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Fourier-Mukai transform for $\hat{\mathcal{D}}^{(0)}$ -modules over formal abelian schemes

Florian VIGUIER

4 juillet 2022

Abstract

In 1996, Rothstein and Laumon simultaneously constructed a Fourier-Mukai transform for \mathcal{D} -modules over a locally noetherian base of characteristic 0. This functor induces an equivalence of categories between quasi-coherent sheaves of \mathcal{D} -modules over an abelian variety A and quasi-coherent sheaves of \mathcal{O} -modules over its universal vectorial extension A^\natural . In this article, we define a Fourier-Mukai transform for \mathcal{D} -modules on an abelian formal scheme $\mathcal{A}/\mathcal{S} = \text{Spf}(V)$, where V is a discrete valuation ring, and we discuss the extension of the classical results of Fourier-Mukai transform to this arithmetic case.

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Introduction

The Fourier-Mukai transform, first defined by Shigeru Mukai in [Muk81] for \mathcal{O} -modules on an abelian variety over an algebraically closed field, is a very useful tool in the study of quasi-coherent sheaves over an abelian variety. Thanks to the independent work of Rothstein [Rot96] and Laumon [Lau96], it has been extended into a functor acting on \mathcal{D} -modules on an abelian variety in characteristic zero. We will here recall the way it is constructed in [Lau96], as it will be the easier construction to adapt for arithmetic varieties.

First, given an abelian variety A over a base S of characteristic zero, one constructs the functor from the category of noetherian S -scheme to the one of abelian groups:

$$\text{Pic}^{\natural}(A \times \bullet/\bullet) : T \mapsto \text{Pic}^{\natural}(A \times T/T),$$

where $\text{Pic}^{\natural}(X/T)$ is the abelian group of sheaves of $\mathcal{D}_{X/T}$ -modules \mathcal{O}_X -invertible \mathcal{L} such that

$$m^{\flat} \mathcal{L} \simeq p_1^{\flat} \mathcal{L} \otimes p_2^{\flat} \mathcal{L},$$

with $m, p_1, p_2 : X \times_T X \rightarrow X$ being respectively the multiplication and the canonical projections and f^b the inverse image functor of \mathcal{D} -modules without the shift. Such a sheaf \mathcal{L} is said to verify the theorem of the square. Note that the group $Pic^h(A/S)$ is also the group of rigid extensions of A (see [MM74] for more details).

The functor $Pic^h(A \times \bullet / \bullet)$ is representable by an abelian group scheme over S of dimension $2dim(A)$ and denoted A^h . It is the universal vectorial extension of A . The representability of this functor leads to the construction of a universal element \mathcal{P} of $Pic^h(A \times A^h/A^h)$, corresponding to the morphism $id_{A^h} \in Hom(A^h, A^h)$. The sheaf \mathcal{P} is thus a $\mathcal{D}_{A \times A^h/A^h}$ -module, $\mathcal{O}_{A \times A^h}$ -invertible, satisfying the theorem of the square, and it is called the Poincaré sheaf. Note that it is not the same Poincaré sheaf as the one defined on the dual abelian variety. These sheaves are linked together in a way that will be explained later.

The classical Fourier-Mukai transform is then defined as the functor \mathcal{F} from the derived category of quasi-coherent $\mathcal{D}_{A/S}$ -modules to the derived category of quasi-coherent \mathcal{O}_{A^h} -modules as follows

$$\mathcal{F}(\mathcal{E}^\bullet) = Rp_*^h(\mathcal{P} \otimes^{\mathbb{L}} p^* \mathcal{E}^\bullet),$$

where $p : A \times A^h \rightarrow A$ and $p^\vee : A \times A^h \rightarrow A^h$ are the canonical projections. One can define in the same way the dual Fourier-Mukai transform $\mathcal{F}^h : D_{qcoh}^b(\mathcal{O}_{A^h}) \rightarrow D_{qcoh}^b(\mathcal{D}_{A/S})$ as follows

$$\mathcal{F}^h(\mathcal{E}^\bullet) = Rp_*^h(\mathcal{P} \otimes^{\mathbb{L}} p^{h*} \mathcal{E}^\bullet).$$

The most important property of the Fourier-Mukai transform for \mathcal{D} -modules is the following

Theorem (Rothstein-Laumon). *The Fourier-Mukai transform induces an equivalence of categories between $D_{qcoh}^b(\mathcal{D}_{A/S})$ and $D_{qcoh}^b(\mathcal{O}_{A^h})$.*

More precisely, it is involutive, in the sense that

$$\mathcal{F}^h \circ \mathcal{F} \simeq \langle -1 \rangle^! \bullet [-dim(A)]$$

and

$$\mathcal{F} \circ \mathcal{F}^h \simeq \langle -1 \rangle^{h*} \bullet [-dim(A)],$$

with $\langle -1 \rangle$ and $\langle -1 \rangle^h$ being the inverse morphisms of A and A^h respectively.

Roughly speaking, a consequence of this theorem is that a complex \mathcal{E}^\bullet of \mathcal{D} -modules over A is characterized by the solutions of all the twists of \mathcal{E}^\bullet with invertible sheaves with connexions.

The goal of any extension of the Fourier-Mukai transform is to preserve this property.

One of the motivations of this work comes from an article of Schnell ([Sch15]) where he uses the Fourier-Mukai transform to establish some structure results for holonomic \mathcal{D} -modules.

In the arithmetic case we fix V a discrete valuation ring of uniformizer π with $p = char(V/\pi) > 0$, V_i the reduction modulo π^i of V , (A_i) an inductive system of abelian schemes over the $S_i = Spec(V_i)$ which are compatible with the reductions modulo π^j (e.g. the reduction modulo π^i of an abelian variety A over S) and $\mathcal{A} = \varinjlim A_i$, which is a formal abelian scheme.

The goal of this article is to define a Fourier-Mukai transform on \mathcal{A} and to study it in order to find the expected involutivity formulas, as it has been previously done for $\mathcal{O}_{\mathcal{A}}$ -modules in [Vig22]. Note that all the previous constructions will adapt without any problem in the arithmetic case, the only work will be to define the functors correctly. However, the proof of the involutivity formulas strongly requires the characteristic zero hypothesis, so that a lot of work is needed in order to prove results in the arithmetic case. The main result of this article is theorem 2.17 where we prove that our arithmetic Fourier-Mukai functor is essentially surjective.

The first chapter will be dedicated to the definitions of "relative" \mathcal{D} -modules and their fundamental properties. Indeed, as \mathcal{P} is a $\mathcal{D}_{A \times A^h/A^h}$ -module, all the \mathcal{D} -modules will be considered as relative

to A^\natural . It is important to study their behavior, especially their push-forward by the projection $A \times A^\natural \rightarrow A$.

These discussions will also lead to an interesting version of the seesaw principle for \mathcal{D} -modules (property 1.18).

The second chapter will focus on the definition of the dual abelian variety \mathcal{A}^\natural , the Poincaré sheaf \mathcal{P} and the Fourier-Mukai transform \mathcal{F} as well as their properties, the most important being the essential surjectivity of \mathcal{F} (theorem 2.17).

I want to thank Christine Huyghe for the numerous conversations we had on this topic and for all of her precious advices.

1 Classic and relative $\hat{\mathcal{D}}^{(0)}$ -modules

For this section, let V be a discrete valuation ring with uniformizer π , $S = \text{Spec}(V)$, for all i $V_i = V/\pi^i V$ and $S_i = \text{Spec} V_i$ and $\mathcal{S} = \text{Spf} S$. For all i let X_i be a variety over S_i such that $X_{i+1} \times_{S_{i+1}} S_i \simeq X_i$ and $\mathcal{X} = \varprojlim X_i$ is smooth (one can think of the X_i as the reduction modulo π^i of a smooth variety X over S).

By the end of this article, the X_i will be considered to be abelian schemes, but as the following definitions can be written in a more general context, we will only assume \mathcal{X} is smooth.

1.1 Definitions

The construction of the sheaves $\mathcal{D}^{(0)}$ won't be recalled in this article. For more information, see [Ber02] (chapter 1). Simply recall that taking a left $\mathcal{D}^{(0)}$ -module structure on a \mathcal{O} -module \mathcal{E} is equivalent to take an integrable connexion on \mathcal{E} (this is the main reason this article will only talk about $\mathcal{D}^{(0)}$ -modules and not $\mathcal{D}^{(m)}$ -modules). Moreover, locally

$$\Gamma(U, \mathcal{D}_X^{(0)}) = \left\{ \sum_{\underline{k}, \text{ finite}} a_{\underline{k}} \partial^{\underline{k}} \mid a_{\underline{k}} \in \Gamma(U, \mathcal{O}_X) \right\},$$

where the ∂_i are the derivation associated to a coordinate system t_i of U (note that these coordinates, and so $\mathcal{D}_X^{(0)}$, depend on the base S considered).

One can define the module $\mathcal{D}_{\mathcal{X}}^{(0)}$ the same way on the formal scheme \mathcal{X} .

Definition 1.1. *Let \mathcal{X} be a formal scheme on \mathcal{S} . $\hat{\mathcal{D}}_{\mathcal{X}/\mathcal{S}}^{(0)}$ is the π -adic completion of $\mathcal{D}_{\mathcal{X}}^{(0)}$:*

$$\hat{\mathcal{D}}_{\mathcal{X}/\mathcal{S}}^{(0)} = \varprojlim_i \mathcal{D}_{\mathcal{X}}^{(0)}/\pi^i \simeq \varprojlim_i \mathcal{D}_{X_i}^{(0)}.$$

Remark. *The subscript \mathcal{X}/\mathcal{S} is here to point out that \mathcal{X} is viewed as a \mathcal{S} -formal scheme. Later in this paper we will have to consider the product of two formal schemes $\mathcal{X} \times \mathcal{Y}$ as a scheme over \mathcal{Y} , and so we will have to consider $\hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Y}/\mathcal{Y}}^{(0)}$ -modules instead of $\hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Y}/\mathcal{S}}^{(0)}$ -modules. In that case, we will call them 'relative' $\hat{\mathcal{D}}^{(0)}$ -modules.*

In the following we may however use the notation $\hat{\mathcal{D}}_{\mathcal{X}}^{(0)}$ instead of $\hat{\mathcal{D}}_{\mathcal{X}/\mathcal{S}}^{(0)}$ if there is no confusion.

Locally,

$$\Gamma(U, \hat{\mathcal{D}}_{\mathcal{X}/\mathcal{S}}^{(0)}) = \left\{ \sum_{\underline{k}} a_{\underline{k}} \partial^{\underline{k}} \mid a_{\underline{k}} \in \Gamma(U, \mathcal{O}_{\mathcal{X}}) \text{ and } v_{\pi}(a_{\underline{k}}) \rightarrow 0 \right\}.$$

Quasi-coherent $\hat{\mathcal{D}}^{(0)}$ -modules

We will take the following definition for quasi-coherent sheaves of $\hat{\mathcal{D}}^{(0)}$ -modules over \mathcal{X} :

Definition 1.2. Let $\mathcal{E} \in D^-(\hat{\mathcal{D}}_{\mathcal{X}}^{(0)})$. For all i define $\mathcal{E}_i = \mathcal{O}_{X_i} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{\mathcal{X}}} \mathcal{E}$.
 \mathcal{E} is said quasi-coherent if

- $\mathcal{E}_0 \in D_{qcoh}^-(\mathcal{D}_{X_0}^{(0)})$,
- There is a canonical isomorphism $\mathcal{E} \simeq R\varprojlim \mathcal{E}_i$.

Denote by $D_{qcoh}^-(\hat{\mathcal{D}}_{\mathcal{X}}^{(0)})$ the full sub-category of $D^-(\hat{\mathcal{D}}_{\mathcal{X}}^{(0)})$ whose objects are the quasi-coherents.

The following characterization can be find in [Ber02].

Theorem 1.3 (Berthelot). If $(\mathcal{E}_i)_i$ is a projective system of complexes such that the i -th element is in $D^b(\mathcal{D}_{X_i}^{(0)})$ and verifying

- $\mathcal{E}_0 \in D_{qcoh}^b(\mathcal{D}_{X_0}^{(0)})$,
- For all i the canonical morphism

$$\mathcal{D}_{X_i}^{(0)} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_{X_{i+1}}^{(0)}} \mathcal{E}_{i+1} \rightarrow \mathcal{E}_i$$

is an isomorphism,

then $\mathcal{E} = R\varprojlim \mathcal{E}_i$ is an object of $D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{X}}^{(0)})$.

Tensor product

We will only recall the definition given in [Ber02] (chapter 3) and some basic properties.

Definition 1.4. Let $\mathcal{E} \in D_{qcoh}^-(\hat{\mathcal{D}}_{\mathcal{X}}^{(0) r})$ (resp. $\mathcal{F} \in D_{qcoh}^-(\hat{\mathcal{D}}_{\mathcal{X}}^{(0) l})$) be a complex of right (resp. left) $\hat{\mathcal{D}}_{\mathcal{X}}^{(0)}$ -module.

As usual, denote $\mathcal{E}_i = \mathcal{O}_{X_i} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{\mathcal{X}}} \mathcal{E}$. Idem for \mathcal{F}_i .

The completed tensor product of \mathcal{E} and \mathcal{F} is defined as

$$\mathcal{E} \hat{\otimes}_{\hat{\mathcal{D}}_{\mathcal{X}}^{(0)}}^{\mathbb{L}} \mathcal{F} = R\varprojlim_i \left(\mathcal{E}_i \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_{X_i}^{(0)}} \mathcal{F}_i \right).$$

It is a $\hat{\mathcal{D}}_{\mathcal{X}}^{(0)}$ -module.

Moreover, if the complexes have bounded cohomology, $\mathcal{E} \hat{\otimes}_{\hat{\mathcal{D}}_{\mathcal{X}}^{(0)}}^{\mathbb{L}} \mathcal{F} \in D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{X}}^{(0)})$.

Finally, if one of the complexes is coherent, there is a canonical isomorphism

$$\mathcal{E} \overset{\mathbb{L}}{\otimes}_{\hat{\mathcal{D}}_{\mathcal{X}}^{(0)}} \mathcal{F} \simeq \mathcal{E} \hat{\otimes}_{\hat{\mathcal{D}}_{\mathcal{X}}^{(0)}}^{\mathbb{L}} \mathcal{F}.$$

Inverse image

The following definition can be found in [Ber02] (chapter 3).

Definition 1.5. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of smooth formal \mathcal{S} -schemes and $\mathcal{E} \in D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{Y}}^{(0)})$. For all i , denote by $f_i : X_i \rightarrow Y_i$ the morphism of S_i -schemes associated to f .

