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# A super Robinson–Schensted–Knuth correspondence with symmetry and the super Littlewood–Richardson rule

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# A ROBINSON–SCHENSTED–KNUTH CORRESPONDENCE FOR SUPER TABLEAUX WITH SYMMETRY AND THE SUPER LITTLEWOOD–RICHARDSON RULE

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NOHRA HAGE

**Abstract** – Robinson–Schensted–Knuth (RSK) correspondence is a bijective correspondence between two-rowed arrays of non-negative integers and pairs of same-shape semistandard tableaux. This correspondence satisfies the symmetry property, that is, exchanging the rows of a two-rowed array is equivalent to exchanging the positions of the corresponding pair of semistandard tableaux. In this article, we introduce a super analogue of the RSK correspondence for super tableaux over a signed alphabet using a super version of Schensted’s insertion algorithms. We give a geometrical interpretation of the super-RSK correspondence by a matrix-ball construction, showing the symmetry property in complete generality. We deduce a combinatorial version of the super Littlewood–Richardson rule on super Schur functions over a finite signed alphabet. Finally, we introduce the notion of super Littlewood–Richardson skew tableaux and we give another combinatorial interpretation of the super Littlewood–Richardson rule.

**Keywords** – super jeu de taquin, super Young tableaux, super plactic monoids, super-RSK correspondences, super Littlewood–Richardson rule.

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### 1. INTRODUCTION

Schensted introduced in [27] a bijection between permutations over the totally ordered alphabet  $[n] := \{1 < \dots < n\}$  and pairs of same-shape standard Young tableaux over  $[n]$  in order to compute the length of the longest decreasing subsequence of a given permutation over  $[n]$ . This correspondence is described using Schensted's insertion procedure that constructs a first standard Young tableau by successively inserting the elements of the given permutation according to a specific rule, while the second standard Young tableau records the evolution of the shape during the insertion. This correspondence had been also described, in a rather different form, much earlier by Robinson in [25] in an attempt to give a first correct proof of the Littlewood–Richardson rule that provides an explicit combinatorial description for expressing a skew Schur function or a product of two Schur functions as a linear combination of Schur functions. This correspondence is then referred to as the *Robinson–Schensted (RS) correspondence*. Knuth generalized in [18] this correspondence to a bijection between two-rowed arrays of elements of  $[n]$  and pairs of same-shape semistandard Young tableaux over  $[n]$ . Knuth's bijection is also constructed using Schensted's insertion algorithm that constructs a first semistandard Young tableau by successively inserting the elements of the second row of the given two-rowed array from left to right, while the second semistandard Young tableau records the evolution of the shape during this insertion using the elements of the first row of the two-rowed array. This bijection is then known as the *Robinson–Schensted–Knuth (RSK) correspondence*. An essential property of this correspondence is that it satisfies the *symmetry property*, that is, under the RSK correspondence, exchanging the rows of a two-rowed array is equivalent to exchanging the positions of the corresponding pair of semistandard Young tableaux, [18]. This property is also proved by Viennot in [33] for the RS correspondence, and by Fulton in [9] for the RSK correspondence, using geometrical interpretations of these correspondences. Since then, the RSK correspondence has found rich applications on representation theory, algebraic combinatorics and probabilistic combinatorics, [8, 9, 17, 32], and it has found many generalizations on other structures of tableaux, [2, 3, 5, 24, 26, 30, 31].

Bonetti, Senato and Venezia introduced in [5] a super-RSK correspondence using super Schensted's right and left insertion on super Young tableaux over a signed alphabet. However, the symmetry property holds only in special cases under this bijection as shown in [20]. A question was to find a super-RSK correspondence satisfying the symmetry property in complete generality and leading to the classical RSK correspondence as a particular case. Muth introduced in [24] a super-RSK correspondence using Haiman's mixed insertion algorithm on super Young tableaux and proved that this correspondence satisfies the symmetry property in complete generality. However, this correspondence is not related to the super plactic monoid of type A and to the representations of the general linear Lie superalgebra, and then it does not yield to a combinatorial description of the super Littlewood–Richardson rule. It is worth noting that Haiman's mixed insertion algorithm is used to define the shifted plactic monoid and allows to give a combinatorial version of the shifted Littlewood–Richardson rule for shifted tableaux, [29]. We introduce in this article a super version of the RSK correspondence on super Young tableaux that satisfies the symmetry property in complete generality, using super Schensted's insertion algorithms and the properties of the super plactic monoid of type A. We deduce a combinatorial description of the super Littlewood–Richardson rule for super tableaux over a signed alphabet.

**Super plactic monoid and insertion.** A *signed alphabet* is a finite or countable totally ordered set  $\mathcal{S}$  which is the disjoint union of two subsets  $\mathcal{S}_0$  and  $\mathcal{S}_1$ . The *super plactic monoid* over  $\mathcal{S}$ , denoted by  $\mathbf{P}(\mathcal{S})$ , is the quotient of the free monoid  $\mathcal{S}^*$  over  $\mathcal{S}$  by the congruence  $\sim_{\mathbf{P}(\mathcal{S})}$  generated by the following family of *super Knuth-like relations*, [20]:

$$\begin{aligned} xzy &= zxy, & \text{with } x = y & \text{ only if } y \in \mathcal{S}_0 \text{ and } y = z \text{ only if } y \in \mathcal{S}_1, \\ yxz &= yzx, & \text{with } x = y & \text{ only if } y \in \mathcal{S}_1 \text{ and } y = z \text{ only if } y \in \mathcal{S}_0, \end{aligned}$$

for any  $x \leq y \leq z$  of elements of  $\mathcal{S}$ . When  $\mathcal{S} = \mathcal{S}_0 = [n]$ , we recover the notion of the *plactic monoid of type A* introduced in [21] by Lascoux and Schützenberger, following the works of Schensted [27] and Knuth [18] on the RSK correspondence. Schützenberger introduced in [28] the jeu de taquin procedure on the structure of Young tableaux in order to give a correct proof of the Littlewood–Richardson rule using the properties of the plactic monoid. Since then, plactic monoids have found several applications in algebraic combinatorics, representation theory, probabilistic combinatorics and rewriting theory, [4, 6–9, 11–16, 19, 23]. When  $\mathcal{S}_0 = \{\bar{m} < \dots < \bar{1}\}$  and  $\mathcal{S}_1 = [n]$  with  $\bar{1} < 1$ , we recover the reverse of the super Knuth relations obtained in [1] where the super plactic congruence is described using Kashiwara’s theory of crystal bases for the representations of the general linear Lie superalgebra  $\mathfrak{gl}_{m,n}$ . The monoid  $\mathbf{P}(\mathcal{S})$  also appeared in [22] as a deformation of the parastatistics algebra which is a superalgebra with even parafermi and odd parbose creation and annihilation operators. Moreover, super algebraic structures have found many applications as combinatorial tools in the study of the invariant theory of superalgebras, the representation theory of general Lie super algebras, and algebras satisfying identities, [2, 3, 5, 10].

A *partition* of a positive integer  $n$  is a weakly decreasing sequence  $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{N}^k$  such that  $\sum \lambda_i = n$ . The transposed diagram  $\{(j, i) \mid (i, j) \in \mathcal{Y}(\lambda)\}$  defines another partition denoted by  $\tilde{\lambda}$ , called the *conjugate partition* of  $\lambda$ . The *Young diagram* of a partition  $\lambda = (\lambda_1, \dots, \lambda_k)$  is the set

$$\mathcal{Y}(\lambda) := \{(i, j) \mid 1 \leq i \leq k, 1 \leq j \leq \lambda_i\},$$

that can be represented by a diagram by drawing a box for each pair  $(i, j)$ . A *super tableau of shape  $\lambda$*  over  $\mathcal{S}$  is a Young diagram of  $\lambda$  filled by elements of  $\mathcal{S}$  such that the entries in each row are weakly increasing allowing the repetition only of elements in  $\mathcal{S}_0$  and the entries in each column are weakly increasing allowing the repetition only of elements in  $\mathcal{S}_1$ . Denote by  $\text{Yt}(\mathcal{S})$  the set of all super tableaux over  $\mathcal{S}$ . Note that, when  $\mathcal{S} = \mathcal{S}_0$  and  $\mathcal{S} = \mathcal{S}_1$ , we recover the notion of row-strict and column-strict semistandard tableaux of type A, respectively, [9]. Denote by  $\mathbf{R} : \text{Yt}(\mathcal{S}) \rightarrow \mathcal{S}^*$  the *column reading map* that reads a super tableau column-wise from bottom to top and from left to right. Schensted-like left and right insertion algorithms are introduced in [20], and consist in inserting elements of  $\mathcal{S}$  into super tableaux by rows and columns, respectively. For any word  $w = x_1 \dots x_k$  over  $\mathcal{S}$ , a super tableau  $\mathbf{T}(w)$  is computed from  $w$  by inserting its letters iteratively from left to right using the right insertion  $\leftarrow$  starting from the empty super tableau:

$$\mathbf{T}(w) := (\emptyset \leftarrow w) = ((\dots (\emptyset \leftarrow x_1) \leftarrow \dots) \leftarrow x_k).$$

Two words  $v$  and  $w$  over  $\mathcal{S}$  are super plactic congruent if and only if the equality  $\mathbf{T}(w) = \mathbf{T}(v)$  holds, [20]. This is the *cross-section property* of super tableaux with respect to  $\sim_{\mathbf{P}(\mathcal{S})}$ . We deduce that the internal product  $\star_{\text{Yt}(\mathcal{S})}$  defined on  $\text{Yt}(\mathcal{S})$  by  $t \star_{\text{Yt}(\mathcal{S})} t' := (t \leftarrow \mathbf{R}(t'))$ , for all  $t$  and  $t'$  in  $\text{Yt}(\mathcal{S})$ , is associative, and the monoids  $(\text{Yt}(\mathcal{S}), \star_{\text{Yt}(\mathcal{S})})$  and  $\mathbf{P}(\mathcal{S})$  are isomorphic.

## 1. Introduction

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**Super jeu de taquin.** Let  $\lambda$  and  $\mu$  be two partitions such that the Young diagram of  $\mu$  is contained in that of  $\lambda$ . A *super skew tableau of shape  $\lambda/\mu$*  over  $\mathcal{S}$  is a Young diagram of the following form

$$\mathcal{Y}(\lambda/\mu) := \{(i, j) \mid 1 \leq i \leq k, \mu_i < j \leq \lambda_i\}$$

filled by elements of  $\mathcal{S}$  such that the entries in each row are weakly increasing allowing the repetition only of elements in  $\mathcal{S}_0$  and the entries in each column are weakly increasing allowing the repetition only of elements in  $\mathcal{S}_1$ . A super skew tableau of shape  $\lambda/\mu$  which is not a super tableau has at least one inner corner. An *inner corner* is a box in the diagram  $\mathcal{Y}(\mu)$  such that the boxes below and to the right do not belong to  $\mathcal{Y}(\mu)$ . An *outer corner* is a box such that neither box below or to the right is in  $\mathcal{Y}(\lambda/\mu)$ . We introduce in [13] the *super jeu taquin* as an algorithm on the structure of super tableaux that consists in applying successively *forward sliding operations* on a super skew tableau that move an inner corner into an outer position by keeping the rows and the columns weakly increasing until no more inner corners remain in the initial super skew tableau. We prove that the *rectification* of a super skew tableau  $S$  by the super jeu de taquin is the unique super tableau whose reading is equivalent to the reading of  $S$  with respect to the super plactic congruence. We deduce that the resulting super tableau does not depend on the order in which we choose inner corners in the forward sliding procedure. Moreover, we relate the super jeu de taquin to the insertion algorithms and we show how we can insert a super tableau into another one by taquin. This interpretation of the insertion product  $\star_{\text{Yt}(\mathcal{S})}$  by taquin will allow us to give in Section 4 a combinatorial description of the super Littlewood–Richardson coefficient. Moreover, we introduce in [13] the super analogue of the Schützenberger’s evacuation procedure which transforms a super tableau  $t$  over a signed alphabet  $\mathcal{S}$  into an *opposite tableau*, denoted by  $t^{\text{op}}$ , over the *opposite alphabet*  $\mathcal{S}^{\text{op}}$  obtained from  $\mathcal{S}$  by reversing its order. We show that the super tableaux  $t$  and  $t^{\text{op}}$  have the same shape and that the map  $t \mapsto t^{\text{op}}$  is an involution on  $\text{Yt}(\mathcal{S})$  that is compatible with the super plactic congruence. The super evacuation procedure will allow us to construct in Subsection 3.4 a dual version of the super-RSK correspondence using the left insertion algorithm on super tableaux.

### ORGANIZATION AND MAIN RESULTS OF THE ARTICLE

We begin by recalling in Section 2 the notions of super Young tableaux, the super plactic monoid and the super jeu de taquin from [13, 20]. We also give a super version of the Robinson–Schensted correspondence for super tableaux over a signed alphabet.

**A super-RSK correspondence.** We introduce in Subsection 3.1 a super version of the RSK correspondence over a signed alphabet. Let  $\mathcal{S}$  and  $\mathcal{S}'$  be signed alphabets. A *signed two-rowed array*  $w$  on  $\mathcal{S}$  and  $\mathcal{S}'$  is a  $2 \times k$  matrix

$$w := \begin{pmatrix} x_1 \dots x_k \\ y_1 \dots y_k \end{pmatrix}$$

with  $x_i$  in  $\mathcal{S}$  and  $y_i$  in  $\mathcal{S}'$ , for all  $i = 1, \dots, k$ , that satisfies Conditions (6) and (7). Starting from a signed two-rowed array  $w$  on  $\mathcal{S}$  and  $\mathcal{S}'$ , Algorithm 3.1.1 computes a pair of same-shape

super tableaux  $(\mathbf{T}(w), \mathbf{Q}(w))$  whose entries are the ones of the second and the first row of  $w$ , respectively. More precisely, the super tableau  $\mathbf{T}(w)$  is equal to  $\mathbf{T}(y_1 \dots y_k)$ , and  $\mathbf{Q}(w)$  is the super tableau obtained by successively adding  $x_1, \dots, x_k$  in the same places as the boxes added when computing  $\mathbf{T}(w)$  starting from an empty tableau. Moreover, Algorithm 3.1.3 allows us to recover the initial signed two-rowed array  $w$  starting from the same-shape pair of super tableaux  $(\mathbf{T}(w), \mathbf{Q}(w))$ . This sets up a one-to-one correspondence between signed two-rowed arrays on signed alphabets  $\mathcal{S}$  and  $\mathcal{S}'$  and pairs of same-shape super tableaux on  $\mathcal{S}$  and  $\mathcal{S}'$ , that we denote by

$$\text{sRSK} : w \mapsto (\mathbf{T}(w), \mathbf{Q}(w)).$$

Hence, we obtain the following first main result of the article:

**Theorem 3.1.5[Super-RSK correspondence].** *Let  $\mathcal{S}$  and  $\mathcal{S}'$  be signed alphabets. The map sRSK defines a one-to-one correspondence between signed two-rowed arrays and pairs of super tableaux on  $\mathcal{S}$  and  $\mathcal{S}'$ , such that for any signed two-rowed array  $w$  on  $\mathcal{S}$  and  $\mathcal{S}'$ , we have that  $\mathbf{T}(w)$  and  $\mathbf{Q}(w)$  are same-shape super tableaux whose entries are the ones of the second and the first row of  $w$ , respectively.*

**The symmetry property.** Using a matrix-ball construction, we show in Subsection 3.2 that the super-RSK correspondence satisfies the symmetry property in complete generality. Let  $w$  be a signed two-rowed array on signed alphabets  $\mathcal{S}$  and  $\mathcal{S}'$ . The *inverse* of  $w$ , denoted by  $w^{\text{inv}}$ , is the signed two-rowed array on  $\mathcal{S}'$  and  $\mathcal{S}$  obtained from  $w$  by *exchanging* the rows of  $w$ , and by sorting the new couples on  $\mathcal{S}' \times \mathcal{S}$  according to Conditions (6) and (7). The signed two-rowed array  $w$  has *symmetry* with respect to the map sRSK if it satisfies the following property:

$$\text{if } \text{sRSK}(w) = (\mathbf{T}(w), \mathbf{Q}(w)) \quad \text{then} \quad \text{sRSK}(w^{\text{inv}}) = (\mathbf{Q}(w), \mathbf{T}(w)).$$

A *signed ball array* on signed alphabets  $\mathcal{S}$  and  $\mathcal{S}'$  is a rectangular array of balls filled with positive integers, whose rows and columns are indexed with elements of  $\mathcal{S}$  and  $\mathcal{S}'$ , respectively, allowing the repetition only of elements in  $\mathcal{S}_1$  and  $\mathcal{S}'_1$ , respectively, and where many balls can occur in the same position. We show in 3.2.2 how to correspond to each signed two-rowed array  $w$  on  $\mathcal{S}$  and  $\mathcal{S}'$ , a signed ball array, denoted by  $\text{Ba}(w)$ , whose rows and columns are indexed with the elements of the first and second row of  $w$ , respectively, from the smaller to the bigger one, and where the indices from  $\mathcal{S}_1$  and  $\mathcal{S}'_1$  are repeated as many times as they appear in  $w$ . First, we associate to each couple in  $w$  a ball in an empty signed ball array and then we order and number these balls with positive integers. Secondly, we add new balls to the given signed ball array, we order and number them and we repeat the same procedure until no more balls can be added. This geometrical construction allows us to prove the symmetry property using the symmetry of the resulting signed ball array  $\text{Ba}(w)$ . More precisely, we denote by  $\mathbf{T}(\text{Ba}(w))$  and  $\mathbf{Q}(\text{Ba}(w))$  the super tableau obtained from  $\text{Ba}(w)$  whose  $k$ -th row lists the indices of the leftmost columns and top-most rows, respectively, where each integer number occurs in the new added balls. We show in Proposition 3.2.4 that  $(\mathbf{T}(\text{Ba}(w)), \mathbf{Q}(\text{Ba}(w))) = (\mathbf{T}(w), \mathbf{Q}(w))$ . Since the matrix-ball construction is symmetric in the rows and columns of the resulting signed ball array, we deduce the following result:

## 1. Introduction

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**Theorem 3.1.10[Symmetry of the super-RSK correspondence].** *Let  $\mathcal{S}$  and  $\mathcal{S}'$  be signed alphabets. All signed two-rowed arrays on  $\mathcal{S}$  and  $\mathcal{S}'$  have symmetry with respect to the super-RSK correspondence map sRSK.*

We end Subsection 3.4 by giving a dual way to construct the pair of super tableaux corresponding to a signed two-rowed array with respect to the map sRSK using the left insertion algorithm and the super evacuation procedure on super tableaux.

