mathbb{R}_{++})$  be a square matrix. If  $\lambda \in \mathbb{R}_{++}$  is a  $\boxtimes$ -eigenvalue then  $\lambda \in [P_{A,-}^\infty + P_{A,+}^\infty = 0]$ . Moreover, if  $\lambda$  is maximal in  $[P_{A,-}^\infty + P_{A,+}^\infty = 0]$ , then it is a  $\boxtimes$ -eigenvalue and  $\lambda = \lim_{q \rightarrow \infty} \rho_A^{(p_q)}$ .*

**Proof:** From Proposition 6.0.3, we have  $\lambda = \rho_A^\infty$ , we deduce that

$$\lambda \in \text{Lim}_{p \rightarrow \infty} [P_A^{(p)} = 0] = [P_{A,-}^\infty \leq 0] \cap [P_{A,+}^\infty \geq 0] = [P_{A,-}^\infty + P_{A,+}^\infty = 0].$$

Conversely, suppose that  $\lambda \in [P_{A,-}^\infty \leq 0] \cap [P_{A,+}^\infty \geq 0]$  and assume that  $\lambda$  is maximal in  $[P_{A,-}^\infty + P_{A,+}^\infty = 0]$ . Then there is a real sequence  $\{\lambda^{(p)}\}_{p \in \mathbb{N}}$  with  $\lambda^{(p)} \in [P_A^{(p)} = 0]$  for all natural numbers  $p$  and such that  $\lim_{p \rightarrow \infty} \lambda^{(p)} = \lambda$ . Let  $\{\rho_A^{(p)}\}_{p \in \mathbb{N}}$  such that  $\rho_A^{(p)}$  is a  $\varphi_p$ -Perron-Frobenius eigenvalue for all  $p$ . From Proposition 6.0.3, there is a  $\boxtimes$ -eigenvalue  $\mu$  such that  $\mu = \lim_{p \rightarrow \infty} \rho_A^{(p)}$ . It follows that  $\mu \in [P_{A,-}^\infty \leq 0] \cap [P_{A,+}^\infty \geq 0] = [P_{A,-}^\infty + P_{A,+}^\infty = 0]$ . Suppose that  $\mu \neq \lambda$  and let us show a contradiction. Since  $\lambda$  is maximal, this implies that  $\mu < \lambda$ . However, this also implies that there is some  $p_0 \in \mathbb{N}$  such that for all  $p > p_0$ ,  $\lambda^{(p)} > \rho_A^{(p)}$ , which is a contradiction. Consequently,  $\mu = \lambda$  and it follows that  $\lambda$  is a  $\boxtimes$ -eigenvalue, which ends the proof.  $\square$

In the following, we propose several examples that we relate to the literature on tropical matrix algebra. For that we are going to [transfer](#) in a Max-Times context several notions used in Max-Plus algebras. In particular, we study the notion of algebraic eigenvalue analysed in [13] and [2]. A  $\boxplus$ -polynomial function is a map  $P_{\boxplus} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  of the form  $P_{\boxplus}(\lambda) = \boxplus_{i \in [n]} p_i \lambda^i = \max_{i \in [n]} p_i \lambda^i$ . If  $p_n \neq 0$ ,  $P_{\boxplus}$  is of degree  $n$ . Suppose that all the entries of  $A$  are positive. The roots of a polynomial function  $P_{\boxplus}$  are the points of non-differentiability of  $P_{\boxplus}$ . All these points can be seen as exponential transformations of the corner points defined in [13]. The notion of maximum cycle mean arising in Max-Plus algebra can be [transferred](#) in a Max-Times context. It is the maximum of the geometric weight-to-length ratio over all cycles (see [13] and [2] in a Max-Plus context). It can be seen as the spectral radius of a matrix in the context of the Max-Times semi-ring:

$$\rho(A) = \max_{k \geq 1} \max_{i_1, \dots, i_k} (a_{i_1, i_2} \times \dots \times a_{i_k, i_1})^{\frac{1}{k}}. \quad (6.6)$$

The permanent polynomial characteristic of  $A$  is

$$\chi_A(\lambda) = \text{per}(A \boxplus \lambda I_n). \quad (6.7)$$

It is shown in [13] that the greatest corner (tropical roots) of the Max-Plus permanent polynomial is the corresponding maximum cycle mean in Max-Plus algebra (Theorem 5.3.4; see also [2]). As the exponential function is increasing, the ranking of the roots is not modified in a Max-Times context.

