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Philippe Gaucher

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# COMPARING CUBICAL AND GLOBULAR DIRECTED PATHS

PHILIPPE GAUCHER

**ABSTRACT.** A flow is a directed space structure over a homotopy type. It is already known that the underlying homotopy type of the realization of a precubical set as a flow is homotopy equivalent to the realization of the precubical set as a topological space. This realization depends on the non-canonical choice of a  $q$ -cofibrant replacement. We construct a new realization functor from precubical sets to flows which is homotopy equivalent to the previous one and which does not depend on the choice of any cofibrant replacement functor. The main tool is the notion of natural  $d$ -path introduced by Raussen. The flow we obtain for a given precubical set is not anymore  $q$ -cofibrant but is still  $m$ -cofibrant. As an application, we prove that the space of execution paths of the realization of a precubical set as a flow is homotopy equivalent to the space of  $d$ -paths of the geometric realization of the precubical set as a Grandis  $d$ -space.

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## 1. INTRODUCTION

**Presentation.** Precubical sets are a prominent geometric model for concurrency theory [5]. The  $n$ -cube represents the concurrent execution of  $n$  actions. The space of  $d$ -paths of the geometric realization of a precubical set as a Grandis  $d$ -space in the sense of [16, 26], is studied in many papers, such as a series of papers [21–23, 25, 26] by Raussen and Ziemiański. Precubical sets can also be realized as flows in the sense of [6]. The realization functor of a precubical set as a flow is first introduced in [10, Definition 7.2].

The two approaches (let us call them the cubical one of Raussen and Ziemiański and the globular one of the author) do not coincide up to homeomorphism. In the cubical approach, the  $d$ -paths from the initial to the final states of the topological  $n$ -cube  $[0, 1]^n$  are the continuous paths from  $(0, \dots, 0)$  to  $(1, \dots, 1)$  which are nondecreasing with respect to each axis of coordinates. Raussen and Ziemiański study also several variants (tame,

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strict, natural etc...) which give rise to homotopy equivalent spaces of paths between two fixed vertices of a general precubical set viewed as a Grandis  $d$ -space. None of these definitions give rise to spaces of  $d$ -paths which are homeomorphic to the spaces of execution paths from the initial to the final states of the  $n$ -cube  $[0, 1]^n$  viewed as a flow. In the latter case, the space of execution paths from the initial to the final states of the  $n$ -cube is the  $(n - 1)$ -dimensional disk  $\mathbf{D}^{n-1}$  (see Theorem 3.11). It means that the latter space depends on a non-canonical choice of an achronal slice in the middle of the topological  $n$ -cube and on a non-canonical choice of a homeomorphism between this achronal slice and  $\mathbf{D}^{n-1}$ .

The underlying homotopy type of a flow is the homotopy type obtained after removing its execution paths. It is defined in [8, Section 6] and a more conceptual construction is provided in [12, Proposition 8.16] using Moore flows. It is already known in full generality that the underlying homotopy type of the realization of a precubical set as a flow is homotopy equivalent to the realization of a precubical set as a topological space [9, Theorem 6.2.1], which is of course the underlying space of the realization of the precubical set as a Grandis  $d$ -space. The purpose of this paper is to prove the directed version of this result.

At first, using the notion of natural  $d$ -path introduced by Raussen in [21, Definition 2.14], we improve the realization functor from precubical sets to flows  $|-|_q : \square^{op}\mathbf{Set} \rightarrow \mathbf{Flow}$  introduced in [10, Definition 7.2] as follows.

**Theorem.** (Theorem 5.8) *There exist a colimit preserving functor*

$$|-|_{nat} : \square^{op}\mathbf{Set} \longrightarrow \mathbf{Flow}$$

*which does not depend on any cofibrant replacement and a natural transformation  $\mu : |-|_q \Rightarrow |-|_{nat}$  such that for all precubical sets  $K$ , the natural map  $\mu_K : |K|_q \rightarrow |K|_{nat}$  induces a bijection on states and a homotopy equivalence  $\mathbb{P}_{\alpha,\beta}|K|_q \simeq \mathbb{P}_{\alpha,\beta}|K|_{nat}$  for all  $\alpha, \beta \in K_0$ .*

Theorem 5.8 implies that  $\mathbb{P}_{\alpha,\beta}|K|_{nat}$  is m-cofibrant,  $\mathbb{P}_{\alpha,\beta}|K|_q$  being q-cofibrant. The interest of the natural realization functor is that it does not depend anymore on the arbitrary choice on any cofibrant replacement functor for the category of flows. Surprisingly, it does not even depend on a m-cofibrant replacement or on a h-cofibrant replacement of the category of flows. The geometric properties of the natural  $d$ -paths enable us in Section 6 to give another description of the natural realization functor using Ziemiański's notion of cube chain. As an application of Theorem 5.8 and of Section 6, we prove the following two theorems:

**Theorem.** (Theorem 7.5) *Let  $K$  be a precubical set. Let  $\alpha, \beta \in K_0$ . The topology of  $\mathbb{P}_{\alpha,\beta}|K|_{nat}$  is the correct topology, i.e. the  $\Delta$ -kelleyfication of the relative topology induced by the compact-open topology.*

The underlying set of  $\mathbb{P}_{\alpha,\beta}|K|_{nat}$  being exactly the set of tame natural  $d$ -paths of  $K$  from  $\alpha$  to  $\beta$ , Theorem 7.5 provides another point of view on its topology.

**Theorem.** (Corollary 7.6) *Let  $K$  be a precubical set. Let  $\alpha, \beta \in K_0$ . Then the space of execution paths  $\mathbb{P}_{\alpha,\beta}|K|_q$  is homotopy equivalent to the space of  $d$ -paths from  $\alpha$  to  $\beta$*

equipped with the  $\Delta$ -kelleyfication of the compact-open topology in the geometric realization of  $K$  as a Grandis  $d$ -space.

Corollary 7.6 is not surprising. However, until a proof was known, it was not sure that the statement was true for all precubical sets in full generality, and not only e.g. for non-positively curved precubical sets in the sense of [15], notion which brings together the properties satisfied by the precubical sets coming from real concurrent systems.

**Prerequisites.** The main tools used in this paper are the  $\{q, m, h\}$ -model structures of flows [14], the homotopical results of [9] about the realization functors of precubical sets as flows, and some topological results due to Ziemiański's about natural  $d$ -paths and the technique of cube chains coming from [26]. At the very end of this paper, [26, Theorem 7.5 and Theorem 7.6] are used for the proof of Corollary 7.6. The necessary reminders are made throughout the paper.

**Outline of the paper.** Section 2 is a reminder about the three model structures of flows: Quillen (q), Hurewicz (h) and mixed (m) introduced in [14]. It contains, as a new and easy remark, the proof that these three model structures on flows are simplicial. Section 3 recalls some basic facts about cocubical objects, gives the definition of a  $r$ -realization functor with  $r \in \{q, m, h\}$  in Definition 3.6, adapts in Theorem 3.8 some tools coming from [9], and finally gives the example of the q-realization functor expounded in [10]. Section 4 recalls the notion of natural (tame)  $d$ -path of a precubical set and proves some basic facts about their topology, in relation with the  $\Delta$ -generated spaces which are the setting of this work. Section 5 expounds the construction of the natural realization functor. It does not depend on any cofibrant replacement functor. It is a new realization functor which is proved to be equivalent in some sense to the one of [10] in Theorem 5.8. This section also proves that this new realization functor is a m-realization functor. Section 6 gives an equivalent definition of the natural realization functor in terms of cube chains in the sense of Ziemiański. This construction is possible thanks to a specific property of the natural realization functor, namely Proposition 5.1 which is used in the proofs of Proposition 6.2 and Theorem 6.3. Finally, Section 7 gives the applications of these results, namely an alternative description of the topology of the space of natural  $d$ -paths of a precubical set in Theorem 7.5 and a proof that all these spaces of execution paths coincide up to homotopy with the spaces of  $d$ -paths of the geometric realization of the precubical set as a Grandis  $d$ -space in Corollary 7.6.

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## 2. THREE SIMPLICIAL MODEL STRUCTURES OF FLOWS

We work with a locally presentable convenient category of topological spaces **Top** for doing algebraic topology. The category of  $\Delta$ -generated spaces or of  $\Delta$ -Hausdorff  $\Delta$ -generated spaces (cf. [13, Section 2 and Appendix B]) are two such examples. The category **Top** is equipped with its q-model structure (we use the terminology of [20]). The m-model structure [4] and the h-model structure [1] of **Top** are also used in various places of the paper. Compact means compact Hausdorff (French convention). The initial object of a category is denoted by  $\emptyset$ . The terminal object of a category is denoted by **1**. A *cellular object* of a combinatorial model category is an object  $X$  such that the

canonical map  $\emptyset \rightarrow X$  is a transfinite composition of pushouts of generating cofibrations. We summarize some basic properties of **Top** used in this paper for the convenience of the reader:

- **Top** is locally presentable.
- All objects of **Top** are sequential topological spaces.
- A closed subset of a  $\Delta$ -generated space equipped with the relative topology is not necessarily  $\Delta$ -generated (e.g. the Cantor set), but it is always sequential.
- All locally path-connected first-countable topological spaces are  $\Delta$ -generated by [3, Proposition 3.11], in particular all locally path-connected metrizable topological spaces are  $\Delta$ -generated.
- The inclusion functor from the full subcategory of  $\Delta$ -generated spaces to the category of general topological spaces together with the continuous maps has a right adjoint called the  $\Delta$ -kelleyfication functor. The latter functor does not change the underlying set.
- Let  $A \subset B$  be a subset of a space  $B$  of **Top**. Then  $A$  equipped with the  $\Delta$ -kelleyfication of the relative topology belongs to **Top**.
- The colimit in **Top** is given by the final topology in the following situations:
  - A transfinite compositions of one-to-one maps.
  - A pushout along a closed inclusion.
  - A quotient by a closed subset or by an equivalence relation having a closed graph.
In these cases, the underlying set of the colimit is therefore the colimit of the underlying sets. In particular, the CW-complexes, and more generally all cellular spaces are equipped with the final topology.
- Cellular spaces are Hausdorff.
- **Top** is cartesian closed. The internal hom  $\mathbf{TOP}(X, Y)$  is given by taking the  $\Delta$ -kelleyfication of the compact-open topology on the set  $\mathcal{TOP}(X, Y)$  of all continuous maps from  $X$  to  $Y$ .

