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A SUPER ROBINSON–SCHENSTED–KNUTH CORRESPONDENCE WITH SYMMETRY AND THE SUPER LITTLEWOOD–RICHARDSON RULE

NOHRA HAGE

Abstract – Robinson–Schensted–Knuth (RSK) correspondence is a bijective correspondence between two-rowed arrays of non-negative integers and pairs of same-shape semistandard tableaux. This correspondence satisfies the symmetry property, that is, exchanging the rows of a two-rowed array is equivalent to exchanging the positions of the corresponding pair of semistandard tableaux. In this article, we introduce a super analogue of the RSK correspondence for super tableaux over a signed alphabet using a super version of Schensted’s insertion algorithms. We give a geometrical interpretation of the super-RSK correspondence by a matrix-ball construction, showing the symmetry property in complete generality. We deduce a combinatorial version of the super Littlewood–Richardson rule on super Schur functions over a finite signed alphabet. Finally, we introduce the notion of super Littlewood–Richardson skew tableaux and we give another combinatorial interpretation of the super Littlewood–Richardson rule.

Keywords – super jeu de taquin, super Young tableaux, super plactic monoids, super-RSK correspondences, super Littlewood–Richardson rule.

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1. INTRODUCTION

Schensted introduced in [27] a bijection between permutations over the totally ordered alphabet $[n] := \{1 < \dots < n\}$ and pairs of same-shape standard Young tableaux over $[n]$ in order to compute the length of the longest decreasing subsequence of a given permutation over $[n]$. This correspondence is described using Schensted’s insertion procedure that constructs a first standard Young tableau by successively inserting the elements of the given permutation according to a specific rule, while the second standard Young tableau records the evolution of the shape during the insertion. This correspondence had been also described, in a rather different form, much earlier by Robinson in [25] in an attempt to give a first correct proof of the Littlewood–Richardson rule that provides an explicit combinatorial description for expressing a skew Schur function or a

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product of two Schur functions as a linear combination of Schur functions. This correspondence is then referred to as the *Robinson–Schensted (RS) correspondence*. Knuth generalized in [18] the RS correspondence to a bijection between two-rowed arrays of elements of $[n]$ and pairs of same-shape semistandard Young tableaux over $[n]$. Knuth’s bijection is also described using Schensted’s insertion algorithm that constructs a first semistandard Young tableau by successively inserting the elements of the second row of the given two-rowed array from left to right, while the second semistandard Young tableau records the evolution of the shape during this insertion using the elements of the first row of the two-rowed array. This bijection is then known as the *Robinson–Schensted–Knuth (RSK) correspondence*. An essential property of this correspondence is that it satisfies the *symmetry property*, that is, under the RSK correspondence, exchanging the rows of a two-rowed array is equivalent to exchanging the positions of the corresponding pair of semistandard Young tableaux, [18]. This property is also proved by Viennot in [33] for the RS correspondence, and by Fulton in [9] for the RSK correspondence, using geometrical interpretations of these correspondences. Since then, the RSK correspondence has found rich applications on representation theory, algebraic combinatorics and probabilistic combinatorics, [8, 9, 17, 32], and has found many generalizations on other structures of tableaux, [2, 3, 5, 24, 26, 30, 31].

Bonetti, Senato and Venezia introduced in [5] a super-RSK correspondence on super Young tableaux over a signed alphabet using super Schensted’s right and left insertion. However, the symmetry property holds only in special cases under this bijection as shown in [20]. A question was to find a super-RSK correspondence satisfying the symmetry property in complete generality and leading to the classical RSK correspondence as a particular case. Muth introduced in [24] a super-RSK correspondence using Haiman’s mixed insertion algorithm on super Young tableaux and proved that this correspondence satisfies the symmetry property in complete generality. However, this correspondence is not related to the super plactic monoid of type A and to the representations of the general linear Lie superalgebra, and then it does not yield to a combinatorial description of the super Littlewood–Richardson rule. It is worth noting that Haiman’s mixed insertion algorithm is used to define the shifted plactic monoid and allows to give a combinatorial version of the shifted Littlewood–Richardson rule for shifted tableaux, [29]. We introduce in this article a super version of the RSK correspondence on super Young tableaux that satisfies the symmetry property in complete generality, using super Schensted’s insertion algorithms and the super plactic monoid of type A. We deduce combinatorial descriptions of the super Littlewood–Richardson rule for super Young tableaux over a finite signed alphabet.

Super plactic monoids, insertion and taquin. A *signed alphabet* is a finite or countable totally ordered set \mathcal{S} which is the disjoint union of two subsets \mathcal{S}_0 and \mathcal{S}_1 . The *super plactic monoid* $\mathbf{P}(\mathcal{S})$ over a signed alphabet \mathcal{S} , is the quotient of the free monoid \mathcal{S}^* over \mathcal{S} by the congruence relation $\sim_{\mathbf{P}(\mathcal{S})}$ generated by the following family of *super Knuth relations*, [20]:

$$\begin{aligned} xzy &= zxy, & \text{with } x = y & \text{ only if } y \in \mathcal{S}_0 \text{ and } y = z \text{ only if } y \in \mathcal{S}_1, \\ yxz &= yzx, & \text{with } x = y & \text{ only if } y \in \mathcal{S}_1 \text{ and } y = z \text{ only if } y \in \mathcal{S}_0, \end{aligned}$$

for any $x \leq y \leq z$ of elements of \mathcal{S} . When $\mathcal{S} = \mathcal{S}_0 = [n]$, we recover the *plactic monoid of type A* introduced by Lascoux and Schützenberger in [21], following the works of Schensted, [27], and Knuth, [18], on the RSK correspondence. Plactic monoids have found several applications in algebraic combinatorics, representation theory, probabilistic combinatorics and rewriting

theory, [4, 6–9, 11–16, 19, 23]. When $\mathcal{S} = \{\bar{m} < \dots < \bar{1} < 1 < \dots < n\}$, we recover the reverse of the super Knuth relations obtained in [1] where the super plactic congruence is described using Kashiwara’s theory of crystal bases for the representations of the general linear Lie superalgebra $\mathfrak{gl}_{m,n}$. It is worth noting that the super plactic monoid also appeared in [22] in the study of the parastatistics algebra. Note finally that the super algebraic structures have been used as combinatorial tools in the study of the invariant theory of superalgebras, the representation theory of super algebras, and algebras satisfying identities, [2, 3, 5, 10].

A *partition* of a positive integer n is a weakly decreasing sequence $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{Z}_{>0}^k$ such that $\sum \lambda_i = n$. The *Young diagram of a partition* $\lambda = (\lambda_1, \dots, \lambda_k)$ is the set $\mathcal{Y}(\lambda)$ of pairs (i, j) such that $1 \leq i \leq k$ and $1 \leq j \leq \lambda_i$, that can be represented by a diagram by drawing a box for each pair (i, j) . The transposed diagram $\{(j, i) \mid (i, j) \in \mathcal{Y}(\lambda)\}$ defines another partition $\bar{\lambda}$, called the *conjugate partition* of λ . A *super tableau of shape* λ over \mathcal{S} is a Young diagram of λ filled with elements of \mathcal{S} such that the entries in each row are weakly increasing allowing the repetition only of elements in \mathcal{S}_0 and the entries in each column are weakly increasing allowing the repetition only of elements in \mathcal{S}_1 . Denote by $\text{Yt}(\mathcal{S})$ the set of all super tableaux over \mathcal{S} . Note that, when $\mathcal{S} = \mathcal{S}_0$ and $\mathcal{S} = \mathcal{S}_1$, we recover the notion of row-strict and column-strict tableaux, respectively, [9]. Denote by $\mathbf{R} : \text{Yt}(\mathcal{S}) \rightarrow \mathcal{S}^*$ the *column reading map* that reads a super tableau column-wise from bottom to top and from left to right. Super versions of Schensted left and right insertion algorithms are introduced in [20] on super tableaux, and consist in inserting elements of \mathcal{S} into super tableaux by rows and columns, respectively. For any word $w = x_1 \dots x_k$ over \mathcal{S} , a super tableau $\mathbf{T}(w)$ is computed by inserting the letters of w iteratively from left to right using the right insertion \leftarrow , and starting from the empty super tableau:

$$\mathbf{T}(w) := (\emptyset \leftarrow w) = ((\dots (\emptyset \leftarrow x_1) \leftarrow \dots) \leftarrow x_k).$$

Two words over \mathcal{S} are equivalent with respect to $\sim_{\mathbf{P}(\mathcal{S})}$ if and only if they lead to the same super tableau after insertion, [20]. This is the *cross-section property* of super tableaux with respect to $\sim_{\mathbf{P}(\mathcal{S})}$. We deduce that the internal product $\star_{\text{Yt}(\mathcal{S})}$ defined on $\text{Yt}(\mathcal{S})$ by $t \star_{\text{Yt}(\mathcal{S})} t' := (t \leftarrow \mathbf{R}(t'))$, for all t and t' in $\text{Yt}(\mathcal{S})$, is associative, and the monoids $(\text{Yt}(\mathcal{S}), \star_{\text{Yt}(\mathcal{S})})$ and $\mathbf{P}(\mathcal{S})$ are isomorphic.

Let λ and μ be partitions such that $\mathcal{Y}(\mu)$ is contained in $\mathcal{Y}(\lambda)$. A *super skew tableau of shape* λ/μ over \mathcal{S} is a Young diagram of the following form

$$\mathcal{Y}(\lambda/\mu) := \{(i, j) \mid 1 \leq i \leq k, \mu_i < j \leq \lambda_i\}$$

filled with elements of \mathcal{S} such that the entries in each row are weakly increasing allowing the repetition only of elements in \mathcal{S}_0 and the entries in each column are weakly increasing allowing the repetition only of elements in \mathcal{S}_1 . An *inner corner* of a super skew tableau of shape λ/μ is a box in $\mathcal{Y}(\mu)$ such that the boxes below and to the right are not in $\mathcal{Y}(\mu)$, and an *outer corner* is a box such that neither box below or to the right is in $\mathcal{Y}(\lambda/\mu)$. Schützenberger introduced in [28] the *jeu de taquin procedure* on Young tableaux in order to give a proof of the Littlewood–Richardson rule using the properties of the plactic monoid of type A. We introduce in [13] the *super jeu taquin procedure* on super tableaux that consists in moving inner corners from a super skew tableau into outer corners by keeping the rows and the columns weakly increasing until no more inner corners remain in the initial super skew tableau. We prove that the *rectification* of a super skew tableau S by the super jeu de taquin is the unique super tableau whose reading is equivalent to the one of S

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with respect to $\sim_{\mathcal{P}(\mathcal{S})}$. We deduce that the resulting super tableau does not depend on the order in which we choose inner corners. We also relate the super jeu de taquin to the insertion algorithms and we show how we can insert a super tableau into another one by taquin. This interpretation of the insertion product $\star_{\text{Yt}(\mathcal{S})}$ by taquin allows us to give in Section 4 a combinatorial description of the super Littlewood–Richardson coefficient. Moreover, we introduce in [13] the super analogue of the *Schützenberger’s evacuation procedure* which transforms a super tableau t over a signed alphabet \mathcal{S} into an *opposite tableau* t^{op} , over the opposite alphabet obtained from \mathcal{S} by reversing its order. We show that the super tableaux t and t^{op} have the same shape and that the map $t \mapsto t^{\text{op}}$ is an involution on $\text{Yt}(\mathcal{S})$ that is compatible with $\sim_{\mathcal{P}(\mathcal{S})}$. The super evacuation procedure allows us to construct in Subsection 3.4 a dual version of the super-RSK correspondence using the left insertion algorithm on super tableaux.

ORGANIZATION AND MAIN RESULTS OF THE ARTICLE

We begin by recalling in Section 2 the notions of super Young tableaux, the super plactic monoid and the super jeu de taquin from [13, 20]. Moreover, we give a super version of the Robinson–Schensted correspondence for super tableaux over a signed alphabet.

A super-RSK correspondence with symmetry. We introduce in Subsection 3.1 a super version of the RSK correspondence over a signed alphabet. Let \mathcal{S} and \mathcal{S}' be signed alphabets. A *signed two-rowed array* w on \mathcal{S} and \mathcal{S}' is a $2 \times k$ matrix

$$w := \begin{pmatrix} x_1 \dots x_k \\ y_1 \dots y_k \end{pmatrix}$$

with x_i in \mathcal{S} and y_i in \mathcal{S}' , for all $i = 1, \dots, k$, that satisfies Conditions (6) and (7). Starting from a signed two-rowed array w on \mathcal{S} and \mathcal{S}' , Algorithm 3.1.1 computes a pair of same-shape super tableaux $(\mathbf{T}(w), \mathbf{Q}(w))$ whose entries are the ones of the second and the first row of w , respectively. More precisely, the super tableau $\mathbf{T}(w)$ is equal to $\mathbf{T}(y_1 \dots y_k)$, and $\mathbf{Q}(w)$ is the super tableau obtained by successively adding x_1, \dots, x_k in the same places as the boxes added when computing $\mathbf{T}(w)$ starting from an empty super tableau. Moreover, Algorithm 3.1.3 allows us to recover the initial signed two-rowed array w starting from the same-shape pair of super tableaux $(\mathbf{T}(w), \mathbf{Q}(w))$. This sets up a one-to-one correspondence between signed two-rowed arrays and pairs of same-shape super tableaux on \mathcal{S} and \mathcal{S}' , that we denote by

$$\text{sRSK} : w \mapsto (\mathbf{T}(w), \mathbf{Q}(w)).$$

Hence, we obtain the following first main result of the article:

Theorem 3.1.5[Super-RSK correspondence]. *Let \mathcal{S} and \mathcal{S}' be signed alphabets. The map sRSK defines a one-to-one correspondence between signed two-rowed arrays and pairs of super tableaux on \mathcal{S} and \mathcal{S}' , such that for any signed two-rowed array w on \mathcal{S} and \mathcal{S}' , we have that $\mathbf{T}(w)$ and $\mathbf{Q}(w)$ are same-shape super tableaux whose entries are the ones of the second and the first row of w , respectively.*

Using a matrix-ball construction, we show in Subsection 3.2 that the super-RSK correspondence satisfies the symmetry property in complete generality. Let w be a signed two-rowed array on signed alphabets \mathcal{S} and \mathcal{S}' . The *inverse* of w , denoted by w^{inv} , is the signed two-rowed array on \mathcal{S}' and \mathcal{S} obtained from w by *exchanging* the rows of w , and by sorting the new couples on $\mathcal{S}' \times \mathcal{S}$ according to Conditions (6) and (7). The signed two-rowed array w has *symmetry with respect to the map sRSK* if it satisfies the following property:

$$\text{if } \text{sRSK}(w) = (\mathbf{T}(w), \mathbf{Q}(w)) \quad \text{then} \quad \text{sRSK}(w^{\text{inv}}) = (\mathbf{Q}(w), \mathbf{T}(w)).$$

A *signed ball array* on signed alphabets \mathcal{S} and \mathcal{S}' is a rectangular array of balls filled with positive integers, whose rows and columns are indexed with elements of \mathcal{S} and \mathcal{S}' , respectively, allowing the repetition only of elements in \mathcal{S}_1 and \mathcal{S}'_1 , respectively, and where many balls can occur in the same position. We show in 3.2.1 how to correspond to each signed two-rowed array w on \mathcal{S} and \mathcal{S}' , a signed ball array, denoted by $\text{Ba}(w)$, whose rows and columns are indexed with the elements of the first and second row of w , respectively, from the smaller to the bigger one, and where only the indices from \mathcal{S}_1 and \mathcal{S}'_1 are repeated as many times as they appear in w . First, we associate to each couple in w a ball in an empty signed ball array and then we order and number these balls with positive integers. Secondly, we add new balls to the given signed ball array, we order and number them and we repeat the same procedure until no more balls can be added. This geometrical construction allows us to prove the symmetry property using the symmetry of the resulting signed ball array $\text{Ba}(w)$. More precisely, we denote by $\mathbf{T}(\text{Ba}(w))$ and $\mathbf{Q}(\text{Ba}(w))$ the super tableau obtained from $\text{Ba}(w)$ whose k -th row lists the indices of the leftmost columns and top-most rows, respectively, where each integer number occurs in the new added balls. Proposition 3.2.3 shows that $(\mathbf{T}(\text{Ba}(w)), \mathbf{Q}(\text{Ba}(w))) = (\mathbf{T}(w), \mathbf{Q}(w))$. Since the matrix-ball construction is symmetric in the rows and columns of the resulting signed ball array, we deduce the following result:

Theorem 3.1.10[Symmetry of the super-RSK correspondence]. *Let \mathcal{S} and \mathcal{S}' be signed alphabets. All signed two-rowed arrays on \mathcal{S} and \mathcal{S}' have symmetry with respect to the super-RSK correspondence map sRSK.*

We end Subsection 3.4 by giving a dual way to construct the pair of super tableaux corresponding to a signed two-rowed array with respect to the map sRSK using the left insertion algorithm and the super evacuation procedure on super tableaux.