**Example 6.0.5** Let us consider the matrix  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ . The geometric mean of elementary cycles of length 1, are 2, 2. The geometric mean of the cycle of length 2 is 1. The permanent polynomial characteristic of  $A$  is  $\chi_A(\lambda) = (2 \boxplus \lambda)^2 \boxplus 1 = \max\{\max\{2, \lambda\}^2, 1\} = (\lambda \boxplus 2)^2$ . Clearly  $\rho = 2$  is a  $\boxtimes$ -eigenvalue and  $v = (1, 1)$  is a  $\boxtimes$ -eigenvector, since  $A \boxtimes v = 2v$ . Moreover  $\rho = 2$  is also an algebraic eigenvalue of  $A$  in the idempotent semiring Max-Times. The  $\varphi_p$  Perron-Frobenius eigenvalue is  $\rho_A^{(p)} = (2^{2p+1} + 1^{2p+1})^{\frac{1}{2p+1}}$  and we have  $\lim_{p \rightarrow \infty} \rho_A^{(p)} = 2$ . We have

$$P_A^{(p)}(\lambda) = ((\lambda^2)^{2p+1} - (2\lambda)^{2p+1} - (2\lambda)^{2p+1} + 4^{2p+1} - 1)^{\frac{1}{2p+1}}.$$

Hence, taking the limit yields:

$$P_A^\infty(\lambda) = (\lambda^2) \boxplus (-2\lambda) \boxplus (-2\lambda) \boxplus 4 \boxplus (-1).$$

Moreover

$$P_{A,-}^\infty(\lambda) = (\lambda^2) \ominus (-2\lambda) \ominus (-2\lambda) \ominus 4 \ominus (-1)$$

and

$$P_{A,+}^\infty(\lambda) = (\lambda^2) \oplus (-2\lambda) \oplus (-2\lambda) \oplus 4 \oplus (-1).$$

We have  $P_{A,-}^\infty(2) = -4 \leq 0$  and  $P_{A,+}^\infty(2) = 4 \geq 0$ .

**Example 6.0.6** Let us consider the matrix  $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 9 \\ 1 & 1 & 3 \end{pmatrix}$ . The geometric

mean of elementary cycles of length 1, are 1, 2, 3, of length 2 are 1 2, 3, of length 3 are  $18^{\frac{1}{3}}, 2^{\frac{1}{3}}$ . Clearly  $\rho = 3$  is a  $\boxtimes$ -eigenvalue and  $v = (2, 3, 1)$  is a  $\boxtimes$ -eigenvector, since  $A \boxtimes v = 3v$ . The permanent polynomial characteristic of  $A$  is

$$\chi_A(\lambda) = (1 \boxplus \lambda)(2 \boxplus \lambda)(3 \boxplus \lambda) \boxplus 2 \boxplus 18 \boxplus (2 \boxplus \lambda) \boxplus 4(3 \boxplus \lambda) \boxplus 9(1 \boxplus \lambda).$$

An elementary simplification yields

$$\chi_A(\lambda) = \lambda^3 \boxplus 3\lambda^2 \boxplus 9\lambda \boxplus 18.$$

The factorization (see the resolution algorithm proposed in [13], p. 108) is:

$$\chi_A(\lambda) = (\lambda \boxplus 3)^2(\lambda \boxplus 2).$$

It follows that the roots of the permanent characteristic polynomial are  $\rho = 3$  (the greatest root) and  $\mu = 2$ . These roots are the algebraic eigenvalues of  $A$ . However only  $\rho = 3$  is a geometric eigenvalue. We have

$$\begin{aligned} P_A^{(p)}(\lambda) = & \left( -(\lambda^3)^{2p+1} + [(2\lambda^2)^{2p+1} + (1\lambda^2)^{2p+1} + (3\lambda^2)^{2p+1}] \right. \\ & - [(2 \cdot 3 \cdot \lambda)^{2p+1} - (1 \cdot 9 \cdot \lambda)^{2p+1} - (1 \cdot 2 \cdot \lambda)^{2p+1} \\ & \quad \left. + (2 \cdot 2 \cdot \lambda)^{2p+1} - (1 \cdot 3 \cdot \lambda)^{2p+1} + (1 \cdot 1 \cdot \lambda)^{2p+1}] \right. \\ & + [(1 \cdot 2 \cdot 3)^{2p+1} + (2 \cdot 1 \cdot 1)^{2p+1} + (2 \cdot 9 \cdot 1)^{2p+1} \\ & \quad \left. - (1 \cdot 2 \cdot 1)^{2p+1} - (2 \cdot 2 \cdot 3)^{2p+1} - (1 \cdot 9 \cdot 1)^{2p+1}] \right)^{\frac{1}{2p+1}}. \end{aligned}$$

