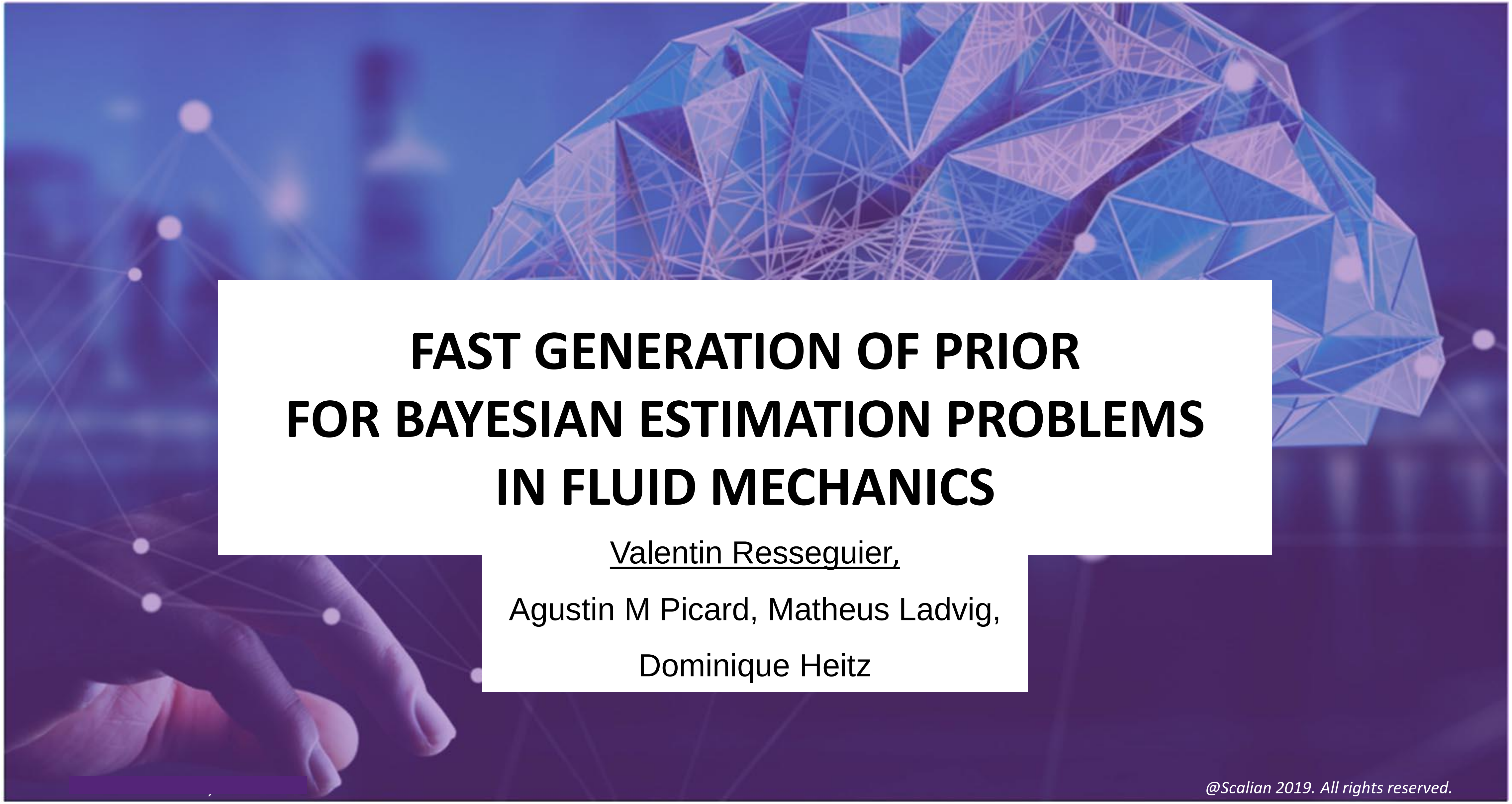

FAST GENERATION OF PRIOR FOR BAYESIAN ESTIMATION PROBLEMS IN FLUID MECHANICS

Valentin Resseguier,

Agustin M Picard, Matheus Ladvig,

Dominique Heitz

CONTENT

I. Context
II. State of the art
  a. Intrusive reduced order model (ROM)
  b. Data assimilation
III. Reduced location uncertainty models
  a. Multiscale modeling
  b. Location uncertainty models (LUM)
  c. Reduced LUM
IV. Numerical results
  a. Uncertainty quantification (Prior)
  b. Data assimilation (Posterior)

PART I

CONTEXT :

OBSERVER FOR WIND TURBINE APPLICATIONS

CONTEXT

Observer for wind turbine application

Application: Real-time estimation and prediction of 3D fluid flow using strongly-limited computational resources & few sensors

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Wind turbine blade

CONTEXT

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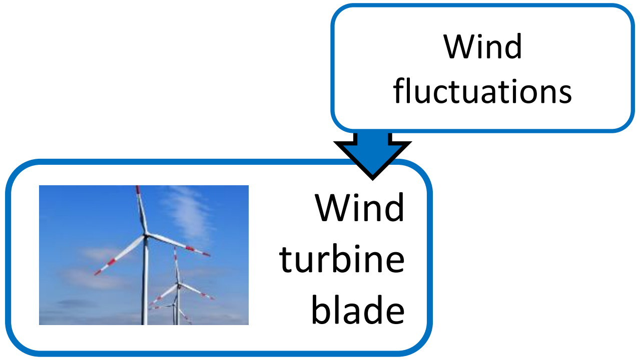

Wind fluctuations

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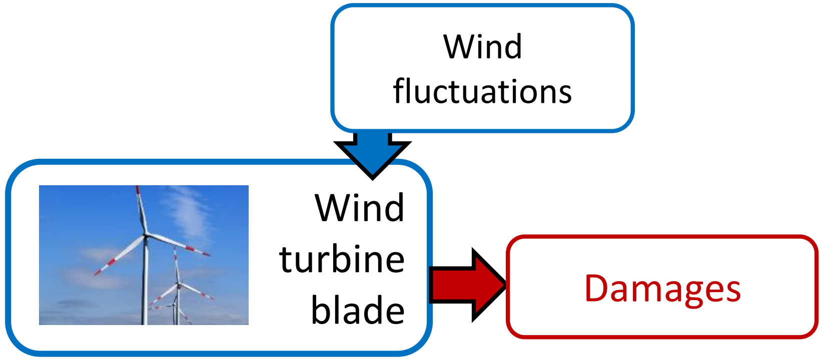

Wind fluctuations

Wind turbine blade

Damages

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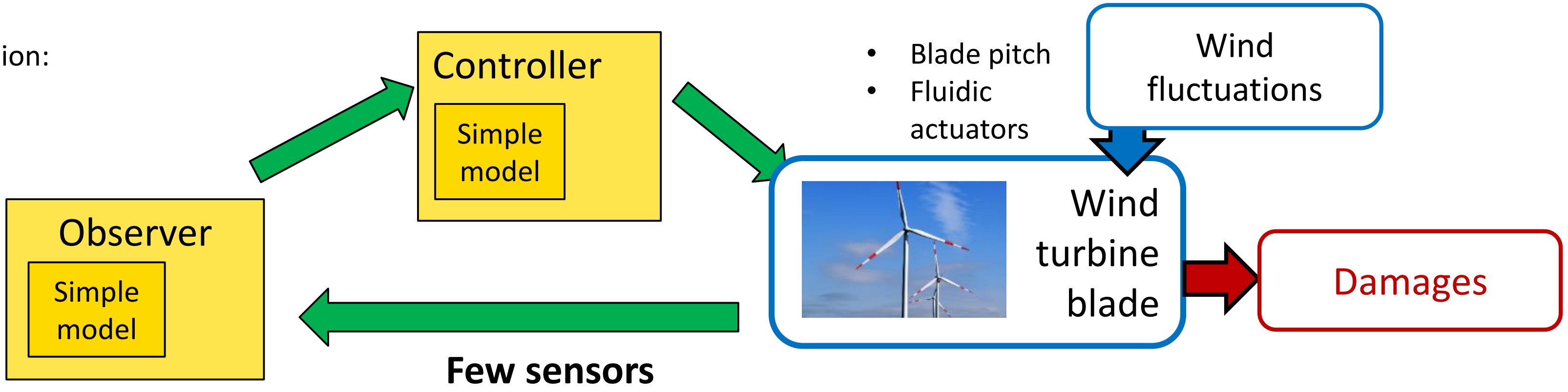
Estimation and prediction:
- Air flow
- Lift, drag, inflow
- …



Controller

Simple model

- Blade pitch
- Fluidic actuators

Wind fluctuations

Observer

Simple model

Wind turbine blade

Damages

Few sensors

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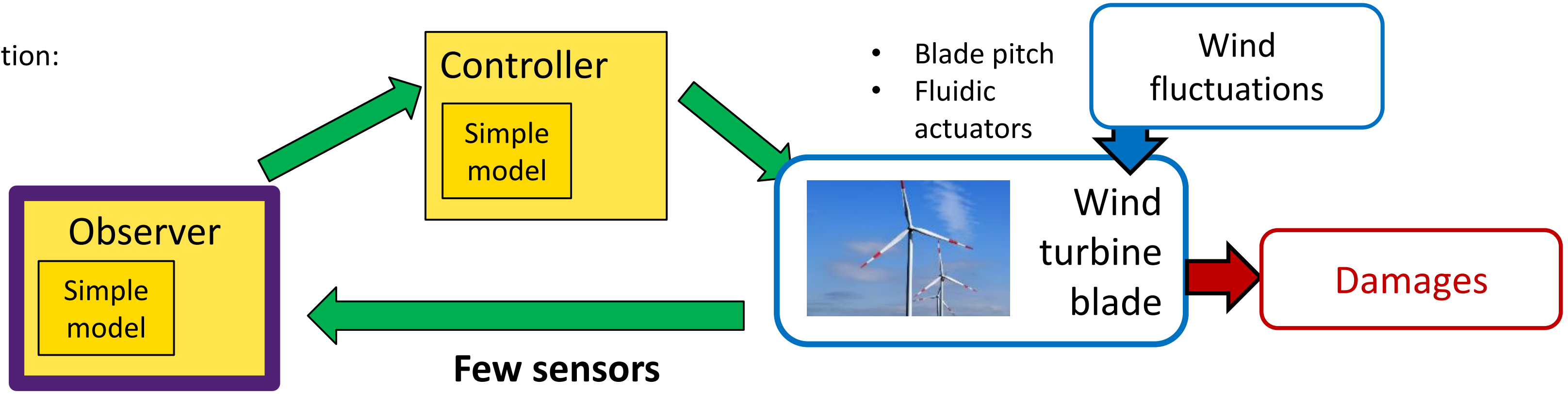

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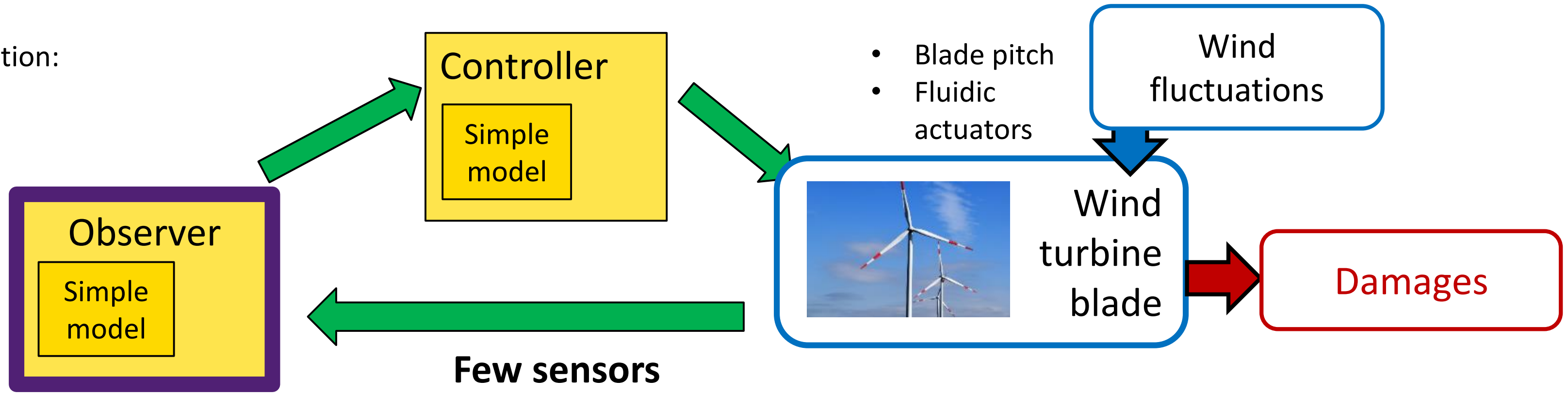

Which simple model? How to combine model & measurements?

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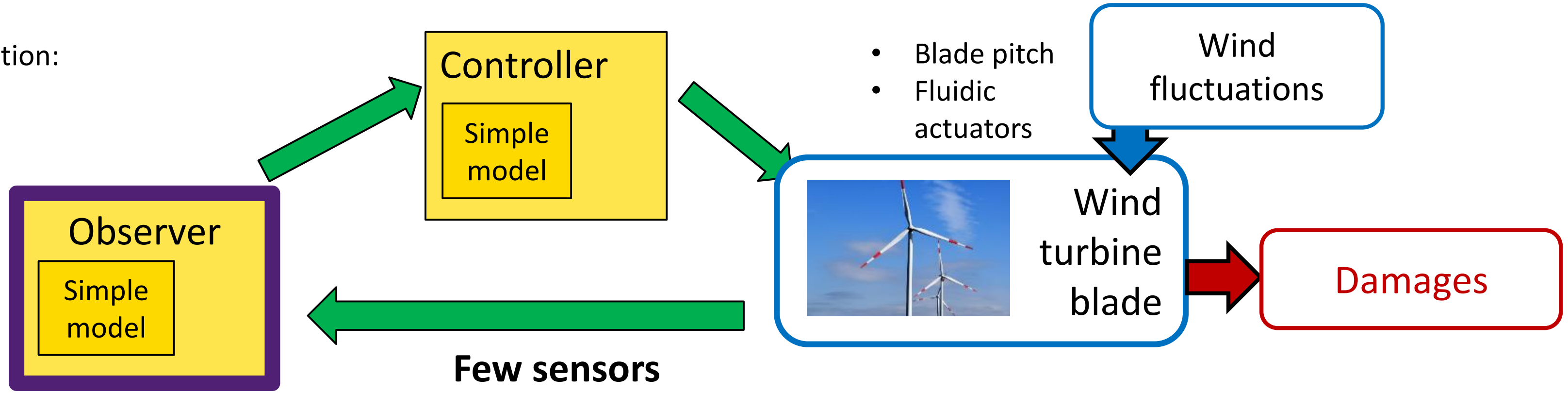

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Few sensors

Which simple model? How to combine model & measurements?

Scientific problem :
Simulation & data assimilation under severe dimensional reduction
typically, $10^7 \rightarrow O(10)$ degrees of freedom

PART II

STATE OF THE ART

a. Intrusive reduced order model (ROM)

b. Data assimilation

REDUCED ORDER MODEL (ROM)

Solution of an PDE with the form:

$$v(x,t,\alpha) \approx \sum_{i=o}^{n} b_i(t)\ \phi_i(x)\ \gamma_i(\alpha)$$

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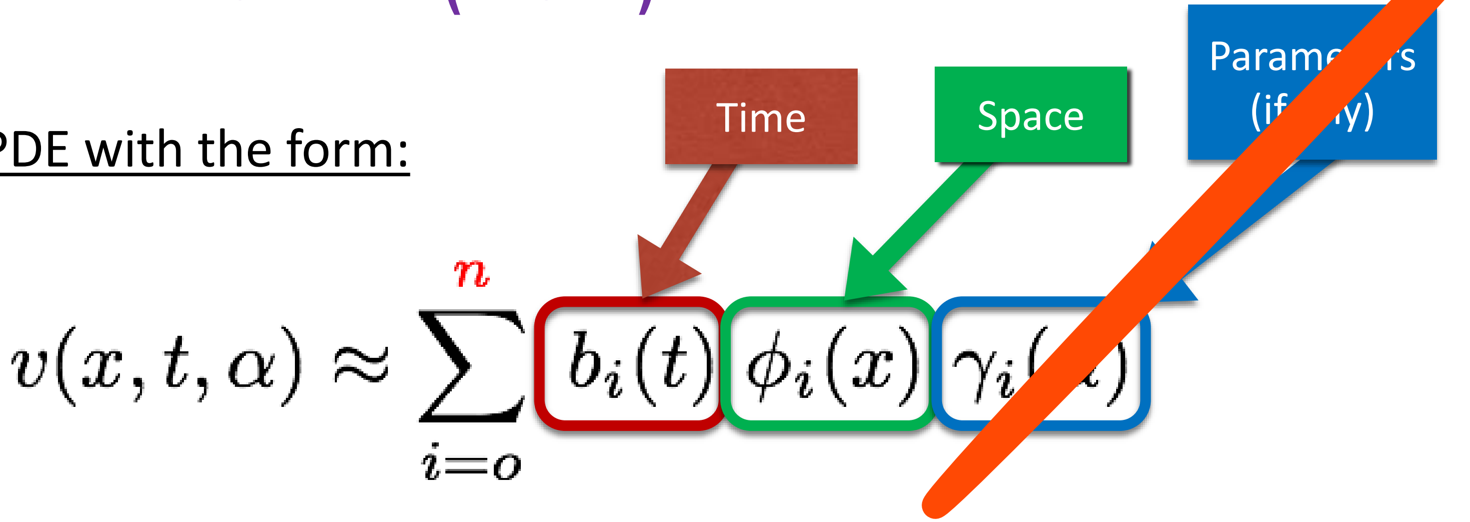

Time

Space

~~Parameters (if any)~~

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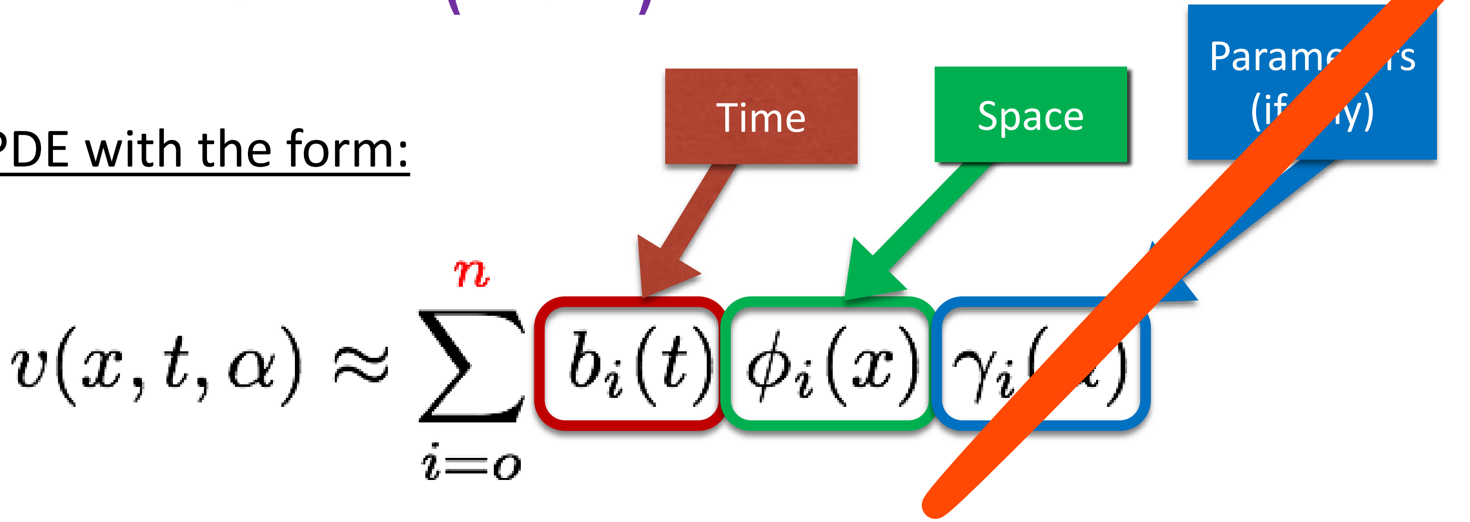

$$v(x,t,\alpha) \approx \sum_{i=o}^{n} \boxed{b_i(t)}\boxed{\phi_i(x)}\cancel{\boxed{\gamma_i(\alpha)}}$$

Order of magnitude examples in CFD

	Full space	Reduced space
Solution coordinates	$v_q(x_i,t))_{qi}$	$(b_i(t))_i$
Dimension	$M \times d \sim 10^7$	$n \sim 10 - 100$

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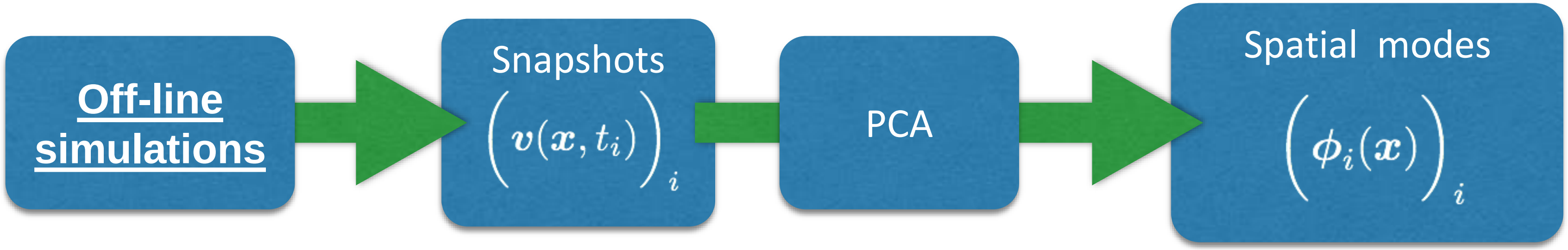