**Super Littlewood–Richardson rule for super tableaux.** We give in Section 4 a combinatorial interpretation of the super Littlewood–Richardson rule using the super-RSK correspondence. Let  $\lambda$ ,  $\mu$  and  $\nu$  be partitions such that the Young diagram of  $\lambda$  is contained in that of  $\nu$ . We show that the number  $c_{\lambda,\mu}^\nu$  of ways a given super tableau  $t$  of shape  $\nu$  can be written as the product of a super tableau  $t'$  of shape  $\lambda$  and a super tableau  $t''$  of shape  $\mu$ , does not depend on  $t$ , and depends only on the partitions  $\lambda$ ,  $\mu$  and  $\nu$ . Moreover, we show that  $c_{\lambda,\mu}^\nu$  is equal to the number of super skew tableaux of shape  $\nu/\lambda$  whose rectification is a given tableau of shape  $\mu$ . Hence, we obtain the following result:

**Theorem 4.2.1[The super Littlewood–Richardson rule].** *Let  $\lambda$ ,  $\mu$  and  $\nu$  be partitions such that the Young diagram of  $\lambda$  is contained in that of  $\nu$ . The following identities*

$$S_\lambda S_\mu = \sum_{\nu} c_{\lambda,\mu}^\nu S_\nu \quad \text{and} \quad S_{\nu/\lambda} = \sum_{\mu} c_{\lambda,\mu}^\nu S_\mu$$

*hold in the tableau  $\mathbb{Z}$ -algebra rising from the super plactic monoid, where  $S_{\nu/\lambda}$ ,  $S_\lambda$ ,  $S_\mu$  and  $S_\nu$  denote respectively the formal sum of all super tableaux of shape  $\nu/\lambda$ ,  $\lambda$ ,  $\mu$  and  $\nu$  over a finite signed alphabet.*

**Super Littlewood–Richardson tableaux.** We suppose in this final part that  $\mathcal{S}_1 = \{1 < \dots < n\}$  and  $\mathcal{S}_0 = \{\bar{m} < \dots < \bar{1}\}$  with  $\bar{1} < 1$ . For any  $w$  in  $\mathcal{S}^*$ , we denote by  $|w|_i$  the number of times the element  $i$  of  $\mathcal{S}$  appears in  $w$ , by  $\underline{|w|}$  the number of elements of  $\mathcal{S}_1$  that appear in  $w$  such that each element is counted only once, and by  $\text{wt}(w)$  the  $(n+m)$ -uplet  $(|w|_{\bar{m}}, \dots, |w|_{\bar{1}}, |w|_1, \dots, |w|_n)$ . For any super skew tableau  $S$  over  $\mathcal{S}$ , we will call  $\text{wt}(\mathbf{R}(S))$  the *weight* of  $S$ . A word  $w$  over  $\mathcal{S}$  is a *super Yamanouchi word* if for every right subword  $w'$  of  $w$ , we have  $|w'|_{\bar{m}} \geq \dots \geq |w'|_{\bar{2}} \geq |w'|_{\bar{1}} \geq \underline{|w'|}$ , and for every left subword  $w'$  of  $w$ , the following property  $|w'|_1 \geq |w'|_2 \geq \dots \geq |w'|_n$  holds. A super skew tableau  $S$  over  $\mathcal{S}$  is called a *super Littlewood–Richardson skew tableau* if its reading word  $\mathbf{R}(S)$  is a super Yamanouchi word. For any partition  $\mu \in \mathbb{N}^k$ , define  $\check{\mu} := (\mu_1, \bar{\mu}_2)$  where  $\mu_1$  is the partition formed by the first  $m$  parts of  $\mu$  and  $\mu_2$  is the partition formed by its last  $k-m$  parts, such that if  $k < m+n$  then some parts of  $\mu_2$  are zero and if  $k < m$  then some parts of  $\mu_1$  are zero and all the parts of  $\mu_2$  are zero.

Using properties of the super plactic congruence and following Theorem 4.2.1, we deduce the following result:

**Theorem 4.3.5.** *The super Littlewood–Richardson coefficient  $c_{\lambda,\mu}^\nu$  is equal to the number of super Littlewood–Richardson skew tableaux of shape  $\nu/\lambda$  and of weight  $\check{\mu}$ .*

This theorem allows us to compute the super Littlewood–Richardson coefficient  $c_{\lambda,\mu}^\nu$  by a new combinatorial method as shown in Example 4.3.6.

**Notation.** Let  $\mathcal{A}$  be a totally ordered alphabet. We will denote by  $\mathcal{A}^*$  the free monoid of words over  $\mathcal{A}$ , the product being concatenation of words, and the identity being the empty word. When  $\mathcal{A}$  is finite, we will denote by  $\#\mathcal{A}$  its cardinal number. We will denote by  $w = x_1 \dots x_k$  a word in  $\mathcal{A}^*$  of length  $k$ , where  $x_1, \dots, x_k$  belong to  $\mathcal{A}$ . The length of a word  $w$  will be denoted by  $|w|$ . Let  $w = x_1 \dots x_k$  be a word in  $\mathcal{A}^*$ . For any  $i \leq j$ , a *subword* of  $w$  is a word  $w' = x_i \dots x_j$  made up of consecutive letters of  $w$ . We will denote by  $[n]$  the ordered set  $\{1 < 2 < \dots < n\}$  for  $n$  in  $\mathbb{Z}_{>0}$ . Let  $\mathcal{S}$  be a finite or countable totally ordered set and  $\|\cdot\| : \mathcal{S} \rightarrow \mathbb{Z}_2$  be any map, where  $\mathbb{Z}_2 = \{0, 1\}$  denotes the additive cyclic group of order 2. The ordered pair  $(\mathcal{S}, \|\cdot\|)$  is called a *signed alphabet* and the map  $\|\cdot\|$  is called the *signature map* of  $\mathcal{S}$ . We also denote  $\mathcal{S}_0 = \{a \in \mathcal{S} \mid \|a\| = 0\}$  and  $\mathcal{S}_1 = \{a \in \mathcal{S} \mid \|a\| = 1\}$ . A monoid  $\mathbf{M}$  is said a  $\mathbb{Z}_2$ -graded monoid or a *supermonoid* if a map  $\|\cdot\| : \mathbf{M} \rightarrow \mathbb{Z}_2$  is given such that  $\|u.v\| = \|u\| + \|v\|$ , for all  $u$  and  $v$  in  $\mathbf{M}$ . We call  $\|u\|$  the  $\mathbb{Z}_2$ -degree of the element  $u$ . The free monoid  $\mathcal{S}^*$  over  $\mathcal{S}$  is  $\mathbb{Z}_2$ -graded by considering  $\|w\| := \|x_1\| + \dots + \|x_k\|$ , for any word  $w = x_1 \dots x_k$  in  $\mathcal{S}^*$ . In the rest of this article, and if there is no possible confusion,  $\mathcal{S}$  denotes a signed alphabet.

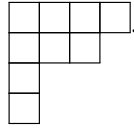
## 2. THE SUPER PLACTIC MONOID OF TYPE A

In this section, we recall the notions of super Young tableaux, the super plactic monoid and the super jeu de taquin from [13, 20]. Moreover, we give a super version of the Robinson–Schensted correspondence for super tableaux over a signed alphabet.

**2.1. Super Young tableaux.** A *partition* of a positive integer  $n$ , denoted by  $\lambda \vdash n$ , is a weakly decreasing sequence  $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{N}^k$  such that  $\sum \lambda_i = n$ . The integer  $k$  is called *number of parts* or *height* of  $\lambda$ . Denote by  $\mathcal{P}_n$  the set of partitions of a positive integer  $n$ , and set  $\mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n$ . The (Ferrers)–Young diagram of a partition  $\lambda = (\lambda_1, \dots, \lambda_k)$  is the set

$$\mathcal{Y}(\lambda) := \{(i, j) \mid 1 \leq i \leq k, 1 \leq j \leq \lambda_i\},$$

that can be represented by a diagram by drawing a box for each pair  $(i, j)$ . For instance, the Young diagram for  $\lambda = (4, 3, 1, 1)$  is the following



The transposed diagram  $\{(j, i) \mid (i, j) \in \mathcal{Y}(\lambda)\}$  defines another partition  $\tilde{\lambda} \vdash n$ , called the *conjugate partition* of  $\lambda$ , whose parts are the lengths of the columns of  $\mathcal{Y}(\lambda)$ . Let  $\lambda \vdash n$  be a partition. A *super semistandard Young tableau*, or *super tableau* for short, over  $\mathcal{S}$  is a pair  $t := (\lambda, \mathcal{T})$  where  $\mathcal{T} : \mathcal{Y}(\lambda) \rightarrow \mathcal{S}$  is a map satisfying the following two conditions:

- i)  $\mathcal{T}(i, j) \leq \mathcal{T}(i, j + 1)$ , with  $\mathcal{T}(i, j) = \mathcal{T}(i, j + 1)$  only if  $\|\mathcal{T}(i, j)\| = 0$ ,
- ii)  $\mathcal{T}(i, j) \leq \mathcal{T}(i + 1, j)$ , with  $\mathcal{T}(i, j) = \mathcal{T}(i + 1, j)$  only if  $\|\mathcal{T}(i, j)\| = 1$ .



## 2. The super plactic monoid of type A

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We will call  $\mathcal{Y}(\lambda)$ ,  $\mathcal{T}$  and  $\lambda$ , the *frame*, the *filing* and the *shape* of the super tableau  $t$  respectively. When the map  $\mathcal{T}$  is injective, we say that  $t$  is a *standard Young tableau*, or *standard tableau* for short, over  $\mathcal{S}$ . Denote by  $\emptyset$  the empty super tableau over  $\mathcal{S}$ . Denote by  $\text{Yt}(\mathcal{S})$  (resp.  $\text{Ys}(\mathcal{S})$ ) the set of all super tableaux (resp. standard tableaux) over  $\mathcal{S}$  and by  $\text{Yt}(\mathcal{S}, \lambda)$  the set of all super tableaux of shape  $\lambda$  over  $\mathcal{S}$ .

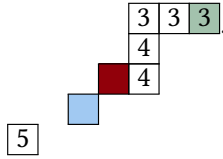
Let  $\lambda$  and  $\mu$  be in  $\mathcal{P}$  with heights  $k$  and  $l$  respectively such that  $l \leq k$ . We write  $\mu \subseteq \lambda$  if  $\mu_i \leq \lambda_i$ , for any  $i$ , that is the Young diagram of  $\mu$  is contained in that of  $\lambda$ . In this case, the set

$$\mathcal{Y}(\lambda/\mu) := \{(i, j) \mid 1 \leq i \leq k, \mu_i < j \leq \lambda_i\}$$

is called a *skew diagram*  $\lambda/\mu$  or a *skew shape*. We denote  $\lambda/0 := \lambda$  the skew shape corresponding to the Young diagram  $\mathcal{Y}(\lambda)$ . Let  $\lambda/\mu$  be a skew shape. A *super semistandard skew tableau*, or *super skew tableau* for short, of shape  $\lambda/\mu$  and frame  $\mathcal{Y}(\lambda/\mu)$  over  $\mathcal{S}$  is a pair  $S := (\lambda/\mu, \mathcal{U})$  where  $\mathcal{U} : \mathcal{Y}(\lambda/\mu) \rightarrow \mathcal{S}$  is a map satisfying the following conditions:

- i)  $\mathcal{U}(i, j) \leq \mathcal{U}(i, j+1)$ , with  $\mathcal{U}(i, j) = \mathcal{U}(i, j+1)$  only if  $\|\mathcal{U}(i, j)\| = 0$ ,
- ii)  $\mathcal{U}(i, j) \leq \mathcal{U}(i+1, j)$ , with  $\mathcal{U}(i, j) = \mathcal{U}(i+1, j)$  only if  $\|\mathcal{U}(i, j)\| = 1$ .

Denote by  $\text{St}(\mathcal{S})$  the set of all super skew tableaux over  $\mathcal{S}$  and by  $\text{St}(\mathcal{S}, \lambda/\mu)$  the set of all super skew tableaux of shape  $\lambda/\mu$  over  $\mathcal{S}$ . Note that we will identify the set of super tableaux of shape  $\lambda$  with the set of super skew tableaux of shape  $\lambda/0$ . A super skew tableau of shape  $\lambda/\mu$  which is not a super tableau has at least one inner corner. An *inner corner* is a box in the diagram  $\mathcal{Y}(\mu)$  such that the boxes below and to the right do not belong to  $\mathcal{Y}(\mu)$ . An *outer corner* is a box such that neither box below or to the right is in  $\mathcal{Y}(\lambda/\mu)$ . Note that in some cases inner corners can be also outer corners. For instance, consider  $\mathcal{S} = \mathbb{N}$  with signature given by  $\mathcal{S}_0$  the set of odd numbers and  $\mathcal{S}_1$  defined consequently. The red box in the following super skew tableau of shape  $(7, 5, 5, 3, 1)/(4, 4, 4, 3)$  is an inner corner, the blue one is an inner and outer corner, and the green one is an outer corner that is not an inner corner:



Denote by  $\mathbf{R} : \text{St}(\mathcal{S}) \rightarrow \mathcal{S}^*$  the *column reading map* that reads a super tableau column-wise from bottom to top and from left to right. For instance, consider the alphabet  $\mathcal{S} = \mathbb{N}$  with signature given by  $\mathcal{S}_0$  the set of even numbers and  $\mathcal{S}_1$  defined consequently. The following diagram is a super tableau of shape  $(3, 2, 2, 1)$  over  $\mathcal{S}$  :

$$t = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 3 & \\ \hline 2 & 4 & \\ \hline 4 & & \\ \hline \end{array} \quad \text{with } \mathbf{R}(t) = 42114323. \quad (1)$$

## 2.2. Super Robinson–Schensted correspondence

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The following diagram is a super skew tableau of shape  $(6, 4, 2, 2, 1, 1)/(3, 2, 2, 1)$  over  $\mathcal{S}$  :

$$S = \begin{array}{|c|c|c|c|c|c|} \hline & & & 1 & 2 & 3 \\ \hline & & 1 & 3 & & \\ \hline & & & & & \\ \hline & & & 4 & & \\ \hline & & 2 & & & \\ \hline & & 4 & & & \\ \hline \end{array} \quad \text{with } \mathbf{R}(S) = 42413123. \quad (2)$$

**2.2. Super Robinson–Schensted correspondence.** From a super tableau we can obtain a word over  $\mathcal{S}$  by taking its column reading. The following algorithm will allow us to invert this process and obtain a super tableau from any word over  $\mathcal{S}$ .

**Algorithm 2.2.1** ([20]). The *right (or row) insertion*, denoted by  $\leftarrow$ , inserts an element  $x$  in  $\mathcal{S}$  into a super tableau  $t$  of  $\text{Yt}(\mathcal{S})$  as follows:

*Input:* A super tableau  $t$  and a letter  $x \in \mathcal{S}$ .

*Output:* A super tableau  $t \leftarrow x$ .

*Method:* If  $t$  is empty, create a box and label it  $x$ . Suppose  $t$  is non-empty. If  $x \in \mathcal{S}_o$  (resp.  $x \in \mathcal{S}_1$ ) is at least as large as (resp. larger than) the last element of the top row of  $t$ , then put  $x$  in a box to the right of this row; otherwise let  $y$  be the smallest element of the top row of  $t$  such that  $y > x$  (resp.  $y \geq x$ ). Then replace  $y$  by  $x$  in this row and recursively insert  $y$  into the super tableau formed by the rows of  $t$  below the topmost. Note that this recursion may end with an insertion into an empty row below the existing rows of  $t$ . Output the resulting super tableau.