The inverse image of \mathcal{E} by f is defined as

$$f_{/\mathcal{S}}^! \mathcal{E} = R\varprojlim_i \left(f_{i/S_i}^! \mathcal{E}_i \right).$$

Remark. The notation $f_{/\mathcal{S}}^!$ is used here to show explicitly the base \mathcal{S} . This notation will make sense in the second half of this article, where there will be schemes which can be seen as \mathcal{S} -schemes or \mathcal{T} -schemes (with \mathcal{T} as \mathcal{S} -scheme). The base scheme is of crucial importance when it comes to \mathcal{D} -modules, as it determines which functions are the constants, and the functors $f_{/\mathcal{S}}^!$ and $f_{/\mathcal{T}}^!$ are different.

However, if there is no confusion possible, we will use the usual notation $f^!$.

Again, if \mathcal{E} is coherent, there is a canonical isomorphism

$$\hat{\mathcal{D}}_{\mathcal{X} \rightarrow \mathcal{Y}}^{(0)} \otimes_{f^{-1}\hat{\mathcal{D}}_{\mathcal{Y}}^{(0)}}^{\mathbb{L}} f^{-1} \mathcal{E} [d_{\mathcal{X}} - d_{\mathcal{Y}}] \simeq f^! \mathcal{E},$$

where the transfer bimodule is defined as $\hat{\mathcal{D}}_{\mathcal{X} \rightarrow \mathcal{Y}}^{(0)} = \varprojlim_{X_i \rightarrow Y_i} \mathcal{D}_{X_i \rightarrow Y_i}^{(0)}$.

The formal functor $f^!$ preserves quasi-coherence and, if f is smooth, preserves coherence. This is because the 'classical' functors $f_i^!$ do and (quasi-)coherence can be tested modulo π^i .

As relative $\hat{\mathcal{D}}^{(0)}$ -modules will be crucial in the article, it is necessary to define the relative inverse image. Let \mathcal{X} , \mathcal{Y} and \mathcal{Z} be three formal schemes over \mathcal{S} , $g : \mathcal{Y} \rightarrow \mathcal{Z}$ and $f = id_{\mathcal{X}} \times g : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{X} \times \mathcal{Z}$ and assume that $\mathcal{E} \in D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Z}/\mathcal{Z}}^{(0)})$ (so only the derivations along \mathcal{X} acts on \mathcal{E}). The goal is to give some sense to the notion of pulling-back \mathcal{E} on $\mathcal{X} \times \mathcal{Y}$. The main idea is to say that we want to preserve the action of the derivations along \mathcal{X} and keep having no action of the derivations along \mathcal{Y} . It can be thought as a base change.

Definition 1.6. For each i , define the transfer bimodule (of level 0) relative to Y_i and Z_i as

$$\mathcal{D}_{X_i \times Y_i \rightarrow X_i \times Z_i / S_i}^{(0)(Y_i, Z_i)} = f_i^* \mathcal{D}_{X_i \times Z_i / Z_i}^{(0)} = \mathcal{O}_{X_i \times Y_i} \otimes_{f_i^{-1} \mathcal{O}_{X_i \times Z_i}} f_i^{-1} \mathcal{D}_{X_i \times Z_i / Z_i}^{(0)}.$$

The transfer bimodule (of level 0) relative to \mathcal{Y} and \mathcal{Z} can then be defined as

$$\hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{X} \times \mathcal{Z} / \mathcal{S}}^{(0)(\mathcal{Y}, \mathcal{Z})} = \varprojlim_i \mathcal{D}_{X_i \times Y_i \rightarrow X_i \times Z_i}^{(0)(Y_i, Z_i)}.$$

For all $\mathcal{E}_i \in D^b(\mathcal{D}_{X_i \times Z_i / Z_i}^{(0)})$, the inverse image relative to Y_i and Z_i of \mathcal{E}_i by f_i is defined by

$$f_{i/S_i}^{!(Y_i, Z_i)} \mathcal{E}_i = \mathcal{D}_{X_i \times Y_i \rightarrow X_i \times Z_i}^{(0)(Y_i, Z_i)} \otimes_{f_i^{-1} \mathcal{D}_{X_i \times Z_i / Z_i}^{(0)}}^{\mathbb{L}} f_i^{-1} \mathcal{E}_i \in D^b(\mathcal{D}_{X_i \times Y_i / Y_i}^{(0)}).$$

Finally, for all $\mathcal{E} \in D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Z}/\mathcal{Z}}^{(0)})$, the inverse image relative to \mathcal{Y} and \mathcal{Z} of \mathcal{E} by f is defined by

$$f_{/\mathcal{S}}^{!(\mathcal{Y}, \mathcal{Z})} \mathcal{E} = R\varprojlim_i \left(f_i^{!(Y_i, Z_i)} \mathcal{E}_i \right).$$

These are exactly the same definitions as in the non-relative case, at the only exception that all the subscripts $/S$ have been changed into subscripts $/Y$ or $/Z$, as now the variety $X \times Y$ (resp. $X \times Z$) is viewed as a Y -variety (resp. Z -variety). Also note that because of that the natural shift that should occur in the definition of the inverse image is $\dim_Y(X \times Y) - \dim_Z(X \times Z) = \dim(X) - \dim(X) = 0$. This explains why there is no shift in the definition.

It is important to point out that these relative functors are actually very close to the functors defined on \mathcal{O} -modules, as they don't modify the structure of $\hat{\mathcal{D}}^{(0)}$ -module of the sheaves they act on. Keeping that in mind, the following results are straightforward.

Property 1.7. • $f_i^{!(Y_i, Z_i)}$ and $f^{!(\mathcal{Y}, \mathcal{Z})}$ verify the transitivity formulas.

- $f_i^{!(Y_i, Z_i)} \mathcal{D}_{X_i \times Z_i / Z_i}^{(0)} \simeq \mathcal{D}_{X_i \times Y_i / Y_i}^{(0)}$.
- $f^{!(\mathcal{Y}, \mathcal{Z})} \hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Z} / \mathcal{Z}}^{(0)} \simeq \hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Y} / \mathcal{Y}}^{(0)}$.
- As $\mathcal{O}_{X_i \times Y_i}$ -modules, $f_i^{!(Y_i, Z_i)} \mathcal{E}_i \simeq Lf_i^* \mathcal{E}_i$.
- As $\mathcal{O}_{\mathcal{X} \times \mathcal{Y}}$ -modules, $f^{!(\mathcal{Y}, \mathcal{Z})} \mathcal{E} \simeq Lf^* \mathcal{E}$, where Lf^* is defined as $Lf^* \mathcal{E} = R\varprojlim Lf_i^* \mathcal{E}_i$.
- If $\mathcal{E}_i \in D_{qcoh}^b(\mathcal{D}_{X_i \times Z_i / Z_i}^{(0)})$, then $f_i^{!(Y_i, Z_i)} \mathcal{E}_i \in D_{qcoh}^b(\mathcal{D}_{X_i \times Y_i / Y_i}^{(0)})$.
- If $\mathcal{E} \in D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Z} / \mathcal{Z}}^{(0)})$, then $f^{!(\mathcal{Y}, \mathcal{Z})} \mathcal{E} \in D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Y} / \mathcal{Y}}^{(0)})$.
- If $\mathcal{E} \in D_{coh}^b(\hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Z} / \mathcal{Z}}^{(0)})$ there is a canonical isomorphism

$$\hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{X} \times \mathcal{Z}}^{(0)(\mathcal{Y}, \mathcal{Z})} \otimes_{f^{-1} \hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Z} / \mathcal{Z}}^{(0)}} f^{-1} \mathcal{E} \simeq f^{!(\mathcal{Y}, \mathcal{Z})} \mathcal{E}.$$

- If f_i (resp. f) is smooth, then $f_i^{!(Y_i, Z_i)}$ (resp. $f^{!(\mathcal{Y}, \mathcal{Z})}$) preserves coherence.

Direct image

As for inverse images, the first definition can be found in [Ber02] (chapter 3).

Definition 1.8. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of smooth formal \mathcal{S} -schemes and $\mathcal{E} \in D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{X}}^{(0)})$. For all i , denote by $f_i : X_i \rightarrow Y_i$ the morphism of S_i -schemes associated to f .

The direct image of \mathcal{E} by f is defined as

$$f_{+/\mathcal{S}} \mathcal{E} = R\varprojlim_i (f_{i+/\mathcal{S}_i} \mathcal{E}_i).$$

Again, if \mathcal{E} is coherent there is a canonical isomorphism

$$Rf_* \left(\hat{\mathcal{D}}_{\mathcal{Y} \leftarrow \mathcal{X}}^{(0)} \otimes_{\hat{\mathcal{D}}_{\mathcal{X}}^{(0)}} \mathcal{E} \right) \simeq f_{+} \mathcal{E},$$

where the transfer bimodule is defined as $\hat{\mathcal{D}}_{\mathcal{Y} \leftarrow \mathcal{X}}^{(0)} = \varprojlim \mathcal{D}_{Y_i \leftarrow X_i}^{(0)}$.

As before, the formal functor f_{+} preserves quasi-coherence and, if f is proper, preserves coherence.

Finally, let \mathcal{X} , \mathcal{Y} and \mathcal{Z} be three formal schemes over \mathcal{S} , $g : \mathcal{Y} \rightarrow \mathcal{Z}$ and $f = id_{\mathcal{X}} \times g : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{X} \times \mathcal{Z}$ and assume that $\mathcal{E} \in D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Y} / \mathcal{Y}}^{(0)})$. One can define the relative direct image of \mathcal{E} using the same idea as before: replacing all the $/S$ in the definition by $/Y$ or $/Z$.

Definition 1.9. For all i , the transfer bimodule (of level 0) relative to Y_i and Z_i is defined by

$$\mathcal{D}_{X_i \times Z_i \leftarrow X_i \times Y_i / S_i}^{(0)(Y_i, Z_i)} = \omega_{X_i \times Y_i / Y_i} \otimes_{\mathcal{O}_{X_i \times Y_i}} \mathcal{D}_{X_i \times Y_i \rightarrow X_i \times Z_i}^{(0)(Y_i, Z_i)} \otimes_{f^{-1}\mathcal{O}_{X_i \times Z_i}} f^{-1}\omega_{X_i \times Z_i / Z_i}^{-1}.$$

The transfer bimodule (of level 0) relative to \mathcal{Y} and \mathcal{Z} is then defined as

$$\hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Z} \leftarrow \mathcal{X} \times \mathcal{Y} / S}^{(0)(\mathcal{Y}, \mathcal{Z})} = \varprojlim_i \mathcal{D}_{X_i \times Z_i \leftarrow X_i \times Y_i}^{(0)(Y_i, Z_i)}.$$

For all $\mathcal{E}_i \in D^b(\mathcal{D}_{X_i \times Y_i / Y_i}^{(0)})$, the direct image relative to Y_i and Z_i of \mathcal{E}_i by f_i is defined as

$$f_{i+}^{(Y_i, Z_i)} \mathcal{E}_i = Rf_{i*} \left(\mathcal{D}_{X_i \times Z_i \leftarrow X_i \times Y_i}^{(0)(Y_i, Z_i)} \otimes_{\mathcal{D}_{X_i \times Y_i / Y_i}^{(0)}} \mathcal{E}_i \right) \in D(\mathcal{D}_{X_i \times Z_i / Z_i}^{(0)}).$$

Finally, for all $\mathcal{E} \in D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Y} / \mathcal{Y}}^{(0)})$, its direct image relative to \mathcal{Y} and \mathcal{Z} is defined by

$$f_{+/S}^{(\mathcal{Y}, \mathcal{Z})} \mathcal{E} = R\varprojlim_i \left(f_{i+}^{(Y_i, Z_i)} \mathcal{E}_i \right).$$

Again, the proofs of the following properties are straightforward.

Property 1.10. • $f_{i+}^{(Y_i, Z_i)}$ and $f_{+/S}^{(\mathcal{Y}, \mathcal{Z})}$ verify the transitivity formulas.

- As $\mathcal{O}_{X_i \times Z_i}$ -modules, $f_{i+}^{(Y_i, Z_i)} \mathcal{E}_i \simeq Rf_{i*} \mathcal{E}_i$.
- As $\mathcal{O}_{\mathcal{X} \times \mathcal{Z}}$ -modules, $f_{+/S}^{(\mathcal{Y}, \mathcal{Z})} \mathcal{E} \simeq Rf_* \mathcal{E}$.
- If $\mathcal{E}_i \in D_{qcoh}^b(\mathcal{D}_{X_{Y_i, i} / Y_i}^{(0)})$, then $f_{i+}^{(Y_i, Z_i)} \mathcal{E}_i \in D_{qcoh}^b(\mathcal{D}_{X_{Z_i, i} / Z_i}^{(0)})$.
- If $\mathcal{E} \in D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{X}_{\mathcal{Y}} / \mathcal{Y}}^{(0)})$, then $f_{+/S}^{(\mathcal{Y}, \mathcal{Z})} \mathcal{E} \in D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{X}_{\mathcal{Z}}}^{(0)})$.
- If $\mathcal{E} \in D_{coh}^b(\hat{\mathcal{D}}_{\mathcal{X}_{\mathcal{Y}} / \mathcal{Y}}^{(0)})$ there is a canonical isomorphism

$$Rf_* \left(\hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Z} \leftarrow \mathcal{X} \times \mathcal{Y}}^{(0)(\mathcal{Y}, \mathcal{Z})} \otimes_{\hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Y} / \mathcal{Y}}^{(0)}} \mathcal{E} \right) \simeq f_{+/S}^{(\mathcal{Y}, \mathcal{Z})} \mathcal{E}.$$

- If f_i (resp. f) is proper, then $f_{i+}^{(Y_i, Z_i)}$ (resp. $f_{+/S}^{(\mathcal{Y}, \mathcal{Z})}$) preserves coherence.

1.2 Fundamental properties

The goal of this section is to give some important 'calculus' results on relative $\hat{\mathcal{D}}^{(0)}$ -modules, such as the projection formula or the base change formula. It is important to note that these results are isomorphisms whose morphisms are constructed using adjunction formulas, hence it is sufficient to prove the adjunction formulas for relative $\hat{\mathcal{D}}^{(0)}$ -modules, as it is sufficient for a morphism of $\hat{\mathcal{D}}^{(0)}$ -modules to be an isomorphism on the underlying \mathcal{O} -modules in order to be an isomorphism of $\hat{\mathcal{D}}^{(0)}$ -modules.

First of all, recall the adjunction formulas for non-relative $\hat{\mathcal{D}}^{(0)}$ -modules. See [Vir04] sections IV.4 et IV.5 for more details.

