FAST GENERATION OF PRIOR FOR BAYESIAN ESTIMATION PROBLEMS IN FLUID MECHANICS

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I. Context

II. State of the art
   a. Intrusive reduced order model (ROM)
   b. Data assimilation

III. Reduced location uncertainty models
   a. Multiscale modeling
   b. Location uncertainty models (LUM)
   c. Reduced LUM

IV. Numerical results
   a. Uncertainty quantification (Prior)
   b. Data assimilation (Posterior)
PART I

CONTEXT :
OBSERVER FOR WIND TURBINE APPLICATIONS
Observer for wind turbine application

*Application*: Real-time estimation and prediction of 3D fluid flow using strongly-limited computational resources & few sensors
Observer for wind turbine application

*Application*: Real-time estimation and prediction of 3D fluid flow using strongly-limited computational resources & few sensors.

- Wind fluctuations
- Wind turbine blade
Observer for wind turbine application

*Application*: Real-time estimation and prediction of 3D fluid flow using strongly-limited computational resources & few sensors
Observer for wind turbine application

Application: Real-time estimation and prediction of 3D fluid flow using strongly-limited computational resources & few sensors

Estimation and prediction:
- Air flow
- Lift, drag, inflow
- ...

Controller
Simple model

Observer
Simple model

Few sensors

Wind turbine blade

Wind fluctuations

• Blade pitch
• Fluidic actuators

Damages
Observer for wind turbine application

Application: Real-time estimation and prediction of 3D fluid flow using strongly-limited computational resources & few sensors

Estimation and prediction:
- Air flow
- Lift, drag, inflow
- ...

Observer
- Simple model

Controller
- Simple model
- Blade pitch
- Fluidic actuators

Wind turbulence
- Wind fluctuations

Few sensors

Damages
Observer for wind turbine application

**Application**: Real-time estimation and prediction of 3D fluid flow using strongly-limited computational resources & few sensors

Estimation and prediction:
- Air flow
- Lift, drag, inflow
- ...

Observer

Controller

Simple model

**Few sensors**

Wind turbine blade

- Blade pitch
- Fluidic actuators

Wind fluctuations

Damages

Which simple model? How to combine model & measurements?
Observer for wind turbine application

Application: Real-time estimation and prediction of 3D fluid flow using strongly-limited computational resources & few sensors

Estimation and prediction:
- Air flow
- Lift, drag, inflow
- ...

Scientific problem:
Simulation & data assimilation under severe dimensional reduction
typically, $10^7 \rightarrow O(10)$ degrees of freedom
PART II

STATE OF THE ART

a. Intrusive reduced order model (ROM)
b. Data assimilation
REDUCED ORDER MODEL (ROM)

Solution of an PDE with the form:

\[ v(x, t, \alpha) \approx \sum_{i=0}^{n} b_i(t) \phi_i(x) \gamma_i(\alpha) \]
REDUCED ORDER MODEL (ROM)

Solution of an PDE with the form:

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**REDUCED ORDER MODEL (ROM)**

Solution of an PDE with the form:

\[ v(x, t, \alpha) \approx \sum_{i=0}^{n} b_i(t) \phi_i(x) \gamma_i(\alpha) \]

<table>
<thead>
<tr>
<th>Solution coordinates</th>
<th>Full space</th>
<th>Reduced space</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ v_q(x_i, t) ]_q_i</td>
<td>( b_i(t) ) _i</td>
<td></td>
</tr>
</tbody>
</table>

**Order of magnitude examples in CFD**

- Dimension: \( M \times d \approx 10^7 \), \( n \approx 10 - 100 \)
Principal Component Analysis (PCA) on a dataset to reduce the dimensionality:

**Approximation:**

\[
v(x, t) \approx \sum_{i=0}^{n} b_i(t) \phi_i(x)
\]
Principal Component Analysis (PCA) on a dataset to reduce the dimensionality:

Approximation:
\[ v(x, t) \approx \sum_{i=0}^{n} b_i(t) \phi_i(x) \]
 Principal Component Analysis (PCA) on a *dataset* to reduce the dimensionality:

\[
v(x, t) \approx \sum_{i=0}^{n} b_i(t) \phi_i(x)
\]

**Approximation:**

---

**INTRUSIVE REDUCED ORDER MODEL**

Combine physical models and learning approaches

- **Resolu**
- **Spatial**
- **PCA**
- **Off-line**
- **Snapshots**
- **PCA**
- **Spatial modes**
- **Off-line**
- **Snapshots**
- **PCA**
- **Spatial modes**

---

**Off-line simulations**

**Snapshots**

**PCA**

**Spatial modes**

---

**Intermediate Reduced Order Model**

Combine physical models and learning approaches

- **Principal Component Analysis (PCA)** on a *dataset* to reduce the dimensionality:

\[
v(x, t) \approx \sum_{i=0}^{n} b_i(t) \phi_i(x)
\]

**Approximation:**
**INTRUSIVE REDUCED ORDER MODEL**
Combine physical models and learning approaches

- **Principal Component Analysis (PCA)** on a *dataset* to reduce the dimensionality:

  ![Diagram showing PCA process]

  - **Off-line simulations** → **Snapshots** → **PCA** → **Spatial modes**

- **Approximation:**

  \[ v(x, t) \approx \sum_{i=0}^{n} b_i(t) \phi_i(x) \]

- **Projection of the “physics” onto the spatial modes (POD-Galerkin):**

  \[ \int_{\Omega} dx \phi_i(x) \cdot (Physical\ equation)\ (e.g.\ Navier-Stokes) \]
**INTRUSIVE REDUCED ORDER MODEL**  
Combine physical models and learning approaches

- **Principal Component Analysis (PCA)** on a *dataset* to reduce the dimensionality:

  ![Diagram showing the process from off-line simulations to snapshots, PCA, and spatial modes](image)

- **Approximation:**
  \[
  v(x, t) \approx \sum_{i=0}^{n} b_i(t) \phi_i(x)
  \]

- **Projection of the “physics”** onto the spatial modes (POD-Galerkin)
  \[
  \int_{\Omega} dx \phi_i(x) \cdot (\text{Physical equation} \text{ (e.g. Navier-Stokes)})
  \]
  \[ \Rightarrow \text{ROM for very fast simulation of temporal modes} \]
INTRUSIVE REDUCED ORDER MODEL
Combine physical models and learning approaches

- Principal Component Analysis (PCA) on a dataset to reduce the dimensionality:

  - Off-line simulations
  - Snapshots $\left(v(x, t)\right)_i$
  - PCA
  - Spatial modes $\left(\phi_i(x)\right)_i$

- Approximation:
  $$v(x, t) \approx \sum_{i=0}^{n} b_i(t) \phi_i(x)$$

- Projection of the “physics” onto the spatial modes (POD-Galerkin)

Don’t work in extrapolation!

$$\int_{\Omega} dx \phi_i(x) \cdot \left(\text{Physical equation} \ (e.g. \text{Navier-Stokes})\right)$$

$\Rightarrow$ ROM for very fast simulation of temporal modes
**INTRUSIVE REDUCED ORDER MODEL**
Combine physical models and learning approaches

- Principal Component Analysis (PCA) on a *dataset* to reduce the dimensionality:
  - **Off-line simulations** → **Snapshots** \( v(x, t) \) → **PCA** → **Spatial modes** \( \phi_i(x) \)

- **Approximation:**
  \[
  v(x, t) \approx \sum_{i=0}^{n} b_i(t) \phi_i(x)
  \]

- **Projection of the “physics”** onto the spatial modes (POD-Galerkin)

Don’t work in extrapolation!

\[
\int_{\Omega} dx \phi_i(x) \cdot (\text{Physical equation} + \text{fitted correction})
\]

→ **ROM for very fast simulation of temporal modes**
**INTRUSIVE REDUCED ORDER MODEL**
Combine physical models and learning approaches

- Principal Component Analysis (PCA) on a *dataset* to reduce the dimensionality:

  - Off-line simulations → Snapshots \( (v(x, t))_i \) \( \rightarrow \) PCA → Spatial modes \( (\phi_i(x))_i \)

- Approximation:
  \[ v(x, t) \approx \sum_{i=0}^{n} b_i(t) \phi_i(x) \]  
  - Resolved modes
  - Don’t work in extrapolation!

