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UNIVERSAL SKOLEM SORTS FOR RANDOMIZATIONS

JORGE MUÑOZ CARVAJAL

ABSTRACT. We show that if a complete theory T in a countable language admits a universal Skolem sort, then its randomization T^R also admits such a sort. We also describe the groupoid associated to the randomization in terms of the groupoid associated to the original theory.

INTRODUCTION

A classical result, attributed to T. Coquand but first published by G. Ahlbrandt and M. Ziegler in [AZ86], states that two classical, complete, ω -categorical theories in countable languages are determined, modulo bi-interpretation, by their automorphism groups. This result was later generalized by I. Ben Yaacov and A. Kaïchouh in [BYK16] to the continuous framework. Namely, two separable ω -categorical continuous structures are bi-interpretable if and only if their automorphism groups are isomorphic as topological groups. They went even further and showed a reconstruction procedure, a way to define a structure from the automorphism group which is bi-interpretable with the original structure. Towards this end, starting from the automorphism group G , they showed that the structure \hat{G} , consisting of the left-completion of the group, \hat{G}_L , together with a distance predicate to each orbit closure of the diagonal action of G on $(\hat{G}_L)^n$, is bi-interpretable with the original structure.

In order to generalize this result and obtain a reconstruction procedure that applies beyond the context of ω -categoricity, I. Ben Yaacov in [BY22] defined the notion of *universal Skolem sort*. To every complete continuous theory T in a countable language admitting such a sort we can associate a topological groupoid $\mathbf{G}(T)$ over $2^{\mathbb{N}}$ which determines T up to bi-interpretability, and from which we can define a theory bi-interpretable with T . We must remark that not every theory has a universal Skolem sort, those that admit it include classical and ω -categorical theories.

Randomizations are basically structures consisting of random functions taking values in structures on a given language. They were first defined by H. J. Keisler in [Kei99] and then reformulated using the framework of continuous logic in [BYK09], and again in [BY13]. Formally, a randomization is a continuous two-sorted structure, one sort for the random functions in the structures and a second sort for $[0, 1]$ -valued random variables. The two sorts are related via maps $\llbracket \varphi(\cdot) \rrbracket$, where $\varphi(x)$ is a formula in the original language, that to each tuple of random elements a assigns a random variable $\llbracket \varphi(a) \rrbracket$. In case the given language is classical we can restrict our attention to $\{0, 1\}$ -valued random variables, which we identify with events.

Randomizations preserve many desirable model-theoretical properties such as completeness, ω -categoricity, stability and NIP. This results can be shown using analytic methods as in [BY13, BY09] or, in the ω -categorical case, dynamically as in [Iba17]. In this context, the correspondence between stable (resp. NIP) formulas and weakly almost periodic (resp. tame) functions, as shown in [Iba16], allows to describe these model theoretic properties in terms of representations of the action of the group of automorphisms on the space of types in a particular class of Banach spaces. Thus, by describing the group of automorphisms of the randomization in terms of the group of automorphisms of the original structure and by constructing a representation of the randomized action from the original representation, T. Ibarlucia shows these preservation results.

Using the notion of universal Skolem sort, and that of associated groupoid, we expect to extend T. Ibarlucia's methods to a broader context. The present article deals with the first of these issues, obtaining the groupoid associated to the randomization in terms of the original groupoid. Our first objective is to show that if a theory T admits a universal Skolem sort, then so it does its randomization T^R . The universal Skolem sort of T^R will be a fiber product between the randomization of the universal Skolem

sort of T and the universal Skolem sort of the theory of random variables which, as an ω -categorical theory, admits such a sort. We then give an explicit description of the groupoid associated to T^R and we show a purely topological construction of this groupoid starting from the groupoid associated to the original theory.

The article is organized as follows, in [Section 1](#) we describe the theories of probability algebras and of random variables. We also define randomizations and give some results on the subject. In [Section 2](#), we present the notions of universal Skolem sort and topological groupoid, and we show how to associate to a theory admitting a universal Skolem sort the topological groupoid that determines it up to bi-interpretation. In [Section 3](#) we exhibit the universal Skolem sort of the randomization theory, assuming the original theory admits such a sort. In [Section 4](#), we study the groupoid \mathbf{G}^Ω consisting of measurable functions from a probability space $(\Omega, \mathcal{B}, \mu)$ into a Polish open topological groupoid \mathbf{G} . We also study $\mathbf{G}^\Omega \rtimes \text{Aut}(\Omega)$, where $\text{Aut}(\Omega)$ is the group of invertible measure preserving transformations of $(\Omega, \mathcal{B}, \mu)$. Finally, in [Section 5](#) we describe the groupoid associated to the randomization and we show that it corresponds to $(\mathbf{G}(T)^\Omega \rtimes \text{Aut}(\Omega))_{2^\mathbb{N}}$, the base change of $\mathbf{G}(T)^\Omega \rtimes \text{Aut}(\Omega)$ to $2^\mathbb{N}$ given by the map used to define the fiber product mentioned above.

Notation. Suppose \mathfrak{A} is a Boolean algebra and $a \in \mathfrak{A}$ then a^0 and a^1 will denote respectively a and $\neg a$. In case $\nu \in 2^m$ and (a_i) is a sequence of elements of \mathfrak{A} of length at least m then a_ν stands for $a_0^{\nu(0)} \wedge \dots \wedge a_{m-1}^{\nu(m-1)}$. For our purposes, \mathfrak{A} will either be an algebra of formulas or a measure algebra. For A a family of sets in a σ -algebra we write $\sigma(A)$ for its generated σ -algebra. If (X, d) is a metric space, then for $x \in X$ and $r > 0$, $B_d(x, r)$ and $\bar{B}_d(x, r)$ will denote, respectively, the open and closed d -balls with radius r centered at x . If the metric is clear from the context we may drop the subindex. Whenever f is a tuple of functions defined on a set Ω and $\psi(x)$ is a classical formula then $\llbracket \psi(f) \rrbracket$ will denote the set $\{\omega \in \Omega ; \psi(f(\omega)) \text{ is true}\}$. Whenever X is a random variable, $\mathbb{E}(X)$ will denote the expectation of X .

1. RANDOMIZATIONS

First, we will describe the theory of atomless probability algebras. The language of probability algebras is defined as $\mathcal{L}_{\text{Pr}} = \{\top, \perp, \neg, \cup, \cap, \mu\}$. If $(\Omega, \mathcal{B}, \mu)$ is a probability space then its measure algebra, $(\bar{\mathcal{B}}, \mu)$, admits an obvious interpretation as an \mathcal{L}_{Pr} -structure. We will denote by Pr the common theory of all measure algebras coming from a probability space. This theory can be axiomatized using the axioms of Boolean algebras, conditions stating that μ is a finitely additive probability measure and that the distance corresponds to the measure of the symmetric difference. If we add to Pr the condition saying that the algebra is atomless we get the theory of atomless probability algebras, which we denote APr . Every atomless probability algebra can be regarded as a model of APr . Conversely, if \mathcal{M} is a model of APr then there is an atomless probability space whose measure algebra is isomorphic to \mathcal{M} . The theory APr is complete, ω -categorical and eliminates quantifiers [[BY13](#), Fact 2.10].

Now we pass to the theory of atomless random variables. Let us define $\mathcal{L}_{\text{RV}} = \{0, \neg, \frac{1}{2}, \dot{\div}\}$, where $\frac{1}{2}$ represents a unary function symbol. For $(\Omega, \mathcal{B}, \mu)$ a probability space, the space of random variables, $L^1((\Omega, \mathcal{B}, \mu), [0, 1])$ together with the L^1 -distance, can be seen as an \mathcal{L}_{RV} -structure by interpreting 0 as the null map and, by setting $(\frac{1}{2}x)(\omega) = \frac{1}{2}x(\omega)$, $(\neg x)(\omega) = 1 - x(\omega)$ and $(x \dot{\div} y)(\omega) = x(\omega) \dot{\div} y(\omega) = \max\{0, x(\omega) - y(\omega)\}$ for $x, y \in L^1((\Omega, \mathcal{B}, \mu), [0, 1])$ and $\omega \in \Omega$. The common theory of such structures will be denoted by RV . Every model of RV is isomorphic to a model of the form $L^1((\Omega, \mathcal{B}, \mu), [0, 1])$ for some probability space $(\Omega, \mathcal{B}, \mu)$. If we add the condition stating that the space must be atomless we get the theory ARV of atomless random variables. As before, ARV is complete, ω -categorical and eliminates quantifiers [[BY13](#), Theorem 2.17].

The theories ARV and APr are bi-interpretable [[BY13](#), Proposition 2.12]. Therefore, for any \mathcal{M} and \mathcal{N} separable models of APr and ARV respectively, $\text{Aut}(\mathcal{M})$ and $\text{Aut}(\mathcal{N})$ are isomorphic as topological groups. Here we give a presentation of such group. Let us begin with the measure algebra on the unit interval, $(\bar{\mathcal{F}}, \lambda)$, or with any separable model of APr for that matter. Let Ω be its Stone space, a compact, Hausdorff topological space. Then, there is an atomless probability measure μ on \mathcal{B} , the measurable subsets of Ω , such that $(\bar{\mathcal{F}}, \lambda)$ is isomorphic to $(\bar{\mathcal{B}}, \mu)$ [[Fre02](#), Paragraph 321.J]. The ideal of μ -negligible subsets of Ω corresponds to the ideal of meager subsets. It follows that μ is complete. Moreover, a subset of Ω is measurable if and only if it differs from a clopen set in a meager set. The group of automorphisms of the measure algebra $(\bar{\mathcal{B}}, \mu)$ will be denoted by $\text{Aut}(\Omega)$ and it consists of the group of invertible measure-preserving transformations of Ω . The action of $\text{Aut}(\Omega)$ on $L^1((\Omega, \mathcal{B}, \mu), [0, 1])$ is

given by $(\tau \cdot f)(\omega) = f(\tau^{-1}(\omega))$ for $\tau \in \text{Aut}(\Omega)$, $f \in L^1((\Omega, \mathcal{B}, \mu), [0, 1])$ and $\omega \in \Omega$. $\text{Aut}(\Omega)$ endowed with the topology of pointwise convergence is a Polish group (see [Kec10, Chapter I]). A compatible left-invariant metric can be constructed as follows: pick $\Xi = (\xi_n)_{n \in \mathbb{N}}$ a sequence of events such that the generated subalgebra is dense in \mathcal{B} , and for $\tau, \rho \in \text{Aut}(\Omega)$ let

$$d_L(\tau, \rho) = \sup_{n \in \mathbb{N}} 2^{-n} \wedge d(\tau(\xi_n), \rho(\xi_n)).$$

In order to define the notion of randomization, we introduce the concept of integration space.

Definition 1.1. An *integration space* is a triple $(\Omega, \mathcal{A}, \mathbb{E})$ such that Ω is any set, $\mathcal{A} \subseteq [0, 1]^\Omega$ with $0 \in \mathcal{A}$ and closed under the maps $\neg, \frac{1}{2}, \dot{+}$, as defined before, and $\mathbb{E} : \mathcal{A} \rightarrow [0, 1]$ satisfies $\mathbb{E}(f + g) = \mathbb{E}(f) + \mathbb{E}(g)$ for $f, g, f + g \in \mathcal{A}$.

If we denote by \mathbb{E} the expected value, namely $\mathbb{E}(f) = \int f \, d\mu$, then $(\Omega, L^1((\Omega, \mathcal{B}, \mu), [0, 1]), \mathbb{E})$ is an integration space. Moreover, by identifying an event with its characteristic function, we get that $(\Omega, \mathcal{B}, \mu)$ is also an integration space.

Definition 1.2. Let \mathcal{L} be a language, let Ω be a non-empty set and let $\{\mathcal{M}_\omega ; \omega \in \Omega\}$ be a family of \mathcal{L} -structures. A *randomization based on* $\{\mathcal{M}_\omega ; \omega \in \Omega\}$ is a triple $(\mathcal{M}, \mathcal{A}, \mathbb{E})$ where $M \subseteq \prod_{\omega \in \Omega} M_\omega$ is non-empty, $\mathcal{A} \subseteq [0, 1]^\Omega$ and $\mathbb{E} : \mathcal{A} \rightarrow [0, 1]$ such that:

- (1) $(\Omega, \mathcal{A}, \mathbb{E})$ is an integration space.
- (2) For every constant symbol c in \mathcal{L} , the map $\omega \mapsto c^{\mathcal{M}_\omega}$ belongs to M .
- (3) For every n -ary function symbol F in \mathcal{L} and every $a_0, \dots, a_{n-1} \in M$, the map

$$\omega \mapsto F^{\mathcal{M}_\omega}(a_0(\omega), \dots, a_{n-1}(\omega))$$

belongs to M .

- (4) For every n -ary predicate symbol P in \mathcal{L} and every $a_0, \dots, a_{n-1} \in M$, the map

$$\omega \mapsto P^{\mathcal{M}_\omega}(a_0(\omega), \dots, a_{n-1}(\omega))$$

belongs to \mathcal{A} .

In case E can be deduced from the context we will omit it and we may simply write $(\mathcal{M}, \mathcal{A})$. Whenever \mathcal{M}_ω is equal to a single structure \mathfrak{M} for every $\omega \in \Omega$, we say that $(\mathcal{M}, \mathcal{A})$ is a randomization of \mathfrak{M} .

Definition 1.3. Given a language \mathcal{L} , the *randomization language*, \mathcal{L}^R , is a continuous language consisting of the same sorts as \mathcal{L} , the home sorts, together with an additional auxiliary sort satisfying:

- (1) For each constant symbol in \mathcal{L} a constant symbol in \mathcal{L}^R belonging to the same sort.
- (2) For each function symbol in \mathcal{L} a function symbol in \mathcal{L}^R between the corresponding main sorts and with the same modulus of uniform continuity as the original symbol.
- (3) For each predicate symbol P in \mathcal{L} a function symbol $\llbracket P() \rrbracket$ from the corresponding main sorts into the auxiliary sort and with the same modulus of uniform continuity.
- (4) If \mathcal{L} is classical, the auxiliary sort will be equipped with the language \mathcal{L}_{Pr} . Otherwise, it will be equipped with the language \mathcal{L}_{RV} .

For simplicity, we will assume that \mathcal{L} consists of a single sort, however the same arguments can be carried out in the many-sorted case with some additional bookkeeping. It follows from the previous two definitions that if $\mathbf{M} = (\mathcal{M}, \mathcal{A}, \mathbb{E})$ is a randomization based on a family of \mathcal{L} -structures $\{\mathcal{M}_\omega ; \omega \in \Omega\}$, then it admits a natural interpretation as an \mathcal{L}^R -prestructure, where the distance on the home sort is given by $d(a, b) = \mathbb{E}[\llbracket d(a, b) \rrbracket]$ for $a, b \in \mathcal{M}$.

The classical example of a randomization is the *Borel randomization* of a structure. Let \mathcal{M} be a separable \mathcal{L} -structure and let $(\Omega, \mathcal{B}, \mu)$ be the standard Borel probability space constructed earlier. Define \mathcal{M}^Ω as the set of measurable functions from Ω into \mathcal{M} and take $(\Omega, \mathcal{A}, \mathbb{E})$ equal to $(\Omega, \mathcal{B}, \mu)$ or $(\Omega, L^1((\Omega, \mathcal{B}, \mu), [0, 1]), \mathbb{E})$, depending on whether \mathcal{L} is classical or not. Then $\mathcal{M}^R = (\mathcal{M}^\Omega, \mathcal{A})$ is a randomization based on \mathcal{M} .

Definition 1.4. Suppose T is an \mathcal{L} -theory. Let T^{Ra} be the common theory of the randomizations based on models of T . The *randomization theory*, denoted T^R , is defined as T^{Ra} together with the condition stating that the underlying integration space is atomless.

As expected, every model of T^{Ra} is isomorphic to a randomization based on models of T . In fact, to every $\mathbf{M} = (\mathcal{M}, \mathcal{A}, \mathbb{E})$ model of T^{Ra} , we can canonically associate a randomization based on models of T whose underlying probability space corresponds to the Stone space of \mathcal{A} . This randomization will be known as

the *canonical representation* and we will identify a model of T^{Ra} with it, see [BY13, Definition 3.9] and the construction thereafter. In particular, whenever \mathbf{M} is a separable model of T^R , we can assume that its underlying probability space is the Stone space $(\Omega, \mathcal{B}, \mu)$ described above.

Given a randomization $\mathbf{M} = (\mathcal{M}, \mathcal{A}, \mathbb{E})$ based on a family of \mathcal{L} -structures $\{\mathcal{M}_\omega ; \omega \in \Omega\}$, for every \mathcal{L} -formula $\varphi(x)$ and every tuple $a \in \mathcal{M}$ we can define a map $\llbracket \varphi(a) \rrbracket : \Omega \rightarrow [0, 1]$ by $\llbracket \varphi(a) \rrbracket(\omega) = \varphi^{\mathcal{M}_\omega}(a(\omega))$ for $\omega \in \Omega$. It turns out that this map belongs to \mathcal{A} and the resulting function $\llbracket \varphi(\cdot) \rrbracket : \mathcal{M}^n \rightarrow \mathcal{A}$ is uniformly definable on randomizations based on \mathcal{L} -structures, see [BY13, Lemma 3.13 and Theorem 3.14]. Moreover, for every \mathcal{L} -formula $\varphi(x, y)$ we have that $\llbracket \inf_y \varphi(x, y) \rrbracket = \inf_y \llbracket \varphi(x, y) \rrbracket$. In particular, if φ is an \mathcal{L} -sentence, then $\llbracket \varphi \rrbracket \in \mathcal{A}$. Therefore,

$$T \vdash \varphi = 0 \text{ implies } T^{Ra} \vdash \llbracket \varphi \rrbracket = 0.$$

This fact is known as the *Transfer Axiom* and will be used constantly, sometimes without explicit mention. The randomization theory T^R eliminates quantifiers down to formulas of the form $\mathbb{E}[\llbracket \varphi(x) \rrbracket]$ [BY13, Theorem 3.32]. Moreover, an explicit axiomatization of T^R is given in [BY13]. Randomizations also preserve many of the properties of the original theory such as, completeness, ω -categoricity, stability and NIP [BY13, Sections 3 and 4]. Notably, if T is ω -categorical and \mathcal{M} is a separable model of T then \mathcal{M}^R , its Borel randomization, is the unique separable model of T^R , modulo isomorphism. The following result is an easy consequence of [BY13, Lemma 3.10].

Fact 1.5. *Let $(\mathcal{M}, \mathcal{A}, \mathbb{E}) \models T^R$. Let A_0, \dots, A_{n-1} be a partition of the underlying probability space and let $a_0, \dots, a_{n-1} \in \mathcal{M}$. Then, there is a unique $c \in \mathcal{M}$ such that c is equal to a_i on A_i for $i < n$.*

The element c in the previous lemma will be denoted by $\langle A_0, \dots, A_{n-1}, a_0, \dots, a_{n-1} \rangle$, or equivalently, by $\langle (A_i)_{i < n}, (a_i)_{i < n} \rangle$. We now recall how definability works in the continuous setting and give some results on definability in the randomization.

Definition 1.6. Let us fix a language \mathcal{L} . Let \mathcal{M} be a \mathcal{L} -structure and $A \subseteq M$, a predicate $P : M^n \rightarrow [0, 1]$ is *definable in \mathcal{M} over A* if there is a sequence of $\mathcal{L}(A)$ -formulas that converges uniformly on M^n to P . A closed set $U \subseteq M^n$ is said to be *definable in \mathcal{M} over A* if $d(x, U)$ is a definable predicate in \mathcal{M} over A . Finally, a map $f : M^n \rightarrow M$ is *definable in \mathcal{M} over A* if the predicate $d(f(x), y)$ on M^{n+1} is definable in \mathcal{M} over A .

As expected these notions are well-behaved with respect to elementary extensions [BYBHU08, Section 9]. Whenever we say that a subset or a function is definable we always mean without parameters. The following result characterizing definability of a set will be used later on when discussing definable closures in randomizations.

Fact 1.7. *Let \mathcal{M} be a structure, U be a closed subset of M^n and A be any subset of M . Then the following assertions are equivalent:*

- (1) U is definable in \mathcal{M} over A .
- (2) U is the set of zeros of an A -definable predicate $\psi(x)$ and $d(x, U) \leq \psi(x)$ on M^n .
- (3) U is the set of zeros of an A -definable predicate $\psi(x)$ and, for every $\epsilon > 0$ there is $\delta > 0$ such that for every $x \in M^n$, $\psi(x) \leq \delta$ implies $d(x, U) \leq \epsilon$.
- (4) For every $\epsilon > 0$ there is a formula with parameters in A , $\psi(x)$, such that

$$U \subseteq \{a \in M^n ; \psi(a) = 0\} \subseteq \{a \in M^n ; \psi(a) < 1\} \subseteq \{a \in M^n ; d(a, U) < \epsilon\}.$$

- (5) For every A -definable predicate $\varphi(x, y)$, the predicate $\psi(y) = \inf_{x \in U} \varphi(x, y)$ is A -definable.