Property 1.11 (Adjunction formulas). Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a proper morphism of smooth formal schemes, $\mathcal{E} \in D_{coh}^b(\hat{\mathcal{D}}_{\mathcal{X}}^{(0)})$ and $\mathcal{F} \in D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{Y}}^{(0)})$. Then for all i

$$Rf_{i*} R\mathcal{H}om_{\mathcal{D}_{X_i}^{(0)}}(\mathcal{E}_i, f_i^! \mathcal{F}_i) \simeq R\mathcal{H}om_{\mathcal{D}_{Y_i}^{(0)}}(f_{i+} \mathcal{E}_i, \mathcal{F}_i)$$

and

$$Rf_* R\mathcal{H}om_{\hat{\mathcal{D}}_{\mathcal{X}}^{(0)}}(\mathcal{E}^\cdot, f^! \mathcal{F}^\cdot) \simeq R\mathcal{H}om_{\hat{\mathcal{D}}_{\mathcal{Y}}^{(0)}}(f_+ \mathcal{E}^\cdot, \mathcal{F}^\cdot).$$

In particular, there are morphisms

$$\mathcal{E}_i \rightarrow f_i^! f_{i+} \mathcal{E}_i \text{ and } \mathcal{E}^\cdot \rightarrow f^! f_+ \mathcal{E}^\cdot$$

and if $f^! \mathcal{F}^\cdot \in D_{\text{coh}}^b(\hat{\mathcal{D}}_{\mathcal{Y}}^{(0)})$, there are morphisms

$$f_{i+} f_i^! \mathcal{F}_i \rightarrow \mathcal{F}_i \text{ and } f_+ f^! \mathcal{F}^\cdot \rightarrow \mathcal{F}^\cdot.$$

There are also adjunction formulas for relative $\hat{\mathcal{D}}^{(0)}$ -modules.

Property 1.12 (Relative adjunction morphisms). *Let \mathcal{X} , \mathcal{Y} and \mathcal{Z} be three formal schemes, $f : \mathcal{X}_{\mathcal{Y}} = \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{X}_{\mathcal{Z}} = \mathcal{X} \times \mathcal{Z}$ a morphism of the form $\text{id}_{\mathcal{X}} \times g$, $\mathcal{E}^\cdot \in D_{\text{qcoh}}^b(\hat{\mathcal{D}}_{\mathcal{X}_{\mathcal{Y}/\mathcal{Y}}}^{(0)})$ and $\mathcal{F}^\cdot \in D_{\text{qcoh}}^b(\mathcal{D}_{\mathcal{X}_{\mathcal{Z}/\mathcal{Z}}})$. The adjunction morphisms*

$$f_i^{!(Y_i, Z_i)} f_{i+}^{(Y_i, Z_i)} \mathcal{E}_i \rightarrow \mathcal{E}_i \text{ et } \mathcal{F}_i \rightarrow f_{i+}^{(Y_i, Z_i)} f_i^{!(Y_i, Z_i)} \mathcal{F}_i$$

are respectively morphisms on the categories $D_{\text{qcoh}}^b(\mathcal{D}_{X_{Y_i/Y_i}}^{(0)})$ and $D_{\text{qcoh}}^b(\mathcal{D}_{X_{Z_i/Z_i}}^{(0)})$.

There are also adjunction morphisms

$$f^{!(\mathcal{Y}, \mathcal{Z})} f_+^{(\mathcal{Y}, \mathcal{Z})} \mathcal{E}^\cdot \rightarrow \mathcal{E}^\cdot \text{ et } \mathcal{F}^\cdot \rightarrow f_+^{(\mathcal{Y}, \mathcal{Z})} f^{!(\mathcal{Y}, \mathcal{Z})} \mathcal{F}^\cdot$$

in the categories $D_{\text{qcoh}}^b(\hat{\mathcal{D}}_{\mathcal{X}_{\mathcal{Y}/\mathcal{Y}}}^{(0)})$ and $D_{\text{qcoh}}^b(\hat{\mathcal{D}}_{\mathcal{X}_{\mathcal{Z}/\mathcal{Z}}}^{(0)})$.

Proof. As the following reasoning will work on all the morphisms $f_i : X_i \times Y_i \rightarrow X_i \times Z_i$, we will simplify the notations and consider a morphism $f : X \times Y \rightarrow X \times Z$.

It is known ([Sta22, Lemme 0096]) that given \mathcal{A} a sheaf of ring on $X \times Y$, \mathcal{B} a sheaf of ring on $X \times Z$ and a structure $\mathcal{B} \rightarrow f_* \mathcal{A}$ then the functor $\mathcal{A} \otimes_{f^{-1} \mathcal{B}} f^{-1} : \text{Mod}(\mathcal{B}) \rightarrow \text{Mod}(\mathcal{A})$ is left adjoint to the functor $f_* : \text{Mod}(\mathcal{A}) \rightarrow \text{Mod}(\mathcal{B})$.

Applying this result with $\mathcal{A} = \mathcal{D}_{X \times Y/Y}^{(0)}$ and $\mathcal{B} = \mathcal{D}_{X \times Z/Z}^{(0)}$ and remarking that $\mathcal{D}_{X \times Z \leftarrow X \times Y}^{(Y, Z)} \simeq \mathcal{D}_{X \times Y/Y}^{(0)}$ (modulo the inversion of the left and right structure), one can say that the functors

$$f_* (\mathcal{D}_{X \times Z \leftarrow X \times Y}^{(Y, Z)} \otimes_{\mathcal{D}_{X \times Y/Y}^{(0)}} \bullet) : \text{Mod}(\mathcal{D}_{X \times Y/Y}^{(0)}) \rightarrow \text{Mod}(\mathcal{D}_{X \times Z/Z}^{(0)})$$

and

$$\mathcal{D}_{X \times Y/Y}^{(0)} \otimes_{f^{-1} \mathcal{D}_{X \times Z/Z}^{(0)}} f^{-1} : \text{Mod}(\mathcal{D}_{X \times Z/Z}^{(0)}) \rightarrow \text{Mod}(\mathcal{D}_{X \times Y/Y}^{(0)})$$

are adjoints. As the first one is left exact ($\mathcal{D}_{X \times Z \leftarrow X \times Y}^{(Y, Z)}$ is $\mathcal{D}_{X \times Y/Y}^{(0)}$ -flat) and the second one is right exact, these functors have derived functors, which are still adjoints. These derived functors are $f_+^{(Y, Z)}$ and $f^{!(Y, Z)}$ respectively.

So, for all i , we have proved the adjunction between $f_i^{!(Y_i, Z_i)}$ and $f_{i+}^{(Y_i, Z_i)}$. For the one between $f^{!(\mathcal{Y}, \mathcal{Z})}$ and $f_+^{(\mathcal{Y}, \mathcal{Z})}$, it is sufficient to apply the inverse limit on the morphisms modulo π^i , as these morphisms are compatible to the reductions modulo π^j . □

Remark. *The adjunction formulas for relative $\hat{\mathcal{D}}^{(0)}$ -modules are not the same as the ones for non-relative $\hat{\mathcal{D}}^{(0)}$ -modules, however, they are the same as the ones for \mathcal{O} -modules. Hence, the following proofs for the results on relative $\hat{\mathcal{D}}^{(0)}$ -modules will be closer to the ones of the results for \mathcal{O} -modules than the ones of the results for non-relative $\hat{\mathcal{D}}^{(0)}$ -modules.*

The projection formula will be an important tool to compute the composition of two Fourier-Mukai transforms. The following statement can be found in [Vir04], section II.4.

Property 1.13 (Projection formula). *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a proper morphism of smooth formal schemes, $\mathcal{E} \in D_{\text{qcoh}}^b(\hat{\mathcal{D}}_{\mathcal{X}/\mathcal{S}}^{(0)})$ and $\mathcal{F} \in D_{\text{qcoh}}^b(\hat{\mathcal{D}}_{\mathcal{Y}/\mathcal{S}}^{(0)})$.*

Then for all i there is a natural isomorphism

$$f_{i+}(f_i^b \mathcal{F}_i \otimes_{\mathcal{O}_{X_i}}^{\mathbb{L}} \mathcal{E}_i) \simeq \mathcal{F}_i \otimes_{\mathcal{O}_{Y_i}}^{\mathbb{L}} f_{i+} \mathcal{E}_i.$$

(Where the functor f_i^b represents the inverse image without the shift.)

This isomorphism is compatible to the reductions modulo π^j , such that it naturally induces an isomorphism

$$f_+(f^b \mathcal{F} \hat{\otimes}_{\mathcal{O}_{\mathcal{X}}}^{\mathbb{L}} \mathcal{E}) \simeq \mathcal{F} \hat{\otimes}_{\mathcal{O}_{\mathcal{Y}}}^{\mathbb{L}} f_+ \mathcal{E}.$$

The same result is true for relative $\hat{\mathcal{D}}^{(0)}$ -modules.

Property 1.14 (Relative projection formula). *Let $f : \mathcal{X}_{\mathcal{Y}} \rightarrow \mathcal{X}_{\mathcal{Z}}$ be a morphism of the form $\text{id}_{\mathcal{X}} \times g$, $\mathcal{E} \in D_{\text{qcoh}}^b(\hat{\mathcal{D}}_{\mathcal{X}_{\mathcal{Z}}/\mathcal{Z}}^{(0)})$ and $\mathcal{F} \in D_{\text{qcoh}}^b(\hat{\mathcal{D}}_{\mathcal{X}_{\mathcal{Y}}/\mathcal{Y}}^{(0)})$. Then*

$$f_{i+}^{(Y_i, Z_i)}(f_i^{!(Y_i, Z_i)} \mathcal{E}_i \otimes_{\mathcal{O}_{X_{Y_i}}}^{\mathbb{L}} \mathcal{F}_i) \simeq \mathcal{E}_i \otimes_{\mathcal{O}_{X_{Z_i}}}^{\mathbb{L}} f_{i+}^{(Y_i, Z_i)} \mathcal{F}_i.$$

Moreover,

$$f_+^{(\mathcal{Y}, \mathcal{Z})}(f^{!(\mathcal{Y}, \mathcal{Z})} \mathcal{E} \hat{\otimes}_{\mathcal{O}_{\mathcal{X}_{\mathcal{Y}}}}^{\mathbb{L}} \mathcal{F}) \simeq \mathcal{E} \hat{\otimes}_{\mathcal{O}_{\mathcal{X}_{\mathcal{Z}}}}^{\mathbb{L}} f_+^{(\mathcal{Y}, \mathcal{Z})} \mathcal{F}.$$

Proof. As explained earlier, as the adjunction formulas are the same for \mathcal{O} -modules and relative $\hat{\mathcal{D}}$ -modules over the $X_i \times Y_i$ and $X_i \times Z_i$, there exists a morphism of $\hat{\mathcal{D}}_{X_i \times Z_i / Z_i}$ -modules

$$\mathcal{E}_i \otimes_{\mathcal{O}_{X_{Z_i}}}^{\mathbb{L}} f_{i+}^{(Y_i, Z_i)} \mathcal{F}_i \rightarrow f_{i+}^{(Y_i, Z_i)}(f_i^{!(Y_i, Z_i)} \mathcal{E}_i \otimes_{\mathcal{O}_{X_{Y_i}}}^{\mathbb{L}} \mathcal{F}_i).$$

To prove that it is an isomorphism, it is sufficient to consider the underlying \mathcal{O} -modules, which are isomorphic as settled by the projection formula for \mathcal{O} -modules.

For the second isomorphism, as adjunction formulas are compatible with reductions modulo π^i , so are the first isomorphisms and one can take their inverse limit. \square

For the base change formula, it works the same way. First recall the non-relative result.

Property 1.15 (Base change). *Let \mathcal{X} , \mathcal{Y} and \mathcal{Z} be three smooth formal schemes, $f : \mathcal{X} \rightarrow \mathcal{Y}$ and $\pi_{\mathcal{X}} : \mathcal{X} \times \mathcal{Z} \rightarrow \mathcal{X}$ and $\pi_{\mathcal{Y}} : \mathcal{Y} \times \mathcal{Z} \rightarrow \mathcal{Y}$ the projections. Consider the cartesian diagram*

$$\begin{array}{ccc} \mathcal{X} \times \mathcal{Z} & \xrightarrow{\pi_{\mathcal{Y}}} & \mathcal{X} \\ f \times \text{id}_{\mathcal{Z}} \downarrow & & \downarrow f \\ \mathcal{Y} \times \mathcal{Z} & \xrightarrow{\pi_{\mathcal{X}}} & \mathcal{Y} \end{array}$$

There is an isomorphism of functors $\pi_{\mathcal{Y}}^! f_+ \simeq (f \times \text{id}_{\mathcal{Z}})_+ \pi_{\mathcal{X}}^!$.

There is also an isomorphism of relative functors $\pi_{\mathcal{Y}}^{!(\mathcal{Z})} f_+ \simeq (f \times \text{id}_{\mathcal{Z}})_{+/Z} \pi_{\mathcal{X}}^{!(\mathcal{Z})}$.

It is important to note that, as we are working on positive characteristic, the usual proof of this result (see [HTT08] e.g.) can't be applied directly as it uses Kashiwara's theorem, which is false in positive characteristic. For that reason, one has to consider only projections for the horizontal arrows, as the usual proof only uses Kashiwara's theorem for closed immersions.

Property 1.16 (Relative base change). *Let \mathcal{X} , \mathcal{Y}_1 , \mathcal{Y}_2 and \mathcal{Z} be four smooth formal schemes, $\pi_1 : \mathcal{X} \times \mathcal{Y}_1 \times \mathcal{Z} \rightarrow \mathcal{X} \times \mathcal{Y}_1$ et $\pi_2 : \mathcal{X} \times \mathcal{Y}_2 \times \mathcal{Z} \rightarrow \mathcal{X} \times \mathcal{Y}_2$ the projections and $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$. Consider $g = id_{\mathcal{X}} \times f$, $h = id_{\mathcal{X}} \times f \times id_{\mathcal{Z}}$ and the following cartesian diagram*

$$\begin{array}{ccc} \mathcal{X} \times \mathcal{Y}_1 \times \mathcal{Z} & \xrightarrow{\pi_1} & \mathcal{X} \times \mathcal{Y}_1 \\ \downarrow h & & \downarrow g \\ \mathcal{X} \times \mathcal{Y}_2 \times \mathcal{Z} & \xrightarrow{\pi_2} & \mathcal{X} \times \mathcal{Y}_2 \end{array}$$

Then there is an isomorphism of functors $\pi_{2/\mathcal{Y}_2}^! g_+^{(\mathcal{Y}_1, \mathcal{Y}_2)} \simeq h_+^{(\mathcal{Y}_1, \mathcal{Y}_2)} \pi_{1/\mathcal{Y}_1}^!$.

There is also an isomorphism of functors $\pi_{2/\mathcal{Y}_2}^! g_+^{(\mathcal{Y}_1, \mathcal{Y}_2)} \simeq h_+^{(\mathcal{Y}_1, \mathcal{Y}_2)} \pi_{1/\mathcal{Y}_1}^! \pi_{1/\mathcal{Z}}^!$.

The idea of the proof is the same as the one for the relative projection formula.