For any word  $w = x_1 \dots x_k$  over  $\mathcal{S}$ , a super tableau  $\mathbf{T}(w)$  is computed from  $w$  by inserting its letters iteratively from left to right using the right insertion starting from the empty super tableau:

$$\mathbf{T}(w) := (\emptyset \leftarrow w) = ((\dots(\emptyset \leftarrow x_1) \leftarrow \dots) \leftarrow x_k).$$

Moreover, for any  $t$  in  $\text{Yt}(\mathcal{S})$ , the equality  $\mathbf{T}(\mathbf{R}(t)) = t$  holds, [20].

**Lemma 2.2.2** (Row-bumping lemma [20]). *Let  $t$  be in  $\text{Yt}(\mathcal{S})$  and  $x, x'$  in  $\mathcal{S}$ . Consider the position  $(i, j)$  (resp.  $(i', j')$ ) of the new box added to the frame of  $t$  (resp.  $(t \leftarrow x)$ ) after computing  $(t \leftarrow x)$  (resp.  $((t \leftarrow x) \leftarrow x')$ ). The following conditions are equivalent:*

- i)  $x \leq x'$ , with  $x = x'$  only if  $\|x\| = 0$ ,
- ii)  $j < j'$ , and  $i \geq i'$ .

**Algorithm 2.2.3.** Let  $w$  be in  $\mathcal{S}$ . When computing the super tableau  $\mathbf{T}(w)$ , a standard tableau  $\mathbf{Q}(w)$  over  $[n]$  can be also computed as follows:

*Input:* A word  $x_1 \dots x_k$  over  $\mathcal{S}$ .

*Output:* A super tableau  $\mathbf{T}(x_1 \dots x_k)$  over  $\mathcal{S}$  and a standard tableau  $\mathbf{Q}(x_1 \dots x_k)$  over  $[n]$ .

*Method:* Start with an empty super tableau  $T_0$  and an empty standard tableau  $Q_0$ . For each  $i = 1, \dots, k$ , compute  $T_{i-1} \leftarrow x_i$  as per Algorithm 2.2.1; let  $T_i$  be the resulting super tableau. Add a box filled with  $i$  to the standard tableau  $Q_{i-1}$  in the same place as the box that belongs to  $T_i$  but not to  $T_{i-1}$ ; let  $Q_i$  be the resulting standard tableau. Output  $T_k$  for  $\mathbf{T}(x_1 \dots x_k)$  and  $Q_k$  as  $\mathbf{Q}(x_1 \dots x_k)$ .

## 2. The super plactic monoid of type A

**Example 2.2.4.** For instance, consider the alphabet  $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$  with signature given by  $\mathcal{S}_0 = \{1, 3, 4\}$  and  $\mathcal{S}_1$  defined consequently. The sequence of pairs produced during the computation of  $\mathbf{T}(w)$  and  $\mathbf{Q}(w)$  starting from the word  $w = 2421456356431541$  is

$$\begin{aligned}
 & (\emptyset, \emptyset), \left( \begin{array}{|c|} \hline 2 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline \end{array} \right), \left( \begin{array}{|c|c|} \hline 2 & 4 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \right), \left( \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 2 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} \right), \left( \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & \\ \hline 2 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline 4 & \\ \hline \end{array} \right), \left( \begin{array}{|c|c|c|} \hline 1 & 4 & 4 \\ \hline 2 & & \\ \hline 2 & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & & \\ \hline 4 & & \\ \hline \end{array} \right), \\
 & \left( \begin{array}{|c|c|c|c|} \hline 1 & 4 & 4 & 5 \\ \hline 2 & & & \\ \hline 2 & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & & & \\ \hline 4 & & & \\ \hline \end{array} \right), \left( \begin{array}{|c|c|c|c|c|} \hline 1 & 4 & 4 & 5 & 6 \\ \hline 2 & & & & \\ \hline 2 & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 5 & 6 & 7 \\ \hline 3 & & & & \\ \hline 4 & & & & \\ \hline \end{array} \right), \left( \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 5 & 6 \\ \hline 2 & 4 & & & \\ \hline 2 & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 5 & 6 & 7 \\ \hline 3 & 8 & & & \\ \hline 4 & & & & \\ \hline \end{array} \right), \\
 & \left( \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 5 & 6 \\ \hline 2 & 4 & 5 & & \\ \hline 2 & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 5 & 6 & 7 \\ \hline 3 & 8 & 9 & & \\ \hline 4 & & & & \\ \hline \end{array} \right), \left( \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 5 & 6 \\ \hline 2 & 4 & 5 & 6 & \\ \hline 2 & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 5 & 6 & 7 \\ \hline 3 & 8 & 9 & 10 & \\ \hline 4 & & & & \\ \hline \end{array} \right), \\
 & \left( \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 4 & 6 \\ \hline 2 & 4 & 5 & 6 & \\ \hline 2 & 5 & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 5 & 6 & 7 \\ \hline 3 & 8 & 9 & 10 & \\ \hline 4 & 11 & & & \\ \hline \end{array} \right), \left( \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 3 & 4 & 6 \\ \hline 2 & 4 & 4 & 6 & \\ \hline 2 & 5 & & & \\ \hline 5 & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 5 & 6 & 7 \\ \hline 3 & 8 & 9 & 10 & \\ \hline 4 & 11 & & & \\ \hline 12 & & & & \\ \hline \end{array} \right), \\
 & \left( \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 3 & 4 & 6 \\ \hline 2 & 3 & 4 & 6 & \\ \hline 2 & 4 & & & \\ \hline 5 & & & & \\ \hline 5 & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 5 & 6 & 7 \\ \hline 3 & 8 & 9 & 10 & \\ \hline 4 & 11 & & & \\ \hline 12 & & & & \\ \hline 13 & & & & \\ \hline \end{array} \right), \left( \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 3 & 4 & 5 \\ \hline 2 & 3 & 4 & 6 & \\ \hline 2 & 4 & 6 & & \\ \hline 5 & & & & \\ \hline 5 & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 5 & 6 & 7 \\ \hline 3 & 8 & 9 & 10 & \\ \hline 4 & 11 & 14 & & \\ \hline 12 & & & & \\ \hline 13 & & & & \\ \hline \end{array} \right), \\
 & \left( \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 3 & 4 & 4 \\ \hline 2 & 3 & 4 & 5 & \\ \hline 2 & 4 & 6 & & \\ \hline 5 & 6 & & & \\ \hline 5 & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 5 & 6 & 7 \\ \hline 3 & 8 & 9 & 10 & \\ \hline 4 & 11 & 14 & & \\ \hline 12 & 15 & & & \\ \hline 13 & & & & \\ \hline \end{array} \right), \left( \mathbf{T}(w) = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 4 & 4 \\ \hline 2 & 3 & 3 & 5 & \\ \hline 2 & 4 & 4 & & \\ \hline 5 & 6 & & & \\ \hline 5 & 6 & & & \\ \hline \end{array}, \mathbf{Q}(w) = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 5 & 6 & 7 \\ \hline 3 & 8 & 9 & 10 & \\ \hline 4 & 11 & 14 & & \\ \hline 12 & 15 & & & \\ \hline 13 & 16 & & & \\ \hline \end{array} \right).
 \end{aligned}$$

**Algorithm 2.2.5** ([20]). Let  $t$  be in  $\text{Yt}(\mathcal{S})$  and  $x$  in  $\mathcal{S}$ . Starting with the super tableau  $t \leftarrow x$ , together with the outer corner that has been added to the frame of  $t$  after insertion, we can recover the original tableau  $t$  and the element  $x$  as follows.

*Input:* A super tableau  $t$  and an outer corner of  $t$ .

*Output:* A super tableau  $t'$  and a letter  $x \in \mathcal{S}$ .

*Method:* Suppose that  $y \in \mathcal{S}_0$  (resp.  $y \in \mathcal{S}_1$ ) is the entry in the outer corner of  $t$ . Find in the row above this outer corner the entry farthest to the right which is strictly smaller (resp. smaller) than  $y$ . Then this entry is replaced by  $y$  and it is bumped up to the next row where the process is repeated until an entry is bumped out of the top row. Output the resulting super tableau for  $t'$  and the last element which is bumped out for  $x$ .

**Algorithm 2.2.6.** Let  $w$  be in  $\mathcal{S}$ . Starting from the pair of super tableaux  $(\mathbf{T}(w), \mathbf{Q}(w))$  obtained by Algorithm 2.2.3 we can recover the initial word  $w$  as follows:

*Input:* A pair  $(T, Q) \in \text{Yt}(\mathcal{S}) \times \text{Ys}([n])$  of same-shape super tableaux containing  $k$  boxes.

*Output:* A word  $x_1 \dots x_k$  over  $\mathcal{S}$ .

*Method:* Start with  $T_k = T$  and  $Q_k = Q$ . For each  $i = k - 1, \dots, 1$ , take the box filled by  $i + 1$  in  $Q_{i+1}$ , and apply Algorithm 2.2.5 to  $T_{i+1}$  with the outer corner corresponding to that box. The

### 2.3. The super plactic monoid of type A

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resulting tableau is  $T_i$ , and the element that is bumped out of the top row of  $T_{i+1}$  is denoted by  $x_{i+1}$ . Remove the box containing  $i + 1$  from  $Q_{i+1}$  and denote the resulting standard tableau by  $Q_i$ . Output the word  $x_1 \dots x_k$ .

This sets up a one-to-one correspondence between words over  $\mathcal{S}$ , and pairs of same-shape tableaux  $(T, Q)$  in  $\text{Yt}(\mathcal{S}) \times \text{Ys}([n])$ :

**Proposition 2.2.7.** *The map  $w \mapsto (\mathbf{T}(w), \mathbf{Q}(w))$  defines a bijection between words over  $\mathcal{S}$  and pairs consisting of a super tableau over  $\mathcal{S}$  and a standard tableau over  $[n]$  of the same shape.*

This is the super version of the *Robinson–Schensted correspondence* between words over  $[n]$  and pairs consisting of a row-strict semistandard tableau and a standard tableau over  $[n]$  of the same shape, [9].

**2.3. The super plactic monoid of type A.** The *super plactic monoid* over  $\mathcal{S}$ , denoted by  $\mathbf{P}(\mathcal{S})$ , is the quotient of the free monoid  $\mathcal{S}^*$  by the congruence generated by the following family of *super Knuth-like relations*, [20]:

$$\begin{aligned} xzy &= zxy, & \text{with } x = y & \text{ only if } \|y\| = 0 \text{ and } y = z \text{ only if } \|y\| = 1, \\ yxz &= yzx, & \text{with } x = y & \text{ only if } \|y\| = 1 \text{ and } y = z \text{ only if } \|y\| = 0, \end{aligned} \quad (3)$$

for any  $x \leq y \leq z$  of elements of  $\mathcal{S}$ . This congruence, denoted by  $\sim_{\mathbf{P}(\mathcal{S})}$ , is called the *super plactic congruence*. Two words over  $\mathcal{S}$  are *super plactic equivalent* if one can be transformed into the other with respect to the congruence  $\sim_{\mathbf{P}(\mathcal{S})}$ . For instance, consider the readings of the super tableau  $t$  presented in (1) and the super skew tableau  $S$  presented in (2), we have

$$\mathbf{R}(t) = 42114323 \sim_{\mathbf{P}(\mathcal{S})} 42141323 \sim_{\mathbf{P}(\mathcal{S})} 42411323 \sim_{\mathbf{P}(\mathcal{S})} 42413123 = \mathbf{R}(S). \quad (4)$$

Since the relations (3) are  $\mathbb{Z}_2$ -homogeneous, we have that  $\mathbf{P}(\mathcal{S})$  is a supermonoid. For any  $w$  in  $\mathcal{S}^*$ , we have  $w \sim_{\mathbf{P}(\mathcal{S})} \mathbf{R}(\mathbf{T}(w))$ , [13]. Moreover, the set  $\text{Yt}(\mathcal{S})$  satisfies the following *cross-section property* for  $\sim_{\mathbf{P}(\mathcal{S})}$ :

**Property 2.3.1** ([20]). For all  $w$  and  $w'$  in  $\mathcal{S}^*$ , we have  $w \sim_{\mathbf{P}(\mathcal{S})} w'$  if and only if  $\mathbf{T}(w) = \mathbf{T}(w')$ .

Define now an internal product  $\star_{\text{Yt}(\mathcal{S})}$  on  $\text{Yt}(\mathcal{S})$  by setting

$$t \star_{\text{Yt}(\mathcal{S})} t' := (t \leftarrow \mathbf{R}(t')) \quad (5)$$

for all  $t, t'$  in  $\text{Yt}(\mathcal{S})$ . By definition the relations  $t \star_{\text{Yt}(\mathcal{S})} \emptyset = t$  and  $\emptyset \star_{\text{Yt}(\mathcal{S})} t = t$  hold, showing that the product  $\star_{\text{Yt}(\mathcal{S})}$  is unitary with respect to  $\emptyset$ . Following Property 2.3.1, we deduce that the product  $\star_{\text{Yt}(\mathcal{S})}$  is associative and the monoids  $(\text{Yt}(\mathcal{S}), \star_{\text{Yt}(\mathcal{S})})$  and  $\mathbf{P}(\mathcal{S})$  are isomorphic.

**Lemma 2.3.2.** *Let  $w$  and  $v$  be two words over  $\mathcal{S}$  and let  $w'$  and  $v'$  be the words obtained respectively from  $w$  and  $v$  after removing the  $i$  largest and the  $j$  smallest letters for any  $i$  and  $j$ . If  $w$  and  $v$  are super plactic congruent then  $w'$  and  $v'$  are so.*

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*Proof.* It is sufficient by induction to prove that the words obtained by removing the largest or the smallest letters from  $w$  and  $v$  are super plactic equivalent. We prove the case of the largest, the other being similar. Suppose that  $w = uxzyu'$  and  $v = uzxyu'$  (resp.  $w = uyxzu'$  and  $v = uyzxu'$ ) for all  $u, u'$  in  $\mathcal{S}^*$  such that  $x \leq y \leq z$  in  $\mathcal{S}$  with  $x = y$  only if  $\|y\| = 0$  (resp.  $\|y\| = 1$ ) and  $y = z$  only if  $\|y\| = 1$  (resp.  $\|y\| = 0$ ). We consider two cases depending on whether or not the element that is removed from  $w$  and  $v$  is one of the letters  $x, y$ , or  $z$ . If this element is not one of the letters  $x, y$ , or  $z$ , then the resulting words are obviously super plactic equivalent. Otherwise, the removed letter must be the letter  $z$  and then the resulting words are equal, showing the claim.  $\square$

**2.4. The super jeu de taquin.** Recall from [13] the super jeu de taquin procedure which transforms super skew tableaux over  $\mathcal{S}$  into super tableaux over  $\mathcal{S}$  as follows. A *sliding operation* belongs to one of the following two operations:

i) *vertical sliding*:  $\begin{array}{|c|c|} \hline \square & y \\ \hline x & \square \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline x & y \\ \hline \square & \square \\ \hline \end{array}$ , for any  $x \leq y$  with  $x = y$  only if  $\|x\| = 0$ ,

ii) *horizontal sliding*:  $\begin{array}{|c|c|} \hline \square & x \\ \hline y & \square \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline x & \square \\ \hline \square & y \\ \hline \end{array}$ , for any  $x \leq y$  with  $x = y$  only if  $\|x\| = 1$ .

