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Mathematical model of anaerobic digestion with leachate recirculation

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Abstract

Landfilled biodegradable materials are subject to biological evolution, namely *biodegradation* or *digestion*, under the action of aerobic bacteria and further anaerobic bacteria. In the absence of special provisions, the liquid flowing through the mass of waste produces leachate, being charged with chemical or biological substances. Therefore, the mathematical modeling is of great interest for making decisions in the field of waste management since it makes possible to predict the behavior of the various complex systems underlying the management of waste. The analysis of the dynamics of the model considered in our work describes the process of anaerobic biodegradation with leachate reinjection, and allows us to predict the amount of biogas produced in the long term. The system admits an infinity of non hyperbolic equilibria but we are able to describe the asymptotic behavior of the solutions depending on the initial condition. Thanks to this model, we can highlight the influence of the leachate recirculation on the production of biogas, with the help of numerical simulations. This recirculation could be a part of the sustainable solutions adapted by recent technologies for the treatment of waste, not only to optimize the total production of biogas, but also to obtain a good quality of the outgoing liquid to meet the discharge standards.

Keywords

Anaerobic digestion ; Mathematical modeling ; Biogas ; Leachate recirculation; Stability analysis, Non-hyperbolic equilibrium.

I INTRODUCTION

The anaerobic digestion process is characterized by a succession of complex reactions. In the literature, there exists different multi-step models but whose mathematical analysis is often difficult or merely not possible [2, 3].

In the Rouez's model [6], a simplification has been proposed in which the process is reduced to two fundamental steps which are the *hydrolysis/acidogenesis* and the *methanogenesis*. This model has been extended in [4] by considering that a part of the dead microorganisms returns to the slowly biodegradable material. On the other hand, in [5] the organic matter is decomposed into two types of substrat, soluble and insoluble. Our model gathers these features, that are

1. the two-steps process,
2. the consideration of both soluble and insoluble substrates,
3. the distribution of the dead microbial matter into a part returning (slowly) to degradable material and another one transformed directly into carbon dioxide.

Our objective is to investigate the influence of leachate recirculation on the performances of biogas production.

The biological scheme underlining the different interactions of the process is depicted on Figure 1.

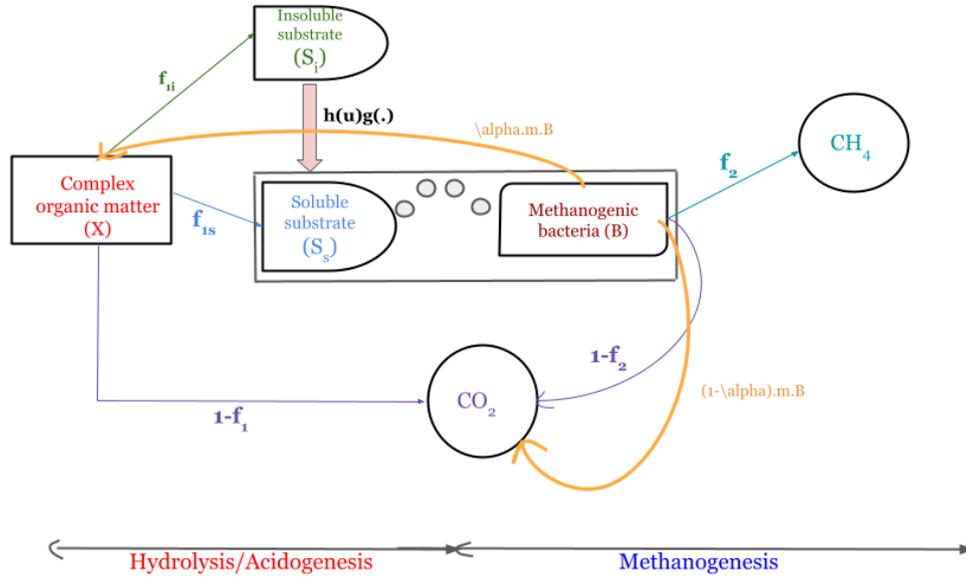


Figure 1: Biological diagram of the interactions of the anaerobic digestion process of organic matter (where $f_1 = f_{1i} + f_{1s}$).