Note that, as Kashiwara's theorem isn't needed here, one can write this result for horizontal arrows being proper and not necessary projections. However, as we won't need more generality in the following this result is enough.

Another important "calculus rule" is that the relative functors are compatible with the non-relative ones. For the sake of clarity the following morphisms will be supposed to be projections, but these results are true for all morphisms as they are nothing more than elementary calculations.

Property 1.17. *Consider the following projections*

$$\begin{array}{ccc} \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} & \xrightarrow{\pi_{\mathcal{Y}\mathcal{Z}}} & \mathcal{Y} \times \mathcal{Z} \\ \downarrow \pi_{\mathcal{X}\mathcal{Z}} & & \downarrow f \\ \mathcal{X} \times \mathcal{Z} & \xrightarrow{g} & \mathcal{Z} \end{array}$$

Consider $\mathcal{E} \in D_{qcoh}^b(\mathcal{D}_{\mathcal{Z}/S})$ and $\mathcal{F} \in D_{qcoh}^b(\mathcal{D}_{\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}/\mathcal{Y}})$. The following isomorphisms hold:

$$\pi_{X_i Z_i / S_i}^{!(Y_i)} \circ g_{i/S_i}^b(\mathcal{E}_i) \simeq \pi_{Y_i Z_i / Y_i}^b \circ f_{i/S_i}^{!(Y_i)}(\mathcal{E}_i) \quad (\text{in the category } D_{qcoh}^b(\mathcal{D}_{X_i \times Y_i \times Z_i / Y_i}^{(0)})),$$

$$g_{i+/S_i} \circ \pi_{X_i Z_i + / S_i}^{(Y_i)}(\mathcal{F}_i) \simeq f_{i+/S_i}^{(Y_i)} \circ \pi_{Y_i Z_i + / Y_i}(\mathcal{F}_i) \quad (\text{in the category } D_{qcoh}^b(\mathcal{D}_{Z_i / S_i}^{(0)})),$$

$$\pi_{\mathcal{X}\mathcal{Z}/S}^{!(\mathcal{Y})} \circ g_{+/S}^b(\mathcal{E}) \simeq \pi_{\mathcal{Y}\mathcal{Z}/\mathcal{Y}}^b \circ f_{+/S}^{!(\mathcal{Y})}(\mathcal{E}) \quad (\text{in the category } D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}/\mathcal{Y}}^{(0)})),$$

$$g_{+/S} \circ \pi_{\mathcal{X}\mathcal{Z}+/S}^{(\mathcal{Y})}(\mathcal{F}) \simeq f_{+/S}^{(\mathcal{Y})} \circ \pi_{\mathcal{Y}\mathcal{Z}+/Y}(\mathcal{F}) \quad (\text{in the category } D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{Z}/S}^{(0)})).$$

The isomorphisms in this section may be troublesome to understand because of the large amount of subscripts and superscript, so keep in mind one thing: if you are writing functors and objects that make sense (especially in term of what are the bases whose sheaves are considered relative to), all the expected isomorphisms will hold. The only thing to be carefull of is that the horizontal arrows for the non-relative base change have to be projections, but as in the following we will work only with projections, there will be no problem.

1.3 Seesaw principle

One interesting application of relative $\hat{\mathcal{D}}^{(0)}$ -modules is that one can find a generalization of the seesaw principle (see [Mum85] corollary 5.6 for the usual version). Even though we won't use it in the rest of this article, it is an interesting result on its own.

Property 1.18 (Seesaw principle). *Let \mathcal{X} , \mathcal{Y} and \mathcal{Z} be three formal schemes, \mathcal{X} and \mathcal{Y} complete and \mathcal{L} a sheaf of $\hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Y} \times \mathcal{Z} / \mathcal{Y} \times \mathcal{Z}}^{(0)}$ -modules $\mathcal{O}_{\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}}$ -invertible.*

Consider $\pi_{\mathcal{X}\mathcal{Y}} : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \rightarrow \mathcal{X} \times \mathcal{Y}$ the projection, for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$,

$$\iota_{(x,y)} : \mathcal{Z} \simeq \{x\} \times \{y\} \times \mathcal{Z} \hookrightarrow \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$$

and for all $z \in \mathcal{Z}$,

$$\tilde{\iota}_z : \mathcal{X} \times \mathcal{Y} \simeq \mathcal{X} \times \mathcal{Y} \times \{z\} \hookrightarrow \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}.$$

- *If $\forall (x, y) \in \mathcal{X} \times \mathcal{Y}$, $\iota_{(x,y)}^* \mathcal{L} \simeq \mathcal{O}_{\mathcal{Z}}$ (as $\mathcal{O}_{\mathcal{Z}}$ -modules), then there exists a $\hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Y} / \mathcal{Y}}^{(0)}$ -module $\mathcal{O}_{\mathcal{X} \times \mathcal{Y}}$ -invertible \mathcal{M} such that $\mathcal{L} \simeq \pi_{\mathcal{X}\mathcal{Y}/\mathcal{Y}}^{!(\mathcal{Z})} \mathcal{M}$ (as $\hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Y} \times \mathcal{Z} / \mathcal{Y} \times \mathcal{Z}}^{(0)}$ -modules).*
- *If moreover there exists $z \in \mathcal{Z}$ such that $\tilde{\iota}_z^{!(\mathcal{Z})} \mathcal{L} \simeq \mathcal{O}_{\mathcal{X} \times \mathcal{Y}}$ (as $\hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Y} / \mathcal{Y}}^{(0)}$ -modules), then $\mathcal{L} \simeq \mathcal{O}_{\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}}$ (as $\hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Y} \times \mathcal{Z} / \mathcal{Y} \times \mathcal{Z}}^{(0)}$ -modules).*

Proof. In order to prove this \mathcal{D} -seesaw-principle, we will first prove it in the case when X , Y and Z are non-formal schemes. The main idea is to adjust the proof of [Mum85] in order to take into account the structure of \mathcal{D} -module of the sheaves.

The first hypothesis allows to apply the seesaw principle for \mathcal{O} -modules: there exists an invertible $\mathcal{O}_{X \times Y}$ -module \mathcal{M} such that $\mathcal{L} \simeq \pi_{XY}^* \mathcal{M}$. Looking in details to the proof of [Mum85] corollary 5.6, one sees that \mathcal{M} can be taken as $\pi_{XY*} \mathcal{L}$. The idea is then to find an appropriate structure of $\mathcal{D}_{X \times Y / Y}$ -module on $\pi_{XY*} \mathcal{L}$. Consider

$$\mathcal{M} = \pi_{XY*}(\mathcal{D}_{X \times Y \leftarrow X \times Y \times Z}^{(Y, Y \times Z)} \otimes_{\mathcal{D}_{X \times Y \times Z / Y \times Z}} \mathcal{L}).$$

It is important to note that none of the functors are derivated, hence \mathcal{M} is a sheaf (not a complex). Its structure of $\mathcal{D}_{X \times Y / Y}$ -module is defined by the structure of $\pi_{XY}^{-1} \mathcal{D}_{X \times Y \times Z / Y \times Z}$ -module of $\mathcal{D}_{X \times Y \leftarrow X \times Y \times Z}^{(Y, Y \times Z)}$ (the same way as it is for $\pi_{XY+Y}^{(Z)}$). Finally, as $\mathcal{O}_{X \times Y}$ -modules, $\mathcal{M} \simeq \pi_{XY*} \mathcal{L}$, so the seesaw principle for \mathcal{O} -modules states that \mathcal{M} is $\mathcal{O}_{X \times Y}$ -invertible and, as $\mathcal{O}_{X \times Y \times Z}$ -modules, $\pi_{XY}^* \mathcal{M} \simeq \mathcal{L}$.

However, there is an adjunction morphism $\pi_{XY/Y}^{!(Z)} \pi_{XY*}(\mathcal{D}_{X \times Y \leftarrow X \times Y \times Z}^{(Y, Y \times Z)} \otimes_{\mathcal{D}_{X \times Y \times Z / Y \times Z}} \mathcal{L}) \rightarrow \mathcal{L}$ in the category of $\mathcal{D}_{X \times Y \times Z / Y \times Z}$ -modules (it is the non-derivated one used in the proof of property 1.12, as $\mathcal{D}_{X \times Y \times Z / Y \times Z} \otimes_{\pi_{XY}^{-1} \mathcal{D}_{X \times Y / Y}} \pi_{XY}^{-1}$ is exact because π_{XY} is a projection). So there is a morphism of $\mathcal{D}_{X \times Y \times Z / Y \times Z}$ -modules $\pi_{XY/Y}^{!(Z)} \mathcal{M} \rightarrow \mathcal{L}$ which is an isomorphism of $\mathcal{O}_{X \times Y \times Z}$ -modules, so it is an isomorphism of $\mathcal{D}_{X \times Y \times Z / Y \times Z}$ -modules.

If moreover there exists $z \in Z$ such that $\tilde{\iota}_z^{!(Z)} \mathcal{L} \simeq \mathcal{O}_{X \times Y}$ as $\mathcal{D}_{X \times Y / Y}$ -modules, then as $\pi_{XY} \circ \tilde{\iota}_z = id_{Y \times Z}$

$$\mathcal{M} \simeq \tilde{\iota}_z^{!(Z)} \pi_{XY/Y}^{!(Z)} \mathcal{M} \simeq \tilde{\iota}_z^{!(Z)} \mathcal{L} \simeq \mathcal{O}_{X \times Y},$$

as $\mathcal{D}_{X \times Y / Y}$ -modules. Hence, $\mathcal{L} \simeq \mathcal{O}_{X \times Y \times Z}$ as $\mathcal{D}_{X \times Y \times Z / Y \times Z}$ -modules.

The seesaw principle is then proved for all the X_i , Y_i and Z_i , reductions modulo π^i of \mathcal{X} , \mathcal{Y} and \mathcal{Z} . In order to prove the seesaw principle for formal schemes, it is only necessary to remark that the isomorphisms expected are obtained via adjunction formulas, and hence are compatible with reductions modulo π^j . Considering the $\hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Y} / \mathcal{Y}}^{(0)}$ -module

$$\mathcal{M} = \varprojlim \mathcal{M}_i = \varprojlim \pi_{X_i Y_i*}(\mathcal{D}_{X_i \times Y_i \leftarrow X_i \times Y_i \times Z_i}^{(Y_i, Y_i \times Z_i)} \otimes_{\mathcal{D}_{X_i \times Y_i \times Z_i / Y_i \times Z_i}^{(0)}} \mathcal{L}_i),$$

one has the expected isomorphism of $\hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Y} \times \mathcal{Z} / \mathcal{Y} \times \mathcal{Z}}^{(0)}$ -modules $\mathcal{L} \simeq \pi_{\mathcal{X}\mathcal{Y}/\mathcal{Y}}^{!(\mathcal{Z})} \mathcal{M}$. Moreover, as the \mathcal{M}_i are \mathcal{O} -invertible, so is \mathcal{M} .

The second part of the seesaw principle is proved the same way as in the non-formal case. \square

Remark. *This result is written in its more general case but can be greatly simplified by considering $\mathcal{Y} = \mathcal{S}$. Moreover, considering $\mathcal{X} = \mathcal{S}$ one can regain the seesaw principle for \mathcal{O} -modules.*

2 Fourier-Mukai transform for $\hat{\mathcal{D}}^{(0)}$ -modules

In this section, we will first construct $\hat{\mathcal{D}}^{(0)}$ -Fourier-Mukai transforms of a given kernel on formal schemes (non necessary abelian) then consider the $\hat{\mathcal{D}}^{(0)}$ -Fourier-Mukai transform of Poincaré kernel on formal abelian schemes.

As in the previous section, let V be a discrete valuation ring with uniformizer π , $S = \text{Spec}(V)$, for all i $V_i = V/\pi^i V$ and $S_i = \text{Spec} V_i$ and $\mathcal{S} = \text{Spf} S$. The non-necessary-abelian formal schemes will be denoted \mathcal{X} , \mathcal{Y} and \mathcal{Z} and the formal abelian scheme will be denoted \mathcal{A} .

2.1 Fourier-Mukai transforms of a given kernel

This section is an adaptation of what is done for \mathcal{O} -modules over classical varieties in [Huy06]. As all the results needed have been previously settled, this section will be close to the section 5.1.

Let \mathcal{X} , \mathcal{Y} and \mathcal{Z} be three quasi-compact smooth formal schemes.

Given \mathcal{X} , \mathcal{Y} and a kernel $\mathcal{P} \in D_{qcoh}^b(\mathcal{O}_{\mathcal{X} \times \mathcal{Y}})$ one can define up to four different Fourier-Mukai transforms depending on the additional structure of $\hat{\mathcal{D}}^{(0)}$ -module of \mathcal{P} we consider.

Definition 2.1. *Let $p : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{X}$ and $q : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Y}$ be the projections, $\mathcal{P} \in D_{qcoh}^b(\mathcal{O}_{\mathcal{X} \times \mathcal{Y}})$, $\mathcal{Q} \in D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Y}/\mathcal{S}}^{(0)})$ and $\mathcal{R} \in D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Y}/\mathcal{Y}}^{(0)})$.*

- *The Fourier-Mukai transform of kernel \mathcal{P} is the functor $\Phi_{\mathcal{P}} : D_{qcoh}^b(\mathcal{O}_{\mathcal{X}}) \rightarrow D_{qcoh}^b(\mathcal{O}_{\mathcal{Y}})$ defined as*

$$\Phi_{\mathcal{P}}(\mathcal{E}^\bullet) = Rq_*(\mathcal{P} \hat{\otimes}_{\mathcal{O}_{\mathcal{X} \times \mathcal{Y}}}^{\mathbb{L}} p^* \mathcal{E}^\bullet),$$

where the functor p^* is defined as $p^* \mathcal{E}^\bullet = R\varprojlim p_i^* \mathcal{E}_i^\bullet$.

- *The Fourier-Mukai transform from \mathcal{X} to \mathcal{Y} of kernel \mathcal{Q} is the functor $\Phi_{\mathcal{Q}}^{\hat{\mathcal{D}}^{(0)}} : D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{X}/\mathcal{S}}^{(0)}) \rightarrow D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{Y}/\mathcal{S}}^{(0)})$ defined as*

$$\Phi_{\mathcal{Q}}^{\hat{\mathcal{D}}^{(0)}}(\mathcal{E}^\bullet) = q_+(\mathcal{Q} \hat{\otimes}_{\mathcal{O}_{\mathcal{X} \times \mathcal{Y}}}^{\mathbb{L}} p^\flat \mathcal{E}^\bullet),$$

where the functor p^\flat is the inverse image without the shift: $p^\flat \mathcal{E} = p^\flat \mathcal{E}[dim(\mathcal{X}) - dim(\mathcal{Y})]$.