Note that if  $x$  or  $y$  in i) and ii) are empty, the following operations are performed:

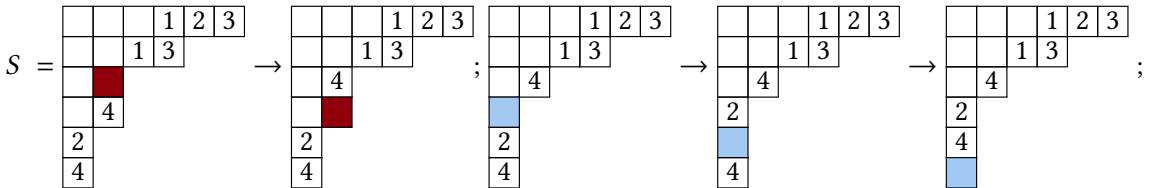
$$\begin{array}{|c|c|} \hline \square & \square \\ \hline x & \square \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline x & \square \\ \hline \square & \square \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline \square & x \\ \hline \square & \square \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline x & \square \\ \hline \square & \square \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \quad \text{for any } x.$$

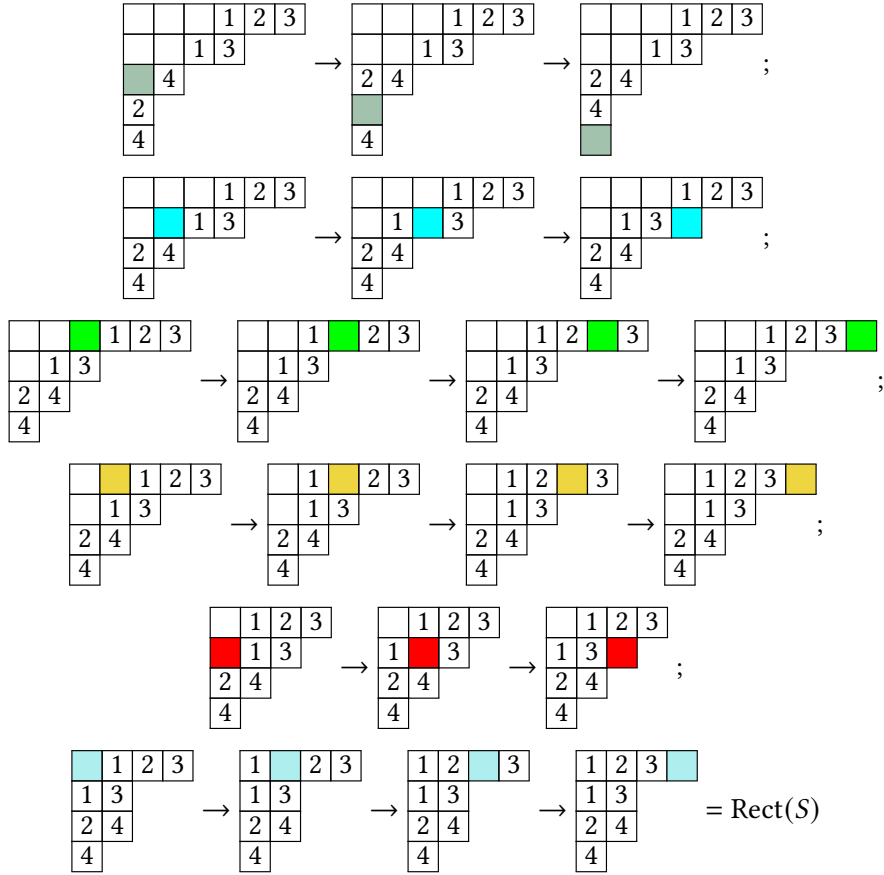
A *forward sliding* is a sequence of sliding operations starting from a super skew tableau and one of its inner corners, and moving the empty box until it becomes an outer corner. The *super jeu de taquin* on a super skew tableau  $S$  consists in applying successively the forward sliding algorithm starting from  $S$  until we get a diagram without inner corners. The resulting diagram, denoted by  $\text{Rect}(S)$ , is a super tableau called the *rectification* of  $S$ , [13]. Two super skew tableaux  $S$  and  $S'$  are *super jeu de taquin equivalent* if and only if  $S$  can be obtained from  $S'$  by a sequence of sliding operations. We show that if two super skew tableaux  $S$  and  $S'$  are super jeu de taquin equivalent, then the words  $\mathbf{R}(S)$  and  $\mathbf{R}(S')$  are super plactic equivalent, [13]. Moreover, we show the following property:

**Property 2.4.1** ([13]). For all  $S, S' \in \text{St}(\mathcal{S})$ ,  $\text{Rect}(S) = \text{Rect}(S')$  if and only if  $\mathbf{R}(S) \sim_{\mathbf{P}(\mathcal{S})} \mathbf{R}(S')$ .

As a consequence, we deduce that starting with a given super skew tableau, all choices of inner corners lead to the same rectified super tableau. This is the *confluence property* of the super jeu de taquin.

**Example 2.4.2.** For instance, the super jeu de taquin on the super skew tableau  $S$  in (2) applies eight occurrences of forward sliding:





with  $R(S) = 42413123 \sim_{P(S)} 42114323 = R(\text{Rect}(S))$ , as shown in (4).

**2.4.3. The reverse sliding.** A reverse sliding operation belongs to one of the following operations, [13]:

- i) vertical reverse sliding:  $\begin{array}{|c|} \hline y \\ \hline x \\ \hline \end{array} \rightarrow \begin{array}{|c|} \hline \\ \hline x \\ \hline y \\ \hline \end{array}$ , for any  $x \leq y$  with  $x = y$  only if  $\|x\| = 0$ ,
- ii) horizontal reverse sliding:  $\begin{array}{|c|} \hline x \\ \hline y \\ \hline \end{array} \rightarrow \begin{array}{|c|} \hline x \\ \hline \\ \hline y \\ \hline \end{array}$ , for any  $x \leq y$  with  $x = y$  only if  $\|x\| = 1$ .

A reverse sliding is a sequence of reverse sliding operations starting from a skew tableau and one of its empty outer corners, and moving this box until it becomes an inner corner. Starting from the resulting super skew tableau of a forward sliding, together with the outer corner that was removed, and if we apply the reverse sliding, then we will obtain the initial super skew tableau with the chosen inner corner.

**2.4.4. Super jeu de taquin and insertion.** Let  $S$  and  $S'$  be super skew tableaux in  $\text{St}(\mathcal{S})$  of shape  $(\lambda_1, \dots, \lambda_k)/(\lambda'_1, \dots, \lambda'_{k'})$  and  $(\mu_1, \dots, \mu_l)/(\mu'_1, \dots, \mu'_{l'})$ , respectively. We will denote by  $[S, S']$  the super skew tableau of shape

$$(\mu_1 + \lambda_1, \dots, \mu_l + \lambda_1, \lambda_1, \dots, \lambda_k)/(\mu'_1 + \lambda_1, \dots, \mu'_{l'} + \lambda_1, \lambda_1, \dots, \lambda_1, \lambda'_1, \dots, \lambda'_{k'}),$$



**Algorithm 3.1.1.** Let  $\mathcal{S}$  and  $\mathcal{S}'$  be signed alphabets. Starting from a signed two-rowed array  $w$  on  $\mathcal{S}$  and  $\mathcal{S}'$ , we compute a pair of same-shape super tableaux  $(\mathbf{T}(w), \mathbf{Q}(w))$  whose entries are the ones of the second and the first row of  $w$ , respectively, as follows:

*Input:* A signed two-rowed array  $w = \begin{pmatrix} x_1 \dots x_k \\ y_1 \dots y_k \end{pmatrix}$  on  $\mathcal{S}$  and  $\mathcal{S}'$ .

*Output:* A pair  $(\mathbf{T}(w), \mathbf{Q}(w)) \in \text{Yt}(\mathcal{S}') \times \text{Yt}(\mathcal{S})$  of same-shape super tableaux with  $k$  boxes.

*Method:* Start with an empty super tableau  $T_0$  and an empty super tableau  $Q_0$ . For each  $i = 1, \dots, k$ , compute  $T_{i-1} \leftarrow y_i$  as per Algorithm 2.2.1 and let  $T_i$  be the resulting super tableau. Add a box filled with  $x_i$  to the super tableau  $Q_{i-1}$  in the same place as the box that belongs to  $T_i$  but not to  $T_{i-1}$ ; let  $Q_i$  be the resulting super tableau. Output  $T_k$  for  $\mathbf{T}(w)$  and  $Q_k$  as  $\mathbf{Q}(w)$ .

**Proposition 3.1.2.** *Algorithm 3.1.1 always halts with the correct output.*

*Proof.* By construction, we have that  $\mathbf{T}(w)$  is a super tableau over  $\mathcal{S}'$  since it is equal to the super tableau  $\mathbf{T}(y_1 \dots y_k)$ . We still have to show that  $\mathbf{Q}(w)$  is a super tableau over  $\mathcal{S}$ . It is sufficient to show by induction that each  $Q_i$ , for  $i = 1, \dots, k$ , is a super tableau over  $\mathcal{S}$ . By definition of the signed two-rowed array, if  $x_i \in \mathcal{S}_0$  is placed to the right of (resp. under or to the right of) an entry  $x_l \in \mathcal{S}_0$  (resp.  $x_l \in \mathcal{S}_1$ ) in  $Q_{i-1}$ , then  $x_i$  is larger (resp. strictly larger) than  $x_l$ . Similarly, if  $x_i \in \mathcal{S}_1$  is placed under (resp. under or to the right of) an entry  $x_l \in \mathcal{S}_1$  (resp.  $x_l \in \mathcal{S}_0$ ) in  $Q_{i-1}$ , then  $x_i$  is larger (resp. strictly larger) than  $x_l$ . Let us now show that if  $x_i \in \mathcal{S}_0$  is placed under an entry  $x_l \in \mathcal{S}_0$  in  $Q_{i-1}$ , then  $x_i$  is strictly larger than  $x_l$ . Suppose the contrary. Then, we have  $x_i = x_l$  and thus by Conditions (6) and (7) we have:

$$y_l \leq y_{l+1} \leq \dots \leq y_i,$$

with  $y_l = y_{l+1} = \dots = y_i$  only if  $\|y_l\| = 0$ . Hence, following the row insertion algorithm, all the boxes added starting from the super tableau  $T_l$  to the super tableau  $T_i$  must be in different columns which contradicts the fact that  $x_i$  and  $x_l$  belong to the same column. Similarly, we show that if  $x_i \in \mathcal{S}_1$  is placed to the right of an entry  $x_l \in \mathcal{S}_1$  in  $Q_{i-1}$ , then  $x_i$  is strictly larger than  $x_l$ . Note finally that by construction the super tableaux  $\mathbf{T}(w)$  and  $\mathbf{Q}(w)$  are of the same shape and contains  $k$  boxes.  $\square$

**Algorithm 3.1.3.** Let  $w$  be a signed two-rowed array on two signed alphabets  $\mathcal{S}$  and  $\mathcal{S}'$ . Starting from the pair of super tableaux  $(\mathbf{T}(w), \mathbf{Q}(w))$  constructed by Algorithm 3.1.1, we can recover the initial signed two-rowed array  $w$  as follows:

*Input:* A pair  $(T, Q) \in \text{Yt}(\mathcal{S}') \times \text{Yt}(\mathcal{S})$  of same-shape super tableaux containing  $k$  boxes.

*Output:* A signed two-rowed array  $w = \begin{pmatrix} x_1 \dots x_k \\ y_1 \dots y_k \end{pmatrix}$  on  $\mathcal{S}$  and  $\mathcal{S}'$ .

*Method:* Start with  $T_k = T$  and  $Q_k = Q$ . For each  $i = k - 1, \dots, 1$ , take the box filled by the largest entry  $x_{i+1}$  in  $Q_{i+1}$  and if there are several equal entries in  $\mathcal{S}_0$  (resp.  $\mathcal{S}_1$ ), the box that is farthest to the right (resp. left) is selected; apply Algorithm 2.2.5 to  $T_{i+1}$  with the outer corner corresponding to that box. Let  $T_i$  be the resulting super tableau and  $y_{i+1}$  be the element that is bumped out of the top row of  $T_{i+1}$ . Remove the box containing  $x_{i+1}$  from  $Q_{i+1}$  and denote the resulting super tableau by  $Q_i$ . Output the  $2 \times k$  matrix  $\begin{pmatrix} x_1 \dots x_k \\ y_1 \dots y_k \end{pmatrix}$ .



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**Proposition 3.1.4.** *Algorithm 3.1.3 always halts with the correct output.*

*Proof.* By construction, we have  $x_1 \leq \dots \leq x_k$ . Suppose now that  $x_{i-1} = x_i$ , for  $i \in [k]$  with  $\|x_i\| = 0$ . Then by constructing, the box that is removed from  $T_i$  lies strictly to the right of the box that is removed from  $T_{i-1}$  in the next step. Following Lemma 2.2.2, the entry  $y_i$  removed first is at least as large as the entry  $y_{i-1}$  removed second, with  $y_i = y_{i-1}$ , only if  $\|y_i\| = 0$ . Suppose now that  $x_{i-1} = x_i$ , for  $i \in [k]$  with  $\|x_i\| = 1$ . Then by constructing, the box that is removed from  $T_i$  lies strictly below the box that is removed from  $T_{i-1}$  in the next step. Following Lemma 2.2.2, we obtain that  $y_{i-1} \geq y_i$ , with  $y_i = y_{i-1}$ , only if  $\|y_i\| = 1$ . Hence, the output  $2 \times k$  matrix satisfies Conditions (6) and (7), and then it is a signed two-rowed array on  $\mathcal{S}$  and  $\mathcal{S}'$ .  $\square$

It is clear from the constructions that the two processes described in Algorithm 3.1.1 and Algorithm 3.1.3 are inverse to each other. This sets up a one-to-one correspondence between signed two-rowed arrays on signed alphabets  $\mathcal{S}$  and  $\mathcal{S}'$  and pairs of same-shape super tableaux on  $\mathcal{S}$  and  $\mathcal{S}'$ . We will denote by

$$\text{sRSK} : w \mapsto (\mathbf{T}(w), \mathbf{Q}(w))$$

this mapping. Hence, we obtain the following first main result of this article:

**Theorem 3.1.5** (Super-RSK correspondence). *Let  $\mathcal{S}$  and  $\mathcal{S}'$  be signed alphabets. The map sRSK defines a one-to-one correspondence between signed two-rowed arrays and pairs of super tableaux on  $\mathcal{S}$  and  $\mathcal{S}'$ , such that for any signed two-rowed array  $w$  on  $\mathcal{S}$  and  $\mathcal{S}'$ , we have that  $\mathbf{T}(w)$  and  $\mathbf{Q}(w)$  are same-shape super tableaux whose entries are the ones of the second and the first row of  $w$ , respectively.*

This is the super version of the *Robinson–Schensted–Knuth correspondence* between two-rowed arrays of non-negative integers and pairs of same-shape semistandard tableaux, [9]. More precisely, when  $\mathcal{S} = \mathcal{S}' = \mathcal{S}_0 = \mathcal{S}'_0 = [n]$ , we recover the classical RSK correspondence on the structure of Young tableaux of type A.

**Example 3.1.6.** Consider  $\mathcal{S} = \mathcal{S}' = [6]$  with signature given by  $\mathcal{S}_0 = \mathcal{S}'_0 = \{1, 3, 4\}$  and  $\mathcal{S}_1 = \mathcal{S}'_1$  defined consequently. The sequence of pairs produced during the computation of  $\mathbf{T}(w)$  and  $\mathbf{Q}(w)$  starting from the following signed two-rowed array:

$$w = \begin{pmatrix} 1 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 6 & 6 & 6 \\ 2 & 4 & 2 & 1 & 4 & 5 & 6 & 3 & 5 & 6 & 4 & 3 & 1 & 5 & 4 & 1 \end{pmatrix}$$

on  $\mathcal{S}$  and  $\mathcal{S}'$  is the following:

$$\begin{aligned} & (\emptyset, \emptyset), (\boxed{2}, \boxed{1}), (\boxed{2 \ 4}, \boxed{1 \ 2}), \left( \begin{array}{|c|c|} \hline \boxed{2} & \boxed{4} \\ \hline \boxed{2} & \\ \hline \end{array}, \begin{array}{|c|c|} \hline \boxed{1} & \boxed{2} \\ \hline \boxed{2} & \\ \hline \end{array} \right), \left( \begin{array}{|c|c|} \hline \boxed{1} & \boxed{4} \\ \hline \boxed{2} & \\ \hline \end{array}, \begin{array}{|c|c|} \hline \boxed{1} & \boxed{2} \\ \hline \boxed{2} & \\ \hline \end{array} \right), \left( \begin{array}{|c|c|c|} \hline \boxed{1} & \boxed{4} & \boxed{4} \\ \hline \boxed{2} & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \boxed{1} & \boxed{2} & \boxed{3} \\ \hline \boxed{2} & & \\ \hline \end{array} \right), \\ & \left( \begin{array}{|c|c|c|c|} \hline \boxed{1} & \boxed{4} & \boxed{4} & \boxed{5} \\ \hline \boxed{2} & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \boxed{1} & \boxed{2} & \boxed{3} & \boxed{3} \\ \hline \boxed{2} & & & \\ \hline \end{array} \right), \left( \begin{array}{|c|c|c|c|c|} \hline \boxed{1} & \boxed{4} & \boxed{4} & \boxed{5} & \boxed{6} \\ \hline \boxed{2} & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline \boxed{1} & \boxed{2} & \boxed{3} & \boxed{3} & \boxed{3} \\ \hline \boxed{2} & & & & \\ \hline \end{array} \right), \left( \begin{array}{|c|c|c|c|c|} \hline \boxed{1} & \boxed{3} & \boxed{4} & \boxed{5} & \boxed{6} \\ \hline \boxed{2} & \boxed{4} & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \hline \boxed{1} & \boxed{2} & \boxed{3} & \boxed{3} & \boxed{3} \\ \hline \boxed{2} & \boxed{4} & & & \\ \hline \end{array} \right), \end{aligned}$$