II MODEL AND ASSUMPTIONS

2.1 Model equations

The process described in Figure 1 is modeled by the following dynamical system:

$$\begin{cases} \dot{X} = -\delta X + \alpha m B \\ \dot{S}_i = f_{1i} \delta X - h(u)g(S_i) \\ \dot{S}_s = f_{1s} \delta X + h(u)g(S_i) - \frac{1}{Y} \mu(S_s) B \\ \dot{B} = (\mu(S_s) - m) B \\ \dot{C}O_2 = (1 - f_1) \delta X + (1 - f_2) \frac{1-Y}{Y} \mu(S_s) B + (1 - \alpha) m B \\ \dot{C}H_4 = f_2 \frac{1-Y}{Y} \mu(S_s) B \end{cases} \quad (1)$$

where we posit

$$f_1 = f_{1i} + f_{1s}.$$

The state variables are the concentrations :

- $X(t)$ of complex organic matter at time t ,
- $S_i(t)$ of insoluble substrate at time t ,
- $S_s(t)$ of soluble substrate at time t ,
- $B(t)$ of methanogenic bacteria at time t ,

and the output variables are the amounts produced per unit of volume :

$CO_2(t)$ of carbon dioxide at time t ,

$CH_4(t)$ of methane at time t .

The function $\mu(\cdot)$ is the specific growth rate of the biological reaction, while the function $(u, S_i) \mapsto h(u)g(S_i)$ describes the conversion rate of insoluble substrate into soluble one, where u is the recirculation rate. As the conversion rate of insoluble substrate into soluble one is a function of two variables that is equal to 0 in absence of insoluble substrate or null recirculation, we have assumed that it can be written as a product of two functions $h(\cdot)$, $g(\cdot)$ of each variable (equal to 0 when their argument is zero).

The parameters of the model are :

α : the proportion of nutrient recycling,

u : the leachate recirculation rate (that can be a manipulating variable),

δ : the degradation rate of the organic matter,

m : the mortality rate of the biomass,

Y : the conversion rate of the substrate,

f_{1_i}, f_{1_s}, f_2 : the stoichiometric coefficients.

2.2 Hypotheses

H₁. The function $\mu(\cdot)$ is \mathcal{C}^1 on \mathbb{R}_+ , non-negative and bounded with $\mu(0) = 0$.

H₂. The functions $g(\cdot)$ and $h(\cdot)$ are \mathcal{C}^1 on \mathbb{R}_+ , increasing with bounded derivative and equal to 0 at zero.

H₃. The parameters $\alpha, u, m, Y, f_{1_i}, f_{1_s}, f_2$ satisfy the following properties:

$$\delta > 0,$$

$$0 < \alpha \leq 1 \text{ and } 0 < u \leq u_{max} < +\infty,$$

$$0 < m < \max_{S_s > 0} \mu(S_s),$$

$$0 < Y < 1,$$

$$0 < f_{1_i}, f_{1_s}, f_2 < 1 \text{ (with } f_{1_i} + f_{1_s} = f_1).$$

III MATHEMATICAL RESULTS

In this section, we present some results, their proofs are given in the appendix.

3.1 Existence and uniqueness of solutions

Proposition 1. The vector field associated to the system of equations (1) is \mathcal{C}^1 in \mathbb{R}^6 . Therefore, for any non-negative initial condition, the Cauchy problem admits a unique solution defined for any time (see for instance [7]). Moreover, the solution remains non-negative for any positive time.

3.2 Asymptotic behavior

Theorem 1. Under hypotheses **H₁**, **H₂**, **H₃**, for any non-negative initial condition, the solution of (1) is bounded. In addition, one has

$$\begin{aligned} \lim_{t \rightarrow +\infty} X(t) &= \lim_{t \rightarrow +\infty} B(t) = 0, \\ \lim_{t \rightarrow +\infty} S_i(t) &= 0, \quad \lim_{t \rightarrow +\infty} S_s(t) := S_s^* \geq 0, \end{aligned}$$

with

$$S_s^* \leq \frac{X(0)}{\alpha Y} + (S_i(0) + S_s(0)) + \frac{B(0)}{Y}.$$

Moreover, if $X(0)$, $S_i(0)$ or $S_s(0)$ is non-null, then $S_s^* > 0$.

3.3 Attractor

Let us define the set

$$\mathcal{E} := \{S_s \in \mathbb{R}_+ : \mu(S_s) \leq m\}.$$

Proposition 2. Under hypotheses \mathbf{H}_1 , \mathbf{H}_2 , \mathbf{H}_3 , for any non-negative initial condition, one has

$$S_s^* = \lim_{t \rightarrow +\infty} S_s(t) \in \mathcal{E}.$$

The system (1) admits a continuum of equilibria which are all non hyperbolic. Therefore, one cannot conclude about their stability by studying the single linearization of the dynamics (and the Center Manifold Theorem is not enough informative). We use the same technique as in [4] which consists in considering a (non-linear) change of coordinates parameterized by $S_s^* \in \mathcal{E}$, such that the dynamics admits a unique equilibrium in these coordinates (see Annex 2). For each $S_s^* \in \mathcal{E}$, this equilibrium is a hyperbolic saddle point, with a stable manifold of dimension three that matches all trajectories of (1) with $\lim_{t \rightarrow +\infty} S_s(t) = S_s^*$. Then, we obtain the following result.