**2.1. Definition.** [6, Definition 4.11] *A flow is a small semicategory enriched over the closed monoidal category  $(\mathbf{Top}, \times)$ . The corresponding category is denoted by **Flow**.*

A flow  $X$  consists of a topological space  $\mathbb{P}X$  of *execution paths*, a discrete space  $X^0$  of *states*, two continuous maps  $s$  and  $t$  from  $\mathbb{P}X$  to  $X^0$  called the source and target map respectively, and a continuous and associative map  $*$  :  $\{(x, y) \in \mathbb{P}X \times \mathbb{P}X; t(x) = s(y)\} \rightarrow \mathbb{P}X$  such that  $s(x * y) = s(x)$  and  $t(x * y) = t(y)$ . Let  $\mathbb{P}_{\alpha, \beta}X = \{x \in \mathbb{P}X \mid s(x) = \alpha \text{ and } t(x) = \beta\}$ : it is the space of execution paths from  $\alpha$  to  $\beta$ ,  $\alpha$  is called the initial state and  $\beta$  is called the final state. Note that the composition is denoted by  $x * y$ , not by  $y \circ x$ . The category **Flow** is locally presentable.

Every set can be viewed as a flow with an empty space of execution paths. Every poset can be viewed as a flow with one execution path from  $\alpha$  to  $\beta$  if and only if  $\alpha < \beta$ . The obvious functor **Set**  $\subset$  **Flow** from the category of sets to that of flows is limit-preserving and colimit-preserving. The following example of flows is important for the sequel:

**2.2. Example.** *For a topological space  $Z$ , let  $\mathbf{Glob}(Z)$  be the flow defined by*

$$\mathbf{Glob}(Z)^0 = \{0, 1\}, \mathbb{P}\mathbf{Glob}(Z) = \mathbb{P}_{0,1}\mathbf{Glob}(Z) = Z, s = 0, t = 1.$$

*This flow has no composition law.*

**2.3. Notation.** Let  $n \geq 1$ . Denote by  $\mathbf{D}^n = \{b \in \mathbb{R}^n, |b| \leq 1\}$  the  $n$ -dimensional disk, and by  $\mathbf{S}^{n-1} = \{b \in \mathbb{R}^n, |b| = 1\}$  the  $(n-1)$ -dimensional sphere. By convention, let  $\mathbf{D}^0 = \{0\}$  and  $\mathbf{S}^{-1} = \emptyset$ .

We need to recall:

**2.4. Theorem.** With  $r \in \{q, m, h\}$ . Then there exists a unique model structure on **Flow** such that:

- A map of flows  $f : X \rightarrow Y$  is a weak equivalence if and only if  $f^0 : X^0 \rightarrow Y^0$  is a bijection and for all  $(\alpha, \beta) \in X^0 \times X^0$ , the continuous map  $\mathbb{P}_{\alpha, \beta} X \rightarrow \mathbb{P}_{f(\alpha), f(\beta)} Y$  is a weak equivalence of the  $r$ -model structure of **Top**.
- A map of flows  $f : X \rightarrow Y$  is a fibration if and only if for all  $(\alpha, \beta) \in X^0 \times X^0$ , the continuous map  $\mathbb{P}_{\alpha, \beta} X \rightarrow \mathbb{P}_{f(\alpha), f(\beta)} Y$  is a fibration of the  $r$ -model structure of **Top**.

This model structure is accessible and all objects are fibrant. Moreover, this model structure is simplicial. It is called the  $r$ -model structure of **Flow**.

*Proof.* It is [14, Theorem 7.4] except the last sentence. The fact that **Flow** is enriched, tensored and cotensored over simplicial sets is proved in [9, Section 3.3]. The  $q$ -model structure of flows is simplicial by [9, Theorem 3.3.15]. It remains to prove the compatibility with the  $m$ -model structure and the  $h$ -model structure. It suffice to prove (see the very end of the proof of [9, Proposition 3.3.14]) that the lift  $k'$  of the commutative square

$$\begin{array}{ccc} (\mathbf{D}^n \times |\Delta[1]|) \cup (\mathbf{D}^n \times [0, 1] \times \{-1, 1\}) & \longrightarrow & \mathbb{P}_{\alpha, \beta} X \\ \downarrow & \nearrow^{k'} & \downarrow \\ \mathbf{D}^n \times [0, 1] \times |\Delta[1]| & \longrightarrow & \mathbb{P}_{\alpha, \beta} Y \end{array}$$

exists if the map  $\mathbb{P}_{\alpha, \beta} X \rightarrow \mathbb{P}_{\alpha, \beta} Y$  is a  $m$ -fibration of spaces or a  $h$ -fibration of spaces. Since every  $m$ -fibration of spaces and every  $h$ -fibration of spaces is a  $q$ -fibration of spaces, the proof is complete.  $\square$

By [14, Theorem 7.7], the  $m$ -model structure is the mixing of the  $q$ -model structure and the  $h$ -model structure in the sense of [4, Theorem 2.1]. The  $q$ -model structure is not only accessible, but also combinatorial. A set of generating cofibrations is the set of maps  $\{\text{Glob}(\mathbf{S}^{n-1}) \subset \text{Glob}(\mathbf{D}^n) \mid n \geq 0\} \cup \{C : \emptyset \rightarrow \{0\}, R : \{0, 1\} \rightarrow \{0\}\}$  by e.g. [14, Theorem 7.6]. Every  $q$ -cofibration of flows is a  $m$ -cofibration and every  $m$ -cofibration of flows is a  $h$ -cofibration by [4, Proposition 3.6].

There exists a flow which is not cofibrant in any of the three model structures by [4, Proposition 7.9]. This behaviour differs from the behaviour of the  $h$ -model structure of topological spaces for which all spaces are  $h$ -cofibrant (and  $h$ -fibrant). The reason is that the  $h$ -model structure of flows does not coincide with the Hurewicz model structure given by [1, Corollary 5.23]. This one exists as well because **Flow** satisfies the monomorphism hypothesis, being locally presentable, and because **Flow** is topologically bicomplete (the proof is similar to the proof that it is simplicial as given in [9, Section 3.3]) since a  $\Delta$ -generated space is homeomorphic to the disjoint sum of its path-connected components by [11, Proposition 2.8]. This Hurewicz model structure is not used in this paper.

### 3. REALIZATION FUNCTORS FROM PRECUBICAL SETS TO FLOWS

**3.1. Notation.** Let  $[0] = \{()\}$  and  $[n] = \{0, 1\}^n$  for  $n \geq 1$ . By convention, one has  $\{0, 1\}^0 = [0] = \{()\}$ . The set  $[n]$  is equipped with the product ordering  $\{0 < 1\}^n$ . Let  $0_n = (0, \dots, 0) \in \{0, 1\}^n$  and  $1_n = (1, \dots, 1) \in \{0, 1\}^n$

Let  $\delta_i^\alpha : [n-1] \rightarrow [n]$  be the set map defined for  $1 \leq i \leq n$  and  $\alpha \in \{0, 1\}$  by  $\delta_i^\alpha(\epsilon_1, \dots, \epsilon_{n-1}) = (\epsilon_1, \dots, \epsilon_{i-1}, \alpha, \epsilon_i, \dots, \epsilon_{n-1})$ . The small category  $\square$  is by definition the subcategory of the category of sets with the set of objects  $\{[n], n \geq 0\}$  and generated by the morphisms  $\delta_i^\alpha$ . They satisfy the cocubical relations  $\delta_j^\beta \delta_i^\alpha = \delta_i^\alpha \delta_{j-1}^\beta$  for  $i < j$  and for all  $(\alpha, \beta) \in \{0, 1\}^2$ . If  $p > q \geq 0$ , then the set of morphisms  $\square([p], [q])$  is empty. If  $p = q$ , then the set  $\square([p], [p])$  is the singleton  $\{\text{Id}_{[p]}\}$ . For  $0 \leq p \leq q$ , all maps of  $\square$  from  $[p]$  to  $[q]$  are one-to-one. A good reference for presheaves is [19].

**3.2. Definition.** [2] The category of presheaves over  $\square$ , denoted by  $\square^{op}\mathbf{Set}$ , is called the category of precubical sets. A precubical set  $K$  consists of a family of sets  $(K_n)_{n \geq 0}$  and of set maps  $\partial_i^\alpha : K_n \rightarrow K_{n-1}$  with  $1 \leq i \leq n$  and  $\alpha \in \{0, 1\}$  satisfying the cubical relations  $\partial_i^\alpha \partial_j^\beta = \partial_{j-1}^\beta \partial_i^\alpha$  for any  $\alpha, \beta \in \{0, 1\}$  and for  $i < j$ . An element of  $K_n$  is called a  $n$ -cube. Let  $\dim(x) = n$  if  $x \in K_n$ .

Let  $\square[n] := \square(-, [n])$ . The boundary of  $\square[n]$  is the precubical set denoted by  $\partial\square[n]$  defined by removing the interior of  $\square[n]$ :  $(\partial\square[n])_k := (\square[n])_k$  for  $k < n$  and  $(\partial\square[n])_k = \emptyset$  for  $k \geq n$ . In particular, one has  $\partial\square[0] = \emptyset$ .

**3.3. Definition.** A cocubical object of a category  $\mathcal{C}$  is a functor  $\square \rightarrow \mathcal{C}$ .

**3.4. Notation.** Let  $\mathcal{C}$  be a cocomplete category. Let  $X : \square \rightarrow \mathcal{C}$  be a functor. Let

$$\widehat{X}(K) = \varinjlim_{\square[n] \rightarrow K} X(\square[n]).$$

**3.5. Proposition.** [9, Proposition 2.3.2] Let  $\mathcal{C}$  be a cocomplete category. The mapping  $X \mapsto \widehat{X}$  induces an equivalence of categories between the category of cocubical objects of  $\mathcal{C}$  and the colimit-preserving preserving functors from  $\square^{op}\mathbf{Set}$  to  $\mathcal{C}$ .