$$\begin{aligned}
 & \left( \begin{array}{|c|c|c|c|c|c|} \hline 1 & 3 & 4 & 5 & 6 & \\ \hline 2 & 4 & 5 & & & \\ \hline 2 & & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 3 & 3 & \\ \hline 2 & 4 & 4 & & & \\ \hline 2 & & & & & \\ \hline \end{array} \right), \left( \begin{array}{|c|c|c|c|c|c|} \hline 1 & 3 & 4 & 5 & 6 & \\ \hline 2 & 4 & 5 & 6 & & \\ \hline 2 & & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 3 & 3 & \\ \hline 2 & 4 & 4 & 4 & & \\ \hline 2 & & & & & \\ \hline \end{array} \right), \\
 & \left( \begin{array}{|c|c|c|c|c|c|} \hline 1 & 3 & 4 & 4 & 6 & \\ \hline 2 & 4 & 5 & 6 & & \\ \hline 2 & 5 & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 3 & 3 & \\ \hline 2 & 4 & 4 & 4 & & \\ \hline 2 & 5 & & & & \\ \hline \end{array} \right), \left( \begin{array}{|c|c|c|c|c|c|} \hline 1 & 3 & 3 & 4 & 6 & \\ \hline 2 & 4 & 4 & 6 & & \\ \hline 2 & 5 & & & & \\ \hline 5 & & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 3 & 3 & \\ \hline 2 & 4 & 4 & 4 & & \\ \hline 2 & 5 & & & & \\ \hline 5 & & & & & \\ \hline \end{array} \right), \\
 & \left( \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 3 & 4 & 6 & \\ \hline 2 & 3 & 4 & 6 & & \\ \hline 2 & 4 & & & & \\ \hline 5 & & & & & \\ \hline 5 & & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 3 & 3 & \\ \hline 2 & 4 & 4 & 4 & & \\ \hline 2 & 5 & & & & \\ \hline 5 & & & & & \\ \hline 5 & & & & & \\ \hline \end{array} \right), \left( \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 3 & 4 & 5 & \\ \hline 2 & 3 & 4 & 6 & & \\ \hline 2 & 4 & 6 & & & \\ \hline 5 & & & & & \\ \hline 5 & & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 3 & 3 & \\ \hline 2 & 4 & 4 & 4 & & \\ \hline 2 & 5 & 6 & & & \\ \hline 5 & & & & & \\ \hline 5 & & & & & \\ \hline \end{array} \right), \\
 & \left( \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 3 & 4 & 4 & \\ \hline 2 & 3 & 4 & 5 & & \\ \hline 2 & 4 & 6 & & & \\ \hline 5 & 6 & & & & \\ \hline 5 & & & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 3 & 3 & \\ \hline 2 & 4 & 4 & 4 & & \\ \hline 2 & 5 & 6 & & & \\ \hline 5 & 6 & & & & \\ \hline 5 & & & & & \\ \hline \end{array} \right), \left( \mathbf{T}(w) = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 4 & 4 & \\ \hline 2 & 3 & 3 & 5 & & \\ \hline 2 & 4 & 4 & & & \\ \hline 5 & 6 & & & & \\ \hline 5 & 6 & & & & \\ \hline \end{array}, \mathbf{Q}(w) = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 3 & 3 & \\ \hline 2 & 4 & 4 & 4 & & \\ \hline 2 & 5 & 6 & & & \\ \hline 5 & 6 & & & & \\ \hline 5 & 6 & & & & \\ \hline \end{array} \right).
 \end{aligned}$$

Moreover, if we apply Algorithm 3.1.3 on the pair  $(\mathbf{T}(w), \mathbf{Q}(w))$ , we recover the initial signed two-rowed array  $w$ .

**Example 3.1.7.** Let  $\mathcal{S} = \mathcal{S}' = \mathbb{N}$  with signature given by  $\mathcal{S}_0$  (resp  $\mathcal{S}'_1$ ) the set of even numbers and  $\mathcal{S}_1$  (resp  $\mathcal{S}'_0$ ) defined consequently. The sequence of pairs produced during the computation of  $\mathbf{T}(w)$  and  $\mathbf{Q}(w)$  starting from the following signed two-rowed array

$$w = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 3 & 2 & 1 & 2 & 4 & 3 & 1 & 2 \end{pmatrix}$$

on  $\mathcal{S}$  and  $\mathcal{S}'$  is the following:

$$\begin{aligned}
 & (\emptyset, \emptyset), (\boxed{3}, \boxed{1}), \left( \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline \end{array} \right), \left( \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline 2 \\ \hline \end{array} \right), \left( \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 1 & \\ \hline 2 & \\ \hline \end{array} \right), \left( \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 2 & & \\ \hline 3 & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & & \\ \hline 2 & & \\ \hline \end{array} \right), \\
 & \left( \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 4 & \\ \hline 3 & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 3 & \\ \hline 2 & & \\ \hline \end{array} \right), \left( \begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 4 & \\ \hline 2 & & \\ \hline 3 & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 3 & \\ \hline 2 & & \\ \hline 4 & & \\ \hline \end{array} \right), \left( \mathbf{T}(w) = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & \\ \hline 2 & 4 & \\ \hline 3 & & \\ \hline \end{array}, \mathbf{Q}(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 3 & \\ \hline 2 & 4 & \\ \hline 4 & & \\ \hline \end{array} \right).
 \end{aligned}$$

Moreover, if we apply Algorithm 3.1.3 on the pair  $(\mathbf{T}(w), \mathbf{Q}(w))$ , then we recover the initial signed two-rowed array  $w$ .

**3.1.8. The symmetry property.** Let  $w$  be a signed two-rowed array on signed alphabets  $\mathcal{S}$  and  $\mathcal{S}'$ . The *inverse* of  $w$ , denoted by  $w^{\text{inv}}$ , is the signed two-rowed array on  $\mathcal{S}'$  and  $\mathcal{S}$  obtained from  $w$  by *exchanging* the rows of  $w$ , that is writing the second row of  $w$  as the first row and the first row of  $w$  as the second row, and by sorting the new couples on  $\mathcal{S}' \times \mathcal{S}$  according to

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Conditions (6) and (7). For instance, the inverse of the signed two-rowed array  $w$  of Example 3.1.6 is the following

$$w^{\text{inv}} = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 4 & 4 & 5 & 5 & 5 & 6 & 6 \\ 2 & 5 & 6 & 2 & 1 & 4 & 5 & 2 & 3 & 5 & 6 & 6 & 4 & 3 & 4 & 3 \end{pmatrix}$$

and the inverse of the signed two-rowed array  $w$  of Example 3.1.7 is the following

$$w^{\text{inv}} = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\ 2 & 4 & 4 & 2 & 1 & 1 & 3 & 3 \end{pmatrix}.$$

We say that the signed two-rowed array  $w$  has *symmetry* with respect to the super-RSK correspondence map  $\text{sRSK}$  if it satisfies the following property:

**Property 3.1.9.** If  $\text{sRSK}(w) = (\mathbf{T}(w), \mathbf{Q}(w))$  then  $\text{sRSK}(w^{\text{inv}}) = (\mathbf{Q}(w), \mathbf{T}(w))$ .

We show the following second main theorem of the article:

**Theorem 3.1.10** (Symmetry of the super-RSK correspondence). *Let  $\mathcal{S}$  and  $\mathcal{S}'$  be signed alphabets. All signed two-rowed arrays on  $\mathcal{S}$  and  $\mathcal{S}'$  have symmetry with respect to the super-RSK correspondence map  $\text{sRSK}$ .*

The rest of this section is devoted to prove this result. We will give a geometrical construction of the super-RSK correspondence based on Fulton's matrix-ball construction, [9], for the non-signed case. This construction will allow us to prove Theorem 3.1.10 using the symmetry of the resulting signed ball array corresponding to a signed two-rowed array.

**3.2. Super matrix-ball construction.** Let  $\mathcal{S}$  and  $\mathcal{S}'$  be two signed alphabets. A *signed ball array* on  $\mathcal{S}$  and  $\mathcal{S}'$  is a rectangular array of balls filled with positive integers, whose rows (resp. columns) are indexed with elements of  $\mathcal{S}$  (resp.  $\mathcal{S}'$ ), from the smaller to the bigger one, allowing the repetition only of elements in  $\mathcal{S}_1$  (resp.  $\mathcal{S}'_1$ ) and where many balls can occur in the same position. A signed ball array on  $\mathcal{S}$  and  $\mathcal{S}'$  is *empty* if it does not contain any ball. For instance, consider  $\mathcal{S} = \mathcal{S}' = [4]$  with signature given by  $\mathcal{S}_0$  (resp.  $\mathcal{S}'_1$ ) the set of even numbers and  $\mathcal{S}_1$  (resp.  $\mathcal{S}'_0$ ) defined consequently. The rectangular array in Figure 1 is a signed ball array on  $\mathcal{S}$  and  $\mathcal{S}'$ .

**3.2.1. Positions in a signed ball array.** We will use the following notations to describe the relative positions of two boxes in a signed ball array. A box  $b'$  is *West* (resp. *west*) of a box  $b$  if the column of  $b'$  is strictly to the left of (resp. left of or equal to) the column of  $b$ . A box  $b'$  is *North* (resp. *north*) of a box  $b$  if the row of  $b'$  is strictly above (resp. above or equal to) the row of  $b$ . Similarly, we define the other positions corresponding to the east and south directions using capital and small letters to denote strict and weak positions. A box  $b'$  is *Northwest* of a box  $b$  if the row of  $b'$  is strictly above to the row of  $b$ , and the column of  $b'$  is left or equal to the column of  $b$ . Similarly, we define the other combinations of positions corresponding to the four cardinal directions using capital and small letters to denote strict and weak inequalities.

	1	2	2	2	3	4
1	(2)				(1)	
1				(1)	(1)	
2	(1)		(2)	(1)	(1)	
3		(3)	(4)			(3)
3					(3)	(2)
4	(2)	(3)	(1)	(1)	(2)	(1)

Figure 1: Example of a signed ball array.

**3.2.2. The matrix-ball construction.** Let  $\mathcal{S}$  and  $\mathcal{S}'$  be two signed alphabets. We will correspond to each signed two-rowed array  $w = \begin{pmatrix} x_1 \cdots x_k \\ y_1 \cdots y_k \end{pmatrix}$  on  $\mathcal{S}$  and  $\mathcal{S}'$ , a signed ball array, denoted by  $\text{Ba}(w)$ , whose rows (resp. columns) are indexed with the elements of the first (resp. second) row of  $w$  and where the indices from  $\mathcal{S}_1$  and  $\mathcal{S}'_1$  are repeated as many times as they appear in  $w$ , as described in the following three steps:

**Step 1.** We start with an empty signed ball array, whose rows (resp. columns) are indexed with the elements of the first (resp. second) row of  $w$  and where the indices from  $\mathcal{S}_1$  and  $\mathcal{S}'_1$  are repeated as many times as they appear in  $w$ . We then associate to each couple  $(x_i, y_i)$  in  $w$  for  $i = 1, \dots, k$ , a ball in a box of the initial empty signed ball array according to the following four cases:

- i)** Suppose  $(x_i, y_i) \in \mathcal{S}_0 \times \mathcal{S}'_0$ . Following Condition (7), equal couples of this form can occur in  $w$ . For each couple  $(x_i, y_i)$  we associate a ball in the box corresponding to the row indexed with  $x_i$  and to the column indexed with  $y_i$  in the signed ball array. In this case, the number of balls in the same position  $(x_i, y_i)$  is equal to the multiplicity of the couple  $(x_i, y_i)$  in  $w$ .
- ii)** Suppose  $(x_i, y_i) \in \mathcal{S}_1 \times \mathcal{S}'_1$ . Following Condition (7), equal couples of this form can occur in  $w$ . For each couple  $(x_i, y_i)$  we associate a ball in the empty box of the signed ball array corresponding to the topmost row indexed with  $x_i$  and the rightmost column indexed with  $y_i$  such that only one ball can occur in the same position, and if there are many rows (resp. columns) indexed with  $x_i$  (resp.  $y_i$ ) we choose the topmost (resp. rightmost) row (resp. column) that does not contain any ball.
- iii)** Suppose  $(x_i, y_i) \in \mathcal{S}_0 \times \mathcal{S}'_1$ . Following Condition (7), we can not have equal couples of this form in  $w$ . We add a ball in the empty box of the signed ball array that corresponds to the row indexed with  $x_i$  and to the rightmost column indexed with  $y_i$ , and if there are many columns that are indexed with  $y_i$  we choose the rightmost one that does not contain any ball.

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iv) Suppose  $(x_i, y_i) \in \mathcal{S}_1 \times \mathcal{S}'_0$ . Following Condition (7), we can not have equal couples of this form in  $w$ . We add a ball in the empty box of the signed ball array that corresponds to the topmost row indexed with  $x_i$  and to the column indexed with  $y_i$ , and if there are many rows that are indexed with  $x_i$ , we choose the topmost one that does not contain any ball.

**Step 2.** If many balls occur in the same position, then we order them arbitrarily by arranging them diagonally from NorthWest to SouthEast. A ball is *northwest* of another one if it is in the same position and NorthWest in this arrangement, or its row and column positions are less than or equal to those of the second ball with at least one inequality strict.

Working from the top-leftmost ball to the bottom-rightmost ball and starting with the first row of the resulting signed ball array, we number all the balls with positive integers by filling each ball by the smallest integer that is larger than all the integers occurring in the balls to the northwest, such that the balls in the same position are numbered with consecutive integers. More precisely, a ball is numbered with 1 if there are no balls northwest of it. A ball is numbered with a positive integer  $i$  if the preceding ball in the same position is numbered with the integer  $i - 1$ , or if the ball is the first one in a given position and the largest number occurring in a ball northwest of the given position is the integer  $i - 1$ . The resulting signed ball array is denoted by  $Ba^1(w)$ .

For instance, Figure 2 (resp. Figure 3) represents the signed ball array  $Ba^1(w)$  corresponding to the signed two-rowed array  $w$  of Example 3.1.6 (resp. Example 3.1.7).

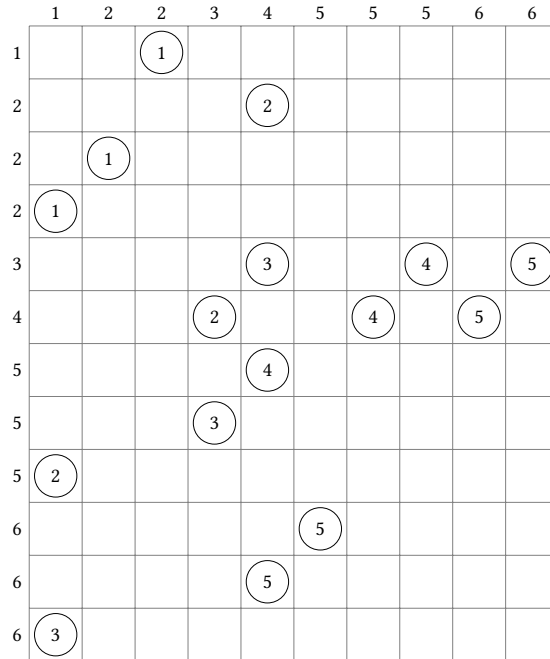


Figure 2:  $Ba^1(w)$  corresponding to  $w$  of Example 3.1.6.

**Step 3.** If there are  $k > 1$  balls filled with the same integer  $i$  in the resulting signed ball array  $Ba^1(w)$ , then they belong by construction to a string from SouthWest to NorthEast. We then introduce new  $k - 1$  balls by putting a ball to the right of each ball in the string but the last,

### 3.2. Super matrix-ball construction

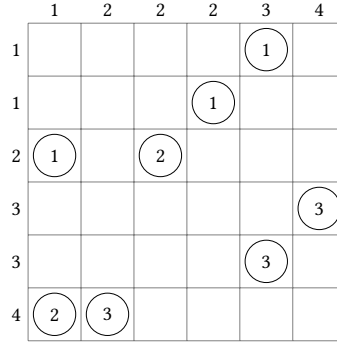


Figure 3:  $Ba^1(w)$  corresponding to  $w$  of Example 3.1.7.

directly under the next ball. We will use a new color for the added balls as illustrated in Figure 4. We do the same for all the balls filled with the same integer. We then number the new added balls

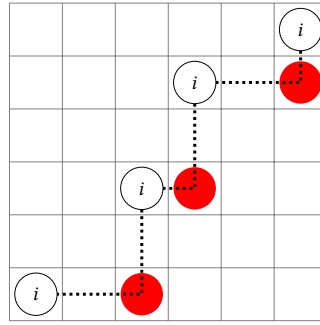


Figure 4: String for new colored balls

as per **Step 2** without taking into consideration the numbering of the initial non-colored balls and by just acting on the new balls from the top-leftmost ball to the bottom-rightmost one. We obtain a new signed ball array, denoted by  $Ba^2(w)$ , that contains the initial non-colored balls and the new colored ones. We repeat the same process on the new added balls and we construct  $Ba^3(w)$  from  $Ba^2(w)$  by adding new colored balls and by numbering them, and so on, stopping when no two balls appear in  $Ba^k(w)$  for any  $k > 1$  with the same number. The resulting signed ball array is denoted by  $Ba(w)$ . We will denote by  $Ba_0(w)$  the signed ball array obtained from  $Ba(w)$  by eliminating the initial non colored balls and by keeping only the new colored ones. Note that we will use the same color for all the colored balls added to compute  $Ba^k(w)$  from  $Ba^{k-1}(w)$  for any  $k > 1$ . We will denote by  $Ba_0^1(w)$  the signed ball array obtained from  $Ba_0(w)$  by keeping only the colored balls added to compute  $Ba^2(w)$  from  $Ba^1(w)$  and by eliminating all the other colored balls. The *bottom-rightmost* ball of a signed ball array is the ball in the position that corresponds to the bottom-most row and to the rightmost column of the given signed ball array, and if many balls occur in this position it corresponds to the last one to the southeast in the corresponding diagonal arrangement.