Proposition 3. Under the same hypotheses than before,

$$\mathcal{A} := \{0\} \times \{0\} \times \mathcal{E} \times \{0\}$$

is a forward attractor of any non-negative initial condition of (1).

3.4 Asymptotic production of biogas

Let us denote by $G(t)$ the total amount of biogas per unit of volume that is produced at time t , that is

$$G(t) = CO_2(t) + CH_4(t).$$

Proposition 4. Under the same hypotheses than before, for any non-negative initial condition, the solution of the system (1) verifies

$$\lim_{t \rightarrow +\infty} G(t) = G(0) + X(0) + aB(0) + S_i(0) + S_s(0) - S_s^*$$

where

$$a := \frac{1 - \alpha f_1 Y}{1 - f_1 Y} > 0.$$

IV NUMERICAL SIMULATIONS

For the growth function μ , we have chosen the Haldane law [1] which modeled a growth inhibition under large concentration of substrate

$$\mu(S_s) = \bar{\mu} \frac{S_s}{K_s + K_i + \frac{S_s^2}{K_i}}$$

where $\bar{\mu}$ is the maximum growth rate, K_s is the half-saturation constant and K_i the inhibition constant, and the simplest functions for g and h :

$$g(S_i) = S_i, \quad h(u) = u.$$

The values of parameters are inspired from [4] and given in Table 1.

$\bar{\mu}$	K_s	K_i	δ	m	Y	α	f_{1_i}	f_{1_s}	f_2
0.3 h^{-1}	160 mg.l^{-1}	10 mg.l^{-1}	0.176 h^{-1}	0.02 h^{-1}	0.05	0.9	0.4	0.3	0.76

Table 1: Parameters values

For these values, the set \mathcal{E} of attracting values for S_s is computed numerically as

$$\mathcal{E} \simeq [0, 14.09] \cup [125.90, +\infty).$$

Initial condition and parameter u for the recirculation are given in Table 2.

$X(0)$	$S_i(0)$	$S_s(0)$	$B(0)$	u
300 mg.l^{-1}	0	0	2 mg.l^{-1}	0.3

Table 2: Operating conditions

The time evolution of the variables and the production of biogas are depicted on Figure 2.

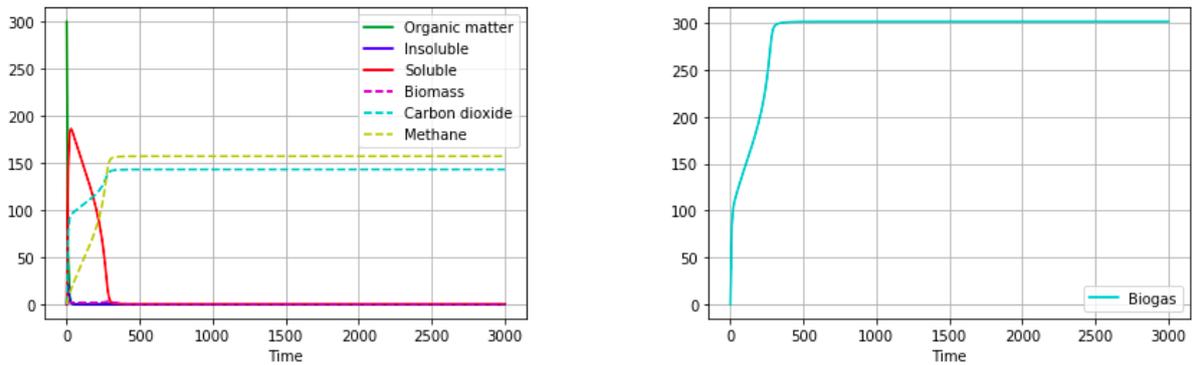


Figure 2: Time evolution of the state variables (left) and total biogas production (right).

To have insights on the impacts of changing operating conditions, we have performed different simulations:

1. Figure 3: soluble substrate for different load of organic matter. As expected, the asymptotic value of S_s never reaches the interval $(14.09, 125.90)$.

2. Figure 4: effect of absence of recirculation. When $u = 0$, the concentration S_i is carrying on increasing all the time, differently to the case $u > 0$ for which we have proved that it converges asymptotically to 0.
3. Figure 5: effect of leachate recirculation and initial load of organic matter on the biogas production. This shows that a minimum of recirculation is necessary to obtain good performances. Moreover, the inhibition in the growth (which implies that the set \mathcal{E} is not connected) induces a discontinuity in the biogas production as a function of $X(0)$. There exists a threshold on the initial load of organic matter above which the biogas production falls down drastically.