Off-line simulations

Snapshots $\left(\boldsymbol{v}(\boldsymbol{x}, t_i)\right)_i$

PCA

Spatial modes $\left(\phi_i(\boldsymbol{x})\right)_i$

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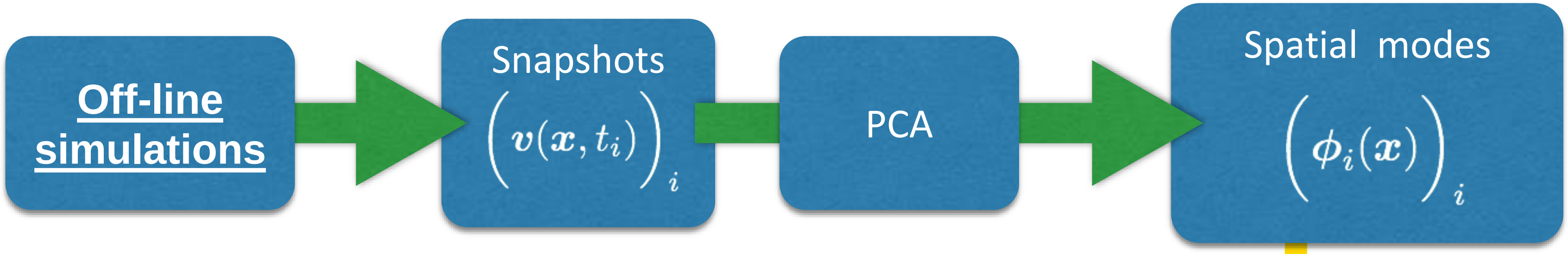

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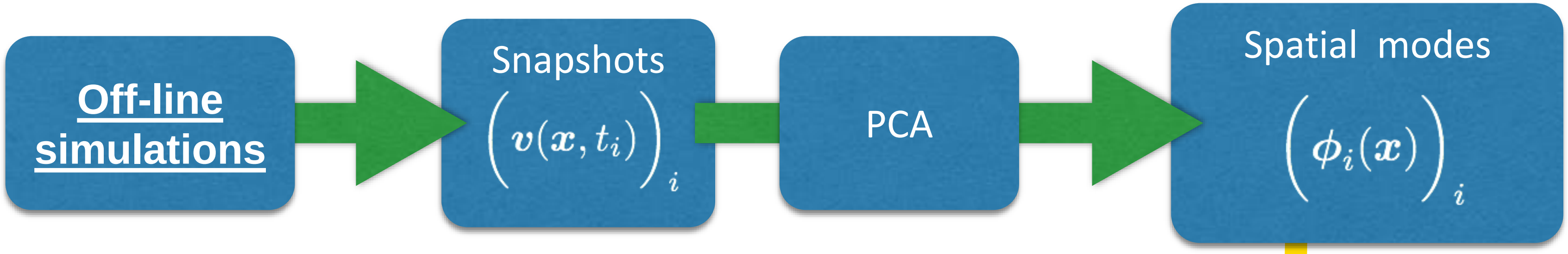

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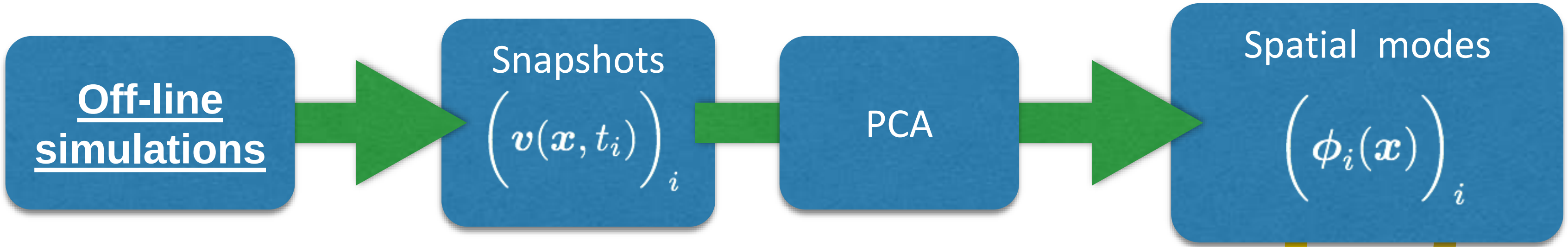

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$\int_{\Omega} dx\, \phi_i(x) \cdot$ (*Physical equation* (e.g. Navier-Stokes))

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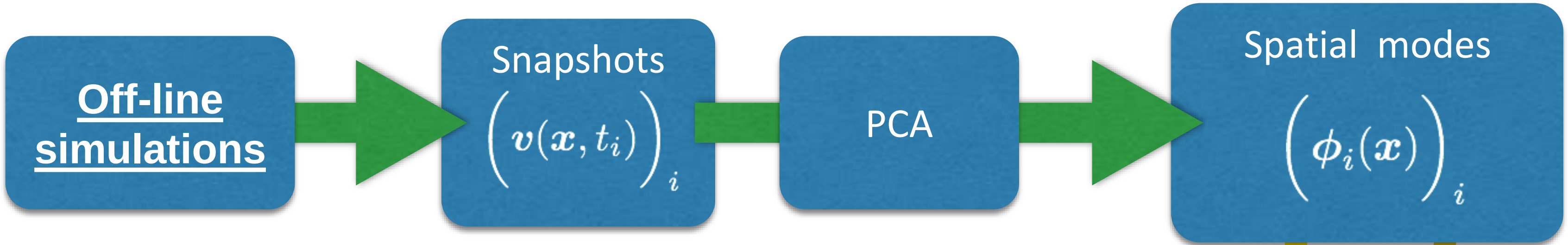

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→ **ROM for very fast simulation of temporal modes**

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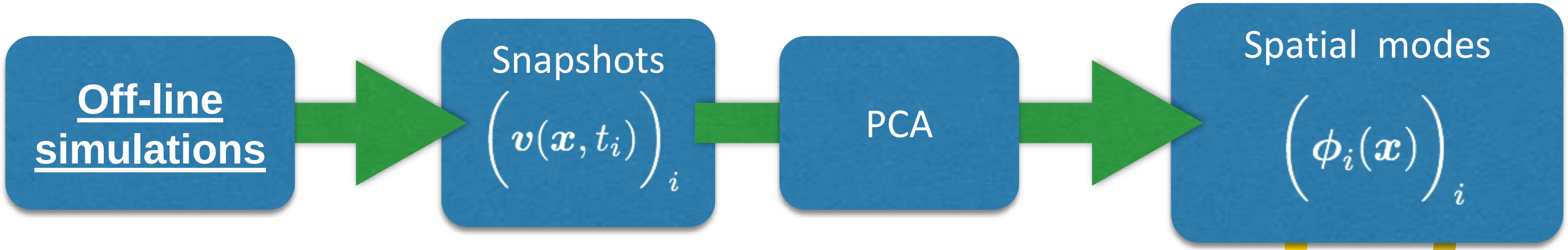

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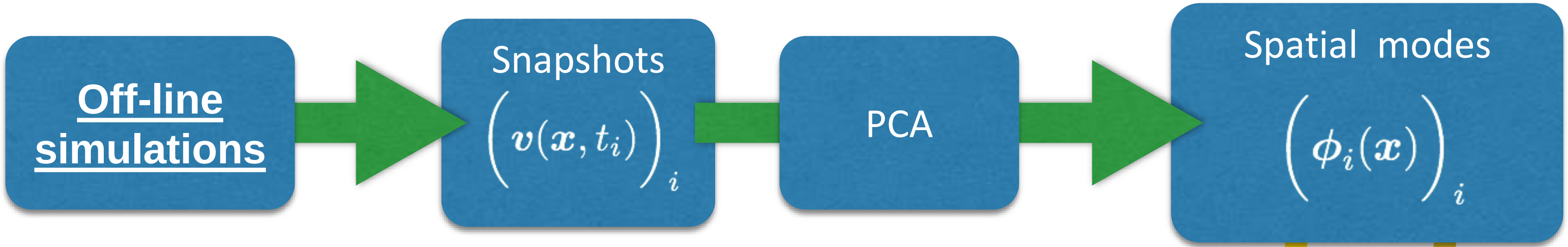

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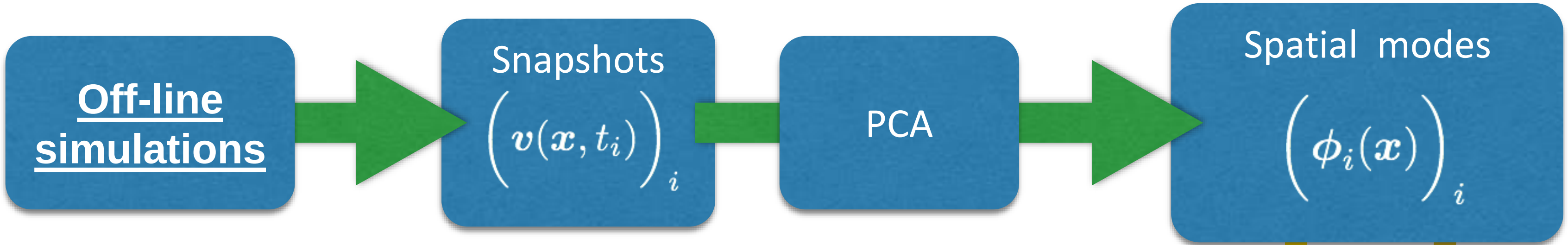

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Resolved modes

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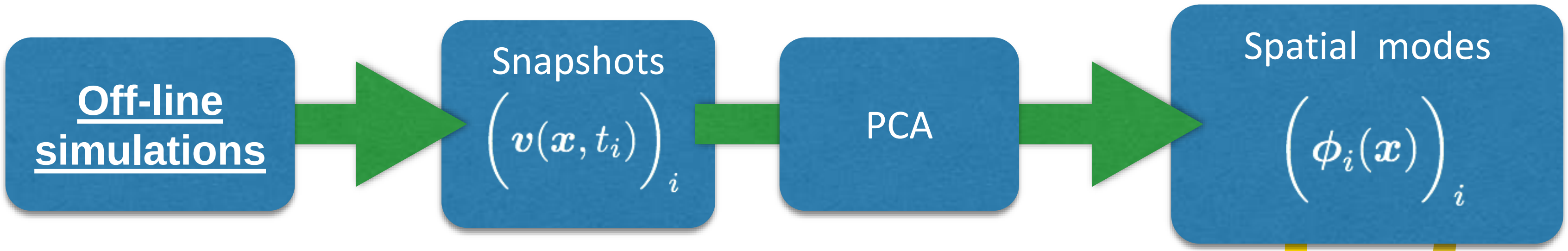

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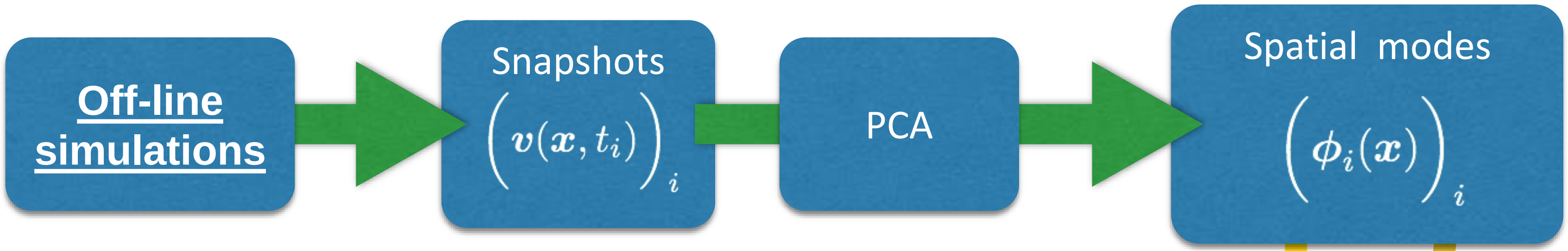

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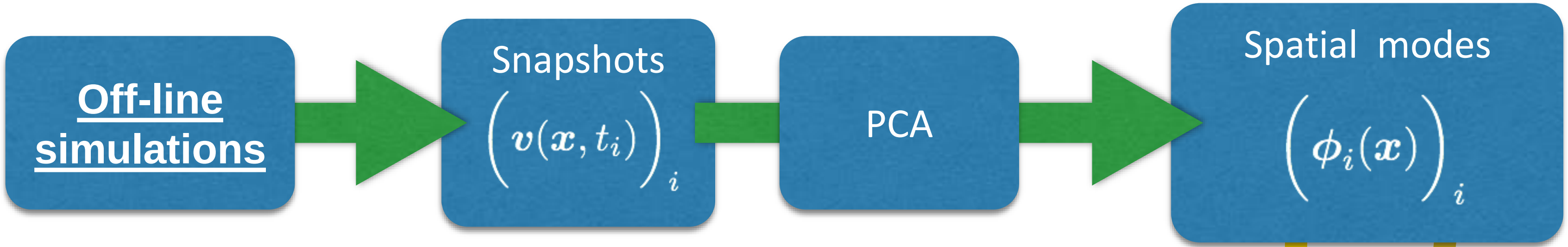

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SPOILER

$$\int_\Omega dx\, \phi_i(x) \cdot$$ (Randomized Navier-Stokes)

→ **ROM for very fast simulation of temporal modes**

DATA ASSIMILATION

= Coupling simulations and measurements y

Numerical Simulation (ROM)

→ erroneous

On-line measurements

→ incomplete
→ possibly noisy

Velocity

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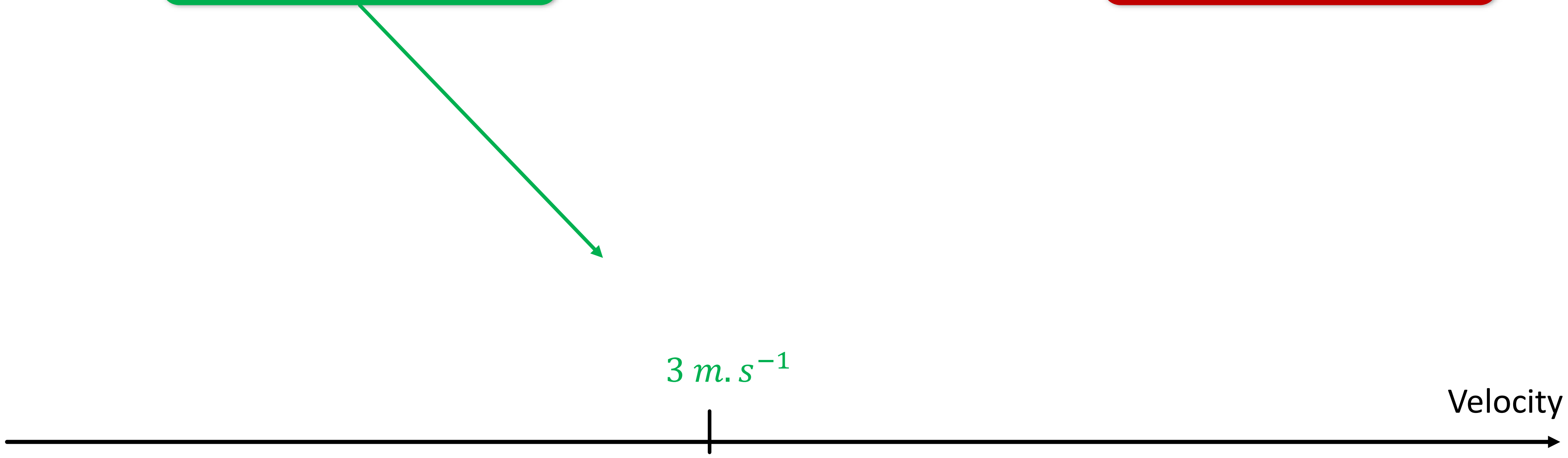

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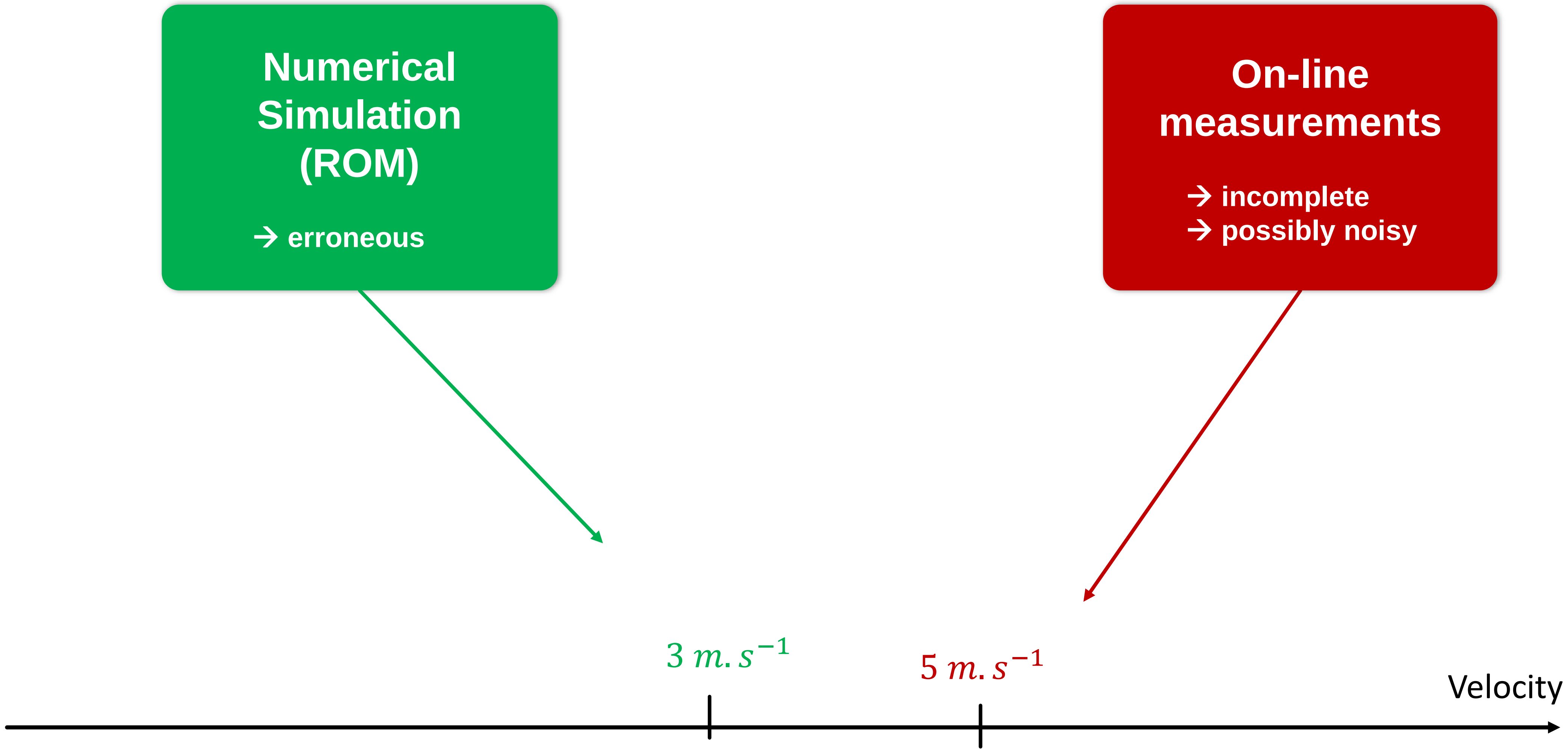

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$3\ m.s^{-1}$

$5\ m.s^{-1}$

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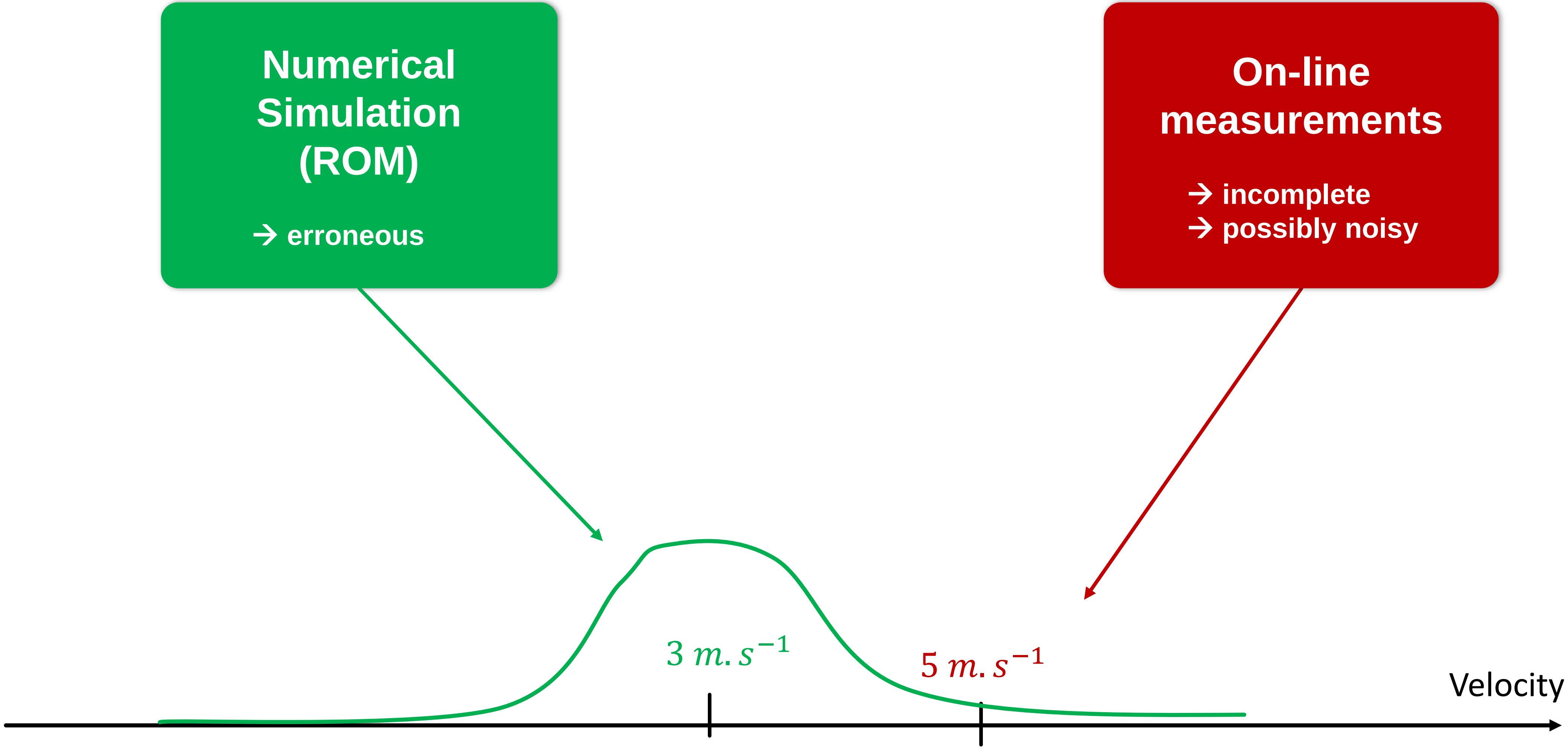