For instance, Figure 5 (resp. Figure 6 ) represents the signed ball array  $Ba(w)$  corresponding

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to the signed two-rowed array  $w$  of Example 3.1.6 (resp. Example 3.1.7).

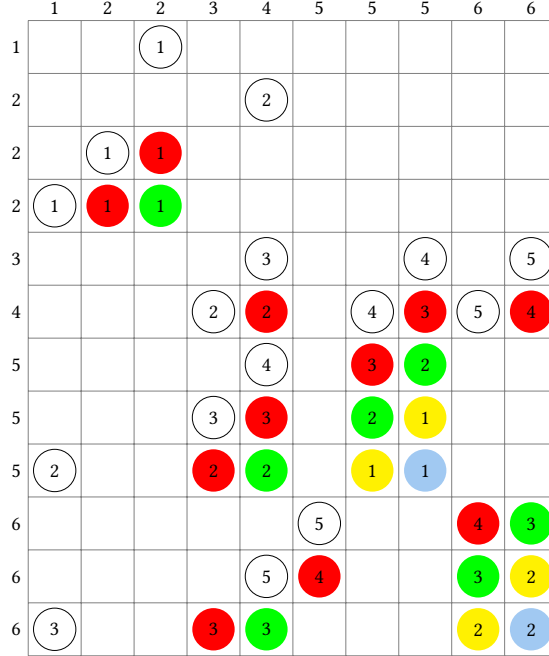


Figure 5:  $Ba(w)$  corresponding to  $w$  of Example 3.1.6.

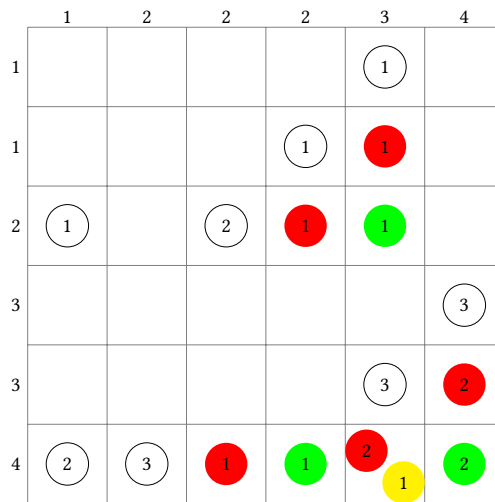


Figure 6:  $Ba(w)$  corresponding to  $w$  of Example 3.1.7.

**3.2.3. Super tableaux for a signed ball array.** Let  $w$  be a signed two-rowed array on two signed alphabets  $\mathcal{S}$  and  $\mathcal{S}'$  and let  $Ba(w)$  be its corresponding signed ball array. We denote by  $T(Ba(w))$  (resp.  $Q(Ba(w))$ ) the super tableau obtained from  $Ba(w)$  whose  $k$ -th row lists the

### 3.2. Super matrix-ball construction

indices of the leftmost columns (resp. top-most rows) of  $Ba^k(w)$  where each integer number occurs in the new added balls, that is, the  $i$ -th entry of its  $k$ -th row is equal to the index of the leftmost column (resp. top-most row) in  $Ba^k(w)$  where a new colored ball filled with  $i$  occurs. Note that the  $i$ -th entry of the first row of  $T(Ba(w))$  (resp.  $Q(Ba(w))$ ) is the index of the leftmost column (resp. top-most row) in  $Ba^1(w)$  where a ball filled with  $i$  occurs.

**Proposition 3.2.4.** *Let  $w$  be a signed two-rowed array on two signed alphabets  $S$  and  $S'$ . The following equality  $(T(Ba(w)), Q(Ba(w))) = (T(w), Q(w))$  holds.*

Theorem 3.1.10 is then a direct consequence of this proposition, since the matrix-ball construction is symmetric in the rows and columns of the resulting signed ball array. In the rest of this subsection we will prove Proposition 3.2.4. But before that, we give the following example that illustrates the symmetry property of the super-RSK correspondence for super tableaux.

**Example 3.2.5.** Consider the signed two-rowed array  $w$  of Example 3.1.6. The super tableaux associated to its signed ball array  $Ba(w)$  illustrated in Figure 5 are the following

$$\left( T(Ba(w)) = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 4 & 4 \\ \hline 2 & 3 & 3 & 5 & \\ \hline 2 & 4 & 4 & & \\ \hline 5 & 6 & & & \\ \hline 5 & 6 & & & \\ \hline \end{array}, Q(Ba(w)) = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 3 & 3 \\ \hline 2 & 4 & 4 & 4 & \\ \hline 2 & 5 & 6 & & \\ \hline 5 & 6 & & & \\ \hline 5 & 6 & & & \\ \hline \end{array} \right)$$

which are equal to the ones of  $w$  already computed in Example 3.1.6:

$$w = \begin{pmatrix} 1 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 6 & 6 & 6 \\ 2 & 4 & 2 & 1 & 4 & 5 & 6 & 3 & 5 & 6 & 4 & 3 & 1 & 5 & 4 & 1 \end{pmatrix} \xleftrightarrow{\text{sRSK}} \left( T(w) = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 4 & 4 \\ \hline 2 & 3 & 3 & 5 & \\ \hline 2 & 4 & 4 & & \\ \hline 5 & 6 & & & \\ \hline 5 & 6 & & & \\ \hline \end{array}, Q(w) = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 3 & 3 \\ \hline 2 & 4 & 4 & 4 & \\ \hline 2 & 5 & 6 & & \\ \hline 5 & 6 & & & \\ \hline 5 & 6 & & & \\ \hline \end{array} \right).$$

Moreover, using the matrix-ball construction, we obtain:

$$w^{\text{inv}} = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 6 & 6 \\ 2 & 5 & 6 & 2 & 1 & 4 & 5 & 2 & 3 & 5 & 6 & 6 & 4 & 3 & 4 & 3 \end{pmatrix} \xleftrightarrow{\text{sRSK}} \left( Q(w) = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 3 & 3 \\ \hline 2 & 4 & 4 & 4 & \\ \hline 2 & 5 & 6 & & \\ \hline 5 & 6 & & & \\ \hline 5 & 6 & & & \\ \hline \end{array}, T(w) = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 4 & 4 \\ \hline 2 & 3 & 3 & 5 & \\ \hline 2 & 4 & 4 & & \\ \hline 5 & 6 & & & \\ \hline 5 & 6 & & & \\ \hline \end{array} \right)$$

by switching the roles of the rows and the columns of the signed ball array of Figure 5.

Consider now the signed two-rowed array  $w$  of Example 3.1.7. The super tableaux associated to its signed ball array  $Ba(w)$  illustrated in Figure 6 are the following

$$\left( T(Ba(w)) = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & \\ \hline 2 & 4 & \\ \hline 3 & & \\ \hline \end{array}, Q(Ba(w)) = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 3 & \\ \hline 2 & 4 & \\ \hline 4 & & \\ \hline \end{array} \right)$$

which are equal to the ones of  $w$  already computed in Example 3.1.7:

$$w = \begin{pmatrix} 1 & 1 & 2 & 3 & 3 & 4 & 4 \\ 3 & 2 & 1 & 2 & 4 & 3 & 1 & 2 \end{pmatrix} \xleftrightarrow{\text{sRSK}} \left( T(w) = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & \\ \hline 2 & 4 & \\ \hline 3 & & \\ \hline \end{array}, Q(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 3 & \\ \hline 2 & 4 & \\ \hline 4 & & \\ \hline \end{array} \right).$$



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Moreover, using the super matrix-ball construction, we obtain:

$$w^{\text{inv}} = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\ 2 & 4 & 4 & 2 & 1 & 1 & 3 & 3 \end{pmatrix} \xleftrightarrow{\text{sRSK}} \left( \mathbf{Q}(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 3 & \\ \hline 2 & 4 & \\ \hline 4 & & \\ \hline \end{array}, \mathbf{T}(w) = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & \\ \hline 2 & 4 & \\ \hline 3 & & \\ \hline \end{array} \right)$$

by switching the roles of the rows and the columns of the signed ball array of Figure 6.

**3.2.6. Proof of Proposition 3.2.4.** Let  $w = \begin{pmatrix} x_1 \dots x_k \\ y_1 \dots y_k \end{pmatrix}$  be a signed two-rowed array on two signed alphabets  $\mathcal{S}$  and  $\mathcal{S}'$ . We will show the result by induction on the number of couples in  $w$ , which is equal to the number of balls in  $\text{Ba}^1(w)$ . The result is obvious when  $w$  contains zero or one couple. Let

$$w_0 := \begin{pmatrix} x_1 \dots x_{k-1} \\ y_1 \dots y_{k-1} \end{pmatrix}$$

be the signed two-rowed array obtained from  $w$  by eliminating its rightmost couple  $(x_k, y_k)$ . By the induction hypothesis, we have  $\mathbf{T}(w_0) = \mathbf{T}(\text{Ba}(w_0))$  and  $\mathbf{Q}(w_0) = \mathbf{Q}(\text{Ba}(w_0))$ . Then it is sufficient to prove the following property.

**Property 3.2.7.** The following equality

$$\mathbf{T}(\text{Ba}(w)) = \mathbf{T}(\text{Ba}(w_0)) \leftarrow y_k$$

holds in  $\text{Yt}(\mathcal{S}')$ , and the super tableau  $\mathbf{Q}(\text{Ba}(w))$  is obtained from  $\mathbf{Q}(\text{Ba}(w_0))$  by placing  $x_k$  in the box that belongs to  $\mathbf{T}(\text{Ba}(w))$  but not to  $\mathbf{T}(\text{Ba}(w_0))$ .

By construction, the signed ball array  $\text{Ba}^1(w)$  contains one ball that is not in  $\text{Ba}^1(w_0)$ , and which we denote by  $B$ , and we suppose is filled with an integer  $j$ . This ball is the bottom-rightmost ball of  $\text{Ba}^1(w)$  which belongs to the position corresponding to the bottom-most row indexed by  $x_k$  and to the rightmost (resp. leftmost) column indexed by  $y_k$  if  $y_k \in \mathcal{S}_0$  (resp.  $y_k \in \mathcal{S}_1$ ). Moreover, the integer  $j$  is larger than all the numbers of the balls in its position.

Suppose first that there are no more balls filled with  $j$  in  $\text{Ba}^1(w)$ . In this case, all the remaining balls in  $\text{Ba}^1(w)$  are to the NorthWest of  $B$ , and then they are all filled with integers that are strictly smaller than  $j$ . Hence when computing  $\text{Ba}^i(w)$ , for  $i > 1$ , no colored balls will be added using  $B$ , and all the rows of  $\mathbf{T}(w)$  and  $\mathbf{T}(w_0)$  (resp.  $\mathbf{Q}(w)$  and  $\mathbf{Q}(w_0)$ ) below the top-most row are the same. Moreover, the first row of  $\mathbf{T}(\text{Ba}(w))$  is obtained from that of  $\mathbf{T}(\text{Ba}(w_0))$  by adding  $y_k$  to the end, since  $j$  is the largest integer in  $\text{Ba}^1(w)$ , and then  $\mathbf{Q}(w)$  is obtained from  $\mathbf{Q}(w_0)$  by adding  $x_k$  to the end of its first row, showing Property 3.2.7.

Suppose now that there are other balls in  $\text{Ba}^1(w)$  filled with  $j$ . By construction, all these balls are to the NorthEast of  $B$ . Consider the ball filled with  $j$  in the position corresponding to the row indexed with  $x$  and the column indexed with  $y$  such that  $x \leq x_k$  is maximal with  $x = x_k$  only if  $x \in \mathcal{S}'_1$  and  $y \geq y_k$  is minimal with  $y = y_k$  only if  $y \in \mathcal{S}_1$ . When  $y_k$  is inserted in the first row of  $\mathbf{T}(\text{Ba}(w_0))$ , the letter  $y$  is then bumped from the  $j$ -th box of this row. Indeed, the entries of the first row of  $\mathbf{T}(\text{Ba}(w_0))$  are the indices of the left-most columns of  $\text{Ba}^1(w_0)$  that contains balls filled with  $1, 2, \dots$ . The first  $j - 1$  entries of this row are less than or equal to  $y_k$ , but the

### 3.3. Super-RSK correspondence and taquin

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$j$ -th entry is  $y$  which is the minimal index that satisfies  $y \geq y_k$  with  $y = y_k$  only if  $y \in \mathcal{S}_1$ , then the entry  $y$  is bumped from the  $j$ -th box when inserting  $y_k$  in the first row of  $T(\text{Ba}(w_0))$ . Hence the top-most row of  $T(\text{Ba}(w))$  is the top-most row of  $T(\text{Ba}(w_0)) \leftarrow y_k$ . Moreover, the super tableau formed by the rows of  $T(\text{Ba}(w))$  below its top-most row is by construction the super tableau  $T(\text{Ba}_0(w))$ , and the super tableau formed by the rows of  $T(\text{Ba}(w_0))$  below its top-most row is the super tableau  $T(\text{Ba}_0(w_0))$ . Then it is sufficient to prove the following property.

**Property 3.2.8.** The following equality

$$T(\text{Ba}_0(w)) = T(\text{Ba}_0(w_0)) \leftarrow y$$

holds in  $\text{Yt}(\mathcal{S}')$ , and the new box from this insertion is the box that belongs to  $\mathcal{Q}(\text{Ba}_0(w))$  but not to  $\mathcal{Q}(\text{Ba}_0(w_0))$ .

This property follows from Property 3.2.7, assumed by induction for the signed ball array  $\text{Ba}^1(w_0)$ , provided with the fact that  $(x_k, y)$  is the position of the bottom-rightmost ball of the signed ball array  $\text{Ba}_0^1(w)$ . Indeed, by construction, there is no entries below the  $x_k$ -th row of  $\text{Ba}_0^1(w)$ . Moreover, if there is other ball in the  $x_k$ -th row of  $\text{Ba}_0^1(w)$ , then this ball is added using two balls of  $\text{Ba}^1(w)$  filled with an integer  $i < j$ . The first ball belongs to the row indexed by  $x_k$ , and the second one lies NorthEast of it, but it is NorthWest of the bottom-rightmost ball filled with  $j$ , and then this second ball cannot lie in a column indexed larger than  $y$ , showing the claim.

**3.3. Super-RSK correspondence and taquin.** Before we proceed to the construction of the dual super-RSK correspondence, we will prove the following result which relates the super-RSK correspondence to the super jeu de taquin. This result will be useful for the sequel.

**Proposition 3.3.1.** Let  $w = \begin{pmatrix} x_1 \dots x_k \\ y_1 \dots y_k \end{pmatrix}$  be a signed two-rowed array on two signed alphabets  $\mathcal{S}$  and  $\mathcal{S}'$  such that  $\text{sRSK}(w) = (T(w), \mathcal{Q}(w))$ , and let  $t$  be any super tableau in  $\text{Yt}(\mathcal{S}')$ . Compute the following super tableau over  $\text{Yt}(\mathcal{S}')$ :

$$((\dots (t \leftarrow y_1) \leftarrow \dots) \leftarrow y_k)$$

and add  $x_1, \dots, x_k$  successively in the new boxes starting with an empty Young diagram of the same shape as  $t$ . Then the entries  $x_1, \dots, x_k$  form a super skew tableau  $S$  such that  $\text{Rect}(S) = \mathcal{Q}(w)$ .