- *The Fourier-Mukai transform from \mathcal{X} to \mathcal{Y} of kernel \mathcal{R} (relative to \mathcal{Y}) is the functor $\Phi_{\mathcal{R}}^{\mathcal{X} \rightarrow \mathcal{Y}} : D_{qcoh}^b(\mathcal{D}_{\mathcal{X}/\mathcal{S}}) \rightarrow D_{qcoh}^b(\mathcal{O}_{\mathcal{Y}})$ defined as*

$$\Phi_{\mathcal{R}}^{\mathcal{X} \rightarrow \mathcal{Y}}(\mathcal{E}^\bullet) = q_{+/Y}(\mathcal{R} \hat{\otimes}_{\mathcal{O}_{\mathcal{X} \times \mathcal{Y}}}^{\mathbb{L}} p^{!(\mathcal{Y})} \mathcal{E}^\bullet).$$

- *The Fourier-Mukai transform from \mathcal{Y} to \mathcal{X} of kernel \mathcal{R} (relative to \mathcal{Y}) is the functor $\Phi_{\mathcal{R}}^{\mathcal{X} \leftarrow \mathcal{Y}} : D_{qcoh}^b(\mathcal{O}_{\mathcal{Y}}) \rightarrow D_{qcoh}^b(\mathcal{D}_{\mathcal{X}/\mathcal{S}})$ defined as*

$$\Phi_{\mathcal{R}}^{\mathcal{X} \leftarrow \mathcal{Y}}(\mathcal{E}^\bullet) = p_+^{(\mathcal{Y})}(\mathcal{Q} \hat{\otimes}_{\mathcal{O}_{\mathcal{X} \times \mathcal{Y}}}^{\mathbb{L}} q_{/\mathcal{Y}}^\flat \mathcal{E}^\bullet).$$

As expected, the composition of two Fourier-Mukai transforms is still a Fourier-Mukai transform. However, given \mathcal{X} , \mathcal{Y} and \mathcal{Z} three formal schemes, there are eight possible compositions of two Fourier-Mukai transforms, depending on which formal scheme the derivations are relative to.

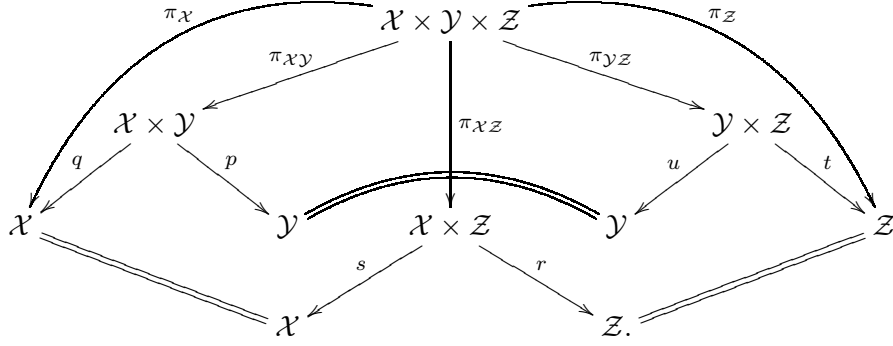
Property 2.2. *Let \mathcal{X} , \mathcal{Y} and \mathcal{Z} be three smooth quasi-compact formal schemes and $\pi_{\mathcal{X}\mathcal{Y}}$, $\pi_{\mathcal{X}\mathcal{Z}}$ and $\pi_{\mathcal{Y}\mathcal{Z}}$ the projections from $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ to $\mathcal{X} \times \mathcal{Y}$, $\mathcal{X} \times \mathcal{Z}$ and $\mathcal{Y} \times \mathcal{Z}$ respectively.*

Proof. The proof is only a succession of base changes and projection formulas and the only difference between the proofs of the eight properties are which version of base change and projection formula is used. Hence, we will only prove the seventh one as an exemple.

Asume then $\mathcal{P} \in D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Y}/\mathcal{Y}}^{(0)})$, $\mathcal{Q} \in D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{Y} \times \mathcal{Z}/\mathcal{Y}}^{(0)})$ and

$$\mathcal{R} = \pi_{\mathcal{X}\mathcal{Z}+}^{(\mathcal{Y})}(\pi_{\mathcal{X}\mathcal{Y}/\mathcal{Y}}^b \mathcal{P} \hat{\otimes}_{\mathcal{O}_{\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}}}^{\mathbb{L}} \pi_{\mathcal{Y}\mathcal{Z}/\mathcal{Y}}^b \mathcal{Q}) \in D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{X} \times \mathcal{Z}}^{(0)}).$$

Using the following notations



one has the following isomorphisms

$$\begin{aligned} \Phi_{\mathcal{R}}^{\hat{\mathcal{D}}^{(0)}}(\mathcal{E}^\cdot) &= r_+(\mathcal{R} \hat{\otimes} s^b \mathcal{E}^\cdot) \\ &= r_+(\pi_{\mathcal{X}\mathcal{Z}+}^{(\mathcal{Y})}(\pi_{\mathcal{X}\mathcal{Y}/\mathcal{Y}}^b \mathcal{P} \hat{\otimes} \pi_{\mathcal{Y}\mathcal{Z}/\mathcal{Y}}^b \mathcal{Q}) \hat{\otimes} s^b \mathcal{E}^\cdot) \\ &\simeq r_+(\pi_{\mathcal{X}\mathcal{Z}+}^{(\mathcal{Y})}(\pi_{\mathcal{X}\mathcal{Y}/\mathcal{Y}}^b \mathcal{P} \hat{\otimes} \pi_{\mathcal{Y}\mathcal{Z}/\mathcal{Y}}^b \mathcal{Q} \hat{\otimes} \pi_{\mathcal{X}\mathcal{Z}}^{!(\mathcal{Y})} s^b \mathcal{E}^\cdot)) && \text{(Projection ; Property 1.14)} \\ &\simeq r_+\pi_{\mathcal{X}\mathcal{Z}+}^{(\mathcal{Y})}(\pi_{\mathcal{X}\mathcal{Y}/\mathcal{Y}}^b \mathcal{P} \hat{\otimes} \pi_{\mathcal{Y}\mathcal{Z}/\mathcal{Y}}^b \mathcal{Q} \hat{\otimes} \pi_{\mathcal{X}\mathcal{Y}/\mathcal{Y}}^b q^{!(\mathcal{Y})} \mathcal{E}^\cdot) && \text{(Property 1.17)} \\ &\simeq r_+\pi_{\mathcal{X}\mathcal{Z}+}^{(\mathcal{Y})}(\pi_{\mathcal{Y}\mathcal{Z}/\mathcal{Y}}^b \mathcal{Q} \hat{\otimes} \pi_{\mathcal{X}\mathcal{Y}/\mathcal{Y}}^b (\mathcal{P} \hat{\otimes} q^{!(\mathcal{Y})} \mathcal{E}^\cdot)) \\ &\simeq t_+^{(\mathcal{Y})} \pi_{\mathcal{Y}\mathcal{Z}+\mathcal{Y}}(\pi_{\mathcal{Y}\mathcal{Z}/\mathcal{Y}}^b \mathcal{Q} \hat{\otimes} \pi_{\mathcal{X}\mathcal{Y}/\mathcal{Y}}^b (\mathcal{P} \hat{\otimes} q^{!(\mathcal{Y})} \mathcal{E}^\cdot)) && \text{(Property 1.17)} \\ &\simeq t_+^{(\mathcal{Y})}(\mathcal{Q} \hat{\otimes} \pi_{\mathcal{Y}\mathcal{Z}+\mathcal{Y}} \pi_{\mathcal{X}\mathcal{Y}/\mathcal{Y}}^b (\mathcal{P} \hat{\otimes} q^{!(\mathcal{Y})} \mathcal{E}^\cdot)) && \text{(Projection ; Property 1.13)} \\ &\simeq t_+^{(\mathcal{Y})}(\mathcal{Q} \hat{\otimes} u^b p_{+\mathcal{Y}}(\mathcal{P} \hat{\otimes} q^{!(\mathcal{Y})} \mathcal{E}^\cdot)) && \text{(Base change ; Property 1.15)} \\ &= \Phi_{\mathcal{Q}}^{\mathcal{Z}+\mathcal{Y}} \circ \Phi_{\mathcal{P}}^{\mathcal{X} \rightarrow \mathcal{Y}}(\mathcal{E}^\cdot). \end{aligned}$$

□

The most important thing to recall is that the composition of two Fourier-Mukai transforms of kernel \mathcal{P} and \mathcal{Q} is a Fourier-Mukai transform of kernel \mathcal{R} obtained by pulling back \mathcal{P} and \mathcal{Q} on $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$, taking their tensor product, and finally pushing it forward on $\mathcal{X} \times \mathcal{Z}$, while making sure to use the good functors regarding the structures of $\hat{\mathcal{D}}^{(0)}$ -modules of \mathcal{P} and \mathcal{Q} .

2.2 Fourier-Mukai transform of Poincaré kernel

From now on, let \mathcal{A} be a formal abelian scheme over \mathcal{S} of dimension d and denote by A_i its reduction modulo π^i . Let also m (resp. m_i), ϵ (resp. ϵ_i) and $\langle -1 \rangle$ (resp. $\langle -1 \rangle_i$) be respectively the multiplication, neutral element and inverse morphisms of \mathcal{A} (resp. A_i).

As the goal of the Fourier-Mukai transform on $\hat{\mathcal{D}}^{(0)}$ -modules is to take into account the structure of $\hat{\mathcal{D}}^{(0)}$ -module of the sheaves over \mathcal{A} , the usual definition of dual abelian scheme as the one representing the Picard functor can't be a good one. The idea is to build another functor, Pic^{\natural} ,

which will take into account the structure of $\widehat{\mathcal{D}}^{(0)}$ -module of the sheaves over \mathcal{A} . Namely, $Pic^{\natural}(\mathcal{A}/\mathcal{S})$ will be the group of rigid extensions of \mathcal{A} .

This construction and the one of the Poincaré sheaf are very similar to what is done in the case of non-arithmetical \mathcal{D} -modules in [Lau96], as it is a summary of the results already proved in [MM74] (chapter I).

Definition 2.3. Let $p_1, p_2 : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ be the canonical projections. $Pic^{\natural}(\mathcal{A}/\mathcal{S})$ is defined as the group of sheaves of $\widehat{\mathcal{D}}_{\mathcal{A}/\mathcal{S}}^{(0)}$ -modules $\mathcal{O}_{\mathcal{A}}$ -invertible satisfying the theorem of the square, i.e.

$$m^{\flat} \mathcal{L} \simeq p_1^{\flat} \mathcal{L} \otimes p_2^{\flat} \mathcal{L},$$

where m^{\flat} is the inverse image for $\widehat{\mathcal{D}}_{\mathcal{A}/\mathcal{S}}^{(0)}$ -modules without the shift.

Property 2.4. The functor $Pic^{\natural}(\mathcal{A} \times \bullet/\bullet) : \{\text{Noetherian } \mathcal{S}\text{-schemes}\} \rightarrow \{\text{groups}\}$ is representable by a smooth group scheme over \mathcal{S} of dimension $2d$, denoted \mathcal{A}^{\natural} .

For the proof, see [MM74], chapter I. It is important to note that the proof uses the fact that we are looking at sheaves with integrable connection, so that one has to consider $\widehat{\mathcal{D}}^{(0)}$ -modules and not $\mathcal{D}^{(m)}$ -modules with $m > 0$. This is the main reason why the following results may be harder to prove, if even true, in a more general case.

Definition 2.5. The element of $Pic^{\natural}(\mathcal{A} \times \mathcal{A}^{\natural}/\mathcal{A}^{\natural})$ associated to the identity $id_{\mathcal{A}^{\natural}} \in Hom(\mathcal{A}^{\natural}, \mathcal{A}^{\natural})$ is called the Poincaré sheaf and denoted by \mathcal{P} .

Remark. As it is an element of $Pic^{\natural}(\mathcal{A} \times \mathcal{A}^{\natural}/\mathcal{A}^{\natural})$, \mathcal{P} has a structure of $\widehat{\mathcal{D}}_{\mathcal{A} \times \mathcal{A}^{\natural}/\mathcal{A}^{\natural}}^{(0)}$ -module, hence a structure of relative $\widehat{\mathcal{D}}^{(0)}$ -module. That fact implies that $\mathcal{A} \times \mathcal{A}^{\natural}$ and \mathcal{A}^{\natural} have to be considered as formal schemes over \mathcal{A}^{\natural} , so that the Fourier-Mukai transform applied to a $\widehat{\mathcal{D}}_{\mathcal{A}}^{(0)}$ -module can only give rise to a $\mathcal{O}_{\mathcal{A}^{\natural}}$ -module.

The previous definitions can also be written on the reductions modulo π^i , leading to $\mathcal{D}^{(0)}$ -dual group schemes \mathcal{A}_i^{\natural} and Poincaré sheaves \mathcal{P}_i . As these definitions are compatible with reductions modulo π^i , the following result is immediate.

Property 2.6. There are canonical isomorphisms $\mathcal{A}^{\natural} \simeq \varinjlim \mathcal{A}_i^{\natural}$ and $\mathcal{P} \simeq \varprojlim \mathcal{P}_i$.

One can use the representability of $Pic^{\natural}(\mathcal{A} \times \bullet/\bullet)$ to find some useful results, such as the following.

Property 2.7. Given $\mathcal{L} \in Pic^{\natural}(\mathcal{A} \times \mathcal{T}/\mathcal{T})$, there exists only one morphism $f : \mathcal{T} \rightarrow \mathcal{A}^{\natural}$ such that

$$\mathcal{L} \simeq (id_{\mathcal{A}} \times f)^{\natural}(\mathcal{T}, \mathcal{A}^{\natural}) \mathcal{P}.$$

In particular, the points of \mathcal{A}^{\natural} are in bijection with the $\widehat{\mathcal{D}}_{\mathcal{A}/\mathcal{S}}^{(0)}$ -modules $\mathcal{O}_{\mathcal{A}}$ -invertible verifying the theorem of the square.

Moreover, $(id_{\mathcal{A}} \times \epsilon^{\natural})^{\natural}(\mathcal{A}^{\natural}) \mathcal{P} \simeq \mathcal{O}_{\mathcal{A}}$.

There is also a kind of compatibility result with the dual abelian formal scheme \mathcal{A}^{\vee} .