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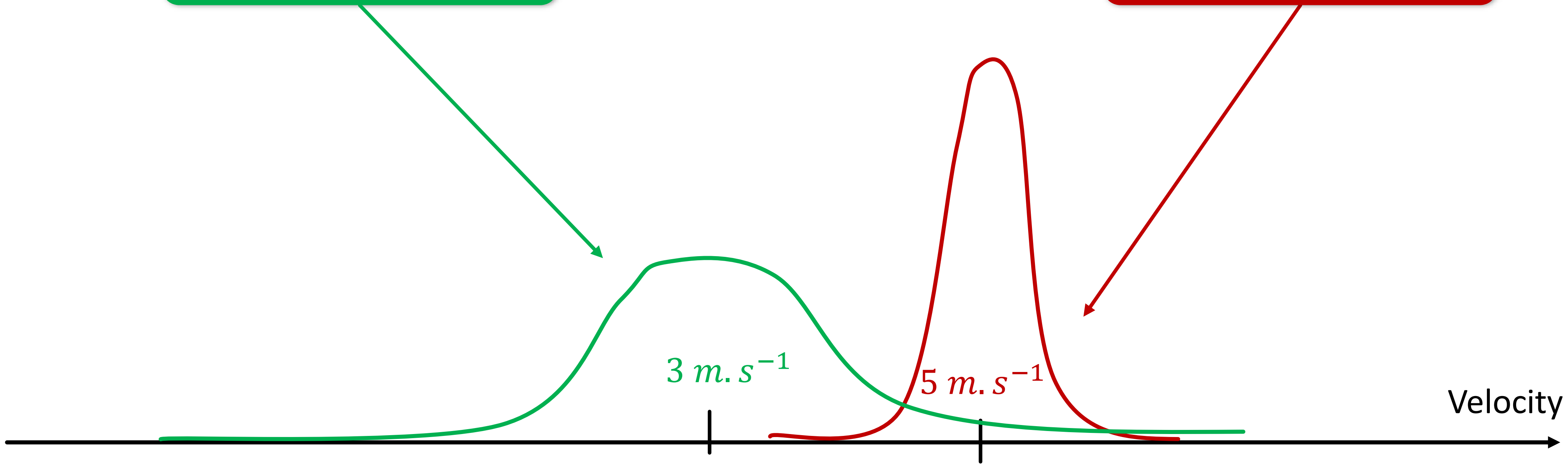

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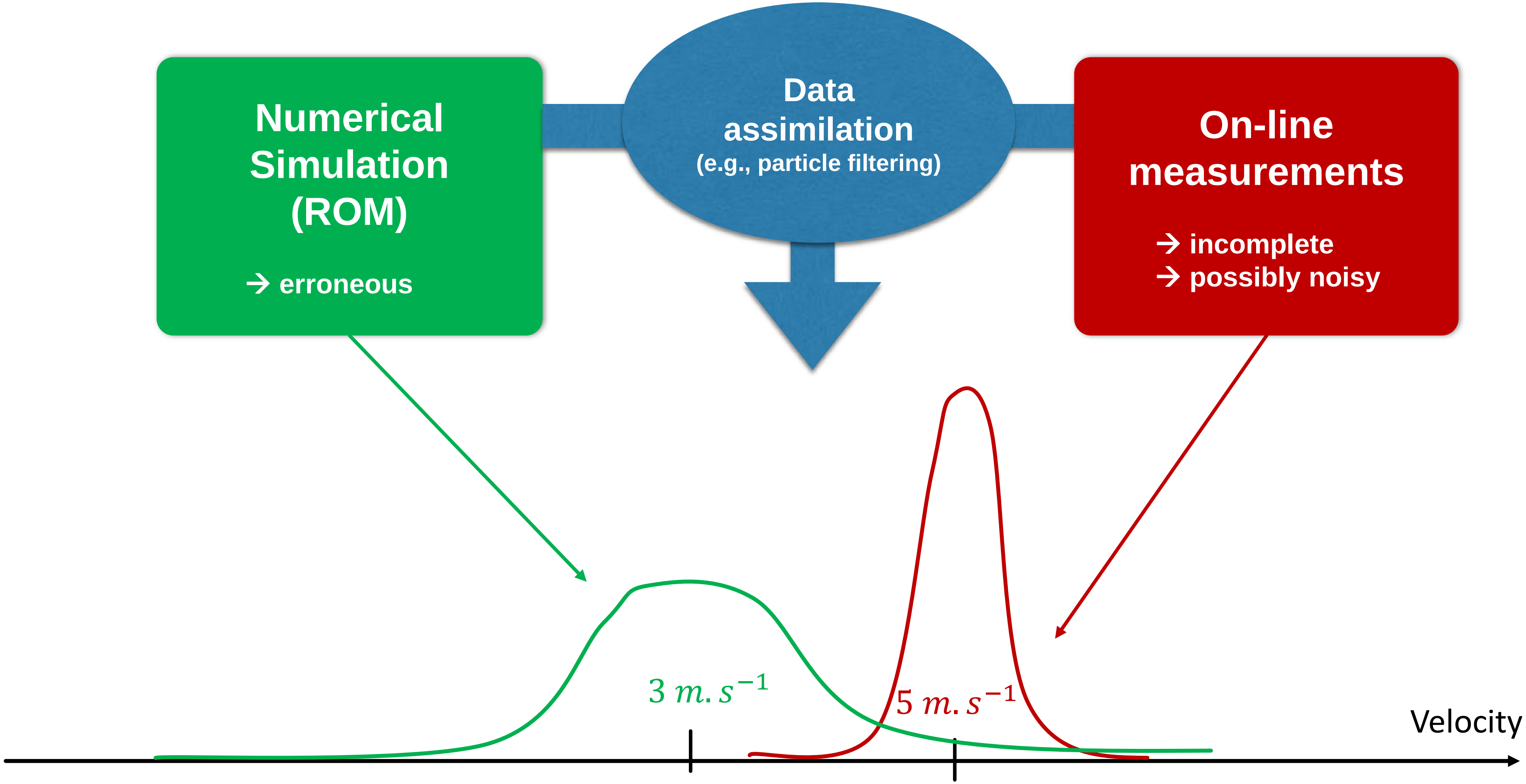

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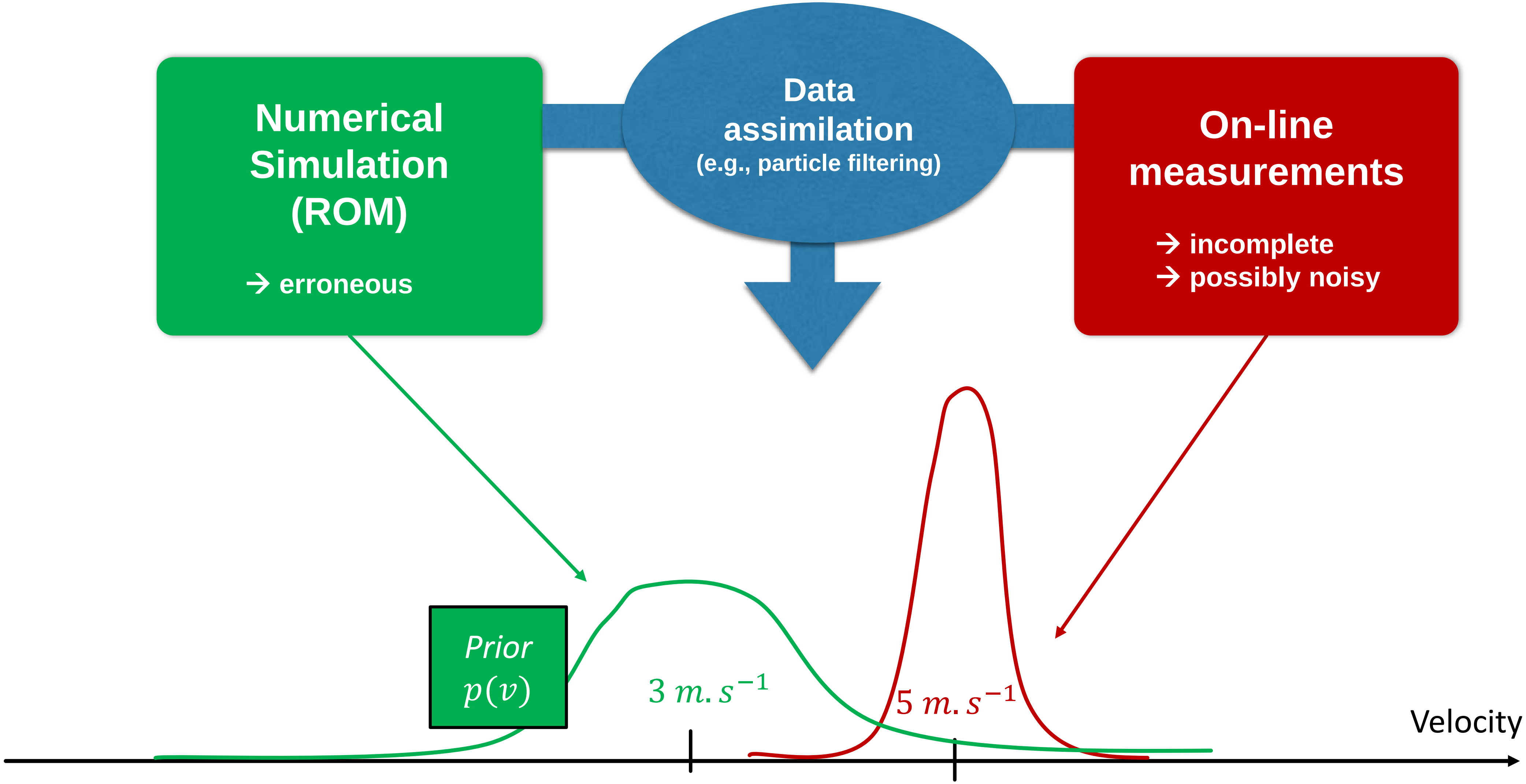

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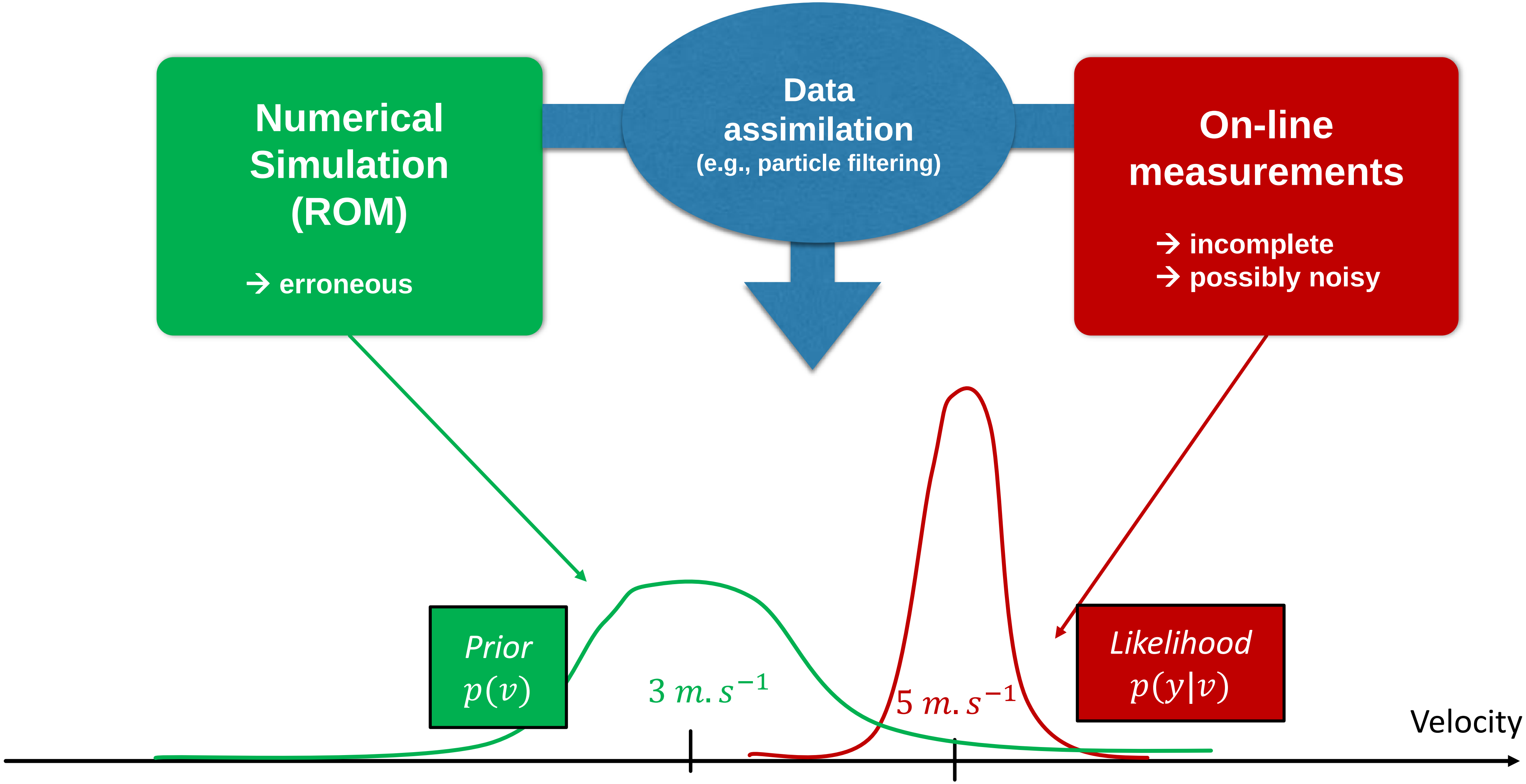

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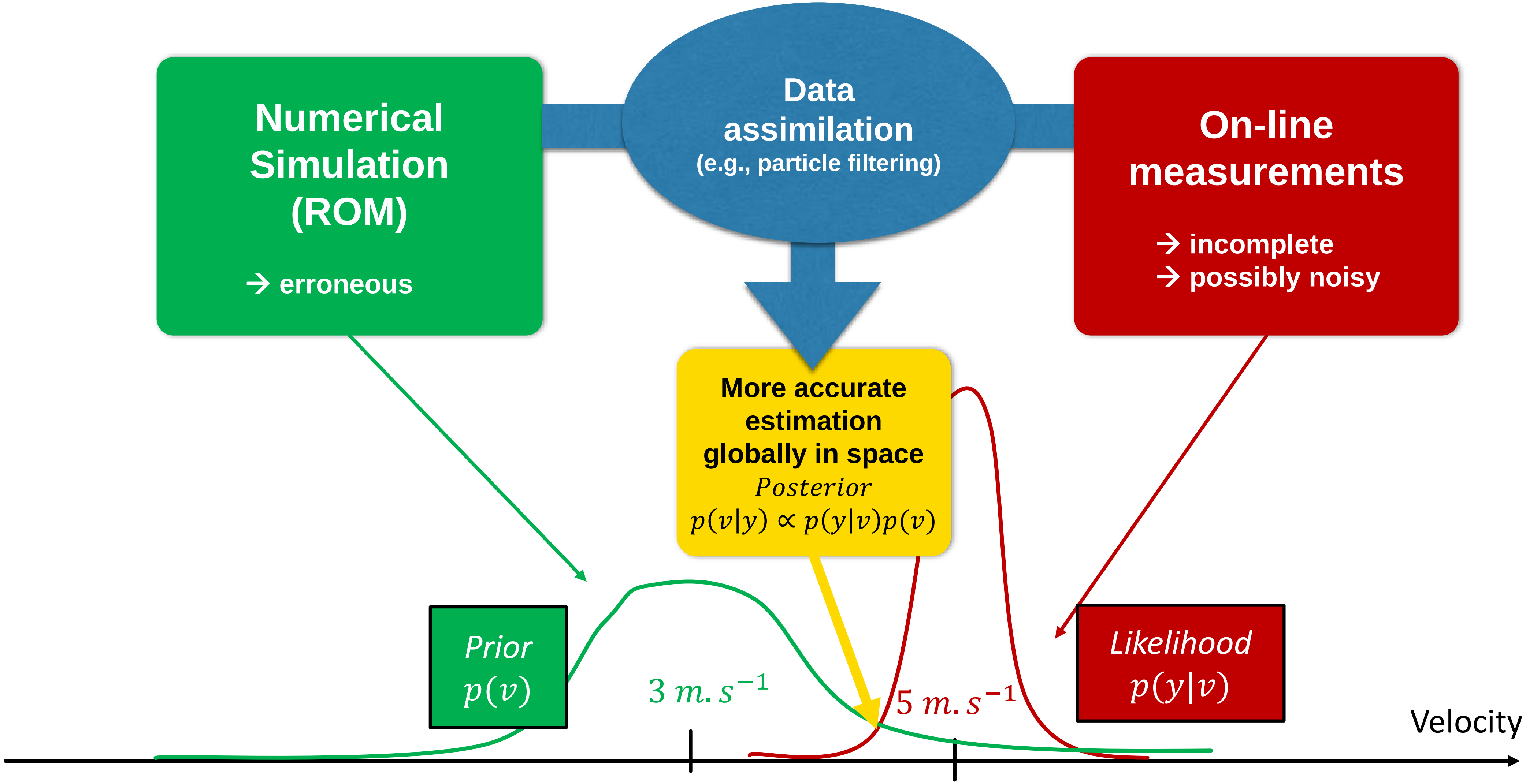

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More accurate estimation globally in space

Posterior

$p(v|y) \propto p(y|v)p(v)$

Prior

$p(v)$

$3\ m.s^{-1}$

$5\ m.s^{-1}$

Likelihood

$p(y|v)$

Velocity

DATA ASSIMILATION

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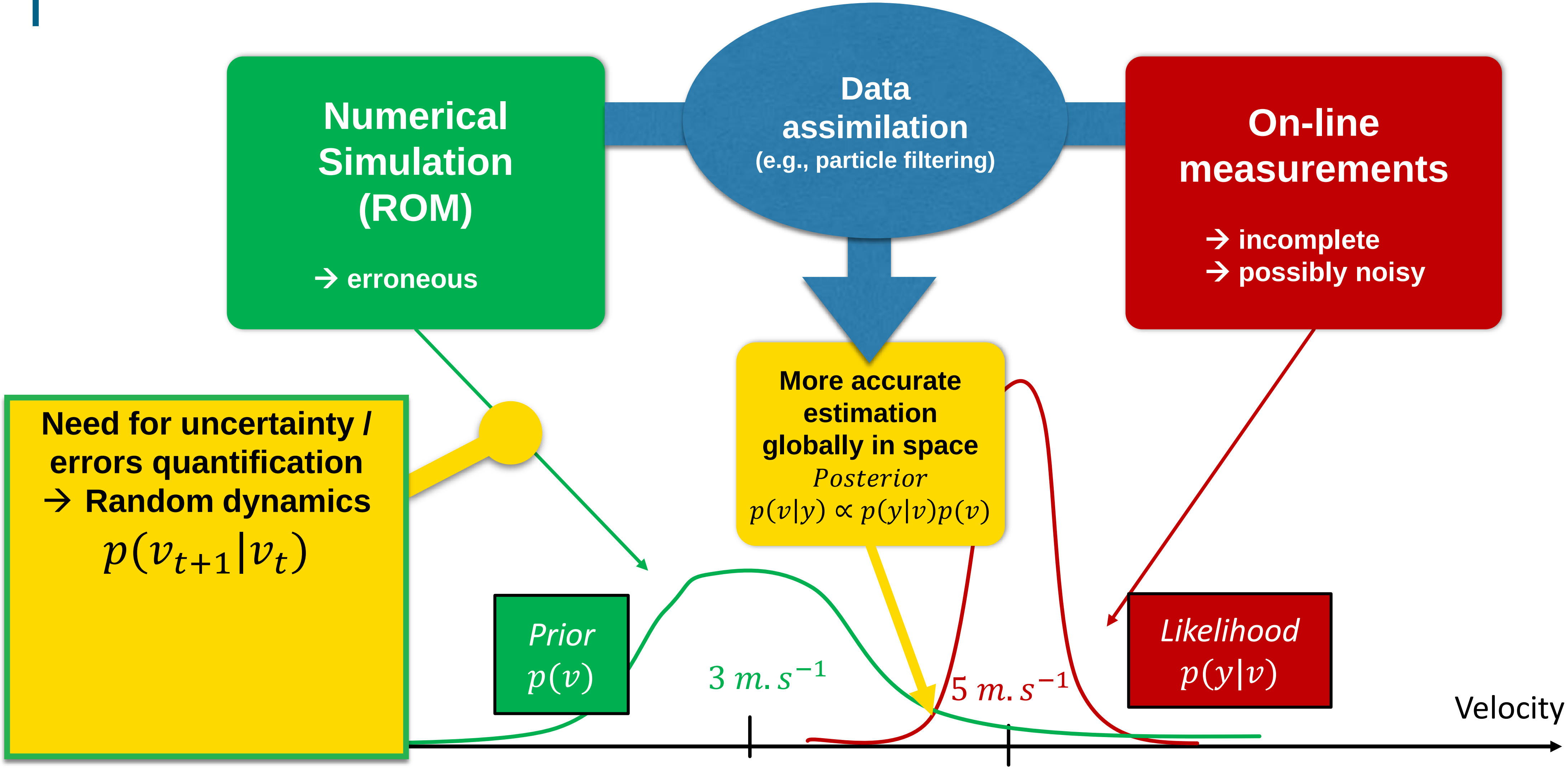