Hence, taking the limit yields:

$$\begin{aligned} P_A^\infty(\lambda) &= -\lambda^3 \boxplus 2\lambda^2 \boxplus \lambda^2 \boxplus 3\lambda^2 \\ &\quad \boxplus 6\lambda \boxplus (-9\lambda) \boxplus (-2\lambda) \boxplus 4\lambda \boxplus (-3\lambda) \boxplus \lambda \\ &\quad \boxplus 3 \boxplus 2 \boxplus 18 \boxplus (-2) \boxplus (-12) \boxplus (-9). \end{aligned}$$

Moreover

$$\begin{aligned} P_{A,-}^\infty(\lambda) &= -\lambda^3 \ominus 2\lambda^2 \ominus \lambda^2 \ominus 3\lambda^2 \\ &\quad \ominus 6\lambda \ominus (-9\lambda) \ominus (-2\lambda) \ominus 4\lambda \ominus (-3\lambda) \ominus \lambda \\ &\quad \ominus 3 \ominus 2 \ominus 18 \ominus (-2) \ominus (-12) \ominus (-9). \end{aligned}$$

and

$$\begin{aligned} P_{A,+}^\infty(\lambda) &= -\lambda^3 \oplus 2\lambda^2 \oplus \lambda^2 \oplus 3\lambda^2 \\ &\quad \oplus 6\lambda \oplus (-9\lambda) \oplus (-2\lambda) \oplus 4\lambda \oplus (-3\lambda) \oplus \lambda \\ &\quad \oplus 3 \oplus 2 \oplus 18 \oplus (-2) \oplus (-12) \oplus (-9). \end{aligned}$$

We have  $P_{A,-}^\infty(3) = -27 \leq 0$  and  $P_{A,+}^\infty(3) = 27 \geq 0$ .  $\mu = 2$  is an algebraic eigenvalue and also satisfies the condition  $P_{A,-}^\infty(2) = -18 \leq 0$  and  $P_{A,+}^\infty(2) = 18 \geq 0$ . Therefore  $\mu$  is also an eigenvalue in limit.

In the next example, we consider a case where there is some  $\mu \in [P_{A,-}^\infty \leq 0] \cap [P_{A,+}^\infty \geq 0] = [P_{A,-}^\infty + P_{A,+}^\infty = 0]$  that is an eigenvalue in limit but is not an algebraic eigenvalue over the Max-Times idempotent semi-ring.

**Example 6.0.7** Let us consider the matrix  $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ .  $\rho = 1$  is a  $\boxtimes$ -eigenvalue associated to  $v = (1, 1)$  since  $A \boxtimes v = 1 \cdot v$ . The permanent is  $(\lambda \boxplus 1)^2$ . The  $\varphi_p$ -Perron-Frobenius eigenvalue is  $\rho_A^{(p)} = 2^{\frac{1}{2p+1}}$  and we have  $\lim_{p \rightarrow \infty} \rho_A^{(p)} = 1$ . For all  $p$ , there is another eigenvalue  $\mu^{(p)} = 0$ . We have

$$P_A^{(p)}(\lambda) = (\lambda^2)^{2p+1} - \lambda^{2p+1} - \lambda^{2p+1} + 1^{2p+1} + (-1)^{2p+1}.$$

Taking the limit yields:

$$P_A^\infty(\lambda) = (\lambda^2) \boxplus (-\lambda) \boxplus (-\lambda) \boxplus 1 \boxplus (-1).$$

Therefore

$$P_{A,-}^\infty(\lambda) = (\lambda^2) \ominus (-\lambda) \ominus (-\lambda) \ominus 1 \ominus (-1)$$

and

$$P_{A,+}^\infty(\lambda) = (\lambda^2) \oplus (-\lambda) \oplus (-\lambda) \oplus 1 \oplus (-1).$$