The super Littlewood–Richardson rule. We give in Section 4 a combinatorial interpretation of the super Littlewood–Richardson rule using the super-RSK correspondence. Let λ , μ and ν be partitions such that $\mathcal{Y}(\lambda)$ is contained in $\mathcal{Y}(\nu)$ and let \mathcal{S} be a finite signed alphabet. We show that the number $c_{\lambda, \mu}^{\nu}$ of ways a given super tableau t of shape ν over \mathcal{S} can be written as the product of a super tableau t' of shape λ and a super tableau t'' of shape μ over \mathcal{S} , does not depend on t , and depends only on the partitions λ , μ and ν . Moreover, we show that $c_{\lambda, \mu}^{\nu}$ is equal to the number of super skew tableaux of shape ν/λ whose rectification is a given tableau of shape μ . Hence, we obtain the following result:

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Theorem 4.2.1[The super Littlewood–Richardson rule]. *Let λ, μ and ν be partitions such that $\mathcal{Y}(\lambda)$ is contained in $\mathcal{Y}(\nu)$. The following identities*

$$S_\lambda S_\mu = \sum_{\nu} c_{\lambda, \mu}^{\nu} S_{\nu} \quad \text{and} \quad S_{\nu/\lambda} = \sum_{\mu} c_{\lambda, \mu}^{\nu} S_{\mu}$$

hold in the tableau \mathbb{Z} -algebra rising from $\mathbf{P}(\mathcal{S})$, where $S_{\nu/\lambda}, S_\lambda, S_\mu$ and S_ν denote respectively the formal sum of all super tableaux of shape $\nu/\lambda, \lambda, \mu$ and ν over a finite signed alphabet \mathcal{S} .

We suppose finally that $\mathcal{S}_1 = [n]$ and $\mathcal{S}_0 = \{\bar{m} < \dots < \bar{1}\}$ with $\bar{1} < 1$. Let w be in \mathcal{S}^* . We denote by $|w|_i$ the number of times the element i of \mathcal{S} appears in w , and by $\overline{|w|}$ the number of elements of \mathcal{S}_1 that appear in w such that each element is counted only once. The *weight* of w is the following $(n+m)$ -uplet

$$\text{wt}(w) := (|w|_{\bar{m}}, \dots, |w|_{\bar{1}}, |w|_1, \dots, |w|_n).$$

The word w is a *super Yamanouchi word* if for every right subword w' of w , the following property $|w'|_{\bar{m}} \geq \dots \geq |w'|_{\bar{2}} \geq |w'|_{\bar{1}} \geq |w'|$ holds, and for every left subword w' of w , the following property $|w'|_1 \geq |w'|_2 \geq \dots \geq |w'|_n$ holds. We call $\text{wt}(\mathbf{R}(S))$ the *weight* of a super skew tableau S over \mathcal{S} , and when $\mathbf{R}(S)$ is a super Yamanouchi word, we call S a *super Littlewood–Richardson skew tableau*. For any partition $\mu \in \mathbb{Z}_{>0}^k$, define the $(n+m)$ -uplet $\check{\mu} := (\mu_1, \check{\mu}_2)$, where μ_1 is the partition formed by the first m parts of μ and μ_2 is the partition formed by its last $k-m$ parts, such that if $m \leq k < m+n$ then the last $(m+n) - k$ parts of μ_2 are zero and if $k < m$ then the last $m - k$ parts of μ_1 are zero and all the parts of μ_2 are zero. Using the super plactic congruence and following Theorem 4.2.1, we deduce the following result:

Theorem 4.3.5. *Let λ, μ and ν be partitions such that $\mathcal{Y}(\lambda)$ is contained in $\mathcal{Y}(\nu)$. The super Littlewood–Richardson coefficient $c_{\lambda, \mu}^{\nu}$ is equal to the number of super Littlewood–Richardson skew tableaux of shape ν/λ and of weight $\check{\mu}$.*

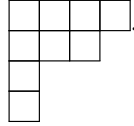
This theorem allows us to compute the super Littlewood–Richardson coefficient $c_{\lambda, \mu}^{\nu}$ using a new combinatorial method as shown in Example 4.3.6.

Notation. We denote by \mathcal{A}^* the free monoid of *words* over a totally ordered alphabet \mathcal{A} , the product being concatenation of words and the identity being the empty word, and by $\#\mathcal{A}$ the cardinal number of \mathcal{A} when it is finite. For any word w in \mathcal{A}^* , the length of w is denoted by $|w|$. For any $i \leq j$, a *subword* of a word $w = x_1 \dots x_k$ in \mathcal{A}^* is a word $w' = x_i \dots x_j$ made up of consecutive letters of w . We denote by $[n]$ the ordered set $\{1 < 2 < \dots < n\}$ for n in $\mathbb{Z}_{>0}$. Let \mathcal{S} be a finite or countable totally ordered set and $\|\cdot\| : \mathcal{S} \rightarrow \mathbb{Z}_2$ be any map, where $\mathbb{Z}_2 = \{0, 1\}$ denotes the additive cyclic group of order 2. We call the ordered pair $(\mathcal{S}, \|\cdot\|)$ a *signed alphabet* and the map $\|\cdot\|$ the *signature map* of \mathcal{S} . We denote $\mathcal{S}_0 = \{a \in \mathcal{S} \mid \|a\| = 0\}$ and $\mathcal{S}_1 = \{a \in \mathcal{S} \mid \|a\| = 1\}$. A monoid \mathbf{M} is a \mathbb{Z}_2 -graded monoid or a *supermonoid* if a map $\|\cdot\| : \mathbf{M} \rightarrow \mathbb{Z}_2$ is given such that $\|u.v\| = \|u\| + \|v\|$, for all u, v in \mathbf{M} . We call $\|u\|$ the \mathbb{Z}_2 -degree of u . The free monoid \mathcal{S}^* over \mathcal{S} is \mathbb{Z}_2 -graded by considering $\|w\| := \|x_1\| + \dots + \|x_k\|$, for any $w = x_1 \dots x_k$ in \mathcal{S}^* . In the rest of this article, and if there is no possible confusion, \mathcal{S} denotes a signed alphabet.

2. THE SUPER PLACTIC MONOID OF TYPE A

In this section, we recall the notions of super Young tableaux, the super plactic monoid and the super jeu de taquin from [13, 20]. Moreover, we give a super version of the Robinson–Schensted correspondence for super tableaux over a signed alphabet.

2.1. Super Young tableaux. Let n be a positive integer. A *partition* of n is a weakly decreasing sequence $\lambda = (\lambda_1, \dots, \lambda_k)$ in $\mathbb{Z}_{>0}^k$ such that $\sum \lambda_i = n$. We call the integer k *number of parts* or *height* of λ . We denote by \mathcal{P}_n the set of partitions of n and we set $\mathcal{P} = \bigcup_{n \in \mathbb{Z}_{>0}} \mathcal{P}_n$. The (Ferrers)–Young diagram of a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of n , denoted by $\mathcal{Y}(\lambda)$, is the set of pairs (i, j) such that $1 \leq i \leq k$ and $1 \leq j \leq \lambda_i$, that can be represented by a diagram by drawing a box for each pair (i, j) . For instance, the Young diagram $\mathcal{Y}((4, 3, 1, 1))$ is represented by the following diagram:



Let λ be in \mathcal{P} . The *conjugate partition* of λ , denoted by $\tilde{\lambda}$, is the partition corresponding to the transposed diagram $\{(j, i) \mid (i, j) \in \mathcal{Y}(\lambda)\}$, whose parts are the number of boxes of the columns of $\mathcal{Y}(\lambda)$. A *super semistandard Young tableau*, or *super tableau* for short, of *shape* λ over \mathcal{S} , is a pair $t := (\lambda, \mathcal{T})$, where $\mathcal{T} : \mathcal{Y}(\lambda) \rightarrow \mathcal{S}$ is a map satisfying the following conditions:

$$\begin{aligned} \mathcal{T}(i, j) &\leq \mathcal{T}(i, j+1), \text{ with } \mathcal{T}(i, j) = \mathcal{T}(i, j+1) \text{ only if } \|\mathcal{T}(i, j)\| = 0, \\ \mathcal{T}(i, j) &\leq \mathcal{T}(i+1, j), \text{ with } \mathcal{T}(i, j) = \mathcal{T}(i+1, j) \text{ only if } \|\mathcal{T}(i, j)\| = 1. \end{aligned} \quad (1)$$

We call $\mathcal{Y}(\lambda)$ and \mathcal{T} the *frame* and the *filling* of t , respectively. We call t a *standard tableau* over \mathcal{S} if \mathcal{T} is injective. We denote by \emptyset the empty super tableau, by $\text{Ys}(\mathcal{S})$ the set of all standard tableaux, and by $\text{Yt}(\mathcal{S})$ (resp. $\text{Yt}(\mathcal{S}, \lambda)$) the set of all super tableaux (resp. of shape λ) over \mathcal{S} .

Let λ and μ be in \mathcal{P} of heights k and l , respectively, such that $l \leq k$. We denote $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for any i , that is, the Young diagram of μ is contained in that of λ , and we call the following set

$$\mathcal{Y}(\lambda/\mu) := \{(i, j) \mid 1 \leq i \leq k, \mu_i < j \leq \lambda_i\}$$

a *skew diagram* λ/μ or a *skew shape*. We denote $\lambda/0 := \lambda$ the skew shape corresponding to the Young diagram $\mathcal{Y}(\lambda)$. Let λ/μ be a skew shape. A *super semistandard skew tableau*, or *super skew tableau* for short, of *shape* λ/μ over \mathcal{S} , is a pair $S := (\lambda/\mu, \mathcal{U})$, where $\mathcal{U} : \mathcal{Y}(\lambda/\mu) \rightarrow \mathcal{S}$ is a map satisfying the conditions (1). We call $\mathcal{Y}(\lambda/\mu)$ and \mathcal{U} the *frame* and the *filling* of S , respectively. We denote by $\text{St}(\mathcal{S})$ (resp. $\text{St}(\mathcal{S}, \lambda/\mu)$) the set of all super skew tableaux (resp. of shape λ/μ) over \mathcal{S} . Note that we will identify the set $\text{Yt}(\mathcal{S}, \lambda)$ with the set $\text{St}(\mathcal{S}, \lambda/0)$. Let S be in $\text{St}(\mathcal{S}, \lambda/\mu)$. If S is not a super tableau, then it has at least an *inner corner*, that is, a box in $\mathcal{Y}(\mu)$ such that the boxes below and to the right are not in $\mathcal{Y}(\mu)$. An *outer corner* of S is a box such that neither box below or to the right is in $\mathcal{Y}(\lambda/\mu)$. Note that, in some cases, inner corners of S are also outer corners of it. We denote by $\mathbf{R} : \text{St}(\mathcal{S}) \rightarrow \mathcal{S}^*$ the *column reading map* that reads a super tableau column-wise from bottom to top and from left to right.

2. The super plactic monoid of type A

Example 2.1.1. Consider the alphabet $\mathcal{S} = \mathbb{Z}_{>0}$ with signature given by \mathcal{S}_0 the set of even numbers and \mathcal{S}_1 defined consequently. The following diagram is a super tableau of shape $(3, 2, 2, 1)$ over \mathcal{S} :

$$t = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 3 & \\ \hline 2 & 4 & \\ \hline 4 & & \\ \hline \end{array} \quad \text{with } \mathbf{R}(t) = 42114323. \quad (2)$$

The following diagram is a super skew tableau of shape $(6, 4, 2, 2, 1, 1)/(3, 2, 2, 1)$ over \mathcal{S} :

$$S = \begin{array}{|c|c|c|c|c|c|} \hline & & & \color{blue}{\square} & 1 & 2 & 3 \\ \hline & & & 1 & \color{blue}{3} & & \\ \hline & & & & & & \\ \hline & & & 4 & & & \\ \hline & & & & 2 & & \\ \hline & & & & 4 & & \\ \hline \end{array} \quad \text{with } \mathbf{R}(S) = 42413123, \quad (3)$$

where the empty blue box and the blue one filled with 3 denote respectively an inner corner and an outer corner of S .

2.2. A super Robinson–Schensted correspondence. From a super tableau we can obtain a word over \mathcal{S} by taking its column reading. The following algorithm will allow us to invert this process and obtain a super tableau from any word over \mathcal{S} .

Algorithm 2.2.1 ([20]). The *right (or row) insertion*, denoted by \leftarrow , inserts an element x in \mathcal{S} into a super tableau t of $\text{Yt}(\mathcal{S})$ as follows:

Input: A super tableau t and a letter $x \in \mathcal{S}$.

Output: A super tableau $t \leftarrow x$.

Method: If t is empty, create a box and fill it with x . Suppose t is non-empty. If $x \in \mathcal{S}_0$ (resp. $x \in \mathcal{S}_1$) is at least as large as (resp. larger than) the last element of the top row of t , then put x in a box to the right of this row; otherwise let y be the smallest element of the top row of t such that $y > x$ (resp. $y \geq x$). Then replace y by x in this row and recursively insert y into the super tableau formed by the rows of t below the topmost. Note that this recursion may end with an insertion into an empty row below the existing rows of t . Output the resulting super tableau.

For any word $w = x_1 \dots x_k$ over \mathcal{S} , a super tableau $\mathbf{T}(w)$ is computed from w by inserting its letters iteratively from left to right starting from the empty super tableau, as follows:

$$\mathbf{T}(w) := (\emptyset \leftarrow w) = ((\dots (\emptyset \leftarrow x_1) \leftarrow \dots) \leftarrow x_k).$$

Note that, for any t in $\text{Yt}(\mathcal{S})$, the equality $\mathbf{T}(\mathbf{R}(t)) = t$ holds, [20].

Lemma 2.2.2 (Row-bumping lemma, [20]). *Let t be in $\text{Yt}(\mathcal{S})$ and x, x' be in \mathcal{S} . Consider the position (i, j) (resp. (i', j')) of the new box added to the frame of t (resp. $(t \leftarrow x)$) after computing $(t \leftarrow x)$ (resp. $((t \leftarrow x) \leftarrow x')$). The following conditions are equivalent:*

i) $x \leq x'$, with $x = x'$ only if $\|x\| = 0$,

ii) $j < j'$ and $i \geq i'$.

2.2. A super Robinson–Schensted correspondence

Algorithm 2.2.3. Let w be in \mathcal{S} . When computing the super tableau $T(w)$, a standard tableau $Q(w)$ over $[n]$ can be also computed as follows:

Input: A word $x_1 \dots x_k$ over \mathcal{S} .

Output: A super tableau $T(x_1 \dots x_k)$ over \mathcal{S} and a standard tableau $Q(x_1 \dots x_k)$ over $[n]$.

Method: Start with an empty super tableau T_0 and an empty standard tableau Q_0 . For each $i = 1, \dots, k$, compute $T_{i-1} \leftarrow x_i$ as per Algorithm 2.2.1; let T_i be the resulting super tableau. Add a box filled with i to the standard tableau Q_{i-1} in the same place as the box that belongs to T_i but not to T_{i-1} ; let Q_i be the resulting standard tableau. Output T_k for $T(x_1 \dots x_k)$ and Q_k as $Q(x_1 \dots x_k)$.