- Projection of the “physics” onto the spatial modes (POD-Galerkin)

\[ \int_{\Omega} d\bar{x} \phi_i(x) \cdot ( \text{Physical equation} ) + \text{fitted correction} + \text{additive noise} \]

\( \rightarrow \) ROM for very fast simulation of temporal modes
**INTRUSIVE REDUCED ORDER MODEL**
Combine physical models and learning approaches

- **Principal Component Analysis (PCA)** on a *dataset* to reduce the dimensionality:

- **Approximation:**
  \[ v(x, t) \approx \sum_{i=0}^{n} b_i(t) \phi_i(x) \]

- **Projection of the “physics”** onto the spatial modes (POD-Galerkin)

Don’t work in extrapolation!

\[ \int_{\Omega} dx \phi_i(x) \cdot (\text{Physical equation} \ (\text{e.g. Navier-Stokes})) \]

→ ROM for very fast simulation of temporal modes
**INTRUSIVE REDUCED ORDER MODEL**

Combine physical models and learning approaches

- **Principal Component Analysis (PCA)** on a *dataset* to reduce the dimensionality:

  \[
  \begin{align*}
  \text{Off-line simulations} & \quad \rightarrow \quad \text{Snapshots} \quad \left( v(x, t_i) \right)_i \\
  \text{PCA} & \quad \rightarrow \quad \text{Spatial modes} \quad \left( \phi_i(x) \right)_i \\
  \end{align*}
  \]

- **Approximation**:

  \[
  \begin{align*}
  v(x, t) & \approx \sum_{i=0}^{n} b_i(t) \phi_i(x) \\
  \end{align*}
  \]

- **Projection of the “physics” onto the spatial modes (POD-Galerkin)**

  \[
  \int_{\Omega} dx \phi_i(x) \cdot (\text{Physical equation} \ (\text{e.g. Navier-Stokes})) \Rightarrow \text{ROM for very fast simulation of temporal modes}
  \]

- **Resolved modes**

  - 8
**INTRUSIVE REDUCED ORDER MODEL**

Combine physical models and learning approaches

- Principal Component Analysis (PCA) on a *dataset* to reduce the dimensionality:

  ![Diagram showing PCA process]

  - **Off-line simulations**
  - **Snapshots** 
  - **PCA**
  - **Spatial modes**

- **Approximation:**

  \[ v(x, t) \approx \sum_{i=0}^{n} b_i(t) \phi_i(x) \]

- **Projection of the “physics” onto the spatial modes (POD-Galerkin):**

  \[ \int_{\Omega} dx \phi_i(x) \cdot (\text{Randomized Navier-Stokes}) \]

  ➔ **ROM for very fast simulation of temporal modes**
DATA ASSIMILATION
= Coupling simulations and measurements $\gamma$

Numerical Simulation (ROM)
→ erroneous

On-line measurements
→ incomplete
→ possibly noisy
DATA ASSIMILATION

= Coupling simulations and measurements $\gamma$

Numerical Simulation (ROM) → erroneous

On-line measurements

→ incomplete
→ possibly noisy

$3 \text{ m. s}^{-1}$

Velocity
DATA ASSIMILATION
= Coupling simulations and measurements $\gamma$

Numerical Simulation (ROM)
→ erroneous

On-line measurements
→ incomplete
→ possibly noisy

Velocity
3 m. s$^{-1}$ 5 m. s$^{-1}$
DATA ASSIMILATION
= Coupling simulations and measurements $\gamma$

- Numerical Simulation (ROM) → erroneous
- On-line measurements → incomplete, possibly noisy

Velocity

- $3 \text{ m.s}^{-1}$
- $5 \text{ m.s}^{-1}$
DATA ASSIMILATION
= Coupling simulations and measurements $\gamma$

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3 m. s$^{-1}$
5 m. s$^{-1}$
DATA ASSIMILATION
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- Numerical Simulation (ROM) → erroneous
- Data assimilation (e.g., particle filtering)
- On-line measurements → incomplete → possibly noisy

Velocity

3 m. s$^{-1}$
5 m. s$^{-1}$
DATA ASSIMILATION
= Coupling simulations and measurements $\gamma$

Numerical Simulation (ROM) → erroneous

Data assimilation (e.g., particle filtering)

On-line measurements → incomplete → possibly noisy

Prior $p(v)$

Velocity

3 m. s$^{-1}$

5 m. s$^{-1}$
DATA ASSIMILATION
= Coupling simulations and measurements $\gamma$

Numerical Simulation (ROM)
→ erroneous

Data assimilation (e.g., particle filtering)

On-line measurements
→ incomplete
→ possibly noisy

Prior $p(v)$

Likelihood $p(y|v)$

Velocity

$3 \text{ m.s}^{-1}$

$5 \text{ m.s}^{-1}$
DATA ASSIMILATION
= Coupling simulations and measurements $\gamma$

Numerical Simulation (ROM) $\rightarrow$ erroneous

Data assimilation (e.g., particle filtering)

On-line measurements $\rightarrow$ incomplete $\rightarrow$ possibly noisy

More accurate estimation globally in space 

Posterior $p(v|y) \propto p(y|v)p(v)$

Prior $p(v)$

Likelihood $p(y|v)$

Velocity

3 m. s$^{-1}$

5 m. s$^{-1}$
DATA ASSIMILATION
= Coupling simulations and measurements $y$

Numerical Simulation (ROM)
→ erroneous

Data assimilation (e.g., particle filtering)

More accurate estimation globally in space
Posterior
$p(v|y) \propto p(y|v)p(v)$

On-line measurements
→ incomplete
→ possibly noisy

Need for uncertainty / errors quantification
→ Random dynamics
$p(v_{t+1}|v_t)$

Prior
$p(v)$

Likelihood
$p(y|v)$

3 m. s$^{-1}$
5 m. s$^{-1}$

Velocity
DATA ASSIMILATION
= Coupling simulations and measurements $y$

Numerical Simulation (ROM)

→ erroneous

Data assimilation (e.g., particle filtering)

More accurate estimation globally in space
$Posterior\ p(v|y) \propto p(y|v)p(v)$

On-line measurements

→ incomplete
→ possibly noisy

Need for uncertainty / errors quantification
→ Random dynamics
$p(v_{t+1}|v_t)$
while minimizing
→ errors (closure)
→ CPU

Prior $p(v)$

Likelihood $p(y|v)$

3 m. s$^{-1}$
5 m. s$^{-1}$

Velocity
DATA ASSIMILATION

Example: the Particle Filter (PF) generates an ensemble $\sim p(v | y)$

- Initialization
  $v_{t=0}^{(j)} \sim \mathcal{N}(0, \Sigma)$

- Loop over time $t$
  
  **Importance sampling**
  
  - $v_t^{(j)} = M \left( v_{t-1}^{(j)}, \text{noise}(t-1) \right)$ \hspace{1cm} Forecast (“Prior” or “background”)
  
  - If an observation $y_t$ is available at the current time $t$
    
    - $W_j(t) \propto p(y_t | v_t^{(j)})$ \hspace{2cm} Likelihood evaluation, up to a constant
    
    - $W_j(t) = \frac{W_j(t)}{\sum_{k=1}^{N_p} W_k(t)}$ \hspace{2cm} Normalization

  **Resampling**
    
    - Each new $v_t^{(j)}$ is replaced by one of the old particles $v_t^{(1)}, ..., v_t^{(N_p)}$ with probability $W_1(t), ..., W_{N_p}(t)$, respectively.

- Final posterior distribution
  
  $p(v_t | y_{t_1}, ..., y_{t_K}) \approx \sum_{k=1}^{N_p} \frac{1}{N_p} \delta \left( v_t - v_t^{(k)} \right)$
DATA ASSIMILATION

Example: the Particle Filter (PF) generates an ensemble $\sim p(v|y)$

- Initialization
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  - If an observation $y_t$ is available at the current time $t$
    
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      Likelihood evaluation, up to a constant
    
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      Normalization

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  $p(v_t|y_{t1}, \ldots, y_{tK}) \approx \sum_{k=1}^{N_p} \frac{1}{N_p} \delta(v_t - v_t^{(k)})$
PART III

REDUCED LOCATION UNCERTAINTY MODELS

a. Multiscale modeling
b. Location uncertainty models (LUM)
c. Reduced LUM (Red LUM)
MULTISCALE MODELING

Vorticity

- Large-scale eddy
- Medium-scale eddy
- Small-scale eddy

[Image of vorticity map with color gradient from blue to yellow, indicating different scales of eddies.]
MULTISCALE MODELING

Fluids are multiscale $\rightarrow$ Many coupled degrees of freedom

- Large-scale eddy
- Medium-scale eddy
- Large-scale eddy
- Small-scale eddy

Vorticity

-5 0 5
Fluids are multiscale  

Many coupled degrees of freedom

We cannot simulate (or observe) every scale.