Proof. The equivalence of (1), (2), (4) and (5) is due to [BY10, Fact 1.7]. Equivalence between (1) and (3) corresponds to [BYBHU08, Remark 9.20]. \square

Proposition 1.8 (Proposition 2.3 in [BYMC21]). *Let T be an \mathcal{L} -theory and D a definable subset of T . Then, for any $\mathbf{M} \models T^R$,*

$$D^R(\mathbf{M}) = \{a \in \mathcal{M}^n ; \mu[\llbracket a \in D \rrbracket] = 1\}.$$

is definable in \mathbf{M} . This definition is moreover uniform across all models of T^R .

The next result, although not explicitly mentioned, is what is essentially shown in the proof of [BYMC21, Proposition 2.6].

Proposition 1.9. *Let T be an \mathcal{L} -theory and f an n -ary definable map in T . Then, for every $\mathbf{M} \models T^R$ the map $f^R : \mathcal{M}^n \rightarrow \mathcal{M}$ given by $f^R(a)(\omega) = f^{\mathcal{M}_\omega}(a(\omega))$ for $a \in \mathcal{M}^n$ and $\omega \in \Omega$ is a definable map in \mathbf{M} . This definition is uniform across all models of T^R .*

2. UNIVERSAL SKOLEM SORTS

Most of the results and definitions given in this section come from [BY22]. We fix a theory T in a countable language.

Definition 2.1. The *metric sorts* of T is the family obtained after closing the set of home sorts of T under the following three operations:

- Infinite product: If $\{F_n ; n \in \mathbb{N}\}$ is a countable family of sorts of T then $F = \prod_{n \in \mathbb{N}} F_n$ is a sort with the metric given by $d(x, y) = \sup_{n \in \mathbb{N}} 2^{-n} \wedge d_{F_n}(x_n, y_n)$.
- Definable subset: If F is a sort of T and E is a definable subset of F then E together with the metric it inherits from F is a sort of T .
- Metric quotient: If F is a sort of T and ρ is a definable pseudo-metric on F then E , the completion of the quotient of F induced by ρ , is a sort of T whose metric is ρ .

In other words, a metric sort of a theory is a definable subset of an imaginary sort. The formulas on an infinite product are the uniform limits of formulas on a finite subproduct. On the other hand, the formulas on a definable subset are just the restriction of formulas to the subset. Finally, the formulas on a metric quotient of F are the formulas on F that are uniformly continuous with respect to the pseudo-metric. The next results presents some natural sorts of the randomizations and characterizes their space of types.

Proposition 2.2. *Let F be a sort of T . Then F^R , defined as the set of random functions on F together with the metric given by $d_{FR}(x, y) = \mathbb{E}[d_F(x, y)]$, is a sort of T^R .*

Proof. If F is a home sort, then by definition F^R with the metric described above is a sort of T^R . Let $\{F_n ; n \in \mathbb{N}\}$ be a countable family of sorts of T such that each F_n^R is a sort of T^R and let $F = \prod_{n \in \mathbb{N}} F_n$. It follows that $D = \prod_{n \in \mathbb{N}} F_n^R$ with the metric given by $d_D(x, y) = \sup_{n \in \mathbb{N}} 2^{-n} \wedge d_{F_n^R}(x_n, y_n)$ is a sort of T^R . Note that D can be naturally identified with F^R . However, the metric d_D does not agree with d_{FR} . Nevertheless, it can be easily seen that d_{FR} is definable. It follows that the two metrics are uniformly equivalent and thus we may take d_{FR} as the canonical metric on $D = F^R$. By Proposition 1.8, if F is a sort of T and E is a definable subset of F then E^R is a definable subset of F^R . Hence, E^R is a sort of T^R . Finally, suppose that F is a sort of T , that ρ is a definable pseudo-metric on F and that E is the quotient sort. We have that $\rho^R(x, y) = \mathbb{E}[\rho(x, y)]$ is a definable pseudo-metric on F^R and that the quotient of F^R induced by ρ^R corresponds to E^R . \square

Given a compact Hausdorff space K , we denote by $\mathfrak{R}(K)$ the space of Borel probability measures on K . Note that $\mathfrak{R}(K)$ is a subset of the unit ball of $C(K)^*$ and as such it is naturally endowed with the weak* topology. Since $\mathfrak{R}(K)$ is closed, it is a compact space. The proof of the following result is completely analogous to that of [BY13, Theorem 3.32] but considering elements in the sort F rather than tuples of elements.

Fact 2.3. *Let F be a sort of T . Then $S_{FR}(T^R)$ is homeomorphic to $\mathfrak{R}(S_F(T))$.*

We will now define the notions of universal and Skolem sorts.

Definition 2.4. Let D and F be sorts of T , we say that D is a *Skolem sort for F* if for every formula $\varphi(x, y)$ in $D \times F$ and every $\epsilon > 0$ there is a definable map $\sigma : D \rightarrow F$ such that for every $x \in D$, $\varphi(x, \sigma(x)) \leq \inf_y \varphi(x, y) + \epsilon$. We say that D is a *universal sort for F* if for every $\epsilon > 0$ there is a definable map $\sigma : D \rightarrow F$ such that the image of every ϵ -ball in D is ϵ -dense in F .

A sort D is said to be *Skolem* if it is a Skolem sort for every sort F . Similarly, it is *universal* if it is a universal sort for any sort F . Notice that by considering $\varphi(x, y) - \inf_z \varphi(x, z)$ in the previous definition, we may restrict our attention to the formulas satisfying $\sup_x \inf_y \varphi(x, y) = 0$. We can further restrict ourselves to a dense subset of such formulas.

Notation. Whenever we have a condition $\psi(x) = 0$ and we have not quantified over the variable x , we will assume that there is an implicit universal quantifier. So, for instance, $\inf_y \varphi(x, y) = 0$ will stand for $\sup_x \inf_y \varphi(x, y) = 0$.

The next result shows that in order to prove that a sort is universal Skolem it suffices to verify it for a small set of sorts.

Proposition 2.5. [BY22, Proposition 4.9] *A sort D is universal Skolem if and only if it is Skolem for every home sort and universal for every finite product of home sorts.*

Between every two universal Skolem sorts there is a definable bijection [BY22, Theorem 4.10]. The next result is a kind of converse of this result.

Lemma 2.6. *If D is a universal Skolem sort and $\eta : D \rightarrow E$ is a definable bijection then E is a universal Skolem sort.*

Proof. Take F any sort, first we will prove that E is Skolem for F . Let $\varphi(x, y)$ be a formula in $E \times F$ such that $\inf_y \varphi(x, y) = 0$ and let $\epsilon > 0$. Let z be a variable of sort D and define $\psi(z, y) = \varphi(\eta(z), y)$. Then, $\inf_y \psi(z, y) = 0$ and, since D is a Skolem sort for F , there is a definable map $\sigma : D \rightarrow F$ such that $\psi(z, \sigma(z)) < \epsilon$. It follows that $\tilde{\sigma} = \sigma \circ \eta^{-1}$ is an ϵ -Skolem map for φ . To show that E is weakly universal for F , take $\epsilon > 0$ and choose $0 < \delta < \epsilon$ such that for all $z_1, z_2 \in D$, $d(z_1, z_2) < \delta$ implies $d(\eta(z_1), \eta(z_2)) < \epsilon$. We know that there is a definable map $\sigma : D \rightarrow F$ such that the image of any δ -ball in D is δ -dense in F and let us define $\tilde{\sigma} = \sigma \circ \eta^{-1}$. Take $x \in E$ and $y \in F$ arbitrary, and let $z = \eta^{-1}(x)$. We know that there is some $z_0 \in D$ such that $d(z_0, z) < \delta$ and $d(\sigma(z_0), \sigma(z)) < \delta$. If $x_0 = \eta(z_0)$ then $d(x_0, x) < \epsilon$ and $d(\tilde{\sigma}(x_0), \tilde{\sigma}(x)) < \epsilon$, which completes the proof. \square

Not every theory admits a universal Skolem sort [BY22, Example 4.15]. However, all classical and all ω -categorical theories admit such a sort.

Fact 2.7. [BY22, Proposition 4.17] *Suppose T is an ω -categorical theory and take \mathcal{M} a separable model of T . Pick a dense enumeration of \mathcal{M} , then $D_0 = \text{tp}(a)$ is a sort of T and $D = D_0 \times 2^{\mathbb{N}}$ is a universal Skolem sort*

We will now exhibit another universal Skolem sort for an ω -categorical theory.

Lemma 2.8. *Assume that T is ω -categorical. If b enumerates a sequence whose definable closure is a model $M \models T$ and D_0 is the type of b then $D = D_0 \times 2^{\mathbb{N}}$ is a universal Skolem sort.*

Proof. Let a be an enumeration of a dense subset of M and let E_0 be the sort defined by the type of a , we denote by S the sort of M . We will build a definable bijection between E_0 and D_0 . Since a is definable over b , there is a formula $\varphi(x, y)$ in $D_0 \times S^{\mathbb{N}}$ such that $\varphi(b, y) = d(y, a)$. Then, for any $b' \in D_0$, $\varphi(b', y)$ is to a distance predicate on $S^{\mathbb{N}}$. Furthermore, if $\varphi(b', a') = 0$ then $a' \in E_0$. Thus, we obtain a definable map $f : D_0 \rightarrow E_0$ where $d(f(x), y) = \varphi(x, y)$. On the other hand, since a is dense in M , for every $n \in \mathbb{N}$, there is a subsequence $(a_{n,k})_{k \in \mathbb{N}}$ of a such that $\lim_{k \rightarrow \infty} a_{n,k} = b_n$. Then, $g(y) = (\lim_{k \rightarrow \infty} y_{n,k})_{n \in \mathbb{N}}$ is a definable map from E_0 into D_0 . It can be easily verified that $f \circ g = \text{Id}_{E_0}$ and that $g \circ f = \text{Id}_{D_0}$. Therefore, $f \times \text{Id}_{2^{\mathbb{N}}}$ is a definable bijection between D and $E_0 \times 2^{\mathbb{N}}$, the result follows from Lemma 2.6. \square

In the case $T = \text{APr}$, we take Ξ to be a sequence of independent subsets of Ω of measure $1/2$ whose generated algebra is dense in $(\bar{\mathcal{B}}, \mu)$. It follows that if $D_{\text{APr},0} = \text{tp}(\Xi)$, then $D_{\text{APr}} = D_{\text{APr},0} \times 2^{\mathbb{N}}$ is a universal Skolem sort for APr, as well as for RV by bi-interpretability.

2.1. Topological groupoids.

We will now give some results on topological groupoids. Groupoids are usually defined as categories in which every morphism is invertible. What we call the base is just the set of objects of the category. However, for our purposes, it is more convenient to take an algebraic approach and see them as groups with a partially defined operation, as in [ADR00].

Definition 2.9. A *groupoid* is a non-empty set, \mathbf{G} , together with a partially defined binary operation and an inverse map such that:

- Whenever $f, g, h \in \mathbf{G}$ and, fg and gh are defined, then $f(gh)$ and $(fg)h$ are both defined and they agree.
- For every $g \in \mathbf{G}$, gg^{-1} and $g^{-1}g$ are both defined.
- For every $f, g \in \mathbf{G}$ if fg is defined then $fgg^{-1} = f$ and $f^{-1}fg = g$.

For $g \in \mathbf{G}$ we define the *target* and the *source* of g as $t_g = gg^{-1}$ and $s_g = g^{-1}g$ respectively. An element $e \in \mathbf{G}$ is *neutral* if $e^2 = e$, the set of neutral elements of \mathbf{G} is known as the *base* of \mathbf{G} and it will be denoted by \mathbf{B} . It is easy to verify that $(g^{-1})^{-1} = g$, t_g and s_g are neutral, $s_{g^{-1}} = t_g$ and $t_{g^{-1}} = s_g$, and fg is defined if and only if $s_f = t_g$. We remark that the target and the source define maps from \mathbf{G} onto \mathbf{B} .

Definition 2.10. A *topological groupoid* is a groupoid \mathbf{G} together with a Hausdorff topology on \mathbf{G} such that the inverse and the product, where defined, are continuous.

A topological groupoid is said to be *open* if the source $s : \mathbf{G} \rightarrow \mathbf{B}$, or equivalently the target $t : \mathbf{G} \rightarrow \mathbf{B}$, is an open map. Now we introduce the notion of a groupoid action.

Definition 2.11. Let \mathbf{B} be a topological space, a *space over \mathbf{B}* is a couple (X, ρ) where X is a topological space and $\rho: X \rightarrow \mathbf{B}$ is a continuous, surjective map.

Note that a topological groupoid is a space over its base, via the target or the source map.

Definition 2.12. Let \mathbf{G} be a topological groupoid of base \mathbf{B} and let (X, ρ) be a space over \mathbf{B} . The *fiber product between \mathbf{G} and X* is defined as

$$\mathbf{G} \times_{\mathbf{B}} X = \{(g, x) \in \mathbf{G} \times X ; s_g = \rho(x)\}.$$

A *continuous action of \mathbf{G} on X* is a continuous map $\cdot : \mathbf{G} \times_{\mathbf{B}} X \rightarrow X$ such that whenever fg and $g \cdot x$ are defined then $f \cdot (g \cdot x) = (fg) \cdot x$, meaning both sides are defined and they agree.

In this case we say that (X, ρ) is a *\mathbf{G} -spacer over \mathbf{B}* and we write $\mathbf{G} \curvearrowright X$. The dual notion of fiber product between X and \mathbf{G} , $X \times_{\mathbf{B}} \mathbf{G}$, is defined analogously but with the target map instead of the source, from which the notions of right action and right \mathbf{G} -space can be easily defined.

Considering \mathbf{G} as a space over \mathbf{B} , we have that the domain of the groupoid operation is the fiber product $\mathbf{G} \times_{\mathbf{B}} \mathbf{G} = \{(f, g) \in \mathbf{G}^2 ; s_f = t_g\}$. Furthermore, the groupoid operation $\mathbf{G} \times_{\mathbf{B}} \mathbf{G} \rightarrow \mathbf{G}$ is simultaneously a left and a right action of \mathbf{G} on itself. Finally, seeing $(\mathbf{B}, \text{id}_{\mathbf{B}})$ as a space over itself, we have a natural continuous action of \mathbf{G} on \mathbf{B} given by $(g, s_g) \mapsto t_g$ (do not mistake with the groupoid operation). The next fact characterizes open groupoids in terms of actions.

Fact 2.13. [BY22, Fact 1.5] *Let \mathbf{G} be a topological groupoid of base \mathbf{B} . Then, the following are equivalent:*

- (1) \mathbf{G} is an open groupoid.
- (2) For any (X, ρ) space over \mathbf{B} , the projection $\mathbf{G} \times_{\mathbf{B}} X \rightarrow X$ is open.
- (3) For any \mathbf{G} -space (X, ρ) , the action law $\mathbf{G} \times_{\mathbf{B}} X \rightarrow X$ is open.
- (4) The groupoid operation $\mathbf{G} \times_{\mathbf{B}} \mathbf{G} \rightarrow \mathbf{G}$ is open.

Finally we introduce the notion of a minimal action.

Definition 2.14. Let \mathbf{G} be a topological groupoid over \mathbf{B} and let (X, ρ) be a \mathbf{G} -space over \mathbf{B} . We say that the action $\mathbf{G} \curvearrowright X$ is *minimal* if every orbit is dense. In particular, when $\mathbf{G} \curvearrowright \mathbf{B}$ is minimal we say that \mathbf{G} itself is *minimal*.

In other words, \mathbf{G} is minimal if and only if for every $e \in \mathbf{B}$ and W open subset of \mathbf{B} there is $g \in \mathbf{G}$ with $t_g = e$ and $s_g \in W$, or equivalently, there is some $h \in \mathbf{G}$ with $s_h = e$ and $t_h \in W$.

2.2. Groupoid associated to a theory.

Now we will show how to define the groupoid associated to a theory admitting a universal Skolem sort following [BY22, Section 5]. Given a sort D , by $S_{nD}(T)$ we mean the space of types on D^n . Suppose D_T is a universal Skolem sort of T . The *groupoid associated to T* , denoted $\mathbf{G}(T)$, is defined as

$$\mathbf{G}(T) = \{\text{tp}(a, b) \in S_{2D_T}(T) ; \text{dcl}(a) = \text{dcl}(b)\}.$$

Its base, $\mathbf{B}(T)$, is the set of types of the form $\text{tp}(a, a)$ with $a \in D_T$, which we identify with $\text{tp}(a)$, and thus $\mathbf{B}(T) = S_{D_T}(T)$, which we know to be homeomorphic to $2^{\mathbb{N}}$ [BY22, Lemma 4.11]. If $p = \text{tp}(a, b)$ then its inverse is given by $p^{-1} = \text{tp}(b, a)$. If $q = \text{tp}(b', c')$ and $\text{tp}(b) = \text{tp}(b')$, then there is a unique $c \in D_T$ such that $\text{tp}(b, c) = q$, and we define $pq = \text{tp}(a, c)$.

Theorem 2.15. [BY22, Proposition 5.3 and Theorem 5.14] *Let T be a complete theory admitting a universal Skolem sort. Then, $\mathbf{G}(T)$, endowed with the topology it inherits as a subset of $S_{2D_T}(T)$, is a Polish open minimal topological groupoid whose base $\mathbf{B}(T)$ is homeomorphic to $2^{\mathbb{N}}$. Furthermore, if T' is another complete theory admitting a universal Skolem sort then $\mathbf{G}(T)$ is isomorphic to $\mathbf{G}(T')$, as topological groupoids, if and only if T and T' are bi-interpretable.*

Assume that T is ω -categorical and let \mathcal{M} be a separable model of T . Recall from Lemma 2.8 that if a enumerates a sequence satisfying $\text{dcl}(a) = \mathcal{M}$ and $D_0 = \text{tp}(a)$, then $D = D_0 \times 2^{\mathbb{N}}$ is a universal Skolem sort. It follows that the groupoid associated to T is $\mathbf{G}(T) = 2^{\mathbb{N}} \times \text{Aut}(\mathcal{M}) \times 2^{\mathbb{N}}$. In particular, when $T = \text{APr}$ the associated groupoid is $\mathbf{G}(\text{APr}) = 2^{\mathbb{N}} \times \text{Aut}(\Omega) \times 2^{\mathbb{N}}$.

3. UNIVERSAL SKOLEM SORT OF THE RANDOMIZATION

From now on, we assume that T is a complete continuous theory in a countable language that admits a universal Skolem sort D_T . In this section we will exhibit a universal Skolem sort for T^R . Assuming T has only one home sort, the randomization theory is two-sorted, the home sort and the sort of random variables. Therefore, it is natural to think that a universal Skolem sort for T^R must involve D_T^R , the randomization of the universal Skolem sort of T , and D_{APr} , the universal Skolem sort of RV, as defined after [Lemma 2.8](#). The home sort and the random variable sort are related through the maps $\llbracket \varphi(\cdot) \rrbracket$, where $\varphi(x)$ is an \mathcal{L} -formula. In order to capture this relation we will define a fiber product between D_T^R and D_{APr} .

Let us fix some notation, $(\Omega, \mathcal{B}, \mu)$ will refer to the probability space obtained from the Stone space of the usual Borel algebra on the interval $[0, 1]$, as described in [Section 1](#). E and E_{RV} will stand respectively for the sort of events and the sort of random variables in ARV. $E(\Omega)$ and $E_{\text{RV}}(\Omega)$ will refer to the sort of events and random variables in $(\Omega, \mathcal{B}, \mu)$. We will identify a member of $E(\Omega)$ with the corresponding clopen subset of Ω . x will denote a variable of sort D_T , \bar{x} a variable of sort $D_{\text{APr},0}$, and \dot{x} a variable of sort $2^{\mathbb{N}}$.

We know that $S_{D_T}(T)$ is homeomorphic to $2^{\mathbb{N}}$. Hence, by considering $2^{\mathbb{N}}$ as a sort of T , the type function $\text{tp} : D_T \rightarrow 2^{\mathbb{N}}$ is definable. By [Proposition 1.9](#), the corresponding map on the randomization $\text{tp}^R : D_T^R \rightarrow (2^{\mathbb{N}})^R$ is also definable. On the other hand, the function $\iota : 2^R \rightarrow E$, that assigns to every $f \in 2^R$ the event whose characteristic function is f , establishes a definable isometric bijection. Following the argument given in the proof of [Proposition 2.2](#), $(2^R)^{\mathbb{N}}$ can be identified with $(2^{\mathbb{N}})^R$. Hence, we have a definable bijection $\psi : (2^{\mathbb{N}})^R \rightarrow E^{\mathbb{N}}$ given by $\psi(f)_n = \llbracket f_n = 0 \rrbracket$ for $f \in (2^{\mathbb{N}})^R$ and $n \in \mathbb{N}$. Recall from [Proposition 2.2](#) that the metric on $(2^{\mathbb{N}})^R$ is given by the expectation of the product metric on $2^{\mathbb{N}}$. It follows that the two metrics on $(2^{\mathbb{N}})^R$ given by

$$d(f, g) = \mathbb{E}[\llbracket d(f, g) \rrbracket] \quad \text{and} \quad d'(f, g) = \sup_{n \in \mathbb{N}} 2^{-n} \wedge \mu(\llbracket f_n = 0 \rrbracket \triangle \llbracket g_n = 0 \rrbracket),$$

are uniformly equivalent. So, when convenient, we can consider tp^R as a map from D_T^R into $E^{\mathbb{N}}$. Moreover, in the separable case, the previous identification commutes with the action of $\text{Aut}(\Omega)$ i.e. for any $\tau \in \text{Aut}(\Omega)$ and any $f \in (2^{\mathbb{N}})^R$, $\psi(\tau \cdot f) = \tau(\psi(f))$.