Example 2.2.4. For instance, consider the alphabet $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$ with signature given by $\mathcal{S}_0 = \{1, 3, 4\}$ and \mathcal{S}_1 defined consequently. The sequence of pairs produced during the computation of $T(w)$ and $Q(w)$ starting from the word $w = 2421456356431541$ is

$$\begin{aligned}
 & (\emptyset, \emptyset), (\boxed{2}, \boxed{1}), (\boxed{2\ 4}, \boxed{1\ 2}), \left(\begin{array}{|c|c|} \hline \boxed{2} & \boxed{4} \\ \hline \boxed{2} & \\ \hline \end{array}, \begin{array}{|c|c|} \hline \boxed{1} & \boxed{2} \\ \hline \boxed{3} & \\ \hline \end{array} \right), \left(\begin{array}{|c|c|} \hline \boxed{1} & \boxed{4} \\ \hline \boxed{2} & \\ \hline \boxed{2} & \\ \hline \end{array}, \begin{array}{|c|c|} \hline \boxed{1} & \boxed{2} \\ \hline \boxed{3} & \\ \hline \boxed{4} & \\ \hline \end{array} \right), \left(\begin{array}{|c|c|c|} \hline \boxed{1} & \boxed{4} & \boxed{4} \\ \hline \boxed{2} & & \\ \hline \boxed{2} & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \boxed{1} & \boxed{2} & \boxed{5} \\ \hline \boxed{3} & & \\ \hline \boxed{4} & & \\ \hline \end{array} \right), \\
 & \left(\begin{array}{|c|c|c|c|} \hline \boxed{1} & \boxed{4} & \boxed{4} & \boxed{5} \\ \hline \boxed{2} & & & \\ \hline \boxed{2} & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \boxed{1} & \boxed{2} & \boxed{5} & \boxed{6} \\ \hline \boxed{3} & & & \\ \hline \boxed{4} & & & \\ \hline \end{array} \right), \left(\begin{array}{|c|c|c|c|c|} \hline \boxed{1} & \boxed{4} & \boxed{4} & \boxed{5} & \boxed{6} \\ \hline \boxed{2} & & & & \\ \hline \boxed{2} & & & & \\ \hline \boxed{4} & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline \boxed{1} & \boxed{2} & \boxed{5} & \boxed{6} & \boxed{7} \\ \hline \boxed{3} & & & & \\ \hline \boxed{4} & & & & \\ \hline \end{array} \right), \left(\begin{array}{|c|c|c|c|c|} \hline \boxed{1} & \boxed{3} & \boxed{4} & \boxed{5} & \boxed{6} \\ \hline \boxed{2} & \boxed{4} & & & \\ \hline \boxed{2} & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline \boxed{1} & \boxed{2} & \boxed{5} & \boxed{6} & \boxed{7} \\ \hline \boxed{3} & \boxed{8} & & & \\ \hline \boxed{4} & & & & \\ \hline \end{array} \right), \\
 & \left(\begin{array}{|c|c|c|c|c|} \hline \boxed{1} & \boxed{3} & \boxed{4} & \boxed{5} & \boxed{6} \\ \hline \boxed{2} & \boxed{4} & \boxed{5} & & \\ \hline \boxed{2} & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline \boxed{1} & \boxed{2} & \boxed{5} & \boxed{6} & \boxed{7} \\ \hline \boxed{3} & \boxed{8} & \boxed{9} & & \\ \hline \boxed{4} & & & & \\ \hline \end{array} \right), \left(\begin{array}{|c|c|c|c|c|} \hline \boxed{1} & \boxed{3} & \boxed{4} & \boxed{5} & \boxed{6} \\ \hline \boxed{2} & \boxed{4} & \boxed{5} & \boxed{6} & \\ \hline \boxed{2} & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline \boxed{1} & \boxed{2} & \boxed{5} & \boxed{6} & \boxed{7} \\ \hline \boxed{3} & \boxed{8} & \boxed{9} & \boxed{10} & \\ \hline \boxed{4} & & & & \\ \hline \end{array} \right), \\
 & \left(\begin{array}{|c|c|c|c|c|} \hline \boxed{1} & \boxed{3} & \boxed{4} & \boxed{4} & \boxed{6} \\ \hline \boxed{2} & \boxed{4} & \boxed{5} & \boxed{6} & \\ \hline \boxed{2} & \boxed{5} & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline \boxed{1} & \boxed{2} & \boxed{5} & \boxed{6} & \boxed{7} \\ \hline \boxed{3} & \boxed{8} & \boxed{9} & \boxed{10} & \\ \hline \boxed{4} & \boxed{11} & & & \\ \hline \end{array} \right), \left(\begin{array}{|c|c|c|c|c|} \hline \boxed{1} & \boxed{3} & \boxed{3} & \boxed{4} & \boxed{6} \\ \hline \boxed{2} & \boxed{4} & \boxed{4} & \boxed{6} & \\ \hline \boxed{2} & \boxed{5} & & & \\ \hline \boxed{5} & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline \boxed{1} & \boxed{2} & \boxed{5} & \boxed{6} & \boxed{7} \\ \hline \boxed{3} & \boxed{8} & \boxed{9} & \boxed{10} & \\ \hline \boxed{4} & \boxed{11} & & & \\ \hline \boxed{12} & & & & \\ \hline \end{array} \right), \\
 & \left(\begin{array}{|c|c|c|c|c|} \hline \boxed{1} & \boxed{1} & \boxed{3} & \boxed{4} & \boxed{6} \\ \hline \boxed{2} & \boxed{3} & \boxed{4} & \boxed{6} & \\ \hline \boxed{2} & \boxed{4} & & & \\ \hline \boxed{5} & & & & \\ \hline \boxed{5} & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline \boxed{1} & \boxed{2} & \boxed{5} & \boxed{6} & \boxed{7} \\ \hline \boxed{3} & \boxed{8} & \boxed{9} & \boxed{10} & \\ \hline \boxed{4} & \boxed{11} & & & \\ \hline \boxed{12} & & & & \\ \hline \boxed{13} & & & & \\ \hline \end{array} \right), \left(\begin{array}{|c|c|c|c|c|} \hline \boxed{1} & \boxed{1} & \boxed{3} & \boxed{4} & \boxed{5} \\ \hline \boxed{2} & \boxed{3} & \boxed{4} & \boxed{6} & \\ \hline \boxed{2} & \boxed{4} & \boxed{6} & & \\ \hline \boxed{5} & & & & \\ \hline \boxed{5} & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline \boxed{1} & \boxed{2} & \boxed{5} & \boxed{6} & \boxed{7} \\ \hline \boxed{3} & \boxed{8} & \boxed{9} & \boxed{10} & \\ \hline \boxed{4} & \boxed{11} & \boxed{14} & & \\ \hline \boxed{12} & & & & \\ \hline \boxed{13} & & & & \\ \hline \end{array} \right), \\
 & \left(\begin{array}{|c|c|c|c|c|} \hline \boxed{1} & \boxed{1} & \boxed{3} & \boxed{4} & \boxed{4} \\ \hline \boxed{2} & \boxed{3} & \boxed{4} & \boxed{5} & \\ \hline \boxed{2} & \boxed{4} & \boxed{6} & & \\ \hline \boxed{5} & \boxed{6} & & & \\ \hline \boxed{5} & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline \boxed{1} & \boxed{2} & \boxed{5} & \boxed{6} & \boxed{7} \\ \hline \boxed{3} & \boxed{8} & \boxed{9} & \boxed{10} & \\ \hline \boxed{4} & \boxed{11} & \boxed{14} & & \\ \hline \boxed{12} & \boxed{15} & & & \\ \hline \boxed{13} & & & & \\ \hline \end{array} \right), \left(T(w) = \begin{array}{|c|c|c|c|c|} \hline \boxed{1} & \boxed{1} & \boxed{1} & \boxed{4} & \boxed{4} \\ \hline \boxed{2} & \boxed{3} & \boxed{3} & \boxed{5} & \\ \hline \boxed{2} & \boxed{4} & \boxed{4} & & \\ \hline \boxed{5} & \boxed{6} & & & \\ \hline \boxed{5} & \boxed{6} & & & \\ \hline \end{array}, Q(w) = \begin{array}{|c|c|c|c|c|} \hline \boxed{1} & \boxed{2} & \boxed{5} & \boxed{6} & \boxed{7} \\ \hline \boxed{3} & \boxed{8} & \boxed{9} & \boxed{10} & \\ \hline \boxed{4} & \boxed{11} & \boxed{14} & & \\ \hline \boxed{12} & \boxed{15} & & & \\ \hline \boxed{13} & \boxed{16} & & & \\ \hline \end{array} \right).
 \end{aligned}$$

Algorithm 2.2.5 ([20]). Let t be in $\text{Yt}(\mathcal{S})$ and x in \mathcal{S} . Starting with the super tableau $t \leftarrow x$, together with the outer corner that has been added to the frame of t after insertion, we can recover the original tableau t and the element x as follows:

Input: A super tableau t and an outer corner of t .

Output: A super tableau t' and a letter $x \in \mathcal{S}$.

Method: Suppose that $y \in \mathcal{S}_0$ (resp. $y \in \mathcal{S}_1$) is the entry in the outer corner of t . Find in the row above this outer corner the entry farthest to the right which is strictly smaller (resp. smaller) than y . Then this entry is replaced by y and it is bumped up to the next row where the process is

2. The super plactic monoid of type A

repeated until an entry is bumped out of the top row. Output the resulting super tableau for t' and the last element which is bumped out for x .

Algorithm 2.2.6. Let w be in \mathcal{S} . Starting from the pair of super tableaux $(T(w), Q(w))$ obtained by Algorithm 2.2.3 we can recover the initial word w as follows:

Input: A pair $(T, Q) \in \text{Yt}(\mathcal{S}) \times \text{Ys}([n])$ of same-shape super tableaux containing k boxes.

Output: A word $x_1 \dots x_k$ over \mathcal{S} .

Method: Start with $T_k = T$ and $Q_k = Q$. For each $i = k - 1, \dots, 1$, take the box filled with $i + 1$ in Q_{i+1} , and apply Algorithm 2.2.5 to T_{i+1} with the outer corner corresponding to that box. The resulting tableau is T_i , and the element that is bumped out of the top row of T_{i+1} is denoted by x_{i+1} . Remove the box containing $i + 1$ from Q_{i+1} and denote the resulting standard tableau by Q_i . Output the word $x_1 \dots x_k$.

This sets up a one-to-one correspondence between words over \mathcal{S} and pairs of same-shape tableaux (T, Q) in $\text{Yt}(\mathcal{S}) \times \text{Ys}([n])$:

Proposition 2.2.7. *The map $w \mapsto (T(w), Q(w))$ defines a bijection between words over \mathcal{S} and pairs consisting of a super tableau over \mathcal{S} and a standard tableau over $[n]$ of the same shape.*

This is the super version of the *Robinson–Schensted correspondence* between words over $[n]$ and pairs consisting of a semistandard tableau and a standard tableau over $[n]$ of the same shape, [9].

2.3. The super plactic monoid of type A. The *super plactic monoid* over \mathcal{S} is the monoid, denoted by $\mathbf{P}(\mathcal{S})$, generated by the set \mathcal{S} and subject to the following family of *super Knuth relations*, [20]:

$$\begin{aligned} xzy &= zxy, & \text{with } x = y & \text{ only if } \|y\| = 0 \text{ and } y = z \text{ only if } \|y\| = 1, \\ yxz &= yzx, & \text{with } x = y & \text{ only if } \|y\| = 1 \text{ and } y = z \text{ only if } \|y\| = 0, \end{aligned} \quad (4)$$

for any $x \leq y \leq z$ of elements of \mathcal{S} . The congruence generated by the relations (4), is denoted by $\sim_{\mathbf{P}(\mathcal{S})}$, and called the *super plactic congruence*. Two words over \mathcal{S} are *super plactic equivalent* if one can be transformed into the other with respect $\sim_{\mathbf{P}(\mathcal{S})}$. For instance, consider the readings of the super tableau t presented in (2) and the super skew tableau S presented in (3), we have

$$\mathbf{R}(t) = 42114323 \sim_{\mathbf{P}(\mathcal{S})} 42141323 \sim_{\mathbf{P}(\mathcal{S})} 42411323 \sim_{\mathbf{P}(\mathcal{S})} 42413123 = \mathbf{R}(S). \quad (5)$$

Since the relations (4) are \mathbb{Z}_2 -homogeneous, the monoid $\mathbf{P}(\mathcal{S})$ is a supermonoid. For any w in \mathcal{S}^* , we have $w \sim_{\mathbf{P}(\mathcal{S})} \mathbf{R}(T(w))$, [13]. Moreover, the set $\text{Yt}(\mathcal{S})$ satisfies the *cross-section property* for $\sim_{\mathbf{P}(\mathcal{S})}$, that is:

Property 2.3.1 ([20]). For all w and w' in \mathcal{S}^* , we have $w \sim_{\mathbf{P}(\mathcal{S})} w'$ if and only if $T(w) = T(w')$.

We define an internal product $\star_{\text{Yt}(\mathcal{S})}$ on $\text{Yt}(\mathcal{S})$ by setting $t \star_{\text{Yt}(\mathcal{S})} t' := (t \leftarrow \mathbf{R}(t'))$, for all t, t' in $\text{Yt}(\mathcal{S})$. By definition the relations $t \star_{\text{Yt}(\mathcal{S})} \emptyset = t$ and $\emptyset \star_{\text{Yt}(\mathcal{S})} t = t$ hold, showing that the product $\star_{\text{Yt}(\mathcal{S})}$ is unitary with respect to \emptyset . Following Property 2.3.1, we deduce that the product $\star_{\text{Yt}(\mathcal{S})}$ is associative and the monoids $(\text{Yt}(\mathcal{S}), \star_{\text{Yt}(\mathcal{S})})$ and $\mathbf{P}(\mathcal{S})$ are isomorphic.

Lemma 2.3.2. *Let w and v be two words over \mathcal{S} and let w' and v' be the words obtained respectively from w and v after removing the i largest and the j smallest letters for any i and j . If w and v are super plactic equivalent, then w' and v' are so.*

Proof. It is sufficient by induction to prove that the words obtained by removing the largest or the smallest letters from w and v are super plactic equivalent. We prove the case of the largest, the other being similar. Suppose that $w = uxzyu'$ and $v = uzxyu'$ (resp. $w = uyxzu'$ and $v = uyzxu'$) for all u, u' in \mathcal{S}^* such that $x \leq y \leq z$ in \mathcal{S} with $x = y$ only if $\|y\| = 0$ (resp. $\|y\| = 1$) and $y = z$ only if $\|y\| = 1$ (resp. $\|y\| = 0$). We consider two cases depending on whether or not the element that is removed from w and v is one of the letters x, y , or z . If this element is not one of these letters, then the resulting words are obviously super plactic equivalent. Otherwise, the removed letter must be the letter z and then the resulting words are equal, showing the claim. \square

2.4. The super jeu de taquin. We recall from [13] the super jeu de taquin procedure which transforms super skew tableaux into super tableaux over \mathcal{S} as follows. A *forward sliding* is a sequence of the following *sliding operations*:

$$\begin{array}{c} \boxed{y} \\ \boxed{x} \end{array} \rightarrow \begin{array}{c} \boxed{x} \boxed{y} \\ \boxed{} \end{array}, \text{ for any } x \leq y \text{ with } x = y \text{ only if } \|x\| = 0,$$

$$\begin{array}{c} \boxed{x} \\ \boxed{y} \end{array} \rightarrow \begin{array}{c} \boxed{x} \\ \boxed{y} \end{array} \boxed{}, \text{ for any } x \leq y \text{ with } x = y \text{ only if } \|x\| = 1,$$

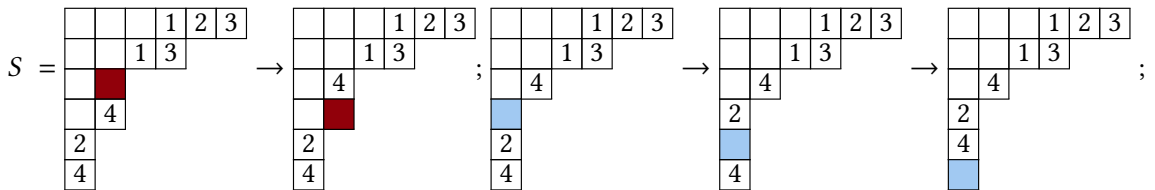
$$\begin{array}{c} \boxed{} \\ \boxed{x} \end{array} \rightarrow \begin{array}{c} \boxed{x} \\ \boxed{} \end{array}, \quad \begin{array}{c} \boxed{x} \\ \boxed{} \end{array} \rightarrow \begin{array}{c} \boxed{x} \\ \boxed{} \end{array} \boxed{}, \quad \begin{array}{c} \boxed{} \\ \boxed{} \end{array} \rightarrow \begin{array}{c} \boxed{} \\ \boxed{} \end{array} \boxed{}, \text{ for any } x,$$

starting from a super skew tableau and one of its inner corners, and moving the empty box until it becomes an outer corner. The *super jeu de taquin* on a super skew tableau S consists in applying successively the forward sliding algorithm starting from S until we obtain a super tableau, denoted by $\text{Rect}(S)$, and called the *rectification* of S , [13]. Two super skew tableaux are *super jeu de taquin equivalent* if one can be obtained from the other by a sequence of sliding operations. We show that if two super skew tableaux are super jeu de taquin equivalent, then their reading words are super plactic equivalent, [13]. Moreover, we show the following property:

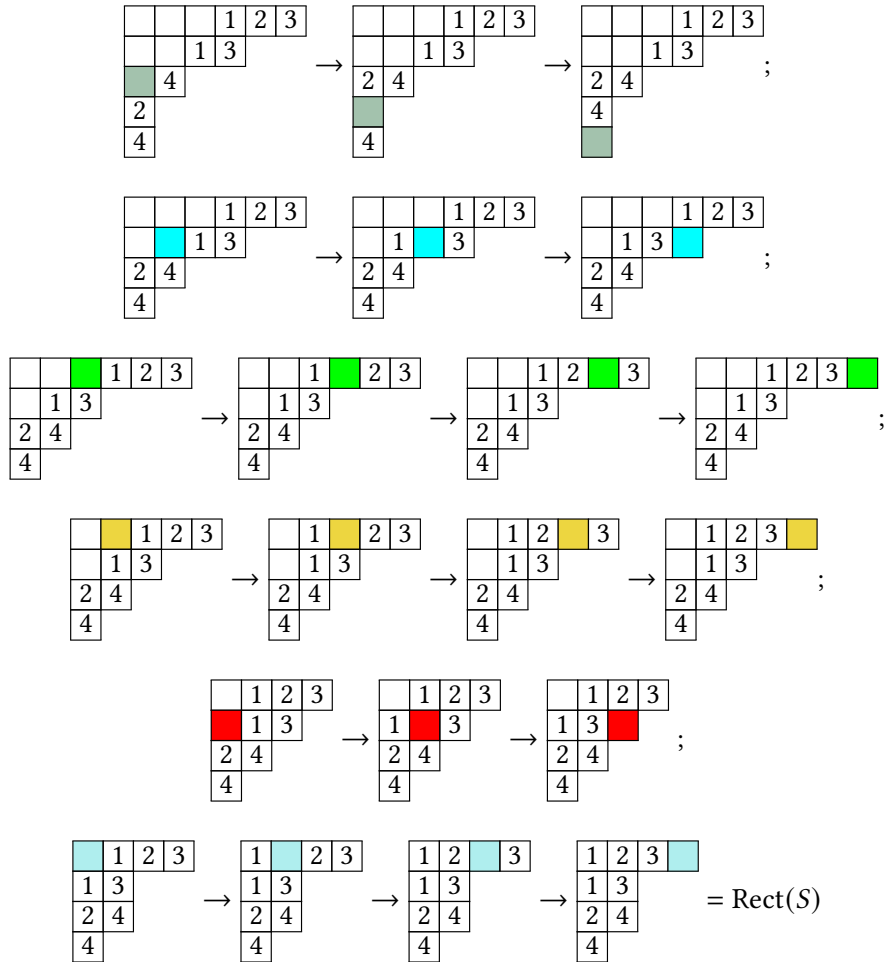
Property 2.4.1 ([13]). For all $S, S' \in \text{St}(\mathcal{S})$, $\text{Rect}(S) = \text{Rect}(S')$ if and only if $\mathbf{R}(S) \sim_{\mathcal{P}(\mathcal{S})} \mathbf{R}(S')$.

We deduce that starting with a given super skew tableau, all choices of inner corners lead to the same rectified super tableau, [13]. This is the *confluence property* of the super jeu de taquin.

Example 2.4.2. For instance, the super jeu de taquin on the super skew tableau S in (3) applies eight occurrences of forward sliding:

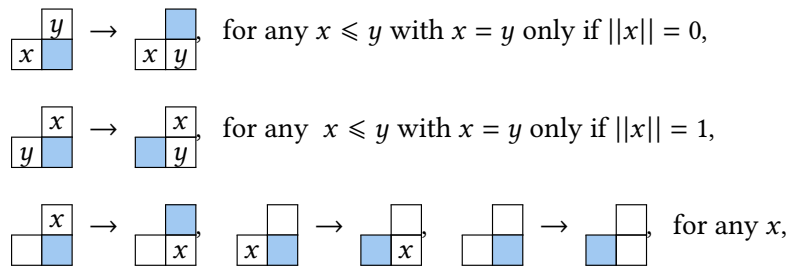


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with $\mathbf{R}(S) = 42413123 \sim_{\mathbf{P}(S)} 42114323 = \mathbf{R}(\text{Rect}(S))$, as shown in (5).

2.4.3. The reverse sliding. A *reverse sliding* is a sequence of the following *reverse sliding operations*, [13]:



starting from a super skew tableau and one of its empty outer corners, and moving this box until it becomes an inner corner. Starting from the resulting super skew tableau of a forward sliding, together with the outer corner that was removed, and by applying the reverse sliding, we recover the initial super skew tableau with the chosen inner corner.

3. A super-RSK correspondence with symmetry

Output: A pair $(\mathbf{T}(w), \mathbf{Q}(w)) \in \text{Yt}(\mathcal{S}') \times \text{Yt}(\mathcal{S})$ of same-shape super tableaux with k boxes.

Method: Start with an empty super tableau T_0 and an empty super tableau Q_0 . For each $i = 1, \dots, k$, compute $T_{i-1} \leftarrow y_i$ as per Algorithm 2.2.1 and let T_i be the resulting super tableau. Add a box filled with x_i to the super tableau Q_{i-1} in the same place as the box that belongs to T_i but not to T_{i-1} ; let Q_i be the resulting super tableau. Output T_k for $\mathbf{T}(w)$ and Q_k as $\mathbf{Q}(w)$.

Proposition 3.1.2. *Algorithm 3.1.1 always halts with the correct output.*

Proof. By construction, we have that $\mathbf{T}(w)$ is a super tableau over \mathcal{S}' since it is equal to the super tableau $\mathbf{T}(y_1 \dots y_k)$. We still have to show that $\mathbf{Q}(w)$ is a super tableau over \mathcal{S} . It is sufficient to show by induction that each Q_i , for $i = 1, \dots, k$, is a super tableau over \mathcal{S} . By definition of the signed two-rowed array, if $x_i \in \mathcal{S}_0$ is placed to the right of (resp. under or to the right of) an entry $x_l \in \mathcal{S}_0$ (resp. $x_l \in \mathcal{S}_1$) in Q_{i-1} , then x_i is larger (resp. strictly larger) than x_l . Similarly, if $x_i \in \mathcal{S}_1$ is placed under (resp. under or to the right of) an entry $x_l \in \mathcal{S}_1$ (resp. $x_l \in \mathcal{S}_0$) in Q_{i-1} , then x_i is larger (resp. strictly larger) than x_l . Let us now show that if $x_i \in \mathcal{S}_0$ is placed under of an entry $x_l \in \mathcal{S}_0$ in Q_{i-1} , then x_i is strictly larger than x_l . Suppose the contrary. Then, we have $x_i = x_l$ and thus by Conditions (6) and (7) we have:

$$y_l \leq y_{l+1} \leq \dots \leq y_i,$$

with $y_l = y_{l+1} = \dots = y_i$ only if $\|y_l\| = 0$. Hence, following Algorithm 2.2.1, all the boxes added starting from T_l to T_i must be in different columns which contradicts the fact that x_i and x_l belong to the same column. Similarly, we show that if $x_i \in \mathcal{S}_1$ is placed to the right of an entry $x_l \in \mathcal{S}_1$ in Q_{i-1} , then x_i is strictly larger than x_l . Note finally that, by construction, the super tableaux $\mathbf{T}(w)$ and $\mathbf{Q}(w)$ are of the same shape and contains k boxes. \square

Algorithm 3.1.3. Let w be a signed two-rowed array on signed alphabets \mathcal{S} and \mathcal{S}' . Starting from the pair $(\mathbf{T}(w), \mathbf{Q}(w))$ constructed by Algorithm 3.1.1, we can recover the initial signed two-rowed array w as follows:

Input: A pair $(T, Q) \in \text{Yt}(\mathcal{S}') \times \text{Yt}(\mathcal{S})$ of same-shape super tableaux containing k boxes.

Output: A signed two-rowed array $w = \begin{pmatrix} x_1 \dots x_k \\ y_1 \dots y_k \end{pmatrix}$ on \mathcal{S} and \mathcal{S}' .

Method: Start with $T_k = T$ and $Q_k = Q$. For each $i = k - 1, \dots, 1$, take the box filled with the largest entry x_{i+1} in Q_{i+1} and if there are several equal entries in \mathcal{S}_0 (resp. \mathcal{S}_1), the box that is farthest to the right (resp. left) is selected; apply Algorithm 2.2.5 to T_{i+1} with the outer corner corresponding to that box. Let T_i be the resulting super tableau and y_{i+1} be the element that is bumped out of the top row of T_{i+1} . Remove the box containing x_{i+1} from Q_{i+1} and denote the resulting super tableau by Q_i . Output the $2 \times k$ matrix $\begin{pmatrix} x_1 \dots x_k \\ y_1 \dots y_k \end{pmatrix}$.

Proposition 3.1.4. *Algorithm 3.1.3 always halts with the correct output.*

Proof. By construction, we have $x_1 \leq \dots \leq x_k$. Suppose now that $x_{i-1} = x_i$, for $i \in [k]$ with $\|x_i\| = 0$. Then by constructing, the box that is removed from T_i lies strictly to the right of the box that is removed from T_{i-1} in the next step. Following Lemma 2.2.2, the entry y_i removed first is at least as large as the entry y_{i-1} removed second, with $y_i = y_{i-1}$, only if $\|y_i\| = 0$.

3. A super-RSK correspondence with symmetry

$$\left(\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 3 & 4 & 6 \\ \hline 2 & 3 & 4 & 6 & \\ \hline 2 & 4 & & & \\ \hline 5 & & & & \\ \hline 5 & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 3 & 3 \\ \hline 2 & 4 & 4 & 4 & \\ \hline 2 & 5 & & & \\ \hline 5 & & & & \\ \hline 5 & & & & \\ \hline \end{array} \right), \left(\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 3 & 4 & 5 \\ \hline 2 & 3 & 4 & 6 & \\ \hline 2 & 4 & 6 & & \\ \hline 5 & & & & \\ \hline 5 & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 3 & 3 \\ \hline 2 & 4 & 4 & 4 & \\ \hline 2 & 5 & 6 & & \\ \hline 5 & & & & \\ \hline 5 & & & & \\ \hline \end{array} \right),$$

$$\left(\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 3 & 4 & 4 \\ \hline 2 & 3 & 4 & 5 & \\ \hline 2 & 4 & 6 & & \\ \hline 5 & 6 & & & \\ \hline 5 & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 3 & 3 \\ \hline 2 & 4 & 4 & 4 & \\ \hline 2 & 5 & 6 & & \\ \hline 5 & 6 & & & \\ \hline 5 & & & & \\ \hline \end{array} \right), \left(\mathbf{T}(w) = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 4 & 4 \\ \hline 2 & 3 & 3 & 5 & \\ \hline 2 & 4 & 4 & & \\ \hline 5 & 6 & & & \\ \hline 5 & 6 & & & \\ \hline \end{array}, \mathbf{Q}(w) = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 3 & 3 \\ \hline 2 & 4 & 4 & 4 & \\ \hline 2 & 5 & 6 & & \\ \hline 5 & 6 & & & \\ \hline 5 & 6 & & & \\ \hline \end{array} \right).$$

By applying Algorithm 3.1.3 on $(\mathbf{T}(w), \mathbf{Q}(w))$, we recover the initial signed two-rowed array w .

Example 3.1.7. Let $\mathcal{S} = \mathcal{S}' = \mathbb{Z}_{>0}$ with signature given by \mathcal{S}_0 (resp \mathcal{S}'_1) the set of even numbers and \mathcal{S}_1 (resp. \mathcal{S}'_0) defined consequently. The sequence of pairs produced during the computation of $(\mathbf{T}(w)$ and $\mathbf{Q}(w)$ starting from the signed two-rowed array $w = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 3 & 2 & 1 & 2 & 4 & 3 & 1 & 2 \end{pmatrix}$ on \mathcal{S} and \mathcal{S}' is the following:

$$(\emptyset, \emptyset), (\boxed{3}, \boxed{1}), \left(\begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline \end{array} \right), \left(\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline 2 \\ \hline \end{array} \right), \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 1 & \\ \hline 2 & \\ \hline \end{array} \right), \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 2 & & \\ \hline 3 & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & & \\ \hline 2 & & \\ \hline \end{array} \right),$$

$$\left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 4 & \\ \hline 3 & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 3 & \\ \hline 2 & & \\ \hline \end{array} \right), \left(\begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 4 & \\ \hline 2 & & \\ \hline 3 & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 3 & \\ \hline 2 & & \\ \hline 4 & & \\ \hline \end{array} \right), \left(\mathbf{T}(w) = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & \\ \hline 2 & 4 & \\ \hline 3 & & \\ \hline \end{array}, \mathbf{Q}(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 3 & \\ \hline 2 & 4 & \\ \hline 4 & & \\ \hline \end{array} \right).$$

By applying Algorithm 3.1.3 on $(\mathbf{T}(w), \mathbf{Q}(w))$, we recover the initial signed two-rowed array w .

3.1.8. The symmetry property. Let w be a signed two-rowed array on signed alphabets \mathcal{S} and \mathcal{S}' . The *inverse* of w , denoted by w^{inv} , is the signed two-rowed array on \mathcal{S}' and \mathcal{S} obtained from w by *exchanging* the rows of w , that is, writing the second row of w as the first row and the first row of w as the second row, and by sorting the new couples on $\mathcal{S}' \times \mathcal{S}$ according to Conditions (6) and (7). For instance, the inverse of the signed two-rowed array w of Example 3.1.6 (resp. Example 3.1.7) is the following:

$$w^{\text{inv}} = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 4 & 4 & 5 & 5 & 6 & 6 \\ 2 & 5 & 6 & 2 & 1 & 4 & 5 & 2 & 3 & 5 & 6 & 6 & 4 & 3 & 4 & 3 \end{pmatrix} \quad (\text{resp. } w^{\text{inv}} = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 & 4 \\ 2 & 4 & 4 & 2 & 1 & 1 & 3 & 3 \end{pmatrix}).$$

We say that w has *symmetry with respect to the map sRSK* if it satisfies the following property:

Property 3.1.9. If $\text{sRSK}(w) = (\mathbf{T}(w), \mathbf{Q}(w))$ then $\text{sRSK}(w^{\text{inv}}) = (\mathbf{Q}(w), \mathbf{T}(w))$.

Theorem 3.1.10 (Symmetry of the super-RSK correspondence). *Let \mathcal{S} and \mathcal{S}' be signed alphabets. All signed two-rowed arrays on \mathcal{S} and \mathcal{S}' have symmetry with respect to the super-RSK correspondence map sRSK.*

The rest of this section is devoted to prove this result. We give a geometrical interpretation of the super-RSK correspondence following Fulton's matrix-ball construction, [9], for the non-signed case. This construction will allow us to prove Theorem 3.1.10 using the symmetry of the resulting signed ball array corresponding to a signed two-rowed array.

3.2. Super matrix-ball construction. Let \mathcal{S} and \mathcal{S}' be signed alphabets. A *signed ball array* on \mathcal{S} and \mathcal{S}' is a rectangular array of balls filled with positive integers, whose rows (resp. columns) are indexed with elements of \mathcal{S} (resp. \mathcal{S}'), from the smaller to the bigger one, allowing the repetition only of elements in \mathcal{S}_1 (resp. \mathcal{S}'_1) and where many balls can occur in the same position. A signed ball array is *empty* if it does not contain any ball. For instance, consider $\mathcal{S} = \mathcal{S}' = [4]$ with signature given by \mathcal{S}_0 (resp. \mathcal{S}'_1) the set of even numbers and \mathcal{S}_1 (resp. \mathcal{S}'_0) defined consequently. The rectangular array in Figure 1 is a signed ball array on \mathcal{S} and \mathcal{S}' .

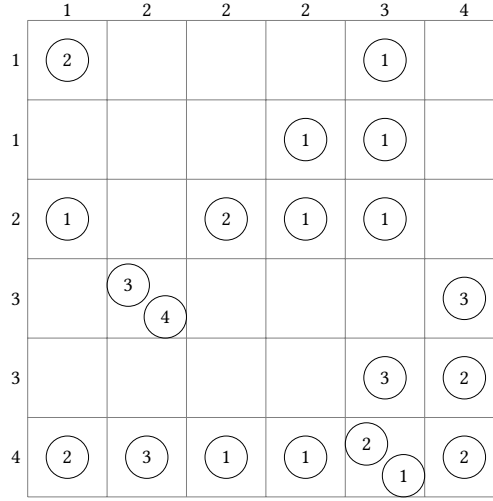


Figure 1: Example of a signed ball array.