Generally, authors

- simulate large scales $w$,
- model the effect of small scales $v'$ in the equations (closure).

Here, we

- model the small scales $v'$ through stochastic functions, parametrized from data and/or from physical scale symmetries.
- inject those in physical equations for physical understanding, simulations & data assimilation.
LOCATION UNCERTAINTY MODELS (LUM)

\[ \nu = w + \nu' \]

Resolved fluid velocity:
\[ w = \sum_{i=0}^{n} b_i \phi_i \]

Unresolved fluid velocity:
\[ \nu' = \sigma dB_t/dt \]

Assumed (conditionally-)Gaussian & white in time (non-stationary in space)
LOCATION UNCERTAINTY MODELS (LUM)

\[ v = w + v' \]

Resolved fluid velocity:
\[ w = \sum_{i=0}^{n} b_i \phi_i \]

Unresolved fluid velocity:
\[ v' = \sigma_d B_t \frac{dt}{dt} \]

Assumed
(conditionally-)Gaussian & white in time
(non-stationary in space)

Randomized Navier-Stokes model
- Good closure
- Good model error quantification
  for data assimilation
Randomized Navier-Stokes model

- Good closure
- Good model error quantification for data assimilation

\[
\nu = w + \nu'
\]

Resolved fluid velocity:
\[
w = \sum_{i=0}^{n} b_i \phi_i
\]

Unresolved fluid velocity:
\[
\nu' = \frac{\sigma d B_t}{dt}
\]

Assumed (conditionally-)Gaussian & white in time (non-stationary in space)

**LUM**
- Memin, 2014
- Resseguier et al. 2017 a, b, c, d
- Cai et al. 2017
- Chapron et al. 2018
- Yang & Memin 2019

**SALT**
- Holm, 2015
- Holm and Tyranowski, 2016
- Arnaudon et al. 2017
- Crisan et al., 2017
- Gay-Balmaz & Holm 2017
- Cotter and al. 2018 a, b
- Cotter and al. 2019

**References**
- Mikulevicius & Rozovskii, 2004
- Flandoli, 2011
- Cotter and al. 2017
- Resseguier et al. 2020 a, b
Randomized Navier-Stokes model

- Good closure
- Good model error quantification
  for data assimilation

\[ \nu = \nu_x + \nu_y \]

Resolved fluid velocity:
\[ \nu_x = \sum_{i=0}^{\infty} b_i \phi_i \]

Unresolved fluid velocity:
\[ \nu_y = \frac{\sigma dB_t}{dt} \]

Assumed (conditionally-)Gaussian & white in time (non-stationary in space)

References:
- Memin, 2014
- Resseguier et al. 2017 a, b, c, d
- Cai et al. 2017
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- Yang & Memin 2019

LUM

- Holm, 2015
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SALT

- Crisan et al., 2017
- Gay-Balmaz & Holm 2017
- Cotter and al. 2018 a, b
- Cotter and al. 2019
LOCATION UNCERTAINTY MODELS (LUM), Randomized Navier-Stokes

\[
\begin{align*}
  v &= w + v' \\
  \text{Resolved fluid velocity:} & \quad w \\
  \text{Unresolved fluid velocity:} & \quad v' = \frac{\sigma dB_t}{dt} \quad \text{(Gaussian, white wrt t)} \\
  \text{(assuming } \nabla \cdot w = 0 \text{ and } \nabla \cdot v' = 0) \\

\text{Momentum conservation} \\
\frac{d}{dt}(w(t, X_t)) &= dF \quad \text{(Forces)} \\

\text{Positions of fluid parcels } X_t: \\
\frac{dX_t}{dt} &= w(t, X_t)dt + \sigma(t, X_t)dB_t \\
\text{Gaussian process} \\
\text{white in time}
\end{align*}
\]
\[ d_t w + w^* \, dt \cdot \nabla w + \sigma dB_t \cdot \nabla w - \nabla \cdot \left( \frac{1}{2} a \nabla w \right) \, dt = dF \]

**Resolved fluid velocity:** \( w \)

**Unresolved fluid velocity:** \( v' = \frac{\sigma dB_t}{dt} \) (Gaussian, white wrt \( t \))

(assuming \( \nabla \cdot w = 0 \) and \( \nabla \cdot v' = 0 \))

**Variance tensor:**

\[
a(x, x) = \frac{\mathbb{E}[(\sigma dB_t)(\sigma dB_t)^T]}{dt}
\]

From Ito-Wentzell formula (Kunita 1990) with Ito notations

*LOCATION UNCERTAINTY MODELS (LUM), Randomized Navier-Stokes*

From Ito-Wentzell formula (Kunita 1990) with Ito notations

\[ \frac{\partial}{\partial t} w + w^* \frac{\partial}{\partial t} \nabla w + \sigma dB_t \cdot \nabla w - \nabla \cdot \left( \frac{1}{2} \sigma a \nabla w \right) dt = dF \]

Resolved fluid velocity: \( w \)

Unresolved fluid velocity: \( v' = \frac{\sigma dB_t}{dt} \) (Gaussian, white wrt \( t \))

(assuming \( \nabla \cdot w = 0 \) and \( \nabla \cdot v' = 0 \))

Variance tensor:
\[ a(x, x) = \frac{\mathbb{E} \{ (\sigma dB_t)(\sigma dB_t)^T \} }{dt} \]

**LOCATION UNCERTAINTY MODELS (LUM), Randomized Navier-Stokes**

\[ v = w + v' \]

Resolved fluid velocity: \( w \)

Unresolved fluid velocity:
\[ v' = \frac{\sigma dB_t}{dt} \]

(Gaussian, white wrt \( t \))

(assuming \( \nabla \cdot w = 0 \) and \( \nabla \cdot v' = 0 \))

Variance tensor:
\[ a(x, x) = \frac{\mathbb{E}\{[\sigma dB_t][\sigma dB_t]^T\}}{dt} \]

\[ d_t w + w^* dt \cdot \nabla w + \sigma dB_t \cdot \nabla w \]

\[ - \nabla \cdot \left( \frac{1}{2} a \nabla w \right) dt = dF \]

From Ito-Wentzell formula (Kunita 1990) with Ito notations

LOCATION UNCERTAINTY MODELS (LUM), Randomized Navier-Stokes

\[ d_t w + w^* dt \cdot \nabla w + \sigma d B_t \cdot \nabla w - \nabla \cdot \left( \frac{1}{2} a \nabla w \right) dt = dF \]

\[ v = w + v' \]

Resolved fluid velocity: \( w \)

Unresolved fluid velocity: \( v' = \frac{\sigma d B_t}{dt} \) (Gaussian, white wrt \( t \))

(assuming \( \nabla \cdot w = 0 \) and \( \nabla \cdot v' = 0 \))

Variance tensor:
\[ a(x, x) = \frac{\mathbb{E}\{(\sigma d B_t)(\sigma d B_t)^T\}}{dt} \]

From Ito-Wentzell formula (Kunita 1990) with Ito notations

LOCATION UNCERTAINTY MODELS (LUM), Randomized Navier-Stokes

\[ v = w + v' \]

Resolved fluid velocity: \( w \)

Unresolved fluid velocity:
\[ v' = \frac{\sigma dB_t}{dt} \] (Gaussian, white wrt \( t \))

(assuming \( \nabla \cdot w = 0 \) and \( \nabla \cdot v' = 0 \))

Variance tensor:
\[ a(x, x) = \frac{\mathbb{E}\{\sigma dB_t (\sigma dB_t)^T\}}{dt} \]

From Ito-Wentzell formula (Kunita 1990) with Ito notations

\[ d_t w + w^* dt \cdot \nabla w + \sigma dB_t \cdot \nabla w = -\nabla \cdot \left( \frac{1}{2} a \nabla w \right) dt + dF \]

Usual terms

LOCATION UNCERTAINTY MODELS (LUM), Randomized Navier-Stokes

\[ \frac{d}{dt} \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{w} + \sigma dB_t \cdot \nabla \mathbf{w} - \nabla \cdot \left( \frac{1}{2} a \nabla \mathbf{w} \right) dt = dF \]

Resolved fluid velocity: \( \mathbf{w} \)