Definition 3.6 is new. Moreover, only  $q$ -realization functors are implicitly studied in [10] because the  $h$ -model structure and the  $m$ -model structure of flows were not yet known: they are introduced 13 years later in [14].

**3.6. Definition.** With  $r \in \{q, m, h\}$ . A functor  $F : \square^{op}\mathbf{Set} \rightarrow \mathbf{Flow}$  is a  $r$ -realization functor if it satisfies the following properties:

- $F$  is colimit-preserving.
- For all  $n \geq 0$ , the map of flows  $F(\partial\square[n]) \rightarrow F(\square[n])$  is a  $r$ -cofibration.
- There is an objectwise weak equivalence of cocubical flows  $F(\square[*]) \rightarrow \{0 < 1\}^*$  in the  $r$ -model structure of  $\mathbf{Flow}$ .

**3.7. Proposition.** With  $r \in \{q, m, h\}$ . Let  $F : \square^{op}\mathbf{Set} \rightarrow \mathbf{Flow}$  be a  $r$ -realization functor. Then for all precubical sets  $K$ , the flow  $F(K)$  is  $r$ -cofibrant and there is a natural bijection  $K_0 \cong F(K)^0$ .

*Proof.* Let  $K$  be a precubical set. Then the canonical map  $\emptyset \rightarrow K$  is a transfinite composition of pushouts of the maps  $\partial\square[n] \rightarrow \square[n]$  for  $n \geq 0$ . Consequently, the canonical map

$\emptyset \rightarrow F(K)$  is a transfinite composition of pushouts of the maps  $F(\partial\Box[n]) \rightarrow F(\Box[n])$  for  $n \geq 0$ . It implies that  $F(K)$  is r-cofibrant. From the objectwise weak equivalence of cocubical flows  $F(\Box[*]) \rightarrow \{0 < 1\}^*$ , we deduce the objectwise bijection of cocubical sets  $F(\Box[*])^0 \cong \{0, 1\}^* \cong \Box[*]_0$ . We obtain the natural bijection  $F(K)^0 \cong K_0$ .  $\square$

**3.8. Theorem.** *With  $r \in \{q, m, h\}$ . Consider two r-realization functors*

$$F_1, F_2 : \Box^{op}\mathbf{Set} \longrightarrow \mathbf{Flow}.$$

*Then there exists a natural transformation  $\mu : F_1 \Rightarrow F_2$  such that there is a commutative diagram of cocubical flows*

$$\begin{array}{ccc} F_1(\Box[*]) & \xrightarrow{\mu_{\Box[*]}} & F_2(\Box[*]) \\ \downarrow & & \downarrow \\ \{0 < 1\}^* & \xlongequal{\quad} & \{0 < 1\}^* \end{array}$$

*and such that for all precubical sets  $K$ , the map  $\mu_K : F_1(K) \rightarrow F_2(K)$  natural with respect to  $K$  is a weak equivalence of the r-model structure of  $\mathbf{Flow}$ . Moreover, for all  $(\alpha, \beta) \in K_0 \times K_0$ , the natural map*

$$\mathbb{P}_{\alpha, \beta} F_1(K) \xrightarrow{\simeq} \mathbb{P}_{\mu_K(\alpha), \mu_K(\beta)} F_2(K)$$

*is a homotopy equivalence of spaces.*

*Proof.* The maps of cocubical flows  $F_i(\Box[*]) \rightarrow \{0 < 1\}^*$  for  $i = 1, 2$  are objectwise fibrations since  $\mathbb{P}_{\alpha, \beta}\{0 < 1\}^*$  is empty or equal to a singleton and because all topological spaces are fibrant. Consequently, they are objectwise trivial fibrations by definition of a realization functor. By [9, Theorem 2.3.3], there exists a natural transformation  $\mu : F_1 \Rightarrow F_2$  such that there is the commutative diagram of cocubical flows depicted in the statement of the theorem and such that, for all precubical sets  $K$ , the natural map  $\mu_K : F_1(K) \rightarrow F_2(K)$  is a weak equivalence of the r-model structure between two r-cofibrant flows. If  $r = h$ , then the natural map  $\mathbb{P}_{\alpha, \beta} F_1(K) \rightarrow \mathbb{P}_{\mu_K(\alpha), \mu_K(\beta)} F_2(K)$  is a homotopy equivalence of spaces by definition of the weak equivalences of the h-model structure of flows. If  $r = q$ , then the flows  $F_1(K)$  and  $F_2(K)$  are q-cofibrant by Proposition 3.7. Therefore, the spaces  $\mathbb{P}_{\alpha, \beta} F_1(K)$  and  $\mathbb{P}_{\mu_K(\alpha), \mu_K(\beta)} F_2(K)$  are q-cofibrant by [13, Theorem 5.7]. Using Whitehead [18, Theorem 7.5.10], we deduce that the natural map  $\mathbb{P}_{\alpha, \beta} F_1(K) \rightarrow \mathbb{P}_{\mu_K(\alpha), \mu_K(\beta)} F_2(K)$  is a homotopy equivalence of spaces. It remains the case  $r = m$ . The flows  $F_1(K)$  and  $F_2(K)$  are m-cofibrant by Proposition 3.7. We deduce that the spaces  $\mathbb{P}_{\alpha, \beta} F_1(K)$  and  $\mathbb{P}_{\mu_K(\alpha), \mu_K(\beta)} F_2(K)$  are m-cofibrant by [14, Theorem 8.7]. By [4, Corollary 3.4], we deduce that the weak homotopy equivalence  $\mathbb{P}_{\alpha, \beta} F_1(K) \rightarrow \mathbb{P}_{\mu_K(\alpha), \mu_K(\beta)} F_2(K)$  is a homotopy equivalence of spaces as well.  $\square$

**3.9. Theorem.** *There exists a q-realization functor  $|-|_q : \Box^{op}\mathbf{Set} \rightarrow \mathbf{Flow}$ .*

*Proof.* Let  $(-)^{cof}$  be a q-cofibrant replacement functor of  $\mathbf{Flow}$ . Let

$$|K|_q := \varinjlim_{\Box[n] \rightarrow K} (\{0 < 1\}^n)^{cof}.$$

It is a q-realization functor by [10, Proposition 7.4].  $\square$



3.10. **Remark.** *The functor*

$$|-|_{bad} : K \mapsto \varinjlim_{\square[n] \rightarrow K} \{0 < 1\}^n$$

is not a  $q$ -realization functor since the map of flows  $|\partial\square[2]|_{bad} \rightarrow |\square[2]|_{bad}$  is not a  $q$ -cofibration of flows.

Moreover, there is the isomorphism  $|\partial\square[n]|_{bad} \cong |\partial\square[n]|_{bad}$  for all  $n \geq 3$  by [10, Theorem 7.1], which is not the expected behavior for a realization functor.

Every  $q$ -realization functor is a  $m$ -realization functor and every  $m$ -realization functor is a  $h$ -realization functor. The drawback of the construction of Theorem 3.9 is that it depends on the non-canonical choice of a  $q$ -cofibrant replacement. It is one of the purpose of the paper to fix this issue.

The following theorem is not used in the sequel. It helps the reader to understand the geometric contents of a  $q$ -realization functor.

3.11. **Theorem.** [9, Theorem 4.2.4 and Theorem 4.2.6] *For all  $n \geq 1$ , there is a homotopy pushout diagram of flows for the  $q$ -model structure*

$$\begin{array}{ccc} \text{Glob}(\mathbf{S}^{n-2}) & \xrightarrow{\begin{smallmatrix} 0 \mapsto 0_n \\ 1 \mapsto 1_n \end{smallmatrix}} & |\partial\square[n]|_q \\ \downarrow & & \downarrow \\ \text{Glob}(\mathbf{D}^{n-1}) & \xrightarrow{\quad h \quad} & |\square[n]|_q. \end{array}$$

There exists a  $q$ -realization functor such that the pushout diagram above is strict.

#### 4. NATURAL $d$ -PATHS

We want to use the notion of natural  $d$ -path introduced by Raussen in [21, Definition 2.14] to build the natural realization functor from precubical sets to flows.

This new realization functor is natural in the sense that it uses natural  $d$ -paths, and also natural in the sense that it is more canonical than the  $q$ -realization functor of Theorem 3.9. Indeed, the latter depends on the non-canonical choice of a  $q$ -cofibrant replacement functor for the category of flows. The new one is independent of such a non-canonical choice.

4.1. **Notation.** *Let  $\delta_i^\alpha : [0, 1]^{n-1} \rightarrow [0, 1]^n$  be the continuous map defined for  $1 \leq i \leq n$  and  $\alpha \in \{0, 1\}$  by  $\delta_i^\alpha(\epsilon_1, \dots, \epsilon_{n-1}) = (\epsilon_1, \dots, \epsilon_{i-1}, \alpha, \epsilon_i, \dots, \epsilon_{n-1})$ . By convention, let  $[0, 1]^0 = \{()\}$ . We obtain a cocubical topological space  $[0, 1]^*$  and the associated colimit-preserving functor from precubical sets to topological spaces is denoted by*

$$|K|_{geom} = \varinjlim_{\square[n] \rightarrow K} [0, 1]^n.$$

The topological space  $|K|_{geom}$  is a cellular topological space. Every point of  $|K|_{geom} \setminus K_0$  admits a unique presentation  $[c; x]$  where  $c$  is a cube of  $K$  and such that  $x \in ]0, 1[^{\dim(c)}$ . A point of  $|K|_{geom}$  may belong to several cubes and therefore admits several presentations  $[c, x]$  with  $x \in [0, 1]^{\dim(c)}$ .

**4.2. Definition.** Let  $U$  be a topological space. A (Moore) path of  $U$  consists of a continuous map  $[0, \ell] \rightarrow U$  with  $\ell > 0$ . The real number  $\ell > 0$  is called the length of the path.