Lemma 3.1. *For every $\epsilon > 0$ and every $k \in \mathbb{N}$ there is $\delta > 0$ such that for every $x \in D_T^R$ and every $A \in E^{\mathbb{N}}$ satisfying $d(\text{tp}^R(x), A) < \delta$, there exists $x' \in D_T^R$ such that $d(x, x') < \epsilon$ and $\text{tp}_n^R(x') = A_n$ for $n < k$.*

Proof. First note that, given $\eta \in \{0, 1\}^k$, there is $a_\eta \in D_T^R$ such that $\text{tp}_\eta^R(a_\eta) = \top \in E$. Take $\delta < \min\{2^{-k}, \epsilon/k\}$ and suppose that $d(\text{tp}^R(x), A) < \delta$. For $n < k$ define $D_n = \text{tp}_n^R(x) \triangle A_n$, and denote by \mathcal{A} the algebra generated by $\{A_n; n < k\} \cup \{\text{tp}_n^R(x); n < k\}$. Let B_0, \dots, B_m be the atoms of \mathcal{A} contained in $D_0 \cup \dots \cup D_{k-1}$. We know that for each $i < m$ there is $\eta_i \in \{0, 1\}^k$ such that $B_i \subseteq A_{\eta_i}$. Define $x' \in D_T^R$ such that x' agrees with a_{η_i} on B_i for $i < m$, and agrees with x on $\neg(D_0 \cup \dots \cup D_{k-1})$. Then $\text{tp}_n^R(x') = A_n$ for $n < k$ and

$$d(x, x') \leq \mu(D_0 \cup \dots \cup D_{k-1}) \leq \sum_{n < k} \mu(D_n) < k\delta < \epsilon. \quad \square$$

We want a fiber product between D_T^R and D_{APr} over $E^{\mathbb{N}}$ to be definable. We already have the map $\text{tp}^R : D_T^R \rightarrow E^{\mathbb{N}}$ and we want to find an appropriate map $\pi : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$. In order to prove definability we will use numeral (3) of [Fact 1.7](#) with the predicate $d(\text{tp}^R(x), \pi(\bar{x}, \dot{x}))$. Assume $d(\text{tp}^R(x), \pi(\bar{x}, \dot{x}))$ is small, the previous lemma shows that we can take an x' close to x so that the first few elements of the sequences $\text{tp}(x')$ and $\pi(\bar{x}, \dot{x})$ coincide. The following definition will ensure that we can find (\bar{x}', \dot{x}') close to (\bar{x}, \dot{x}) so that $\text{tp}^R(x') = \pi(\bar{x}', \dot{x}')$. Surprisingly, at least for the author, this property will be enough to show that the fiber product is in fact a universal Skolem sort of T^R .

Definition 3.2. A map $\pi : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$ is said to be *weakly open* if it is definable and for any given $\epsilon > 0$ and $N \in \mathbb{N}$, there is some $k \in \mathbb{N}$ such that for every $(\bar{x}, \dot{x}) \in D_{\text{APr}}$, every separable $\mathcal{A} \models \text{APr}$ containing \bar{x} and every $A \in E^{\mathbb{N}}(\mathcal{A})$ with $\pi_n(\bar{x}, \dot{x}) = A_n$ for every $n < k$, there exists $(\bar{x}', \dot{x}') \in D_{\text{APr}}(\mathcal{A})$ satisfying $d(\bar{x}, \bar{x}') < \epsilon$, $\dot{x} \upharpoonright_N = \dot{x}' \upharpoonright_N$, $\pi(\bar{x}', \dot{x}') = A$. If under the same conditions (\bar{x}', \dot{x}') can be chosen so that $\sigma(\bar{x}') = \mathcal{A}$ we say that π is *pseudo-open*.

Lemma 3.3. *Let $\pi^0 : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$ be weakly open. Define $\pi : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$ by $\pi_n(\bar{x}, \dot{x}) = \pi_{2n}^0(\bar{x}, \dot{x})$ for $n \in \mathbb{N}$ and $(\bar{x}, \dot{x}) \in D_{\text{APr}}$. Then, π is pseudo-open.*

Proof. Given $\epsilon > 0$ and $N \in \mathbb{N}$, take $k \in \mathbb{N}$ as in the definition of weakly open for π^0 . We can assume that k is pair, so $k = 2m$ for some $m \in \mathbb{N}$. Take $(\bar{x}, \dot{x}) \in D_{\text{APr}}$, $\mathcal{A} \models \text{APr}$ separable with $\bar{x} \subseteq \mathcal{A}$ and $A \in \mathbb{N}$ satisfying $\pi_n(\bar{x}, \dot{x}) = A_n$ for every $n < m$. Let $(a_l)_{l \in \mathbb{N}}$ be an enumeration of a dense subset of \mathcal{A} . For $n \in \mathbb{N}$ we set

$$B_n = \begin{cases} A_l & \text{if } n = 2l \text{ with } l \in \mathbb{N} \\ \pi_n^0(\bar{x}, \dot{x}) & \text{if } n = 2l + 1, \text{ with } l < m \\ a_l & \text{if } n = k + 2l + 1 \text{ with } l \in \mathbb{N}. \end{cases}$$

It follows that $\pi_n^0(\bar{x}, \dot{x}) = B_n$ for $n < k$. Therefore, there is $(\bar{x}', \dot{x}') \in D_{\text{APr}}(\mathcal{A})$ satisfying $d(\bar{x}, \bar{x}') < \epsilon$, $\dot{x} \upharpoonright_N = \dot{x}' \upharpoonright_N$ and $\pi^0(\bar{x}', \dot{x}') = B$. Hence, $\pi(\bar{x}', \dot{x}') = A$ and $\mathcal{A} = \sigma((a_l)_{l \in \mathbb{N}}) \subseteq \sigma(B) \subseteq \sigma(\bar{x}')$. We conclude that π is pseudo-open. \square

Lemma 3.4. *Suppose that $\pi : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$ is pseudo-open. Then, the fiber product*

$$D_T^R \times_{E^{\mathbb{N}}} D_{\text{APr}} = \{(x, \bar{x}, \dot{x}) \in D_T^R \times D_{\text{APr}}; \text{tp}^R(x) = \pi(\bar{x}, \dot{x})\}.$$

is a definable subset of $D_T^R \times D_{\text{APr}}$.

Proof. We will show that for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $(x, \bar{x}, \dot{x}) \in D_T^R \times D_{\text{APr}}$ if $d(\text{tp}^R(x), \pi(\bar{x}, \dot{x})) < \delta$ then $d((x, \bar{x}, \dot{x}), D_T^R \times_{E^{\mathbb{N}}} D_{\text{APr}}) < \epsilon$. Fix $\epsilon > 0$ and take $N \in \mathbb{N}$ with $2^{-N} < \epsilon$. For this ϵ and this N let $k \in \mathbb{N}$ be as in the definition of pseudo-open. Now, for such ϵ and k take $\delta > 0$ as given by Lemma 3.1. Suppose that $(x, \bar{x}, \dot{x}) \in D_T^R \times D_{\text{APr}}$ and that $d(\text{tp}^R(x), \pi(\bar{x}, \dot{x})) < \delta$. Using Lemma 3.1, we find $x' \in D_T^R$ such that $d(x, x') < \epsilon$ and $\text{tp}_n^R(x') = \pi_n(\bar{x}, \dot{x})$ for $n < k$. As π is pseudo-open, there is $(\bar{x}', \dot{x}') \in D_{\text{APr}}$ with $\pi(\bar{x}', \dot{x}') = \text{tp}^R(x')$ such that $d(\bar{x}, \bar{x}') < \epsilon$ and $\dot{x} \upharpoonright_N = \dot{x}' \upharpoonright_N$. Thus, $(x', \bar{x}', \dot{x}') \in D_T^R \times_{E^{\mathbb{N}}} D_{\text{APr}}$ and $d((x, \bar{x}, \dot{x}), (x', \bar{x}', \dot{x}')) < \epsilon$. \square

Assuming that a pseudo-open map π exists, as it will be the case, the previous lemma shows that $D_{T^R} = D_T^R \times_{E^{\mathbb{N}}} D_{\text{APr}}$ is a sort of T^R . Now we want to show that it is in fact a universal Skolem sort. By Proposition 2.5, it suffices to show that for any sort F of T , D_{T^R} is universal Skolem for F^R . So, from now on, we will assume that a pseudo-open map $\pi : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$ exists and that F is a sort of T . We let z be a variable of sort F and $\hat{x} = (x, \bar{x}, \dot{x})$ be a variable of sort D_{T^R} . First we will prove that D_{T^R} is universal for F^R . We will denote by d_∞ the supremum metric.

Lemma 3.5. *For any given $\epsilon > 0$ there is $\epsilon' > 0$ such that for every $\hat{x} \in D_{T^R}$ and every $y \in D_T^R$, if $d_\infty(x, y) < \epsilon'$ then there is $(\bar{y}, \dot{y}) \in D_{\text{APr}}$ such that $\hat{y} = (y, \bar{y}, \dot{y}) \in D_{T^R}$ and $d(\hat{x}, \hat{y}) < \epsilon$.*

Proof. Fix $\epsilon > 0$, let $N \in \mathbb{N}$ verify $2^{-N} < \epsilon$, and take $k \in \mathbb{N}$ as in the definition of being pseudo-open. We know that there is $\delta > 0$ such that for every $x_0, x_1 \in D_T$, $d(x_0, x_1) < \delta$ implies $\text{tp}_i(x_0) = \text{tp}_i(x_1)$ for every $i < k$. Let $\epsilon' = \min\{\epsilon, \delta\}$, take any $\hat{x} \in D_T^R \times_{E^{\mathbb{N}}} D_{\text{APr}}$ and take $y \in D_T^R$ with $d_\infty(x, y) < \epsilon'$. So, $\text{tp}_i^R(y) = \text{tp}_i^R(x) = \pi_i(\bar{x}, \dot{x})$ for $i < k$. Since π is pseudo-open, there is some $(\bar{y}, \dot{y}) \in D_{\text{APr}}$ such that $d((\bar{x}, \dot{x}), (\bar{y}, \dot{y})) < \epsilon$ and $\pi(\bar{y}, \dot{y}) = \text{tp}^R(y)$, completing the proof. \square

Lemma 3.6. *Let F be a sort of T , then the sort D_{T^R} is universal for F^R .*

Proof. Fix $\epsilon > 0$, and take $0 < \epsilon' < \epsilon$ as in the previous lemma. Since D_T is universal for F , there exists a definable map $\sigma : D_T \rightarrow F$ such that the image of any $\epsilon'/2$ -ball in D_T is $\epsilon'/2$ -dense in F . Let $\sigma^R : D_T^R \rightarrow F^R$ be the corresponding map in the randomization. Define the map $\tau : D_{T^R} \rightarrow F^R$ by $\tau(\hat{x}) = \sigma^R(x)$. Pick any $\hat{x} \in D_{T^R}$ and $z \in F^R$. We have that $T^R \models \forall x \forall z \exists y [d(x, y) \vee d(z, \sigma^R(y))] \leq \epsilon'/2$. By [BY13, Lemma 3.13], there is some $y \in D_T^R$ satisfying $d_\infty(x, y) < \epsilon'$ and $d_\infty(\sigma^R(y), z) < \epsilon$. By the previous lemma, there is $(\bar{y}, \dot{y}) \in D_{\text{APr}}$ such that $\hat{y} = (y, \bar{y}, \dot{y}) \in D_{T^R}$ and $d(\hat{x}, \hat{y}) < \epsilon$. Thus, $d(\tau(\hat{y}), z) < \epsilon$, showing that the image via τ of $B(\hat{x}, \epsilon)$ is ϵ -dense. \square

The next lemma says that as a result of having defined D_{T^R} as a fiber product over $E^{\mathbb{N}}$ with the type function, the map $\llbracket \varphi(x) \rrbracket$ can be obtained in a definable manner from (\bar{x}, \dot{x}) .

Lemma 3.7. *Suppose that $\varphi(x)$ is an \mathcal{L} -formula in D_T . Then there is a definable map $\sigma : D_{\text{APr}} \rightarrow E_{RV}$ such that for every $\hat{x} \in D_{T^R}$, $\llbracket \varphi(x) \rrbracket = \sigma(\bar{x}, \dot{x})$.*

Proof. We know that there exists a continuous function $f : 2^{\mathbb{N}} \rightarrow [0, 1]$ such that $\varphi(x) = f(\text{tp}(x))$ for every $x \in D_T$. Identifying $(2^{\mathbb{N}})^R$ with $E^{\mathbb{N}}$, the map $f^R : E^{\mathbb{N}} \rightarrow E_{RV}$ is definable. Since for every $x \in D_T^R$, $\llbracket \varphi(x) \rrbracket = f^R(\text{tp}^R(x))$, we conclude that for every $\hat{x} \in D_{T^R}$, $\llbracket \varphi(x) \rrbracket = f^R(\pi(\bar{x}, \dot{x}))$, which is a definable map on $(\bar{x}, \dot{x}) \in D_{\text{APr}}$. \square

Assume that T is a classical theory. Suppose that for $i < m$, $\sigma_i : D_T \rightarrow F$ and $\eta_i : D_{\text{APr}} \rightarrow E$ are definable. Furthermore, assume that for every $(\bar{x}, \dot{x}) \in D_{\text{APr}}$, $\{\eta_0(\bar{x}, \dot{x}), \dots, \eta_{m-1}(\bar{x}, \dot{x})\}$ is a partition of the probability space. Then, the map $\rho : D_{T^R} \rightarrow F^R$ given by

$$\rho(\hat{x}) = \langle \eta_0(\bar{x}, \dot{x}), \dots, \eta_{m-1}(\bar{x}, \dot{x}), \sigma_0^R(x), \dots, \sigma_{m-1}^R(x) \rangle$$

is definable, as

$$d(\rho(\hat{x}), y) = \sum_{i < m} \mu(\llbracket y \neq \sigma_i^R(x) \rrbracket \cap \eta_i(\bar{x}, \dot{x})).$$

Lemma 3.8. *Suppose that T is a classical theory and that F is a sort of T . Then, D_{T^R} is a Skolem sort for F^R .*

Proof. Fix $\varphi(\hat{x}, y)$ a formula in $D_{T^R} \times F^R$ with $\inf_y \varphi(\hat{x}, y) = 0$ and let $\epsilon > 0$. We can assume that there is a \mathcal{L}_{Pr} -formula $\psi(z, \bar{x}, \dot{x})$ in $E^m \times D_{\text{APr}}$ such that

$$\varphi(\hat{x}, y) = \psi(\llbracket \varphi_0(x, y) \rrbracket, \dots, \llbracket \varphi_{m-1}(x, y) \rrbracket, \bar{x}, \dot{x})$$

where $\varphi_i(x, y)$ is a \mathcal{L} -formula in $D_T \times F$ for $i < m$. Take $\delta < \epsilon$ such that for every $A, B \in E^m$ and every $(\bar{x}, \dot{x}) \in D_{\text{APr}}$, $d(A, B) < \delta$ implies $|\psi(A, \bar{x}, \dot{x}) - \psi(B, \bar{x}, \dot{x})| < \epsilon/2$. Since the sort D_T is Skolem for F , for each $\nu \in 2^m$ there exists a definable map $\sigma_\nu : D_T \rightarrow F$ such that $\llbracket \exists u \varphi_\nu(x, u) \rrbracket = \llbracket \varphi_\nu(x, \sigma_\nu(x)) \rrbracket$. In addition, by [Lemma 3.7](#), for $\nu \in 2^m$ there are $\eta_\nu, \lambda_\nu : D_{\text{APr}} \rightarrow E$ such that for every $\hat{x} \in D_{T^R}$, $\llbracket \exists u \varphi_\nu(x, u) \rrbracket = \eta_\nu(\bar{x}, \dot{x})$ and $\llbracket \forall u \varphi_\nu(x, u) \rrbracket = \lambda_\nu(\bar{x}, \dot{x})$. We define the \mathcal{L}_{Pr} -formula

$$\chi(\bar{x}, \dot{x}, z) = \psi(z, \bar{x}, \dot{x}) \vee \left(\sum_{\nu \in 2^m} \mu(z_\nu \setminus \eta_\nu(\bar{x}, \dot{x})) \vee \mu(\lambda_\nu(\bar{x}, \dot{x}) \setminus z_\nu) \right).$$

Note that the condition $\chi(\bar{x}, \dot{x}, z) = 0$ essentially says that $\psi(z, \bar{x}, \dot{x}) = 0$ and z is a tuple of the form $(\llbracket \varphi_i(x, y) \rrbracket)_{i < m}$. It follows that, $\inf_z \chi(\bar{x}, \dot{x}, z) = 0$. Therefore, there is a definable map $\xi : D_{\text{APr}} \rightarrow E^m$ such that $\chi(\bar{x}, \dot{x}, \xi(\bar{x}, \dot{x})) < \delta/2^m$. Let $\rho : D_{T^R} \rightarrow F^R$ be the definable application given by

$$\rho(\hat{x}) = \langle (\xi_\nu(\bar{x}, \dot{x}) ; \nu \in 2^m), (\sigma_\nu^R(x) ; \nu \in 2^m) \rangle.$$

We will show that $\varphi(\hat{x}, \rho(\hat{x})) < \epsilon$. First, note that for $i < m$

$$\begin{aligned} d(\llbracket \varphi_i(x, \rho(\hat{x})) \rrbracket, \xi_i(\bar{x}, \dot{x})) &= d(\llbracket \xi_\nu(\bar{x}, \dot{x}) \cap \llbracket \exists u \varphi_\nu(x, u) \rrbracket ; \nu \in 2^m, \nu(i) = 0 \rrbracket, \xi_i(\bar{x}, \dot{x})) \\ &= \sum_{\nu(i)=0} \mu(\xi_\nu(\bar{x}, \dot{x}) \setminus \llbracket \exists u \varphi_\nu(x, u) \rrbracket) \\ &< \sum_{\nu(i)=0} \delta/2^m \\ &< \delta. \end{aligned}$$

Thus,

$$\begin{aligned} \phi(\hat{x}, \rho(\hat{x})) &= \psi(\llbracket \varphi_0(x, \rho(\hat{x})) \rrbracket, \dots, \llbracket \varphi_{m-1}(x, \rho(\hat{x})) \rrbracket, \bar{x}, \dot{x}) \\ &< \psi(\xi(\bar{x}, \dot{x}), \bar{x}, \dot{x}) + \epsilon/2 \\ &\leq \chi(\bar{x}, \dot{x}, \xi(\bar{x}, \dot{x})) + \epsilon/2 \\ &< \epsilon. \end{aligned} \quad \square$$

We will now study the continuous case, which will also cover the classical one. This means that we will no longer work with events but with random variables instead. This poses two main difficulties. Firstly, it is not so easy to say in terms of (\bar{x}, \dot{x}) that z is a tuple of the form $(\llbracket \varphi_i(x, y) \rrbracket)_{i < m}$, as we did in the formula $\chi(\bar{x}, \dot{x}, z)$. Secondly, the map ξ will give us a tuple of random variables that we must use to define the approximate Skolem map. In this case, instead of a partition of the probability space in measurable sets we do a partition of the constant map $\mathbf{1}$ in random variables.

Fix $n \in \mathbb{N}$, for $i < 2^n$ we will define continuous functions (in each case we truncate the functions at 1) $\lambda_{n,i} : [0, 1] \rightarrow [0, 1]$ by:

$$\begin{aligned}\lambda_{n,0}(x) &= 2^{2n}(1/2^n \dot{-} x), \\ \lambda_{n,i}(x) &= 2^{2n}[(x \dot{-} (i/2^n - 1/2^{2n})) \wedge ((i+1)/2^n \dot{-} x)] \text{ for } 0 < i < 2^n - 1, \\ \lambda_{n,2^n-1}(x) &= 2^{2n}(x \dot{-} ((2^n - 1)/2^n - 1/2^{2n})).\end{aligned}$$

These mappings give rise to definable functions $\lambda_{n,i} : E_{RV} \rightarrow E_{RV}$ defined by $\lambda_{n,i}(z) = \lambda_{n,i} \circ z$. Since for every $x \in [0, 1]$, $\sum_{i < 2^n} \lambda_{n,i}(x) = 1$, then for every $z \in E_{RV}$, $\{\lambda_{n,i}(z) ; i < 2^n\}$ is a partition of the constant map **1**. We also remark that $\lambda_{n,i}(z) = 1$ exactly when $i/2^n \leq z \leq (i+1)/2^n - 1/2^{2n}$ and, $\lambda_{n,i}(z) = 0$ when $z \leq i/2^n - 1/2^{2n}$ or $(i+1)/2^n \leq z$. Now, for $m \geq 1$ and $\bar{i} \in (2^n)^m$, we define $\lambda_{n,\bar{i}}^m : [0, 1]^m \rightarrow [0, 1]$ by $\lambda_{n,\bar{i}}^m(u) = \prod_{k < m} \lambda_{n,i_k}(u_k)$.

The following two lemmas solve the first problem mentioned above.