Property 2.8. Define $i : H^0(\mathcal{A}, \Omega_{\mathcal{A}/\mathcal{S}}^1) \rightarrow Pic^{\natural}(\mathcal{A}/\mathcal{S})$ and $p : Pic^{\natural}(\mathcal{A}/\mathcal{S}) \rightarrow Pic^0(\mathcal{A}/\mathcal{S})$ by $i(\omega) = \mathcal{O}_{\mathcal{A}}$ with the integrable connection $d + \omega$ and $p(\mathcal{L}) = \mathcal{L}$ seen as a $\mathcal{O}_{\mathcal{A}}$ -module. Then there is an exact sequence of abelian schemes:

$$E(\mathcal{A}/\mathcal{S}) = (0 \rightarrow H^0(\mathcal{A}, \Omega_{\mathcal{A}/\mathcal{S}}^1) \rightarrow Pic^{\natural}(\mathcal{A}/\mathcal{S}) \rightarrow Pic^0(\mathcal{A}/\mathcal{S})).$$

Moreover, if \mathcal{S} is affine, the morphism p is surjective.

One then has a functor $\mathcal{T} \mapsto E(\mathcal{A} \times \mathcal{T}/\mathcal{T}) : \{\text{Locally noetherian } \mathcal{S}\text{-schemes}\} \rightarrow \{\text{Exact sequences of abelian schemes}\}$. This functor is representable by an exact sequence of abelian schemes over \mathcal{S} :

$$0 \rightarrow \mathbb{V}(\epsilon^* \mathcal{T}_{\mathcal{A}/\mathcal{S}}) \rightarrow \mathcal{A}^\natural \rightarrow \mathcal{A}^\vee \rightarrow 0,$$

with $\mathbb{V}(\mathcal{E}) = \text{Spec}(\text{Sym}(\mathcal{E}))$. Denote by θ the morphism from \mathcal{A}^\natural to \mathcal{A}^\vee .

Proof. This is proved on the reductions modulo π^i , using results of sections 2.6 and 3.2.3 of chapter I of [MM74]. Then simply take the inverse limit. □

Property 2.9. *There is an isomorphism of $\mathcal{O}_{\mathcal{A} \times \mathcal{A}^\natural}$ -modules*

$$\mathcal{P} \simeq (id_{\mathcal{A}} \times \theta)^* \mathcal{P}^\vee,$$

where \mathcal{P}^\vee is the Poincaré sheaf on $\mathcal{A} \times \mathcal{A}^\vee$.

Proof. The representability of $\mathcal{T} \mapsto (Pic^\natural(\mathcal{A} \times \mathcal{T}/\mathcal{T}) \rightarrow Pic^0(\mathcal{A} \times \mathcal{T}/\mathcal{T}))$ leads to the following commutative diagram

$$\begin{array}{ccc} Pic^\natural(\mathcal{A} \times \mathcal{A}^\natural/\mathcal{A}^\natural) & \xrightarrow{\theta_{\mathcal{A}^\natural}} & Pic^0(\mathcal{A} \times \mathcal{A}^\natural/\mathcal{A}^\natural) \\ \uparrow \Phi_{\mathcal{A}^\natural}^\natural & & \uparrow \Phi_{\mathcal{A}^\natural}^0 \\ Hom(\mathcal{A}^\natural, \mathcal{A}^\natural) & \xrightarrow{\theta_{\bullet}} & Hom(\mathcal{A}^\natural, \mathcal{A}^\vee) \end{array}$$

where $\theta_{\mathcal{A}^\natural}$ is the forgetting of the structure of $\mathcal{D}_{\mathcal{A} \times \mathcal{A}^\natural/\mathcal{A}^\natural}$ -module and $\Phi_{\mathcal{A}^\natural}^\natural$ and $\Phi_{\mathcal{A}^\natural}^0$ are the representation isomorphisms. Evaluating in the element $id_{\mathcal{A}^\natural} \in Hom(\mathcal{A}^\natural, \mathcal{A}^\natural)$, one obtains

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\quad} & \mathcal{P} \\ \uparrow & & \uparrow \\ id_{\mathcal{A}^\natural} & \xrightarrow{\quad} & \theta \end{array}$$

where the left \mathcal{P} is a \mathcal{D} -module and the right one is a \mathcal{O} -module.

On the other hand, as $\mathcal{P}^\vee = \Phi_{\mathcal{A}^\vee}^0(id_{\mathcal{A}^\vee})$, Yoneda's lemma states that

$$\Phi_{\mathcal{A}^\natural}^0(\theta) = \left(Pic^0(\mathcal{A} \times \bullet/\bullet)(\theta) \right) (\mathcal{P}^\vee) = (id_{\mathcal{A}} \times \theta)^* \mathcal{P}^\vee.$$

Hence, $\mathcal{P} = \Phi_{\mathcal{A}^\natural}^0(\theta) = (id_{\mathcal{A}} \times \theta)^* \mathcal{P}^\vee$, up to an isomorphism and as \mathcal{O} -modules. □

Property 2.10. *The morphism $\theta : \mathcal{A}^\natural \rightarrow \mathcal{A}^\vee$ is a torsor, hence affine.*

Moreover, the sheaf $\theta_* \mathcal{O}_{\mathcal{A}^\natural}$ has an exhaustive filtration by locally free sub- $\mathcal{O}_{\mathcal{A}^\vee}$ -modules of finite rank which makes it a filtrated $\mathcal{O}_{\mathcal{A}^\vee}$ -algebra, with an isomorphism of graduated $\mathcal{O}_{\mathcal{A}^\vee}$ -algebras

$$\varphi_\bullet : gr_\bullet(\theta_* \mathcal{O}_{\mathcal{A}^\natural}) \rightarrow \pi^{\vee*} \text{Sym}_\bullet^{\mathcal{O}_{\mathcal{S}}}(\epsilon^* \mathcal{T}_{\mathcal{A}/\mathcal{S}}).$$

Proof. It is proved in the same way as in [Lau96], part 2.3, for the reductions modulo π^i and one can then take the inverse limit of these results modulo π^i . □

The Fourier-Mukai transform on \mathcal{A} for $\hat{\mathcal{D}}^{(0)}$ -modules is defined as the Fourier-Mukai transform with kernel the Poincaré (relative) sheaf.

Definition 2.11. Denote by $p : \mathcal{A} \times \mathcal{A}^\natural \rightarrow \mathcal{A}$ and $p^\natural : \mathcal{A} \times \mathcal{A}^\natural \rightarrow \mathcal{A}^\natural$ the canonical projections.

The Fourier-Mukai transform on \mathcal{A} for $\hat{\mathcal{D}}^{(0)}$ -modules is defined as the functor $\mathcal{F} : D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{A}/S}^{(0)}) \rightarrow D_{qcoh}^b(\mathcal{O}_{\mathcal{A}^\natural})$ given by

$$\mathcal{F}(\mathcal{E}^\cdot) = \Phi_{\mathcal{P}}^{\mathcal{A} \rightarrow \mathcal{A}^\natural}(\mathcal{E}^\cdot) = p_{+/A^\natural}^\natural(\mathcal{P} \otimes_{\mathcal{A} \times \mathcal{A}^\natural}^{\mathbb{L}} p^{!(\mathcal{A}^\natural)} \mathcal{E}^\cdot).$$

The \mathcal{D} -dual Fourier-Mukai transform is then defined as the functor $\mathcal{F}^\natural : D_{qcoh}^b(\mathcal{O}_{\mathcal{A}^\natural}) \rightarrow D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{A}/S}^{(0)})$ given by

$$\mathcal{F}^\natural(\mathcal{E}^\cdot) = \Phi_{\mathcal{P}}^{\mathcal{A} \leftarrow \mathcal{A}^\natural}(\mathcal{E}^\cdot) = p_+^{(\mathcal{A}^\natural)}(\mathcal{P} \otimes_{\mathcal{A} \times \mathcal{A}^\natural}^{\mathbb{L}} p_{\mathcal{A}^\natural}^{\natural\flat} \mathcal{E}^\cdot).$$

2.3 Essential surjectivity of the Fourier-Mukai transform for $\hat{\mathcal{D}}^{(0)}$ -modules

As stated in the introduction, the goal of this Fourier-Mukai transform is to be involutive and hence to give an equivalence of categories between $D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{A}/S}^{(0)})$ and $D_{qcoh}^b(\mathcal{O}_{\mathcal{A}^\natural})$. However, the proof of the involutivity in the non-arithmetic case, which can be found in [Lau96] for instance, will not fully apply in the arithmetic case, mostly because of the lack of Kashiwara's theorem for $\hat{\mathcal{D}}^{(0)}$ -modules.

Still, it is good to have an idea of the proof to know where we are going, so let us recall here the main steps. First, it is known that the composition of two Fourier-Mukai transform is a Fourier-Mukai transform, hence $\mathcal{F}^\natural \circ \mathcal{F}$ (or $\mathcal{F} \circ \mathcal{F}^\natural$) is a Fourier-Mukai transform of kernel \mathcal{R} from the category $D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{A}/S}^{(0)})$ (or $D_{qcoh}^b(\mathcal{O}_{\mathcal{A}^\natural})$) into itself. The goal is to prove that the Fourier-Mukai transform of kernel \mathcal{R} is an isomorphism, and hence understand the kernel itself. After some transformations using the base change theorem, the seesaw principle and some basic properties of the elements of $Pic^\natural(A \times \mathcal{A}^\natural / \mathcal{A}^\natural)$, \mathcal{R} is proved to be isomorphic to the pullback by the multiplication of $p_+^{(\mathcal{A}^\natural)} \mathcal{P}$ (or $p_{+/A^\natural}^\natural \mathcal{P}$). A very difficult lemma states that this sheaf is isomorphic to the push-forward of \mathcal{O}_S by the neutral element, modulo a shift. Once this lemma proven, the end of the proof is straightforward using base change and projection formulas.

So, the key argument is the understanding of the push-forward of \mathcal{P} by the two projections. The main idea will be to adapt the proofs in [Lau96] to the arithmetic case. If one of the two push-forward is computable using some more arguments than in the non-arithmetic case, the other one is more complicated to compute, hence the following lemma.

Lemma 2.12. $p_{+/A^\natural}^\natural \mathcal{P} \simeq \epsilon_* \mathcal{O}_S[-d] \in D_{qcoh}^b(\mathcal{O}_{\mathcal{A}^\natural})$.

Remark. The expected second isomorphism should be $p_+^{(\mathcal{A}^\natural)} \mathcal{P} \simeq \epsilon_+ \mathcal{O}_S[-d] \in D_{qcoh}^b(\hat{\mathcal{D}}_{\mathcal{A}/S}^{(0)})$, however it can't be proved using the same method as in [Lau96] (or, at least, not without a lot of work). The main reason is that the proof of this isomorphism in the non-arithmetic case uses the Kashiwara's equivalence theorem, which is known to be false for $\hat{\mathcal{D}}^{(0)}$ -modules. However, as there exists a version of this result for $\mathcal{D}_{\mathbb{Q}}^\dagger$ -modules (see e.g. [Car09]), one could expect an isomorphism of the form

$$\mathcal{D}_{\mathcal{A}/S, \mathbb{Q}}^\dagger \otimes_{\hat{\mathcal{D}}_{\mathcal{A}/S, \mathbb{Q}}^{(0)}} p_+^{(\mathcal{A}^\natural)} \mathcal{P}_{\mathbb{Q}} \simeq \epsilon_+ \mathcal{O}_{S, \mathbb{Q}}[-d] \in D_{qcoh}^b(\mathcal{D}_{\mathcal{A}/S, \mathbb{Q}}^\dagger).$$

But even if this isomorphism were true it will not be of use for the proof of the involutivity of the Fourier-Mukai transform, hence we will not try to prove it.

Proof. In order to prove the isomorphism, the first thing to do is to find a morphism of $\mathcal{O}_{\mathcal{A}^\natural}$ -modules between $p_{+/A^\natural}^\natural \mathcal{P}$ and $\epsilon_* \mathcal{O}_S[-d]$. Note that, as \mathcal{P} is invertible, hence coherent, and p^\natural and ϵ^\natural are proper, the sheaves $p_{+/A^\natural}^\natural \mathcal{P}$ and $\epsilon_* \mathcal{O}_S$ are coherent.

The goal is to construct the morphism $p_{+/\mathcal{A}^\natural}^\natural \mathcal{P} \rightarrow \epsilon^\natural_* \mathcal{O}_S[-d]$ by adjunction from a morphism $\epsilon^\natural_* p_{+/\mathcal{A}^\natural}^\natural \mathcal{P} \rightarrow \mathcal{O}_S[-d]$ (recall that the complex of \mathcal{O}_S -modules $\epsilon^\natural_* \mathcal{E}$ is defined as $R\varprojlim_i \epsilon_i^{\natural*} \mathcal{E}_i$). Consider the following base change diagram

$$\begin{array}{ccc} \mathcal{A}^c & \xrightarrow{id_{\mathcal{A}} \times \epsilon^\natural} & \mathcal{A} \times \mathcal{A}^\natural \\ \pi \downarrow & & \downarrow p^\natural \\ \mathcal{S}^c & \xrightarrow{\epsilon^\natural} & \mathcal{A}^\natural. \end{array}$$

Using the base change formula for \mathcal{O} -modules in the previous diagram, one finds the following isomorphisms.

$$\epsilon^\natural_* p_{+/\mathcal{A}^\natural}^\natural \mathcal{P} = \epsilon^\natural_* R p_{*}^\natural (\hat{\mathcal{D}}_{\mathcal{A}^\natural \leftarrow \mathcal{A} \times \mathcal{A}^\natural / \mathcal{A}^\natural}^{(0)} \otimes_{\hat{\mathcal{D}}_{\mathcal{A} \times \mathcal{A}^\natural / \mathcal{A}^\natural}^{(0)}} \mathcal{P}) \simeq R \pi_* (id_{\mathcal{A}} \times \epsilon^\natural)^* (\hat{\mathcal{D}}_{\mathcal{A}^\natural \leftarrow \mathcal{A} \times \mathcal{A}^\natural / \mathcal{A}^\natural}^{(0)} \otimes_{\hat{\mathcal{D}}_{\mathcal{A} \times \mathcal{A}^\natural / \mathcal{A}^\natural}^{(0)}} \mathcal{P}).$$

Denote by $DR_{\mathcal{A} \times \mathcal{A}^\natural / \mathcal{A}^\natural}(\mathcal{P})$ the de Rham complex of \mathcal{P} (between degrees 0 and d):

$$DR_{\mathcal{A} \times \mathcal{A}^\natural / \mathcal{A}^\natural}(\mathcal{P}) : 0 \rightarrow \mathcal{P} \rightarrow \Omega_{\mathcal{A} \times \mathcal{A}^\natural / \mathcal{A}^\natural}^1 \otimes \mathcal{P} \rightarrow \dots \rightarrow \Omega_{\mathcal{A} \times \mathcal{A}^\natural / \mathcal{A}^\natural}^d \otimes \mathcal{P} \rightarrow 0$$