We will use the following notations to describe the relative positions of two boxes in a signed ball array. A box b' is *West* (resp. *west*) of a box b if the column of b' is strictly to the left of (resp. left of or equal to) the column of b . A box b' is *North* (resp. *north*) of a box b if the row of b' is strictly above (resp. above or equal to) the row of b . Similarly, we define the other positions corresponding to the east and south directions using capital and small letters to denote strict and weak positions. A box b' is *Northwest* of a box b if the row of b' is strictly above to the row of b , and the column of b' is left or equal to the column of b . Similarly, we define the other combinations of positions corresponding to the four cardinal directions using capital and small letters to denote strict and weak inequalities.

3.2.1. The matrix-ball construction. Let \mathcal{S} and \mathcal{S}' be signed alphabets. We will correspond to each signed two-rowed array $w = \begin{pmatrix} x_1 \dots x_k \\ y_1 \dots y_k \end{pmatrix}$ on \mathcal{S} and \mathcal{S}' , a signed ball array, denoted by $\text{Ba}(w)$, whose rows (resp. columns) are indexed with the elements of the first (resp. second) row of w and where the indices from \mathcal{S}_1 and \mathcal{S}'_1 are repeated as many times as they appear in w , as described in the following three steps:

Step 1. We start with an empty signed ball array, whose rows (resp. columns) are indexed with the elements of the first (resp. second) row of w and where the indices from \mathcal{S}_1 and \mathcal{S}'_1 are repeated as many times as they appear in w . We then associate to each couple (x_i, y_i) in w for $i = 1, \dots, k$, a

3. A super-RSK correspondence with symmetry

ball in a box of the initial empty signed ball array according to the following four cases:

- i) Suppose $(x_i, y_i) \in \mathcal{S}_0 \times \mathcal{S}'_0$. Following Condition (7), equal couples of this form can occur in w . For each couple (x_i, y_i) we associate a ball in the box corresponding to the row indexed with x_i and to the column indexed with y_i in the signed ball array. In this case, the number of balls in the same position (x_i, y_i) is equal to the multiplicity of the couple (x_i, y_i) in w .
- ii) Suppose $(x_i, y_i) \in \mathcal{S}_1 \times \mathcal{S}'_1$. Following Condition (7), equal couples of this form can occur in w . For each couple (x_i, y_i) we associate a ball in the empty box of the signed ball array corresponding to the topmost row indexed with x_i and the rightmost column indexed with y_i such that only one ball can occur in the same position, and if there are many rows (resp. columns) indexed with x_i (resp. y_i) we choose the topmost (resp. rightmost) row (resp. column) that does not contain any ball.
- iii) Suppose $(x_i, y_i) \in \mathcal{S}_0 \times \mathcal{S}'_1$. Following Condition (7), we can not have equal couples of this form in w . We add a ball in the empty box of the signed ball array that corresponds to the row indexed with x_i and to the rightmost column indexed with y_i , and if there are many columns that are indexed with y_i we choose the rightmost one that does not contain any ball.
- iv) Suppose $(x_i, y_i) \in \mathcal{S}_1 \times \mathcal{S}'_0$. Following Condition (7), we can not have equal couples of this form in w . We add a ball in the empty box of the signed ball array that corresponds to the topmost row indexed with x_i and to the column indexed with y_i , and if there are many rows that are indexed with x_i , we choose the topmost one that does not contain any ball.

Step 2. If many balls occur in the same position, then we order them arbitrarily by arranging them diagonally from NorthWest to SouthEast. A ball is *northwest* of another one if it is in the same position and NorthWest in this arrangement, or its row and column positions are less than or equal to those of the second ball with at least one inequality strict. The *bottom-rightmost* ball of a signed ball array is the ball in the position that corresponds to the bottom-most row and to the rightmost column of the given signed ball array, and if many balls occur in this position it corresponds to the last one to the southeast in the corresponding diagonal arrangement. Similarly, we define the *top-leftmost ball* of a signed ball array.

Working from the top-leftmost ball to the bottom-rightmost ball and starting with the first row of the resulting signed ball array, we number all the balls with positive integers by filling each ball by the smallest integer that is larger than all the integers occurring in the balls to the northwest, such that the balls in the same position are numbered with consecutive integers. More precisely, a ball is numbered with 1 if there are no balls northwest of it. A ball is numbered with a positive integer i if the preceding ball in the same position is numbered with the integer $i - 1$, or if the ball is the first one in a given position and the largest number occurring in a ball northwest of the given position is the integer $i - 1$. The resulting signed ball array is denoted by $\text{Ba}^1(w)$.

For instance, Figure 2 (resp. Figure 3) represents the signed ball array $\text{Ba}^1(w)$ corresponding to the signed two-rowed array w of Example 3.1.6 (resp. Example 3.1.7).

Step 3. If there are $k > 1$ balls filled with the same integer i in the resulting signed ball array $\text{Ba}^1(w)$, then they belong by construction to a string from SouthWest to NorthEast. We then introduce new $k - 1$ balls by putting a ball to the right of each ball in the string but the last, directly under the next ball. We will use a new color for the added balls as illustrated in Figure 4.

3.2. Super matrix-ball construction

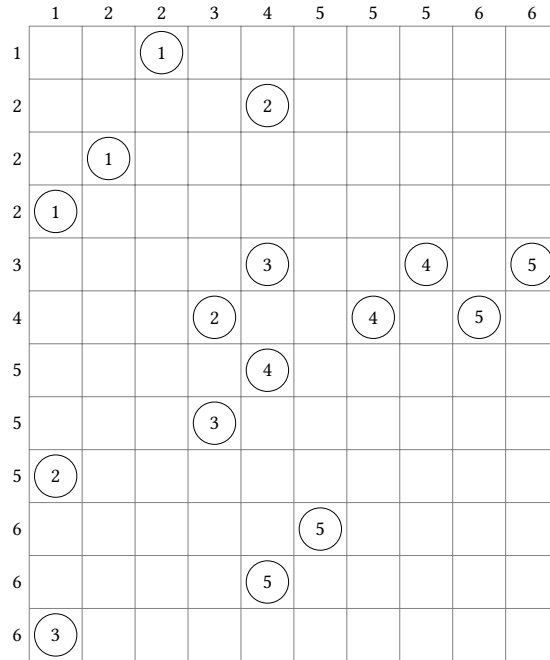


Figure 2: $Ba^1(w)$ corresponding to w of Example 3.1.6.

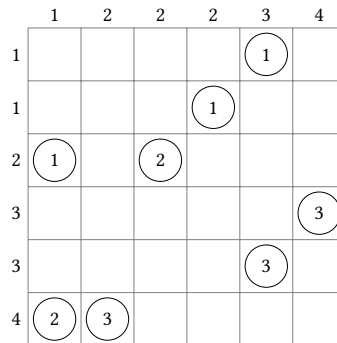


Figure 3: $Ba^1(w)$ corresponding to w of Example 3.1.7.

We do the same for all the balls filled with the same integer. We then number the new added balls as per **Step 2** without taking into consideration the numbering of the initial non-colored balls and by just acting on the new balls from the top-leftmost ball to the bottom-rightmost one. We obtain a new signed ball array, denoted by $Ba^2(w)$, that contains the initial non-colored balls and the new colored ones. We repeat the same process on the new added balls and we construct $Ba^3(w)$ from $Ba^2(w)$ by adding new colored balls and by numbering them, and so on, stopping when no two balls appear in $Ba^k(w)$ for any $k > 1$ with the same number. The resulting signed ball array is denoted by $Ba(w)$. Note that we will use the same color for all the colored balls added to compute $Ba^k(w)$ from $Ba^{k-1}(w)$ for any $k > 1$. We denote by $Ba_0(w)$ the signed ball array

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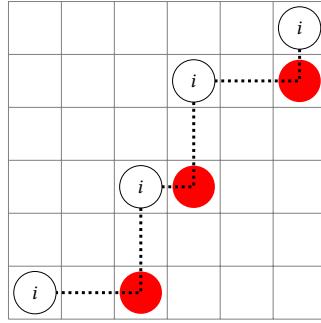


Figure 4: String for new colored balls

obtained from $Ba(w)$ by eliminating the initial non colored balls and by keeping only the new colored ones, and by $Ba_0^1(w)$ the signed ball array obtained from $Ba_0(w)$ by keeping only the colored balls added to compute $Ba^2(w)$ from $Ba^1(w)$ and by eliminating all the other colored balls. For instance, Figure 5 (resp. Figure 6) represents the signed ball array $Ba(w)$ corresponding to the signed two-rowed array w of Example 3.1.6 (resp. Example 3.1.7).

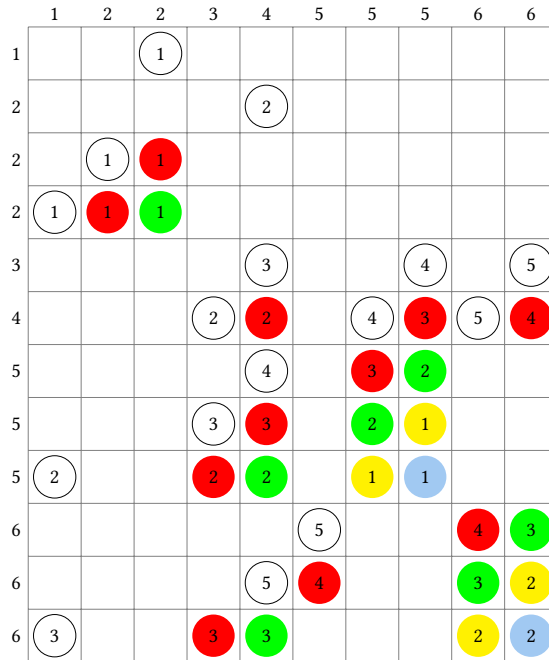


Figure 5: $Ba(w)$ corresponding to w of Example 3.1.6.

3.2.2. Super tableaux for signed ball arrays. Let w be a signed two-rowed array on signed alphabets \mathcal{S} and \mathcal{S}' . We denote by $T(Ba(w))$ (resp. $Q(Ba(w))$) the super tableau obtained from $Ba(w)$ such that its k -th row lists the indices of the leftmost columns (resp. top-most rows) of $Ba^k(w)$ where each integer number occurs in the new added balls. That is, the i -th entry of

	1	2	2	2	3	4
1					(1)	
1				(1)	(1)	
2	(1)		(2)	(1)	(1)	
3						(3)
3				(3)		(2)
4	(2)	(3)	(1)	(1)	(2)	(1)

Figure 6: $Ba(w)$ corresponding to w of Example 3.1.7.

the first row of $T(Ba(w))$ (resp. $Q(Ba(w))$) is the index of the leftmost column (resp. top-most row) in $Ba^1(w)$ where a ball filled with i occurs, and the i -th entry of the k -th row of $T(Ba(w))$ (resp. $Q(Ba(w))$), for $k > 1$, is equal to the index of the leftmost column (resp. top-most row) in $Ba^k(w)$ where a new colored ball filled with i occurs.

Proposition 3.2.3. *Let w be a signed two-rowed array on signed alphabets S and S' . The following equality $(T(Ba(w)), Q(Ba(w))) = (T(w), Q(w))$ holds.*

Theorem 3.1.10 is then a direct consequence of this proposition, since the matrix-ball construction is symmetric in the rows and columns of the resulting signed ball array. In the rest of this subsection we will prove Proposition 3.2.3. Before that, we give the following example that illustrates the symmetry property of the super-RSK correspondence for super tableaux.

Example 3.2.4. Consider the signed two-rowed array w of Example 3.1.6. The super tableaux associated to its signed ball array $Ba(w)$ illustrated in Figure 5 are the following:

$$T(Ba(w)) = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 4 & 4 \\ \hline 2 & 3 & 3 & 5 & \\ \hline 2 & 4 & 4 & & \\ \hline 5 & 6 & & & \\ \hline 5 & 6 & & & \\ \hline \end{array}, \quad Q(Ba(w)) = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 3 & 3 \\ \hline 2 & 4 & 4 & 4 & \\ \hline 2 & 5 & 6 & & \\ \hline 5 & 6 & & & \\ \hline 5 & 6 & & & \\ \hline \end{array}$$

which are equal to the ones of w already computed in Example 3.1.6:

$$w = \begin{pmatrix} 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 5 & 6 & 6 & 6 \\ 2 & 4 & 2 & 1 & 4 & 5 & 6 & 3 & 5 & 6 & 4 & 3 & 1 & 5 & 4 & 1 \end{pmatrix} \xleftrightarrow{\text{sRSK}} \left(T(w) = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 4 & 4 \\ \hline 2 & 3 & 3 & 5 & \\ \hline 2 & 4 & 4 & & \\ \hline 5 & 6 & & & \\ \hline 5 & 6 & & & \\ \hline \end{array}, \quad Q(w) = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 3 & 3 \\ \hline 2 & 4 & 4 & 4 & \\ \hline 2 & 5 & 6 & & \\ \hline 5 & 6 & & & \\ \hline 5 & 6 & & & \\ \hline \end{array} \right).$$

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Moreover, using the matrix-ball construction, we obtain:

$$w^{\text{inv}} = \left(\begin{array}{cccccccccccc} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 4 & 4 & 5 & 5 & 6 & 6 \\ 2 & 5 & 6 & 2 & 1 & 4 & 5 & 2 & 3 & 5 & 6 & 6 & 4 & 3 & 4 & 3 \end{array} \right) \xleftrightarrow{\text{sRSK}} \left(\mathbf{Q}(w) = \begin{array}{cccc} \boxed{1} & \boxed{2} & \boxed{3} & \boxed{3} & \boxed{3} \\ \boxed{2} & \boxed{4} & \boxed{4} & \boxed{4} & \\ \boxed{2} & \boxed{5} & \boxed{6} & & \\ \boxed{5} & \boxed{6} & & & \\ \boxed{5} & \boxed{6} & & & \end{array}, \mathbf{T}(w) = \begin{array}{cccc} \boxed{1} & \boxed{1} & \boxed{1} & \boxed{4} & \boxed{4} \\ \boxed{2} & \boxed{3} & \boxed{3} & \boxed{5} & \\ \boxed{2} & \boxed{4} & \boxed{4} & & \\ \boxed{5} & \boxed{6} & & & \\ \boxed{5} & \boxed{6} & & & \end{array} \right)$$

by switching the roles of the rows and the columns of the signed ball array of Figure 5.

Consider now the signed two-rowed array w of Example 3.1.7. The super tableaux associated to its signed ball array $\text{Ba}(w)$ illustrated in Figure 6 are the following:

$$\mathbf{T}(\text{Ba}(w)) = \begin{array}{ccc} \boxed{1} & \boxed{1} & \boxed{2} \\ \boxed{2} & \boxed{3} & \\ \boxed{2} & \boxed{4} & \\ \boxed{3} & & \end{array}, \quad \mathbf{Q}(\text{Ba}(w)) = \begin{array}{ccc} \boxed{1} & \boxed{2} & \boxed{3} \\ \boxed{1} & \boxed{3} & \\ \boxed{2} & \boxed{4} & \\ \boxed{4} & & \end{array}$$

which are equal to the ones of w already computed in Example 3.1.7:

$$w = \left(\begin{array}{cccccccc} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 3 & 2 & 1 & 2 & 4 & 3 & 1 & 2 \end{array} \right) \xleftrightarrow{\text{sRSK}} \left(\mathbf{T}(w) = \begin{array}{ccc} \boxed{1} & \boxed{1} & \boxed{2} \\ \boxed{2} & \boxed{3} & \\ \boxed{2} & \boxed{4} & \\ \boxed{3} & & \end{array}, \mathbf{Q}(w) = \begin{array}{ccc} \boxed{1} & \boxed{2} & \boxed{3} \\ \boxed{1} & \boxed{3} & \\ \boxed{2} & \boxed{4} & \\ \boxed{4} & & \end{array} \right).$$

Moreover, using the super matrix-ball construction, we obtain:

$$w^{\text{inv}} = \left(\begin{array}{cccccccc} 1 & 1 & 2 & 2 & 3 & 3 & 4 \\ 2 & 4 & 4 & 2 & 1 & 1 & 3 & 3 \end{array} \right) \xleftrightarrow{\text{sRSK}} \left(\mathbf{Q}(w) = \begin{array}{ccc} \boxed{1} & \boxed{2} & \boxed{3} \\ \boxed{1} & \boxed{3} & \\ \boxed{2} & \boxed{4} & \\ \boxed{4} & & \end{array}, \mathbf{T}(w) = \begin{array}{ccc} \boxed{1} & \boxed{1} & \boxed{2} \\ \boxed{2} & \boxed{3} & \\ \boxed{2} & \boxed{4} & \\ \boxed{3} & & \end{array} \right)$$

by switching the roles of the rows and the columns of the signed ball array of Figure 6.