Lemma 3.9. *Let F be a sort of T and let $\varphi(x, y)$ be a formula in $D_T \times F$. For $x \in D_T^R$, let $A(x) = \{\llbracket \varphi(x, y) \rrbracket ; y \in F^R\}$. Then there is a definable predicate $P : E_{RV} \times D_{APr} \rightarrow [0, 1]$ such that for every $\hat{x} \in D_{TR}$ and every $z \in E_{RV}$, $d(z, A(x)) = P(z, \bar{x}, \hat{x})$.*

Proof. Take $n \in \mathbb{N}$, for $i < 2^n$ let $\varphi_{n,i}(x, y) = |i/2^n - \varphi(x, y)|$. Now, let us define

$$\tau_n(z, x) = \sum_{i < 2^n} \lambda_{n,i}(z) \llbracket \inf_y \varphi_{n,i}(x, y) \rrbracket, \quad P_n(z, x) = \mathbb{E}[\tau_n(z, x)].$$

We have that $d(z, A(x)) = \inf_y d(z, \llbracket \varphi(x, y) \rrbracket) = \inf_y \mathbb{E}[|z - \llbracket \varphi(x, y) \rrbracket|] = \mathbb{E}[\inf_y |z - \llbracket \varphi(x, y) \rrbracket|]$, and let us denote by $\tau(z, x)$ this last function. Note that $|\tau(z, x) - \llbracket \inf_y \varphi_{n,i}(x, y) \rrbracket| \leq 1/2^n$ on $\llbracket i/2^n \leq z \leq (i+1)/2^n \rrbracket$. Thus, on this set,

$$\begin{aligned}|\tau(z, x) - \tau_n(z, x)| &= |\tau(z, x) - \lambda_{n,i}(z) \llbracket \inf_y \varphi_{n,i}(x, y) \rrbracket - \lambda_{n,i+1}(z) \llbracket \inf_y \varphi_{n,i+1}(x, y) \rrbracket| \\ &\leq \left| \llbracket \inf_y \varphi_{n,i}(x, y) \rrbracket - \lambda_{n,i}(z) \llbracket \inf_y \varphi_{n,i}(x, y) \rrbracket - \lambda_{n,i+1}(z) \llbracket \inf_y \varphi_{n,i+1}(x, y) \rrbracket \right| + 1/2^n \\ &\leq \left| \llbracket \inf_y \varphi_{n,i}(x, y) \rrbracket - \llbracket \inf_y \varphi_{n,i+1}(x, y) \rrbracket \right| + 1/2^n \\ &\leq 1/2^{n-1}.\end{aligned}$$

Hence, $d(\tau(z, x), \tau_n(z, x)) \leq 1/2^{n-1}$. In this way, $d(z, A(x)) = \lim_{n \rightarrow \infty} P_n(z, x)$. By **Lemma 3.7**, for each $n \in \mathbb{N}$ and each $i < 2^n$, there is a definable map $\sigma_{n,i} : D_{APr} \rightarrow E_{RV}$ such that for every $\hat{x} \in D_{TR}$, $\llbracket \inf_y \varphi_{n,i}(x, y) \rrbracket = \sigma_{n,i}(\bar{x}, \hat{x})$. Therefore, if $P(z, \bar{x}, \hat{x}) = \lim_{n \rightarrow \infty} \mathbb{E} \left[\sum_{i < 2^n} \lambda_{n,i}(z) \sigma_{n,i}(\bar{x}, \hat{x}) \right]$ then $d(z, A(x)) = P(z, \bar{x}, \hat{x})$. \square

If we generalize the proof of the previous lemma and use the random variables $\{\lambda_{n,\bar{i}}^m ; \bar{i} \in (2^n)^m\}$ in order to consider multiple formulas simultaneously we get the following result.

Lemma 3.10. *Let F be a sort of T and let $\varphi_0(x, y), \dots, \varphi_{m-1}(x, y)$ be formulas in $D_T \times F$. For $x \in D_T^R$, let*

$$A(x) = \{(\llbracket \varphi_0(x, y) \rrbracket, \dots, \llbracket \varphi_{m-1}(x, y) \rrbracket) ; y \in F^R\}.$$

Then, there is a definable predicate $P : E_{RV}^m \times D_{APr} \rightarrow [0, 1]$ such that for every $\hat{x} \in D_{TR}$ and every $z \in E_{RV}^m$, $d(z, A(x)) = P(z, \bar{x}, \hat{x})$.

As we will use the maps $\{\lambda_{n,i} ; i < 2^n\}$ to define our approximate Skolem functions, we want to ensure that our definable map $\xi : D_{APr} \rightarrow E_{RV}$ gives us witnesses where the overlap between $\lambda_{n,i}(\xi(z))$ and $\lambda_{n,i+1}(\xi(z))$ occurs on a set of small measure. The next technical lemmas will guarantee precisely that.

Lemma 3.11. *Let $a_0 < b_0 < \dots < a_{n-1} < b_{n-1}$ be real numbers in $[0, 1]$ and let $\delta = \max_{i < n} \{b_i - a_i\}$. Then, for any $r > 0$, the set $\{z \in E_{RV} ; \sum_{i < n} \mu[a_i < z < b_i] \leq r\}$ is δ -dense.*

Proof. Fix $r > 0$ and take $z \in E_{RV}$. For each $i < n$, pick $c_i, d_i \in \mathbb{R}$ such that $a_i < c_i < d_i < b_i$ and $\mu[c_i < z < d_i] \leq r/n$. Also, let g_i be the linear function passing through (c_i, a_i) and (d_i, b_i) . Define the function $f : [0, 1] \rightarrow [0, 1]$ by (we let $b_{-1} = 0$)

$$f(x) = \begin{cases} \min\{x, a_i\} & \text{if } b_{i-1} \leq x < c_i, \ i < n \\ g_i(x) & \text{if } c_i \leq x < d_i, \ i < n \\ b_i & \text{if } d_i \leq x < b_i, \ i < n \\ x & \text{if } x \geq b_{n-1}. \end{cases}$$

We have that for every $x \in [0, 1]$, $|f(x) - x| < \delta$. Thus, if $z_0 = f \circ z$ then $d(z, z_0) < \delta$. Furthermore, for every $i < n$, $\mu[a_i < z_0 < b_i] = \mu[c_i < z < d_i] \leq r/n$. Hence, $\sum_{i < n} \mu[a_i < z_0 < b_i] \leq r$. \square

Take $z, z_0 \in E_{RV}$, from [BY13, Lemma 2.15], we know that $\text{tp}(z) = \text{tp}(z_0)$ if and only if they have the same distribution as random variables. Hence, for any $a, b \in [0, 1]$ with $a < b$, the real-valued map that to each type $p \in S_{E_{RV}}(\text{ARV})$ assigns $\mu[a < z < b]$, where z is any realization of p , is well-defined and we will simply denote it by $\mu[a < p < b]$.

Lemma 3.12. *For any $a, b \in [0, 1]$ with $a < b$, the real-valued map defined on $S_{E_{RV}}(\text{ARV})$ by $p \mapsto \mu[a < p < b]$ is lower semi-continuous.*

Proof. Take $r \in \mathbb{R}$ and define $F = \{p \in S_{E_{RV}}(\text{ARV}); \mu[a < p < b] \leq r\}$, we must prove that F is closed. Suppose $q \in S_{E_{RV}}(\text{ARV}) \setminus F$, so $\mu[a < q < b] > r$. Then, there is $\delta > 0$ such that $r_0 = \mu[a + \delta \leq q \leq b - \delta] > r$. Take $\epsilon = \delta(r_0 - r)$, we will show that $B(q, \epsilon) \cap F = \emptyset$. Let $p \in F$ and pick any $z \models q$ and $z_0 \models p$. If $A = \llbracket a + \delta \leq z \leq b - \delta \rrbracket \setminus \llbracket a < z_0 < b \rrbracket$ then $\mu(A) \geq r_0 - r$ and for $\omega \in A$, $|z(\omega) - z_0(\omega)| \geq \delta$. Hence, $d(z, z_0) \geq \int_A |z - z_0| \geq \epsilon$. Since z and z_0 are arbitrary realizations of q and p , we conclude that $d(q, p) \geq \epsilon$. \square

Corollary 3.13. *Let $a_0 < b_0 < \dots < a_{n-1} < b_{n-1}$ and r be real numbers in $[0, 1]$. The set*

$$\mathcal{C} = \{z \in E_{RV}; \sum_{i < n} \mu[a_i < z < b_i] \leq r\}$$

is definable.

Proof. Since ARV is ω -categorical, it suffices to show that $\{p \in S_{E_{RV}}(\text{ARV}); \sum_{i < n} \mu[a_i < p < b_i] \leq r\}$ is type-definable, i.e. closed. This follows immediately from the previous lemma and the fact that the sum of lower semi-continuous functions is again lower semi-continuous. \square

In order to proof that D_{TR} is a Skolem sort we will find approximations of the desired map. Towards this end we will use the notion of a Katětov function.

Definition 3.14. Let (X, d) be a metric space. A map $\psi : X \rightarrow [0, +\infty]$ is said to be *Katětov* if for all $x, y \in X$, $\psi(x) \leq d(x, y) + \psi(y)$ and $d(x, y) \leq \psi(x) + \psi(y)$.

Lemma 3.15. *Let F be a sort of T , then for every formula $\varphi(\hat{x}, y)$ in $D_{TR} \times F^R$ and every $\epsilon > 0$ there is a formula $\psi_0(\hat{x}, y)$ such that for every $\hat{x} \in D_{TR}$*

- $\psi_0(\hat{x}, y)$ is a Katětov function on y .
- There is $y \in F^R$ such that $\varphi(\hat{x}, y) + \psi_0(\hat{x}, y) < \inf_y \varphi(\hat{x}, y) + \epsilon$.

Proof. First, we remark that it is enough to show the lemma for a dense set of formulas. So, we can assume that there is an \mathcal{L}_{Pr} -formula $\psi(z, \bar{x}, \hat{x})$ in $(E_{RV})^n \times D_{APr}$ such that

$$\varphi(\hat{x}, y) = \psi(\llbracket \theta_1(x, y) \rrbracket, \dots, \llbracket \theta_m(x, y) \rrbracket, \bar{x}, \hat{x})$$

where $\theta_i(x, y)$ is an \mathcal{L} -formula in $D_T \times F$ with $\inf_y \theta_i(\hat{x}, y) = 0$, for $i \leq m$. For simplicity we show the case $m = 1$, i.e. $\varphi(\hat{x}, y) = \psi(\llbracket \theta(x, y) \rrbracket, \bar{x}, \hat{x})$, but the proof generalizes using Lemma 3.10 and the functions $\{\lambda_{n, \bar{i}}^m; \bar{i} \in (2^n)^m\}$. Pick $\delta_1, \delta_0 < \epsilon$ such that for every $z_0, z_1 \in E_{RV}$, $d(z_0, z_1) < \delta_0$ implies $|\psi(z_0, \bar{x}, \hat{x}) - \psi(z_1, \bar{x}, \hat{x})| < \epsilon/3$ and $d(z_0, z_1) < \delta_1$ implies $|\psi(z_0, \bar{x}, \hat{x}) - \psi(z_1, \bar{x}, \hat{x})| < \delta_0/5$. Also, let $P(z, \bar{x}, \hat{x})$ be the predicate as per Lemma 3.9 for the formula $\theta(x, y)$. Now, take $n \in \mathbb{N}$ such that $2^{-n+2} < \min\{\delta_1, \delta_0/5\}$. For $i < 2^n$, define $\theta_i(x, y) = |i/2^n - \theta(x, y)|$. Since D_T is a Skolem sort for F , for each $i < 2^n$, there exist a definable function $\sigma_i : D_T \rightarrow F$ such that $\theta_i(x, \sigma_i(x)) \leq \inf_y \theta_i(x, y) + \delta_0/5$. Additionally, by Lemma 3.11 and Corollary 3.13, the set

$$\mathcal{C} = \left\{ z \in E_{RV}; \sum_{i < 2^n} \mu \left[\frac{i+1}{2^n} - \frac{1}{2^{2n}} < z < \frac{i+1}{2^n} \right] \leq \delta_0/5 \right\}$$

is a sort of T^R which is furthermore 2^{-2n} -dense in E_{RV} . Next, define $\chi(\bar{x}, \hat{x}, z) = \psi(z, \bar{x}, \hat{x}) \vee P(z, \bar{x}, \hat{x})$ as a formula in $D_{APr} \times \mathcal{C}$. We claim that $\inf_{z \in \mathcal{C}} \chi(\bar{x}, \hat{x}, z) \leq \delta_0/5$. Indeed, take $r > 0$ and let $y \in F^R$ be such that $\psi(\llbracket \theta(x, y) \rrbracket, \bar{x}, \hat{x}) < r$. Pick $z \in \mathcal{C}$ such that $d(\llbracket \theta(x, y) \rrbracket, z) < 2^{-2n}$. So, $P(z, \bar{x}, \hat{x}) < 2^{-2n}$ and

$\psi(z, \bar{x}, \hat{x}) < r + \delta_0/5$, implying that $\chi(\bar{x}, \hat{x}, z) < r + \delta_0/5$. Since r is arbitrary, the result follows. We also know that D_{APr} is a Skolem sort for \mathcal{C} , so there is a definable map $\xi : D_{\text{APr}} \rightarrow \mathcal{C}$ such that $\chi(\bar{x}, \hat{x}, \xi(\bar{x}, \hat{x})) < 2\delta_0/5$. Now, let us define

$$\psi_0(\hat{x}, y) = \mathbb{E} \left[\sum_{i < 2^n} \lambda_{n,i}(\xi(\bar{x}, \hat{x})) \llbracket d(\sigma_i(x), y) \rrbracket \right].$$

It is a routine verification that $\psi_0(\hat{x}, y)$ is Katětov in y . To prove the second part, fix $\hat{x} \in D_{TR}$ and take $y \in F^R$ that agrees with $\sigma_i(x)$ on $A_i = \llbracket i/2^n \leq \xi(\bar{x}, \hat{x}) \leq (i+1)/2^n - 1/2^{2^n} \rrbracket$ for $i < 2^n$. Given $i < 2^n$, let us denote $B_i = \llbracket (i+1)/2^n - 1/2^{2^n} < \xi(\bar{x}, \hat{x}) < (i+1)/2^n \rrbracket$. We have that,

$$\begin{aligned} \psi_0(\hat{x}, y) &= \sum_{i < 2^n} \left[\int_{B_i} (\lambda_{n,i}(\xi(\bar{x}, \hat{x})) \llbracket d(\sigma_i(x), y) \rrbracket + \lambda_{n,i+1}(\xi(\bar{x}, \hat{x})) \llbracket d(\sigma_{i+1}(x), y) \rrbracket) d\mu \right] \\ &\leq \sum_{i < 2^n} \mu(A_i) \\ &\leq \delta_0/5. \end{aligned}$$

We also remark that

$$\begin{aligned} d(\llbracket \theta(x, y) \rrbracket, \xi(\bar{x}, \hat{x})) &\leq \sum_{i < 2^n} \left[\int_{A_i} \llbracket \theta(x, \sigma_i(x)) \rrbracket - \xi(\bar{x}, \hat{x}) \mid d\mu + \int_{B_i} d\mu \right] \\ &\leq \sum_{i < 2^n} \left[\int_{A_i} \llbracket \theta_i(x, \sigma_i(x)) \rrbracket + |i/2^n - \xi(\bar{x}, \hat{x})| d\mu + \mu(B_i) \right] \\ &\leq \sum_{i < 2^n} \left[\int_{A_i} \llbracket \inf_y \theta_i(x, y) \rrbracket \right] + \delta_0/5 + 1/2^n + \delta_0/5 \\ &\leq \mathbb{E} \left[\sum_{i < 2^n} \lambda_{n,i}(\xi(\bar{x}, \hat{x})) \llbracket \inf_y \theta_i(x, y) \rrbracket \right] + 1/2^n + 2\delta_0/5 \\ &< P(\xi(\bar{x}, \hat{x}), \bar{x}, \hat{x}) + 1/2^{n-2} + 2\delta_0/5 \\ &< \delta_0. \end{aligned}$$

Therefore, $\psi(\llbracket \theta(x, y) \rrbracket, \bar{x}, \hat{x}) < \psi(\xi(\bar{x}, \hat{x}), \bar{x}, \hat{x}) + \epsilon/3 \leq 2\delta_0/5 + \epsilon/3$. Thus, $\varphi(\hat{x}, y) + \psi_0(\hat{x}, y) < 3\delta_0/5 + \epsilon/3 < \epsilon$. \square

Theorem 3.16. *Suppose that T is a continuous theory that admits a Skolem sort, D_T , and that F is a sort of T . Then D_{TR} is a Skolem sort for F^R .*

Proof. Let $\varphi(\hat{x}, y)$ be a formula in $D_{TR} \times F^R$ such that $\inf_y \varphi(\hat{x}, y) = 0$ and let $\epsilon > 0$. Take $0 < \delta < \epsilon$ such that $d(y, y') \leq \delta$ implies $|\varphi(\hat{x}, y) - \varphi(\hat{x}, y')| \leq \epsilon/2$. Now define $\delta_n = 2^{-n-3}\delta$. By the previous lemma, there is a formula $\varphi_0(x, y)$, which is Katětov in y , and for every $\hat{x} \in D_{TR}$ there is $y_0 \in F^R$ such that $\varphi(\hat{x}, y_0) + \varphi_0(\hat{x}, y_0) < \delta_0$. Iterating this procedure, we obtain a sequence of formulas $(\varphi_n(\hat{x}, y))_{n \in \mathbb{N}}$, which is Katětov in y , and for each $\hat{x} \in D_{TR}$, a sequence $(y_n)_{n \in \mathbb{N}}$ such that

$$\varphi_n(\hat{x}, y_{n+1}) + \varphi_{n+1}(\hat{x}, y_{n+1}) < \inf_y \varphi_n(\hat{x}, y) + \delta_{n+1} < \delta_n + \delta_{n+1}.$$

Fix $\hat{x} \in D_{TR}$, then $d(y_n, y_{n+1}) \leq \varphi_n(\hat{x}, y_n) + \varphi_n(\hat{x}, y_{n+1}) < 2\delta_n + \delta_{n+1}$. Thus, $(y_n)_{n \in \mathbb{N}}$ is a convergent sequence, say to $\sigma(\hat{x}) \in F^R$. Note that $\varphi_n(\hat{x}, \sigma(\hat{x})) \leq \varphi_n(\hat{x}, y_n) + d(y_n, \sigma(\hat{x})) < \delta_n + d(y_n, \sigma(\hat{x}))$. So, $(\varphi_n(\hat{x}, \sigma(\hat{x})))_{n \in \mathbb{N}}$ converges to 0. Hence, for any $y \in F^R$, we have that $|d(y, \sigma(\hat{x})) - \varphi_n(\hat{x}, y)| \leq \varphi_n(\hat{x}, \sigma(\hat{x}))$ converges to 0. Thus, $(\varphi_n(\hat{x}, y))_{n \in \mathbb{N}}$ converges uniformly to $d(y, \sigma(\hat{x}))$, and the map $\sigma : D_{TR} \rightarrow F^R$ is definable. Since $d(y_0, \sigma(\hat{x})) \leq \delta$, we obtain that

$$\varphi(\hat{x}, \sigma(\hat{x})) \leq \varphi(\hat{x}, y_0) + \epsilon/2 < \delta_0 + \epsilon/2 < \epsilon.$$

Therefore, σ is an ϵ -Skolem map for φ . \square

The previous theorem together with [Lemma 3.6](#) give us the following result.

Corollary 3.17. *Let T be a complete continuous theory admitting a universal Skolem sort, D_T . Then $D_{TR} = D_T^R \times_{E^{\mathbb{N}}} D_{\text{APr}}$ is a universal Skolem sort of T^R .*

Suppose now that T is an ω -categorical theory and let \mathcal{M} be a separable model of T . We know from [Fact 2.7](#) that $D_T = D_0 \times 2^{\mathbb{N}}$, where $D_0 = \text{tp}(a)$ is the type of a dense enumeration of M , is a universal Skolem sort for T . Let $c_a \in D_0^R(M)$ be the map taking constant value a and take $\Xi \in D_{\text{APr},0}(\Omega)$ as before. It is easy to see, using for example [Lemma 5.6](#) below, that $\text{dcl}(c_a, \Xi) = \mathcal{M}^R$. Thus, by [Lemma 2.8](#), $D_0^R \times D_{\text{APr},0} \times 2^{\mathbb{N}} = D_0^R \times D_{\text{APr}}$ is a universal Skolem sort for T^R . On the other hand, $D_{T^R} = (D_0 \times 2^{\mathbb{N}})^R \times_{E^{\mathbb{N}}} D_{\text{APr}}$ is also a universal Skolem sort for T^R . Since $(D_0 \times 2^{\mathbb{N}})^R = D_0^R \times E^{\mathbb{N}}$ and $\text{tp} : D_0 \times 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$ coincides with the natural projection, we have that

$$D_{T^R} = (D_0^R \times E^{\mathbb{N}}) \times_{E^{\mathbb{N}}} D_{\text{APr}} \cong D_0^R \times D_{\text{APr}}.$$

Thus, modulo a definable bijection, the sort D_T^R coincides with the sort we knew beforehand for ω -categorical structures. We finish this section by exhibiting a pseudo-open map $\pi^0 : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$.