**4.3. Definition.** Let  $\gamma_1 : [0, \ell_1] \rightarrow U$  and  $\gamma_2 : [0, \ell_2] \rightarrow U$  be two paths of a topological space  $U$  such that  $\gamma_1(\ell_1) = \gamma_2(0)$ . The Moore composition  $\gamma_1 * \gamma_2 : [0, \ell_1 + \ell_2] \rightarrow U$  is the Moore path defined by

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(t) & \text{for } t \in [0, \ell_1] \\ \gamma_2(t - \ell_1) & \text{for } t \in [\ell_1, \ell_1 + \ell_2]. \end{cases}$$

The Moore composition of Moore paths is strictly associative.

**4.4. Definition.** Let  $n \geq 1$ . A  $d$ -path of  $|\square[n]|_{\text{geom}} = [0, 1]^n$  is a nonconstant continuous map  $\gamma : [0, \ell] \rightarrow [0, 1]^n$  with  $\ell > 0$  such that  $\gamma(0), \gamma(\ell) \in \{0, 1\}^n$  and such that  $\gamma$  is nondecreasing with respect to each axis of coordinates.

**4.5. Definition.** Let  $K$  be a general precubical set. A  $d$ -path of  $K$  is a path  $[0, \ell] \rightarrow |K|_{\text{geom}}$  which is the Moore composition  $\gamma_1 * \dots * \gamma_n$  of  $d$ -paths of cubes of  $|K|_{\text{geom}}$ .  $\gamma(0) \in K_0$  is called the initial state of  $\gamma$  and  $\gamma(\ell) \in K_0$  is called the final state of  $\gamma$ .

**4.6. Remark.** All  $d$ -paths of a precubical set  $K$  start and end at a vertex of  $K$ .

**4.7. Notation.** With the notations of Definition 4.5. A  $d$ -path  $\gamma : [0, \ell] \rightarrow |K|_{\text{geom}}$  can be written  $\gamma = {}^0[c_1; \gamma_1] \overset{t_1}{*} \dots \overset{t_{n-1}}{*} [c_n; \gamma_n]^{t_n}$  or  $\gamma = [c_1; \gamma_1] * \dots * [c_n; \gamma_n]$  with  $0 = t_0 < t_1 < \dots < t_n = \ell$  such that for all  $1 \leq i \leq n$  and  $t \in [t_{i-1}, t_i]$ ,  $\gamma(t) = [c_i; \gamma_i(t)]$  with  $\dim(c_i) \geq 1$  and such that  $\gamma(t_i) \in K_0$  for  $0 \leq i \leq n$ . The sequence  $(c_1, \dots, c_n)$  is called a carrier of  $\gamma$ . The notation  $\text{Carrier}(\gamma)$  means that a carrier of  $\gamma$  is chosen: it is not unique.

An important feature shared by all  $d$ -paths of a precubical set  $K$  is that they have a well-defined  $L_1$ -arc length [21, Section 2.2.1]. Intuitively, the natural  $d$ -paths are the  $d$ -paths whose speed corresponds to the  $L_1$ -arc length. We give an explicit definition of a natural  $d$ -path which is sufficient for this paper by starting from the  $d$ -paths of the topological  $n$ -cube  $[0, 1]^n$ . It is equivalent to Raussen's definition of tame natural  $d$ -paths.

**4.8. Definition.** Let  $n \geq 1$ . A natural  $d$ -path of the topological  $n$ -cube  $[0, 1]^n$  is a  $d$ -path  $\gamma = (\gamma_1, \dots, \gamma_n) : [0, n] \rightarrow [0, 1]^n$  such that for all  $t \in [0, n]$ , one has  $t = \gamma_1(t) + \dots + \gamma_n(t)$ . The set of natural  $d$ -paths of  $[0, 1]^n$  is denoted by  $N_n$ . It is equipped with the compact-open topology.

**4.9. Definition.** A  $d$ -path  $\gamma$  of a precubical set  $K$  is natural if it can be written  $\gamma = [c_1; \gamma_1] * \dots * [c_n; \gamma_n]$  such that each  $\gamma_i$  is a natural  $d$ -path of the cube  $c_i$  for all  $i \in \{1, \dots, n\}$ .

**4.10. Proposition.** Let  $n \geq 1$ . The topological space  $N_n$  is  $\Delta$ -generated and  $\Delta$ -Hausdorff. It is contractible and compact.

*Proof.* The compact-open topology is metrizable with the distance of the uniform convergence. Therefore it is first countable. Consider a ball  $B(\gamma, \epsilon)$  for this metric. Let  $\gamma' \in B(\gamma, \epsilon)$ . Then each convex combination  $(1 - u)\gamma + u\gamma'$  is a natural  $d$ -path since  $(1 - u)t + ut = t$  and for all  $t \in [0, n]$  and all  $i \in \{1, \dots, n\}$ , one has

$$|((1 - u)\gamma_i + u\gamma'_i)(t) - \gamma_i(t)| = u|\gamma'_i(t) - \gamma_i(t)| < u\epsilon \leq \epsilon.$$

It means that the space  $N_n$  is locally path-connected. By [3, Proposition 3.11], it is  $\Delta$ -generated, and also  $\Delta$ -Hausdorff, being metrizable. It is contractible since there is a homotopy  $H : [0, 1] \times N_n \rightarrow N_n$  between the identity of  $N_n$  and the constant map taking each natural  $d$ -path to the natural  $d$ -path  $\delta : t \mapsto (t/n, t/n, \dots, t/n)$  given by the convex combination  $H(u, \gamma) = u\delta + (1 - u)\gamma$ . It is compact by [26, Proposition 9.5] applied to the sequence  $\underline{n} = (n)$ . We want to give a different argument which does not use Lipschitz maps on metric spaces. Let  $(\gamma^k)_{k \geq 0} = (\gamma_1^k, \dots, \gamma_n^k)_{k \geq 0}$  be a sequence of  $N_n$ . By a Cantor diagonalization argument, one can suppose that the sequence  $(\gamma^k(r))_{k \geq 0}$  of  $[0, 1]^n$  converges to  $(\gamma_1^\infty(r), \dots, \gamma_n^\infty(r))$  for all  $r \in \mathbb{Q} \cap [0, n]$ . Let  $\gamma_i^-(x) = \sup\{\gamma_i^\infty(r) \mid r \in \mathbb{Q} \cap [0, x]\}$  and  $\gamma_i^+(x) = \inf\{\gamma_i^\infty(r) \mid r \in \mathbb{Q} \cap [x, n]\}$ . Then, by density of  $\mathbb{Q}$ , for all  $x \in [0, n]$ , one has  $(\gamma_1^+(x) - \gamma_1^-(x)) + \dots + (\gamma_n^+(x) - \gamma_n^-(x)) = 0$ . Thus, for all  $x \in [0, n]$  and for all  $1 \leq i \leq n$ , since  $\gamma_i^+(x) - \gamma_i^-(x) \geq 0$ , we deduce that  $\gamma_i^+(x) = \gamma_i^-(x)$ . It means that  $\gamma_i^\pm = \gamma_i^- : [0, n] \rightarrow [0, 1]$  is continuous for all  $i \in \{1, \dots, n\}$ . Consequently, each sequence  $(\gamma_i^k)_{k \geq 0}$  converges pointwise for  $1 \leq i \leq n$ . By the second Dini theorem, the convergence is uniform. Using [12, Lemma 6.10], we deduce that  $(\gamma^k)_{k \geq 0}$  has a convergent subsequence. We deduce that  $N_n$  is sequentially compact, hence compact, being sequential.  $\square$

**4.11. Notation.** Let  $\underline{x} = (x_1, \dots, x_n)$  and  $\underline{x}' = (x'_1, \dots, x'_n)$  be two elements of  $[0, 1]^n$ . Let

$$d_\infty(\underline{x}, \underline{x}') = \max_{1 \leq i \leq n} |x_i - x'_i|.$$

**4.12. Definition.** Let  $n \geq 2$ . Let  $\mathcal{V}_n = \{0, 1\}^n \setminus \{0_n, 1_n\}$ . Consider the continuous map  $\phi : N_n \rightarrow [0, 1]$  defined by  $\phi(\gamma) = \min_{(t,v) \in [0,n] \times \mathcal{V}_n} d_\infty(\gamma(t), v)$ . Let  $\partial N_n = \phi^{-1}(0)$  equipped with the relative topology.

**4.13. Notation.** Let  $\partial N_0 = N_0 = \partial N_1 = \emptyset$ .

There is the proposition:

**4.14. Proposition.** Let  $n \geq 2$ . The underlying set of  $\partial N_n$  is exactly the set of Moore compositions of natural  $d$ -paths of subcubes of  $[0, 1]^n$ . For every  $\gamma \in \partial N_n$ ,  $\gamma([0, n])$  is included in the boundary of  $[0, 1]^n$ . The topology of  $\partial N_n$  is  $\Delta$ -generated and  $\Delta$ -Hausdorff.