Unresolved fluid velocity: \( \mathbf{v}' = \frac{\sigma dB_t}{dt} \) (Gaussian, white wrt \( t \))

Variance tensor:

\[ a(x, x) = \frac{\mathbb{E} \left\{ (\sigma dB_t)(\sigma dB_t)^T \right\}}{dt} \]

Usual terms

Skew-symmetric multiplicative random forcing

From Ito-Wentzell formula (Kunita 1990) with Ito notations

Forces

\[ dt \frac{\partial w}{\partial t} + w^* dt \cdot \nabla w + \sigma dB_t \cdot \nabla w - \nabla \cdot \left( \frac{1}{2} a \nabla w \right) dt = dF \]

**Resolved fluid velocity:** \( w \)

**Unresolved fluid velocity:** \( v' = \frac{\sigma dB_t}{dt} \) (Gaussian, white wrt \( t \))

(assuming \( \nabla \cdot w = 0 \) and \( \nabla \cdot v' = 0 \))

**Variance tensor:**
\[ a(x,x) = \frac{\mathbb{E}[(\sigma dB_t)(\sigma dB_t)^T]}{dt} \]

**Usual terms**
- Advection
- Diffusion

**Skew-symmetric multiplicative random forcing**

From Ito-Wentzell formula (Kunita 1990) with Ito notations

**LOCATION UNCERTAINTY MODELS (LUM), Randomized Navier-Stokes**

LOCATION UNCERTAINTY MODELS (LUM), Randomized Navier-Stokes

\[ \frac{d}{dt} w + w^* \nabla w + \sigma dB_t \cdot \nabla w - \nabla \cdot \left( \frac{1}{2} a \nabla w \right) dt = dF \]

Resolved fluid velocity: \( w \)

Unresolved fluid velocity: \( v' = \frac{\sigma dB_t}{dt} \) (Gaussian, white wrt \( t \) )

(assuming \( \nabla \cdot w = 0 \) and \( \nabla \cdot v' = 0 \))

Variance tensor:

\[ a(x,x) = \frac{\mathbb{E} [(\sigma dB_t)(\sigma dB_t)^T]}{dt} \]

Usual terms

Skew-symmetric multiplicative random forcing

From Ito-Wentzell formula (Kunita 1990) with Ito notations

Forces
Diffusion
Advection
\[
d_t w + w^* \, dt \cdot \nabla w + \sigma dB_t \cdot \nabla w - \nabla \cdot \left( \frac{1}{2} a \nabla w \right) \, dt = dF
\]

LOCATION UNCERTAINTY MODELS (LUM), Randomized Navier-Stokes

\[ v = w + v' \]
Resolved fluid velocity: \( w \)
Unresolved fluid velocity: \( v' = \sigma dB_t \, dt \) (Gaussian, white wrt \( t \))
(assuming \( \nabla \cdot w = 0 \) and \( \nabla \cdot v' = 0 \))

Variance tensor:
\[ a(x, x) = \mathbb{E}\{\sigma dB_t \sigma dB_t^T\} \, dt \]

Usual terms
Skew-symmetric multiplicative random forcing


From Ito-Wentzell formula (Kunita 1990) with Ito notations
LOCATION UNCERTAINTY MODELS (LUM), Randomized Navier-Stokes

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Variance tensor:
\[ a(x, x) = \frac{\mathbb{E}\{\sigma dB_t \sigma dB_t^T\}}{dt} \]

From Ito-Wentzell formula (Kunita 1990) with Ito notations

\[ d_t w + w^* dt \cdot \nabla w + \sigma dB_t \cdot \nabla w = -\nabla \cdot \left( \frac{1}{2} a \nabla w \right) dt = dF \]

Symmetric negative

Usual terms

Skew-symmetric multiplicative random forcing

Balanced energy fluxes

LOCATION UNCERTAINTY MODELS (LUM), Randomized Navier-Stokes

\[ \nu = w + \nu' \]

Resolved fluid velocity: \( w \)

Unresolved fluid velocity: \( \nu' = \frac{\sigma dB_t}{dt} \) (Gaussian, white with respect to \( t \))

\[ d_t w + C(w, w) dt + C(\sigma dB_t, w) + F(w) dt = dF \]

From Ito-Wentzell formula (Kunita 1990) with Ito notations

Resolved fluid velocity: \( w \)

Unresolved fluid velocity: \( \nu' = \frac{\sigma dB_t}{dt} \) (Gaussian, white with respect to \( t \))

Usual terms

Skew-symmetric multiplicative random forcing

Balanced energy fluxes

Resseguier V. al. (2017), Part I. Geophys. Astro. Fluid. hal-01391420
POD-Galerkin gives SDEs for resolved modes

\[ v = w + v' \]

Resolved fluid velocity:
\[ w(x, t) = \sum_{i=0}^{n} b_i(t) \phi_i(x) \]

Unresolved fluid velocity:
\[ v' = \frac{\sigma dB_t}{dt} \text{ (Gaussian, white wrt } t) \]

\[
\int_{\Omega} \phi_i(x) \cdot (d_t w + C(w, w)dt + F(w)dt + C(\sigma dB_t, w) = dF) \, dx
\]

Full order: \( M \sim 10^7 \)
Reduced order: \( n \sim 10 \)
**REDUCED LUM (RED LUM)**

POD-Galerkin gives SDEs for resolved modes

\[ v = w + v' \]

Resolved fluid velocity:
\[ w(x, t) = \sum_{i=0}^{n} b_i(t) \phi_i(x) \]

Unresolved fluid velocity:
\[ v' = \frac{\sigma dB_t}{dt} \]

(Gaussian, white wrt \( t \))

\[
\int_{\Omega} \phi_i(x) \cdot (d_t w + C(w, w)dt + F(w)dt + C(\sigma dB_t, w) = dF) \, dx
\]

Full order: \( M \sim 10^7 \)
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\[ w(x,t) = \sum_{i=0}^{n} b_i(t) \phi_i(x) \]

**Unresolved fluid velocity:**
\[ v' = \frac{\sigma dB_t}{dt} \] (Gaussian, white wrt t)

\[ \int_{\Omega} \phi_i(x) \cdot (dtw + C(w,w)dt + F(w)dt + C(\sigma dB_t, w) = dF) \, dx \]

\[ db(t) = H(b(t)) \, dt + K(\sigma dB_t) \, b(t) \]

Full order : \( M \sim 10^7 \)
Reduced order : \( n \sim 10 \)

Resseguier et al. (2021). SIAM-ASA J Uncertain. hal- 03169957
Reduced Lum (Red Lum)

POD-Galerkin gives SDEs for resolved modes

\[ v = w + v' \]

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\[ w(x,t) = \sum_{i=0}^{n} b_i(t) \phi_i(x) \]

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\[ v' = \sigma dB_t \text{ (Gaussian, white wrt t)} \]

\[ \int_{\Omega} \phi_i(x) \cdot (d_t w + C(w,w)dt + F(w)dt + C(\sigma dB_t,w) = dF) \, dx \]

\[ db(t) = H\left(b(t)\right) dt + K(\sigma dB_t) \, b(t) \]

2nd order polynomial

Full order: \( M \sim 10^7 \)
Reduced order: \( n \sim 10 \)

Resseguier et al. (2021). SIAM-ASA J Uncertain. hal- 03169957
**REDUCED LUM (RED LUM)**

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\[ v' = \frac{\sigma dB_t}{dt} \] (Gaussian, white wrt \( t \))

\[
\int_{\Omega} \phi_i(x) \cdot (d_t w + C(w, w)dt + F(w)dt + C(\sigma dB_t, w) = dF) \ dx
\]

\[ db(t) = H(b(t))dt + K(\sigma dB_t) b(t) \]

Multiplicative skew-symmetric noise

2\textsuperscript{nd} order polynomial

Full order: \( M \sim 10^7 \)
Reduced order: \( n \sim 10 \)

Resseguier et al. (2021). SIAM-ASA J Uncertain. hal- 03169957
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\[ v' = \frac{\sigma dB_t}{dt} \] (Gaussian, white wrt t)

\[ \int_{\Omega} \phi_i(x) \cdot (d_t w + C(w, w)dt + F(w)dt + C(\sigma dB_t, w) = dF) \, dx \]

\[ db(t) = H(b(t)) \, dt + K(\sigma dB_t) b(t) \]

- **Multiplicative skew-symmetric noise**
- **2\textsuperscript{nd} order polynomial**