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Posterior

$p(v|y) \propto p(y|v)p(v)$

Need for uncertainty / errors quantification

→ Random dynamics

$p(v_{t+1}|v_t)$

Prior

$p(v)$

Likelihood

$p(y|v)$

DATA ASSIMILATION

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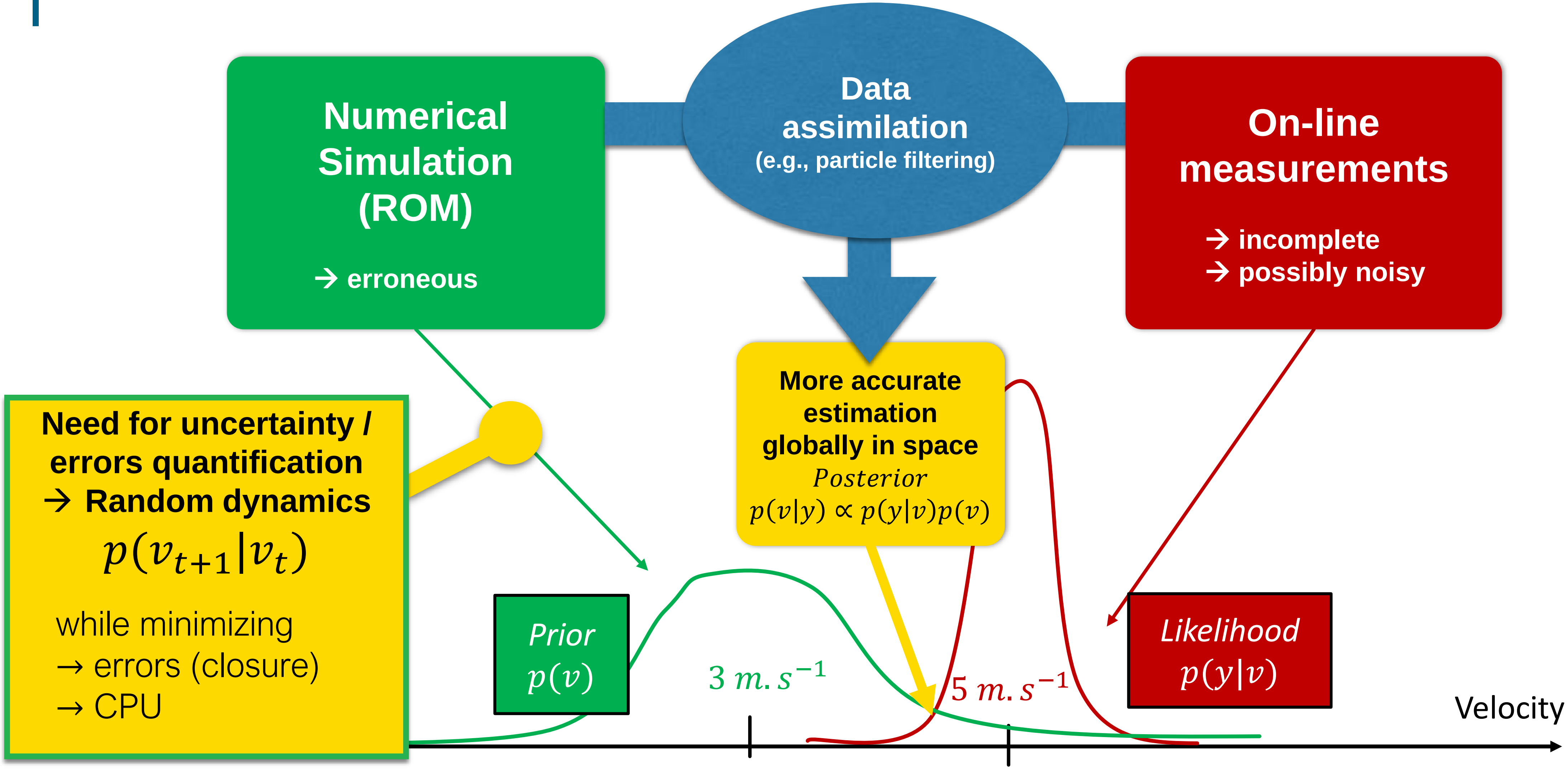

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More accurate estimation globally in space
Posterior
$p(v|y) \propto p(y|v)p(v)$

Need for uncertainty / errors quantification
→ Random dynamics

$$p(v_{t+1}|v_t)$$

while minimizing
→ errors (closure)
→ CPU

DATA ASSIMILATION

Example : the Particle Filter (PF) generates an ensemble $\sim p(v|y)$

- Initialization

  $v_{t=0}^{(j)} \sim \mathcal{N}(0, \Sigma)$

- Loop over time t

  Importance sampling

  - $v_t^{(j)} = M\left(v_{t-1}^{(j)}, noise(t-1)\right)$ Forecast (“Prior” or “backgroud”)
  - If an observation y_t is available at the current time t
    - ○ $W_j(t) \propto p\left(y_t \middle| v_t^{(j)}\right)$ Likelihood evaluation, up to a constant
    - ○ $\mathbf{W}_j(t) = \frac{W_j(t)}{\sum_{k=1}^{N_p} W_k(t)}$ Normalization

  Resampling

    - ○ Each new $v_t^{(j)}$ is replaced by one of the old particles $v_t^{(1)}, \ldots, v_t^{(N_p)}$ with probability $\mathbf{W}_1(t), \ldots, \mathbf{W}_{N_p}(t)$, respectively.

- Final posterior distribution

$$p\left(v_t \middle| y_{t_1}, \ldots, y_{t_K}\right) \approx \sum_{k=1}^{N_p} \frac{1}{N_p} \delta\left(v_t - v_t^{(k)}\right)$$

DATA ASSIMILATION

Example : the Particle Filter (PF) generates an ensemble $\sim p(v|y)$

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    Resampling

    - ○ Each new $v_t^{(j)}$ is replaced by one of the old particles $v_t^{(1)}, \dots, v_t^{(N_p)}$ with probability $\mathbf{W}_1(t), \dots, \mathbf{W}_{N_p}(t)$, respectively.
- Final posterior distribution

$$p\left(v_t \middle| y_{t_1}, \dots, y_{t_K}\right) \approx \sum_{k=1}^{N_p} \frac{1}{N_p} \delta\left(v_t - v_t^{(k)}\right)$$

PART III

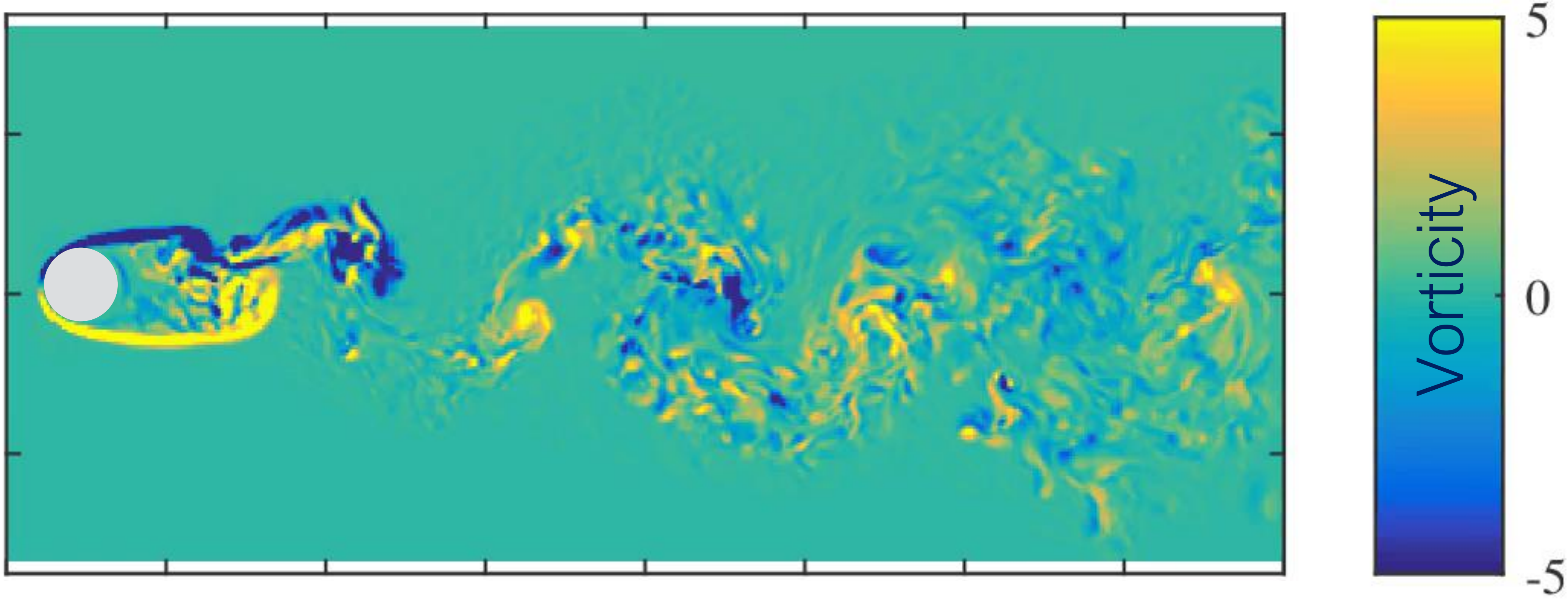
REDUCED LOCATION UNCERTAINTY MODELS

a. Multiscale modeling

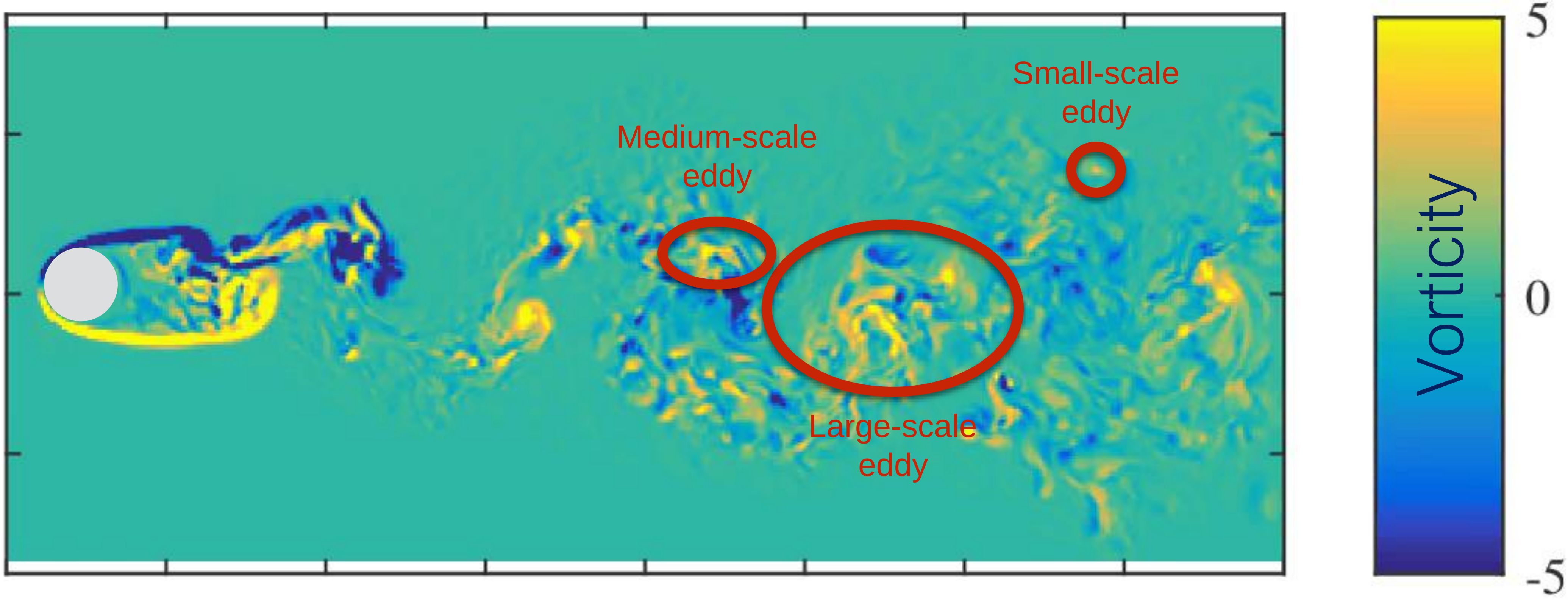
b. Location uncertainty models (LUM)

c. Reduced LUM (Red LUM)

MULTISCALE MODELING

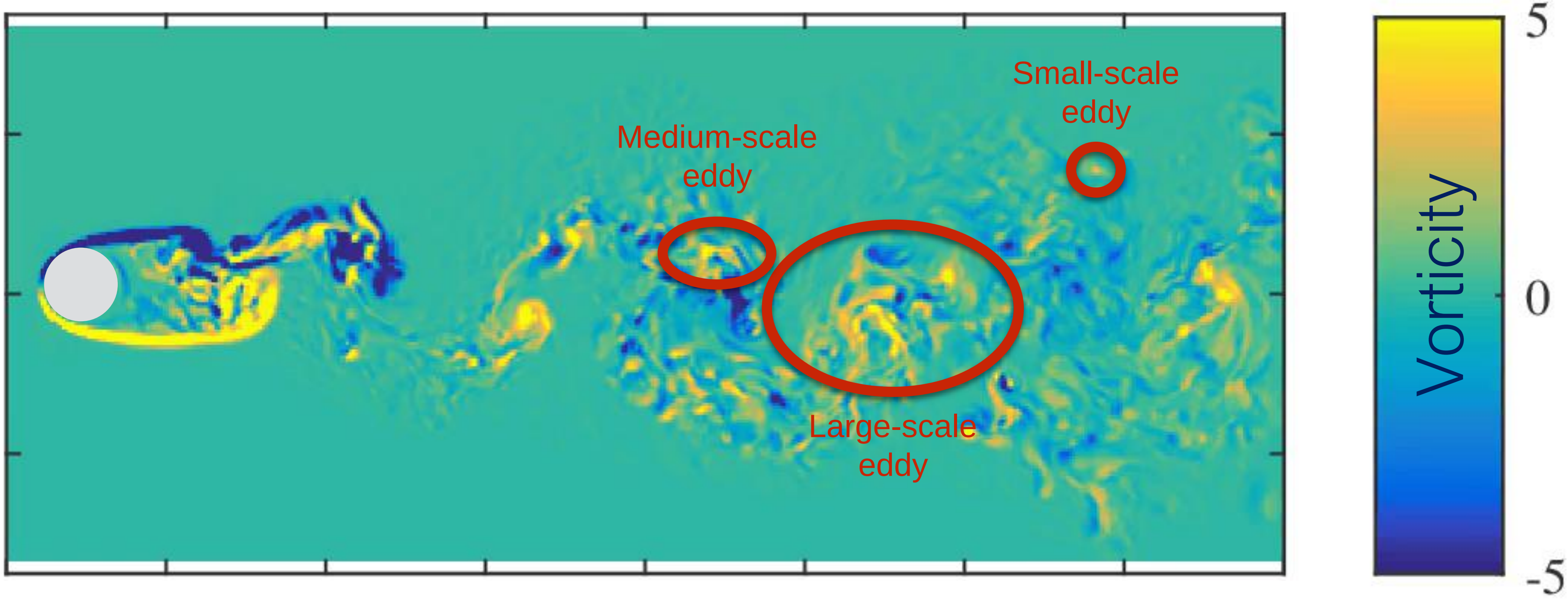


MULTISCALE MODELING



MULTISCALE MODELING

Fluids are multiscale → Many coupled degrees of freedom



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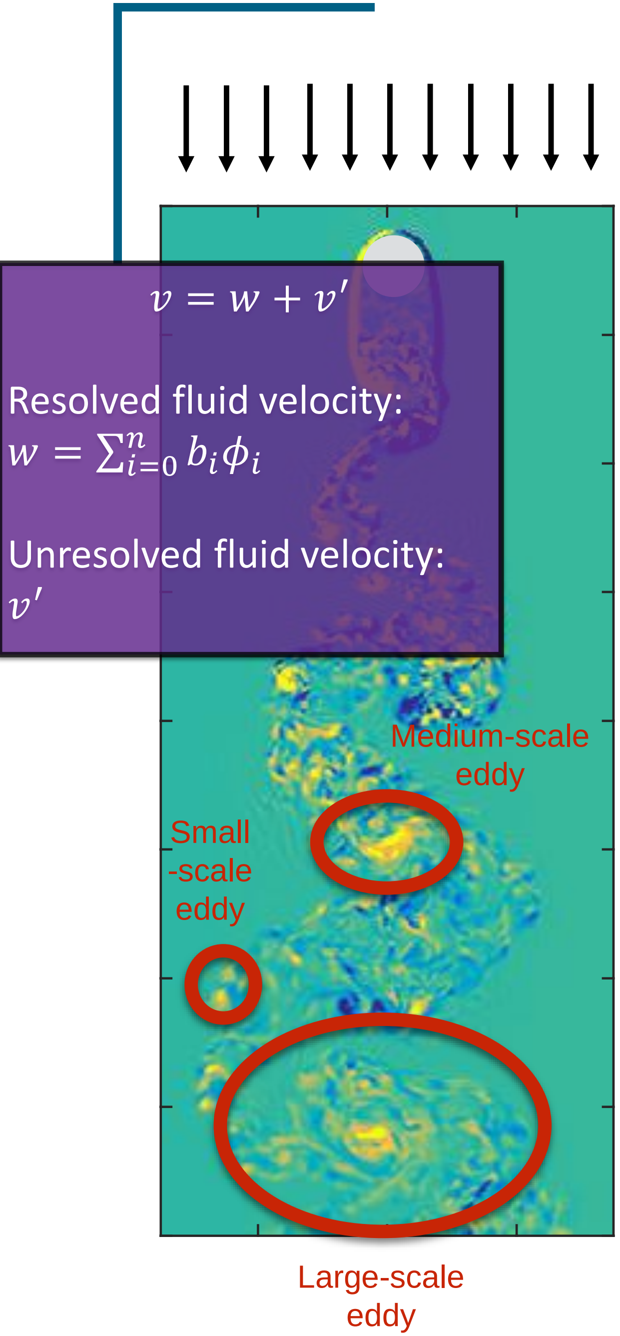

$v = w + v'$

Resolved fluid velocity:
$w = \sum_{i=0}^{n} b_i \phi_i$

Unresolved fluid velocity:
v'

Fluids are multiscale → Many coupled degrees of freedom

We cannot simulate (or observe) every scales.

Generally, authors

- simulate large scales w,
- model *the effect of* small scales v' in the equations (closure).

Here, we

- **model the small scales v' through *stochastic functions*,**
  parametrized from data and/or from physical scale symmetries.
- **inject those in *physical* equations**
  for physical understanding, simulations & data assimilation.