For $i, j \in \mathbb{N}$ with $i \leq j$ let $n_{i,j} = \frac{j(j+1)}{2} + i$. So, $n_{\cdot,\cdot}$ is a bijection between $\{(i, j) \in \mathbb{N}^2; i \leq j\}$ and \mathbb{N} . π^0 will be defined recursively, and for $i \in \mathbb{N}$, the steps of the form $n_{i,j}$ will be used to define π_i . Now, for $\nu \in \{0, 1\}^{\mathbb{N}}$ let $\bar{\nu} = \sum_{i=0}^{\infty} \nu_i 2^i$. The integers $\bar{\nu}$ will be used to go through the indices of the elements in the sort $2^{\mathbb{N}}$.

Definition 3.18. We define the map $\pi^0 : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$ in a recursive manner. Fix $(\bar{a}, \alpha) \in D_{\text{APr}}$, let $\mathcal{A}_0 = \{\perp, \top\}$, and for $n > 0$ let \mathcal{A}_n be the algebra generated by $\{a_0, \dots, a_{n-1}\}$. Suppose we have already carried out our construction up to the step $n-1$ and we have already used $\alpha_0, \dots, \alpha_{2^{n-2}}$. We know that there exist $i, j \in \mathbb{N}$ with $i \leq j$ such that $n = n_{i,j}$. If the construction, up to the step $n-1$, neither implies $\pi_i(\bar{a}, \alpha) \cap a_\nu = \emptyset$ nor $a_\nu \subseteq \pi_i(\bar{a}, \alpha)$, then:

- in case $\alpha_{2^{n-1+\bar{\nu}}} = 0$, we impose the condition $\pi_i(\bar{a}, \alpha) \cap a_\nu \subseteq a_\nu \cap a_n$,
- in case $\alpha_{2^{n-1+\bar{\nu}}} = 1$, we impose the condition $a_\nu \cap a_n \subseteq \pi_i(\bar{a}, \alpha) \cap a_\nu$.

Since the sequence \bar{a} is independent, the measure of the atoms tend to zero. Thus, the previous conditions define a unique sequence $A \in E^{\mathbb{N}}$. Furthermore, each element in the sequence is a uniform limit of \mathcal{L}_{APr} -terms in (\bar{a}, α) . It follows that the map π^0 is definable.

Lemma 3.19. *The map $\pi^0 : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$ is weakly open.*

Proof. Given $\epsilon > 0$ and $N \in \mathbb{N}$, take $k \in \mathbb{N}$ such that $2^{-n_{k,k}} < \epsilon$ and $2^{-n_{k,k}} - 1 \geq N$. We will work inside a fixed separable $\mathcal{A} \models \text{APr}$. Suppose that $(\bar{a}, \alpha) \in D_{\text{APr}}$ and $A \in E^{\mathbb{N}}$ satisfy $\pi_n^0(\bar{a}, \alpha) = A_n$ for every $n < k$. For $i < n_{k,k}$ take $b_i = a_i$ and for $l < 2^{n_{k,k}} - 1$ let $\beta_l = \alpha_l$. So, independent of the choice of the remaining b_i and β_l , $d(\bar{a}, \bar{b}) < \epsilon$ and $\alpha \upharpoonright_N = \beta \upharpoonright_N$. Take $n \geq n_{k,k}$ and suppose that we have already defined b_0, \dots, b_{n-1} and $\beta_0, \dots, \beta_{2^{n-2}}$. Assume that $n = n_{i,j}$ for some $i, j \in \mathbb{N}$. For $\nu \in \{0, 1\}^{\mathbb{N}}$, if $\mu(b_\nu \cap A_i) \leq \mu(b_\nu)/2$ put $\beta_{2^{n-1+\bar{\nu}}} = 0$ and take $D_\nu \in \mathcal{A}$ with $b_\nu \cap A_i \subseteq D_\nu \subseteq b_\nu$ and $\mu(D_\nu) = \mu(b_\nu)/2$. In case $\mu(b_\nu \cap A_i) > \mu(b_\nu)/2$, put $\beta_{2^{n-1+\bar{\nu}}} = 1$ and pick $D_\nu \in \mathcal{A}$ with $D_\nu \subseteq b_\nu \cap A_i$ and $\mu(D_\nu) = \mu(b_\nu)/2$. Finally, define $b_n = \bigcup_{\nu \in \{0,1\}^{\mathbb{N}}} D_\nu$.

By the previous process we obtain an element $(\bar{b}, \beta) \in D_{\text{APr}}$. By construction, for every $i \in \mathbb{N}$, every $j \geq i$ and every $\nu \in 2^{n_{i,j}}$, $\mu(b_\nu \cap A_i) \leq \mu(b_\nu)/2$ if and only if $\mu(b_\nu \cap \pi_i^0(\bar{b}, \beta)) \leq \mu(b_\nu)/2$. Hence, $\pi^0(\bar{b}, \beta) = A$. \square

[Lemma 3.3](#) yields the following result.

Corollary 3.20. *The map $\pi : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$ defined by $\pi_n(\bar{x}, \dot{x}) = \pi_{2^n}^0(\bar{x}, \dot{x})$ for $n \in \mathbb{N}$ and $(\bar{x}, \dot{x}) \in D_{\text{APr}}$ is pseudo-open.*

4. RANDOM FUNCTIONS ON GROUPOIDS

In this section we will define the groupoid $\mathbf{G}^\Omega \rtimes \text{Aut}(\Omega)$ for \mathbf{G} a Polish groupoid. We will show that several properties of \mathbf{G} are preserved when passing to $\mathbf{G}^\Omega \rtimes \text{Aut}(\Omega)$ such as being open or minimal. Finally, from the map $\pi : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$ we will define a base change of $\mathbf{G}^\Omega \rtimes \text{Aut}(\Omega)$ to $2^{\mathbb{N}}$. In [Section 5](#), we will see how this construction relates to the groupoid associated to the randomization. We begin this section with some results on the space X^Ω for a Polish space X .

4.1. The topology on the space of random functions.

Let X be a Polish space and let d be a compatible metric on X bounded by 1. Recall that $(\Omega, \mathcal{B}, \mu)$ denotes the Stone space of the usual measure algebra of $[0, 1]$. The space $X^\Omega = L^0((\Omega, \mathcal{B}, \mu), X)$ admits a natural metric, the integration of the metric on X . Namely, for $f, g \in X^\Omega$ let

$$d^\Omega(f, g) = \int d(f(\omega), g(\omega)) d\mu(\omega).$$

On the other hand, convergence in measure induces a topology on X^Ω which is moreover metrizable.

Fact 4.1. [Fan44] *Let X be a Polish space and let d be a compatible metric on X bounded by 1. Then, the topology of convergence in measure on X^Ω is induced by the distance d_0 defined as*

$$d_0(f, g) = \inf\{\epsilon > 0 ; \mu[\|d(f, g) \geq \epsilon\|] < \epsilon\} \text{ for } f, g \in X^\Omega.$$

Given $A \subseteq \Omega$, $U \subseteq X$ and $\epsilon > 0$ let

$$\mathcal{O}(A, U, \epsilon) = \{f \in X^\Omega ; \mu(\{\omega \in A ; f(\omega) \notin U\}) < \epsilon\}.$$

We will denote by τ the topology on X^Ω having as a subbase the sets of the form $\mathcal{O}(A, U, \epsilon)$ where $A \in \mathcal{B}$, $U \subseteq X$ open and $\epsilon > 0$. For $A_0, \dots, A_{n-1} \subseteq \Omega$, $U_0, \dots, U_{n-1} \subseteq X$ and $\epsilon > 0$ we set

$$\mathcal{O}((A_i)_{i < n}, (U_i)_{i < n}, \epsilon) = \bigcap_{i < n} \mathcal{O}(A_i, U_i, \epsilon).$$

The next lemma, whose proof is an easy but tedious exercise, shows that by only considering the sets defined by partitions of Ω we obtain a basis of τ .

Lemma 4.2. *Let X be a Polish space, then the family of sets of the form $\mathcal{O}((A_i)_{i < n}, (U_i)_{i < n}, \epsilon)$ where $n \in \mathbb{N}$, A_0, \dots, A_{n-1} is a partition of Ω , U_0, \dots, U_{n-1} are open subsets of X and $\epsilon > 0$, is a base for the topology τ on X^Ω .*

Since every measurable set in Ω differs from a clopen set in a set of null measure, we may only consider clopen subsets of Ω . The three topologies defined thus far agree, as will be shown below. However, we find it convenient to exhibit a base of this topology that does not explicitly mention the metric on X .

Lemma 4.3. *Let X be a Polish space and let d be a compatible metric on X bounded by 1. Then the following topologies on X^Ω agree:*

- (1) τ .
- (2) The topology induced by d^Ω .
- (3) The topology of convergence in measure.

Moreover, a sequence $(f_n)_{n \in \mathbb{N}}$ in X^Ω converges to f if and only if for every subsequence of $(f_n)_{n \in \mathbb{N}}$ there is a further subsequence that converges to f almost everywhere. Finally, X^Ω , endowed with this topology, is a Polish space.

Proof. The equivalence between (2) and (3), and the moreover part correspond to [Moo76, Proposition 6]. It is a well known fact that (X^Ω, d^Ω) is a Polish space, see for example [Moo76, Proposition 7]. It remains to show the equivalence between τ and the other topologies.

3. \subseteq 1. Pick any $f \in X^\Omega$ and any $\epsilon > 0$. Let $\{W_i ; i \in \mathbb{N}\}$ be a countable base of the topology of X consisting of open sets of diameter less than ϵ . For $i \in \mathbb{N}$ we set $A_i = f^{-1}(W_i)$, then there is some $n \in \mathbb{N}$ such that $\mu(\bigcup_{i < n} A_i) \geq 1 - \epsilon/2$. We have that $f \in \mathcal{O}((A_i)_{i < n}, (W_i)_{i < n}, \epsilon/2n) \subseteq B_{d_0}(f, \epsilon)$.

1. \subseteq 3. Let $A \subseteq \Omega$ be measurable, $U \subseteq X$ be open and $\epsilon > 0$. Suppose that $f \in \mathcal{O}(A, U, \epsilon)$, let us denote $A_0 = \{\omega \in A ; f(\omega) \in U\}$ and take $r > 0$ satisfying $\mu(A \setminus A_0) + r < \epsilon$. By Lusin's theorem there is $A_1 \subseteq A_0$ compact such that $f \upharpoonright_{A_1}$ is continuous and $\mu(A_0 \setminus A_1) < r/2$. Since $f(A_1) \subseteq U$, by continuity, there is $\delta < r/2$ such that $B_d(f(A_1), \delta) \subseteq U$. It follows that $f \in B_{d_0}(f, \delta) \subseteq \mathcal{O}(A, U, \epsilon)$. \square

For $\tau \in \text{Aut}(\Omega)$ and $f \in X^\Omega$ define $(\tau \cdot f)(\omega) = f(\tau^{-1}(\omega))$ for $\omega \in \Omega$. It is easy to see that $\tau \cdot f \in X^\Omega$ and a straightforward verification shows that this operation defines a continuous action by d^Ω -isometries (as well as d_0 -isometries) of $\text{Aut}(\Omega)$ on X^Ω .

4.2. The groupoid of random functions.

Let \mathbf{G} be a Polish open topological groupoid of base \mathbf{B} . Our first goal is to show that \mathbf{G}^Ω is also a Polish open topological groupoid.

Proposition 4.4. *Let \mathbf{G} be a Polish topological groupoid. Then, \mathbf{G}^Ω , with the topology of convergence in measure and operations defined pointwise, is a Polish topological groupoid, .*

Proof. Take $g, h \in \mathbf{G}^\Omega$ such that $g(\omega)h(\omega)$ is defined for every $\omega \in \Omega$. Take $U \subseteq \mathbf{G}$ open and let $V \subseteq \mathbf{G}^2$ be the inverse image of U under the product map. So, V is the intersection of a closed and an open set, hence a measurable subset of \mathbf{G}^2 . On the other hand, the map $f : \Omega \rightarrow \mathbf{G}^2$ defined by $f(\omega) = (g(\omega), h(\omega))$ is measurable. Therefore, $(gh)^{-1}(U) = f^{-1}(V)$ is measurable and $gh \in \mathbf{G}^\Omega$, i.e. \mathbf{G}^Ω is closed under pointwise multiplication. Clearly, \mathbf{G}^Ω is also closed under pointwise inverses and we conclude that it is in fact a groupoid.

Take $g, h \in \mathbf{G}^\Omega$ such that gh is defined. Suppose moreover that we also have sequences $(g_n)_{n \in \mathbb{N}}, (h_n)_{n \in \mathbb{N}}$ converging to g and h , respectively such that $g_n h_n$ is defined for every $n \in \mathbb{N}$. Let $(n_k)_{k \in \mathbb{N}}$ be any subsequence, take $(m_i)_{i \in \mathbb{N}}$ a further subsequence such that $g_{m_i}(\omega)$ converges to $g(\omega)$, when $i \rightarrow \infty$, for almost every $\omega \in \Omega$. Then, there is an even further subsequence $(m_{i_j})_{j \in \mathbb{N}}$ such that $h_{m_{i_j}}(\omega)$ converges to $h(\omega)$, when $j \rightarrow \infty$, for almost every $\omega \in \Omega$. Therefore, $\lim_{j \rightarrow \infty} g_{m_{i_j}} h_{m_{i_j}}(\omega) = gh(\omega)$ for almost every $\omega \in \Omega$. It follows, that $(g_n h_n)_{n \in \mathbb{N}}$ converges to gh and that the product in \mathbf{G}^Ω is continuous. A similar argument shows that the inverse is also continuous. Hence, \mathbf{G}^Ω is a topological groupoid and, by [Lemma 4.3](#), it is Polish. \square

Note that the base of \mathbf{G}^Ω is \mathbf{B}^Ω . The source and target maps will be denoted $s^\Omega : \mathbf{G}^\Omega \rightarrow \mathbf{B}^\Omega$ and $t^\Omega : \mathbf{G}^\Omega \rightarrow \mathbf{B}^\Omega$ respectively, and they are given by $s^\Omega(g)(\omega) = s_{g(\omega)}$ and $t^\Omega(g)(\omega) = t_{g(\omega)}$ for $\omega \in \Omega$ and $g \in \mathbf{G}^\Omega$. To show that if \mathbf{G} is open then \mathbf{G}^Ω is also open, we need a few technical lemmas. We assume that every topological space is endowed with the Borel σ -algebra.

Theorem 4.5. [[Kec95](#), Theorem 15.1] *Let X, Y be Polish spaces and $f : X \rightarrow Y$ be continuous. If $A \subseteq X$ is Borel and $f \upharpoonright A$ is injective, then $f(A)$ is Borel.*

Theorem 4.6. [[Kec95](#), Theorem 12.16] *Let X be a Polish space and E an equivalence relation such that every equivalence class is closed. Suppose that either the saturation of any open set is Borel, or the saturation of any closed set is Borel. Then E admits a Borel transversal.*

Lemma 4.7. *Let X and Y be Polish spaces and let $\pi : X \rightarrow Y$ be a continuous, open, surjective map. Then there is $f : Y \rightarrow X$ measurable such that $\pi \circ f = \text{id}_Y$.*

Proof. Define the equivalence relation E on X by xEx' if and only if $\pi(x) = \pi(x')$. For $x \in X$, $[x]_E = \pi^{-1}(\pi(x))$, a closed set. For $U \subseteq X$ open, since π is open and continuous, its saturation $[U]_E = \pi^{-1}(\pi(U))$ is open. By [Theorem 4.6](#), there is $T \subseteq X$ Borel that meets every equivalence class in exactly one point. Now we define $f : Y \rightarrow X$, for $y \in Y$ we set $f(y)$ as the only element in $T \cap \pi^{-1}(\{y\})$. Take $U \subseteq X$ Borel, note that $f^{-1}(U) = \pi(U \cap T)$ and $\pi \upharpoonright (U \cap T)$ is injective. Using [Theorem 4.5](#), we obtain that $f^{-1}(U)$ is Borel. Therefore, f is measurable. \square

Corollary 4.8. *Let X, Y be Polish spaces and let Z be a topological space. Suppose that $\pi : X \rightarrow Y$ is a continuous, open, surjective map and that $f : Z \rightarrow Y$ is a measurable map. Then, there is a measurable application $g : Z \rightarrow X$ such that $f = \pi \circ g$.*

Proof. By the previous lemma, there is $h : Y \rightarrow X$ measurable with $\pi \circ h = \text{id}_Y$, then $g = h \circ f$ is the desired function. \square

Corollary 4.9. *Let X, Y be Polish spaces and let Z be a topological space. Suppose that $\pi : X \rightarrow Y$ is a continuous, open map and that $f : Z \rightarrow Y$ is a measurable map such that $f(Z) \subseteq \pi(U)$ for some U , open subset of X . Then, there is a measurable application $g : Z \rightarrow U$ such that $f = \pi \circ g$.*

Proof. Use the previous corollary replacing X with U and Y with $\pi(U)$. \square

We borrow the following notation from randomizations, for A_0, \dots, A_{n-1} a partition of Ω and $f_0, \dots, f_{n-1} \in X^\Omega$, $\langle (A_i)_{i < n}, (f_i)_{i < n} \rangle$ will denote the element in X^Ω that agrees with f_i on A_i for $i < n$.

Lemma 4.10. *Let \mathbf{G} be a Polish open topological groupoid. Let A_0, \dots, A_{n-1} be a partition of Ω and let U_0, \dots, U_{n-1} be open subsets of \mathbf{G} . Suppose that $e \in \mathcal{O}(\langle (A_i)_{i < n}, (t(U_i))_{i < n} \rangle)$, then there is $g \in \mathcal{O}(\langle (A_i)_{i < n}, (U_i)_{i < n} \rangle)$ such that $t^\Omega(g) = e$.*

Proof. Fix $i \leq n$, take $a_i \in t(U_i)$ and let $e_i = \langle A_i, \Omega \setminus A_i, e, a_i \rangle$. Then, $e_i : \Omega \rightarrow \mathbf{B}$ is a measurable map whose image is contained in $t(U_i)$. By the previous lemma, there is a measurable map $g_i : \Omega \rightarrow \mathbf{G}$ whose image is contained in U_i and such that $t^\Omega(g_i) = t \circ g_i = e_i$. Then $g = \langle (A_i)_{i < n}, (g_i)_{i < n} \rangle$ is the desired map. \square

Theorem 4.11. *Let \mathbf{G} be a Polish open topological groupoid, then \mathbf{G}^Ω is also a Polish open topological groupoid.*

Proof. By [Proposition 4.4](#), it remains to show that it is in fact an open groupoid. We claim that for any A_1, \dots, A_n partition of Ω , U_1, \dots, U_n open subsets of \mathbf{G} and $\epsilon > 0$, $t^\Omega(\mathcal{O}((A_i)_{i < n}, (U_i)_{i < n}, \epsilon)) = \mathcal{O}((A_i)_{i < n}, (t(U_i))_{i < n}, \epsilon)$. Pick $e = t^\Omega(g)$ with $g \in \mathcal{O}((A_i)_{i < n}, (U_i)_{i < n}, \epsilon)$. Notice that for $i < n$, if $\omega \in A_i$ and $g(\omega) \in U_i$ then $e(\omega) \in t(U_i)$. Thus, for $i < n$,

$$\mu(\{\omega \in A_i ; e(\omega) \notin t(U_i)\}) \leq \mu(\{\omega \in A_i ; g(\omega) \notin U_i\}) < \epsilon$$

and $e \in \mathcal{O}((A_i)_{i < n}, (t(U_i))_{i < n}, \epsilon)$. Now take $e \in \mathcal{O}((A_i)_{i < n}, (t(U_i))_{i < n}, \epsilon)$, for $i < n$ define $B_i = \{\omega \in A_i ; e(\omega) \notin t(U_i)\}$. We have that $e \in \mathcal{O}(A_1 \setminus B_1, B_1, \dots, A_n \setminus B_n, B_n, t(U_1), \mathbf{B}, \dots, t(U_n), \mathbf{B})$, so, by the previous lemma, there is $g \in \mathcal{O}(A_1 \setminus B_1, B_1, \dots, A_n \setminus B_n, B_n, U_1, \mathbf{G}, \dots, U_n, \mathbf{G})$ such that $t^\Omega(g) = e$. It is true that for $i < n$, $\{\omega \in \Omega ; g(\omega) \notin U_i\} \subseteq B_i$. Therefore, $g \in \mathcal{O}((A_i)_{i < n}, (U_i)_{i < n}, \epsilon)$, which completes the proof. \square

Let $\varphi : X \rightarrow Y$ be a continuous map between Polish spaces. Define the application $\varphi^\Omega : X^\Omega \rightarrow Y^\Omega$ by $\varphi^\Omega(x) = \varphi \circ x$ for $x \in X^\Omega$. This map is continuous since for any $A \subseteq \Omega$, V open subset of Y and $\epsilon > 0$

$$(\varphi^\Omega)^{-1}(\mathcal{O}(A, V, \epsilon)) = \mathcal{O}(A, \varphi^{-1}(V), \epsilon)$$

It follows that if (X, ρ) is a space over \mathbf{B} then (X^Ω, ρ^Ω) is a space over \mathbf{B}^Ω .