*Proof.* The set  $\partial N_n$  is exactly the set of natural paths of  $[0, 1]^n$  whose image intersects  $\mathcal{V}_n$ . Let  $\gamma = (\gamma_1, \dots, \gamma_n) \in \partial N_n$  and let  $t_0 \in ]0, n[$  such that  $\gamma(t_0) = (\epsilon_1, \dots, \epsilon_n) \in \mathcal{V}_n$ . Since  $\gamma$  is natural, one has  $t_0 = \epsilon_1 + \dots + \epsilon_n$  which is therefore an integer between 1 and  $n - 1$ . Then  $\gamma = \gamma^a * \gamma^b$  with  $\gamma^a(0) = 0_n$ ,  $\gamma^a(t_0) = \gamma^b(0) \in \mathcal{V}_n$  and  $\gamma^b(n - t_0) = 1_n$ . Therefore, for all  $t \in [0, t_0]$ ,  $t = \gamma_1(t) + \dots + \gamma_n(t) = \gamma_1^a(t) + \dots + \gamma_n^a(t)$ . Let  $J = \{j \in \{1, \dots, n\} \mid \epsilon_j = 0\}$ . Since the paths are nondecreasing with respect to each axis of coordinates, it implies that  $\gamma_j^a(t) = 0$  for all  $j \in J$ . Thus for all  $t \in [0, t_0]$ ,  $t = \sum_{j \notin J} \gamma_j^a(t)$ . It means that  $\gamma^a$  is a natural path of the subcube from  $0_n$  to  $\gamma^a(t_0)$ . For all  $t \in [t_0, n]$ , on has  $t = \gamma_1(t) + \dots + \gamma_n(t) = \gamma_1^b(t - t_0) + \dots + \gamma_n^b(t - t_0)$ , the first equality since  $\gamma$  is natural, the second equality by definition of the Moore composition of paths. We deduce that  $t - t_0 = (\gamma_1^b(t - t_0) - \epsilon_1) + \dots + (\gamma_n^b(t - t_0) - \epsilon_n)$  for all  $t \in [t_0, n]$ . If for some  $i \in \{1, \dots, n\}$ ,  $\epsilon_i = 1$ , then  $\gamma_i^b = 1$  since the paths are nondecreasing with respect to each axis of coordinates. We obtain  $t - t_0 = \sum_{j \in J} \gamma_j^b(t - t_0)$  for all  $t \in [t_0, n]$ . It means that  $\gamma^b$  is a natural path of the subcube going from  $\gamma(t_0)$  to  $1_n$ . We deduce that the underlying set of  $\partial N_n$  is exactly the set of Moore compositions of natural paths of

subcubes of  $[0, 1]^n$ . The second assertion is a consequence of this fact. Consider  $\gamma \in \partial N_n$ . There exists  $t_0 \in ]0, n[$  such that  $\gamma(t_0) \in \mathcal{V}_n$ . Since  $\mathcal{V}_n$  is discrete, there exists an open  $U$  of  $[0, 1]^n$  such that  $U \cap \mathcal{V}_n = \{\gamma(t_0)\}$ . Then  $W(\{t_0\}, U) = \{\gamma' \in \partial N_n \mid \gamma'(t_0) \in U\}$  is an open subset of  $\partial N_n$  for the compact-open topology. The latter being metrizable, take a ball  $B(\gamma, \epsilon) \subset W(\{t_0\}, U)$  and repeat the reasoning of Proposition 4.10: we deduce that  $\partial N_n$  is locally path-connected as well, hence  $\Delta$ -generated (and also  $\Delta$ -Hausdorff, being metrizable) by [3, Proposition 3.11].  $\square$

In other terms,  $\partial N_n$  is a closed subset of  $N_n$  which remains  $\Delta$ -generated and also  $\Delta$ -Hausdorff when equipped with the relative topology. Both  $\partial N_n$  and  $N_n$  are equipped with the compact-open topology and are metrizable and sequentially compact.

## 5. NATURAL REALIZATION FROM PRECUBICAL SETS TO FLOWS

We define a flow  $|\square[n]|_{nat}$  for  $n \geq 0$  called *the natural  $n$ -cube* as follows. The set of states is  $\{0, 1\}^n$ . Let  $n \geq 1$  and  $\alpha, \beta \in \{0, 1\}^n$ . Let  $\alpha = (\alpha_1, \dots, \alpha_n)$  and  $\beta = (\beta_1, \dots, \beta_n)$ . Assume that  $\alpha < \beta$ . Let  $I = \{i \in \{1, \dots, n\} \mid \alpha_i \neq \beta_i\}$ . By hypothesis,  $I$  is nonempty. Let  $m$  be the cardinal of  $I$ . Then  $\alpha$  ( $\beta$  resp.) is the initial (final resp.) state of a  $m$ -subcube of  $\square[n]$ . Then let  $\mathbb{P}_{\alpha, \beta}|\square[n]|_{nat} = N_m$  viewed as the space of natural  $d$ -paths in the  $m$ -subcube from  $\alpha$  to  $\beta$ . Assume that  $\alpha \geq \beta$ . Let  $\mathbb{P}_{\alpha, \beta}|\square[n]|_{nat} = \emptyset$ . The composition law is defined by the Moore composition of natural  $d$ -paths, which is still a natural  $d$ -path.

**5.1. Proposition.** *Let  $\phi : [n] \rightarrow [n + 1]$  be a map of the small category  $\square$ . Then the continuous map  $\mathbb{P}_{0_n, 1_n}|\square[n]|_{nat} \rightarrow \mathbb{P}_{\phi(0_n), \phi(1_n)}|\square[n + 1]|_{nat}$  induced by  $\phi$  is the identity of  $N_n$ .*

*Proof.* It is a straightforward consequence of the definitions.  $\square$

**5.2. Corollary.** *We obtain a well-defined cocubical flow  $|\square[*]|_{nat}$ .*

*Proof.* Consider an algebraic relation  $\phi_1\phi_2 = \psi_1\psi_2 : [n] \rightarrow [n + 2]$  in the small category  $\square$ . Consider the diagram of topological spaces

$$\begin{array}{ccccc} \mathbb{P}_{0_n, 1_n}|\square[n]|_{nat} & \longrightarrow & \mathbb{P}_{\phi_2(0_n), \phi_2(1_n)}|\square[n + 1]|_{nat} & \longrightarrow & \mathbb{P}_{\phi_1\phi_2(0_n), \phi_1\phi_2(1_n)}|\square[n + 2]|_{nat} \\ \parallel & & & & \parallel \\ \mathbb{P}_{0_n, 1_n}|\square[n]|_{nat} & \longrightarrow & \mathbb{P}_{\psi_2(0_n), \psi_2(1_n)}|\square[n + 1]|_{nat} & \longrightarrow & \mathbb{P}_{\psi_1\psi_2(0_n), \psi_1\psi_2(1_n)}|\square[n + 2]|_{nat} \end{array}$$

By Proposition 5.1 and by definition of  $|\_ |_{nat}$ , the two horizontal composite maps are equal to the identity of  $N_n$ . It means that the diagram is commutative and that the cocubical relations are satisfied.  $\square$

Using Proposition 3.5, we obtain:

**5.3. Definition.** *Let  $K$  be a precubical set. Consider the colimit-preserving functor*

$$|K|_{nat} = \varinjlim_{\square[n] \rightarrow K} |\square[n]|_{nat}.$$

*It is called the natural realization of  $K$  as a flow.*

**5.4. Proposition.** *Let  $n \geq 0$ . There is a homeomorphism  $\partial N_n \cong \mathbb{P}_{0_n, 1_n}|\partial\square[n]|_{nat}$ .*

*Proof.* Using [26, Proposition 10.2] applied with the sequence  $(n)$ , we deduce that this map is a homeomorphism: the idea of the proof is that there is a continuous bijection from  $\mathbb{P}_{0_n,1_n}|\partial\Box[n]|_{nat}$  to  $\partial N_n$  and that both  $\mathbb{P}_{0_n,1_n}|\partial\Box[n]|_{nat}$  and  $\partial N_n$  are compact.  $\square$

**5.5. Proposition.** *The continuous map  $\partial N_n \subset N_n$  is a h-cofibration of spaces for all  $n \geq 0$ .*

*Proof.* Using [26, Proposition 10.3] applied with the sequence  $(n)$ , we deduce that this map is a strong neighborhood deformation retract, i.e. a h-cofibration by [24, Theorem 2].  $\square$

**5.6. Corollary.** *The map of flows  $\text{Glob}(\partial N_n) \subset \text{Glob}(N_n)$  is a h-cofibration of flows for all  $n \geq 0$ .*

*Proof.* A map of flows of the form  $\text{Glob}(U) \rightarrow \text{Glob}(V)$  satisfies the left lifting property with respect to a map of flows  $f : X \rightarrow Y$  if and only if the map  $U \rightarrow V$  satisfies the left lifting property with respect to all maps  $\mathbb{P}_{\alpha,\beta}X \rightarrow \mathbb{P}_{f(\alpha),f(\beta)}Y$  for all  $(\alpha, \beta) \in X^0 \times X^0$ . Using the characterization of the trivial h-fibrations of flows (see Theorem 2.4) and Proposition 5.5, we deduce that the map  $\text{Glob}(\partial N_n) \rightarrow \text{Glob}(N_n)$  is a h-cofibration of flows.  $\square$

**5.7. Proposition.** *For all  $n \geq 0$ , the map  $|\partial\Box[n]|_{nat} \rightarrow |\Box[n]|_{nat}$  is a h-cofibration of flows.*

*Proof.* From the homeomorphism of Proposition 5.4 and by definition of  $N_n$ , we deduce that the commutative diagram of spaces

$$\begin{array}{ccc} \partial N_n & \xrightarrow{\cong} & \mathbb{P}_{0_n,1_n}|\partial\Box[n]|_{nat} \\ \downarrow & & \downarrow \\ N_n & \xrightarrow{\cong} & \mathbb{P}_{0_n,1_n}|\Box[n]|_{nat} \end{array}$$

is a pushout diagram of spaces. The top homeomorphism yields a map of flows

$$\text{Glob}(\partial N_n) \longrightarrow |\partial\Box[n]|_{nat}$$

taking 0 to  $0_n$  and 1 to  $1_n$  for all  $n \geq 0$ . We obtain the pushout diagram of flows

$$\begin{array}{ccc} \text{Glob}(\partial N_n) & \longrightarrow & |\partial\Box[n]|_{nat} \\ \downarrow & & \downarrow \\ \text{Glob}(N_n) & \longrightarrow & |\Box[n]|_{nat} \end{array}$$

Using Corollary 5.6, we deduce that the map  $|\partial\Box[n]|_{nat} \rightarrow |\Box[n]|_{nat}$  is a h-cofibration of flows for all  $n \geq 0$ .  $\square$

**5.8. Theorem.** *There exists a natural transformation  $\mu : | - |_q \Rightarrow | - |_{nat}$  such that for all precubical sets  $K$ , the natural map  $\mu_K : |K|_q \rightarrow |K|_{nat}$  induces a bijection on states and a homotopy equivalence  $\mathbb{P}_{\alpha,\beta}|K|_q \simeq \mathbb{P}_{\alpha,\beta}|K|_{nat}$  for all  $\alpha, \beta \in K_0$ .*

*Proof.* The map  $|\Box[*]|_{nat} \rightarrow \{0 < 1\}^*$  is an objectwise weak equivalence for the h-model structure of **Flow** since all spaces  $N_n$  for  $n \geq 1$  are contractible by Proposition 4.10. By

Proposition 5.7, the natural realization functor is then a h-realization functor. The proof is complete thanks to Theorem 3.8.  $\square$

The statement of Theorem 3.8 being symmetric, there is also a natural transformation  $\nu : |-|_{nat} \Rightarrow |-|_q$  such that, for all precubical sets  $K$ , the natural map  $\nu_K : |K|_{nat} \rightarrow |K|_q$  induces a bijection on states and a homotopy equivalence  $\mathbb{P}_{\alpha,\beta}|K|_{nat} \simeq \mathbb{P}_{\alpha,\beta}|K|_q$  for all  $\alpha, \beta \in K_0$ . This statement is less intuitive because, morally speaking, the natural realization contains more execution paths than the q-realization.