Full order: \( M \sim 10^7 \)
Reduced order: \( n \sim 10 \)

Ressegui et al. (2021). SIAM-ASA J Uncertain. hal- 03169957
REDUCED LUM (RED LUM)
POD-Galerkin gives SDEs for resolved modes

\[ v = w + v' \]

Resolved fluid velocity:
\[ w(x, t) = \sum_{i=0}^{n} b_i(t) \phi_i(x) \]

Unresolved fluid velocity:
\[ v' = \frac{\sigma dB_t}{dt} \quad \text{(Gaussian, white wrt t)} \]

Variance tensor:
\[ a(x, x) = \frac{\mathbb{E}\{(\sigma dB_t)(\sigma dB_t)^T\}}{dt} \]

\[ \int_{\Omega} \phi_i(x) \cdot (d_t w + C(w, w)dt + F(w)dt + C(\sigma dB_t, w) = dF) \ dx \]

\[ db(t) = H(b(t)) \ dt + K(\sigma dB_t) \ b(t) \]

\[ K_{fq}[\xi] = -\int_{\Omega} \phi_f \cdot C(\xi, \phi_q) \]

Full order : \( M \sim 10^7 \)
Reduced order : \( n \sim 10 \)

Resseguier et al. (2021). SIAM-ASA J Uncertain. hal-03169957
### Reduced Lum (Red Lum)

POD-Galerkin gives SDEs for resolved modes

\[ v = w + v' \]

Resolved fluid velocity:
\[ w(x, t) = \sum_{i=0}^{n} b_i(t) \phi_i(x) \]

Unresolved fluid velocity:
\[ v' = \frac{\sigma dB_t}{dt} \quad \text{(Gaussian, white wrt } t) \]

Variance tensor:
\[ a(x, x) = \frac{\mathbb{E}\{(\sigma dB_t)(\sigma dB_t)^T\}}{dt} \]

2\textsuperscript{nd} order polynomial

Coefficients given by:
- Randomized Navier-Stokes
- \( (\phi_j)_j \)
- \( a(x) \approx \Delta t \, v'(v')^T \)

\[ \bar{f} = \frac{1}{T} \int_0^T f \]

\[ K_{jq} [\xi] = -\int_{\Omega} \phi_j \cdot C(\xi, \Phi_q) \]

Full order: \( M \sim 10^7 \)
Reduced order: \( n \sim 10 \)

Resseguier et al. (2021). SIAM-ASA J Uncertain. hal-03169957
**Reduced LUM (RED LUM)**

POD-Galerkin gives SDEs for resolved modes

\[ v = w + v' \]

Resolved fluid velocity:
\[ w(x, t) = \sum_{i=0}^{n} b_i(t) \phi_i(x) \]

Unresolved fluid velocity:
\[ v' = \frac{\sigma dB_t}{dt} \] (Gaussian, white wrt \( t \))

Variance tensor:
\[ a(x, x) = \frac{\mathbb{E}\{(\sigma dB_t)(\sigma dB_t)^T\}}{dt} \]

\[ \bar{f} = \frac{1}{T} \int_0^T f \]

\[ \int_{\Omega} \phi_i(x) \cdot (d_t w + C(w, w)dt + F(w)dt + C(\sigma dB_t, w) = dF) \ dx \]

\[ db(t) = H(b(t)) dt + K(\sigma dB_t) b(t) \]

2\textsuperscript{nd} order polynomial

Coefficients given by:
- Randomized Navier-Stokes
- \( (\phi_j)_j \)
- \( a(x) \approx \Delta t \overline{v' (v')} \)

Randomized Navier-Stokes

Resseguier et al. (2021). SIAM-ASA J Uncertain. hal-03169957
**REDUCED LUM (RED LUM)**

POD-Galerkin gives SDEs for resolved modes

\[
\phi_i(x) \cdot \left( dt \, w + C(w,w)dt + F(w)dt + C(\sigma dB_t, w) = dF \right) dx
\]

2\textsuperscript{nd} order polynomial

Resolved fluid velocity:
\[
w(x,t) = \sum_{i=0}^{n} b_i(t) \phi_i(x)
\]

Unresolved fluid velocity:
\[
v' = \frac{\sigma dB_t}{dt} \quad \text{(Gaussian, white wrt } t)\]

Variance tensor:
\[
a(x,x) = \frac{\mathbb{E}[(\sigma dB_t)(\sigma dB_t)^T]}{dt}
\]

Coefficients given by:
- Randomized Navier-Stokes
- \((\phi_j)_j\)
- \(a(x) \approx \Delta t \, v' (v')^T\)

\[
\bar{f} = \frac{1}{T} \int_0^T f
\]

Randomized Navier-Stokes

PCA modes

\[
K_{jq}[\xi] = -\int_\Omega \phi_j \cdot C(\xi, \phi_q)
\]

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**REDUCED LUM (RED LUM)**

POD-Galerkin gives SDEs for resolved modes

\[ v = w + v' \]

Resolved fluid velocity:
\[ w(x, t) = \sum_{i=0}^{n} b_i(t) \phi_i(x) \]

Unresolved fluid velocity:
\[ v' = \frac{\sigma dB_t}{dt} \] (Gaussian, white wrt t)

Variance tensor:
\[ a(x, x) = \frac{\mathbb{E}[(\sigma dB_t)(\sigma dB_t)^T]}{dt} \]

\[ f = \frac{1}{T} \int_0^{T} f \]

2nd order polynomial

\[ \int_{\Omega} \phi_i(x) \cdot (d_t w + C(w, w)dt + F(w)dt + C(\sigma dB_t, w) = dF) \ dx \]

\[ db(t) = H(b(t)) \ dt + K(\sigma dB_t) \ b(t) \]

Coefficients given by:
- Randomized Navier-Stokes
- \( (\phi_j) \)_{j}
- \( a(x) \approx \Delta t \overline{v'} (v')^T \)

Full order: \( M \sim 10^7 \)
Reduced order: \( n \sim 10 \)

Resseguier et al. (2021), SIAM-ASA J Uncertain, hal-03169957
**REDUCED LUM (RED LUM)**

POD-Galerkin gives SDEs for resolved modes

\[ v = w + v' \]

Resolved fluid velocity:
\[ w(x, t) = \sum_{i=0}^{n} b_i(t) \phi_i(x) \]

Unresolved fluid velocity:
\[ v' = \sigma dB_t (\text{Gaussian, white wrt } t) \]

Variance tensor:
\[ a(x, x) = \frac{\mathbb{E}(\sigma dB_t(\sigma dB_t)^T)}{dt} \]

**Coefficients given by:**
- Randomized Navier-Stokes
- \( (\phi_j)_j \)
- \( a(x) \approx \Delta t \overline{v'} (\overline{v'})^T \)

**Reduced order** : \( n \sim 10 \)

**Full order** : \( M \sim 10^7 \)

\[ f = \frac{1}{T} \int_0^T f \]

\[ \int_{\Omega} \phi_i(x) \cdot (dt w + C(w, w)dt + F(w)dt + C(\sigma dB_t, w) = dF) \, dx \]

\[ db(t) = H(b(t)) \, dt + K(\sigma dB_t) \, b(t) \]

**Multiplicative skew-symmetric noise**

\[ K_{f_q}[\xi] = - \int_{\Omega} \phi_j \cdot C(\xi, \phi_q) \]
**Reduced LUM (RED LUM)**

POD-Galerkin gives SDEs for resolved modes

\[ v = w + v' \]

Resolved fluid velocity:
\[ w(x,t) = \sum_{i=0}^{n} b_i(t) \phi_i(x) \]

Unresolved fluid velocity:
\[ v' = \frac{\sigma dB_t}{dt} \]

(Gaussian, white wrt t)

Variance tensor:
\[ a(x,x) = \frac{\mathbb{E}[(\sigma dB_t)(\sigma dB_t)^T]}{dt} \]

2\textsuperscript{nd} order polynomial

\[ db(t) = H(b(t)) dt + K(\sigma dB_t) b(t) \]

\[ f = \frac{1}{T} \int_0^T f \]

\[ \int_{\Omega} \phi_i(x) \cdot (dt w + C(w,w)dt + F(w)dt + C(\sigma dB_t,w) = dF) dx \]

Multiplicative skew-symmetric noise

Full order: \( M \sim 10^7 \)
Reduced order: \( n \sim 10 \)

Coefficients given by:
- Randomized Navier-Stokes
- \( (\phi_j)_j \)
- \( a(x) \approx \Delta t \leftrightarrow (v')^T \)