Lemma 4.12. *Let \mathbf{G} be a Polish groupoid and let (X, ρ) be a Polish \mathbf{G} -space over \mathbf{B} . Then, (X^Ω, ρ^Ω) is a Polish \mathbf{G}^Ω -space over \mathbf{B}^Ω .*

Proof. The action $\mathbf{G}^\Omega \curvearrowright X^\Omega$ is defined by $(g \cdot x)(\omega) = g(\omega) \cdot x(\omega)$ whenever $t^\Omega(g) = \rho^\Omega(x)$. The fact that $g \cdot x$ belongs to X^Ω and that the action is continuous is showed in the same way as in the case of Polish groups, see [\[Iba17, Definition 2.1\]](#) and the discussion thereafter. \square

Lemma 4.13. *Let \mathbf{G} be a Polish open groupoid acting minimally on (X, ρ) , space over \mathbf{B} . Let $V \subseteq X$ be open. Then, for any $x \in X^\Omega$, there is $g \in \mathbf{G}^\Omega$ such that $g \cdot x \in \mathcal{O}(\Omega, V)$.*

Proof. Let us fix $x \in X^\Omega$. By [Fact 2.13](#), the projection $\eta : \mathbf{G} \times_{\mathbf{B}} X \rightarrow X$ is open. Let W be the inverse image of V under the action law, so W is an open subset of $\mathbf{G} \times_{\mathbf{B}} X$. Since $\mathbf{G} \curvearrowright X$ is minimal, for every $\omega \in \Omega$ there is some $h \in \mathbf{G}$ such that $h \cdot x(\omega) \in V$. Thus, $(g, x(\omega)) \in W$, which implies $x(\omega) \in \eta(W)$. Since this is true for every $\omega \in \Omega$, we obtain that $x(\Omega) \subseteq \eta(W)$. Hence, by [Corollary 4.9](#), there is a measurable map $\tilde{g} : \Omega \rightarrow W$ such that $x = \eta \circ \tilde{g}$. Then, the projection of \tilde{g} onto the first coordinate is the desired map. \square

Corollary 4.14. *Suppose \mathbf{G} is a Polish open groupoid acting on (X, ρ) , Polish space over \mathbf{B} . Then, $\mathbf{G} \curvearrowright X$ is minimal if and only if $\mathbf{G}^\Omega \curvearrowright X^\Omega$ is minimal. In particular, \mathbf{G} is minimal if and only if \mathbf{G}^Ω is minimal.*

Proof. Assume first that $\mathbf{G}^\Omega \curvearrowright X^\Omega$ is minimal. Fix $x \in X$ and $V \subseteq X$ open. Let $c_x \in X^\Omega$ be the constant map of value x and consider the open set $\mathcal{O}(\Omega, V, 1/2)$. Using minimality, we get $g \in \mathbf{G}^\Omega$ such that $g \cdot c_x \in \mathcal{O}(\Omega, V, 1/2)$. Therefore, for some $\omega \in \Omega$, $g(\omega) \cdot x \in V$, which shows minimality. Suppose now that $\mathbf{G} \curvearrowright X$ is minimal. Let A_0, \dots, A_{n-1} be a partition of Ω , V_0, \dots, V_{n-1} be open subsets of X and $\epsilon > 0$. Fix $x \in X^\Omega$, by the previous lemma, for every $i < n$, there is $g_i \in \mathbf{G}^\Omega$ satisfying $g_i \cdot x \in \mathcal{O}(\Omega, V_i)$. Define $g = ((A_i)_{i < n}, (g_i)_{i < n})$, it follows that $g \cdot x \in \mathcal{O}((A_i)_{i < n}, (V_i)_{i < n}, \epsilon)$, finishing the proof. \square

We will now study the semidirect product between a groupoid and a group.

Definition 4.15. Let \mathbf{G} be a groupoid and Γ a group. An *action of Γ on \mathbf{G}* is a map $\cdot : \Gamma \times \mathbf{G} \rightarrow \mathbf{G}$ such that for every $\gamma, \delta \in \Gamma$ and $g, h \in \mathbf{G}$

$$1_\Gamma \cdot g = g, \quad \gamma \cdot (\delta \cdot g) = (\gamma\delta) \cdot g, \quad \gamma \cdot (gh) = (\gamma \cdot g)(\gamma \cdot h).$$

The last equality meaning that whenever gh is defined then $(\gamma \cdot g)(\gamma \cdot h)$ is also defined and both sides agree. Thus, for any $e \in \mathbf{B}$ and $\gamma \in \Gamma$, we must have $\gamma e \in \mathbf{B}$. From this, we deduce that for every $\gamma \in \Gamma$ and every $g \in \mathbf{G}$, $(\gamma \cdot g)^{-1} = \gamma \cdot g^{-1}$, $s_{\gamma \cdot g} = \gamma \cdot s_g$ and $t_{\gamma \cdot g} = \gamma \cdot t_g$.

Definition 4.16. Let Γ be a group acting on a groupoid \mathbf{G} . We define the *semidirect product $\mathbf{G} \rtimes \Gamma$* as the groupoid whose underlying set is $\mathbf{G} \times \Gamma$ and whose operations are given by

$$(g, \gamma)(h, \delta) = (g(\gamma \cdot h), \gamma\delta), \quad (g, \gamma)^{-1} = (\gamma^{-1} \cdot g^{-1}, \gamma^{-1}).$$

In this case the source and target maps are given by

$$t(g, \gamma) = (t_g, 1_\Gamma), \quad s(g, \gamma) = (\gamma^{-1} s_g, 1_\Gamma).$$

The base of $\mathbf{G} \rtimes \Gamma$ is $\mathbf{B} \times \{1_\Gamma\} \cong \mathbf{B}$.

Lemma 4.17. *Let \mathbf{G} be a topological groupoid and let Γ be a topological group acting continuously on \mathbf{G} . Then, $\mathbf{G} \rtimes \Gamma$, endowed with the product topology, is a topological groupoid. If \mathbf{G} is open then $\mathbf{G} \rtimes \Gamma$ is also open. Finally, if \mathbf{G} and Γ are Polish then so it is $\mathbf{G} \rtimes \Gamma$.*

Proof. Since the action and the inversion are continuous maps, the application $\eta : \mathbf{G} \times \Gamma \longrightarrow \mathbf{G}$ defined by $\eta(g, \gamma) = \gamma^{-1} \cdot g^{-1}$ is also continuous. Let $U \subseteq \mathbf{G}$ and $V \subseteq \Gamma$ be open, we claim that $(U \times V)^{-1} = \eta^{-1}(U) \cap (\mathbf{G} \times V^{-1})$. Indeed, $(g, \gamma) \in (U \times V)^{-1}$ if and only if $(\gamma^{-1} \cdot g^{-1}, \gamma^{-1}) \in U \times V$, which occurs exactly when $\eta(g, \gamma) \in U$ and $\gamma \in V^{-1}$, i.e. $(g, \gamma) \in \eta^{-1}(U) \cap (\mathbf{G} \times V^{-1})$.

To show that the groupoid operation is continuous, take $(g, \gamma), (h, \delta) \in \mathbf{G} \rtimes \Gamma$ whose product is defined and take $U \subseteq \mathbf{G}$, $V \subseteq \Gamma$ open sets with $(g(\gamma \cdot h), \gamma\delta) \in U \times V$. By the continuity of the group operation, there are open sets $V_0, V_1 \subseteq \Gamma$ with $\gamma \in V_0$, $\delta \in V_1$ and $V_0 V_1 \subseteq V$. Using the continuity of the groupoid operation, we find U_0 and U_1 open subsets of \mathbf{G} , with $g \in U_0$, $\gamma \cdot h \in U_1$ and $U_0 U_1 \subseteq U$. Finally, since the action is continuous, there are $U_2 \subseteq \mathbf{G}$ and $V_2 \subseteq V_0$ open sets with $h \in U_2$, $\gamma \in V_2$ and $V_2 \cdot U_2 \subseteq U_1$. In this way, $(g, \gamma) \in U_0 \times V_2$, $(h, \delta) \in U_2 \times V_1$ and $(U_0 \times V_2)(U_2 \times V_1) \subseteq U \times V$, showing that the groupoid operation is continuous.

Now assume that \mathbf{G} is open. Let $U \subseteq \mathbf{G}$ and $V \subseteq \Gamma$ be open, we have that $t(U \times V) = t(U)$, which is an open subset of \mathbf{B} . Finally, if \mathbf{G} and Γ are Polish, it is a well-known fact that their product is also a Polish space. \square

Suppose that (X, ρ) is a \mathbf{G} -space and that Γ is a group together with continuous actions $\Gamma \curvearrowright \mathbf{G}$ and $\Gamma \curvearrowright X$ such that $\gamma \cdot (g \cdot x) = (\gamma \cdot g) \cdot (\gamma \cdot x)$ for every $\gamma \in \Gamma$, $g \in \mathbf{G}$ and $x \in X$ for which $g \cdot x$ is defined. In particular, this implies that $\rho(\gamma \cdot x) = \gamma \cdot \rho(x)$. It is a straightforward verification that $(g, \gamma) \cdot x = g \cdot (\gamma \cdot x)$, whenever $s_g = \rho(\gamma \cdot x)$, defines a continuous action of $\mathbf{G} \rtimes \Gamma$ on X . Furthermore, if $\mathbf{G} \curvearrowright X$ is minimal then $(\mathbf{G} \rtimes \Gamma) \curvearrowright X$ is also minimal. Indeed, if $x \in X$ and $V \subseteq X$ is open, then there is $g \in \mathbf{G}$ such that $g \cdot x \in V$. It follows that $(g, 1_\Gamma) \cdot x \in V$. Note that the action of $\text{Aut}(\Omega)$ on X^Ω satisfies $\tau \cdot (g \cdot x) = (\tau \cdot g) \cdot (\tau \cdot x)$ for every $\tau \in \text{Aut}(\Omega)$, $g \in \mathbf{G}^\Omega$ and $x \in X^\Omega$ for which $g \cdot x$ is defined. This fact, together with [Corollary 4.14](#) and [Lemma 4.17](#), implies the following result.

Corollary 4.18. *If \mathbf{G} is a Polish open topological groupoid over \mathbf{B} then $\mathbf{G} \wr \Omega = \mathbf{G}^\Omega \rtimes \text{Aut}(\Omega)$ is a Polish open topological groupoid over \mathbf{B}^Ω . If furthermore \mathbf{G} is minimal then $\mathbf{G}^\Omega \rtimes \text{Aut}(\Omega)$ is minimal.*

We will now discuss the notion of base change.

Definition 4.19. Suppose \mathbf{G} is a groupoid of base \mathbf{B} , \mathbf{C} is a set and $\theta : \mathbf{C} \longrightarrow \mathbf{B}$ is any map. The *base change* of \mathbf{G} to \mathbf{C} is the groupoid whose underlying set is

$$\mathbf{G}_\theta = \{(b, g, a) \in \mathbf{C} \times \mathbf{G} \times \mathbf{C} ; \theta(a) = t_g \text{ and } \theta(b) = s_g\},$$

and whose operations are given by

$$(c, h, b)(b, g, a) = (c, hg, a), \quad (b, g, a)^{-1} = (a, g^{-1}, b).$$

Proposition 4.20. *Suppose \mathbf{G} is a topological groupoid of base \mathbf{B} , \mathbf{C} is a topological space and $\theta : \mathbf{C} \longrightarrow \mathbf{B}$ is a continuous map. Then \mathbf{G}_θ , endowed with the product topology, is a topological groupoid whose base is homeomorphic to \mathbf{C} via $c \longmapsto (c, \theta(c), c)$. If moreover \mathbf{G} and \mathbf{C} are Polish then \mathbf{G}_θ is also Polish*

Proof. The inverse map is continuous since for $U_0, U_1 \subseteq \mathbf{C}$ and $V \subseteq \mathbf{G}$ open sets,

$$((U_0 \times V \times U_1) \cap \mathbf{G}_\theta)^{-1} = (U_1 \times V^{-1} \times U_0) \cap \mathbf{G}_\theta.$$

To see that the product is continuous, pick $(c, h, b), (b, g, a) \in \mathbf{G}_\theta$ and suppose that $(c, hg, a) \in U_0 \times V \times U_1$ for some open sets $U_0, U_1 \subseteq \mathbf{C}$ and $V \subseteq \mathbf{G}$. Let W_0 and W_1 be open neighborhoods of h and g , respectively, satisfying $W_0 W_1 \subseteq V$. It follows that

$$((U_0 \times W_0 \times \mathbf{C}) \cap \mathbf{G}_\theta)((\mathbf{C} \times W_1 \times U_1) \cap \mathbf{G}_\theta) \subseteq U_0 \times V \times U_1.$$

By the definition of \mathbf{G}_θ , the neutral elements are those of the form $(c, \theta(c), c)$ for $c \in \mathbf{C}$. The map $c \longmapsto (c, \theta(c), c)$ is clearly a homeomorphism. The moreover part follows from the fact that \mathbf{G}_θ is a closed subspace of the product $\mathbf{C} \times \mathbf{G} \times \mathbf{C}$. \square

Having in mind that we want to build the groupoid associated to the randomization theory, we must obtain a groupoid whose base is the Cantor space. However, if \mathbf{G} denotes the groupoid of T , its base is $2^{\mathbb{N}}$ and thus the base of $\mathbf{G} \wr \Omega$ is $(2^{\mathbb{N}})^{\Omega}$. So, we will use the map $\pi : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$ to define a base change to $2^{\mathbb{N}}$. If the map $\theta : 2^{\mathbb{N}} \rightarrow \mathbf{B}^{\Omega}$ is continuous and open then the base change of $\mathbf{G} \wr \Omega$ to $2^{\mathbb{N}}$ is an open groupoid. However, in our case, the function θ need not be open. So, we will show another sufficient condition for the base change to $2^{\mathbb{N}}$ to be open and minimal.

Recall the identification $E^{\mathbb{N}} \cong (2^{\mathbb{N}})^R$. In the separable case we have that $(2^{\mathbb{N}})^R$ agrees with $(2^{\mathbb{N}})^{\Omega}$. Hence, the pseudo-open map π , obtained from [Corollary 3.20](#), when considered inside the separable model $(\bar{\mathcal{B}}, \mu) \models \text{APr}$, can be written as $\pi : D_{\text{APr}}(\Omega) \rightarrow (2^{\mathbb{N}})^{\Omega}$. Recall that Ξ is the sequence of independent subsets of measure $1/2$ used to define the universal Skolem sort of APr , as after [Lemma 2.8](#). Ξ defines moreover a left-invariant metric d_L on $\text{Aut}(\Omega)$, as in [Section 1](#). Given $\alpha \in 2^{\mathbb{N}}$ let us write $\pi(\alpha)$ for $\pi(\Xi, \alpha)$, this defines a continuous map $\pi : 2^{\mathbb{N}} \rightarrow (2^{\mathbb{N}})^{\Omega}$. The intended use of the map π , as a function defined on D_{APr} or on $2^{\mathbb{N}}$, will be clear from the context.

Pick $A, B \in E^{\mathbb{N}}(\Omega)$ and suppose they are identified with $f_A, f_B \in (2^{\mathbb{N}})^{\Omega}$. Then, given $k \in \mathbb{N}$, $A_n = B_n$ for every $n < k$ if and only if $d_{\infty}(f_A, f_B) < 2^{-k}$. Now suppose that $\bar{a} \in D_{\text{APr}_0}(\Omega)$ generates $\bar{\mathcal{B}}$. Then, as Ξ also generates $\bar{\mathcal{B}}$, the map $\Xi \mapsto \bar{a}$ defines an automorphism of $(\bar{\mathcal{B}}, \mu)$ which corresponds to some $\tau \in \text{Aut}(\Omega)$. It follows that $d(\Xi, \bar{a}) = d(\Xi, \tau(\Xi)) = d_L(\text{id}_{\Omega}, \tau)$. Therefore, the map $\pi : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$ being pseudo-open translates to the following property.

Definition 4.21. A map $\theta : 2^{\mathbb{N}} \rightarrow (2^{\mathbb{N}})^{\Omega}$ is said to be *almost uniformly open* if for every $\epsilon > 0$ and $N \in \mathbb{N}$ there is $\delta > 0$ such that for every $\alpha \in 2^{\mathbb{N}}$

$$B_{d_{\infty}}(\theta(\alpha), \delta) \subseteq B_{d_L}(\text{id}_{\Omega}, \epsilon) \cdot \theta([\alpha \upharpoonright_N]).$$

Since the action of $\text{Aut}(\Omega)$ on $(2^{\mathbb{N}})^{\Omega}$ is by d_{∞} -isometries and the metric d_L is invariant on the left, being almost uniformly open implies that for every $\epsilon > 0$ and $N \in \mathbb{N}$ there is $\delta > 0$ such that for every $\alpha \in 2^{\mathbb{N}}$ and $\tau \in \text{Aut}(\Omega)$,

$$B_{d_{\infty}}(\tau \cdot \theta(\alpha), \delta) \subseteq B_{d_L}(\tau, \epsilon) \cdot \theta([\alpha \upharpoonright_N]).$$

The name almost uniformly open comes from the fact that the map θ itself is not necessarily uniformly open, when considering the metric d_{∞} on $(2^{\mathbb{N}})^{\Omega}$. However, once we enlarge the image of θ by the action of $\text{Aut}(\Omega)$ we get uniform openness. We will now state a technical lemma in order to show that if the original groupoid is open and minimal, and the base change map is almost uniformly open, then the base change to $2^{\mathbb{N}}$ is also open and minimal.

Lemma 4.22. *Let \mathbf{G} be a Polish open minimal groupoid over $2^{\mathbb{N}}$. Take $g \in \mathbf{G}^{\Omega}$, $e \in (2^{\mathbb{N}})^{\Omega}$ and $\epsilon > 0$. Then, there exists $h \in \mathbf{G}^{\Omega}$ such that $t^{\Omega}(h) = t^{\Omega}(g)$, $\llbracket g \neq h \rrbracket = \llbracket d(e, s^{\Omega}(g)) \geq \epsilon \rrbracket$ and $d_{\infty}(s^{\Omega}(h), e) < \epsilon$.*

Proof. Let $\{V_1, \dots, V_k\}$ be a cover of $2^{\mathbb{N}}$ in disjoint clopen sets of diameter less than ϵ . For $i \leq k$, we define $A_i = e^{-1}(V_i)$. So, $\{A_1, \dots, A_k\}$ is a partition of Ω , by [Lemma 4.13](#), there is $f \in \mathbf{G}^{\Omega}$ such that $t^{\Omega}(f) = t^{\Omega}(g)$ and $s^{\Omega}(f) \in \mathcal{O}(A_1, \dots, A_k, V_1, \dots, V_k)$. Let $B = \llbracket d(e, s^{\Omega}(g)) < \epsilon \rrbracket$ and define $h = \langle B, \Omega \setminus B, g, f \rangle$. Clearly, $t^{\Omega}(h) = t^{\Omega}(g)$ and $\llbracket g \neq h \rrbracket = \llbracket d(e, s^{\Omega}(g)) \geq \epsilon \rrbracket$. Now take $\omega \in \Omega$, if $\omega \in B$ then $d(s^{\Omega}(h)_{\omega}, e_{\omega}) = d(s^{\Omega}(g)_{\omega}, e_{\omega}) < \epsilon$. In case $\omega \in \Omega \setminus B$, take $i \leq k$ such that $\omega \in A_i \cap B$. We have that $s^{\Omega}(h)_{\omega} = s^{\Omega}(f)_{\omega} \in V_i$, since the diameter of V_i is less than ϵ and $e_{\omega} \in V_i$, we get that $d(s^{\Omega}(h)_{\omega}, e_{\omega}) < \epsilon$. We conclude that $d_{\infty}(s^{\Omega}(h), e) < \epsilon$. \square

Theorem 4.23. *Let \mathbf{G} be a Polish open minimal groupoid over $2^{\mathbb{N}}$ and let $\theta : 2^{\mathbb{N}} \rightarrow (2^{\mathbb{N}})^{\Omega}$ be a continuous almost uniformly open map. Then, $(\mathbf{G} \wr \Omega)_{\theta}$, the base change of $\mathbf{G} \wr \Omega$ to $2^{\mathbb{N}}$, is a Polish open minimal groupoid.*

Proof. We already know that $(\mathbf{G} \wr \Omega)_{\theta}$ is a Polish groupoid. It remains to show that it is open and minimal. First we will show that for any $(\alpha, g, \tau, \beta) \in (\mathbf{G} \wr \Omega)_{\theta}$ and \mathcal{U} open neighborhood of (α, g, τ, β) in $(\mathbf{G} \wr \Omega)_{\theta}$, $t(\mathcal{U})$ is a neighborhood of α . We can assume that there are some $\epsilon > 0$, $N \in \mathbb{N}$ and $W \subseteq G^{\Omega}$ open neighborhood of g such that

$$\mathcal{U} = ([\alpha \upharpoonright_N] \times W \times B_{d_L}(\tau, \epsilon) \times [\beta \upharpoonright_N]) \cap (\mathbf{G} \wr \Omega)_{\theta}.$$

For such ϵ and N take $\delta > 0$ as in the definition of almost uniformly open. Now, take $W_0 \subseteq W$ open set containing g such that for every $f \in W_0$, $d_0(s^{\Omega}(f), s^{\Omega}(g)) < \delta$. We may further assume that if $f \in W_0$ then $B_{d_0}(f, \delta) \subseteq W$. Finally, take $K \geq N$ such that $[\alpha \upharpoonright_K] \subseteq \theta^{-1}(t^{\Omega}(W_0))$. We claim that

$[\alpha \upharpoonright_K] \subseteq t(\mathcal{U})$. Indeed, take $\alpha' \in [\alpha \upharpoonright_K]$, then there is $f \in W_0$ such that $t^\Omega(f) = \theta(\alpha')$. In particular, we have that $d_0(s^\Omega(g), s^\Omega(f)) < \delta$. Using the previous lemma, we obtain $h \in \mathbf{G}^\Omega$ such that $t^\Omega(h) = t^\Omega(f)$, $\llbracket h \neq f \rrbracket = \llbracket d(s^\Omega(f), s^\Omega(g)) \geq \delta \rrbracket$ and $d_\infty(s^\Omega(h), s^\Omega(g)) < \delta$. This implies that $d_0(f, h) < \delta$ and so $h \in W$. Since $s^\Omega(g) = \tau \cdot \theta(\beta)$ and θ is almost uniformly open, we obtain that $s^\Omega(h) \in B_{d_L}(\tau, \epsilon) \cdot \theta([\beta \upharpoonright_N])$. Hence, there is $\rho \in B_{d_L}(\tau, \epsilon)$ and there is $\beta' \in [\beta \upharpoonright_N]$ such that $s^\Omega(h) = \rho \cdot \theta(\beta')$. Therefore, $(\alpha', h, \rho, \beta') \in \mathcal{U}$ and $t(\alpha', h, \rho, \beta') = \alpha'$.