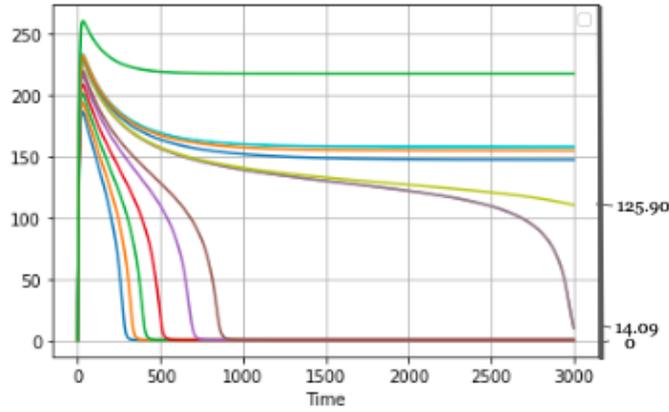


Figure 3: Time evolution of the soluble substrate for different values of $X(0)$.

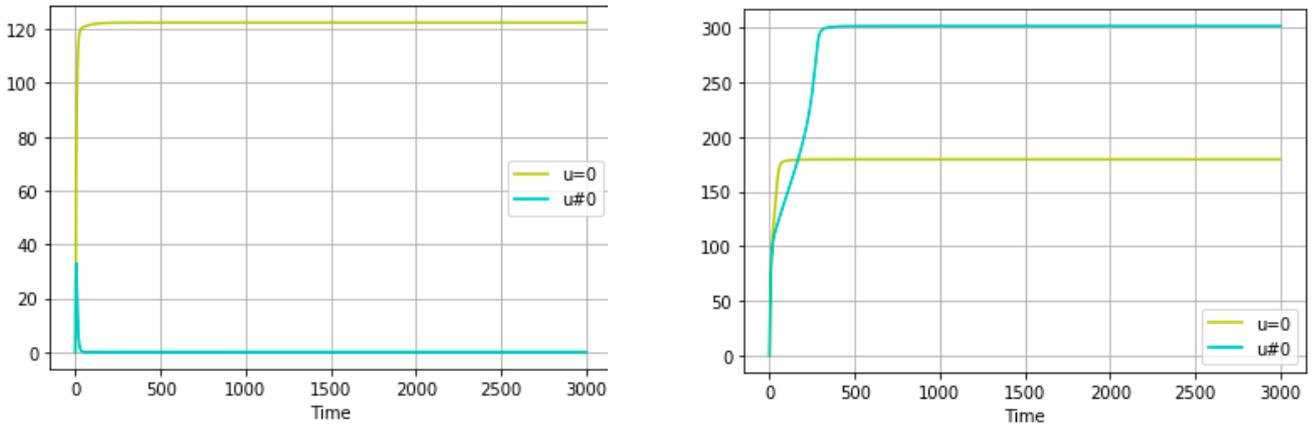


Figure 4: Insoluble substrate (left) and biogas production (right) with and without recirculation.

V CONCLUSIONS

- The theoretical results are confirmed by the numerical simulations.
- With Haldane growth function, the biogas production is a non monotonic function of the initial load of organic matter $X(0)$ (it increases and then decreases with a discontinuity at the maximal value).
- Future perspectives of this work concern the maximization of the biogas production:
 1. for large amounts of organic matter to be treated, the optimal supply of organic matter over time,
 2. the optimal control of the recirculation rate u .