The transfer bimodule $\hat{\mathcal{D}}_{\mathcal{A}^\natural \leftarrow \mathcal{A} \times \mathcal{A}^\natural / \mathcal{A}^\natural}^{(0)}$ is canonically isomorphic to its Spencer complex

$$0 \rightarrow \hat{\mathcal{D}}_{\mathcal{A} \times \mathcal{A}^\natural / \mathcal{A}^\natural}^{(0)} \rightarrow \Omega_{\mathcal{A} \times \mathcal{A}^\natural / \mathcal{A}^\natural} \otimes \hat{\mathcal{D}}_{\mathcal{A} \times \mathcal{A}^\natural / \mathcal{A}^\natural}^{(0)} \rightarrow \dots \rightarrow \Omega_{\mathcal{A} \times \mathcal{A}^\natural / \mathcal{A}^\natural}^{\dim(\mathcal{A})} \otimes \hat{\mathcal{D}}_{\mathcal{A} \times \mathcal{A}^\natural / \mathcal{A}^\natural}^{(0)} \rightarrow 0,$$

hence there is a canonical isomorphism

$$\hat{\mathcal{D}}_{\mathcal{A}^\natural \leftarrow \mathcal{A} \times \mathcal{A}^\natural / \mathcal{A}^\natural}^{(0)} \otimes_{\hat{\mathcal{D}}_{\mathcal{A} \times \mathcal{A}^\natural / \mathcal{A}^\natural}^{(0)}} \mathcal{P} \simeq DR_{\mathcal{A} \times \mathcal{A}^\natural / \mathcal{A}^\natural}(\mathcal{P})[d],$$

where the second complex is then between degrees $-d$ and 0. There is then an isomorphism

$$\epsilon^\natural_* p_{+/\mathcal{A}^\natural}^\natural \mathcal{P} \simeq R \pi_* (id_{\mathcal{A}} \times \epsilon^\natural)^* DR_{\mathcal{A} \times \mathcal{A}^\natural / \mathcal{A}^\natural}(\mathcal{P})[d].$$

As the elements of the de Rham complex of \mathcal{P} are flats, they are acyclic for $(id_{\mathcal{A}} \times \epsilon^\natural)^*$ and $(id_{\mathcal{A}} \times \epsilon^\natural)^* \Omega_{\mathcal{A} \times \mathcal{A}^\natural / \mathcal{A}^\natural}^k \simeq \Omega_{\mathcal{A}/\mathcal{S}}^k$. Hence,

$$(id_{\mathcal{A}} \times \epsilon^\natural)^* DR_{\mathcal{A} \times \mathcal{A}^\natural / \mathcal{A}^\natural}(\mathcal{P}) \simeq DR_{\mathcal{A}/\mathcal{S}}((id_{\mathcal{A}} \times \epsilon^\natural)^* \mathcal{P}).$$

As previously settled (property 2.7), the $\mathcal{O}_{\mathcal{A}}$ -module $(id_{\mathcal{A}} \times \epsilon^\natural)^* \mathcal{P}$ is well known: in fact, as $\mathcal{D}_{\mathcal{A}/\mathcal{S}}$ -modules, $(id_{\mathcal{A}} \times \epsilon^\natural)^{\natural} \mathcal{P} \simeq \mathcal{O}_{\mathcal{A}}$. Their underlying $\mathcal{O}_{\mathcal{A}}$ -modules are then isomorphic too.

There is then a canonical isomorphism

$$\epsilon^\natural_* p_{+/\mathcal{A}^\natural}^\natural \mathcal{P} \simeq R \pi_* DR_{\mathcal{A}/\mathcal{S}}(\mathcal{O}_{\mathcal{A}})[d] \simeq R \pi_* DR(\mathcal{A}/\mathcal{S})[d],$$

which is then a complex in degrees between $-d$ and 0.

As the $2d$ -th cohomology group of $R \pi_* DR(\mathcal{A}/\mathcal{S})$ is $\mathcal{H}_{dR}^{2d}(\mathcal{A}/\mathcal{S})$, it will be useful to understand the cohomology of \mathcal{A} .

Lemma 2.13.

$$H_{dR}^n(\mathcal{A}/\mathcal{S}) \simeq \varprojlim H_{dR}^n(A_i/S_i) \simeq \bigwedge^n Lie(\mathcal{A}^\natural).$$

Proof. It is known that the functors $R\Gamma$ and $R\varprojlim$ commute, hence

$$R\Gamma R\varprojlim DR(A_i/S_i) \simeq R\varprojlim R\Gamma DR(A_i/S_i).$$

As the elements of $DR(A_i/S_i)$ are the reductions modulo π^i of the elements of $DR(A_{i+1}/S_{i+1})$, $R\varprojlim DR(A_i/S_i)$ is the complex which elements of the projective limits of the elements of the $DR(A_i/S_i)$, i.e. $DR(\mathcal{A}/\mathcal{S})$. Hence, the cohomology groups of the left hand side are the $H_{dR}^n(\mathcal{A}/\mathcal{S})$. It is then sufficient to compute the cohomology groups of the right hand side.

A priori, the spectral sequence of the composed functors gives two terms for the n -th cohomology group of the right hand side: $\varprojlim H_{dR}^n(A_i/S_i)$ and $R^1\varprojlim H_{dR}^{n-1}(A_i/S_i)$, the functor $R\varprojlim$ being of cohomological dimension 1. However, it is known that $H_{dR}^n(A_i/S_i) \simeq \epsilon^{\natural*}\Omega_{A_i^{\natural}/S_i}^n$ ([Col98], theorem 2.2), hence the $R^1\varprojlim$ is trivial and that conclude the proof of the first isomorphism.

The second isomorphism is a direct consequence of the isomorphism $H^1(A_i/S_i) \simeq Lie(A_i^{\natural})$ (see [MM74], parts I.4.1 and I.4.2). □

As $H_{dR}^1(\mathcal{A}/\mathcal{S}) \simeq Lie(\mathcal{A}^{\natural})$ is a free \mathcal{V} -module of rank $2d$, it is clear that $H_{dR}^{2d}(\mathcal{A}/\mathcal{S}) \simeq \mathcal{V}$. Hence, denoting $F^\bullet = R\pi_* DR(\mathcal{A}/\mathcal{S})[d]$, one finds a morphism of complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & F^{-d} & \longrightarrow & \dots & \longrightarrow & F^{d-1} & \xrightarrow{d_{d-1}} & F^d & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & F^d / \text{Im}(d_{d-1}) & \longrightarrow & 0 \\ & & & & & & & & \parallel & & \\ & & & & & & & & \mathcal{H}^d(F^\bullet) & \simeq & \mathcal{O}_{\mathcal{S}}, \end{array}$$

i.e. a morphism

$$\epsilon^{\natural*} p_{+/\mathcal{A}^{\natural}}^{\natural} \mathcal{P} \rightarrow \mathcal{O}_{\mathcal{S}}[-d].$$

By adjunction, one finds the expected morphism

$$p_{+/\mathcal{A}^{\natural}}^{\natural} \mathcal{P} \rightarrow \epsilon_*^{\natural} \mathcal{O}_{\mathcal{S}}[-d].$$

The goal is then to prove it is an isomorphism. In order to do that we will proceed as follow: First we will understand more precisely the points of \mathcal{A}^{\natural} , then prove the expected isomorphism on every point of \mathcal{A}^{\natural} , next we will use the coherence of the sheaves and Nakayama's lemma to prove the isomorphism on the varieties A_i^{\natural} , and finally deduce the expected isomorphism on the formal scheme \mathcal{A} .

Step 1: Recall that the points of \mathcal{A}^{\natural} are the elements of $Pic^{\natural}(\mathcal{A}/\mathcal{S})$. One have the following lemma.

Lemma 2.14. *For all $\mathcal{L} \in Pic^{\natural}(\mathcal{A}/\mathcal{S})$*

$$H_{dR}^n(\mathcal{L}) \simeq \begin{cases} 0 & \text{if } \mathcal{L} \not\simeq \mathcal{O}_{\mathcal{A}} \\ L_{2d-n} \epsilon^{\natural*} \epsilon_*^{\natural} \mathcal{O}_{\mathcal{S}} & \text{if } \mathcal{L} \simeq \mathcal{O}_{\mathcal{A}} \end{cases}$$

Proof. Take $\mathcal{L} \in Pic^{\natural}(\mathcal{A}/\mathcal{S})$ such that $\mathcal{L} \not\simeq \mathcal{O}_{\mathcal{A}}$. Denote $H^0 = H_{dR}^0(\mathcal{L}_{\mathbb{Q}})$, which is then a $K = \text{Frac}(V)$ -vector space and denote r its dimension. It is known ([Ked10], lemma 5.1.5) that $r = 0$ or $r = 1$ and there is an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathcal{A},\mathbb{Q}} \otimes_K H^0 \rightarrow \mathcal{L}_{\mathbb{Q}} \rightarrow \mathcal{L}' \rightarrow 0,$$

where \mathcal{L}' is a coherent $\mathcal{O}_{\mathcal{A},\mathbb{Q}}$ -module with an integrable connection.

If we denote by $sp : \mathcal{A}_K \rightarrow \mathcal{A}$ the specialization morphism, it is known that the functors sp^* and sp_* achieve an equivalence of categories between the coherent $\mathcal{O}_{\mathcal{A},\mathbb{Q}}$ -modules and the coherent $\mathcal{O}_{\mathcal{A}_K}$ -modules. As settled in [Ber96] (proposition 2.2.3) \mathcal{L}' is then locally free of finite rank. Denote r' its rank. Hence, as $\mathcal{L}_{\mathbb{Q}}$ is invertible, $r + r' = 1$. However, $r' \neq 0$ because if not there will be an isomorphism $\mathcal{L}_{\mathbb{Q}} \simeq \mathcal{O}_{\mathcal{A},\mathbb{Q}} \otimes H^0 \simeq \mathcal{O}_{\mathcal{A},\mathbb{Q}}$. Hence $r' = 1$ and $r = 0$, then $H^0 = 0$.

We then know that $H_{dR}^0(\mathcal{L}_{\mathbb{Q}}) = 0$, hence $H_{dR}^0(\mathcal{L})$ is of torsion. However, as $H_{dR}^0(\mathcal{L}) \subset H^0(\mathcal{L})$ and none of the sections of \mathcal{L} can be of torsion over \mathcal{A} (because locally $\mathcal{L} \simeq \mathcal{O}_{\mathcal{A}}$), $H_{dR}^0(\mathcal{L})$ is torsion-free. Hence $H_{dR}^0(\mathcal{L}) = 0$.

Assume then there is an integer $n > 0$ such that $H_{dR}^n(\mathcal{L}) \neq 0$ and consider the smallest one. As $\mathcal{L} \in \text{Pic}^{\natural}(\mathcal{A}/S)$, it verifies the theorem of the square: $m^*\mathcal{L} \simeq p_1^*\mathcal{L} \otimes p_2^*\mathcal{L}$. Moreover, there exists an isomorphism $\varphi = (m, \langle -1 \rangle \circ p_2)$ of $\mathcal{A} \times \mathcal{A}$ ($\varphi^2 = id$) such that $m = p_1 \circ \varphi$, hence $\varphi^*m^*\mathcal{L} \simeq p_1^*\mathcal{L}$. The Künneth formula then settles

$$H_{dR}^n(m^*\mathcal{L}) \simeq \bigoplus_{k+l=n} H_{dR}^k(\mathcal{L}) \otimes H_{dR}^l(\mathcal{L}) \simeq \bigoplus_{k+l=n} H_{dR}^k(\mathcal{L}) \otimes H_{dR}^l(\mathcal{O}_{\mathcal{A}}).$$

As for all $k < n$, $H_{dR}^k(\mathcal{L}) = 0$, one deduces that

$$0 \simeq H_{dR}^n(\mathcal{L}) \otimes H_{dR}^0(\mathcal{O}_{\mathcal{A}}) \simeq H_{dR}^n(\mathcal{L}),$$

which is a contradiction. Hence for all n , $H_{dR}^n(\mathcal{L}) = 0$.

Let now assume $\mathcal{L} \simeq \mathcal{O}_{\mathcal{A}}$. The goal is to compute

$$H_{dR}^n(\mathcal{A}/S) \simeq R\varprojlim H_{dR}^n(A_i/S_i) \simeq R\varprojlim \bigwedge^n \text{Lie}(A_i^{\natural}).$$

In order to do that, denote by \mathcal{I} the ideal associated to the injection ϵ_i^{\natural} , $\mathcal{I} = V(x_1, \dots, x_{2d})$ and $R = \bigoplus_{i=1}^{2d} \mathcal{O}_{A_i^{\natural}} e_i$ and consider the Koszul complex

$$0 \rightarrow \bigwedge^{2d} R \rightarrow \dots \rightarrow \bigwedge^2 R \rightarrow R \rightarrow \mathcal{O}_{A_i^{\natural}} \rightarrow \epsilon_{i*}^{\natural} \mathcal{O}_{S_i} \rightarrow 0,$$

where the arrow $R \rightarrow \mathcal{O}_{A_i^{\natural}}$ is given by $e_n \mapsto x_n$. Applying $\text{Hom}(\bullet, \epsilon_{i*}^{\natural} \mathcal{O}_{S_i})$, one find the complex whose elements are the $\epsilon_{i*}^{\natural} \mathcal{O}_{S_i} \otimes \left(\bigwedge^n \mathcal{I}/\mathcal{I}^2 \right)^{\vee}$, hence the $\epsilon_{i*}^{\natural} \mathcal{O}_{S_i} \otimes \bigwedge^n \text{Lie}(A_i^{\natural})$, and whose differentials are zero. Hence, the n -th cohomology group of this complex is

$$\mathcal{E}xt^n(\epsilon_{i*}^{\natural} \mathcal{O}_{S_i}, \epsilon_{i*}^{\natural} \mathcal{O}_{S_i}) \simeq \epsilon_{i*}^{\natural} \mathcal{O}_{S_i} \otimes \bigwedge^n \text{Lie}(A_i^{\natural}),$$

then $H_{dR}^n(A_i/S_i) \simeq \mathcal{H}^n((\epsilon_i^{\natural})^{-1} R\mathcal{H}om_{\mathcal{O}_{A_i}}(\epsilon_{i*}^{\natural} \mathcal{O}_{S_i}, \epsilon_{i*}^{\natural} \mathcal{O}_{S_i}))$. Hence, by adjunction

$$H_{dR}^n(A_i/S_i) \simeq \mathcal{H}^n(R\mathcal{H}om_{\mathcal{O}_{S_i}}(L\epsilon_i^{\natural*} \epsilon_{i*}^{\natural} \mathcal{O}_{S_i}, \mathcal{O}_{S_i})) \simeq L_{2d-n} \epsilon_i^{\natural*} \epsilon_{i*}^{\natural} \mathcal{O}_{S_i}.$$

As these isomorphisms are compatible to the reductions modulo π^i , one has the expected isomorphism

$$H_{dR}^n(\mathcal{A}/S) \simeq R\varprojlim H_{dR}^n(A_i/S_i) \simeq L_{2d-n} \epsilon^{\natural*} \epsilon_*^{\natural} \mathcal{O}_S.$$

□

Step 2: As the goal is to use Nakayama's lemma on the schemes A_i^{\natural} , we will need to have an isomorphism with restriction of each closed point of each A_i^{\natural} . We will see that such a point can be viewed as a $\text{Spf}(W(k(x)))$ -point of \mathcal{A}^{\natural} . This is the reason of the hypothesis in the following lemma.