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**REDUCED LUM (RED LUM)**

POD-Galerkin gives SDEs for resolved modes

\[ \nu = w + v' \]

Resolved fluid velocity:
\[ w(x,t) = \sum_{i=0}^{n} b_i(t) \phi_i(x) \]

Unresolved fluid velocity:
\[ v' = \frac{\sigma dB_t}{dt} \]

(Gaussian, white wrt \( t \))

Variance tensor:
\[ a(x,x) = \frac{\mathbb{E}\{(\sigma dB_t)(\sigma dB_t)^T\}}{dt} \]

2\(^{nd}\) order polynomial

\[ \int_{\Omega} \phi_i(x) \cdot (d_tw + C(w,w)dt + F(w)dt + C(\sigma dB_t, w) = dF) \ dx \]

\[ db(t) = H(b(t)) \ dt + K(\sigma dB_t) b(t) \]

Multiplicative skew-symmetric noise

**Covariance to estimate**

Coefficients given by:
- Randomized Navier-Stokes
- \((\phi_j)_j\)
- \(a(x) \approx \Delta t \ v' (v')^T\)

\( f = \frac{1}{T} \int_{0}^{T} f \)

**Reduction**

- Full order: \( M \sim 10^7 \)
- Reduced order: \( n \sim 10 \)

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**REDUCED LUM (RED LUM)**

POD-Galerkin gives SDEs for resolved modes

\[ \phi_i(x) \cdot \left( d_t w + C(w,w)dt + F(w)dt + C(\sigma dB_t,w) = dF \right) dx \]

\[ db(t) = H(b(t)) dt + K(\sigma dB_t) b(t) \]

Resolved fluid velocity:
\[ w(x,t) = \sum_{i=0}^n b_i(t) \phi_i(x) \]

Unresolved fluid velocity:
\[ v' = \sigma dB_t \quad \text{(Gaussian, white wrt } t) \]

Variance tensor:
\[ a(x,x) = \frac{\mathbb{E}\left\{ (\sigma dB_t)(\sigma dB_t)^T \right\}}{dt} \]

2nd order polynomial

Coefficients given by:
- Randomized Navier-Stokes
- \( (\phi_j)_j \)
- \( a(x) \approx \Delta t \overline{v'} (v')^T \)

Multiplicative skew-symmetric noise

Covariance to estimate
\[ \mathbb{E}\left\{ K_{jq}(\sigma dB_t) K_{ip}(\sigma dB_t) \right\}/dt \approx \Delta t K_{jq} \frac{b_p}{b_p^2} \frac{\Delta b_i}{\Delta t} v' \]

Full order: \( M \sim 10^7 \)

Reduced order: \( n \sim 10 \)

Resseguier et al. (2021). SIAM-ASA J Uncertain. hal-03169957
REDUCED LUM (RED LUM)

POD-Galerkin gives SDEs for resolved modes

\[ v = w + v' \]

Resolved fluid velocity:
\[ w(x,t) = \sum_{i=0}^{n} b_i(t) \phi_i(x) \]

Unresolved fluid velocity:
\[ v' = \sigma dB_t \] (Gaussian, white wrt t)

Variance tensor:
\[ a(x,x) = \frac{\mathbb{E}[(\sigma dB_t)(\sigma dB_t)^T]}{dt} \]

\[ \int_{\Omega} \phi_i(x) \cdot (d_t w + C(w,w)dt + F(w)dt + C(\sigma dB_t,w) = dF) \ dx \]

\[ db(t) = H(b(t)) \ dt + K(\sigma dB_t) b(t) \]

Multiplicative skew-symmetric noise

\[ \mathbb{E} \left( K_{jq}(\sigma dB_t) K_{ip}(\sigma dB_t) \right) / dt \approx \Delta t \ K_{jq} \left[ \frac{b_p}{b_p^2} \Delta b_i \ \Delta t \ v' \right] \]

Coefficients given by:
- Randomized Navier-Stokes
- \( (\phi_j)_j \)
- \( a(x) \approx \Delta t \ v' (v')^T \)

\[ f = \frac{1}{T} \int_0^T f \]

Resseguier et al. (2021). SIAM-ASA J Uncertain. hal-03169957
**REDUCED LUM (RED LUM)**

POD-Galerkin gives SDEs for resolved modes

\[ \int_{\Omega} \phi_i(x) \cdot (d_t w + C(w, w) dt + F(w) dt + C(\sigma dB_t, w) = dF) \, dx \]

\[ db(t) = H(b(t)) \, dt + K(\sigma dB_t) b(t) \]

- Resolved fluid velocity: \( w(x, t) = \sum_{i=0}^{n} b_i(t) \phi_i(x) \)
- Unresolved fluid velocity: \( v' = \frac{\sigma dB_t}{dt} \) (Gaussian, white wrt \( t \))
- Variance tensor:
  \[ a(x, x) = \frac{\mathbb{E}[(\sigma dB_t)(\sigma dB_t)^T]}{dt} \]

**Coefficients given by:**
- Randomized Navier-Stokes
- \( (\phi_j)_j \)
- \( a(x) \approx \Delta t \, v' (v')^T \)

\[ \bar{f} = \frac{1}{T} \int_{0}^{T} f \]

**Covariance to estimate**

\[ \mathbb{E} \left( K_{jq}(\sigma dB_t) K_{ip}(\sigma dB_t) \right) / dt \approx \Delta t \, K_{jq} \left[ \begin{array}{c} \frac{b_p}{b_p} \frac{\Delta b_i}{\Delta t} \end{array} \right] \]

- Full order: \( M \sim 10^7 \)
- Reduced order: \( n \sim 10 \)

Resseguier et al. (2021), SIAM-ASA J Uncertain. hal-03169957
**REDUCED LUM (RED LUM)**

POD-Galerkin gives SDEs for resolved modes

\[ \nabla \cdot \phi_j(x) \cdot (dt w + C(w,w)dt + F(w)dt + C(\sigma dB_t, w) = dF) \ dx \]

\[ \int_\Omega \phi_i(x) \cdot dF \]

\[ \mathbf{db}(t) = H\left(b(t)\right)dt + K\left(\sigma dB_t\right)\mathbf{b}(t) \]

**2nd order polynomial**

\[ \begin{align*}
\mathcal{K}_{ij} = & \mathbb{E} \left[ \left( \sigma dB_t \right) \cdot \left( \sigma dB_t \right)^T \right] \\
\Delta \mathbf{b}_i \approx & \mathbb{E} \left[ K_{ij} \left( \sigma dB_t \right) \cdot \mathbf{K}_{ip} \left( \sigma dB_t \right) \right] / dt \approx \Delta t \mathcal{K}_{ij} \\
\mathbf{b}_i \approx & \mathbb{E} \left[ K_{ij} \left( \sigma dB_t \right) \cdot \mathbf{K}_{ip} \left( \sigma dB_t \right) \right] / dt \approx \Delta t \mathcal{K}_{ij} \\
\end{align*} \]

Coefficients given by:
- Randomized Navier-Stokes
- \( \phi_j \)
- \( a(x) \approx \Delta t \mathbf{v}' (\mathbf{v}')^T \)

Resolved fluid velocity:
\( w(x,t) = \sum_{i=0}^{n} b_i(t) \phi_i(x) \)

Unresolved fluid velocity:
\( \mathbf{v}' = \sigma dB_t \) (Gaussian, white wrt. \( t \))

Variance tensor:
\[ a(x,x) = \frac{\mathbb{E}\left[\left(\sigma dB_t\right)\left(\sigma dB_t\right)^T\right]}{dt} \]

Multiplicative skew-symmetric noise

Covariance to estimate

\[ \mathbb{E} \left[ \mathcal{K}_{ij} \left( \sigma dB_t \right) \cdot \mathcal{K}_{ip} \left( \sigma dB_t \right) \right] / dt \approx \Delta t \mathcal{K}_{ij} \]

\[ \begin{align*}
\mathbf{b}_i \approx & \mathbb{E} \left[ K_{ij} \left( \sigma dB_t \right) \cdot \mathbf{K}_{ip} \left( \sigma dB_t \right) \right] / dt \approx \Delta t \mathcal{K}_{ij} \\
\end{align*} \]
**REDUCED LUM (RED LUM)**
POD-Galerkin gives SDEs for resolved modes

\[ v = w + v' \]