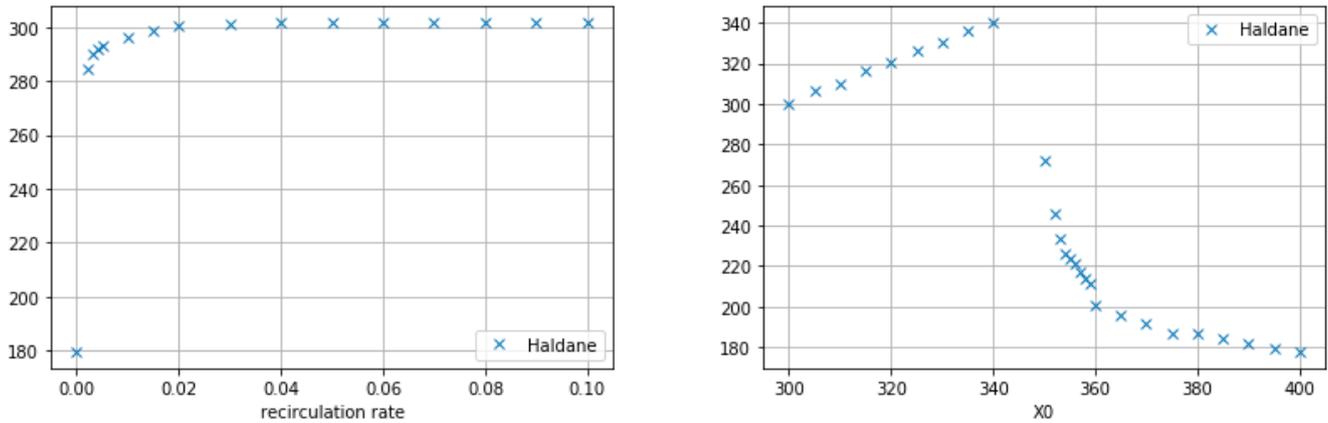


Figure 5: Total production of biogas as a function of the recirculation (left) and of the initial load of organic matter $X(0)$ (right).

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A ANNEX 1. PROOF OF PROPOSITION 1

Under hypotheses $\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3$, for any non-negative initial condition $(X^0, S_i^0, S_s^0, B^0, CO_2^0, CH_4^0)$ we have $\dot{B}(t) = (\mu(S_s) - m)B(t)$, then $B(t) = B^0 \exp(\int_0^t (\mu(S_s(z)) - m)dz)$, therefore

$$B(t) \geq 0, \forall t \geq 0.$$

Moreover, $\dot{X}(t) = -\delta X(t) + \alpha m B(t)$. Since $B(t) \geq 0, \forall t \geq 0$, one has $\dot{X}(t) \geq -\delta X(t), \forall t \geq 0$, and, by Grönwall Lemma [7], $X(t) \geq X^0 \cdot \exp(-\delta t), \forall t \geq 0$. As $X^0 \geq 0$, then

$$X(t) \geq 0, \forall t \geq 0.$$

In addition, $\dot{S}_i(t) = f_{1_i} \delta X(t) - h(u)g(S_i)$. Since $X(t) \geq 0, \forall t \geq 0$, then $\dot{S}_i(t) \geq -h(u)g(S_i) \forall t \geq 0$. Considering the following Cauchy problem:

$$\begin{cases} \dot{\bar{S}}_i(t) = -h(u)g(\bar{S}_i) \\ \bar{S}_i(0) = S_i^0. \end{cases} \quad (2)$$

By Comparison Lemma [7], one has $\bar{S}_i(t) \leq S_i(t), \forall t \geq 0$. Then, it suffices to show that $0 \leq \bar{S}_i(t), \forall t \geq 0$.

★ if $S_i^0 = 0$, as $g(0) = 0$, it's clear that 0 is a solution of the Cauchy problem (2).

★ if $S_i^0 > 0$, therefore $\bar{S}_i(t) > 0$, for all $t \geq 0$, indeed by uniqueness of the solution of the Cauchy problem (2), $\bar{S}_i(\cdot)$ can't cross the axis $\bar{S}_i = 0$. We deduce that

$$\bar{S}_i(t) \geq 0, \forall t \geq 0.$$

In the same way, we show that

$$S_s(t) \in \mathbb{R}_+, \forall t > 0.$$

Let us now prove that

$$CO_2(t) \in \mathbb{R}_+, \forall t \geq 0, \quad \text{and} \quad CH_4(t) \in \mathbb{R}_+, \forall t \geq 0.$$

By integrating between 0 and t the following equations

$$C\dot{O}_2(t) = (1 - f_1)\delta X(t) + (1 - f_2) \cdot \frac{1 - Y}{Y} \mu(S_s)B(t) + (1 - \alpha)mB(t) \quad (3)$$

$$C\dot{H}_4(t) = f_2 \cdot \frac{1 - Y}{Y} \mu(S_s)B(t) \quad (4)$$

and since $X(t), B(t)$ and $\mu(S_s)$ are non negative for all $t \geq 0$, our result is proved.

B ANNEX 2. PROOF OF THEOREM 1

To show the boundedness of the solution of the system (1) we introduce the following function:

$$F(t) = X(t) + \alpha Y(S_i(t) + S_s(t)) + \alpha B(t), \quad \forall t \geq 0. \quad (5)$$

From the equations of the system (1) we obtain:

$$\dot{F}(t) = (\alpha f_1 Y - 1)\delta X(t). \quad (6)$$

Since $\alpha f_1 Y < 1$, and $X(\cdot)$ is non negative, then $F(\cdot)$ is non-increasing, so that $F(t) \leq F(0), \forall t \geq 0$.

And in addition, by the non-negativity of the variables $X(\cdot), S_i(\cdot), S_s(\cdot)$ and $B(\cdot)$, one has:

$$0 \leq F(t) \leq X^0 + \alpha Y(S_i^0 + S_s^0) + \alpha B^0, \quad \forall t \geq 0.$$

Therefore, $F(\cdot)$ is bounded, which implies by the non-negativity of the variables that $X(t), S_i(t), S_s(t)$ and $B(t)$ are bounded.