LOCATION UNCERTAINTY MODELS (LUM)

$$v = w + v'$$

Resolved fluid velocity:
$w = \sum_{i=0}^{n} b_i \phi_i$

Unresolved fluid velocity:
$v' = \frac{\sigma dB_t}{dt}$

Assumed
(conditionally-)Gaussian
& white in time
(non-stationary in space)

LOCATION UNCERTAINTY MODELS (LUM)

$v = w + v'$

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Randomized
Navier-Stokes model

- Good closure
- Good model error quantification

for data assimilation

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References :

Mikulevicius & Rozovskii, 2004
Flandoli, 2011

LUM

Memin, 2014
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Cai et al. 2017
Chapron et al. 2018
Yang & Memin 2019

SALT

Holm, 2015
Holm and Tyranowski, 2016
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Randomized Navier-Stokes

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Unresolved fluid velocity:
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(assuming $\nabla \cdot w = 0$ and $\nabla \cdot v' = 0$)

Momentum conservation

$$\mathrm{d}\big(w(t, X_t)\big) = dF \text{ (Forces)}$$

Positions of fluid parcels X_t :

$$dX_t = w(t, X_t)dt + \underbrace{\sigma(t, X_t)dB_t}_{\substack{\text{Gaussian process} \\ \text{white in time}}}$$

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Variance tensor:
$a(x,x) = \frac{\mathbb{E}\{(\sigma dB_t)(\sigma dB_t)^T\}}{dt}$

From Ito-Wentzell formula (Kunita 1990) with Ito notations

$$d_t w + w^* dt \cdot \nabla w + \sigma dB_t \cdot \nabla w - \nabla \cdot \left(\frac{1}{2} a \nabla w \right) dt = dF$$

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Usual terms

Skew-symmetric multiplicative random forcing

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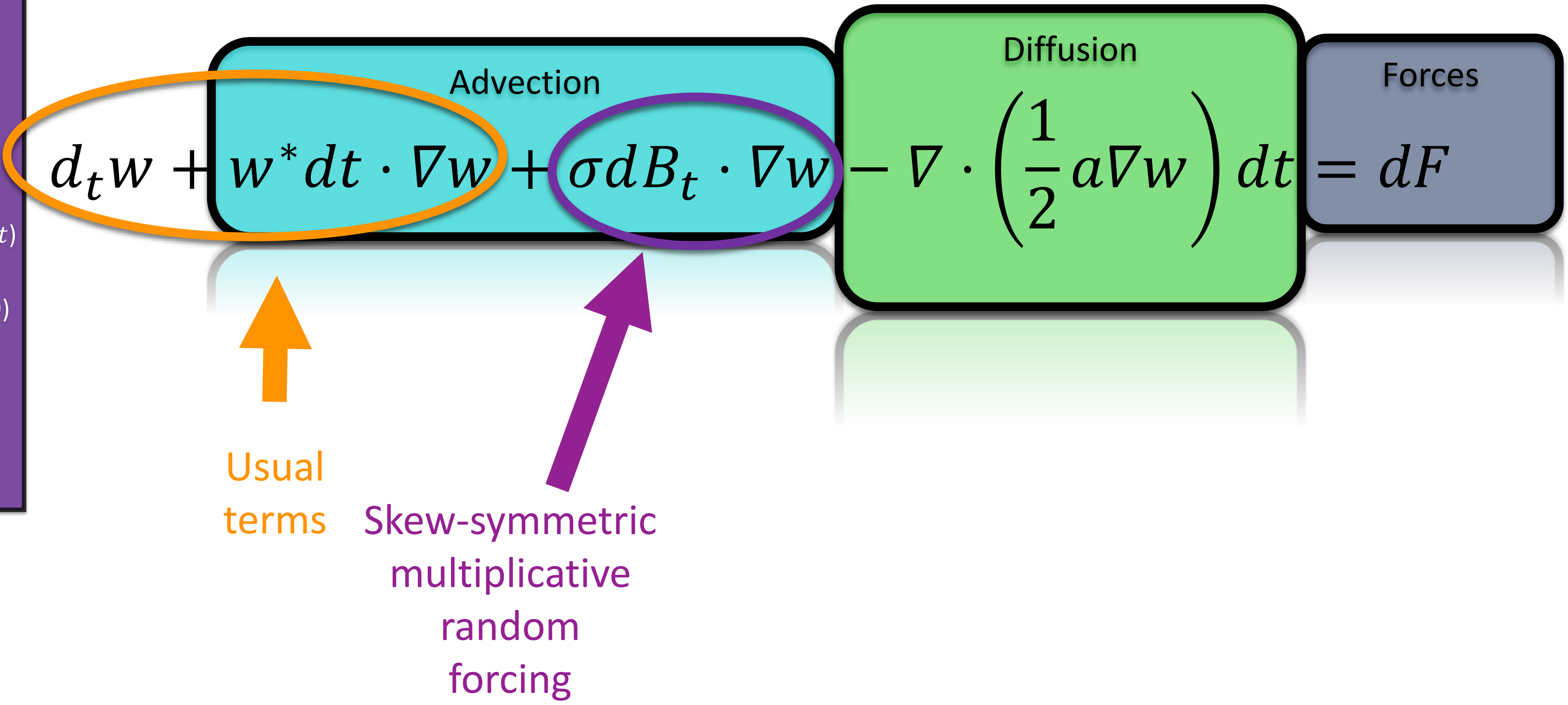

Advection

Diffusion

Forces

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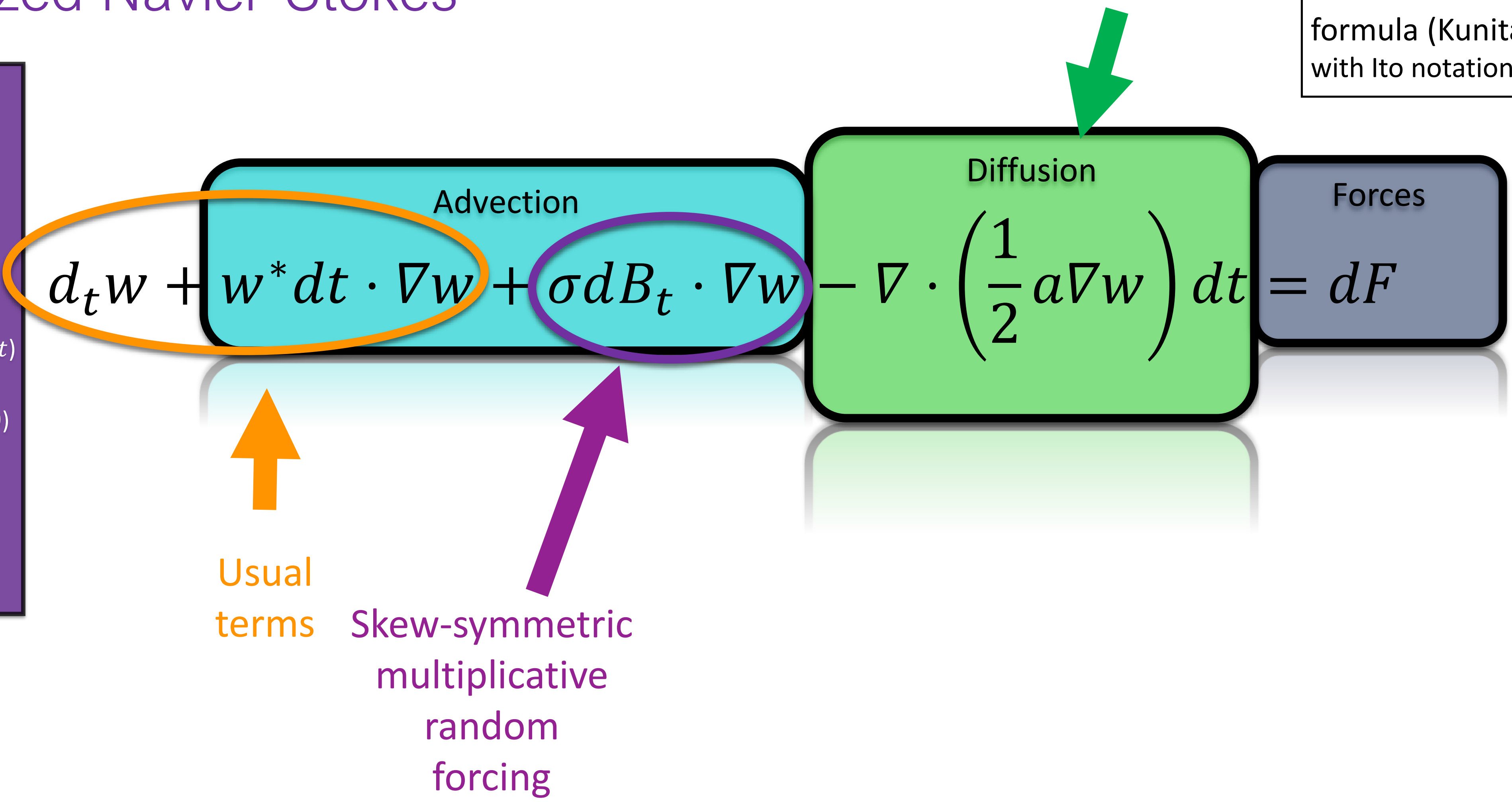

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Forces

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Symmetric negative

Usual terms

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Resseguier V. & al. (2017). Part I. Geophys. Astro. Fluid. hal-01391420

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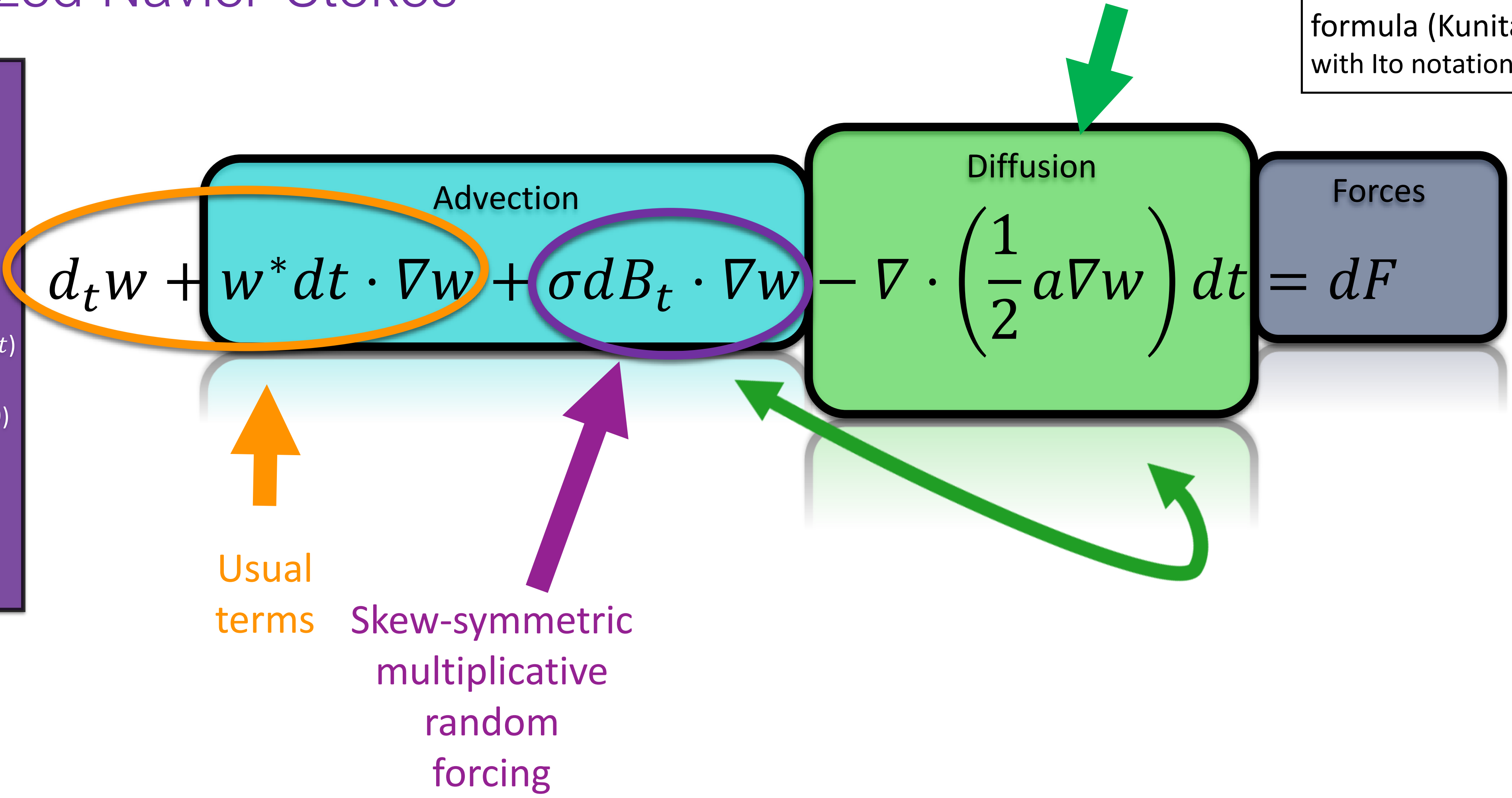

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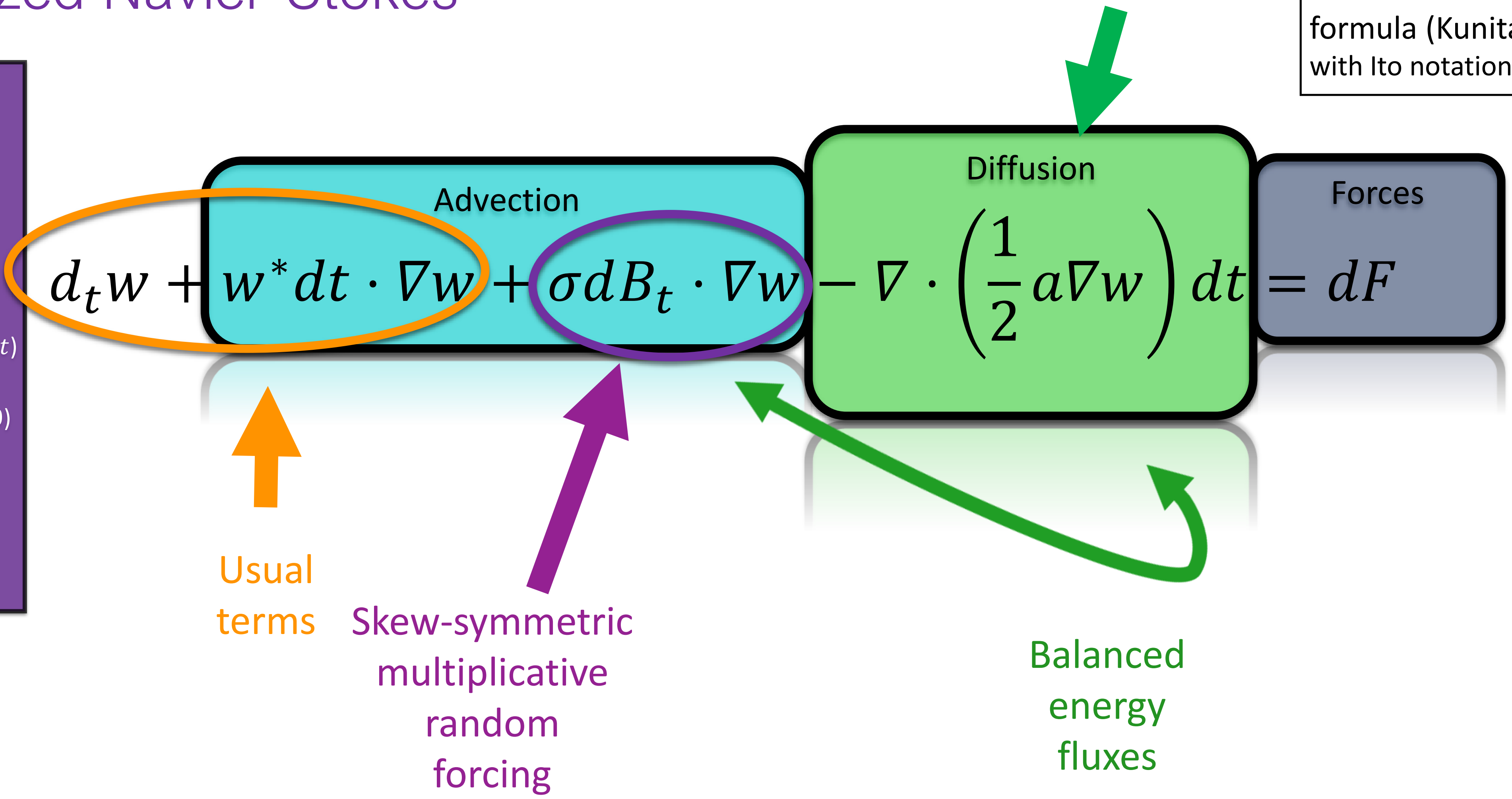

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Skew-symmetric multiplicative random forcing

Balanced energy fluxes

Resseguier V. & al. (2017). Part I. Geophys. Astro. Fluid. hal-01391420

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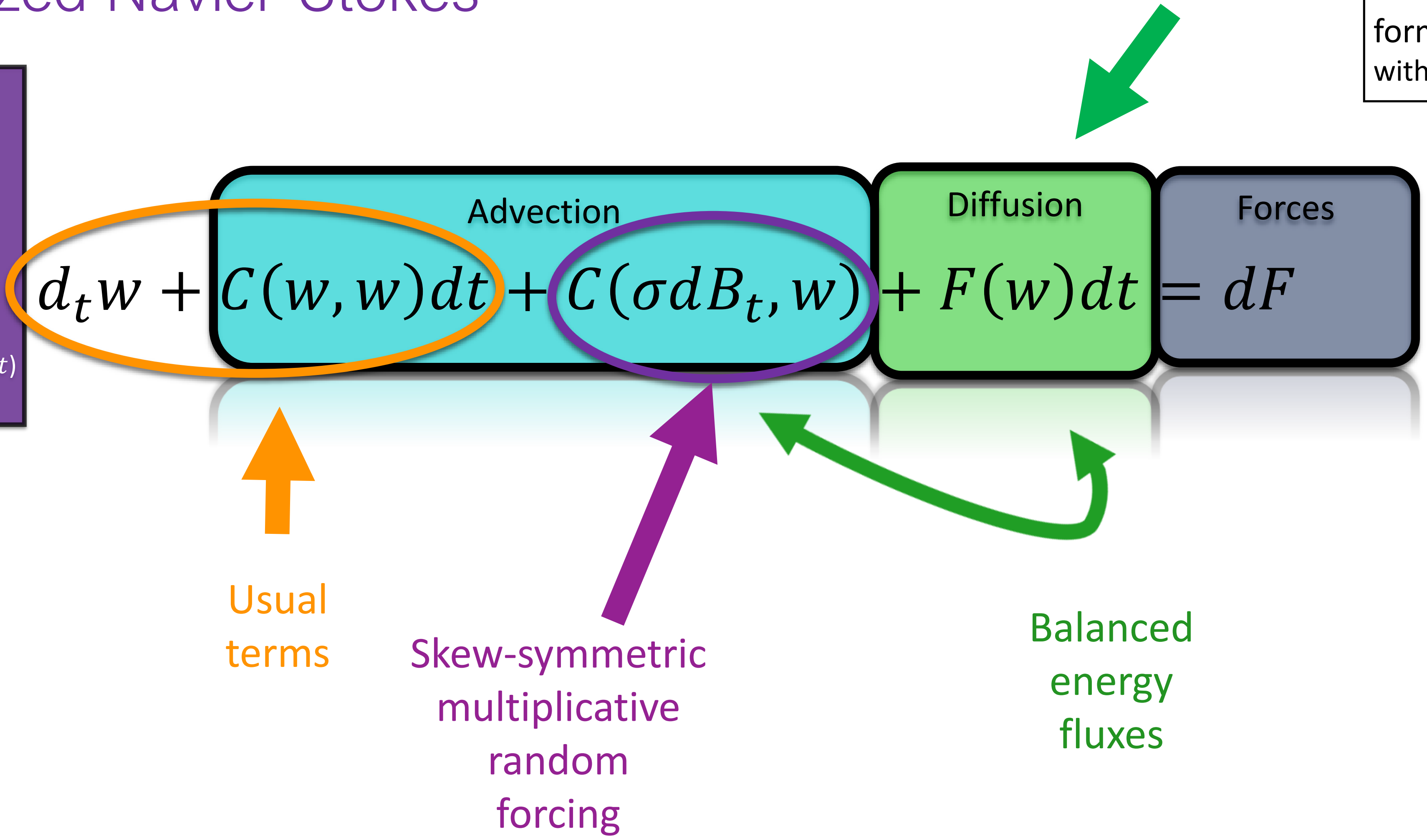

$$d_t w + C(w,w)dt + C(\sigma dB_t, w) + F(w)dt = dF$$

Symmetric negative

Usual terms

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Resseguier V. & al. (2017). Part I. Geophys. Astro. Fluid. hal-01391420

REDUCED LUM (RED LUM)

POD-Galerkin gives SDEs for resolved modes

Full order : $M \sim 10^7$
Reduced order : $n \sim 10$

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Resolved fluid velocity:
$w(x,t) = \sum_{i=0}^{n} b_i(t)\phi_i(x)$

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$$\int_{\Omega} \phi_i(x) \cdot \left(d_t w + C(w,w)dt + F(w)dt + C(\sigma dB_t, w) = dF\right) dx$$

Resseguier et al. (2021). *SIAM-ASA J Uncertain* . hal- 03169957

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$$\Rightarrow \quad db(t) = H\big(b(t)\big)\, dt + K(\sigma dB_t)\, b(t)$$

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$$\Rightarrow \quad db(t) = \boxed{H(b(t))}\, dt + K(\sigma dB_t)\, b(t)$$