*Proof.* Consider a super tableau  $t'$  over  $\text{Yt}(\mathcal{S}^<)$  of the same shape as  $t$ , where  $\mathcal{S}^<$  is a signed alphabet whose elements are all smaller than the ones of  $\mathcal{S}$ . Following Theorem 3.1.5, the pair of super tableaux  $(t, t')$  corresponds to some signed two-rowed array

$$\begin{pmatrix} x'_1 \dots x'_l \\ y'_1 \dots y'_l \end{pmatrix}.$$

Hence, the following signed two-rowed array

$$W := \begin{pmatrix} x'_1 \dots x'_l x_1 \dots x_k \\ y'_1 \dots y'_l y_1 \dots y_k \end{pmatrix}$$

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corresponds to the pair

$$\left( (t \leftarrow \mathbf{R}(\mathbf{T}(w)), Q') \right),$$

where  $Q'$  a super tableau over  $\text{Yt}(\mathcal{S}^< \cup \mathcal{S})$  whose entries  $x'_1, \dots, x'_l$  are the ones of  $t'$ , and whose entries  $x_1, \dots, x_k$  are the ones of  $S$ . Consider now  $W^{\text{inv}}$  the inverse of  $W$ . Following Theorem 3.1.10, the signed two-rowed array  $W^{\text{inv}}$  corresponds to the pair  $\left( Q', (t \leftarrow \mathbf{R}(\mathbf{T}(w))) \right)$ , and the signed two-rowed array obtained from  $W^{\text{inv}}$  by eliminating the couples  $(y'_j, x'_j)$  corresponds to the pair  $(\mathbf{Q}(w), \mathbf{T}(w))$ . The word of the second row of  $W^{\text{inv}}$  is then super plactic equivalent to the word  $\mathbf{R}(Q')$ , and when we remove the  $y'_j$ 's from this word we obtain a word that is super plactic equivalent to the word  $\mathbf{R}(\mathbf{Q}(w))$ . However, by construction of the super tableau  $Q'$ , when we remove the  $l$  smallest letters from  $\mathbf{R}(Q')$ , we recover the word  $\mathbf{R}(S)$ . Hence, we deduce by Lemma 2.3.2 that the words  $\mathbf{R}(S)$  and  $\mathbf{R}(\mathbf{Q}(w))$  are also super plactic equivalent, showing by Property 2.4.1 that the equality  $\text{Rect}(S) = \mathbf{Q}(w)$  holds in  $\text{Yt}(\mathcal{S})$ .  $\square$

**Example 3.3.2.** Consider the signed two-rowed array  $w$  of Example 3.1.7. We have shown the following:

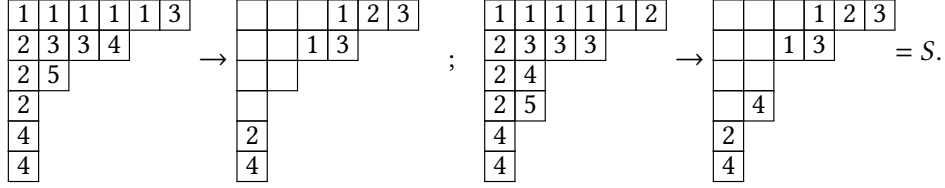
$$w = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 3 & 2 & 1 & 2 & 4 & 3 & 1 & 2 \end{pmatrix} \xleftrightarrow{\text{sRSK}} \left( \mathbf{T}(w) = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & \\ \hline 2 & 4 & \\ \hline 3 & & \\ \hline \end{array}, \mathbf{Q}(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 3 & \\ \hline 2 & 4 & \\ \hline 4 & & \\ \hline \end{array} \right).$$

Consider now the following super tableau

$$t = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 3 & \\ \hline 4 & 5 & \\ \hline 4 & & \\ \hline \end{array}.$$

We insert the elements of the second row of  $w$  into the super tableau  $t$ , and we place the elements of the first row of  $w$  in the new added boxes starting from an empty tableau of the same shape as  $t$ . We obtain:

$$\begin{array}{c} \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 3 & & \\ \hline 4 & 5 & & \\ \hline 4 & & & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} ; \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 3 & 3 & \\ \hline 4 & 5 & & \\ \hline 4 & & & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} ; \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 3 & 3 & \\ \hline 2 & 5 & & \\ \hline 4 & & & \\ \hline 4 & & & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} ; \\ \\ \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 \\ \hline 2 & 3 & 3 & & \\ \hline 2 & 5 & & & \\ \hline 4 & & & & \\ \hline 4 & & & & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|c|} \hline & & & 1 & 2 \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array} ; \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 & 4 \\ \hline 2 & 3 & 3 & & & \\ \hline 2 & 5 & & & & \\ \hline 4 & & & & & \\ \hline 4 & & & & & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|c|c|} \hline & & & 1 & 2 & 3 \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline \end{array} ; \\ \\ \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 3 & 4 & & \\ \hline 2 & 5 & & & & \\ \hline 4 & & & & & \\ \hline 4 & & & & & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|c|c|} \hline & & & 1 & 2 & 3 \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline \end{array} ; \end{array}$$



Moreover, we have shown in Example 2.4.2 that  $\text{Rect}(S) = \mathcal{Q}(w)$ .

**3.4. A dual super-RSK correspondence.** We recall in this subsection the left insertion algorithm on super tableaux which gives us a dual way to construct the pair of super tableaux corresponding to a signed two-rowed array with respect to the super-RSK correspondence map. We first begin by recalling the super evacuation procedure on super tableaux introduced in [13].

**3.4.1. Super evacuation on super tableaux.** Denote by  $\mathcal{S}^{\text{op}}$  the *opposite alphabet* obtained from a signed alphabet  $\mathcal{S}$  by reversing its order. Denote by  $x^*$  the letter in  $\mathcal{S}^{\text{op}}$  corresponding to  $x$  in  $\mathcal{S}$  where  $\|x^*\| = 0$  (resp.  $\|x^*\| = 1$ ) if  $\|x\| = 0$  (resp.  $\|x\| = 1$ ). For all  $x, y$  in  $\mathcal{S}$ , we have  $x < y$  if and only if  $x^* > y^*$ . For any word  $w = x_1 \dots x_k$  over  $\mathcal{S}$ , denote by  $w^* = x_k^* \dots x_1^*$  the corresponding word over  $\mathcal{S}^{\text{op}}$ , called the *opposite* of  $w$ . Then, for all  $v$  and  $w$  in  $\mathcal{S}^*$ , the equality  $(vw)^* = w^*v^*$  holds, inducing an anti-isomorphism between the free monoids over  $\mathcal{S}^{\text{op}}$  and  $\mathcal{S}$ . By identifying  $(\mathcal{S}^{\text{op}})^{\text{op}}$  with  $\mathcal{S}$ , the equality  $(w^*)^* = w$  holds, for any  $w$  in  $\mathcal{S}^*$ . Moreover, we show the following property:

**Property 3.4.2** ([13]). For all  $v$  and  $w$  in  $\mathcal{S}$ , the following equivalence holds:

$$v \sim_{\mathcal{P}(\mathcal{S})} w \quad \text{if, and only, if} \quad v^* \sim_{\mathcal{P}(\mathcal{S}^{\text{op}})} w^*. \quad (8)$$

**Algorithm 3.4.3** ([13]). Let  $t$  be in  $\text{Yt}(\mathcal{S})$ . An *opposite tableau* in  $\text{Yt}(\mathcal{S}^{\text{op}})$ , denoted by  $t^{\text{op}}$ , is constructed from  $t$  using the super jeu de taquin as follows:

*Input:* A super tableau  $t$  in  $\text{Yt}(\mathcal{S})$ .

*Output:* A super tableau  $t^{\text{op}}$  in  $\text{Yt}(\mathcal{S}^{\text{op}})$  with the same shape as  $t$ .

*Method:* Start with an empty Young diagram with the same frame as  $t$ . Remove the box containing the top-leftmost element  $x$  in  $t$ , and perform a forward sliding on the resulting super skew tableau. We obtain a super tableau, denoted by  $t^*$ , whose frame has one box removed from the frame of  $t$ . Put the letter  $x^*$  in the initial empty Young diagram in the same place as the box that was removed from the frame of  $t$ . Repeat the algorithm on  $t^*$  and continue until all the elements of  $t$  have been removed and the initial empty Young diagram has been filled by their corresponding letters in  $\mathcal{S}^{\text{op}}$ . Output the resulting super tableau for  $t^{\text{op}}$ .

The procedure of construction  $t^{\text{op}}$  from  $t$  is called the *super evacuation*. This is the super analogue of the Schützenberger’s evacuation procedure, [28]. Moreover, we show the following property:

**Property 3.4.4** ([13]). For any  $t$  in  $\text{Yt}(\mathcal{S})$ , the super tableau  $t^{\text{op}}$  satisfies the following equivalence:

$$(\mathbf{R}(t))^* \sim_{\mathcal{P}(\mathcal{S}^{\text{op}})} \mathbf{R}(t^{\text{op}}). \quad (9)$$

Moreover, the map  $t \mapsto t^{\text{op}}$  is an involution on  $\text{Yt}(\mathcal{S})$ .

### 3. A super-RSK correspondence with symmetry

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Let  $w = \begin{pmatrix} x_1 \dots x_k \\ y_1 \dots y_k \end{pmatrix}$  be a signed two-rowed array on two signed alphabets  $\mathcal{S}$  and  $\mathcal{S}'$ . Define

$$w^* := \begin{pmatrix} x_k^* \dots x_1^* \\ y_k^* \dots y_1^* \end{pmatrix}$$

to be the signed two-rowed array on  $\mathcal{S}^{\text{op}}$  and  $(\mathcal{S}')^{\text{op}}$  whose first and second rows are the opposites of the ones of  $w$ . In particular, by definition, the equality  $(w^*)^{\text{inv}} = (w^{\text{inv}})^*$  holds.

**Proposition 3.4.5.** *Let  $\mathcal{S}$  and  $\mathcal{S}'$  be signed alphabets and  $w$  be a signed two-rowed array on  $\mathcal{S}$  and  $\mathcal{S}'$ . If  $\text{sRSK}(w) = (\mathbf{T}(w), \mathbf{Q}(w))$  then  $\text{sRSK}(w^*) = (\mathbf{T}(w)^{\text{op}}, \mathbf{Q}(w)^{\text{op}})$ .*

*Proof.* Consider  $w = \begin{pmatrix} x_1 \dots x_k \\ y_1 \dots y_k \end{pmatrix}$  a signed two-rowed array on  $\mathcal{S}$  and  $\mathcal{S}'$ . Following Subsection 2.3, the following equivalence  $\mathbf{R}(\mathbf{T}(w)) \sim_{\mathbf{P}(\mathcal{S})} y_1 \dots y_k$  holds. Then we have

$$\mathbf{R}(\mathbf{T}(w^*)) \sim_{\mathbf{P}(\mathcal{S}^{\text{op}})} y_k^* \dots y_1^* = (y_1 \dots y_k)^* \sim_{\mathbf{P}(\mathcal{S}^{\text{op}})}^{(8)} (\mathbf{R}(\mathbf{T}(w)))^* \sim_{\mathbf{P}(\mathcal{S}^{\text{op}})}^{(9)} \mathbf{R}(\mathbf{T}(w)^{\text{op}}).$$

We deduce by Property 2.3.1 that the equality  $\mathbf{T}(w^*) = \mathbf{T}(w)^{\text{op}}$  holds in  $\text{Yt}((\mathcal{S}')^{\text{op}})$ . Hence, we obtain  $\text{sRSK}(w^*) = (\mathbf{T}(w)^{\text{op}}, \mathbf{Q}(w^*))$ . Similarly, we show that

$$\text{sRSK}((w^*)^{\text{inv}}) = \text{sRSK}((w^{\text{inv}})^*) = (\mathbf{Q}(w)^{\text{op}}, \mathbf{T}(w^*)).$$

We deduce by Theorem 3.1.10 that  $(\mathbf{Q}(w)^{\text{op}}, \mathbf{T}(w^*)) = (\mathbf{Q}(w^*), \mathbf{T}(w)^{\text{op}})$ , showing the claim.  $\square$

**3.4.6. A dual construction of the super RSK-correspondence.** We will present a dual way to construct the super RSK-correspondence using the following left insertion algorithm on super tableaux.

**Algorithm 3.4.7** ([20]). The *left (or column) insertion*, denoted by  $\rightsquigarrow$ , inserts an element  $x$  in  $\mathcal{S}$  into a super tableau  $t$  of  $\text{Yt}(\mathcal{S})$  as follows:

*Input:* A super tableau  $t$  and a letter  $x \in \mathcal{S}$ .

*Output:* A super tableau  $x \rightsquigarrow t$ .

*Method:* If  $t$  is empty, create a box and label it  $x$ . Suppose  $t$  is non-empty. If  $x \in \mathcal{S}_0$  (resp.  $x \in \mathcal{S}_1$ ) is larger than (resp. at least as large as) the bottom element of the leftmost column of  $t$ , then put  $x$  in a box to the bottom of this column; Otherwise, let  $y$  be the smallest element of the leftmost column of  $t$  such that  $y \geq x$  (resp.  $y > x$ ). Then replace  $y$  by  $x$  in this column and recursively insert  $y$  into the super tableau formed by the columns of  $t$  to the right of the leftmost. Note that this recursion may end with an insertion into an empty column to the right of the existing columns of  $t$ . Output the resulting super tableau.

**Property 3.4.8.** As a consequence of Property 2.3.1, the following *commutation property* holds in  $\text{Yt}(\mathcal{S})$ , for all  $t$  in  $\text{Yt}(\mathcal{S})$  and  $x, y$  in  $\mathcal{S}$ :

$$y \rightsquigarrow (t \leftarrow x) = (y \rightsquigarrow t) \leftarrow x.$$

In particular, for any word  $w = x_1 \dots x_k$  in  $\mathcal{S}^*$  the super tableau  $\mathbf{T}(w)$  can be also computed by inserting its elements iteratively from right to left using the left insertion starting from the empty super tableau:

$$\mathbf{T}(w) = (w \rightsquigarrow \emptyset) := (x_1 \rightsquigarrow (\dots \rightsquigarrow (x_k \rightsquigarrow \emptyset) \dots)).$$





whose rectification is  $t$ . We will prove that the integer  $c_{\lambda, \mu}^v$  is equal to the number of super skew tableaux in  $\text{St}(\mathcal{S}, \nu/\lambda)$  whose rectification is a given tableau of shape  $\mu$ .

For any  $T$  in  $\text{Yt}(\mathcal{S}, \mu)$ , we set

$$\text{St}(\mathcal{S}, \nu/\lambda, \dashv T) := \{S \in \text{St}(\mathcal{S}, \nu/\lambda) \mid \text{Rect}(S) = T\}.$$

We prove in Theorem 4.2.1 that there is a canonical one-to-one correspondence between the sets  $\text{Yt}(\mathcal{S}, \lambda, \mu, \dashv t)$  and  $\text{St}(\mathcal{S}, \nu/\lambda, \dashv T)$ . We deduce that the super Littlewood–Richardson coefficient  $c_{\lambda, \mu}^v$  does not depend on  $t$  and  $T$ , and depends only on  $\lambda$ ,  $\mu$  and  $\nu$ . Note that the later property is proved in [13] using an interpretation of the super jeu de taquin in terms of Fomin’s growth diagrams.

**4.2. Super Littlewood–Richardson rule.** Starting from the super plactic monoid  $\mathbf{P}(\mathcal{S})$ , we construct a  $\mathbb{Z}$ -algebra, denoted by  $R_{\mathcal{S}}$ , whose linear generators are the monomials in the monoid  $\mathbf{P}(\mathcal{S})$ . This is an associative and unitary ring that is not commutative. A generic element in  $R_{\mathcal{S}}$  can be realized by a formal sum of super plactic classes with coefficients from  $\mathbb{Z}$ . Since every super plactic class can be represented by a super tableau, a typical element in  $R_{\mathcal{S}}$  is a formal sum of super tableaux. There is a canonical homomorphism from  $R_{\mathcal{S}}$  onto the ring of polynomials  $\mathbb{Z}[X]$  that takes a super tableau  $t$  to its monomial  $x^t$ , where  $x^t$  is the product of the variables  $x_i$ , each occurring as many times in  $x^t$  as  $i$  occurs in  $t$ . For instance, the following monomial

$$x^{\mathbf{T}(w)} = x_1^3 x_2^2 x_3^2 x_4^4 x_5^3 x_6^2 \quad (\text{resp. } x^{\mathbf{Q}(w)} = x_1^1 x_2^3 x_3^3 x_4^3 x_5^3 x_6^3)$$

corresponds to the super tableau  $\mathbf{T}(w)$  (resp.  $\mathbf{Q}(w)$ ) computed in Example 3.1.6. Moreover, the following monomial

$$x^{\mathbf{T}(w)} = x_1^2 x_2^2 x_3^2 x_4 \quad (\text{resp. } x^{\mathbf{Q}(w)} = x_1^2 x_2^2 x_3^2 x_4^2)$$

corresponds to the super tableau  $\mathbf{T}(w)$  (resp.  $\mathbf{Q}(w)$ ) computed in Example 3.1.7.

Define  $S_{\lambda}$  (resp.  $S_{\lambda/\mu}$ ) in  $R_{\mathcal{S}}$  to be the sum of all super tableaux (resp. super skew tableaux) of shape  $\lambda$  (resp.  $\lambda/\mu$ ) and entries in  $\mathcal{S}$ , with  $\lambda \in \mathcal{P}$  (resp.  $\lambda/\mu$  is a skew shape). The image of  $S_{\lambda}$  (resp.  $S_{\lambda/\mu}$ ) in  $\mathbb{Z}[X]$  gives rise to the so-called *super Schur function* (resp. *super skew Schur function*)  $s_{\lambda}(X)$  (resp.  $s_{\lambda/\mu}(X)$ ).