3.2.5. Proof of Proposition 3.2.3. Let $w = \begin{pmatrix} x_1 \dots x_k \\ y_1 \dots y_k \end{pmatrix}$ be a signed two-rowed array on signed alphabets \mathcal{S} and \mathcal{S}' . We show the result by induction on the number of couples in w , which is equal to the number of balls in $\text{Ba}^1(w)$. The result is obvious when w contains zero or one couple. Let

$$w_0 := \begin{pmatrix} x_1 \dots x_{k-1} \\ y_1 \dots y_{k-1} \end{pmatrix}$$

be the signed two-rowed array obtained from w by eliminating its rightmost couple (x_k, y_k) . By the induction hypothesis, we have $\mathbf{T}(w_0) = \mathbf{T}(\text{Ba}(w_0))$ and $\mathbf{Q}(w_0) = \mathbf{Q}(\text{Ba}(w_0))$. Then it is sufficient to prove the following property:

Property 3.2.6. The equality $\mathbf{T}(\text{Ba}(w)) = \mathbf{T}(\text{Ba}(w_0)) \leftarrow y_k$ holds in $\text{Yt}(\mathcal{S}')$, and the super tableau $\mathbf{Q}(\text{Ba}(w))$ is obtained from $\mathbf{Q}(\text{Ba}(w_0))$ by placing x_k in the box that belongs to $\mathbf{T}(\text{Ba}(w))$ but not to $\mathbf{T}(\text{Ba}(w_0))$.

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By construction, the signed ball array $Ba^1(w)$ contains one ball that is not in $Ba^1(w_0)$, which we denote by B , and we suppose is filled with an integer j . This ball is the bottom-rightmost ball of $Ba^1(w)$ which belongs to the position corresponding to the bottom-most row indexed by x_k and to the rightmost (resp. leftmost) column indexed by y_k if $y_k \in \mathcal{S}_0$ (resp. $y_k \in \mathcal{S}_1$). Moreover, the integer j is larger than all the numbers of the balls in its position.

Suppose first that there are no more balls filled with j in $Ba^1(w)$. In this case, all the remaining balls in $Ba^1(w)$ are to the NorthWest of B , and then they are all filled with integers that are strictly smaller than j . Hence when computing $Ba^i(w)$, for $i > 1$, no colored balls will be added using B , and all the rows of $T(w)$ and $T(w_0)$ (resp. $Q(w)$ and $Q(w_0)$) below the top-most row are the same. Moreover, the first row of $T(Ba(w))$ is obtained from that of $T(Ba(w_0))$ by adding y_k to the end, since j is the largest integer in $Ba^1(w)$, and then $Q(w)$ is obtained from $Q(w_0)$ by adding x_k to the end of its first row, showing Property 3.2.6.

Suppose now that there are other balls in $Ba^1(w)$ filled with j . By construction, all these balls are to the NorthEast of B . Consider the ball filled with j in the position corresponding to the row indexed with x and the column indexed with y such that $x \leq x_k$ is maximal with $x = x_k$ only if $x \in \mathcal{S}'_1$ and $y \geq y_k$ is minimal with $y = y_k$ only if $y \in \mathcal{S}_1$. When y_k is inserted in the first row of $T(Ba(w_0))$, the letter y is then bumped from the j -th box of this row. Indeed, the entries of the first row of $T(Ba(w_0))$ are the indices of the left-most columns of $Ba^1(w_0)$ that contains balls filled with $1, 2, \dots$. The first $j - 1$ entries of this row are less than or equal to y_k , but the j -th entry is y which is the minimal index that satisfies $y \geq y_k$ with $y = y_k$ only if $y \in \mathcal{S}_1$, then the entry y is bumped from the j -th box when inserting y_k in the first row of $T(Ba(w_0))$. Hence the top-most row of $T(Ba(w))$ is the top-most row of $T(Ba(w_0)) \leftarrow y_k$. Moreover, the super tableau formed by the rows of $T(Ba(w))$ below its top-most row is by construction the super tableau $T(Ba_0(w))$, and the super tableau formed by the rows of $T(Ba(w_0))$ below its top-most row is the super tableau $T(Ba_0(w_0))$. Then it is sufficient to prove that the equality $T(Ba_0(w)) = T(Ba_0(w_0)) \leftarrow y$ holds in $Yt(\mathcal{S}')$, and the new box from this insertion is the box that belongs to $Q(Ba_0(w))$ but not to $Q(Ba_0(w_0))$. This follows by induction from Property 3.2.6, provided with the fact that (x_k, y) is the position of the bottom-rightmost ball of the signed ball array $Ba_0^1(w)$. Indeed, by construction, there is no entries below the x_k -th row of $Ba_0^1(w)$. Moreover, if there is other ball in the x_k -th row of $Ba_0^1(w)$, then this ball is added using two balls of $Ba^1(w)$ filled with an integer $i < j$. The first ball belongs to the row indexed by x_k , and the second one lies NorthEast of it, but it is NorthWest of the bottom-rightmost ball filled with j , and then this second ball cannot lie in a column indexed larger than y , showing the claim.

3.3. Super-RSK correspondence and taquin. Before we proceed to the construction of the dual super-RSK correspondence, we prove the following result which relates the super-RSK correspondence to the super jeu de taquin. This result will be useful for the sequel.

Proposition 3.3.1. *Let $w = \begin{pmatrix} x_1 \dots x_k \\ y_1 \dots y_k \end{pmatrix}$ be a signed two-rowed array on signed alphabets \mathcal{S} and \mathcal{S}' such that $sRSK(w) = (T(w), Q(w))$, and let t be any super tableau in $Yt(\mathcal{S}')$. If we compute the super tableau $((\dots (t \leftarrow y_1) \leftarrow \dots) \leftarrow y_k)$ over $Yt(\mathcal{S}')$, and we add x_1, \dots, x_k successively in the new boxes starting with an empty Young diagram of the same shape as t , then the entries x_1, \dots, x_k form a super skew tableau S such that $Rect(S) = Q(w)$.*

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Proof. Consider a super tableau t' over $\text{Yt}(\mathcal{S}^<)$ of the same shape as t , where $\mathcal{S}^<$ is a signed alphabet whose elements are all smaller than the ones of \mathcal{S} . Following Theorem 3.1.5, the pair (t, t') corresponds to some signed two-rowed array $\begin{pmatrix} x'_1 \dots x'_l \\ y'_1 \dots y'_l \end{pmatrix}$. Hence, the signed two-rowed array $W := \begin{pmatrix} x'_1 \dots x'_l x_1 \dots x_k \\ y'_1 \dots y'_l y_1 \dots y_k \end{pmatrix}$ corresponds to the pair $\left((t \leftarrow \mathbf{R}(\mathbf{T}(w))), Q' \right)$, where Q' is a super tableau over $\text{Yt}(\mathcal{S}^< \cup \mathcal{S})$ whose entries x'_1, \dots, x'_l are the ones of t' , and whose entries x_1, \dots, x_k are the ones of S . Consider now the inverse W^{inv} of W . Following Theorem 3.1.10, the signed two-rowed array W^{inv} corresponds to the pair $\left(Q', (t \leftarrow \mathbf{R}(\mathbf{T}(w))) \right)$, and the signed two-rowed array obtained from W^{inv} by eliminating the couples (y'_i, x'_i) corresponds to the pair $(\mathbf{Q}(w), \mathbf{T}(w))$. The word of the second row of W^{inv} is then super plactic equivalent to the word $\mathbf{R}(Q')$, and when we remove the y'_j 's from this word we obtain a word that is super plactic equivalent to $\mathbf{R}(Q(w))$. However, by construction of Q' , when we remove the l smallest letters from $\mathbf{R}(Q')$, we recover the word $\mathbf{R}(S)$. Hence, we deduce by Lemma 2.3.2 that $\mathbf{R}(S)$ and $\mathbf{R}(Q(w))$ are also super plactic equivalent, showing by Property 2.4.1 that $\text{Rect}(S) = \mathbf{Q}(w)$. \square

Example 3.3.2. We show in Example 3.1.7 the following computation:

$$w = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 3 & 2 & 1 & 2 & 4 & 3 & 1 & 2 \end{pmatrix} \xleftrightarrow{\text{sRSK}} \left(\mathbf{T}(w) = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & \\ \hline 2 & 4 & \\ \hline 3 & & \\ \hline \end{array}, \mathbf{Q}(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 3 & \\ \hline 2 & 4 & \\ \hline 4 & & \\ \hline \end{array} \right).$$

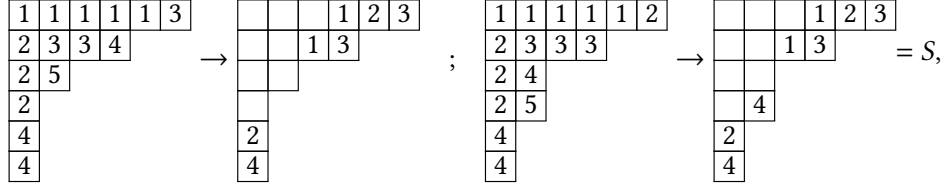
Consider now the super tableau $t = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 3 & \\ \hline 4 & 5 & \\ \hline 4 & & \\ \hline \end{array}$. We insert the elements of the second row of w

into t , and we place the elements of the first row of w in the new added boxes starting from an empty tableau of the same shape as t . Then we obtain the following:

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 3 & & \\ \hline 4 & 5 & & \\ \hline 4 & & & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}; \quad \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 3 & 3 & \\ \hline 4 & 5 & & \\ \hline 4 & & & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}; \quad \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 3 & 3 & \\ \hline 2 & 5 & & \\ \hline 4 & & & \\ \hline 4 & & & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array};$$

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 \\ \hline 2 & 3 & 3 & & \\ \hline 2 & 5 & & & \\ \hline 4 & & & & \\ \hline 4 & & & & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|c|} \hline & & & 1 & 2 \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array}; \quad \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 & 4 \\ \hline 2 & 3 & 3 & & & \\ \hline 2 & 5 & & & & \\ \hline 4 & & & & & \\ \hline 4 & & & & & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|c|c|} \hline & & & 1 & 2 & 3 \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline \end{array};$$

$$\begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 3 & 4 & & \\ \hline 2 & 5 & & & & \\ \hline 4 & & & & & \\ \hline 4 & & & & & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|c|c|} \hline & & & 1 & 2 & 3 \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline \end{array};$$



such that $\text{Rect}(S) = \mathbf{Q}(w)$, as shown in Example 2.4.2.

3.4. A dual super-RSK correspondence. We recall the left insertion algorithm on super tableaux which gives us a dual version of the super-RSK correspondence. We first begin by recalling the super evacuation procedure on super tableaux introduced in [13].

3.4.1. Super evacuation on super tableaux. Let \mathcal{S} be a signed alphabet. We denote by \mathcal{S}^{op} the *opposite alphabet* obtained from \mathcal{S} by reversing its order, and by x^* the letter in \mathcal{S}^{op} corresponding to x in \mathcal{S} where $\|x^*\| = 1$ (resp. $\|x^*\| = 0$) if $\|x\| = 1$ (resp. $\|x\| = 0$). For all x, y in \mathcal{S} , we have $x < y$ if and only if $x^* > y^*$. For any word $w = x_1 \dots x_k$ over \mathcal{S} , we denote by $w^* = x_k^* \dots x_1^*$ the corresponding *opposite word* of w over \mathcal{S}^{op} . Then, for all v and w in \mathcal{S}^* , the equality $(vw)^* = w^*v^*$ holds, inducing an anti-isomorphism between the free monoids over \mathcal{S}^{op} and \mathcal{S} . By identifying $(\mathcal{S}^{\text{op}})^{\text{op}}$ with \mathcal{S} , we have $(w^*)^* = w$, for any w in \mathcal{S}^* . Moreover, we show that, for all v and w in \mathcal{S} , the following equivalence holds, [13]:

$$v \sim_{\mathbf{P}(\mathcal{S})} w \quad \text{if, and only, if} \quad v^* \sim_{\mathbf{P}(\mathcal{S}^{\text{op}})} w^*. \quad (8)$$

Algorithm 3.4.2 ([13]). Let t be in $\text{Yt}(\mathcal{S})$. An *opposite tableau* in $\text{Yt}(\mathcal{S}^{\text{op}})$ can be constructed from t using the super jeu de taquin, as follows:

Input: A super tableau t in $\text{Yt}(\mathcal{S})$.

Output: A super tableau t^{op} in $\text{Yt}(\mathcal{S}^{\text{op}})$ with the same shape as t .

Method: Start with an empty Young diagram with the same frame as t . Remove the box containing the top-leftmost element x in t , and perform the super jeu de taquin procedure (2.4) on the resulting super skew tableau. We obtain a super tableau, denoted by t^* , whose frame has one box removed from the one of t . Put the letter x^* in the initial empty Young diagram in the same place as the box that was removed from the frame of t . Repeat the algorithm on t^* and continue until all the elements of t have been removed and the initial empty Young diagram has been filled with their corresponding letters in \mathcal{S}^{op} . Output the resulting super tableau for t^{op} .

This procedure, called the *super evacuation*, is the super analogue of the *Schützenberger's evacuation procedure*, [28]. We show the following property:

Property 3.4.3 ([13]). For any t in $\text{Yt}(\mathcal{S})$, the super tableau t^{op} satisfies the following equivalence:

$$(\mathbf{R}(t))^* \sim_{\mathbf{P}(\mathcal{S}^{\text{op}})} \mathbf{R}(t^{\text{op}}), \quad (9)$$

and the map $t \mapsto t^{\text{op}}$ is an involution on $\text{Yt}(\mathcal{S})$.

Let \mathcal{S} and \mathcal{S}' be signed alphabets and $w = \begin{pmatrix} x_1 \dots x_k \\ y_1 \dots y_k \end{pmatrix}$ a signed two-rowed array on \mathcal{S} and \mathcal{S}' .

We define $w^* := \begin{pmatrix} x_k^* \dots x_1^* \\ y_k^* \dots y_1^* \end{pmatrix}$ the signed two-rowed array on \mathcal{S}^{op} and $(\mathcal{S}')^{\text{op}}$ whose first and second rows are the opposites of the ones of w . In particular, by definition, we have $(w^*)^{\text{inv}} = (w^{\text{inv}})^*$.

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Proposition 3.4.4. *Let \mathcal{S} and \mathcal{S}' be signed alphabets and w a signed two-rowed array on \mathcal{S} and \mathcal{S}' . If $\text{sRSK}(w) = (\mathbf{T}(w), \mathbf{Q}(w))$ then $\text{sRSK}(w^*) = (\mathbf{T}(w)^{\text{op}}, \mathbf{Q}(w)^{\text{op}})$.*

Proof. Consider $w = \begin{pmatrix} x_1 \dots x_k \\ y_1 \dots y_k \end{pmatrix}$ a signed two-rowed array on \mathcal{S} and \mathcal{S}' . Following Subsection 2.3, the following equivalence $\mathbf{R}(\mathbf{T}(w)) \sim_{\mathbf{P}(\mathcal{S})} y_1 \dots y_k$ holds. Then we obtain

$$\mathbf{R}(\mathbf{T}(w^*)) \sim_{\mathbf{P}(\mathcal{S}^{\text{op}})} y_k^* \dots y_1^* = (y_1 \dots y_k)^* \sim_{\mathbf{P}(\mathcal{S}^{\text{op}})}^{(8)} (\mathbf{R}(\mathbf{T}(w)))^* \sim_{\mathbf{P}(\mathcal{S}^{\text{op}})}^{(9)} \mathbf{R}(\mathbf{T}(w)^{\text{op}}).$$

We deduce by Property 2.3.1 that the equality $\mathbf{T}(w^*) = \mathbf{T}(w)^{\text{op}}$ holds in $\text{Yt}((\mathcal{S}')^{\text{op}})$. Hence, we obtain $\text{sRSK}(w^*) = (\mathbf{T}(w)^{\text{op}}, \mathbf{Q}(w^*))$. Similarly, we show that

$$\text{sRSK}((w^*)^{\text{inv}}) = \text{sRSK}((w^{\text{inv}})^*) = (\mathbf{Q}(w)^{\text{op}}, \mathbf{T}(w^*)).$$

We deduce by Theorem 3.1.10 that $(\mathbf{Q}(w)^{\text{op}}, \mathbf{T}(w^*)) = (\mathbf{Q}(w^*), \mathbf{T}(w)^{\text{op}})$, showing the claim. \square

3.4.5. A dual construction of the super RSK-correspondence. We present a dual way to construct the super RSK-correspondence using the left insertion algorithm on super tableaux.