2nd order polynomial

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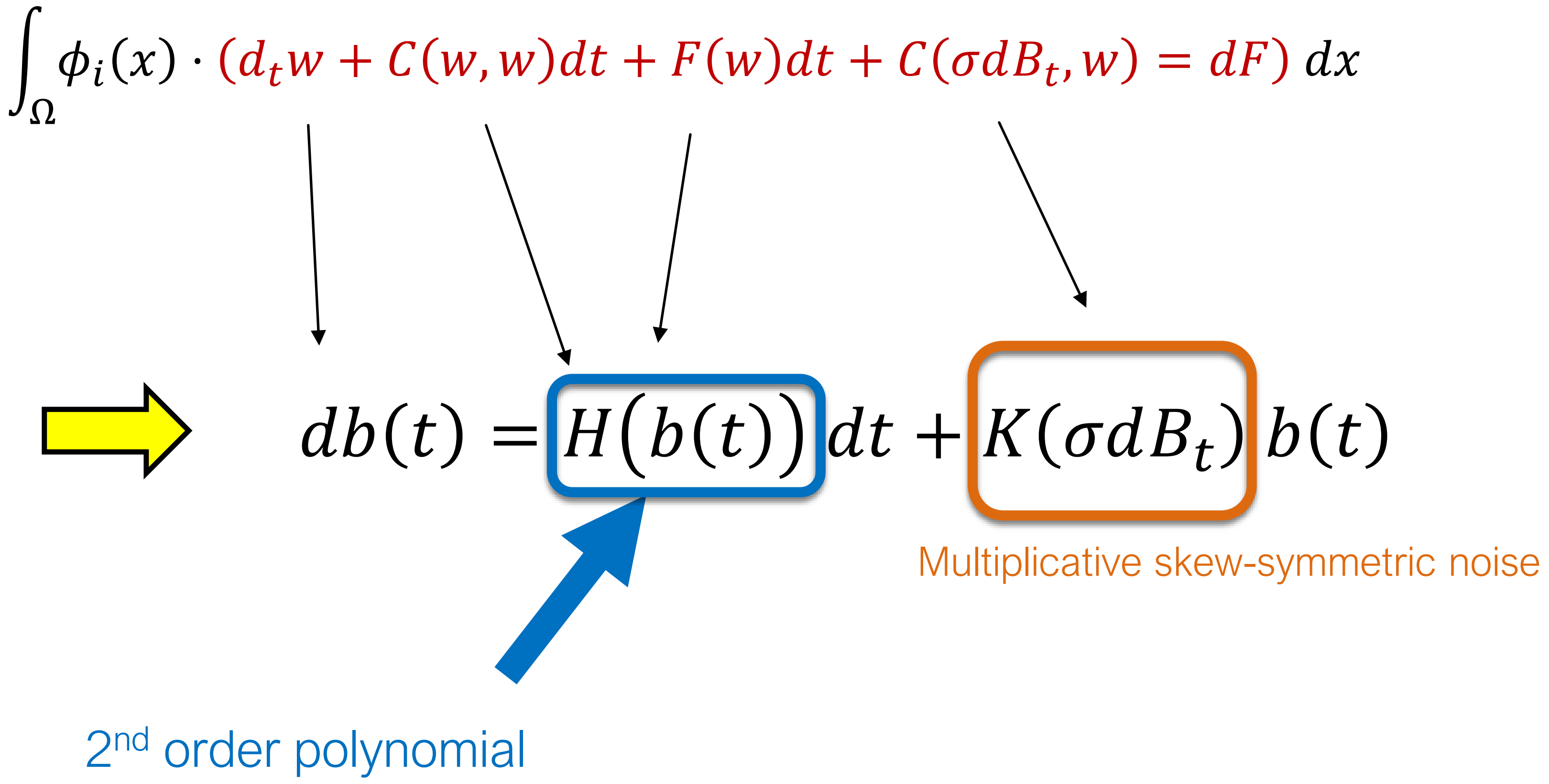

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Multiplicative skew-symmetric noise

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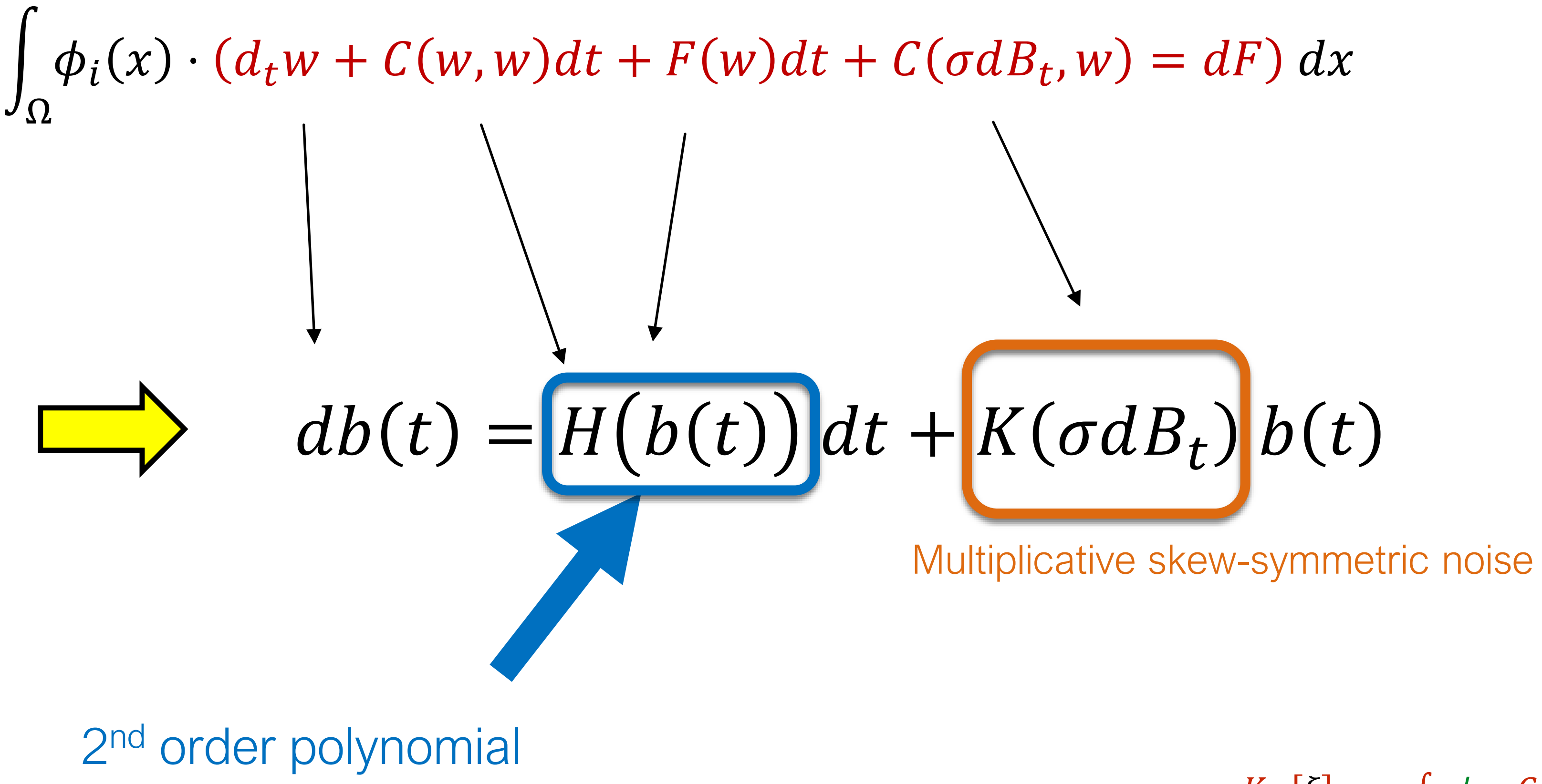

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$K_{jq}[\xi] = -\int_\Omega \phi_j \cdot C(\xi, \phi_q)$

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2nd order polynomial

Coefficients given by :
- Randomized Navier-Stokes
- $(\phi_j)_j$
- $a(x) \approx \Delta t\ \overline{v'\ (v')^T}$

$$\overline{f} = \frac{1}{T}\int_0^T f$$

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$w(x,t) = \sum_{i=0}^{n} b_i(t)\phi_i(x)$

Unresolved fluid velocity:
$v' = \frac{\sigma dB_t}{dt}$ (Gaussian, white wrt t)

Variance tensor:
$a(x,x) = \frac{\mathbb{E}\{(\sigma dB_t)(\sigma dB_t)^T\}}{dt}$

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$$db(t) = \boxed{H\big(b(t)\big)}\, dt + K(\sigma dB_t)\ \ b(t)$$

2nd order polynomial

Coefficients given by :

- Randomized Navier-Stokes
- $(\phi_j)_j$
- $a(x) \approx \Delta t\ \overline{v' \ (v')^T}$

Randomized Navier-Stokes

PCA modes

$$\overline{f} = \frac{1}{T}\int_0^T f$$

$$K_{jq}[\xi] = -\int_\Omega \phi_j \cdot C(\xi, \phi_q)$$

Resseguier et al. (2021). *SIAM-ASA J Uncertain* . hal- 03169957

REDUCED LUM (RED LUM)

POD-Galerkin gives SDEs for resolved modes

Full order : $M \sim 10^7$
Reduced order : $n \sim 10$

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Multiplicative skew-symmetric noise

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2nd order polynomial

Multiplicative skew-symmetric noise

Covariance to estimate

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Multiplicative skew-symmetric noise

Covariance to estimate

$$\mathbb{E}\Big(K_{jq}(\sigma dB_t)\, K_{ip}(\sigma dB_t)\Big)/dt \approx \Delta t\, K_{jq}\left[\overline{\frac{b_p}{\overline{b_p^2}}\,\frac{\Delta b_i}{\Delta t}\, v'}\right]$$

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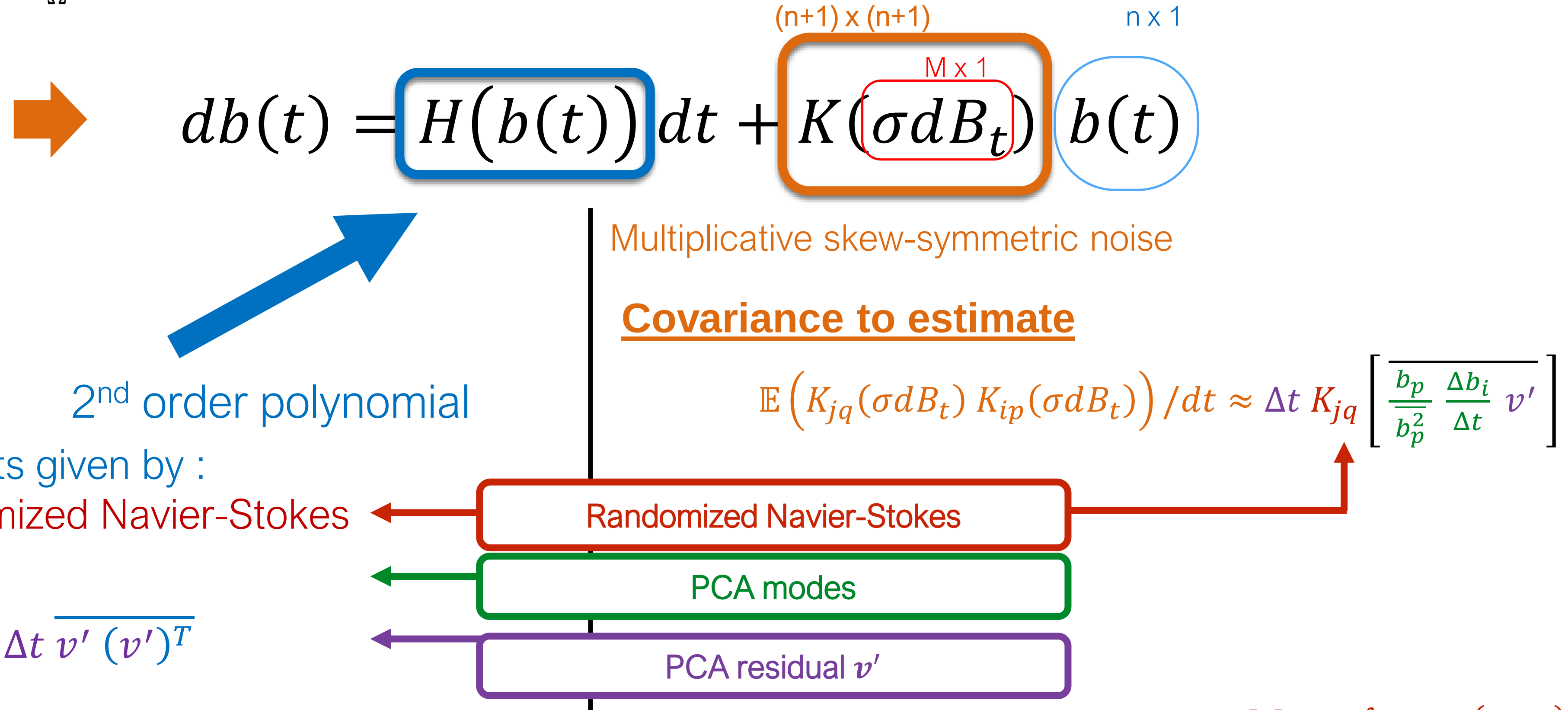

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(n+1) x (n+1)

M x 1

n x 1

2nd order polynomial

Multiplicative skew-symmetric noise

Covariance to estimate

$$\mathbb{E}\left(K_{jq}(\sigma dB_t)\, K_{ip}(\sigma dB_t)\right)/dt \approx \Delta t\, K_{jq}\left[\overline{\frac{b_p}{\overline{b_p^2}} \frac{\Delta b_i}{\Delta t}\, v'}\right]$$

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(n+1) x (n+1)   M x 1   n x 1

$$db(t) = \boxed{H\big(b(t)\big)}\,dt + \boxed{K\big(\boxed{\sigma dB_t}\big)}\,\boxed{b(t)}$$

2nd order polynomial

Multiplicative skew-symmetric noise

Covariance to estimate

$$\mathbb{E}\left(K_{jq}(\sigma dB_t)\,K_{ip}(\sigma dB_t)\right)/dt \approx \Delta t\, K_{jq}\left[\overline{\frac{b_p}{b_p^2}\,\frac{\Delta b_i}{\Delta t}\,v'}\right]$$

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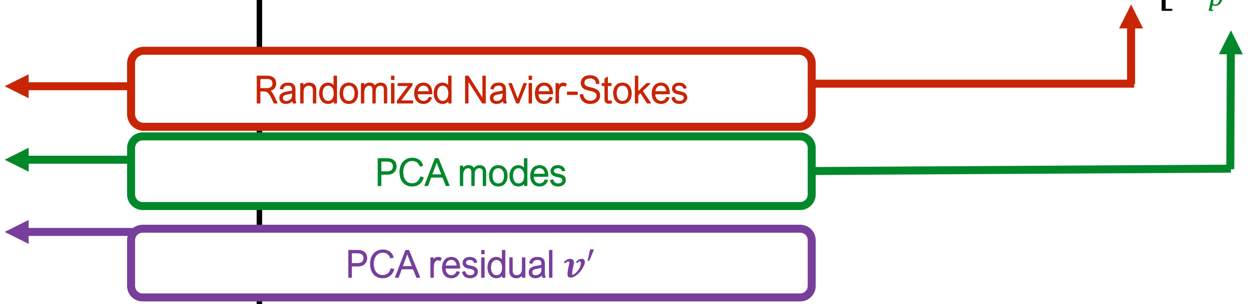

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2nd order polynomial

Multiplicative skew-symmetric noise

Covariance to estimate

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Randomized Navier-Stokes

PCA modes

PCA residual v'

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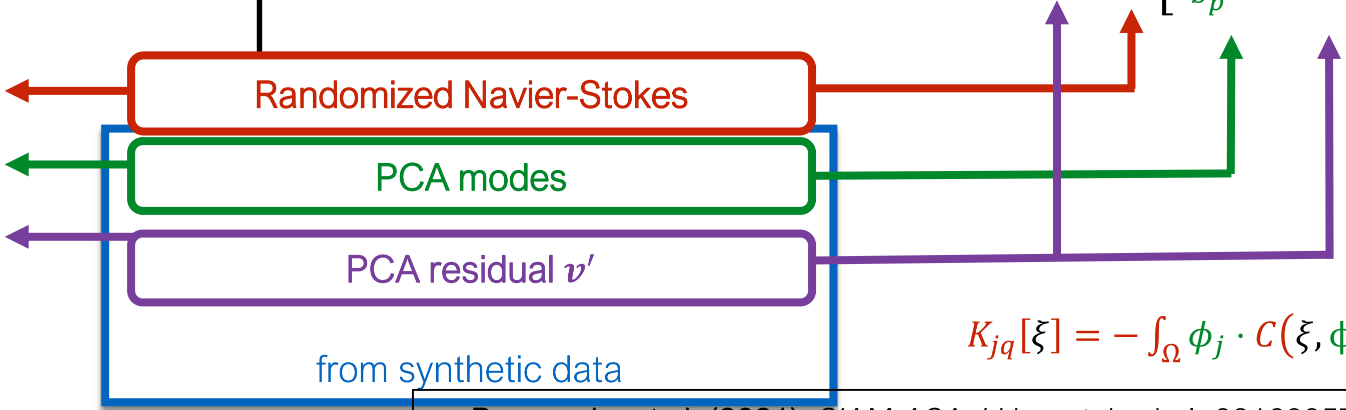
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PCA modes
PCA residual v'
from synthetic data

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New estimator

- Consistency proven ($\Delta t \to 0$)
- Numerically efficient
- Physically-based
  → Robustness in extrapolation

Multiplicative skew-symmetric noise

Covariance to estimate

$$\mathbb{E}\left(K_{jq}(\sigma dB_t)\,K_{ip}(\sigma dB_t)\right)/dt \approx \Delta t\, K_{jq}\left[\overline{\frac{b_p}{\overline{b_p^2}}\,\frac{\Delta b_i}{\Delta t}\,v'}\right]$$

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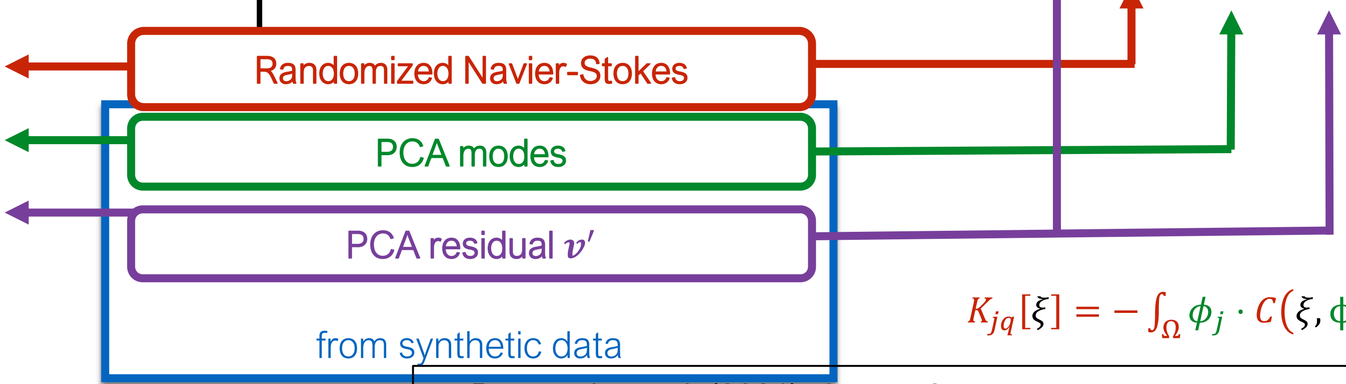

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PCA modes

PCA residual v'

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REDUCED LUM (RED LUM)

Multiplicative noise covariance

Full order (∼ nb spatial grid points): $M \sim 10^7$
Reduced order : $n \sim 10$
Number of time steps : $N \sim 10^4$

$$v = w + v'$$

Resolved fluid velocity:
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PCA modes

PCA residual v'

from synthetic data

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- **Curse of dimensionality**
  - Since σdB_t is white in time,
    $\Sigma_{jq,ip} = \mathbb{E}\Big(K_{jq}(\sigma dB_t)\, K_{ip}(\sigma dB_t)\Big)/dt \approx \Delta t\, \overline{K_{jq}(v')\, K_{ip}(v')}$
  - K is a matrix of integro-differential operators → cannot be evaluated on $v'(x,t)$ at every time t
  - Covariance of $\sigma dB_t \approx \Delta t^2\, \overline{\big(v'(x,t)\big)\big(v'(y,t)\big)^T}$: $M \times M \sim 10^{13}$ coefficients → intractable

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  - **Efficient estimator** $\Sigma_{jq,ip} \approx \Delta t\, K_{jq}\left[\overline{\frac{\overline{b_p}}{\overline{b_p^2}}\, \frac{\Delta b_i}{\Delta t}\, v'}\right]$ (hybrid fitting & physics-based)
    requires only $O(n^2 M)$ correlation estimations and $O(n^2)$ evaluations of K

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    requires only $O(n^2 M)$ correlation estimations and $O(n^2)$ evaluations of K
- **Consistency of our estimator** (convergence in probability for $\Delta t \to 0$, using stochastic calculus and continuity of K)

$$\Delta t\; K_{jq}\left[\overline{b_p\frac{\Delta b_i}{\Delta t}\; v'}\right] = \Delta t\;\overline{b_p\frac{\Delta b_i}{\Delta t}K_{jq}[v']} \approx \frac{1}{T}\int_0^T b_p\; d < b_i, K_{jq}(\sigma B) > = \frac{1}{T}\int_0^T b_p \sum_{\mathrm{r}=0}^{\mathrm{n}} b_r d < K_{ir}(\sigma B), K_{jq}(\sigma B) > = \sum_{\mathrm{r}=0}^{\mathrm{n}} \Sigma_{jq,ir}\;\overline{b_p b_r} = \Sigma_{jq,ip}\;\overline{b_p^2} \quad \text{(orthogonality from PCA)}$$

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PCA residual v'

from synthetic data

$$db(t) = H\big(b(t)\big)\, dt + \underset{(n+1) \times (n+1)}{\boxed{K\big(\overset{M \times 1}{\boxed{\sigma dB_t}}\big)}}\, b(t) \quad \text{with} \quad K_{jq}[\xi] = -\int_\Omega \phi_j \cdot C(\xi, \phi_q)$$