Resolved fluid velocity:
\[ w(x,t) = \sum_{i=0}^{n} b_i(t) \phi_i(x) \]

Unresolved fluid velocity:
\[ v' = \frac{\sigma dB_t}{dt} \] (Gaussian, white wrt t)

Variance tensor:
\[ a(x,x) = \frac{\mathbb{E}((\sigma dB_t)(\sigma dB_t)^T)}{dt} \]

\[ f = \frac{1}{T} \int_0^T f \]

2nd order polynomial

\[ \int_{\Omega} \phi_i(x) \cdot (d_t w + C(w,w)dt + F(w)dt + C(\sigma dB_t,w) = dF) \, dx \]

\[ db(t) = H(b(t)) \, dt + K(\sigma dB_t) \begin{pmatrix} b(t) \end{pmatrix} \]

Multiplicative skew-symmetric noise

\[ \mathbb{E} \left( K_{jq}(\sigma dB_t) K_{ip}(\sigma dB_t) \right) / dt \approx \Delta t \, K_{jq} \begin{pmatrix} \frac{\Delta b_i}{b_p^2} \frac{\Delta b_i}{\Delta t} \end{pmatrix} \]

Covariance to estimate

Coefficients given by:
- Randomized Navier-Stokes
- \( (\phi_j)_j \)
- \[ a(x) \approx \Delta t \, v' (v')^T \]

Full order: \( M \sim 10^7 \)
Reduced order: \( n \sim 10 \)

Randomized Navier-Stokes

PCA modes

PCA residual \( v' \)

from synthetic data

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**REDUCED LUM (RED LUM)**

**POD-Galerkin gives SDEs for resolved modes**

\[ v = w + v' \]

Resolved fluid velocity:
\[ w(x, t) = \sum_{i=0}^{n} b_i(t) \phi_i(x) \]

Unresolved fluid velocity:
\[ v' = \frac{\sigma dB_t}{dt} \] (Gaussian, white wrt t)

Variance tensor:
\[ a(x) = \left\langle (\sigma dB_t)^T (\sigma dB_t) \right\rangle \]

\[ f = \frac{1}{T} \int_0^T f \]

\[ \int_{\Omega} \phi_i(x) \cdot (d_t w + C(w,w) dt + F(w) dt + C(\sigma dB_t, w) = dF) \ dx \]

\[ db(t) = H(b(t)) dt + K(\sigma dB_t) b(t) \]

**New estimator**
- Consistency proven ($\Delta t \to 0$)
- Numerically efficient
- Physically-based → Robustness in extrapolation

\[ \mathbb{E} \left( K_{jq} (\sigma dB_t) K_{ip} (\sigma dB_t) \right) / dt \approx \Delta t \ K_{jq} \left[ \frac{b_p \Delta b_i}{b_p^2} \Delta t \ v' \right] \]

Randomized Navier-Stokes

PCA modes

PCA residual $v'$

from synthetic data

Resseguier et al. (2021). *SIAM-ASA J Uncertain*. hal-03169957

Full order : $M \sim 10^7$
Reduced order : $n \sim 10$
**REduced LUM (RED LUM)**

Multiplicative noise covariance

\[
v = w + v'
\]

Resolved fluid velocity:
\[
w(x, t) = \sum_{i=0}^{n} b_i(t) \phi_i(x)
\]

Unresolved fluid velocity:
\[
v' = \frac{\sigma dB_t}{dt}
\]

(Gaussian, white wrt \(t\))

\[
db(t) = H(b(t)) \, dt + K(\sigma dB_t) \, b(t)
\]

with \(K_{jq} [\xi] = -\int_{\Omega} \phi_j \cdot C(\xi, \phi_q)\)

**Curse of dimensionality**

- Since \(\sigma dB_t\) is white in time,
  \[
  \Sigma_{jq, ip} = \mathbb{E} \left( K_{jq} (\sigma dB_t) K_{ip} (\sigma dB_t) \right) / dt \approx \Delta t \, K_{jq} (v') K_{ip} (v')
  \]
- \(K\) is a matrix of integro-differential operators → cannot be evaluated on \(v'(x, t)\) at every time \(t\)
- Covariance of \(\sigma dB_t \approx \Delta t^2 \langle v'(x, t)\rangle \langle v'(y, t)\rangle^T : M \times M \sim 10^{13}\) coefficients → intractable

\[
\bar{f} = \frac{1}{T} \int_0^T f
\]

Full order (~ nb spatial grid points): \(M \sim 10^7\)

Reduced order: \(n \sim 10\)

Number of time steps: \(N \sim 10^4\)

**New estimator**

- Consistency proven (\(\Delta t \rightarrow 0\))
- Numerically efficient
- Physically-based → Robustness in extrapolation

Resseguier et al. (2021). SIAM-ASA J Uncertain. hal-03169957
**REDUCED LUM (RED LUM)**

Multiplicative noise covariance

\[ \nu = w + \nu' \]

Resolved fluid velocity:
\[ w(x,t) = \sum_{i=0}^{n} b_i(t) \phi_i(x) \]

Unresolved fluid velocity:
\[ \nu' = \frac{\sigma dB_t}{dt} \] (Gaussian, white wrt t)

\[ db(t) = H(b(t)) dt + K(\sigma dB_t) b(t) \] with \[ K_{jq}[\xi] = -\int_{\Omega} \phi_j \cdot C(\xi, \phi_q) \]

**Curse of dimensionality**

- Since \( \sigma dB_t \) is white in time,
  \[ \Sigma_{jq,ip} = \mathbb{E} \left( K_{jq}(\sigma dB_t) K_{ip}(\sigma dB_t) \right) / dt \approx \Delta t K_{jq}(\nu') K_{ip}(\nu') \]
  
- \( K \) is a matrix of integro-differential operators \( \rightarrow \) cannot be evaluated on \( \nu'(x,t) \) at every time t
  
- Covariance of \( \sigma dB_t \approx \Delta t^2 \mathbb{E} \left( \nu'(x,t) \right) \mathbb{E} \left( \nu'(y,t) \right)^T \) : \( M \times M \sim 10^{13} \) coefficients \( \rightarrow \) intractable

- **Efficient estimator** \[ \Sigma_{jq,ip} \approx \Delta t K_{jq} \left[ \frac{\partial p}{\partial \xi} \frac{\Delta b_i}{\Delta t} \nu' \right] \] (hybrid fitting & physics-based)

  requires only \( O(n^2 M) \) correlation estimations and \( O(n^2) \) evaluations of \( K \)

\[ \bar{f} = \frac{1}{T} \int_0^T f \]

Full order (~ nb spatial grid points): \( M \sim 10^7 \)

Reduced order : \( n \sim 10 \)

Number of time steps : \( N \sim 10^4 \)

New estimator

- Consistency proven \((\Delta t \to 0)\)
- Numerically efficient
- Physically-based
  \( \rightarrow \) Robustness in extrapolation

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Ressegui et al. (2021). SIAM-ASA J Uncertain. hal-03169957
**REDUCED LUM (RED LUM)**

Multiplicative noise covariance

\[ v = w + v' \]

Resolved fluid velocity:
\[ w(x, t) = \sum_{i=0}^{n} b_i(t) \phi_i(x) \]

Unresolved fluid velocity:
\[ v' = \frac{\sigma d B_t}{d t} \] (Gaussian, white wrt t)

**db(t) = H(b(t)) dt + K(σdBt) b(t) with**
\[ K_{ijq}[\xi] = -\int_{\Omega} \phi_j \cdot C(\xi, \Phi_q) \]

**Curse of dimensionality**
- Since \( \sigma dB_t \) is white in time,
  \[ \Sigma_{ijq} = \mathbb{E}\left( K_{jq} (\sigma dB_t) K_{ip} (\sigma dB_t) \right) / dt \approx \Delta t K_{jq} (v') K_{ip} (v') \]
  
  - \( K \) is a matrix of integro-differential operators \( \rightarrow \) cannot be evaluated on \( v'(x, t) \) at every time t
  
  - Covariance of \( \sigma dB_t \approx \Delta t^2 \left( v'(x, t) \right) \left( v'(y, t) \right)^T : M \times M \sim 10^{13} \) coefficients \( \rightarrow \) intractable
  
  - **Efficient estimator**
    \[ \Sigma_{ijq,ip} \approx \Delta t K_{jq} \left[ \frac{\partial b_p}{\partial \xi} \frac{\partial b_i}{\partial \xi} \right] v' \] (hybrid fitting & physics-based)
    
    requires only \( O(n^2 M) \) correlation estimations and \( O(n^2) \) evaluations of \( K \)