Algorithm 3.4.6 ([20]). The *left (or column) insertion*, denoted by \rightsquigarrow , inserts an element x in \mathcal{S} into a super tableau t of $\text{Yt}(\mathcal{S})$ as follows:

Input: A super tableau t and a letter $x \in \mathcal{S}$.

Output: A super tableau $x \rightsquigarrow t$.

Method: If t is empty, create a box and label it x . Suppose t is non-empty. If $x \in \mathcal{S}_0$ (resp. $x \in \mathcal{S}_1$) is larger than (resp. at least as large as) the bottom element of the leftmost column of t , then put x in a box to the bottom of this column; Otherwise, let y be the smallest element of the leftmost column of t such that $y \geq x$ (resp. $y > x$). Then replace y by x in this column and recursively insert y into the super tableau formed by the columns of t to the right of the leftmost. Note that this recursion may end with an insertion into an empty column to the right of the existing columns of t . Output the resulting super tableau.

Property 3.4.7. As a consequence of Property 2.3.1, the following *commutation property* holds in $\text{Yt}(\mathcal{S})$, for all t in $\text{Yt}(\mathcal{S})$ and x, y in \mathcal{S} :

$$y \rightsquigarrow (t \leftarrow x) = (y \rightsquigarrow t) \leftarrow x.$$

In particular, for any word $w = x_1 \dots x_k$ in \mathcal{S}^* , the super tableau $\mathbf{T}(w)$ is also computed by inserting its elements iteratively from right to left using the left insertion \rightsquigarrow as follows:

$$\mathbf{T}(w) = (w \rightsquigarrow \emptyset) := (x_1 \rightsquigarrow (\dots \rightsquigarrow (x_k \rightsquigarrow \emptyset) \dots)).$$

Algorithm 3.4.8. Let \mathcal{S} and \mathcal{S}' be signed alphabets. Starting from a signed two-rowed array w on \mathcal{S} and \mathcal{S}' , we can compute the pair of super tableaux $\text{sRSK}(w) = (\mathbf{T}(w), \mathbf{Q}(w))$ using the left insertion, as follows:

Input: A signed two-rowed array $w = \begin{pmatrix} x_1 \dots x_k \\ y_1 \dots y_k \end{pmatrix}$ on \mathcal{S} and \mathcal{S}' .

Output: A pair $(\mathbf{T}'(w), \mathbf{Q}'(w)) \in \text{Yt}(\mathcal{S}') \times \text{Yt}(\mathcal{S})$ of same-shape tableaux containing k boxes.

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Method: Start with an empty super tableau T'_{k+1} and an empty super tableau Q'_{k+1} . For each $i = k, \dots, 1$, compute $y_i \rightsquigarrow T'_{i+1}$ as per Algorithm 3.4.6 and let T'_i be the resulting super tableau. Let Q'_i be super tableau obtained from Q'_{i+1} , by performing the reverse sliding algorithm (2.4.3), using the box that belongs to T'_i but not to T'_{i+1} , and then place x_i in the top-leftmost corner of the result. Output T'_1 for $\mathbf{T}'(w)$ and Q'_1 as $\mathbf{Q}'(w)$.

Proposition 3.4.9. *The output of Algorithm 3.4.8 is equal to $\text{sRSK}(w) = (\mathbf{T}(w), \mathbf{Q}(w))$.*

Proof. Following Property 3.4.7, the super tableaux $\mathbf{T}(w)$ and $\mathbf{T}'(w)$ are equal. We still have to show that $\mathbf{Q}(w) = \mathbf{Q}'(w)$. We will proceed by induction on the number of couples in w . The result is obvious when w contains zero or one couple. Let w' be the signed two-rowed array obtained from w by eliminating its first couple (x_1, y_1) . By the induction hypothesis, we have $\mathbf{Q}(w') = \mathbf{Q}'(w')$. Moreover, on the one hand, and since the super tableau $\mathbf{Q}'(w)$ is computed using the reverse sliding algorithm, we have that $\mathbf{Q}'(w') = (\mathbf{Q}'(w))^*$, with the property that $\mathbf{Q}'(w)$ has x_1 in its top-leftmost corner. On the second hand, by applying Proposition 3.3.1 to the signed two-rowed array w' and the super tableau containing one box filled with y_1 , we obtain that $\mathbf{Q}(w') = (\mathbf{Q}(w))^*$, and then $\mathbf{Q}'(w') = (\mathbf{Q}(w))^*$, with the property that $\mathbf{Q}(w)$ has x_1 in its top-leftmost corner, showing that $\mathbf{Q}(w) = \mathbf{Q}'(w)$. \square

Example 3.4.10. Consider $w = \begin{pmatrix} 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 6 & 6 & 6 \\ 2 & 4 & 2 & 1 & 4 & 5 & 6 & 3 & 5 & 6 & 4 & 3 & 1 & 5 & 4 & 1 \end{pmatrix}$ of Example 3.1.6. The sequence of pairs produced during the computation of $(\mathbf{T}(w), \mathbf{Q}(w))$ as per Algorithm 3.4.8 is the following:

$$\begin{aligned}
 & (\emptyset, \emptyset), \quad (\boxed{1}, \boxed{6}), \quad \left(\begin{array}{|c|} \hline \boxed{1} \\ \hline \boxed{4} \\ \hline \end{array}, \begin{array}{|c|} \hline \boxed{6} \\ \hline \boxed{6} \\ \hline \end{array} \right), \quad \left(\begin{array}{|c|} \hline \boxed{1} \\ \hline \boxed{4} \\ \hline \boxed{5} \\ \hline \end{array}, \begin{array}{|c|} \hline \boxed{6} \\ \hline \boxed{6} \\ \hline \boxed{6} \\ \hline \end{array} \right), \quad \left(\begin{array}{|c|c|} \hline \boxed{1} & \boxed{1} \\ \hline \boxed{4} & \boxed{6} \\ \hline \boxed{5} & \boxed{6} \\ \hline \end{array}, \begin{array}{|c|c|} \hline \boxed{5} & \boxed{6} \\ \hline \boxed{6} & \boxed{6} \\ \hline \end{array} \right), \quad \left(\begin{array}{|c|c|} \hline \boxed{1} & \boxed{1} \\ \hline \boxed{3} & \boxed{4} \\ \hline \boxed{5} & \boxed{6} \\ \hline \end{array}, \begin{array}{|c|c|} \hline \boxed{5} & \boxed{6} \\ \hline \boxed{5} & \boxed{6} \\ \hline \end{array} \right), \\
 & \left(\begin{array}{|c|c|} \hline \boxed{1} & \boxed{1} \\ \hline \boxed{3} & \boxed{4} \\ \hline \boxed{4} & \boxed{5} \\ \hline \end{array}, \begin{array}{|c|c|} \hline \boxed{5} & \boxed{6} \\ \hline \boxed{5} & \boxed{6} \\ \hline \end{array} \right), \quad \left(\begin{array}{|c|c|} \hline \boxed{1} & \boxed{1} \\ \hline \boxed{3} & \boxed{4} \\ \hline \boxed{4} & \boxed{5} \\ \hline \end{array}, \begin{array}{|c|c|} \hline \boxed{4} & \boxed{6} \\ \hline \boxed{5} & \boxed{6} \\ \hline \boxed{5} & \boxed{6} \\ \hline \end{array} \right), \quad \left(\begin{array}{|c|c|} \hline \boxed{1} & \boxed{1} \\ \hline \boxed{3} & \boxed{4} \\ \hline \boxed{4} & \boxed{5} \\ \hline \end{array}, \begin{array}{|c|c|} \hline \boxed{4} & \boxed{4} \\ \hline \boxed{5} & \boxed{6} \\ \hline \boxed{5} & \boxed{6} \\ \hline \end{array} \right), \quad \left(\begin{array}{|c|c|c|} \hline \boxed{1} & \boxed{1} & \boxed{4} \\ \hline \boxed{3} & \boxed{3} & \boxed{4} \\ \hline \boxed{4} & \boxed{5} & \boxed{6} \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \boxed{4} & \boxed{4} & \boxed{4} \\ \hline \boxed{5} & \boxed{6} & \boxed{6} \\ \hline \boxed{5} & \boxed{6} & \boxed{6} \\ \hline \end{array} \right), \\
 & \left(\begin{array}{|c|c|c|} \hline \boxed{1} & \boxed{1} & \boxed{4} \\ \hline \boxed{3} & \boxed{3} & \boxed{4} \\ \hline \boxed{4} & \boxed{5} & \boxed{6} \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \boxed{3} & \boxed{4} & \boxed{4} \\ \hline \boxed{4} & \boxed{6} & \boxed{6} \\ \hline \boxed{5} & \boxed{6} & \boxed{6} \\ \hline \end{array} \right), \quad \left(\begin{array}{|c|c|c|} \hline \boxed{1} & \boxed{1} & \boxed{4} \\ \hline \boxed{3} & \boxed{3} & \boxed{4} \\ \hline \boxed{4} & \boxed{5} & \boxed{6} \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \boxed{3} & \boxed{3} & \boxed{4} \\ \hline \boxed{4} & \boxed{4} & \boxed{4} \\ \hline \boxed{5} & \boxed{6} & \boxed{6} \\ \hline \end{array} \right), \quad \left(\begin{array}{|c|c|c|} \hline \boxed{1} & \boxed{1} & \boxed{4} \\ \hline \boxed{3} & \boxed{3} & \boxed{5} \\ \hline \boxed{4} & \boxed{4} & \boxed{6} \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \boxed{3} & \boxed{3} & \boxed{3} \\ \hline \boxed{4} & \boxed{4} & \boxed{4} \\ \hline \boxed{5} & \boxed{6} & \boxed{6} \\ \hline \end{array} \right), \\
 & \left(\begin{array}{|c|c|c|c|} \hline \boxed{1} & \boxed{1} & \boxed{1} & \boxed{4} \\ \hline \boxed{2} & \boxed{3} & \boxed{3} & \boxed{5} \\ \hline \boxed{3} & \boxed{3} & \boxed{3} & \boxed{5} \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \boxed{2} & \boxed{3} & \boxed{3} & \boxed{3} \\ \hline \boxed{4} & \boxed{4} & \boxed{4} & \boxed{4} \\ \hline \boxed{5} & \boxed{6} & \boxed{6} & \boxed{6} \\ \hline \end{array} \right), \quad \left(\begin{array}{|c|c|c|c|} \hline \boxed{1} & \boxed{1} & \boxed{1} & \boxed{4} \\ \hline \boxed{2} & \boxed{3} & \boxed{3} & \boxed{5} \\ \hline \boxed{3} & \boxed{3} & \boxed{3} & \boxed{5} \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \boxed{2} & \boxed{3} & \boxed{3} & \boxed{3} \\ \hline \boxed{2} & \boxed{4} & \boxed{4} & \boxed{4} \\ \hline \boxed{2} & \boxed{5} & \boxed{6} & \boxed{6} \\ \hline \end{array} \right), \quad \left(\begin{array}{|c|c|c|c|} \hline \boxed{1} & \boxed{1} & \boxed{1} & \boxed{4} \\ \hline \boxed{2} & \boxed{3} & \boxed{3} & \boxed{5} \\ \hline \boxed{3} & \boxed{3} & \boxed{3} & \boxed{5} \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \boxed{2} & \boxed{3} & \boxed{3} & \boxed{3} \\ \hline \boxed{2} & \boxed{4} & \boxed{4} & \boxed{4} \\ \hline \boxed{2} & \boxed{5} & \boxed{6} & \boxed{6} \\ \hline \end{array} \right), \\
 & \left(\begin{array}{|c|c|c|c|} \hline \boxed{1} & \boxed{1} & \boxed{1} & \boxed{4} \\ \hline \boxed{2} & \boxed{3} & \boxed{3} & \boxed{5} \\ \hline \boxed{3} & \boxed{3} & \boxed{3} & \boxed{5} \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \boxed{2} & \boxed{3} & \boxed{3} & \boxed{3} \\ \hline \boxed{2} & \boxed{4} & \boxed{4} & \boxed{4} \\ \hline \boxed{2} & \boxed{5} & \boxed{6} & \boxed{6} \\ \hline \end{array} \right), \\
 & \left(\mathbf{T}(w) = \begin{array}{|c|c|c|c|c|} \hline \boxed{1} & \boxed{1} & \boxed{1} & \boxed{4} & \boxed{4} \\ \hline \boxed{2} & \boxed{3} & \boxed{3} & \boxed{5} & \boxed{6} \\ \hline \boxed{2} & \boxed{4} & \boxed{4} & \boxed{6} & \boxed{6} \\ \hline \boxed{5} & \boxed{6} & \boxed{6} & \boxed{6} & \boxed{6} \\ \hline \end{array}, \mathbf{Q}(w) = \begin{array}{|c|c|c|c|c|} \hline \boxed{1} & \boxed{2} & \boxed{3} & \boxed{3} & \boxed{3} \\ \hline \boxed{2} & \boxed{4} & \boxed{4} & \boxed{4} & \boxed{4} \\ \hline \boxed{2} & \boxed{5} & \boxed{6} & \boxed{6} & \boxed{6} \\ \hline \boxed{5} & \boxed{6} & \boxed{6} & \boxed{6} & \boxed{6} \\ \hline \end{array} \right).
 \end{aligned}$$

We prove in Theorem 4.2.1 that there is a canonical one-to-one correspondence between the sets $\text{Yt}(\mathcal{S}, \lambda, \mu, \dashv t)$ and $\text{St}(\mathcal{S}, \nu/\lambda, \dashv T)$. We deduce that the coefficient $c_{\lambda, \mu}^{\nu}$ does not depend on t and T , and depends only on λ, μ and ν . Note that the later property is proved in [13] using an interpretation of the super jeu de taquin in terms of Fomin’s growth diagrams.

4.2. Super Littlewood–Richardson rule. We denote by $R_{\mathcal{S}}$ the \mathbb{Z} -algebra constructed from the super plactic monoid $\mathbf{P}(\mathcal{S})$ whose linear generators are the monomials in $\mathbf{P}(\mathcal{S})$. This algebra is an associative and unitary ring that is not commutative. A generic element in $R_{\mathcal{S}}$ is realized by a formal sum of super plactic classes with coefficients from \mathbb{Z} . Following Property 2.3.1, a typical element in $R_{\mathcal{S}}$ is a formal sum of super tableaux. A canonical homomorphism from $R_{\mathcal{S}}$ onto the ring of polynomials $\mathbb{Z}[X]$ is obtained by taking each super tableau t to its monomial x^t , where x^t is the product of the variables x_i , each occurring as many times in x^t as i occurs in t . For instance, the following monomial

$$x^{\mathbf{T}(w)} = x_1^3 x_2^2 x_3^2 x_4^4 x_5^3 x_6^2 \quad (\text{resp. } x^{\mathbf{Q}(w)} = x_1^1 x_2^3 x_3^3 x_4^3 x_5^3 x_6^3)$$

corresponds to $\mathbf{T}(w)$ (resp. $\mathbf{Q}(w)$) computed in Example 3.1.6. Moreover, the following monomial

$$x^{\mathbf{T}(w)} = x_1^2 x_2^2 x_3^2 x_4 \quad (\text{resp. } x^{\mathbf{Q}(w)} = x_1^2 x_2^2 x_3^2 x_4^2)$$

corresponds to the super tableau $\mathbf{T}(w)$ (resp. $\mathbf{Q}(w)$) computed in Example 3.1.7.

We define S_{λ} (resp. $S_{\lambda/\mu}$) in $R_{\mathcal{S}}$ to be the sum of all super tableaux (resp. super skew tableaux) of shape λ (resp. λ/μ) and entries in \mathcal{S} , with $\lambda \in \mathcal{P}$ (resp. λ/μ is a skew shape). By taking the image of S_{λ} (resp. $S_{\lambda/\mu}$) in $\mathbb{Z}[X]$, we obtain the so-called *super Schur function* (resp. *super skew Schur function*) $s_{\lambda}(X)$ (resp. $s_{\lambda/\mu}(X)$).

Theorem 4.2.1 (The super Littlewood–Richardson rule). *Let λ, μ and ν be partitions in \mathcal{P} such that ν/λ is a skew shape. The following identities*

$$S_{\lambda} S_{\mu} = \sum_{\nu} c_{\lambda, \mu}^{\nu} S_{\nu} \quad \text{and} \quad S_{\nu/\lambda} = \sum_{\mu} c_{\lambda, \mu}^{\nu} S_{\mu} \quad (10)$$

hold in $R_{\mathcal{S}}$.