- **Curse of dimensionality**
  - Since σdB_t is white in time, $\Sigma_{jq,ip} = \mathbb{E}\Big(K_{jq}(\sigma dB_t)\, K_{ip}(\sigma dB_t)\Big)/dt \approx \Delta t\, \overline{K_{jq}(v')\, K_{ip}(v')}$
  - K is a matrix of integro-differential operators → cannot be evaluated on $v'(x,t)$ at every time t
  - Covariance of $\sigma dB_t \approx \Delta t^2\, \overline{\big(v'(x,t)\big)\big(v'(y,t)\big)^T}$: $M \times M \sim 10^{13}$ coefficients → intractable
  - **Efficient estimator** $\Sigma_{jq,ip} \approx \Delta t\, K_{jq}\left[\overline{\frac{b_p}{\overline{b_p^2}}\, \frac{\Delta b_i}{\Delta t}\, v'}\right]$ (hybrid fitting & physics-based) requires only $O(n^2 M)$ correlation estimations and $O(n^2)$ evaluations of K
- **Consistency of our estimator** (convergence in probability for $\Delta t \to 0$, using stochastic calculus and continuity of K)

$$\Delta t\, K_{jq}\left[\overline{b_p \frac{\Delta b_i}{\Delta t}\, v'}\right] = \Delta t\, \overline{b_p \frac{\Delta b_i}{\Delta t} K_{jq}[v']} \approx \frac{1}{T}\int_0^T b_p\, d < b_i, K_{jq}(\sigma B) > = \frac{1}{T}\int_0^T b_p \sum_{\mathrm{r}=0}^{\mathrm{n}} b_r d < K_{ir}(\sigma B), K_{jq}(\sigma B) > = \sum_{\mathrm{r}=0}^{\mathrm{n}} \Sigma_{jq,ir}\, \overline{b_p b_r} = \Sigma_{jq,ip}\, \overline{b_p^2} \text{ (orthogonality from PCA)}$$

- **Optimal time subsampling at Δt needed** to meet the white assumption

New estimator

- Consistency proven ($\Delta t \to 0$)
- Numerically efficient
- Physically-based

→ Robustness in extrapolation

$$\overline{f} = \frac{1}{T}\int_0^T f$$

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REDUCED LUM (RED LUM)

Multiplicative noise covariance

Full order (~ nb spatial grid points): $M \sim 10^7$
Reduced order : $n \sim 10$
Number of time steps : $N \sim 10^4$

$$v = w + v'$$

Resolved fluid velocity:
$w(x,t) = \sum_{i=0}^{n} b_i(t)\phi_i(x)$

Unresolved fluid velocity:
$v' = \frac{\sigma dB_t}{dt}$ (Gaussian, white wrt t)

Randomized Navier-Stokes

PCA modes

PCA residual v'

from synthetic data

$$db(t) = H\big(b(t)\big)\,dt + \underset{(n+1)\times(n+1)}{\boxed{K\big(\overset{\text{M x 1}}{\boxed{\sigma dB_t}}\big)}}\, b(t) \quad \text{with} \quad K_{jq}[\xi] = -\int_\Omega \phi_j \cdot C(\xi, \phi_q)$$

- **Curse of dimensionality**
  - Since σdB_t is white in time,
    $\Sigma_{jq,ip} = \mathbb{E}\Big(K_{jq}(\sigma dB_t)\; K_{ip}(\sigma dB_t)\Big)/dt \approx \Delta t\; \overline{K_{jq}(v')\; K_{ip}(v')}$
  - K is a matrix of integro-differential operators → cannot be evaluated on $v'(x,t)$ at every time t
  - Covariance of $\sigma dB_t \approx \Delta t^2\; \overline{\big(v'(x,t)\big)\big(v'(y,t)\big)^T}$: $M \times M \sim 10^{13}$ coefficients → intractable
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    requires only $O(n^2 M)$ correlation estimations and $O(n^2)$ evaluations of K
- **Consistency of our estimator** (convergence in probability for $\Delta t \to 0$, using stochastic calculus and continuity of K)

$$\Delta t\; K_{jq}\left[\overline{b_p \frac{\Delta b_i}{\Delta t}\; v'}\right] = \Delta t\; \overline{b_p \frac{\Delta b_i}{\Delta t} K_{jq}[v']} \approx \frac{1}{T}\int_0^T b_p\; d < b_i, K_{jq}(\sigma B) > = \frac{1}{T}\int_0^T b_p \sum_{\mathrm{r}=0}^{\mathrm{n}} b_r d < K_{ir}(\sigma B), K_{jq}(\sigma B) > = \sum_{\mathrm{r}=0}^{\mathrm{n}} \Sigma_{jq,ir}\; \overline{b_p b_r} = \Sigma_{jq,ip}\; \overline{b_p^2} \quad \text{(orthogonality from PCA)}$$

- **Optimal time subsampling at Δt needed** to meet the white assumption
- **Additional reduction for efficient sampling** :
  diagonalization of $\Sigma \;\rightarrow K(\sigma dB_t) \approx \alpha(d\beta_t)$ with a n-dimensional (instead of (n+1)²-dimensional) Brownian motion β

$$\overline{f} = \frac{1}{T}\int_0^T f$$

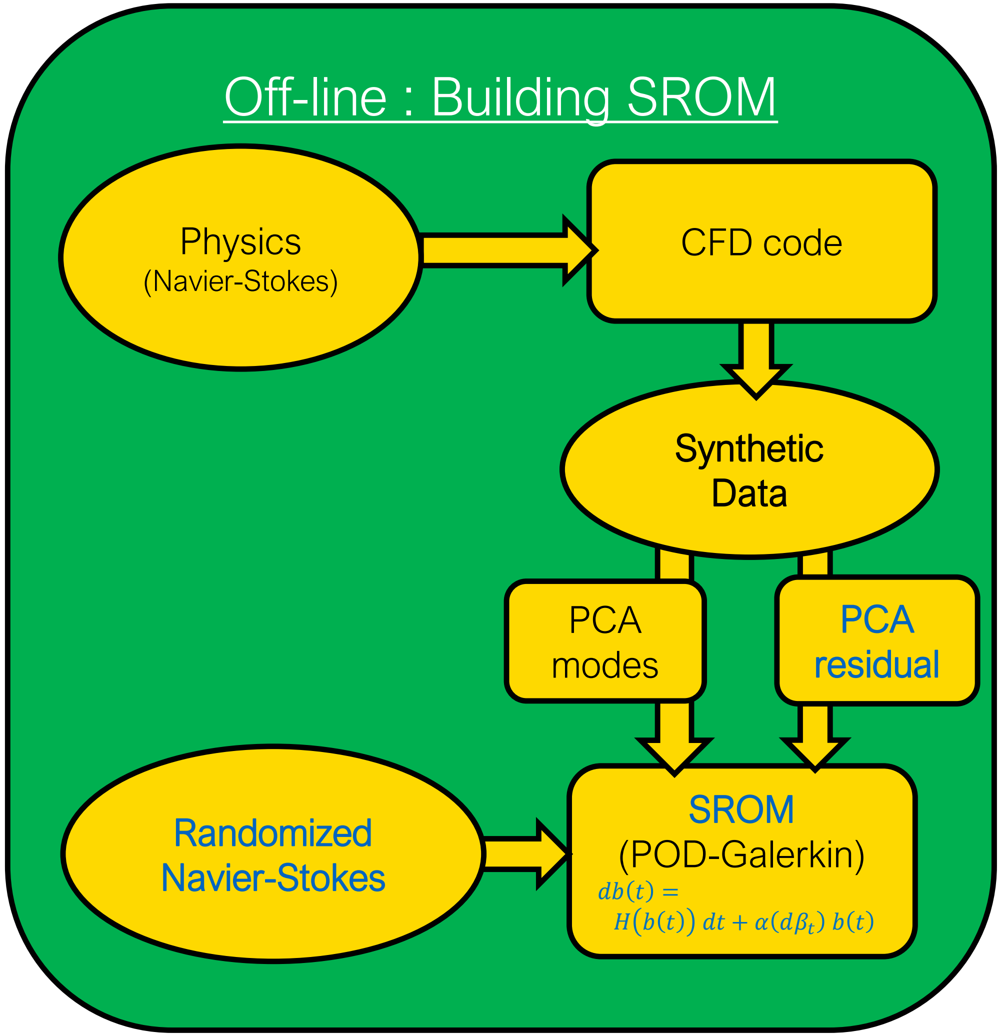
New estimator

- Consistency proven ($\Delta t \to 0$)
- Numerically efficient
- Physically-based
  → Robustness in extrapolation

Resseguier et al. (2021). *SIAM-ASA J Uncertain* . hal- 03169957

SUMMARY

Stochastic ROM + Data assimilation



Off-line : Building SROM

Physics
(Navier-Stokes)

CFD code

Synthetic
Data

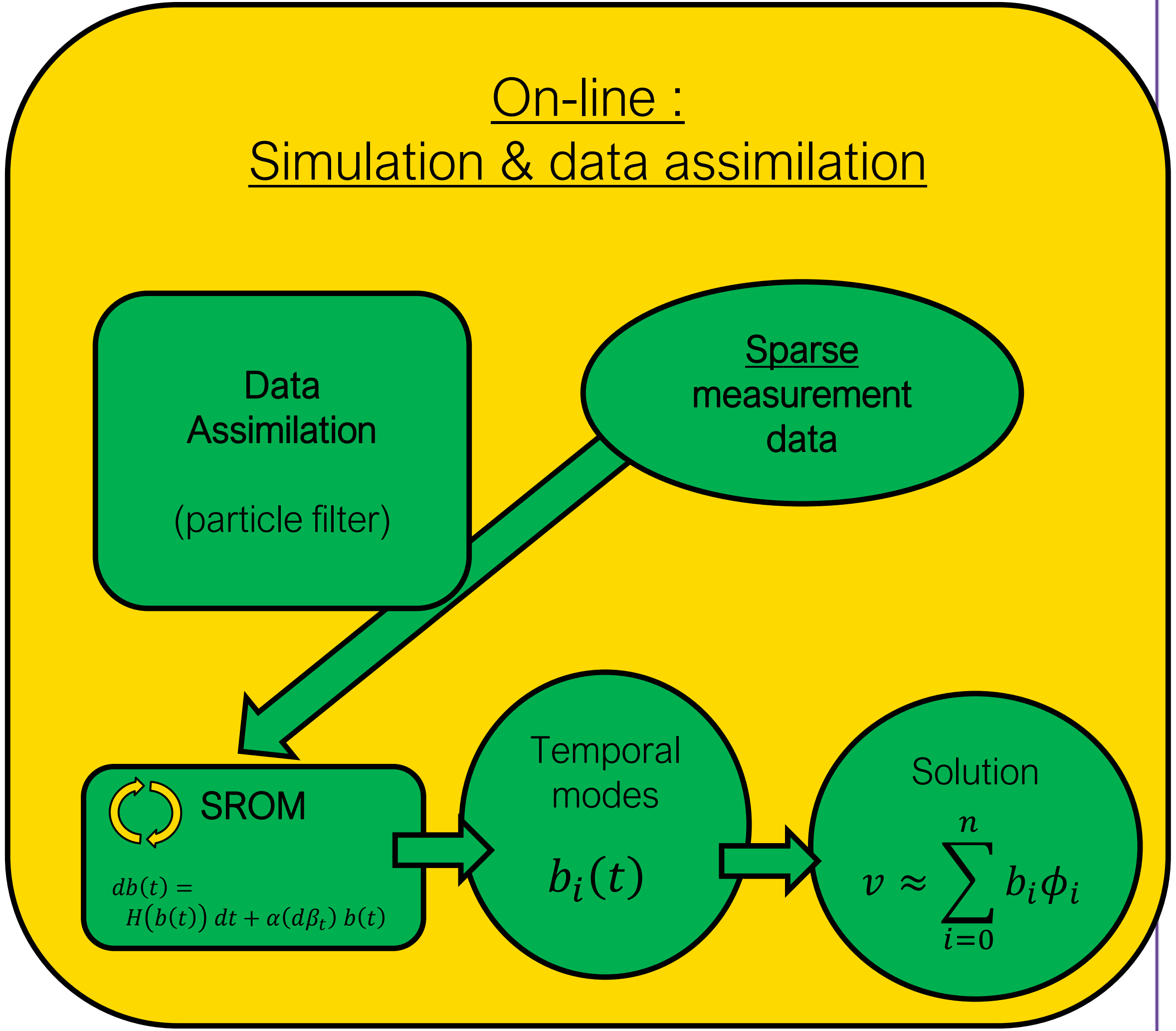
PCA
modes

PCA
residual

Randomized
Navier-Stokes

SROM
(POD-Galerkin)

$$db(t) = H(b(t))\,dt + \alpha(d\beta_t)\,b(t)$$



On-line :
Simulation & data assimilation

Data
Assimilation

(particle filter)

Sparse
measurement
data

SROM

$$db(t) = H(b(t))\,dt + \alpha(d\beta_t)\,b(t)$$

Temporal
modes

$b_i(t)$

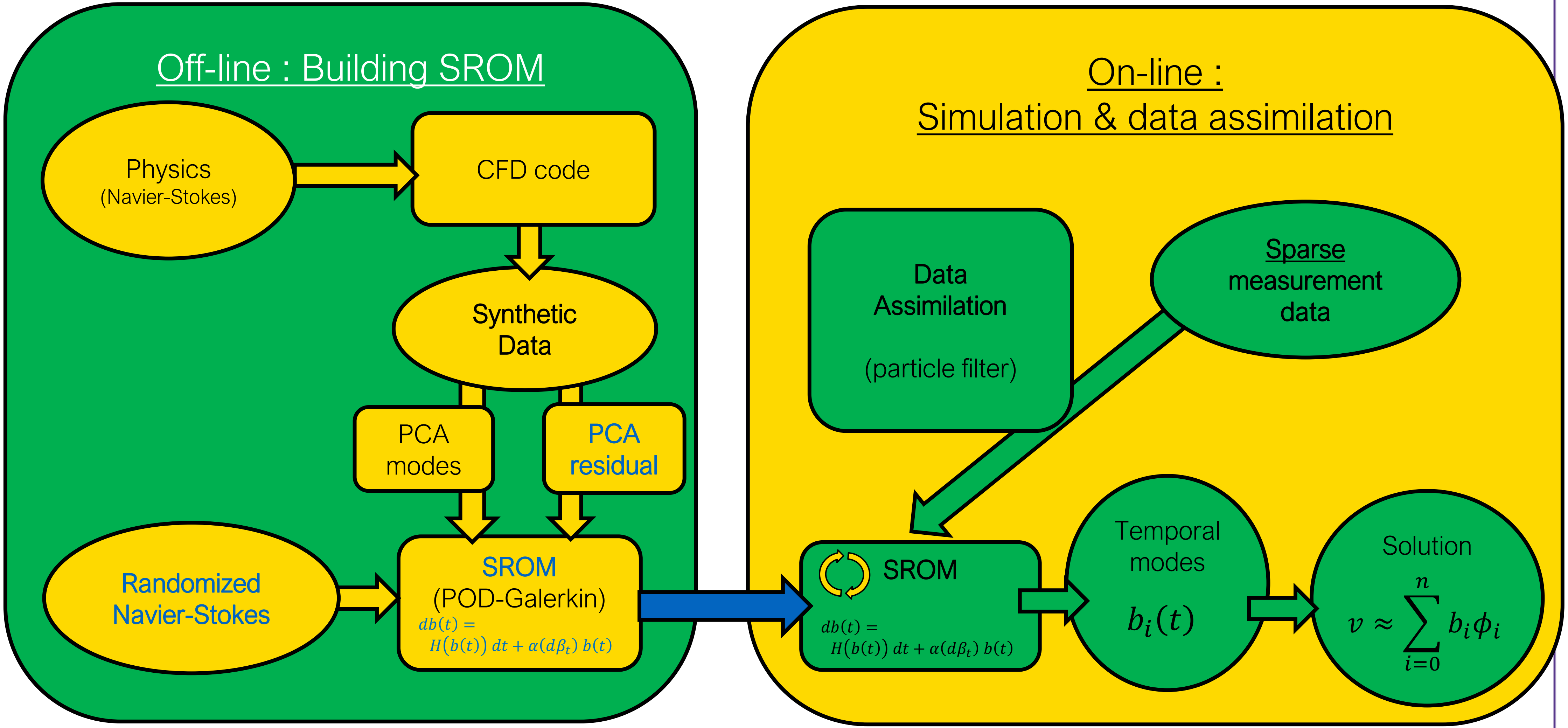
Solution

$$v \approx \sum_{i=0}^{n} b_i \phi_i$$

Resseguier et al. (2022). *J Comp.Phys* . hal-03445455

SUMMARY

Stochastic ROM + Data assimilation



Off-line : Building SROM

Physics
(Navier-Stokes)

CFD code

Synthetic Data

PCA modes

PCA residual

Randomized Navier-Stokes

SROM
(POD-Galerkin)

$db(t) = H(b(t))\,dt + \alpha(d\beta_t)\,b(t)$

On-line :
Simulation & data assimilation

Data Assimilation

(particle filter)

Sparse measurement data

SROM

$db(t) = H(b(t))\,dt + \alpha(d\beta_t)\,b(t)$

Temporal modes

$b_i(t)$

Solution

$v \approx \sum_{i=0}^{n} b_i \phi_i$

Resseguier et al. (2022). *J Comp.Phys* . hal-03445455

PART IV

NUMERICAL RESULTS

a. Uncertainty quantification (Prior)

b. Data assimilation (Posterior)

UNCERTAINTY QUANTIFICATION (PRIOR)

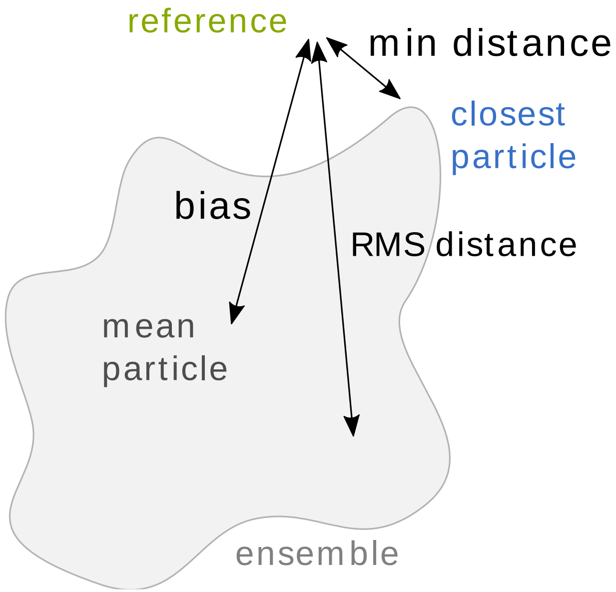
Known initial conditions $b(t = 0)$

From 10^7 to 8 degrees of freedom
No data assimilation
Known initial conditions $b(t = 0)$

$$db(t) = H\big(b(t)\big)\, dt + \alpha(d\beta_t)\, b(t)$$

Metrics choice

- $b_i(t)$ VS reference
- Error metrics



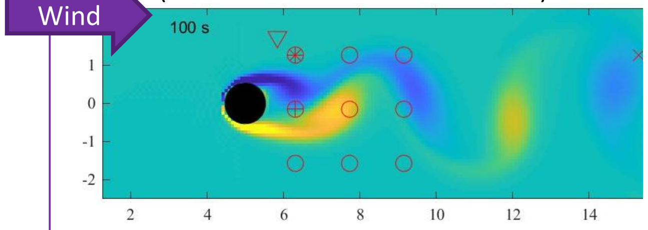
Test cases

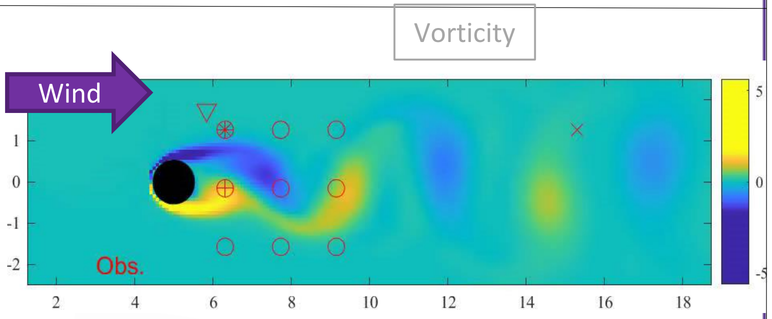
Full-order reference

Reduced-order reference

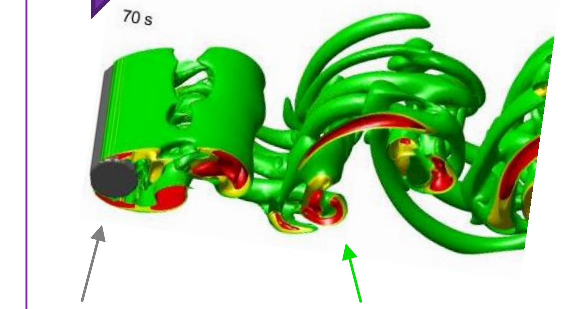
PCA-projection of the full-order simulation
(Optimal from 8-dof linear decomposition)

Reynolds number (Re) = 100 / 2D
(full-order simulation has 10^4 dof)



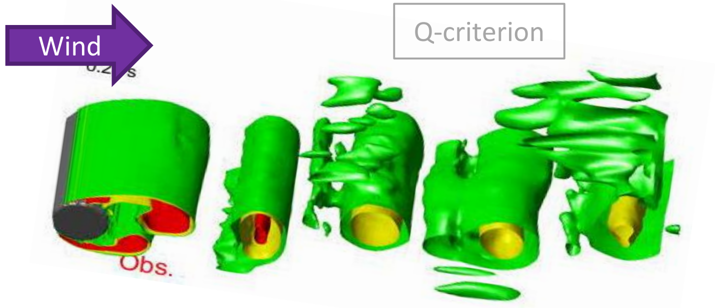


Reynolds number (Re) = 300 3D
(full-order simulation has 10^7 dof)