By integrating the equation (6) between 0 and t, we obtain for all non-negative t,

$$F(t) = F(0) + (\alpha f_1 Y - 1)\delta \int_0^t X(s)ds. \quad (7)$$

As $F(\cdot)$ is bounded then

$$\int_0^t X(s)ds < +\infty. \quad (\star)$$

Moreover, from the first equation of (1), we have

$$X(t) = X(0) - \delta \int_0^t X(s)ds + \alpha m \int_0^t B(s)ds. \quad (8)$$

Therefore, with (\star) we get

$$\int_0^t B(s)ds < +\infty. \quad (\star\star)$$

Then, from the fourth equation of (1), we obtain:

$$B(t) = B(0) + \int_0^t \mu(S_s(s))B(s)ds - m \int_0^t B(s)ds. \quad (9)$$

As $B(\cdot)$ is bounded and with $(\star\star)$, we deduce that

$$\int_0^t \mu(S_s(s))B(s)ds < +\infty. \quad (\star\star\star)$$

In addition, the integration of the last two equations of the system (1) between 0 and t leads to the following formulas

$$CO_2(t) = CO_2(0) + (1-f_1)\delta \int_0^t X(s)ds + (1-f_2)\frac{1-Y}{Y} \int_0^t \mu(S_s(s))B(s)ds + (1-\alpha)m \int_0^t B(s)ds \quad (10)$$

$$CH_4(t) = CH_4(0) + f_2\frac{1-Y}{Y} \int_0^t \mu(S_s(s))B(s)ds. \quad (11)$$

Then we get the boundedness of the variables $CO_2(t)$ and $CH_4(t)$ taking into account (\star) , $(\star\star)$ and $(\star\star\star)$.

Let us show the convergence of $X(\cdot)$, $S_i(\cdot)$, $S_s(\cdot)$ and $B(\cdot)$.

The function $F : [0, +\infty[\rightarrow [0, +\infty[$ is differentiable, non-increasing and bounded by 0, therefore, $F(\cdot)$ converges to a limit denoted by F_{lim} when t tends to $+\infty$.

Moreover, one has:

$$\ddot{F}(t) = (\alpha f_1 Y - 1)\delta(-\delta X(t) + \alpha m B(t)). \quad (12)$$

From (12) and since $X(\cdot)$ and $B(\cdot)$ are bounded, therefore, \ddot{F} is bounded and thus \dot{F} is uniformly continuous on \mathbb{R}_+ . We get by Barbalat's Lemma [9], $\lim_{t \rightarrow +\infty} \dot{F}(t) = 0$. Then from (6), we obtain

$$\lim_{t \rightarrow +\infty} X(t) = 0.$$

In the same way, as $X(\cdot)$, $B(\cdot)$ and $\mu(\cdot)$ are bounded, then \ddot{X} is bounded and thus \dot{X} is uniformly continuous on \mathbb{R}_+ and by Barbalat's Lemma, one has

$$\lim_{t \rightarrow +\infty} B(t) = 0.$$

Similarly, \ddot{S}_i is bounded which implies that \dot{S}_i is uniformly continuous on $]0, +\infty[$, and by Barbalat's lemma, we get

$$\lim_{t \rightarrow +\infty} S_i(t) = 0.$$

In addition, as $F(t)$ converges to F_{lim} , we deduce that

$$\exists S_s^* \in \mathbb{R}_+, \lim_{t \rightarrow +\infty} S_s(t) = S_s^* = \frac{F_{lim}}{\alpha y}.$$

Moreover, from (5) and as $F(\cdot)$ is non increasing, by tending t to ∞ , we conclude that

$$S_s^* \leq \frac{X^0}{\alpha Y} + (S_i^0 + S_s^0) + \frac{B^0}{Y}.$$

Now let us show that $S_s^* \neq 0$, when X^0, S_i^0 or S_s^0 is non null.

Assuming that $S_s^* = 0$, as $\mu(\cdot)$ is continuous, one has $\lim_{t \rightarrow +\infty} \mu(S_s(t)) = 0$.

For $\eta = (f_{1_i} + f_{1_s})\alpha m Y > 0$, one has

$$\exists T > 0, \forall t > T : |\mu(S_s(t))| \leq \eta$$

therefore, from the equations of the system (1), we obtain:

$$(f_{1_i} + f_{1_s})\dot{X}(t) + \dot{S}_i(t) + \dot{S}_s(t) \geq 0, \forall t > T.$$

When $(f_{1_i} + f_{1_s})X(0) + S_i(0) + S_s(0) > 0$, the variables $X(\cdot), S_i(\cdot)$ or $S_s(\cdot)$ cannot reaches 0 in finite time. Then

$$(f_{1_i} + f_{1_s})X(t) + S_i(t) + S_s(t) \geq (f_{1_i} + f_{1_s})X(T) + S_i(T) + S_s(T) > 0, \forall t > T$$

wich is in contradiction with the fact that $\lim_{t \rightarrow +\infty} (f_{1_i} + f_{1_s})X(t) + S_i(t) + S_s(t) = 0$.