Proposition 5.7 means that the natural realization functor is a h-realization functor. In fact, it is possible to prove better. For all precubical sets  $K$  and all  $(\alpha, \beta) \in K_0 \times K_0$ , there is a homotopy equivalence  $\mathbb{P}_{\alpha,\beta}|K|_{nat} \simeq \mathbb{P}_{\alpha,\beta}|K|_q$ . Since  $|K|_q$  is q-cofibrant, the space  $\mathbb{P}_{\alpha,\beta}|K|_q$  is q-cofibrant by [13, Theorem 5.7]. It means that the spaces of execution paths  $\mathbb{P}_{\alpha,\beta}|K|_{nat}$  are m-cofibrant for all  $(\alpha, \beta) \in K_0 \times K_0$ . This suggests that the natural realization  $|K|_{nat}$  is a m-cofibrant flow. Indeed we have the following theorem:

**5.9. Theorem.** *The natural realization functor is a m-realization functor. For any precubical set  $K$ , the flow  $|K|_{nat}$  is m-cofibrant.*

*Proof.* The map  $|\square[*]|_{nat} \rightarrow \{0 < 1\}^*$  is an objectwise weak equivalence for the h-model structure of **Flow**, and therefore for the m-model structure of **Flow** as well. There is a homeomorphism  $\partial N_n \cong \mathbb{P}_{0_n, 1_n}|\partial\square[n]|_{nat}$  (Proposition 5.4) and a homotopy equivalence  $\mathbb{P}_{0_n, 1_n}|\partial\square[n]|_{nat} \simeq \mathbb{P}_{0_n, 1_n}|\partial\square[n]|_q$  (Theorem 3.8). Since  $|\partial\square[n]|_q$  is a q-cofibrant flow by Proposition 3.7, the space  $\mathbb{P}_{0_n, 1_n}|\partial\square[n]|_q$  is q-cofibrant by [13, Theorem 5.7]. Moreover,  $N_n$  is contractible by Proposition 4.10, hence m-cofibrant. It implies that all maps  $\partial N_n \rightarrow N_n$  for  $n \geq 0$  are h-cofibrations of spaces between m-cofibrant spaces [4, Corollary 3.7]. By [4, Corollary 3.12], the maps  $\partial N_n \rightarrow N_n$  are therefore m-cofibrations of spaces for all  $n \geq 0$ . Thus, the map of flows  $\text{Glob}(\partial N_n) \rightarrow \text{Glob}(N_n)$  is a m-cofibration of flows for all  $n \geq 0$  by the same argument as in the proof of Corollary 5.6. Using the pushout diagram in the proof of Proposition 5.7, we deduce that the natural realization functor is a m-realization functor. By Proposition 3.7, we deduce that the flow  $|K|_{nat}$  is m-cofibrant.  $\square$

## 6. NATURAL REALIZATION AND CUBE CHAINS

Cube chains are introduced in [25, Definition 1.1]. We use the presentation given in [26, Section 7] instead. Let  $\text{Seq}(n)$  be the set of sequences of positive integers  $\underline{n} = (n_1, \dots, n_p)$  with  $n_1 + \dots + n_p = n$ . Let  $\underline{n} = (n_1, \dots, n_p) \in \text{Seq}(n)$ . Then  $|\underline{n}| = n$  is the length of  $\underline{n}$  and  $\ell(\underline{n}) = p$  is the number of elements of  $\underline{n}$ . Let  $K$  be a precubical set and  $A = a_1 < \dots < a_k \subset \{1, \dots, n\}$  and  $\epsilon \in \{0, 1\}$ . The *iterated face map* is defined by

$$\partial_A^\epsilon = \partial_{a_1}^\epsilon \partial_{a_2}^\epsilon \dots \partial_{a_k}^\epsilon.$$

**6.1. Definition.** *Let  $\underline{n} \in \text{Seq}(n)$ . The  $\underline{n}$ -cube is the precubical set*

$$\square[\underline{n}] = \square[n_1] * \dots * \square[n_p]$$

*where the notation  $*$  means that the final state  $1_{n_i}$  of the precubical set  $\square[n_i]$  is identified with the initial state  $0_{n_{i+1}}$  of the precubical set  $\square[n_{i+1}]$  for  $1 \leq i \leq p-1$ .*

Let  $K$  be a precubical set. Let  $\alpha, \beta \in K_0$ . Let  $n \geq 1$ . The category  $\text{Ch}_{\alpha,\beta}(K, n)$  is defined as follows. The objects are the maps of precubical sets  $\square[\underline{n}] \rightarrow K$  with  $|\underline{n}| = n$

where the initial state of  $\square[n_1]$  is mapped to  $\alpha$  and the final state of  $\square[n_p]$  is mapped to  $\beta$ . Let  $A \sqcup B = \{1, \dots, m_1 + m_2\}$  be a partition with the cardinal of  $A$  equal to  $m_1 > 0$  and the cardinal of  $B$  equal to  $m_2 > 0$ . Let

$$\phi_{A,B} : \square[m_1] * \square[m_2] \longrightarrow \square[m_1 + m_2]$$

be the unique map of precubical sets such that

$$\begin{aligned} \phi_{A,B}(\text{Id}_{[m_1]}) &= \partial_B^0(\text{Id}_{[m_1+m_2]}), \\ \phi_{A,B}(\text{Id}_{[m_2]}) &= \partial_A^1(\text{Id}_{[m_1+m_2]}). \end{aligned}$$

For  $i \in \{1, \dots, \ell(\underline{n})\}$  and a partition  $A \sqcup B = \{1, \dots, n_i\}$ , let

$$\delta_{i,A,B} = \text{Id}_{\square[n_1]} * \dots * \text{Id}_{\square[n_{i-1}]} * \phi_{A,B} * \text{Id}_{\square[n_{i+1}]} * \dots * \text{Id}_{\square[n_{\ell(\underline{n})}]}.$$

The morphisms are the commutative diagrams

$$\begin{array}{ccc} \square[\underline{n}_a] & \xrightarrow{a} & K \\ \downarrow & & \parallel \\ \square[\underline{n}_b] & \xrightarrow{b} & K \end{array}$$

where the left vertical map is a composite of maps of precubical sets of the form  $\delta_{i,A,B}$ .

From a precubical set  $K$ , we are going to define a flow  $\|K\|$  as follows. The set of states is  $K_0$ . Consider the small diagram of spaces

$$\mathcal{D}_{\alpha,\beta}(K, n) : \text{Ch}_{\alpha,\beta}(K, n) \longrightarrow \mathbf{Top}$$

defined by on objects by

$$\mathcal{D}_{\alpha,\beta}(K, n)(\square[\underline{n}] \rightarrow K) = N_{n_1} \times \dots \times N_{n_p}$$

and on morphisms by using the maps

$$\mathbb{P}|\phi_{A,B}|_{nat} : \mathbb{P}(\square[m_1] * \square[m_2]) \rightarrow \mathbb{P}(\square[m_1 + m_2])$$

which induce maps  $N_{m_1} \times N_{m_2} \rightarrow N_{m_1+m_2}$  given by the Moore composition of natural  $d$ -paths. The set of execution spaces  $\mathbb{P}_{\alpha,\beta}\|K\|$  is defined as follows:

$$\mathbb{P}_{\alpha,\beta}\|K\| = \bigsqcup_{n \geq 1} \varinjlim \mathcal{D}_{\alpha,\beta}(K, n).$$

It is easy to see that the concatenation of tuples induces functors

$$\mathcal{D}_{\alpha,\beta}(K, n_1) \times \mathcal{D}_{\beta,\gamma}(K, n_2) \rightarrow \mathcal{D}_{\alpha,\gamma}(K, n_1 + n_2),$$

and, using [13, Proposition A.4], continuous maps

$$\varinjlim \mathcal{D}_{\alpha,\beta}(K, n_1) \times \varinjlim \mathcal{D}_{\beta,\gamma}(K, n_2) \rightarrow \varinjlim \mathcal{D}_{\alpha,\gamma}(K, n_1 + n_2).$$

We obtain an associative composition map

$$\mathbb{P}_{\alpha,\beta}\|K\| \times \mathbb{P}_{\beta,\gamma}\|K\| \rightarrow \mathbb{P}_{\alpha,\gamma}\|K\|$$

for all  $(\alpha, \beta, \gamma) \in K_0 \times K_0 \times K_0$ .