Proof. Prove that for any t in $\text{Yt}(\mathcal{S}, \nu)$ and any T in $\text{Yt}(\mathcal{S}, \mu)$, there is a canonical one-to-one correspondence between the sets $\text{Yt}(\mathcal{S}, \lambda, \mu, \dashv t)$ and $\text{St}(\mathcal{S}, \nu/\lambda, \dashv T)$. Start with (t', t'') in $\text{Yt}(\mathcal{S}, \lambda, \mu, \dashv t)$ and let

$$\begin{pmatrix} x_1 \dots x_k \\ y_1 \dots y_k \end{pmatrix}$$

be the signed two-rowed array on \mathcal{S} corresponding to the couple (t'', T) . By inserting the letters y_i , for $i = 1, \dots, k$, into the super tableau t' , we form a super skew tableau S by successively placing the letter x_i , for $i = 1, \dots, k$, into the new boxes starting with an empty Young diagram of the same shape as t' . Since the super tableau $t = t' \star_{\text{Yt}(\mathcal{S})} t''$ has shape ν , then the super skew tableau S has shape ν/λ and we deduce by Proposition 3.3.1 that S belongs to $\text{St}(\mathcal{S}, \nu/\lambda, \dashv T)$.

Conversely, start with a super skew tableau S in $\text{St}(\mathcal{S}, \nu/\lambda, \dashv T)$, and let T' be in $\text{Yt}(\mathcal{S}^{\leq}, \lambda)$ where \mathcal{S}^{\leq} is a signed alphabet whose elements are all smaller than the ones of \mathcal{S} . Let T'' be

4. The super Littlewood–Richardson rule

the super tableau in $\text{Yt}(\mathcal{S}^< \cup \mathcal{S}, \nu)$ that contains the super tableau T' such that when we remove T' from T'' we obtain the super skew tableau S . By Theorem 3.1.5, the couple (t, T'') corresponds to a unique signed two-rowed array of the following form $\begin{pmatrix} x'_1 \dots x'_l & x_1 \dots x_k \\ y'_1 \dots y'_l & y_1 \dots y_k \end{pmatrix}$. Then the signed two rowed array $\begin{pmatrix} x'_1 \dots x'_l \\ y'_1 \dots y'_l \end{pmatrix}$ corresponds to the couple (t', T') , and following Proposition 3.3.1 the signed two-rowed array $\begin{pmatrix} x_1 \dots x_k \\ y_1 \dots y_k \end{pmatrix}$ corresponds to the couple (t'', T) , for some super tableaux t' in $\text{Yt}(\mathcal{S}, \lambda)$ and t'' in $\text{Yt}(\mathcal{S}, \mu)$, such that $t' \star_{\text{Yt}(\mathcal{S})} t'' = t$. Hence, we obtain a pair (t', t'') in $\text{Yt}(\mathcal{S}, \lambda, \mu, \vdash t)$. We deduce that the cardinal number of the set $\text{St}(\mathcal{S}, \nu/\lambda, \vdash T)$ is equal to the coefficient $c_{\lambda, \mu}^{\nu}$, showing the claim. \square

Since neither $\text{Yt}(\mathcal{S}, \lambda, \mu, \vdash t)$ nor $\text{St}(\mathcal{S}, \nu/\lambda, \vdash T)$ in the correspondence of the proof of Theorem 4.2.1 depends on the contents of the super tableaux used to define the other, we deduce the following result:

Corollary 4.2.2. *The cardinal number $c_{\lambda, \mu}^{\nu}$ of the sets $\text{Yt}(\mathcal{S}, \lambda, \mu, \vdash t)$ and $\text{St}(\mathcal{S}, \nu/\lambda, \vdash T)$ is independent of choice of t and T and depends only on λ , μ and ν .*

4.3. Super Littlewood–Richardson skew tableaux. We introduce in this final part the notion of super Littlewood–Richardson skew tableaux and we give a new combinatorial description of the super Littlewood–Richardson coefficient. We suppose in the sequel that

$$\mathcal{S} = \{\bar{m} < \dots < \bar{1} < 1 < \dots < n\}$$

with $\mathcal{S}_1 = \{1 < \dots < n\}$ and $\mathcal{S}_0 = \{\bar{m} < \dots < \bar{1}\}$. For any w in \mathcal{S}^* , we denote by $|w|_i$ the number of times the element i of \mathcal{S} appears in w , and by $\underline{|w|}$ the number of elements of \mathcal{S}_1 appearing in w such that each element is counted only once. The *weight map* is the map

$$\text{wt} : \mathcal{S}^* \rightarrow (\mathbb{Z}_{>0} \cup \{0\})^{m+n}$$

defined by $\text{wt}(w) = (|w|_{\bar{m}}, \dots, |w|_{\bar{1}}, |w|_1, \dots, |w|_n)$, for all w in \mathcal{S}^* . We call $\text{wt}(\mathbf{R}(S))$ the *weight* of a super skew tableau S over \mathcal{S} .

4.3.1. Super Yamanouchi words. A word w over \mathcal{S} is a *super Yamanouchi word* if it satisfies the following two conditions:

i) for every right subword w' of w , the following property holds:

$$|w'|_{\bar{m}} \geq \dots \geq |w'|_{\bar{2}} \geq |w'|_{\bar{1}} \geq \underline{|w'|},$$

ii) for every left subword w' of w , the following property holds:

$$|w'|_1 \geq |w'|_2 \geq \dots \geq |w'|_n.$$

4.3. Super Littlewood–Richardson skew tableaux

When $\mathcal{S} = \mathcal{S}_0$, we recover the notion of *Yamanouchi words* that describes the elements of highest weights for the crystal structure of the general Lie algebra of type A, [23].

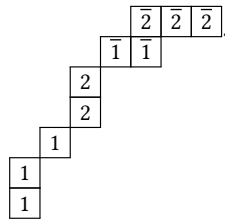
For instance, suppose $\mathcal{S} = \{\bar{2} < \bar{1} < 1 < 2\}$. The word $12\bar{2}\bar{1}\bar{1}\bar{2}\bar{2}$ is a super Yamanouchi word over \mathcal{S} . However, the word $w = \bar{1}\bar{1}\bar{2}\bar{2}\bar{1}\bar{2}\bar{2}$ is not a super Yamanouchi word over \mathcal{S} because for its right subword $w' = \bar{1}\bar{2}\bar{2}\bar{1}\bar{2}\bar{2}$, we have $|w'|_{\bar{1}} < |w'|$.

Lemma 4.3.2. *Let w and v be words over \mathcal{S} that are super plactic equivalent. Then w is a super Yamanouchi word if and only if v is a super Yamanouchi word.*

Proof. Suppose first that $w = uxzyu'$ and $v = uzxyu'$ for all u, u' in \mathcal{S}^* such that $x \leq y \leq z$ in \mathcal{S} with $x = y$ only if $||y|| = 0$ and $y = z$ only if $||y|| = 1$. If $x < y < z$, there is no changes in the numbers of consecutive elements of \mathcal{S}_0 (resp. \mathcal{S}_1) in each right (resp. left) subword of w and v . If $x = \bar{1}$ and $y = z = 1$ or $x = y = \bar{1}$ and $z = 1$ such that w is a super Yamanouchi word. Then $|u'|_{\bar{1}} \geq |u'|$, and hence v is a super Yamanouchi word. Similarly, we show that if v is a super Yamanouchi word, then w is so. If $x = y = k \in \mathcal{S}_0$ and $z = k + 1 \in \mathcal{S}_0$ such that w is a super Yamanouchi word. Then by definition, we have $|u'|_k \geq |u'|_{k+1}$, showing that $zxyu'$ is a super Yamanouchi word, and then v is so. Similarly, we show that if v is a super Yamanouchi word, then w is so. If $x = k \in \mathcal{S}_1$ and $y = z = k + 1 \in \mathcal{S}_1$ such that w is a super Yamanouchi word. Then the following property $|u|_k > |u|_{k+1}$ holds, showing that $uzxy$ is a super Yamanouchi word, and hence v is so. Similarly, we show that if v is a super Yamanouchi word, then w is so.

Suppose now that $w = uyxzu'$ and $v = uyzxu'$ for all u, u' in \mathcal{S}^* such that $x \leq y \leq z$ in \mathcal{S} with $x = y$ only if $||y|| = 1$ and $y = z$ only if $||y|| = 0$. If $x < y < z$, there is no changes in the numbers of consecutive elements of \mathcal{S}_0 (resp. \mathcal{S}_1) in each right (resp. left) subword of w and v . Suppose $x = y = k \in \mathcal{S}_1$ and $z = k + 1 \in \mathcal{S}_1$ such that w is a super Yamanouchi word. Then by definition, we have $|u|_k > |u|_{k+1}$, showing that $uyzx$ is a super Yamanouchi word, and hence v is so. Similarly, we show that if v is a super Yamanouchi word, then w is so. Suppose now that $x = k \in \mathcal{S}_0$ and $y = z = k + 1 \in \mathcal{S}_0$ such that w is a super Yamanouchi word. Then by definition, the following property $|u'|_k > |u'|_{k+1}$ holds, showing that $yzxu'$ is a super Yamanouchi word, and then v is so. Similarly, we show that if v is a super Yamanouchi word, then w is so. \square

4.3.3. Super Littlewood–Richardson skew tableaux. We say that a super skew tableau S over \mathcal{S} is a *super Littlewood–Richardson skew tableau* if its reading word $\mathbf{R}(S)$ is a super Yamanouchi word. For instance, the following super tableau over $\mathcal{S} = \{\bar{2} < \bar{1} < 1 < 2\}$ is a super Littlewood–Richardson skew tableau of shape $(7, 5, 3, 3, 2, 1, 1)/(4, 3, 2, 2, 1)$ and of weight $(3, 2, 3, 2)$:



For any partition μ in \mathcal{P} of height k , we define the $(n + m)$ -uplet $\check{\mu} := (\mu_1, \tilde{\mu}_2)$ where μ_1 is the partition formed by the first m parts of μ and μ_2 is the partition formed by its last $k - m$ parts, such that if $m \leq k < m + n$ then the last $(m + n) - k$ parts of μ_2 are zero and if $k < m$ then the last $m - k$

iii) $c_{\lambda,\mu}^{\nu} = 0$, for other μ .

Hence, we obtain the following decomposition in R_S :

$$S_{(5,4,3,2)/(3,3,1)} = S_{(5,2)} + S_{(5,1,1)} + 2S_{(4,3)} + 2S_{(4,2,1)} + 2S_{(3,2,2)} + S_{(3,3,1)}.$$

Note finally that the readings of the *genuine highest weight super tableaux* introduced in [1] as highest weight vectors for the crystal graph for the representations of the general linear Lie superalgebra of type A are super Yamanouchi words. A future work would be to investigate the combinatorial properties of the representations of the general linear Lie superalgebra $\mathfrak{gl}_{m,n}$ using the constructions developed in this article.

REFERENCES

- [1] Georgia Benkart, Seok-Jin Kang, and Masaki Kashiwara. Crystal bases for the quantum superalgebra $U_q(\mathfrak{gl}(m, n))$. *Journal of the American Mathematical Society*, 13(2):295–331, 2000.
- [2] A Berele and A Regev. Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras. *Advances in Mathematics*, 64(2):118–175, 1987.
- [3] A. Berele and J.B. Remmel. Hook flag characters and their combinatorics. *Journal of Pure and Applied Algebra*, 35:225–245, 1985.
- [4] Leonid A. Bokut, Yuqun Chen, Weiping Chen, and Jing Li. New approaches to plactic monoid via Gröbner–Shirshov bases. *J. Algebra*, 423:301–317, 2015.
- [5] F. Bonetti, D. Senato, and A. Venezia. The Robinson–Schensted correspondence for the fourfold algebra. *Boll. Un. Mat. Ital. B (7)*, 2(3):541–554, 1988.
- [6] Alan J. Cain, Robert D. Gray, and António Malheiro. Finite Gröbner–Shirshov bases for Plactic algebras and biautomatic structures for Plactic monoids. *J. Algebra*, 423:37–53, 2015.
- [7] Alan J. Cain, Robert D. Gray, and António Malheiro. Crystal monoids & crystal bases: rewriting systems and biautomatic structures for plactic monoids of types A_n , B_n , C_n , D_n , and G_2 . *J. Combin. Theory Ser. A*, 162:406–466, 2019.
- [8] Neil O Connell. Conditioned random walks and the RSK correspondence. *Journal of Physics A: Mathematical and General*, 36(12):3049–3066, 2003.
- [9] William Fulton. *Young tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry.
- [10] Frank D Grosshans, Gian-Carlo Rota, and Joel A Stein. *Invariant theory and superalgebras*. Number 69. American Mathematical Soc., 1987.
- [11] Nohra Hage. Finite convergent presentation of plactic monoid for type C. *Internat. J. Algebra Comput.*, 25(8):1239–1263, 2015.
- [12] Nohra Hage. Coherent presentations of super plactic monoids of type A by insertions. submitted, arXiv:2112.09633, December 2021.

REFERENCES

- [13] Nohra Hage. Super jeu de taquin and combinatorics of super tableaux of type A. *International Journal of Algebra and Computation*, 32(05):929–952, 2022.
- [14] Nohra Hage and Philippe Malbos. Knuth’s coherent presentations of plactic monoids of type A. *Algebr. Represent. Theory*, 20(5):1259–1288, 2017.
- [15] Nohra Hage and Philippe Malbos. String of columns rewriting and confluence of the jeu de taquin. submitted, arXiv:2012.15649, December 2020.
- [16] Nohra Hage and Philippe Malbos. Chinese syzygies by insertions. *Semigroup Forum*, 104(1):88–108, 2022.
- [17] Masaki Kashiwara. Crystallizing the q -analogue of universal enveloping algebras. In *Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990)*, pages 791–797. Math. Soc. Japan, Tokyo, 1991.
- [18] Donald E. Knuth. Permutations, matrices, and generalized Young tableaux. *Pacific J. Math.*, 34:709–727, 1970.
- [19] Łukasz Kubat and Jan Okniński. Gröbner-Shirshov bases for plactic algebras. *Algebra Colloq.*, 21(4):591–596, 2014.
- [20] Roberto La Scala, Vincenzo Nardoza, and Domenico Senato. Super RSK-algorithms and super plactic monoid. *Int. J. Algebra Comput.*, 16(2):377–396, 2006.
- [21] Alain Lascoux and Marcel-P. Schützenberger. Le monoïde plaxique. In *Noncommutative structures in algebra and geometric combinatorics (Naples, 1978)*, volume 109 of *Quad. “Ricerca Sci.”*, pages 129–156. CNR, Rome, 1981.
- [22] Jean-Louis Loday and Todor Popov. Parastatistics algebra, Young tableaux and the super plactic monoid. *International Journal of Geometric Methods in Modern Physics*, 05(08):1295–1314, 2008.
- [23] M. Lothaire. *Algebraic combinatorics on words*, volume 90 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2002.
- [24] Robert Muth. Super RSK correspondence with symmetry. *Electron. J. Comb.*, 26(2):2–27, 2019.
- [25] Gilbert de Beauregard Robinson. On the Representations of the Symmetric Group. *Amer. J. Math.*, 60(3):745–760, 1938.
- [26] Bruce E Sagan and Richard P Stanley. Robinson-Schensted algorithms for skew tableaux. *Journal of Combinatorial Theory, Series A*, 55(2):161–193, 1990.
- [27] C. Schensted. Longest increasing and decreasing subsequences. *Canad. J. Math.*, 13:179–191, 1961.
- [28] M.-P. Schützenberger. Quelques remarques sur une construction de Schensted. *MATHEMATICA SCANDINAVICA*, 12:117–128, 1963.
- [29] Luis Serrano. The shifted plactic monoid. *Mathematische Zeitschrift*, 266(2):363–392, 2010.
- [30] Mark Shimozono and Dennis E. White. A color-to-spin domino Schensted algorithm. *Electron. J. Comb.*, 8(1):Research paper R21, 50 p., 2001.
- [31] Marc van Leeuwen. Spin-preserving Knuth correspondences for ribbon tableaux. *Electron. J. Comb.*, 12, 2005.

REFERENCES

- [32] Marc A. A. van Leeuwen. The Littlewood-Richardson rule, and related combinatorics. In *Interaction of combinatorics and representation theory*, volume 11 of *MSJ Mem.*, pages 95–145. Math. Soc. Japan, Tokyo, 2001.
- [33] Gérard Viennot. Une forme géométrique de la correspondance de Robinson-Schensted. In *Combinatoire et représentation du groupe symétrique*, pages 29–58. Springer, 1977.

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