C ANNEX 2. PROOFS OF PROPOSITION 2 AND PROPOSITION 3

If $S_s^* \notin \mathcal{E}$, one has $\lim_{t \rightarrow +\infty} \mu(S_s(t)) = \mu(S_s^*)$, therefore by choosing $\epsilon = \frac{\mu(S_s^*) - m}{2} > 0$ one gets

$$\exists T > 0, \forall t > T : (\mu(S_s(t)) - m) > \epsilon.$$

Consequently, and since $B(\cdot)$ is non negative we obtain

$$\dot{B}(t) > \epsilon B(t), \forall t > T.$$

Since $B(\cdot)$ is bounded and by a standard comparison theorem [7], we deduce that $\lim_{t \rightarrow +\infty} B(t) = +\infty$ wich is in contradiction with the fact that $B(\cdot)$ converges asmpotically to 0. Therefore, we conclude that $S_s(\cdot)$ converges to S_s^* with $S_s^* \in \mathcal{E}$.

Now let us fix $e = (0, 0, S_s^*, 0)$ with $S_s^* \in \text{int } \mathcal{E}$. We consider the variable $R(\cdot)$ defined by:

$$R(t) = \frac{(f_{1_i} + f_{1_s})X(t) + S_i(t) + (S_s(t) - S_s^*)}{B(t)} + c_1, \forall t \geq 0$$

where

$$c_1 = \frac{\mu(S_s^*)/Y - \alpha(f_{1_i} + f_{1_s})m}{\mu(S_s^*) - m}$$

By a direct computation, we get

$$\dot{R}(t) = -r_1(\mu(S_s) - \mu(S_s^*)) - (\mu(S_s) - m)R(t)$$

with

$$r_1 = \frac{(1 - \alpha(f_{1_i} + f_{1_s})Y)m}{Y(m - \mu(S_s^*))} > 0$$

We consider the domain:

$$K = \{(X, S_i, R, B) \in \mathbb{R}_+^2 \times \mathbb{R} \times \mathbb{R}_+^* / S_s^* - S_i - (f_{1_i} + f_{1_s})X + (R - c_1)B \geq 0\}$$

and the application $p(\cdot)$ defined by:

$$\begin{aligned} p : \mathbb{R}_+^4 &\rightarrow K \\ Z &\mapsto p(Z) = (X, S_i, \frac{(f_{1_i} + f_{1_s})X + S_i + (S_s - S_s^*)}{B} + c_1, B) \end{aligned}$$

with $Z = (X, S_i, S_s, B) \in \mathbb{R}_+^4$.

p is a homeomorphism from \mathbb{R}_+^4 to K . Then, the system (1) can be written equivalently on K , as follows:

$$\begin{cases} \dot{X} = -\delta X + \alpha m B \\ \dot{S}_i = f_{1_i} \delta X - h(u)g(S_i) \\ \dot{R} = -r_1(k(X, S_i, R, B) - \mu(S_s^*)) - (k(X, S_i, R, B) - m)R \\ \dot{B} = (k(X, S_i, R, B) - m)B \end{cases} \quad (13)$$

with

$$k(X, S_i, R, B) = \mu(S_s^* - (f_{1_i} + f_{1_s})X + (R - c_1)B)$$

The systems (1) and (13) are locally topologically equivalent. However, trajectories with $B(t) = 0, \forall t \geq 0$ are not considered by the homeomorphism p , therefore, we consider the domain K_0 defined by:

$$K_0 = K \cup I \times \mathbb{R} \times \{0\}$$

with

$$I = \{(X, S_i) \in \mathbb{R}_+^2 / S_s^* - S_i - (f_{1_i} + f_{1_s})X \geq 0\}$$

Since $S_s^* \in \text{int } \mathcal{E}$, the origin $(0, 0, 0, 0)$ is the unique equilibrium point of (13) in K_0 .