**Theorem 4.2.1** (The super Littlewood–Richardson rule). *Let  $\lambda$ ,  $\mu$  and  $\nu$  be partitions in  $\mathcal{P}$  such that  $\nu/\lambda$  is a skew shape. The following identities*

$$S_{\lambda} S_{\mu} = \sum_{\nu} c_{\lambda, \mu}^{\nu} S_{\nu} \quad \text{and} \quad S_{\nu/\lambda} = \sum_{\mu} c_{\lambda, \mu}^{\nu} S_{\mu} \tag{10}$$

hold in  $R_{\mathcal{S}}$ .

*Proof.* Prove that for any super tableau  $t$  in  $\text{Yt}(\mathcal{S}, \nu)$  and any super tableau  $T$  in  $\text{Yt}(\mathcal{S}, \mu)$ , there is a canonical one-to-one correspondence between the sets  $\text{Yt}(\mathcal{S}, \lambda, \mu, \dashv t)$  and  $\text{St}(\mathcal{S}, \nu/\lambda, \dashv T)$ . Start with  $(t', t'')$  in  $\text{Yt}(\mathcal{S}, \lambda, \mu, \dashv t)$  and let

$$\begin{pmatrix} x_1 \dots x_k \\ y_1 \dots y_k \end{pmatrix}$$



#### 4. The super Littlewood–Richardson rule

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be the signed two-rowed array on the signed alphabet  $\mathcal{S}$  corresponding to the couple  $(t'', T)$ . By inserting the letters  $y_i$ , for  $i = 1, \dots, k$ , into the super tableau  $t'$ , we form a super skew tableau  $S$  by successively placing the letter  $x_i$ , for  $i = 1, \dots, k$ , into the new boxes starting with an empty Young diagram of the same shape as  $t'$ . Since the super tableau  $t = t' \star_{\text{Yt}(\mathcal{S})} t''$  has shape  $\nu$ , then the super skew tableau  $S$  has shape  $\nu/\lambda$  and we deduce by Proposition 3.3.1 that  $S$  belongs to  $\text{St}(\mathcal{S}, \nu/\lambda, \dashv T)$ .

Conversely, start with a super skew tableau  $S$  in  $\text{St}(\mathcal{S}, \nu/\lambda, \dashv T)$ , and let  $T'$  be in  $\text{Yt}(\mathcal{S}, \lambda)$  such that all its entries are smaller than the ones of  $S$ . Let  $T''$  be the super tableau in  $\text{Yt}(\mathcal{S}, \nu)$  that contains the super tableau  $T'$  such that when we remove  $T'$  from  $T''$  we obtain the super skew tableau  $S$ . By Theorem 3.1.5, the couple  $(t, T'')$  corresponds to a unique signed two-rowed array of the following form

$$\begin{pmatrix} x'_1 \dots x'_l x_1 \dots x_k \\ y'_1 \dots y'_l y_1 \dots y_k \end{pmatrix}.$$

Then the signed two rowed array

$$\begin{pmatrix} x'_1 \dots x'_l \\ y'_1 \dots y'_l \end{pmatrix}$$

corresponds to the couple  $(t', T')$ , and the signed two-rowed array

$$\begin{pmatrix} x_1 \dots x_k \\ y_1 \dots y_k \end{pmatrix}$$

corresponds to the couple  $(t'', S)$ , for some super tableaux  $t'$  in  $\text{Yt}(\mathcal{S}, \lambda)$  and  $t''$  in  $\text{Yt}(\mathcal{S}, \mu)$ , such that  $t' \star_{\text{Yt}(\mathcal{S})} t'' = t$ . Hence, we obtain a pair  $(t', t'')$  in  $\text{Yt}(\mathcal{S}, \lambda, \mu, \dashv t)$ . We deduce that the cardinal number of the set  $\text{St}(\mathcal{S}, \nu/\lambda, \dashv T)$  is equal to the super Littlewood–Richardson coefficient  $c_{\lambda, \mu}^{\nu}$ , showing the claim.  $\square$

Since neither the set  $\text{Yt}(\mathcal{S}, \lambda, \mu, \dashv t)$  nor the set  $\text{St}(\mathcal{S}, \nu/\lambda, \dashv T)$  in the correspondence of the proof of Theorem 4.2.1 depends on the contents of the super tableaux used to define the other, we deduce the following result:

**Corollary 4.2.2.** *The cardinal number  $c_{\lambda, \mu}^{\nu}$  of the sets  $\text{Yt}(\mathcal{S}, \lambda, \mu, \dashv t)$  and  $\text{St}(\mathcal{S}, \nu/\lambda, \dashv T)$  is independent of choice of  $t$  and  $T$  and depends only on  $\lambda$ ,  $\mu$  and  $\nu$ .*

**4.3. Super Littlewood–Richardson skew tableaux.** In this final part, we introduce the notion of super Littlewood–Richardson skew tableaux and we give a new combinatorial description of the super Littlewood–Richardson coefficient.

We will suppose that

$$\mathcal{S} = \{\overline{m} < \dots < \overline{1} < 1 < \dots < n\}$$

with  $\mathcal{S}_1 = \{1 < \dots < n\}$  and  $\mathcal{S}_0 = \{\overline{m} < \dots < \overline{1}\}$ . For any  $w$  in  $\mathcal{S}^*$ , we will denote by  $|w|_i$  the number of times the element  $i$  of  $\mathcal{S}$  appears in  $w$ , and by  $\overline{|w|}$  the number of elements of  $\mathcal{S}_1$  that appear in  $w$  such that each element is counted only once. The *weight map* is the map

$$\text{wt} : \mathcal{S}^* \rightarrow (\mathbb{N} \cup \{0\})^{m+n}$$

defined by  $\text{wt}(w) = (|w|_{\overline{m}}, \dots, |w|_{\overline{1}}, |w|_1, \dots, |w|_n)$ , for all  $w$  in  $\mathcal{S}^*$ . For any super skew tableau  $S$  over  $\mathcal{S}$ , we will call  $\text{wt}(\mathbf{R}(S))$  the *weight* of  $S$ .

**4.3.1. Super Yamanouchi words.** A word  $w$  over  $\mathcal{S}$  is a *super Yamanouchi word* if it satisfies the following two conditions:

i) for every right subword  $w'$  of  $w$ , the following property holds:

$$|w'|_{\bar{m}} \geq \dots \geq |w'|_{\bar{2}} \geq |w'|_{\bar{1}} \geq \underline{|w'|},$$

ii) for every left subword  $w'$  of  $w$ , the following property holds:

$$|w'|_1 \geq |w'|_2 \geq \dots \geq |w'|_n.$$

When  $\mathcal{S} = \mathcal{S}_0$ , we recover the notion of *Yamanouchi words* that describe the elements of highest weights for the crystal structure of the general Lie algebra of type A, [23].

For instance, suppose  $\mathcal{S} = \{\bar{2} < \bar{1} < 1 < 2\}$ . The word  $12\bar{2}\bar{1}\bar{1}\bar{2}\bar{2}$  is a super Yamanouchi word over  $\mathcal{S}$ . However, the word  $w = \bar{1}\bar{1}\bar{2}\bar{2}\bar{1}\bar{2}\bar{2}$  is not a super Yamanouchi word over  $\mathcal{S}$  because for its right subword  $w' = \bar{1}\bar{2}\bar{2}\bar{1}\bar{2}\bar{2}$ , we have  $|w'|_{\bar{1}} < \underline{|w'|}$ .

**Lemma 4.3.2.** *Let  $w$  and  $v$  be words in  $\mathcal{S}$  that are super plactic equivalent. Then  $w$  is a super Yamanouchi word if and only if  $v$  is a super Yamanouchi word.*

*Proof.* Suppose first that  $w = uxzyu'$  and  $v = uzxyu'$  for all  $u, u'$  in  $\mathcal{S}^*$  such that  $x \leq y \leq z$  in  $\mathcal{S}$  with  $x = y$  only if  $||y|| = 0$  and  $y = z$  only if  $||y|| = 1$ . If  $x < y < z$ , there is no changes in the numbers of consecutive elements of  $\mathcal{S}_0$  (resp.  $\mathcal{S}_1$ ) in each right (resp. left) subword of  $w$  and  $v$ . If  $x = \bar{1}$  and  $y = z = 1$  or  $x = y = \bar{1}$  and  $z = 1$  such that  $w$  is a super Yamanouchi word. Then  $|u'|_{\bar{1}} \geq \underline{|u'|}$ , and hence  $v$  is a super Yamanouchi word. Similarly, we show that if  $v$  is a super Yamanouchi word, then  $w$  is so. If  $x = y = k \in \mathcal{S}_0$ , and  $z = k + 1 \in \mathcal{S}_0$  such that  $w$  is a super Yamanouchi word. Then by definition, the following property  $|u'|_k \geq |u'|_{k+1}$  holds, showing that  $zxuy'$  is a super Yamanouchi word, and then  $v$  is so. Similarly, we show that if  $v$  is a super Yamanouchi word, then  $w$  is so. If  $x = k \in \mathcal{S}_1$  and  $y = z = k + 1 \in \mathcal{S}_1$  such that  $w$  is a super Yamanouchi word. Then the following property  $|u|_k > |u|_{k+1}$  holds, showing that  $uzxy$  is a super Yamanouchi word, and hence  $v$  is so. Similarly, we show that if  $v$  is a super Yamanouchi word, then  $w$  is so.

Suppose now that  $w = uyxzu'$  and  $v = uyzxu'$  for all  $u, u'$  in  $\mathcal{S}^*$  such that  $x \leq y \leq z$  in  $\mathcal{S}$  with  $x = y$  only if  $||y|| = 1$  and  $y = z$  only if  $||y|| = 0$ . If  $x < y < z$ , there is no changes in the numbers of consecutive elements of  $\mathcal{S}_0$  (resp.  $\mathcal{S}_1$ ) in each right (resp. left) subword of  $w$  and  $v$ . Suppose  $x = y = k \in \mathcal{S}_1$ , and  $z = k + 1 \in \mathcal{S}_1$  such that  $w$  is a super Yamanouchi word. Then by definition, the following property  $|u|_k > |u|_{k+1}$  holds, showing that  $uyzx$  is a super Yamanouchi word, and hence  $v$  is so. Similarly, we show that if  $v$  is a super Yamanouchi word, then  $w$  is so. Suppose now that  $x = k \in \mathcal{S}_0$  and  $y = z = k + 1 \in \mathcal{S}_0$  such that  $w$  is a super Yamanouchi word. Then by definition, the following property  $|u'|_k > |u'|_{k+1}$  holds, showing that  $yzxu'$  is a super Yamanouchi word, and then  $v$  is so. Similarly, we show that if  $v$  is a super Yamanouchi word, then  $w$  is so. □

## 4. The super Littlewood–Richardson rule

**4.3.3. Super Littlewood–Richardson skew tableaux.** A super skew tableau  $S$  over  $\mathcal{S}$  is called a *super Littlewood–Richardson skew tableau* if its reading word  $\mathbf{R}(S)$  is a super Yamanouchi word. For instance, the following super tableau over  $\mathcal{S} = \{\bar{2} < \bar{1} < 1 < 2\}$  is a super Littlewood–Richardson skew tableau of shape  $(7, 5, 3, 3, 2, 1, 1)/(4, 3, 2, 2, 1)$  and of weight  $(3, 2, 3, 2)$ :

				$\bar{2}$	$\bar{2}$	$\bar{2}$
			$\bar{1}$	$\bar{1}$		
		$2$	$2$			
	$1$					
$1$						
$1$						

For any partition  $\mu$  of height  $k$ , define

$$\check{\mu} := (\mu_1, \tilde{\mu}_2)$$

where  $\mu_1$  is the partition formed by the first  $m$  parts of  $\mu$  and  $\mu_2$  is the partition formed by its last  $k - m$  parts, such that if  $k < m + n$  then some parts of  $\mu_2$  are zero and if  $k < m$  then some parts of  $\mu_1$  are zero and all the parts of  $\mu_2$  are zero. For instance, suppose  $\mathcal{S} = \{\bar{2} < \bar{1} < 1 < 2\}$  and  $\mu = (3, 2, 2, 2, 1)$ . Then  $\check{\mu} = (3, 2, 3, 2)$  with  $\mu_1 = (3, 2)$ ,  $\mu_2 = (2, 2, 1)$  and  $\tilde{\mu}_2 = (3, 2)$ .

Let  $\mu$  be a partition. Denote by  $T(\mu)$  the super tableau of shape  $\mu$  whose first row contains only  $\bar{m}$ , the second row contains only  $\bar{m} + 1$ , and so on row by row, until the  $m$ -th row that contains only  $\bar{1}$ , and starting from the nonempty  $(m + 1)$ -th box of each column, the corresponding  $i$ -th column contains only  $i$ . Then, the super tableau  $T(\mu)$  can be divided into two super tableaux: the first one of shape  $\mu_1$  that contains only the elements of  $\mathcal{S}_0$  and the second one of shape  $\mu_2$  that contains only the elements of  $\mathcal{S}_1$ , such that the following equality

$$\text{wt}(\mathbf{R}(T(\mu))) = (\mu_1, \tilde{\mu}_2) = \check{\mu}$$

holds. For instance, suppose  $\mathcal{S} = \{\bar{2} < \bar{1} < 1 < 2\}$  and  $\mu = (3, 2, 2, 2, 1)$ . Then the super tableau  $T(\mu)$  is the following:

$\bar{2}$	$\bar{2}$	$\bar{2}$
$\bar{1}$	$\bar{1}$	
$1$	$2$	
$1$	$2$	
$1$		

such that  $\mu_1 = (3, 2)$ ,  $\mu_2 = (2, 2, 1)$  and  $\check{\mu} = (3, 2, 3, 2) = \text{wt}(\mathbf{R}(T(\mu)))$ .

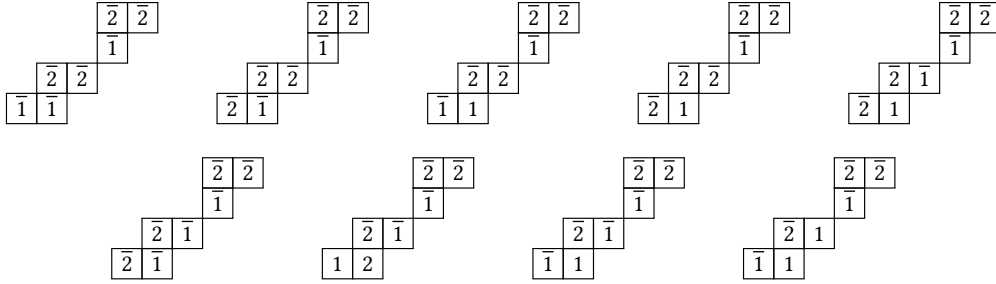
It is clear from the definition of  $T(\mu)$ , that the super tableau  $T(\mu)$  is the only super tableau of shape  $\mu$  whose reading is a super Yamanouchi word. Hence, following Lemma 4.3.2, we deduce the following result.

**Lemma 4.3.4.** *Let  $\mu$  be a partition. A super skew tableau  $S$  is a super Littlewood–Richardson skew tableau of weight  $\check{\mu}$  if and only if its rectification is the super tableau  $T(\mu)$ .*

As a consequence, following Theorem 4.2.1 we deduce the following result.

**Theorem 4.3.5.** *The super Littlewood–Richardson coefficient  $c_{\lambda, \mu}^{\nu}$  is equal to the number of super Littlewood–Richardson skew tableaux of shape  $\nu/\lambda$  and of weight  $\check{\mu}$ .*

**Example 4.3.6.** Suppose  $\mathcal{S} = \{\bar{2} < \bar{1} < 1 < 2\}$ ,  $\nu = (5, 4, 3, 2)$  and  $\lambda = (3, 3, 1)$ . The super Littlewood–Richardson skew tableaux of shape  $(5, 4, 3, 2)/(3, 3, 1)$  are the following:



Following Theorem 4.3.5, we have:

- i)**  $c_{\lambda, \mu}^{\nu} = 1$ , for  $\mu = (5, 2)$ ,  $\mu = (5, 1, 1)$ ,  $\mu = (3, 3, 1)$ ,
- ii)**  $c_{\lambda, \mu}^{\nu} = 2$ , for  $\mu = (4, 3)$ ,  $\mu = (4, 2, 1)$ ,  $\mu = (3, 2, 2)$ ,
- iii)**  $c_{\lambda, \mu}^{\nu} = 0$ , for other  $\mu$ .

Hence, we obtain the following decomposition in  $R_{\mathcal{S}}$ :

$$S_{(5,4,3,2)/(3,3,1)} = S_{(5,2)} + S_{(5,1,1)} + 2S_{(4,3)} + 2S_{(4,2,1)} + 2S_{(3,2,2)} + S_{(3,3,1)}.$$

Note finally that the readings of the *genuine highest weight super tableaux* introduced in [1] as highest weight vectors for the crystal graph for the representations of the general linear Lie superalgebra of type A are super Yamanouchi words. A future work would be to investigate the combinatorial properties of the representations of the general linear Lie superalgebra  $\mathfrak{gl}_{m,n}$  using the constructions developed in this article.

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