Lemma 2.15. *Let k be a finite extension of the residue field of V , $W(k)$ its ring of Witt vectors and $\mathcal{S}' = \text{Spf}(W(k))$. For all \mathcal{S}' -point of \mathcal{A}^{\natural} , if $\iota : \mathcal{S}' \hookrightarrow \mathcal{A}^{\natural}$ is the injection, then*

$$L\iota^* p_{+/\mathcal{A}^{\natural}}^{\natural} \mathcal{P} \simeq L\iota^* \epsilon_{*}^{\natural} \mathcal{O}_{\mathcal{S}}[-d].$$

Proof. First of all, remark that the morphism

$$L\iota^* p_{+/\mathcal{A}^{\natural}}^{\natural} \mathcal{P} \rightarrow L\iota^* \epsilon_{*}^{\natural} \mathcal{O}_{\mathcal{S}}[-d].$$

is well defined, as it is the pullback by ι of the morphism $p_{+/\mathcal{A}^{\natural}}^{\natural} \mathcal{P} \rightarrow \epsilon_{*}^{\natural} \mathcal{O}_{\mathcal{S}}[-d]$ previously built.

Also note that, using some base change in the diagram

$$\begin{array}{ccc} \mathcal{S}' & \longrightarrow & \mathcal{S} \\ \epsilon^{\natural'} \downarrow & & \downarrow \epsilon^{\natural} \\ \mathcal{A}^{\natural'} & \xrightarrow{p^{\natural'}} & \mathcal{A}^{\natural} \end{array}$$

one can assume $\mathcal{S}' = \mathcal{S}$.

In that case, as $\epsilon_{*}^{\natural} \mathcal{O}_{\mathcal{S}}$ has support in $\epsilon^{\natural}(\mathcal{S})$, it is sufficient to prove that

$$\left\{ \begin{array}{l} \forall \iota : \mathcal{S} \hookrightarrow \mathcal{A}^{\natural}, \iota \neq \epsilon^{\natural} \quad L\iota^* p_{+/\mathcal{A}^{\natural}}^{\natural} \mathcal{P} = 0. \\ L\epsilon^{\natural*} p_{+/\mathcal{A}^{\natural}}^{\natural} \mathcal{P} \simeq L\epsilon^{\natural*} \epsilon_{*}^{\natural} \mathcal{O}_{\mathcal{S}}[-d]. \end{array} \right.$$

The base change formula in the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{id_{\mathcal{A}} \times \iota} & \mathcal{A} \times \mathcal{A}^{\natural} \\ \pi \downarrow & & \downarrow p^{\natural} \\ \mathcal{S} & \xrightarrow{\iota} & \mathcal{A}^{\natural} \end{array}$$

gives the isomorphism

$$L\iota^* p_{+/\mathcal{A}^{\natural}}^{\natural} \mathcal{P} \simeq \pi_{+} L(id_{\mathcal{A}} \times \iota)^* \mathcal{P}.$$

However, $L(id_{\mathcal{A}} \times \iota)^* \mathcal{P} = (id_{\mathcal{A}} \times \iota)^* \mathcal{P}$ is by definition of \mathcal{P} the element of $\text{Pic}^{\natural}(\mathcal{A}/\mathcal{S})$ which corresponds to ι . Denote by \mathcal{L} this element. As $\pi_{+} \mathcal{L} \simeq H_{dR}^{\bullet}(\mathcal{L})[d]$, the previous lemma states that

$$L\iota^* p_{+/\mathcal{A}^{\natural}}^{\natural} \mathcal{P} \simeq \begin{cases} 0 & \text{if } \mathcal{L} \not\simeq \mathcal{O}_{\mathcal{A}} \text{ i.e. } \iota \neq \epsilon^{\natural}. \\ L\epsilon^{\natural*} \epsilon_{*}^{\natural} \mathcal{O}_{\mathcal{S}}[-d] & \text{if } \mathcal{L} \simeq \mathcal{O}_{\mathcal{A}} \text{ i.e. } \iota = \epsilon^{\natural}. \end{cases}$$

Hence the isomorphism

$$L\iota^* p_{+/\mathcal{A}^{\natural}}^{\natural} \mathcal{P} \simeq L\iota^* \epsilon_{*}^{\natural} \mathcal{O}_{\mathcal{S}}[-d].$$

□

Step 3: To conclude, we will use the coherence of the sheaves to obtain an isomorphism on the schemes A_i^{\natural} . In order to do so, remark that the morphism

$$p_{+/\mathcal{A}^{\natural}}^{\natural} \mathcal{P} \rightarrow \epsilon_{*}^{\natural} \mathcal{O}_{\mathcal{S}}[-d]$$

induces a morphism

$$p_{+/\mathcal{A}_i^{\natural}}^{\natural} \mathcal{P}_i \rightarrow \epsilon_{i*}^{\natural} \mathcal{O}_{S_i}[-d]$$

on the reductions modulo π^i . Fix a closed point x of A_i^{\natural} , which is then a closed point of A_0^{\natural} . Denote k its residue field, which is then a finite extension of the residue field of V . As k is finite, it is perfect and one can take its Witt vectors ring $W(k)$ and denote by ω its uniformizer. One then has the following diagram

$$\begin{array}{ccc} \text{Spec}(W(k)/\omega) & \hookrightarrow & A_0^{\natural} \\ \downarrow & \searrow & \downarrow \\ \text{Spec}(W(k)/\omega^2) & \xrightarrow{f} & A_1^{\natural} \end{array}$$

with f that exists by formal smoothness. Doing an immediate recurrence, one find the closed point x can be lift up on all the varieties A_i^{\natural} . Denote by $\iota : \text{Spf}(W(k)) \hookrightarrow \mathcal{A}^{\natural}$ this lift. The previous lemma then settles that the sheaves $p_{+/A^{\natural}}^{\natural} \mathcal{P}$ and $\epsilon_{i*}^{\natural} \mathcal{O}_S[-d]$ restricted to ι are isomorphic. Hence, after reduction modulo π^i , the sheaves $p_{i+/A_i^{\natural}}^{\natural} \mathcal{P}_i$ and $\epsilon_{i*}^{\natural} \mathcal{O}_{S_i}[-d]$ are isomorphic in restriction to x , and that for all x .

As the sheaves $p_{+/A_i^{\natural}}^{\natural} \mathcal{P}_i$ and $\epsilon_{i*}^{\natural} \mathcal{O}_{S_i}[-d]$ are coherent on A_i^{\natural} and are isomorphic on each closed point of A_i^{\natural} , Nakayama's lemma allows to claim that one has an isomorphism

$$p_{+/A_i^{\natural}}^{\natural} \mathcal{P}_i \simeq \epsilon_{i*}^{\natural} \mathcal{O}_{S_i}[-d],$$

and that for all i . As all these isomorphisms came from a same morphism on the formal scheme \mathcal{A}^{\natural} , it has to be an isomorphism too, i.e.

$$p_{+/A^{\natural}}^{\natural} \mathcal{P} \simeq \epsilon_{i*}^{\natural} \mathcal{O}_S[-d].$$

□

Property 2.16.

$$\mathcal{F} \circ \mathcal{F}^{\natural} \simeq \langle -1 \rangle^{\natural*} \bullet [-d] : D_{qcoh}^b(\mathcal{O}_{\mathcal{A}^{\natural}}) \rightarrow D_{qcoh}^b(\mathcal{O}_{\mathcal{A}^{\natural}}).$$

Proof. The main result of the previous section states that

$$\mathcal{F} \circ \mathcal{F}^{\natural} = \Phi_{\mathcal{P}}^{A \rightarrow \mathcal{A}^{\natural}} \circ \Phi_{\mathcal{P}}^{A^{\natural} \leftarrow A} \simeq \Phi_{\mathcal{R}},$$

with $\mathcal{R} = \pi_{13+/(1,3)}(\pi_{12/1}^{\natural(3)} \mathcal{P} \otimes \pi_{23/3}^{\natural(1)} \mathcal{P})$, where the π_{ij} are the projections of $\mathcal{A}^{\natural} \times \mathcal{A} \times \mathcal{A}^{\natural}$. In order to understand the kernel \mathcal{R} , we will consider the sheaf

$$\mathcal{Q} = \pi_{12}^* \mathcal{P} \otimes \pi_{23}^* \mathcal{P} \otimes m_{13}^* \mathcal{P}^{-1},$$

with $m_{13} : \mathcal{A}^{\natural} \times \mathcal{A} \times \mathcal{A}^{\natural} \rightarrow \mathcal{A} \times \mathcal{A}^{\natural}$ being defined as $m_{13}(a, b, c) = (b, m^{\natural}(a, c))$. As the kernel \mathcal{R} doesn't have any structure of \mathcal{D} -module, one can forget the structure of \mathcal{D} -module of \mathcal{P} . Remember also that $\varphi^* \mathcal{P}$ is defined as $\varinjlim \varphi_i^* \mathcal{P}_i$ (there is no need to derive the functors here as the three morphisms are flat). Using the property 2.9, one finds the isomorphism

$$\mathcal{Q} \simeq \pi_{12}^*(id_{\mathcal{A}} \times \theta)^* \mathcal{P}^{\vee} \otimes \pi_{23}^*(id_{\mathcal{A}} \times \theta)^* \mathcal{P}^{\vee} \otimes m_{13}^*(id_{\mathcal{A}} \times \theta)^* \mathcal{P}^{\vee-1},$$

where \mathcal{P}^{\vee} is the Poincaré sheaf on $\mathcal{A} \times \mathcal{A}^{\vee}$. Denote by π_{ij}^{\vee} the projections of $\mathcal{A}^{\vee} \times \mathcal{A} \times \mathcal{A}^{\vee}$ and m_{13}^{\vee} the multiplication of the first and third factors. Then

$$\mathcal{Q} \simeq (\theta \times id_{\mathcal{A}} \times \theta)^*(\pi_{12}^{\vee*} \mathcal{P}^{\vee} \otimes \pi_{23}^{\vee*} \mathcal{P}^{\vee} \otimes m_{13}^{\vee*} \mathcal{P}^{\vee-1}).$$

As $\mathcal{A}^{\vee\vee} \simeq \mathcal{A}$, \mathcal{P}^\vee is also the Poincaré sheaf on $\mathcal{A}^\vee \times \mathcal{A}^{\vee\vee}$ and

$$\pi_{12}^{\vee*} \mathcal{P}^\vee \otimes \pi_{23}^{\vee*} \mathcal{P}^\vee \otimes m_{13}^{\vee*} \mathcal{P}^{\vee-1} \simeq \mathcal{O}_{\mathcal{A}^\vee \times \mathcal{A} \times \mathcal{A}^\vee}.$$

This is just saying that \mathcal{P}^\vee verifies the theorem of the square for the $\mathcal{O}_{\mathcal{A}^\vee \times \mathcal{A}^{\vee\vee}}$ -modules. Hence, as $\mathcal{O}_{\mathcal{A}^\natural \times \mathcal{A} \times \mathcal{A}^\natural}$ -modules,

$$\mathcal{Q} \simeq \mathcal{O}_{\mathcal{A}^\natural \times \mathcal{A} \times \mathcal{A}^\natural},$$

hence

$$\pi_{12}^* \mathcal{P} \otimes \pi_{23}^* \mathcal{P} \simeq m_{13}^* \mathcal{P}.$$

Note that one can actually use the seesaw principle for \mathcal{D} -modules (property 1.18) to prove that, as $\mathcal{D}_{\mathcal{A}^\natural \times \mathcal{A} \times \mathcal{A}^\natural / \mathcal{A}^\natural \times \mathcal{A}^\natural}$ -modules, $\pi_{12/1}^{!(3)} \mathcal{P} \otimes \pi_{23/3}^{!(1)} \mathcal{P} \simeq m_{13/3}^{!(1)} \mathcal{P}$.

Rewriting the $\mathcal{O}_{\mathcal{A}^\natural \times \mathcal{A}^\natural}$ -module \mathcal{R} as $\pi_{13+/ (1,3)} m_{13}^* \mathcal{P}$ and using the base change formula, one can easily find that $m^{\natural*} p_{+/\mathcal{A}^\natural} \mathcal{P}$.

The previous lemma allows to find an isomorphism $\mathcal{R} \simeq m^{\natural*} \epsilon_*^\natural \mathcal{O}_S[-d]$. Then, considering the commutative diagram

$$\begin{array}{ccc} \Gamma^\natural & \xrightarrow{\tilde{m}^\natural} & S \\ \tilde{\epsilon}^\natural \downarrow & & \downarrow \epsilon^\natural \\ \mathcal{A}^\natural \times \mathcal{A}^\natural & \xrightarrow{m^\natural} & \mathcal{A}^\natural \end{array}$$

where Γ^\natural is the anti-diagonal of \mathcal{A}^\natural (i.e. the elements of the form $(a, -a)$), another base change leads to the isomorphism $\mathcal{R} \simeq \tilde{\epsilon}_*^\natural \mathcal{O}_{\Gamma^\natural}[-d]$.

Hence, the composition $\mathcal{F} \circ \mathcal{F}^\natural$ is the Fourier-Mukai transform of kernel $\tilde{\epsilon}_*^\natural \mathcal{O}_{\Gamma^\natural}[-d]$. The projection formula then states that

$$\mathcal{F} \circ \mathcal{F}^\natural(\mathcal{E}^\cdot) \simeq (p_1 \circ \tilde{\epsilon}^\natural)_* (p_2 \circ \tilde{\epsilon}^\natural)^* \mathcal{E}^\cdot[-d],$$

with $p_1, p_2 : \mathcal{A}^\natural \times \mathcal{A}^\natural \rightarrow \mathcal{A}^\natural$ the projections. As there is a commutative diagram of isomorphisms

$$\begin{array}{ccc} & \Gamma^\natural & \\ p_1 \circ \tilde{\epsilon}^\natural \swarrow & & \searrow p_2 \circ \tilde{\epsilon}^\natural \\ \mathcal{A}^\natural & \xrightarrow{\langle -1 \rangle^\natural} & \mathcal{A}^\natural \end{array}$$

one finally concludes that

$$\mathcal{F} \circ \mathcal{F}^\natural(\mathcal{E}^\cdot) \simeq \langle -1 \rangle^* \mathcal{E}^\cdot[-d].$$

□

The following result is a direct consequence.

Theorem 2.17. *\mathcal{F} is essentially surjective and \mathcal{F}^\natural is faithful.*

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