**6.2. Proposition.** *There is an isomorphism of cocubical flows*

$$\|\square[*]\| \cong \|\square[*]\|_{nat}.$$

*Proof.* At first, we prove the isomorphism of flows  $|\square[n]| \cong |\square[n]|_{nat}$  by induction on  $n \geq 0$ . The statement is obvious for  $n = 0$ . Let  $n \geq 1$  and  $\alpha, \beta \in \{0, 1\}^n$  with  $\alpha < \beta$ . Let  $\alpha = (\alpha_1, \dots, \alpha_n)$  and  $\beta = (\beta_1, \dots, \beta_n)$ . Let

$$I = \{i \in \{1, \dots, n\} \mid \alpha_i \neq \beta_i\}.$$

By hypothesis,  $I$  is nonempty. Let  $m$  be the cardinal of  $I$ . Then  $\alpha$  ( $\beta$  resp.) is the initial (final resp.) state of a  $m$ -subcube  $\underline{c}$  of  $\square[n]$ . We deduce that the category  $\text{Ch}_{\alpha, \beta}(\square[n], p)$  is empty for  $p \neq m$  and that it has a terminal object  $\underline{c} : \square[m] \rightarrow \square[n]$  for  $p = m$  corresponding to the subcube from  $\alpha$  to  $\beta$ . We deduce the homeomorphisms

$$\begin{aligned} \mathbb{P}_{\alpha, \beta} |\square[n]| &= \varinjlim_{\substack{\underline{n}=(n_1, \dots, n_p), \ell(\underline{n})=m \\ \square[\underline{n}] \rightarrow \square[n] \in \text{Ch}_{\alpha, \beta}(\square[n], m)}} N_{n_1} \times \dots \times N_{n_p} \\ &\cong N_m \\ &= \mathbb{P}_{\alpha, \beta} |\square[n]|_{nat}, \end{aligned}$$

the first equality by definition of  $|\square[n]|$ , the homeomorphism because of the unique map  $\underline{c} : \square[m] \rightarrow \square[n]$  which is the terminal object of  $\text{Ch}_{\alpha, \beta}(\square[n], m)$ , and the last equality by Proposition 5.1 applied to the map  $\underline{c} : \square[m] \rightarrow \square[n]$ . By Proposition 5.1 again, the isomorphism  $|\square[n]| \cong |\square[n]|_{nat}$  is natural with respect to  $[n]$ .  $\square$

We do not know yet that the functor  $||-\|$  is colimit-preserving. An additional argument based on Proposition 5.1 as well is necessary for proving Theorem 6.3.

**6.3. Theorem.** *There is a natural isomorphism of flows  $||K|| \cong |K|_{nat}$  for all precubical sets  $K$ .*

*Proof.* Let  $\underline{n} = (n_1, \dots, n_p) \in \text{Seq}(n)$ . Every map of precubical sets  $\square[\underline{n}] \rightarrow K$  gives rise to a map of flows  $|\square[\underline{n}]|_{nat} \rightarrow |K|_{nat}$ , and therefore to a continuous map

$$N_{n_1} \times \dots \times N_{n_p} \longrightarrow \mathbb{P}|K|_{nat}.$$

Let  $\phi_{A, B} : \square[m_1] * \square[m_2] \rightarrow \square[m_1 + m_2]$  as above. A composite map of precubical sets  $\square[m_1] * \square[m_2] \rightarrow \square[m_1 + m_2] \rightarrow K$  gives rise to the commutative diagram of flows

$$\begin{array}{ccc} |\square[m_1] * \square[m_2]|_{nat} & \longrightarrow & |K|_{nat} \\ \downarrow & & \parallel \\ |\square[m_1 + m_2]|_{nat} & \longrightarrow & |K|_{nat} \end{array}$$

and therefore to the commutative diagram of spaces

$$\begin{array}{ccc} N_{m_1} \times N_{m_2} & \longrightarrow & \mathbb{P}|K|_{nat} \\ \downarrow & & \parallel \\ N_{m_1 + m_2} & \longrightarrow & \mathbb{P}|K|_{nat} \end{array}$$

Consequently, we obtain a cocone

$$(N_{n_1} \times \dots \times N_{n_p}) \begin{array}{c} \square[\underline{n}] \rightarrow K \\ \in \text{Ch}_{\alpha, \beta}(K, n) \end{array} \xrightarrow{\bullet} \mathbb{P}|K|_{nat}$$



and then a map of flows  $\|K\| \rightarrow |K|_{nat}$  which is bijective on states. For each map of precubical sets  $\square[n] \rightarrow K$ , we obtain by Proposition 6.2 a map of flows

$$|\square[n]|_{nat} \cong \|\square[n]\| \longrightarrow \|K\|.$$

Using Proposition 5.1, we obtain a cocone of flows

$$(|\square[n]|_{nat})_{\square[n] \rightarrow K} \xrightarrow{\bullet} \|K\|$$

and therefore a map of flows  $|K|_{nat} \rightarrow \|K\|$  such that the composite map  $|K|_{nat} \rightarrow \|K\| \rightarrow |K|_{nat}$  is the identity. Thus, the map  $\|K\| \rightarrow |K|_{nat}$  is onto on execution paths and the map  $|K|_{nat} \rightarrow \|K\|$  is one-to-one on execution paths. Consider an execution path  $\gamma$  of  $\|K\|$ . It belongs to a colimit and therefore has a representative  $(\gamma_1, \dots, \gamma_p)$  in a space of the form  $N_{n_1} \times \dots \times N_{n_p}$  corresponding to some map of precubical set  $\square[\underline{n}] \rightarrow K$  with  $\underline{n} = (n_1, \dots, n_p)$ . By Proposition 5.1, there exists an execution path  $\overline{\gamma}_1 * \dots * \overline{\gamma}_p$  of  $|K|_{nat}$  which is mapped to  $(\gamma_1, \dots, \gamma_p)$  by the map of flows  $|K|_{nat} \rightarrow \|K\|$ . The latter is therefore surjective on execution paths. And the proof is complete.  $\square$

## 7. APPLICATIONS

We can now give the applications of the previous results. We recall at first some basic facts about  $\Delta$ -inclusions.

**7.1. Definition.** *A one-to-one map of  $\Delta$ -generated spaces  $i : A \rightarrow B$  is a  $\Delta$ -inclusion if for all  $\Delta$ -generated spaces  $Z$ , the set map  $Z \rightarrow A$  is continuous if and only if the composite set map  $Z \rightarrow A \rightarrow B$  is continuous.*

**7.2. Proposition.** *Let  $i : A \rightarrow B$  be a one-to-one continuous map. The following assertions are equivalent:*

- (1)  *$i$  is a  $\Delta$ -inclusion.*
- (2)  *$A$  is homeomorphic to  $i(A)$  equipped with the  $\Delta$ -kelleyfication of the relative topology.*
- (3) *A set map  $[0, 1] \rightarrow A$  is continuous if and only if the composite set map  $[0, 1] \rightarrow A \rightarrow B$  is continuous.*

*Proof.* The proof is similar to the same statement for  $k$ -inclusions of  $k$ -spaces.  $\square$

**7.3. Corollary.** *A continuous bijection  $f : U \rightarrow V$  of **Top** is a homeomorphism if and only if it is a  $\Delta$ -inclusion.*

Let  $K$  be a general precubical set. The underlying set of the space  $\mathbb{P}_{\alpha, \beta}|K|_{nat}$  for  $(\alpha, \beta) \in K_0 \times K_0$  is exactly the set of natural  $d$ -paths of  $K$  from  $\alpha$  to  $\beta$ . The  $L_1$ -arc length of a  $d$ -path of  $\mathbb{P}_{\alpha, \beta}|K|_{nat}$  is always an integer because  $(\alpha, \beta) \in K_0 \times K_0$  and two natural  $d$ -paths which are in the same path-connected component have the same  $L_1$ -arc length. Therefore, the following definition makes sense.

**7.4. Definition.** *Let  $K$  be a precubical set. Let  $(\alpha, \beta) \in K_0 \times K_0$ . Denote by*

$$(\mathbb{P}_{\alpha, \beta}|K|_{nat})_{\Delta}$$

the underlying set of the space  $\mathbb{P}_{\alpha,\beta}|K|_{nat}$  equipped with the  $\Delta$ -kelleyfication of the relative topology induced by the set inclusion

$$\mathbb{P}_{\alpha,\beta}|K|_{nat} \subset \bigsqcup_{n \geq 1} \mathbf{TOP}([0, n], |K|_{geom}).$$

Since the space  $|K|_{geom}$  is Hausdorff, the space  $(\mathbb{P}_{\alpha,\beta}|K|_{nat})_{\Delta}$  is Hausdorff as well.

**7.5. Theorem.** *Let  $K$  be a precubical set. Let  $(\alpha, \beta) \in K_0 \times K_0$ . There is a homeomorphism*

$$\mathbb{P}_{\alpha,\beta}|K|_{nat} \cong (\mathbb{P}_{\alpha,\beta}|K|_{nat})_{\Delta}.$$

*Proof.* By Theorem 6.3, there is a natural isomorphism of flows  $||K|| \cong |K|_{nat}$  for all precubical sets  $K$ . It then suffices to prove that there is a homeomorphism  $\mathbb{P}_{\alpha,\beta}||K|| \cong (\mathbb{P}_{\alpha,\beta}|K|_{nat})_{\Delta}$  for all  $(\alpha, \beta) \in K_0 \times K_0$ . For all  $n \geq 0$ ,  $N_n$  is equipped with the compact-open topology which is  $\Delta$ -generated by Proposition 4.10. The Moore composition of natural  $d$ -paths induces continuous maps

$$\mathcal{D}_{\alpha,\beta}(K, n)(\square[\underline{n}] \rightarrow K) = N_{n_1} \times \dots \times N_{n_p} \longrightarrow \mathbf{TOP}([0, n], |K|_{geom})$$

for all  $\underline{n} = (n_1, \dots, n_p) \in \text{Seq}(n)$ . The morphisms of the small category  $\mathcal{D}_{\alpha,\beta}(K, n)$  are induced by products of maps of the form  $N_{m_1} \times N_{m_2} \rightarrow N_{m_1+m_2}$  induced by the Moore composition of natural  $d$ -paths and by identities of  $N_n$  for  $n \geq 1$ . Therefore, we obtain a well-defined cocone

$$\mathcal{D}_{\alpha,\beta}(K, n) \xrightarrow{\bullet} \mathbf{TOP}([0, n], |K|_{geom})$$

for all  $n \geq 1$ . We obtain a continuous map

$$\mathbb{P}_{\alpha,\beta}|K|_{nat} \cong \varinjlim \mathcal{D}_{\alpha,\beta}(K, n) \subset \bigsqcup_{n \geq 1} \mathbf{TOP}([0, n], |K|_{geom})$$

which yields a continuous bijection  $\mathbb{P}_{\alpha,\beta}|K|_{nat} \rightarrow (\mathbb{P}_{\alpha,\beta}|K|_{nat})_{\Delta}$ . Consider a set map