(round) wind turbine blade

vortices



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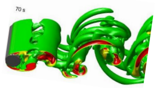
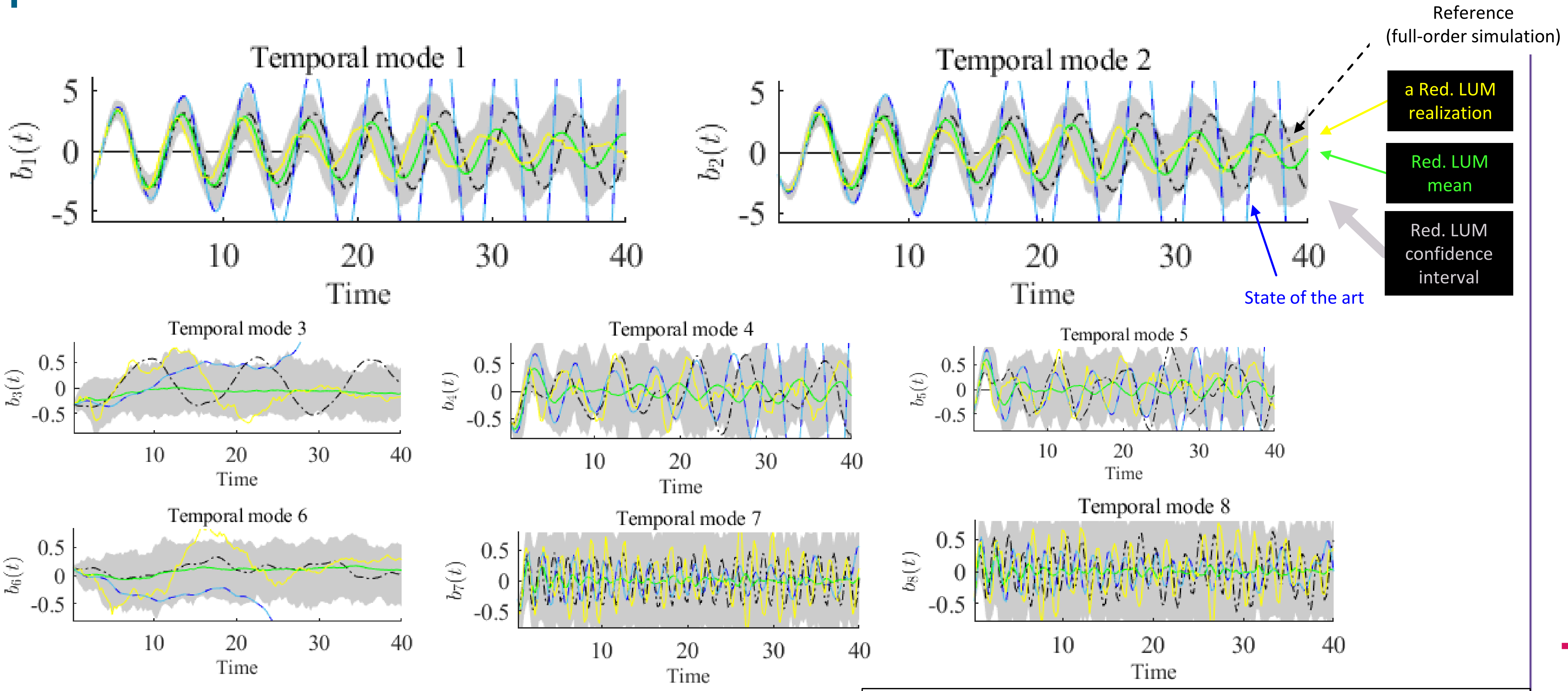
UNCERTAINTY QUANTIFICATION (PRIOR)

$b_i(t)$ VS reference



From 10^7 to 8 degrees of freedom
No data assimilation
Known initial conditions $b(t=0)$



Reference
(full-order simulation)

a Red. LUM realization

Red. LUM mean

Red. LUM confidence interval

State of the art

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UNCERTAINTY QUANTIFICATION (PRIOR)

Error on the reduced solution w

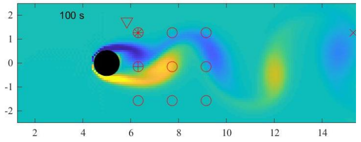
From 10^7 to 8 degrees of freedom
No data assimilation
Known initial conditions $b(t=0)$

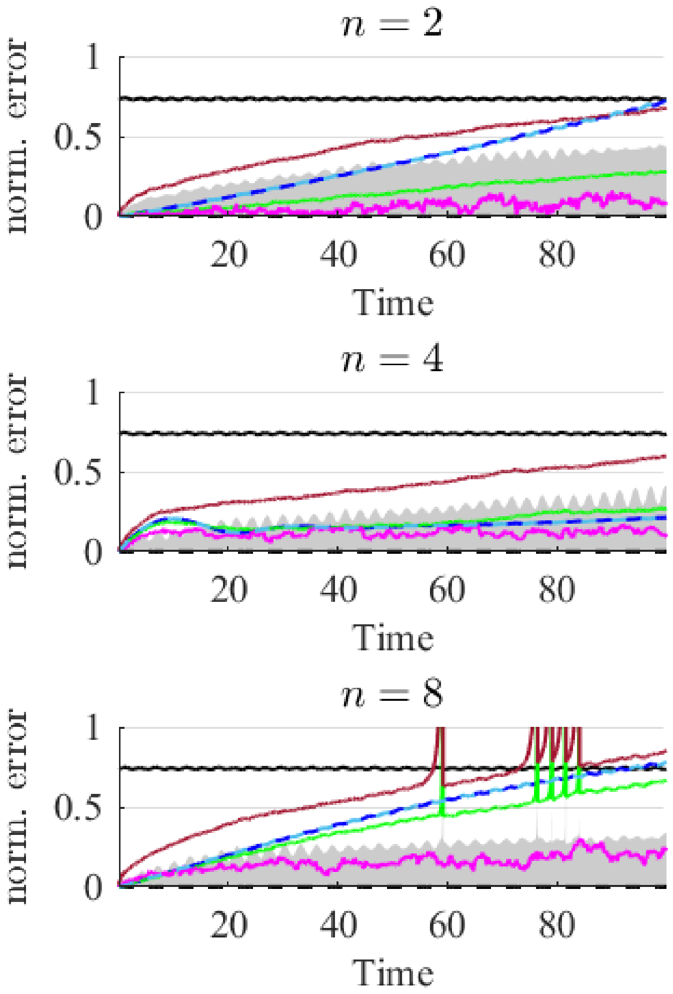
$$v = w + v'$$

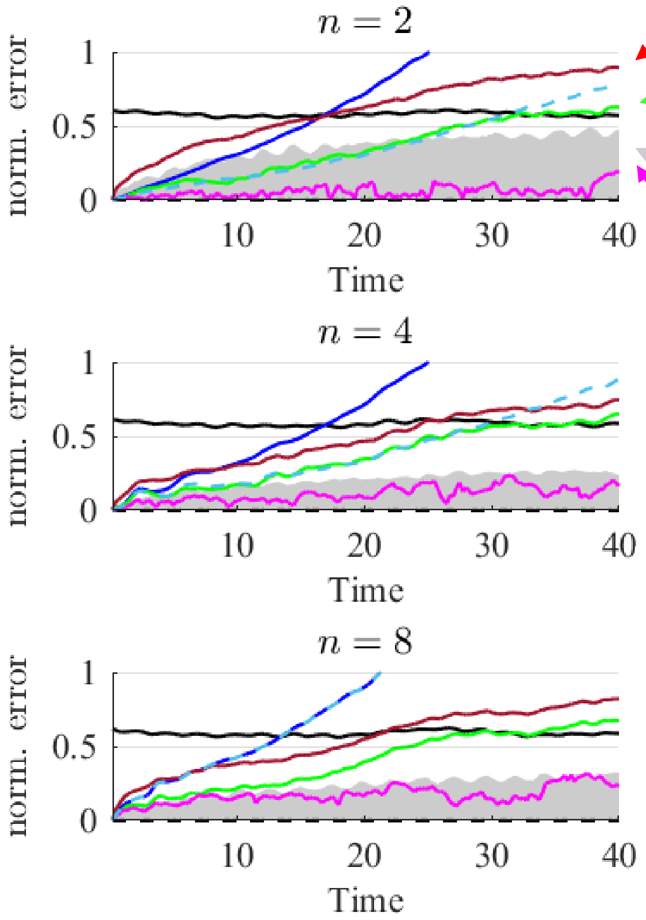
Resolved fluid velocity:
$w = \sum_{i=0}^{n} b_i \phi_i$

Unresolved fluid velocity:
v'

Reynolds number (Re) = 100 / 2D
(full-order simulation has 10^4 dof)





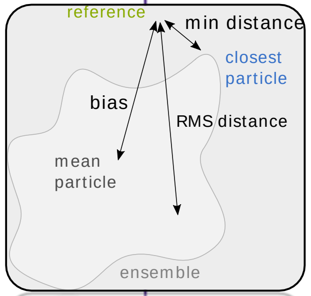


Red. LUM
RMSE

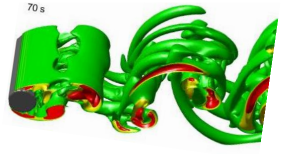
Red. LUM
bias

Red. LUM
std

Red. LUM
ensemble
minimal
distance
to the reference



Reynolds number (Re) = 300 3D
(full-order simulation has 10^7 dof)



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UNCERTAINTY QUANTIFICATION (PRIOR)

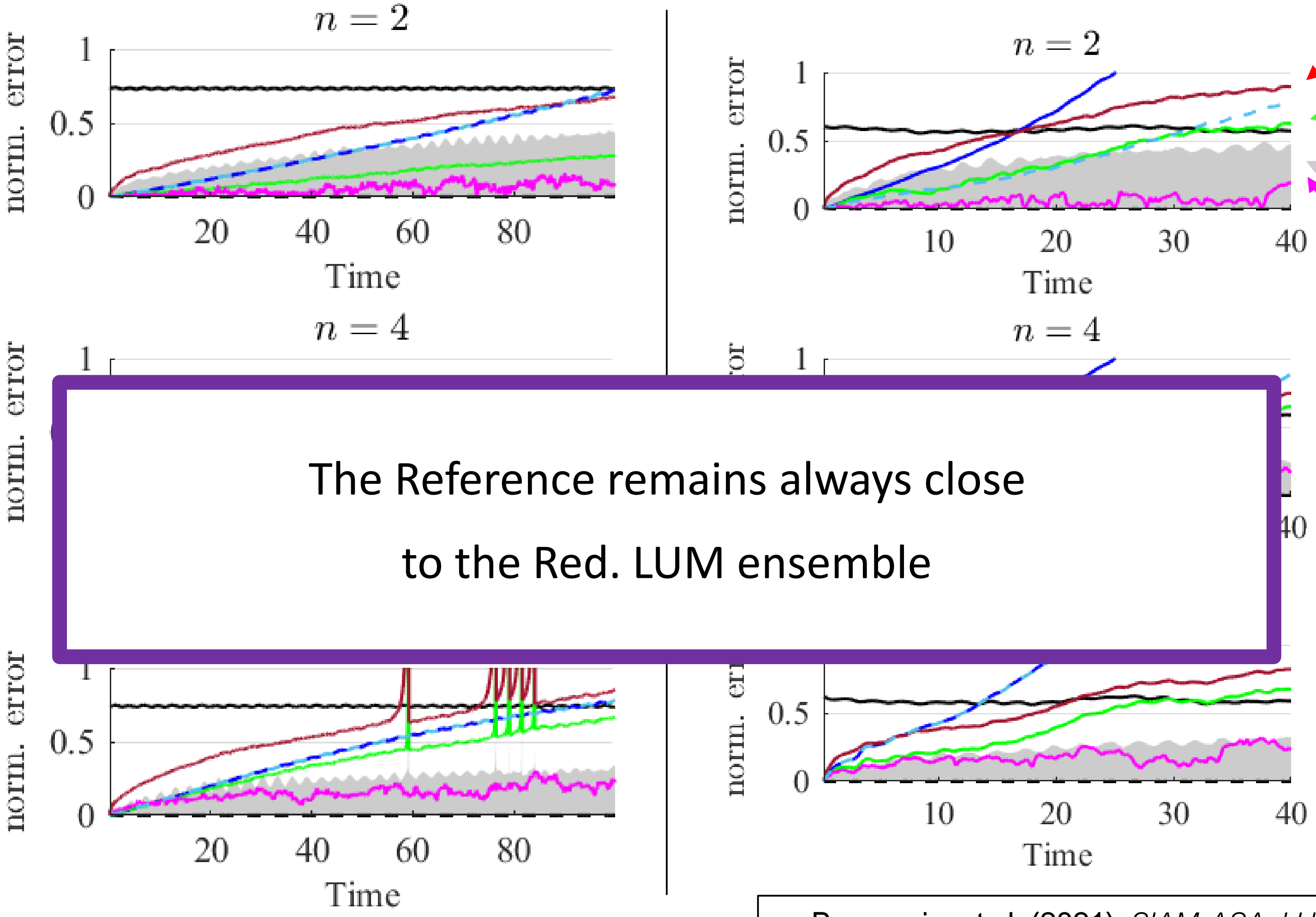
Error on the reduced solution w

From 10^7 to 8 degrees of freedom
No data assimilation
Known initial conditions $b(t = 0)$

$$v = w + v'$$

Resolved fluid velocity:
$w = \sum_{i=0}^{n} b_i \phi_i$

Unresolved fluid velocity:
v'



The Reference remains always close

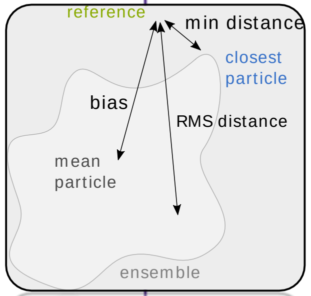
to the Red. LUM ensemble

Red. LUM
RMSE

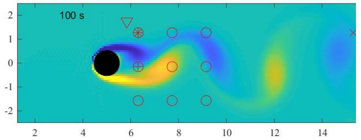
Red. LUM
bias

Red. LUM
std

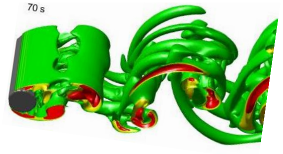
Red. LUM
ensemble
minimal
distance
to the reference



Reynolds number (Re) = 100 / 2D
(full-order simulation has 10^4 dof)



Reynolds number (Re) = 300 3D
(full-order simulation has 10^7 dof)



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DATA ASSIMILATION (POSTERIOR)

On-line estimation of the solution

From 10^7 to 8 degrees of freedom

● Single measurement point (blurred & noisy velocity)

Reference

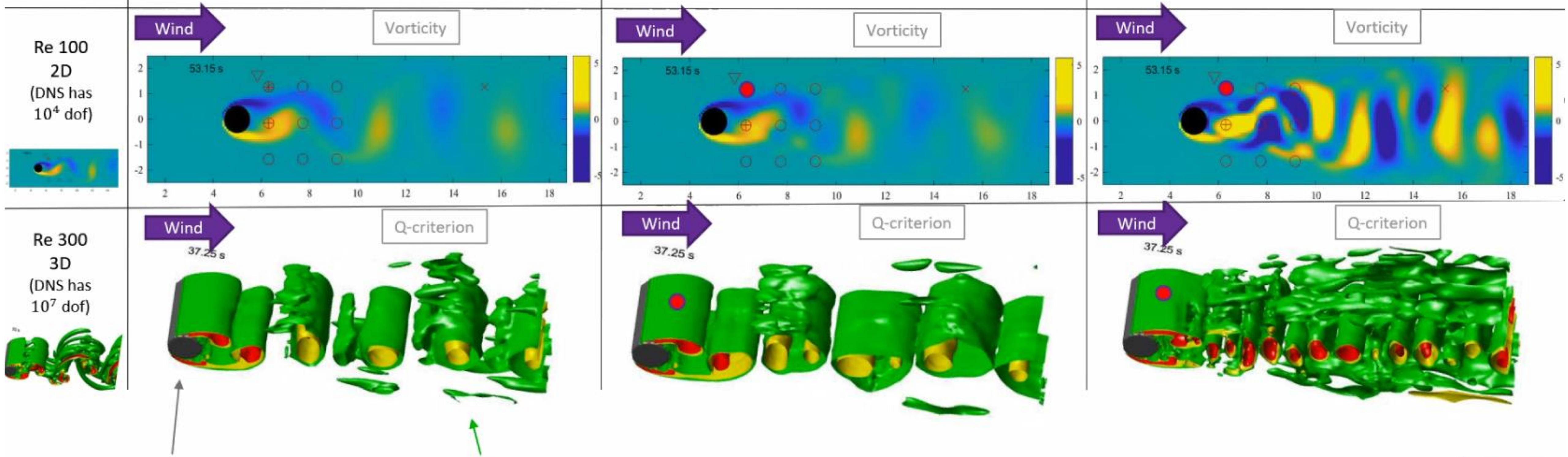
PCA-projection of the DNS
(Optimal from 8-dof linear decomposition)

Our method

POD-Galerkin with Navier-Stokes
under location uncertainty (LUM)

State-of-the-art

POD-Galerkin with Navier-Stokes + optimally
tuned eddy viscosity & additive noise



Re 100
2D
(DNS has
10^4 dof)

Re 300
3D
(DNS has
10^7 dof)

(round) wind turbine blade

vortices

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DATA ASSIMILATION (POSTERIOR)

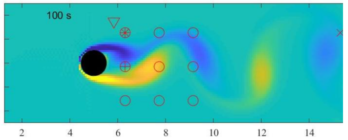
Error on the solution estimation

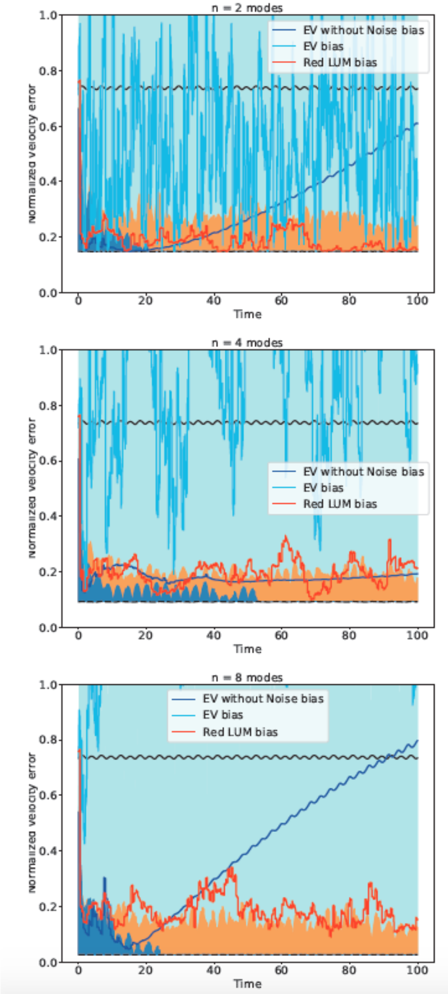
$$v = w + v'$$

Resolved fluid velocity:
$w = \sum_{i=0}^{n} b_i \phi_i$

Unresolved fluid velocity:
v'

Reynolds number (Re) = 100 / 2D
(full-order simulation has 10^4 dof)



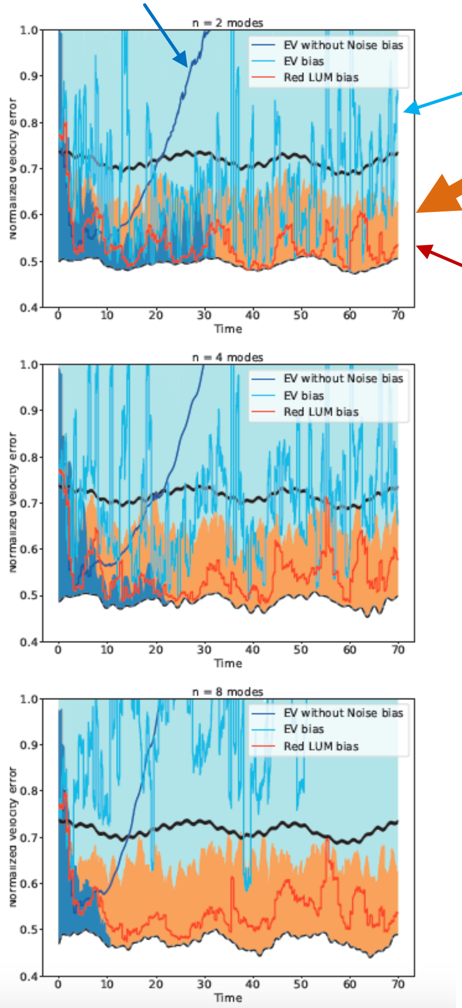


State of the art

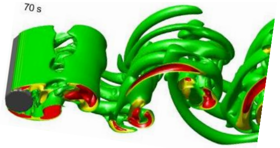
State of the art

Red. LUM std

Red. LUM bias



Reynolds number (Re) = 300 3D
(full-order simulation has 10^7 dof)



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CONCLUSION

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CONCLUSION

- Reduced order model (ROM) : for very fast and robust CFD $(10^7 \rightarrow 8$ degrees of freedom.)
  - Combine data & physics (built off-line)
  - Closure problem handled by LUM
  - Efficient estimator for the multiplicative noise
  - Efficient generation of prior / Model error quantification
- Data assimilation (Bayesian inverse problem) :
  to correct the fast simulation on-line by incomplete/noisy measurements
- First results
  - Optimal unsteady flow estimation/prediction in the whole spatial domain (large-scale structures)
  - Robust far outside the training set

NEXT STEPS

- Real measurements
- Better stochastic closure
- Parametric ROM (unknown inflow)
- Increasing Reynolds
  (ROM of (non-polynomial) turbulence models)

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