The linearization of (13) near the origin is based on the following jacobian matrix:

$$J(0, 0, 0, 0) = \begin{pmatrix} -\delta & 0 & 0 & \delta m \\ f_{1_i} \delta & -h(u)g'(0) & 0 & 0 \\ r_1(f_{1_i} + f_{1_s})\mu'(S_s^*) & r_1\mu'(S_s^*) & m - \mu(S_s^*) & r_1 c_1 \mu'(S_s^*) \\ 0 & 0 & 0 & \mu(S_s^*) - m \end{pmatrix}$$

Its eigenvalues are $\lambda_1 = -\delta < 0$, $\lambda_2 = -h(u)g'(0) < 0$, $\lambda_3 = -(\mu(S_s^*) - m) > 0$ and $\lambda_4 = (\mu(S_s^*) - m) < 0$. Since the origin is an hyperbolic equilibrium of (13) in K_0 , by the Stable Manifold Theorem [8], there exists a stable manifold $W^s(0)$ and an instable manifold $W^u(0)$, such that:

$$\begin{aligned} \dim(W^s(0)) &= 3 \\ \dim(W^u(0)) &= 1 \end{aligned}$$

It's clear that $W^u(0) = \{0\} \times \{0\} \times \mathbb{R} \times \{0\}$.

Now let us show that any point $(X, S_i, R, B) \in W^s(0) \setminus (0, 0, 0, 0)$ has to be such that $B \neq 0$.

If $B^0 = 0$, any solution of (13) verifies $B(t) = 0, \forall t > 0$, since $\lim_{t \rightarrow +\infty} X(t) = 0$ and $\lim_{t \rightarrow +\infty} S_i(t) = 0$, then, as $\mu(\cdot)$ is C^1 on \mathbb{R}_+ , one has

$$\lim_{t \rightarrow +\infty} k(X(t), S_i(t), R(t), B(t)) = \lim_{t \rightarrow +\infty} \mu(S_s^* - S_1(t) - (f_1 + f_2)X(t)) = \mu(S_s^*),$$

For $\eta = \frac{m - \mu(S_s^*)}{2} > 0$, one has

$$\exists T > 0 : \forall t > T : m - \mu(S_s^* - S_i(t) - (f_{1_i} + f_{1_s})X(t)) > \eta$$

From the equation of the system (13), one has

$$\dot{R}(t) > \eta R(t) - r_1(\mu(S_s^* - S_i(t) - (f_{1_i} + f_{1_s})X(t)) - \mu(S_s^*)), \forall t > T.$$

Then $\lim_{t \rightarrow +\infty} R(t) = +\infty$.

K is positively invariant by the dynamics (13), therefore, any trajectory $(X(t), S_i(t), R(t), B(t))$ in $S = W^s(0) \cap K$ is homeomorphic to a trajectory $(X(t), S_i(t), S_s(t), B(t))$ of (1) and converges asymptotically to $(0, 0, S_s^*, 0)$. Finally and with the results of proposition 2 and proposition 3, we conclude that the set $\mathcal{A} := \{0\} \times \{0\} \times \mathcal{E} \times \{0\}$ is a forward attractor of (1).

D ANNEX 3. PROOF OF PROPOSITION 4

From the formulas (8) and (9) and by tending t to ∞ , one has:

$$\delta \int_0^{\infty} X(s)ds = X^0 + B^0 + (\alpha - 1)m \int_0^{\infty} B(s)ds + \int_0^{\infty} \mu(S_s(s))B(s)ds \quad (14)$$

Moreover, by integrating the second and third equation of the system (1) between $t = 0$ and $t = +\infty$ we obtain

$$\int_0^{\infty} \mu(S_s(s))B(s)ds = f_1 Y \delta \int_0^{\infty} X(s)ds + Y(S_i^0 + (S_s^0 - S_s^*)). \quad (15)$$

From a simple calculus, we get the following expressions

$$\int_0^{\infty} X(s)ds = \frac{A+1}{\delta\alpha C} X^0 + \frac{(A+1)}{\delta C} B^0 + \frac{A}{\delta f_1 C} (S_i^0 + (S_s^0 - S_s^*)) \quad (16)$$

$$\int_0^{\infty} B(s)ds = \frac{(A+1) - \alpha C}{\alpha^2 m C} X^0 + \frac{A+1}{\alpha m C} B^0 + \frac{A}{\alpha m f_1 C} (S_i^0 + (S_s^0 - S_s^*)) \quad (17)$$

$$\int_0^{\infty} \mu(S_s(s))B(s)ds = D(X^0 + \delta B^0) + F(S_i^0 + (S_s^0 - S_s^*)) \quad (18)$$

where $A = \frac{f_1 Y}{(1-f_1)Y}$, $C = \frac{\alpha - (A+1)(\alpha-1)}{\alpha}$, $D = \frac{f_1 Y(A+1)}{\alpha C}$ and $F = \frac{Y(A+C)}{C}$.

Therefore, from the formulas (16), (17) and (18) we deduce the asymptotic value of the total biogas $G(t)$ provided in Proposition 4.