We have  $P_{A,-}^\infty(1) = -1 \leq 0$  and  $P_{A,+}^\infty(1) = 1 \geq 0$ . There is another solution  $\mu = 0$ , we have  $P_{A,-}^\infty(0) = -1 \leq 0$  and  $P_{A,+}^\infty(0) = 1 \geq 0$ . Hence  $P_{A,-}^\infty(0) + P_{A,+}^\infty(0) = 0$  and  $0$  is an eigenvalue in limit associated with the vector  $u = (1, -1)$ . We have  $A \boxtimes u = 0$ . However  $u$  is not a nonnegative vector. Moreover,  $0$  is not an algebraic eigenvalue.

Example 6.0.7 shows that an eigenvalue in limit may not be algebraic. However, examples 6.0.5, 6.0.6 and 6.0.7 do not show any situation in which an algebraic eigenvalue is not an eigenvalue in limit. In example 6.0.6,  $\mu = 2$

is an algebraic (non-geometric) eigenvalue which is an eigenvalue in limit. In the following, Example 6.0.8 shows that an algebraic eigenvalue may not be an eigenvalue in limit. However, further investigations would be needed to know if any algebraic eigenvalue of a non-negative matrix is the absolute value of an eigenvalue in limit. In particular, it would be interesting to determine if a factorization of the polynomials in limit is possible and how it is linked to that of the permanent of a non-negative matrix.

**Example 6.0.8** *Let us consider the matrix  $A = \begin{pmatrix} 2 & 7 \\ 1 & 3 \end{pmatrix}$ . The geometric mean of elementary cycles of length 1, are 2, 3. The geometric mean of the cycle of length 2 is  $7^{\frac{1}{2}}$ . Clearly  $\rho = 3$  is a  $\boxtimes$ -eigenvalue. The permanent polynomial characteristic of  $A$  is  $\chi_A(\lambda) = (2\boxplus\lambda)(3\boxplus\lambda)\boxplus 7 = \max\{\max\{2, \lambda\} \cdot \max\{3, \lambda\}, 7\}$ . The factorization is:*

$$\chi_A(\lambda) = (\lambda \boxplus 3)(\lambda \boxplus \frac{7}{3}).$$

Hence,  $\eta = \frac{7}{3}$  is also an algebraic eigenvalue of  $A$  in the idempotent semiring  $\text{Max-Times}$ . We have

$$P_A^{(p)}(\lambda) = ((\lambda^2)^{2p+1} - (2\lambda)^{2p+1} - (3\lambda)^{2p+1} + 6^{2p+1} - 7^{2p+1})^{\frac{1}{2p+1}}.$$

Hence, taking the limit yields:

$$P_A^\infty(\lambda) = (\lambda^2) \boxplus (-2\lambda) \boxplus (-3\lambda) \boxplus 6 \boxplus (-7).$$

Moreover

$$P_{A,-}^\infty(\lambda) = (\lambda^2) \bar{\cup} (-2\lambda) \bar{\cup} (-3\lambda) \bar{\cup} 6 \bar{\cup} (-7)$$

and

$$P_{A,+}^\infty(\lambda) = (\lambda^2) \dot{\cup} (-2\lambda) \dot{\cup} (-3\lambda) \dot{\cup} 6 \dot{\cup} (-7).$$

The geometric eigenvalue  $\rho = 3$  satisfies the relations  $P_{A,-}^\infty(3) = -9 \leq 0$  and  $P_{A,+}^\infty(3) = 9 \geq 0$ . We can check that the algebraic eigenvalue  $\eta = \frac{7}{3}$  is not an eigenvalue in limit. We have  $P_{A,-}^\infty(\frac{7}{3}) = -7 < 0$  and  $P_{A,+}^\infty(\frac{7}{3}) = -7 < 0$ . Therefore  $P_{A,-}^\infty(\frac{7}{3}) + P_{A,+}^\infty(\frac{7}{3}) = -14 \neq 0$ . However,  $\mu = -\frac{7}{3}$  is an eigenvalue in limit since  $P_{A,-}^\infty(-\frac{7}{3}) = -7 \leq 0$  and  $P_{A,+}^\infty(-\frac{7}{3}) = 7 \geq 0$ .

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