$$f : [0, 1] \longrightarrow \mathbb{P}_{\alpha,\beta}|K|_{nat}$$

such that the composite set map

$$[0, 1] \longrightarrow \mathbb{P}_{\alpha,\beta}|K|_{nat} \longrightarrow (\mathbb{P}_{\alpha,\beta}|K|_{nat})_{\Delta}$$

is continuous. Since  $[0, 1]$  is path-connected, and since the  $L_1$ -arc length is constant on a path-connected component, there exists a commutative diagram of spaces of the form

$$\begin{array}{ccccc} [0, 1] & \longrightarrow & \mathbb{P}_{\alpha,\beta}|K|_{nat} & \rightarrow & (\mathbb{P}_{\alpha,\beta}|K|_{nat})_{\Delta} & \longrightarrow & \bigsqcup_{n \geq 1} \mathbf{TOP}([0, n], |K|_{geom}) \\ & & \parallel & & & & \parallel \\ [0, 1] & \longrightarrow & \mathbf{TOP}([0, n_0], |K|_{geom}) & \xrightarrow{c} & \bigsqcup_{n \geq 1} \mathbf{TOP}([0, n], |K|_{geom}) & & \end{array}$$

for some integer  $n_0 \geq 1$ . By adjunction, we obtain a continuous map

$$[0, 1] \times [0, n_0] \longrightarrow |K|_{geom}.$$

The topological space  $|K|_{geom}$  is a cellular topological space. It is Hausdorff and the image of the compact  $[0, 1] \times [0, n_0]$  is a closed compact subset of  $|K|_{geom}$  which, by [17, Proposition A.1], intersects finitely many interiors of cubes and vertices. At this point, one can suppose that  $K$  is a finite precubical set, i.e. that  $\bigcup_{n \geq 0} K_n$  is finite, say that it

has  $N$  elements. We want to prove that the set map  $f : [0, 1] \rightarrow \mathbb{P}_{\alpha, \beta} |K|_{nat}$  is continuous. Since the  $\Delta$ -generated spaces are sequential, it suffices to prove the sequential continuity of  $f : [0, 1] \rightarrow \mathbb{P}_{\alpha, \beta} |K|_{nat}$ . Let  $(t_n)_{n \geq 0}$  be a sequence of  $[0, 1]$  which converges to  $t_\infty \in [0, 1]$ . For all  $t \in [0, 1]$ ,  $\text{Carrier}(f(t))$  is of the form  $(c_1, \dots, c_p)$  with  $\dim(c_1) + \dots + \dim(c_p) = n_0$ . We deduce that the set  $\{\text{Carrier}(\gamma(t)) \mid t \in [0, 1]\}$  has less than  $(N + 1)^{n_0}$  elements, i.e. that it is finite. Thus the sequence of carriers  $(\text{Carrier}(f(t_n)))_{n \geq 0}$  has a constant subsequence. Suppose that the sequence  $(\text{Carrier}(f(t_n)))_{n \geq 0}$  is constant and equal to  $(c_1, \dots, c_p)$ . Then the sequence of paths  $(f(t_n))_{n \geq 0}$  belongs to the image of the continuous map  $N_{\dim(c_1)} \times \dots \times N_{\dim(c_p)} \rightarrow \mathbb{P}_{\alpha, \beta} |K|_{nat}$  induced by the Moore composition of paths. The space  $\mathbb{P}_{\alpha, \beta} |K|_{nat}$  is Hausdorff since there is a continuous bijection from this space to the Hausdorff space  $(\mathbb{P}_{\alpha, \beta} |K|_{nat})_\Delta$ . The product  $N_{\dim(c_1)} \times \dots \times N_{\dim(c_p)}$  is a finite product in **Top** of compact metrizable spaces by Proposition 4.10. By [12, Lemma 6.9], this product coincides with the product taken in the category of general topological spaces. It means that  $N_{\dim(c_1)} \times \dots \times N_{\dim(c_p)}$  is compact. Consequently, the image of  $N_{\dim(c_1)} \times \dots \times N_{\dim(c_p)} \rightarrow \mathbb{P}_{\alpha, \beta} |K|_{nat}$  is compact closed in  $\mathbb{P}_{\alpha, \beta} |K|_{nat}$  which is sequential, being  $\Delta$ -generated. It implies that this image is sequential, as a closed subset of a  $\Delta$ -generated space, and therefore sequentially compact. We deduce that the sequence  $(f(t_n))_{n \geq 0}$  has a limit point which is necessarily  $f(t_\infty)$  by continuity of the composite map  $[0, 1] \rightarrow \mathbb{P}_{\alpha, \beta} |K|_{nat} \rightarrow (\mathbb{P}_{\alpha, \beta} |K|_{nat})_\Delta$ . In fact, we have proved that every subsequence of  $(f(t_n))_{n \geq 0}$  has a subsequence which has a limit point which is necessarily  $f(t_\infty)$ . Suppose that the sequence  $(f(t_n))_{n \geq 0}$  does not converge to  $f(t_\infty)$ . Then there exists an open neighborhood  $V$  of  $f(t_\infty)$  in  $\mathbb{P}_{\alpha, \beta} |K|_{nat}$  such that for some  $M \geq 0$ , and for all  $n \geq M$ ,  $f(t_n) \in V^c$ , the complement of  $V$ , the latter being closed in  $\mathbb{P}_{\alpha, \beta} |K|_{nat}$ . Thus,  $(f(t_n))_{n \geq M}$  cannot have a limit point: contradiction. We deduce that  $f : [0, 1] \rightarrow \mathbb{P}_{\alpha, \beta} |K|_{nat}$  is sequentially continuous, hence continuous. It means that the continuous bijection  $\mathbb{P}_{\alpha, \beta} |K|_{nat} \rightarrow (\mathbb{P}_{\alpha, \beta} |K|_{nat})_\Delta$  is a  $\Delta$ -inclusion. Therefore, the latter is a homeomorphism by Corollary 7.3.  $\square$

The word “weak homotopy equivalence” can be replaced by “homotopy equivalence” in the statements of [26, Theorem 7.5 and Theorem 7.6] because all maps of [26, Equation 7.5] are homotopy equivalences. Indeed, it is proved in [26, Proposition 10.3] that some specific map is a h-cofibration. Therefore the diagram of [26, Proposition 10.4] is Reedy h-cofibrant and the map  $Q_n^K$  is a homotopy equivalence.

The spaces of  $d$ -paths of general precubical sets are equipped in [26] with the compact-open topology instead of some kind of kellyfication of the compact-open topology. The latter is the correct internal hom, both for  $k$ -spaces and  $\Delta$ -generated spaces, except in very specific situations like Proposition 4.10 and Proposition 4.14. Since the  $\Delta$ -kellyfication functor takes (weak resp.) homotopy equivalences to (weak resp.) homotopy equivalences, this point is not an issue.

**7.6. Corollary.** *Let  $K$  be a precubical set. Let  $\alpha, \beta \in K_0$ . Then the space of execution paths  $\mathbb{P}_{\alpha, \beta} |K|_q$  is homotopy equivalent to the space of  $d$ -paths from  $\alpha$  to  $\beta$  equipped with the  $\Delta$ -kellyfication of the compact-open topology in the geometric realization of  $K$  as a Grandis  $d$ -space.*

*Proof.* By Theorem 5.8, there is a homotopy equivalence  $\mathbb{P}_{\alpha, \beta} |K|_q \simeq \mathbb{P}_{\alpha, \beta} |K|_{nat}$ . By Theorem 7.5, there is a homeomorphism  $\mathbb{P}_{\alpha, \beta} |K|_{nat} \cong (\mathbb{P}_{\alpha, \beta} |K|_{nat})_\Delta$ . By [26, Theorem 7.5

and Theorem 7.6], the space  $(\mathbb{P}_{\alpha,\beta}|K|_{nat})_{\Delta}$  and the space of  $d$ -paths from  $\alpha$  to  $\beta$  in the geometric realization of  $K$  as a Grandis  $d$ -space are homotopy equivalent.  $\square$

In fact, we could prove Corollary 7.6 by using only Theorem 5.8 and Theorem 6.3 without using Theorem 7.5. Let us sketch the argument. There is a natural isomorphism of flows  $||K|| \cong |K|_{nat}$ . The category  $\text{Ch}_{\alpha,\beta}(K, n)$  is a poset and a direct Reedy category with the degree function  $\underline{n} \mapsto n - \ell(\underline{n})$  (see [26, Section 10]). Let  $\mathbf{c} = \square[\underline{n}] \rightarrow K$  with  $\ell(\underline{n}) = n$  and  $\underline{n} = (n_1, \dots, n_p)$ . Using [7, Theorem B.3], we prove that the continuous map  $L_{\mathbf{c}}\mathcal{D}_{\alpha,\beta}(K, n) \rightarrow \mathcal{D}_{\alpha,\beta}(K, n)(\mathbf{c})$  is the pushout product of the maps  $\partial N_{n_i} \rightarrow N_{n_i}$  for  $i = 1, \dots, p$ . The latter are h-cofibrations by Proposition 5.5. Thus the diagram  $\mathcal{D}_{\alpha,\beta}(K, n)$  is Reedy h-cofibrant and the colimit  $\varinjlim \mathcal{D}_{\alpha,\beta}(K, n)$  is actually a homotopy colimit in the h-model structure of **Top**. Since all spaces  $N_n$  are contractible by Proposition 4.10, we deduce the homotopy equivalences

$$\mathbb{P}_{\alpha,\beta}|K|_q \simeq \mathbb{P}_{\alpha,\beta}||K|| \simeq \bigsqcup_{n \geq 1} \underset{\in \text{Ch}_{\alpha,\beta}(K, n)}{\text{holim}_{\square[\underline{n}] \rightarrow K} \{0\}} \simeq \bigsqcup_{n \geq 1} |\text{Ch}_{\alpha,\beta}(K, n)|.$$

We conclude using [26, Theorem 7.6].

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UNIVERSITÉ PARIS CITÉ, CNRS, IRIF, F-75013, PARIS, FRANCE

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