It remains to show that the action $(\mathbf{G} \wr \Omega)_\theta \curvearrowright 2^\mathbb{N}$ is minimal. Take $\alpha \in 2^\mathbb{N}$ and $\nu \in 2^N$ with $N \in \mathbb{N}$. For this N and $\epsilon = 1$ take $\delta > 0$ as in the definition of almost uniformly open. Take $\{V_1, \dots, V_k\}$ a cover of $2^\mathbb{N}$ consisting of disjoint clopen sets of diameter less than δ . Take any $\beta \in [\nu]$ and for $i \leq k$ define $A_i = \theta(\beta)^{-1}(V_i)$. In this way, $\{A_1, \dots, A_k\}$ is a partition of Ω , and by [Lemma 4.13](#), there is $g \in \mathbf{G}^\Omega$ such that $t^\Omega(g) = \theta(\alpha)$ and $s^\Omega(g) \in \mathcal{O}(A_1, \dots, A_k, V_1, \dots, V_k)$. This implies that $d_\infty(s^\Omega(g), \theta(\beta)) < \delta$. So, there are $\tau \in \text{Aut}(\Omega)$ and $\beta' \in [\nu]$ such that $s^\Omega(g) = \tau \cdot \theta(\beta')$. Thus, $(\alpha, g, \tau, \beta') \in (\mathbf{G} \wr \Omega)_\theta$, $t(\alpha, g, \tau, \beta') = \alpha$ and $s(\alpha, g, \tau, \beta') \in [\nu]$. \square

5. THE GROUPOID OF THE RANDOMIZATION

Recall the following fact which describes the automorphism group of the Borel randomization.

Theorem 5.1. [[Iba17](#), Theorem 2.18] *Let \mathcal{M} be a separable structure and let \mathcal{M}^R be its Borel randomization. Then,*

$$\text{Aut}(\mathcal{M}^R) = \text{Aut}(\mathcal{M}) \wr \Omega.$$

It follows that if T is an ω -categorical theory and \mathcal{M} is its separable model, then $\mathbf{G}(T^R) = 2^\mathbb{N} \times (\text{Aut}(\mathcal{M}) \wr \Omega) \times 2^\mathbb{N}$. In order to generalize this result to the context of theories admitting universal Skolem sorts we do the construction of the previous section. We will show that the groupoid associated to the randomization of a theory admitting a universal Skolem sort is of the form $(\mathbf{G} \wr \Omega)_\pi$, where \mathbf{G} is the groupoid associated to the original theory and $\pi : 2^\mathbb{N} \rightarrow (2^\mathbb{N})^\Omega$ is the map discussed above.. Furthermore, we show in the ω -categorical case that the two presentations of the groupoid are indeed equivalent. Since an element of the groupoid is the type of a couple where the definable closure of its entries agree, we do a brief study of the definable closure in randomizations.

5.1. Definable closure in randomizations.

An extensive treatment of this topic for classical theories can be found in [[AGK15](#)]. In this section we generalize some of the results therein to the case where the original theory is continuous. During this subsection D and F will denote sorts of T . x and y will denote variables in D and F respectively. As usual, for $a \in M$ we say $a \in \text{dcl}(A)$ if $\{a\}$ is definable in \mathcal{M} over A . For $a \in F$ we let $\text{dcl}_D(a) = \text{dcl}(a) \cap D$. We will denote by $(\Upsilon, \mathcal{F}, \mu)$ the underlying probability space of the randomizations we consider in this subsection. It is also worth recalling that E denotes the sort of events.

Lemma 5.2. *Let D and F be sorts of T . If $a \in F^R$ and $b \in D^R$ then*

$$\llbracket b \in \text{dcl}(a) \rrbracket = \{v \in \Upsilon ; b(v) \in \text{dcl}(a(v))\}$$

is measurable.

Proof. We know that for any formula $\psi(x)$ and any c in the randomization, the set $\llbracket \psi(c) = 0 \rrbracket$ is measurable. On the other hand, from [Fact 1.7](#), we have that $v \in \llbracket a \in \text{dcl}(b) \rrbracket$ if and only if for every $n > 0$ there is a formula $\varphi(x, y)$ such that $\varphi(b(v), a(v)) = 0$ and the diameter of $\{c \in D ; \varphi(c, a(v)) < 1/n\}$ is less or equal than $1/n$. Furthermore, we can assume that the formulas belong to a fixed countable dense set of formulas, which we will call Σ . Given a formula $\varphi(x, y)$ we define

$$\tilde{\varphi}_n(y) = \sup_{z_1, z_2} [(1 - \varphi(z_1, y) \vee \varphi(z_2, y)) \wedge d(z_1, z_2) \div 1/n].$$

Then, $\tilde{\varphi}_n(d) = 0$ precisely when the diameter of $\{c \in D ; \varphi(c, d) < 1/n\}$ is less or equal than $1/n$. Now, we set

$$B_n = \bigcup_{\varphi \in \Sigma} \llbracket \varphi(b, a) \vee \tilde{\varphi}_n(a) = 0 \rrbracket$$

We thus obtain that $\llbracket b \in \text{dcl}(a) \rrbracket = \bigcap_{n \in \mathbb{N}} B_n$ is measurable. \square

Lemma 5.3. *Let D and F be sorts of T . Take $a \in F^R$ and $b \in D^R$. If b is definable over a then $\mu[b \in \text{dcl}(a)] = 1$.*

Proof. Suppose on the contrary that $\mu[b \in \text{dcl}(a)] = 1 - r < 1$. We will show that there exists some element different from b that realizes $\text{tp}(b/a)$. For $\varphi(x, y)$ a formula and $n > 0$ let

$$A_{\varphi, n} = [\varphi(b, a) > 0] \cup \left([\varphi(b, a) = 0] \cap \bigcup_{m>0} [\inf_z [(\varphi(z, a) \dot{-} (1 - 1/m)) \wedge (1/n \dot{-} d(z, b))] = 0] \right).$$

In this way, $v \in A_{\varphi, n}$ when $\varphi(b(v), a(v)) = 0$ implies that there exists z with $\varphi(z, a(v)) < 1$ and $d(b(v), z) \geq 1/n$. Let us fix a countable dense set of formulas Σ and define $A_n = \bigcap_{\varphi \in \Sigma} A_{\varphi, n}$, then, $-\llbracket b \in \text{dcl}(a) \rrbracket = \bigcup_{n>0} A_n$. So, there must be some $N \in \mathbb{N}$ such that $\mu(A_N) \geq r/2$. Now, take any formula $\varphi(x, y)$, any $\epsilon > 0$ and pick $n \in \mathbb{N}$ with $1/n < \epsilon/2$. Given $i < n$, define $B_i = A_i \cap \llbracket i/n \leq \varphi(b, a) < (i+1)/n \rrbracket$ and $\varphi_i(x, y) = |i/n + 1/2n - \varphi(x, y)| \dot{-} 1/2n$. Let $c_i \in D^R$ be such that

$$\llbracket \inf_z [(\varphi_i(z, a) \dot{-} \epsilon/2) \wedge (1/N - d(b, z))] \rrbracket = \llbracket (\varphi_i(c_i, a) \dot{-} \epsilon/2) \wedge (1/N - d(b, c_i)) \rrbracket.$$

Note that for $v \in B_i$, $\varphi_i(b(v), a(v)) = 0$. Hence, $\varphi_i(c_i(v), a(v)) \leq \epsilon/2$ and $d(b(v), c_i(v)) \geq 1/N$. Take $c = (B_0, \dots, B_{n-1}, \Upsilon \setminus A_n, c_0, \dots, c_{n-1}, b)$, then $d(b, c) = \int_{A_n} d(b(v), c(v)) d\mu(v) \geq \mu(A_n)/N = r/2N$ and $|\llbracket \varphi(b, a) \rrbracket - \llbracket \varphi(c, a) \rrbracket| \leq (1/n + \epsilon/2) < \epsilon$. So, $|\mathbb{E}\llbracket \varphi(b, a) \rrbracket - \mathbb{E}\llbracket \varphi(c, a) \rrbracket| < \epsilon$. The same argument generalizes when considering several formulas simultaneously. The result will follow by compactness. \square

Given $A \subseteq E_{\text{RV}}$ we denote by $\sigma(A)$ the minimal σ -sub-algebra of E such that every $f \in A$ is $\sigma(A)$ -measurable. So, $\sigma(A)$ is the σ -algebra generated by $\{\llbracket f < r \rrbracket ; f \in A, r > 0\}$. In particular, when A consists of $\{0, 1\}$ -valued random variables, $\sigma(A)$ is the generated σ -algebra. In case $a \in D^R$, by $\sigma(a)$ we mean $\sigma(\{\llbracket \varphi(a) \rrbracket ; \varphi(x) \text{ formula}\})$.

Fact 5.4. *Let $A \subseteq E_{\text{RV}}$, then $\text{dcl}_E(A) = \sigma(A)$.*

Proof. Assume we are working inside a sufficiently saturated model, by [BY13, Theorem 2.17], for $U, V \in E$, $\text{tp}(U/A) = \text{tp}(V/A)$ if and only if $\text{tp}(U/\sigma(A)) = \text{tp}(V/\sigma(A))$. This implies that, $\text{dcl}_E(A) = \text{dcl}_E(\sigma(A)) = \sigma(A)$. \square

Lemma 5.5. *Let D be a sort of T and let $a \in D^R$, then $\text{dcl}_E(a) = \sigma(a)$.*

Proof. Assume we are working inside a sufficiently saturated model and let $A = \{\llbracket \varphi(a) \rrbracket ; \varphi(x) \text{ formula}\}$. By quantifier elimination, for any $U, V \in E$, $\text{tp}(U/a) = \text{tp}(V/a)$ if and only if $\text{tp}(U/A) = \text{tp}(V/A)$. Hence, $\text{dcl}_E(a) = \text{dcl}_E(A) = \sigma(A) = \sigma(a)$. \square

Lemma 5.6. *Let E_0 be a sort of APr and let D and F be sorts of T . Take $(a, \bar{a}) \in F^R \times E_0$ and $b \in D^R$, then b is definable over (a, \bar{a}) if and only if:*

- (1) $\mu[b \in \text{dcl}(a)] = 1$.
- (2) $\sigma(b, a, \bar{a}) = \sigma(a, \bar{a})$.

Proof. \Rightarrow) The first item corresponds to Lemma 5.3. The second item follows from the fact that $\sigma(a, \bar{a}) \subseteq \sigma(b, a, \bar{a}) = \text{dcl}_{E_{\text{RV}}}(b, a, \bar{a}) \subseteq \text{dcl}_{E_{\text{RV}}}(a, \bar{a}) = \sigma(a, \bar{a})$.

\Leftarrow) We can assume that we work inside a sufficiently saturated model. Let $c \models \text{tp}(b/(a, \bar{a}))$, we will show that $c = b$. Take $\varphi(x, y)$ any formula in $D \times F$, we have that $\llbracket \varphi(b, a) \rrbracket \in \sigma(b, a) \subseteq \sigma(a, \bar{a})$. Hence, there is a predicate $\psi(u, a, \bar{a})$ such that $d(u, \llbracket \varphi(b, a) \rrbracket) = \psi(u, a, \bar{a})$. Since, $c \models \text{tp}(b/(a, \bar{a}))$ then $\psi(\llbracket \varphi(c, a) \rrbracket) = 0$. Therefore, $\llbracket \varphi(b, a) \rrbracket = \llbracket \varphi(c, a) \rrbracket$ for any formula $\varphi(x, y)$. Now fix any $v \in \llbracket b \in \text{dcl}(a) \rrbracket$ and take $\epsilon > 0$, then there is a formula $\varphi(x, y)$ such that $\varphi(b(v), a(v)) = 0$ and $\text{diam}(\{d \in D_2 ; \varphi(d, a(v)) < 1\}) < \epsilon$. Due to the fact that $\llbracket \varphi(b, a) \rrbracket = \llbracket \varphi(c, a) \rrbracket$, we get that $\varphi(c(v), a(v)) = 0$. This implies that $d(b(v), c(v)) < \epsilon$. Since ϵ was arbitrary, $b(v) = c(v)$. It follows that $b(v) = c(v)$ for almost every $v \in \Upsilon$ and we conclude that $b = c$. \square

By taking $E_0 = E$ and $\bar{a} = \perp$ we obtain the following result.

Corollary 5.7. *Let D and F be sorts of T . Let $a \in F^R$ and $b \in D^R$, then b is definable over a if and only if:*

- (1) $\mu[b \in \text{dcl}(a)] = 1$.
- (2) $\sigma(a, b) = \sigma(a)$.

Note that if $\hat{a} \in D_{T^R}$ then, by Lemma 3.7, $\text{dcl}_E(\hat{a}) \subseteq \sigma(\bar{a})$. This fact together with the previous lemma allow us to understand interdefinability in the universal sort of T^R .

Corollary 5.8. *Assume that T admits a universal Skolem sort, D_T . Let $\hat{a}, \hat{b} \in D_{T^R}$, then \hat{b} is definable over \hat{a} if and only if:*

- (1) $\mu[\![b \in \text{dcl}(a)]\!] = 1$.
- (2) $\sigma(a, b, \bar{b}) \subseteq \sigma(\bar{a})$.

Proof. \Rightarrow) Assume $\hat{b} \in \text{dcl}(\hat{a})$, in particular, $b \in \text{dcl}(a, \bar{a})$. By Lemma 5.6, $\mu[\![b \in \text{dcl}(a)]\!] = 1$ and $\sigma(a, b, \bar{b}) \subseteq \sigma(a, \bar{a}, b, \bar{b}) = \sigma(a, \bar{a}) = \sigma(\bar{a})$.

\Leftarrow) Let $\hat{b} = (b, \bar{b}, \beta)$, clearly $\beta \in \text{dcl}(\hat{a})$. By (2), $\bar{b} \subseteq \sigma(\bar{a}) \subseteq \text{dcl}_E(\hat{a})$, which implies $\bar{b} \in \text{dcl}(\hat{a})$. The fact that $b \in \text{dcl}(\hat{a})$ follows from Lemma 5.6. \square

Corollary 5.9. *Assume that T admits a universal Skolem sort, D_T . Let $\hat{a}, \hat{b} \in D_{T^R}$, then $\text{dcl}(\hat{b}) = \text{dcl}(\hat{a})$ if and only if:*

- (1) $\mu[\![\text{dcl}(b) = \text{dcl}(a)]\!] = 1$.
- (2) $\sigma(a, b, \bar{b}, \bar{a}) = \sigma(\bar{a}) = \sigma(\bar{b})$.

Let D be any sort of T and let \mathbf{M} be a separable model of T^R . An element $a \in D^R(\mathbf{M})$ gives rise to a measurable function $g : \Omega \rightarrow S_D(T)$ defined by $g(\omega) = \text{tp}(a(\omega))$ for $\omega \in \Omega$. The next result gives us a converse of this result. From now on we may identify a formula $\varphi(u)$ in D with the continuous map it induces on the space of types $\varphi : S_D(T) \rightarrow [0, 1]$. It follows, that for $g \in S_D(T)^\Omega$, $\varphi \circ g \in L^1((\Omega, \mathcal{B}, \mu), [0, 1])$.

Lemma 5.10. *Let D be a sort of T . Then, $S_{D^R}(T^R)$ is homeomorphic to $S_D(T)^\Omega / \text{Aut}(\Omega)$. Furthermore, for any $f \in S_D(T)^\Omega$, there is $a \in D^R$ such that $f(\omega) = \text{tp}(a(\omega))$ for almost every $\omega \in \Omega$.*

Proof. Recall from Fact 2.3 that we have an identification between $S_{D^R}(T^R)$ and $\mathfrak{R}(S_D(T))$. By [Iba17, Lemma 3.5], the latter is homeomorphic to $S_D(T)^\Omega / \text{Aut}(\Omega)$. For the last part, take $f \in S_D(T)^\Omega$, then, by the previous identification, $[f]_{\text{Aut}(\Omega)}$ corresponds to a type $\mathbf{p} \in S_{D^R}(T^R)$. Let $b \models \mathbf{p}$, it follows that for any set of formulas $\varphi_1(u), \dots, \varphi_n(u)$ in D ,

$$\text{tp}(\varphi_1 \circ f, \dots, \varphi_n \circ f) = \text{tp}([\![\varphi_1(b)]\!], \dots, [\![\varphi_n(b)]\!]).$$

This implies, that for any $\epsilon > 0$ there is $\tau \in \text{Aut}(\Omega)$ such that $d([\![\varphi_i(\tau \cdot b)]\!], \varphi_i \circ f) < \epsilon$. for every $i \leq n$. This means that the set of conditions

$$\Sigma(u) = \{d([\![\varphi(u)]\!], \varphi \circ f) = 0 ; \varphi \text{ formula in } D\}$$

is approximately finitely satisfiable. Hence, a in a separable model satisfying $\Sigma(u)$ is the desired element. \square

Let $\Xi \in D_{\text{APr}, 0}(\Omega)$ denote the sequence of independent sets of measure 1/2 used to define D_{APr} . Take $p \in \mathbf{G}(T^R)$ and let $(\hat{a}, \hat{b}) \models p$ in some separable model \mathbf{M} of T^R . We may assume that $\text{dcl}(\hat{a}) = \text{dcl}(\hat{b}) = M$ and even that $\bar{a} = \Xi$. In this way, $p = \text{tp}((a, \Xi, \alpha), (b, \bar{b}, \beta))$. By Corollary 5.9, $\sigma(\Xi) = \sigma(\bar{b})$. This implies that $\bar{b} = \tau(\Xi)$ for some $\tau \in \text{Aut}(\Omega)$. Thus, every type $p \in \mathbf{G}(T^R)$ is of the form $\text{tp}((a, \Xi, \alpha), (b, \tau(\Xi), \beta))$ for some $a, b \in D_T^R(\mathbf{M})$, where \mathbf{M} is a separable randomization, and $\tau \in \text{Aut}(\Omega)$. Furthermore, since $\text{tp}(a, b) \in S_{2D_T^R}(T^R)$, we obtain a function $g \in S_{2D_T}(T)^\Omega$ defined by $g(\omega) = \text{tp}(a(\omega), b(\omega))$ for $\omega \in \Omega$. By Corollary 5.9, we have that $g \in \mathbf{G}(T)^\Omega$.

Theorem 5.11. *Let T be a complete, countable theory admitting a universal Skolem sort, D_T . Then $\mathbf{G}(T^R)$ is homeomorphic to*

$$\mathbf{H} = \{(g, \alpha, \tau, \beta) \in \mathbf{G}(T)^\Omega \times \mathbf{G}(\text{APr}) ; t^\Omega(g) = \pi(\alpha), s^\Omega(g) = \tau \cdot \pi(\beta)\},$$

where \mathbf{H} is endowed with the subspace topology, as a subset of the product space $\mathbf{G}(T)^\Omega \times \mathbf{G}(\text{APr})$.

Proof. We will define a map $F : \mathbf{G}(T^R) \rightarrow \mathbf{H}$ in the following manner. Given $p \in \mathbf{G}(T^R)$, we know that it is of the form $\text{tp}((a, \Xi, \alpha), (b, \tau(\Xi), \beta))$ for some $a, b \in D_T^R$, $\tau \in \text{Aut}(\Omega)$ and $\alpha, \beta \in 2^\mathbb{N}$. We define $g \in \mathbf{G}(T)^\Omega$ by $g(\omega) = \text{tp}(a(\omega), b(\omega))$, for $\omega \in \Omega$ and we set $F(p) = (g, \alpha, \tau, \beta)$. Note that $t^\Omega(g) = \text{tp}^R(a) = \pi(\Xi, \alpha) = \pi(\alpha)$ and similarly $s^\Omega(g) = \text{tp}^R(b) = \pi(\tau(\Xi), \beta) = \tau \cdot \pi(\beta)$, so the map is well-defined. To show that the map $F : \mathbf{G}(T^R) \rightarrow \mathbf{H}$ is surjective, take $(g, \alpha, \tau, \beta) \in \mathbf{G}(T)^\Omega \times \mathbf{G}(\text{APr})$ with $t^\Omega(g) = \pi(\alpha)$ and $s^\Omega(g) = \tau \cdot \pi(\beta)$. By Lemma 5.10, there is a randomization \mathbf{M} based on Ω and there are $a, b \in D_T^R(\mathbf{M})$ such that $g(\omega) = \text{tp}(a(\omega), b(\omega))$ for every $\omega \in \Omega$. We have that $\text{tp}^R(a) = t^\Omega(g) = \pi(\Xi, \alpha)$ and that $\text{tp}^R(b) = s^\Omega(g) = \pi(\tau(\Xi), \beta)$. So, (a, Ξ, α) and $(b, \tau(\Xi), \beta)$ belong to D_{T^R} .

Since, $\text{tp}(a(\omega), b(\omega))$ belongs to $\mathbf{G}(T)$ for almost every $\omega \in \Omega$, we get $\mu[\text{dcl}(b) = \text{dcl}(a)] = 1$. Furthermore, $\sigma(a, b, \tau(\Xi), \Xi) = \sigma(\Xi) = \sigma(\tau(\Xi))$. Hence, by [Corollary 5.9](#), $p = \text{tp}((a, \Xi, \alpha), (b, \tau(\Xi), \beta)) \in \mathbf{G}(T^R)$ and we obtain that $(g, \alpha, \tau, \beta) = F(p)$. To see that it is injective, take $p, q \in \mathbf{G}(T^R)$ and suppose that $F(p) = F(q) = (g, \alpha, \tau, \beta)$. It follows that $p = \text{tp}((a, \Xi, \alpha), (b, \tau(\Xi), \beta))$ and $q = \text{tp}((c, \Xi, \alpha), (d, \tau(\Xi), \beta))$ for some $a, b, c, d \in D_T^R$. Let $\varphi(x, y)$ be any formula in D_T^2 , then $\llbracket \varphi(a, b) \rrbracket = \varphi \circ g = \llbracket \varphi(c, d) \rrbracket$. This implies that $p = q$.

Next we will show that F is continuous. Here we will use the maps $\{\lambda_{n,i} : [0, 1] \rightarrow [0, 1] ; i < 2^n\}$ defined just before [Lemma 3.9](#). First, note that if $V \subseteq \mathbf{G}(\text{APr})$ is open then $F^{-1}(\mathbf{G}(T)^\Omega \times V)$ is clearly open. So, it suffices to show that for any $A \subseteq \Omega$ clopen, $U \subseteq \mathbf{G}(T)$ open and $\epsilon > 0$, $W = F^{-1}(\mathcal{O}(A, U, \epsilon))$ is open. We can assume, that $U = [\varphi(x, y) < 1]$ for some formula $\varphi(x, y)$. Take $p = \text{tp}((a, \Xi, \alpha), (b, \tau(\Xi), \beta)) \in W$, so $\delta = \mu(A \cap \llbracket \varphi(a, b) = 1 \rrbracket) < \epsilon$. Now choose any $r > 0$ with $\delta + r < \epsilon$. We know that $A \in \text{dcl}(\Xi)$, then there is an \mathcal{L}_{APr} -term $\chi(\bar{x})$ satisfying $d(A, \chi(\Xi)) < r/2$. In particular, $\mu(\chi(\Xi) \cap \llbracket \varphi(a, b) = 1 \rrbracket) < \delta + r/2$. Moreover, since $\llbracket \varphi(a, b) = 1 \rrbracket = \lim_{n \rightarrow \infty} \llbracket \lambda_{n, 2^n - 1}(\varphi(a, b)) \rrbracket$, there is $N \in \mathbb{N}$ such that $\mathbb{E}(\chi(\Xi) \wedge \llbracket \lambda_{N, 2^N - 1}(\varphi(a, b)) \rrbracket) < \delta + r/2$. We define $\psi(\hat{x}, \hat{y}) = \mathbb{E}(\chi(\bar{x}) \wedge \llbracket \lambda_{N, 2^N - 1}(\varphi(x, y)) \rrbracket)$. By definition, $p \in [\psi(\hat{x}, \hat{y}) < \delta + r/2]$, we claim that $[\psi(\hat{x}, \hat{y}) < \delta + r/2] \subseteq W$. Let $q = \text{tp}((c, \Xi, \gamma), (d, \rho(\Xi), \delta)) \in [\psi(\hat{x}, \hat{y}) < \delta + r/2]$, this means that $\mathbb{E}(\chi(\Xi) \wedge \llbracket \lambda_{N, 2^N - 1}(\varphi(c, d)) \rrbracket) < \delta + r/2$. Therefore,

$$\begin{aligned} \mu(A \cap \llbracket \varphi(c, d) = 1 \rrbracket) &< \mu(\chi(\Xi) \cap \llbracket \varphi(c, d) = 1 \rrbracket) + r/2 \\ &\leq \mathbb{E}(\chi(\Xi) \wedge \llbracket \lambda_{N, 2^N - 1}(\varphi(c, d)) \rrbracket) + r/2 \\ &< \delta + r \\ &< \epsilon. \end{aligned}$$

This implies that $q \in W$, which shows that F is continuous. We will now see that F is also open. Take $\psi(\hat{x}, \hat{y})$ a formula in $D_{T^R}^2$, we can assume that it is of the form

$$\psi(\hat{x}, \hat{y}) = \psi^*(\llbracket \varphi_1(x, y) \rrbracket, \dots, \llbracket \varphi_n(x, y) \rrbracket, \bar{x}, \hat{x}, \bar{y}, \hat{y})$$

where $\psi^*(z, \bar{x}, \hat{x}, \bar{y}, \hat{y})$ is a formula in $E_{\text{RV}}^n \times D_{\text{APr}}^2$. Let $p \in [\psi(\hat{x}, \hat{y}) < 1]$, let $(g, \alpha, \tau, \beta) = F(p)$ and pick $r > 0$ with $\psi(\hat{x}, \hat{y})^p + r < 1$. Pick $\epsilon > 0$ such that $d((z, \bar{x}, \hat{x}, \bar{y}, \hat{y}), (z', \bar{x}', \hat{x}', \bar{y}', \hat{y}')) < \epsilon$ implies $|\psi^*(z, \bar{x}, \hat{x}, \bar{y}, \hat{y}) - \psi^*(z', \bar{x}', \hat{x}', \bar{y}', \hat{y}')| < r$. Note that for $i \leq n$, the map $\hat{\varphi}_i : \mathbf{G}(T)^\Omega \rightarrow L^1((\Omega, \mathcal{B}, \mu), [0, 1])$ given by $\hat{\varphi}_i(h) = \varphi_i \circ h$ is continuous. Thus, there is $U \subseteq \mathbf{G}(T)^\Omega$ open with $g \in U$ such that for any $h \in U$, $d(\varphi_i \circ g, \varphi_i \circ h) < \epsilon$ for $i \leq n$. Also, there is $V \subseteq \mathbf{G}(\text{APr})$ open with $(\alpha, \tau, \beta) \in V$ such that for any $(\alpha', \tau' \beta') \in V$, $d(\alpha, \alpha') \vee d(\beta, \beta') \vee d(\tau(\Xi), \tau'(\Xi)) < \epsilon$. We thus get that $(g, \alpha, \tau, \beta) \in (U \times V) \cap \mathbf{H} \subseteq F([\psi(\hat{x}, \hat{y}) < 1])$, which completes the proof. \square

The next step is to determine how the groupoid operations are defined on \mathbf{H} . Take $(a, \Xi, \alpha) \in D_{T^R}$, then $\text{tp}^R(a) = \pi(\alpha)$ and $\text{tp}(a, \Xi, \alpha)$ corresponds to $(\pi(\alpha), \alpha, \text{id}_\Omega, \alpha)$. We thus have an identification between $2^{\mathbb{N}}$ and $\mathbf{B}(T^R)$ via

$$\alpha \in 2^{\mathbb{N}} \mapsto (\pi(\alpha), \alpha, \text{id}_\Omega, \alpha) \in \mathbf{B}(T^R).$$

Now take $(g, \alpha, \tau, \beta) \in \mathbf{H}$, which we identify with $p = \text{tp}((a, \Xi, \alpha), (b, \tau(\Xi), \beta))$. Let $\mathbf{M} = \text{dcl}(a, \Xi, \alpha)$, which as usual we suppose is based on $(\Omega, \mathcal{B}, \mu)$. Given $\omega \in \Omega$ let $\mathcal{A}_\omega = \mathcal{M}_{\tau(\omega)}$. We define a new randomization \mathbf{A} , based on $\{\mathcal{A}_\omega ; \omega \in \Omega\}$, where $x \in \prod_{\omega \in \Omega} \mathcal{A}_\omega$ belongs to \mathbf{A} if and only if there is $a \in \mathbf{M}$ such that $x(\omega) = a(\tau(\omega))$ for every $\omega \in \Omega$. It follows that the function $f_{\tau^{-1}} : \mathbf{M} \rightarrow \mathbf{A}$, that assigns to each $a \in \mathbf{M}$ the map $\omega \mapsto a(\tau(\omega))$, is an isomorphism. Moreover, on the sort of events, $f_{\tau^{-1}} = \tau^{-1}$. Therefore, $p^{-1} = \text{tp}((b, \tau(\Xi), \beta), (a, \Xi, \alpha)) = \text{tp}(f_{\tau^{-1}}(b, \tau(\Xi), \beta), f_{\tau^{-1}}(a, \Xi, \alpha)) = \text{tp}((f_{\tau^{-1}}(b), \Xi, \beta), (f_{\tau^{-1}}(a), \tau^{-1}(\Xi), \alpha))$. It is true that for every $\omega \in \Omega$,

$$\text{tp}((f_{\tau^{-1}}(b)(\omega), (f_{\tau^{-1}}(a)(\omega))) = \text{tp}(b(\tau(\omega)), a(\tau(\omega))) = g(\tau(\omega))^{-1} = (\tau^{-1} \cdot g^{-1})(\omega).$$

Consequently,

$$(g, \alpha, \tau, \beta)^{-1} = (\tau^{-1} \cdot g^{-1}, \beta, \tau^{-1}, \alpha).$$

Let $(g, \alpha, \tau, \beta), (h, \beta, \rho, \gamma) \in \mathbf{H}$ correspond to $p = \text{tp}((a, \Xi, \alpha), (b, \tau(\Xi), \beta))$ and $q = \text{tp}((c, \Xi, \beta), (d, \rho(\Xi), \gamma))$ respectively. Define $\mathbf{A} = \text{dcl}(a, \Xi, \alpha) = \text{dcl}(b, \tau(\Xi), \beta)$ and $\mathbf{M} = \text{dcl}(c, \Xi, \beta) = \text{dcl}(d, \rho(\Xi), \gamma)$. We remark that for almost every $\omega \in \Omega$, $\mathcal{A}_\omega = \text{dcl}(a(\omega)) = \text{dcl}(b(\omega))$ and $\mathcal{M}_\omega = \text{dcl}(c(\omega)) = \text{dcl}(d(\omega))$. Furthermore,

$$\text{tp}(c(\omega)) = t(h_\omega) = \pi(\beta)(\omega) = \tau \cdot \pi(\beta)(\tau(\omega)) = s(g_{\tau(\omega)}) = \text{tp}(b(\tau(\omega))).$$

Hence, the map sending $c(\omega)$ to $b(\tau(\omega))$ defines an isomorphism $f_\omega : M_\omega \rightarrow \mathcal{A}_{\tau(\omega)}$. This implies that the map

$$f : \mathbf{M} \rightarrow \mathbf{A}$$

$$u \mapsto (\omega \mapsto f_{\tau^{-1}(\omega)}(u(\tau^{-1}(\omega))))$$

is an isomorphism sending c to b and agreeing with τ on the sort of events. It follows that $q = \text{tp}((b, \tau(\Xi), \beta), (f(d), \tau\rho(\Xi), \gamma))$ and we get that $pq = \text{tp}((a, \Xi, \alpha), (f(d), \tau\rho(\Xi), \gamma))$. Now, for $\omega \in \Omega$,

$$\begin{aligned} \text{tp}(a(\omega), f(d)(\omega)) &= \text{tp}(a(\omega), f_{\tau^{-1}(\omega)}(d(\tau^{-1}(\omega)))) \\ &= \text{tp}(a(\omega), b(\omega)) \cdot \text{tp}(b(\omega), f_{\tau^{-1}(\omega)}(d(\tau^{-1}(\omega)))) \\ &= \text{tp}(a(\omega), b(\omega)) \cdot \text{tp}(c(\tau^{-1}(\omega)), d(\tau^{-1}(\omega))) \\ &= g(\omega)(\tau \cdot h)(\omega). \end{aligned}$$

We thus conclude that

$$(g, \alpha, \tau, \beta)(h, \beta, \rho, \gamma) = (g(\tau \cdot h), \alpha, \tau\rho, \gamma).$$

It follows that the target and source maps are given by,

$$t(g, \alpha, \tau, \beta) = (\pi(\alpha), \alpha, \text{id}_\Omega, \alpha) \cong \alpha, \quad s(g, \alpha, \tau, \beta) = (\pi(\beta), \beta, \text{id}_\Omega, \beta) \cong \beta.$$

The next result shows that this identification is independent of the choice of $\Xi \in D_{\text{APr},0}(\Omega)$ as long as it generates the whole algebra.

Lemma 5.12. *Fix $\Delta \in D_{\text{APr},0}(\Omega)$ generating $\bar{\mathcal{B}}$ and define $\theta : 2^\mathbb{N} \rightarrow (2^\mathbb{N})^\Omega$ by $\theta(\alpha) = \pi(\Delta, \alpha)$. Then,*

$$\mathbf{H}_\Delta = \{(g, \alpha, \tau, \beta) \in \mathbf{G}(T)^\Omega \times \mathbf{G}(\text{APr}) ; t^\Omega(g) = \theta(\alpha), s^\Omega(g) = \tau \cdot \theta(\beta)\}$$

is homeomorphic to \mathbf{H} via a map that respects the identification with $\mathbf{G}(T^R)$. This means that there is map $\Lambda : \mathbf{H} \rightarrow \mathbf{H}_\Delta$ such that the following diagram commute.

$$\begin{array}{ccc} \mathbf{G}(T^R) & \xrightarrow{F} & \mathbf{H} \\ & \searrow F_\Delta & \downarrow \Lambda \\ & & \mathbf{H}_\Delta \end{array}$$

Proof. Let $\rho \in \text{Aut}(\Omega)$ be such that $\rho(\Xi) = \Delta$. It follows that $\theta(\alpha) = \pi(\rho(\Xi), \alpha) = \rho \cdot \pi(\alpha)$ for every $\alpha \in 2^\mathbb{N}$. Define

$$\Lambda : \mathbf{H} \rightarrow \mathbf{H}_\Delta$$

$$(g, \alpha, \tau, \beta) \mapsto (\rho \cdot g, \alpha, \rho\tau\rho^{-1}, \beta).$$

Note that $s^\Omega(\rho \cdot g) = \rho \cdot s^\Omega(g) = \rho\tau \cdot \pi(\beta) = \rho\tau\rho^{-1} \cdot \theta(\beta)$. Analogously, $t^\Omega(\rho \cdot g) = \theta(\alpha)$, so the map is well-defined. It is a routine verification that Λ is a bijection. Let $A_1, \dots, A_n \subseteq \Omega$, let U_1, \dots, U_n be open subsets of $\mathbf{G}(T)$ and let $\epsilon > 0$, then

$$\rho^{-1} \cdot \mathcal{O}(A_1, \dots, A_n, U_1, \dots, U_n, \epsilon) = \mathcal{O}(\rho^{-1}(A_1), \dots, \rho^{-1}(A_n), U_1, \dots, U_n, \epsilon).$$

Furthermore, if $V \subseteq \mathbf{G}(\text{APr})$ is open then $\rho^{-1}V\rho = \{(\alpha, \rho^{-1}\tau\rho, \beta) \in \mathbf{G}(\text{APr}) ; (\alpha, \tau, \beta) \in V\}$ is also open. Since,

$$\Lambda^{-1}(\mathcal{O}(A_1, \dots, A_n, U_1, \dots, U_n, \epsilon) \times V) = (\rho^{-1}\mathcal{O}(A_1, \dots, A_n, U_1, \dots, U_n, \epsilon) \times \rho^{-1}V\rho) \cap \mathbf{H},$$

we obtain that Λ is continuous. By a similar argument, the map is also open and we conclude that it is in fact a homeomorphism. Now, take $p = \text{tp}((a, \Xi, \alpha), (b, \tau(\Xi), \beta)) \in \mathbf{G}(T^R)$ and let $(g, \alpha, \tau, \beta) = F(p)$. We set $\mathbf{M} = \text{dcl}(a, \Xi, \alpha)$, and for each $\omega \in \Omega$ let $\mathcal{A}_\omega = \mathcal{M}_{\rho^{-1}(\omega)}$. As before, say that $x \in \mathbf{A}$ if and only if for some $a \in \mathbf{M}$, $x(\omega) = a(\rho^{-1}(\omega))$ for every $\omega \in \Omega$. Again, we have an isomorphism $f_\rho : \mathbf{M} \rightarrow \mathbf{A}$ that agrees with ρ on the sort of events. We then have that, $p = \text{tp}((f_\rho(a), \Delta, \alpha), (f_\rho(b), \rho\tau\rho^{-1}(\Delta), \beta))$ and for $\omega \in \Omega$, $\text{tp}(f_\rho(a)(\omega), f_\rho(b)(\omega)) = \text{tp}(a(\rho^{-1}(\omega)), b(\rho^{-1}(\omega))) = (\rho \cdot g)(\omega)$. This implies that $F_\Delta(p) = (\rho \cdot g, \alpha, \rho\tau\rho^{-1}, \beta) = \Lambda(g, \alpha, \tau, \beta)$, which completes the proof. \square

Theorem 5.13. *Let T be a complete, countable theory admitting a universal Skolem sort. Let $\pi : 2^{\mathbb{N}} \rightarrow (2^{\mathbb{N}})^{\Omega}$ be the map defined above. Then, $\mathbf{G}(T^R)$ is isomorphic, as a topological groupoid, to $(\mathbf{G}(T) \wr \Omega)_{\pi}$, the base change of $\mathbf{G}(T) \wr \Omega$ to $2^{\mathbb{N}}$ given by π .*

Proof. We have that

$$(\mathbf{G}(T) \wr \Omega)_{\pi} = \{(\alpha, g, \tau, \beta) \in 2^{\mathbb{N}} \times \mathbf{G}(T)^{\Omega} \times \text{Aut}(\Omega) \times 2^{\mathbb{N}} ; t^{\Omega}(g) = \pi(\alpha), \tau^{-1} \cdot s^{\Omega}(g) = \pi(\beta)\}$$

endowed with the topology induced from the product, which is equal to \mathbf{H} up to a reordering of variables. \square

Suppose T is ω -categorical and \mathcal{M} is its separable model, we know that $\mathbf{G}(T) = 2^{\mathbb{N}} \times \text{Aut}(\mathcal{M}) \times 2^{\mathbb{N}}$. Hence, an element $g \in \mathbf{G}(T)^{\Omega}$ can be identified with $(t^{\Omega}(g), \tilde{g}, s^{\Omega}(g))$ for some $\tilde{g} \in \text{Aut}(\mathcal{M})^{\Omega}$. Thus, if $(\alpha, g, \tau, \beta) \in \mathbf{G}(T^R)$ then

$$(\alpha, g, \tau, \beta) = (\alpha, t^{\Omega}(g), \tilde{g}, s^{\Omega}(g), \tau, \beta) = (\alpha, \pi(\alpha), \tilde{g}, \tau \cdot \pi(\beta), \tau, \beta) \cong (\alpha, \tilde{g}, \tau, \beta),$$

which gives the correspondence between $\mathbf{G}(T^R)$, as constructed here, and, as previously known, i.e. $2^{\mathbb{N}} \times (\text{Aut}(\mathcal{M}) \wr \Omega) \times 2^{\mathbb{N}}$.

6. FURTHER QUESTIONS

The reader may have noticed that the map $\pi : D_{\text{APr}} \rightarrow E^{\mathbb{N}}$ is not canonical. Given any two pseudo-open maps the fiber products they define are universal Skolem sorts. Hence, there is a definable bijection between them which induces an isomorphism between the corresponding groupoids. Nevertheless, we do not know how to explicitly build such an isomorphism and show directly the independence of the choice of the pseudo-open map. Furthermore, the property of being pseudo-open is sufficient for the fiber product to be definable. However, we do not know if it is also necessary and in case it is not, what would be an appropriate characterization of the maps whose fiber product define a universal Skolem sort.

As stated in [BY22, Section 6], the goal is to characterize stability, and NIP, in dynamical terms by means of the action of the groupoid associated to the theory on the space of types. A direct generalization of the results shown in [Iba17]. We already have a topological characterization of the randomization of the action $\mathbf{G}(T) \curvearrowright S_{D_T}(T)$, which will be shown in a subsequent article.

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