  **Consistency of our estimator** (convergence in probability for \( \Delta t \to 0 \), using stochastic calculus and continuity of \( K \))

\[ \Delta t K_{jq} \left[ \frac{\partial b_p}{\partial \xi} \frac{\partial b_i}{\partial \xi} \right] v' = \Delta t b_p \frac{\partial b_i}{\partial \xi} K_{jq} [v'] \approx \frac{1}{T} \int_0^T b_p d < b_i, K_{jq}(\sigma B) > = \frac{1}{T} \int_0^T b_p \sum_{r=0}^{n} b_r d < K_{ir}(\sigma B), K_{jq}(\sigma B) > = \sum_{r=0}^{n} \Sigma_{jq,ir} b_p b_r = \Sigma_{jq,ip} b_p^2 \] (orthogonality from PCA)

**New estimator**
- Consistency proven (\( \Delta t \to 0 \))
- Numerically efficient
- Physically-based
  \( \rightarrow \) Robustness in extrapolation

Full order (\( \sim \) nb spatial grid points): \( M \sim 10^7 \)

Reduced order : \( n \sim 10 \)

Number of time steps : \( N \sim 10^4 \)

Randomized Navier-Stokes

PCA modes

PCA residual \( v' \)

from synthetic data

Resseguier et al. (2021). SIAM-ASA J Uncertain. hal-03169957
**Reduced LUM (Red LUM)**

**Multiplicative noise covariance**

\[ v = w + v' \]

Resolved fluid velocity:
\[ w(x, t) = \sum_{i=0}^{n} b_i(t) \phi_i(x) \]

Unresolved fluid velocity:
\[ v' = \frac{\sigma \sigma_{Bt}}{dt} \] (Gaussian, white wrt t)

\[ db(t) = H(b(t)) \, dt + K(\sigma dB_t) \, b(t) \] with \( K_{jq}[\xi] = -\int_{\Omega} \phi_j \cdot C(\xi, \Phi_q) \)

- **Curse of dimensionality**
  - Since \( \sigma dB_t \) is white in time,
    \[ \Sigma_{jq,ip} = \mathbb{E} \left( K_{jq}(\sigma dB_t) K_{ip}(\sigma dB_t) \right) / dt \approx \Delta t \, K_{jq}(v') K_{ip}(v') \]
  - \( K \) is a matrix of integro-differential operators → cannot be evaluated on \( v'(x, t) \) at every time \( t \)
  - Covariance of \( \sigma dB_t \approx \Delta t^2 \, (v'(x, t))(v'(y, t))^T : M \times M \sim 10^{13} \) coefficients → intractable
  - Efficient estimator \( \Sigma_{jq,ip} \approx \Delta t \, K_{jq} \left[ \frac{b_p}{b_p} \frac{\partial b_i}{\partial t} \right] \) (hybrid fitting & physics-based)
    requires only \( O(n^2 M) \) correlation estimations and \( O(n^2) \) evaluations of \( K \)

- **Consistency of our estimator** (convergence in probability for \( \Delta t \to 0 \), using stochastic calculus and continuity of \( K \))

\[ \Delta t \, K_{jq} \left[ \frac{b_p}{b_p} \frac{\partial b_i}{\partial t} \right] = \Delta t \, b_p \frac{\partial b_i}{\partial t} K_{jq} \left[ v' \right] \approx \frac{1}{T} \int_0^T b_p \, b_i \right] \right] \]

- **Optimal time subsampling at \( \Delta t \) needed** to meet the white assumption

\[ \bar{f} = \frac{1}{T} \int_0^T f \]

Full order (≈ nb spatial grid points): \( M \sim 10^7 \)
Reduced order: \( n \sim 10 \)
Number of time steps: \( N \sim 10^4 \)

New estimator
- Consistency proven (\( \Delta t \to 0 \))
- Numerically efficient
- Physically-based → Robustness in extrapolation

Resseguier et al. (2021). SIAM-ASA J Uncertain. hal-03169957
**Reduced Lum (Red Lum)**

**Multiplicative noise covariance**

\[ v = w + v' \]

- Resolved fluid velocity:
  \[ w(x, t) = \sum_{i=0}^{n} b_i(t) \phi_i(x) \]

- Unresolved fluid velocity:
  \[ v' = \frac{\sigma dB_t}{dt} \] (Gaussian, white wrt \( t \))

\[ db(t) = H(b(t)) \, dt + K(\sigma dB_t) \, b(t) \quad \text{with} \quad K_{jq}[\xi] = - \int_\Omega \phi_j \cdot C(\xi, \phi_q) \]

\[ \sigma dB_t \] is white in time,

\[ \Sigma_{jq,ip} = \mathbb{E}(K_{jq}(\sigma dB_t) K_{ip}(\sigma dB_t)) / \Delta t \approx \Delta t K_{jq}(v') K_{ip}(v') \]

- \( K \) is a matrix of integro-differential operators → cannot be evaluated on \( v'(x, t) \) at every time \( t \)

- Covariance of \( \sigma dB_t \approx \Delta t^2 (v'(x, t))(v'(y, t))' : M \times M \sim 10^{13} \) coefficients → intractable

- Efficient estimator \( \Sigma_{jq,ip} \approx \Delta t K_{jq} \left[ \frac{b_p}{b_p} \frac{\Delta b_i}{\Delta t} v' \right] \) (hybrid fitting & physics-based)

  requires only \( O(n^2 M) \) correlation estimations and \( O(n^2) \) evaluations of \( K \)

- **Consistency of our estimator** (convergence in probability for \( \Delta t \to 0 \), using stochastic calculus and continuity of \( K \))

\[ \Delta t K_{jq} \left[ \frac{b_p}{b_p} \frac{\Delta b_i}{\Delta t} v' \right] = \frac{1}{\Delta t} \int_0^T b_p d \phi_i, K_{jq}(\sigma B_t) > = \frac{1}{\Delta t} \int_0^T b_p \sum_{i=0}^n b_i d < K_{ip}(\sigma B_t), K_{jq}(\sigma B) > = \Sigma_{ij} \Sigma_{jq,ir} \frac{b_p}{b_p} \frac{\Delta b_i}{\Delta t} = \Sigma_{jq,ip} \frac{b_p}{b_p} \frac{\Delta b_i}{\Delta t} \] (orthogonality from PCA)

- **Optimal time subsampling at \( \Delta t \) needed** to meet the white assumption

- **Additional reduction for efficient sampling**: diagonalization of \( \Sigma \to K(\sigma dB_t) \approx \alpha(d\beta_t) \) with a \( n \)-dimensional (instead of \((n+1)^2\)-dimensional) Brownian motion \( \beta \)

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New estimator

- Consistency proven ([\( \Delta t \to 0 \)]
- Numerically efficient
- Physically-based → Robustness in extrapolation

Full order (~ nb spatial grid points): \( M \sim 10^7 \)

Reduced order: \( n \sim 10 \)

Number of time steps: \( N \sim 10^4 \)

Resseguier et al. (2021). SIAM-ASA J Uncertain. hal-03169957
SUMMARY
Stochastic ROM + Data assimilation

Off-line : Building SROM

- Physics (Navier-Stokes)
- CFD code
- Synthetic Data
  - PCA modes
  - PCA residual

Randomized Navier-Stokes

SROM (POD-Galerkin)

\[ \frac{db(t)}{dt} = \frac{H(b(t))}{H(b(t))} \, dt + \alpha d\beta(t) b(t) \]

On-line : Simulation & data assimilation

Data Assimilation
  - (particle filter)

Sparse measurement data

Temporal modes

\[ v \approx \sum_{i=0}^{n} b_i \phi_i \]

Solution

Resseguier et al. (2022). J Comp.Phys : hal-03445455
SUMMARY
Stochastic ROM + Data assimilation

Off-line : Building SROM

Physics (Navier-Stokes) → CFD code → Synthetic Data → PCA modes → Randomized Navier-Stokes → SROM (POD-Galerkin)

\[ \frac{db(t)}{dt} = H(b(t)) \, dt + a(\beta) \, b(t) \]

On-line : Simulation & data assimilation

Data Assimilation (particle filter) → Sparse measurement data → Temporal modes

\[ b_i(t) \]

Solution

\[ v \approx \sum_{i=0}^{n} b_i \phi_i \]

Resseguier et al. (2022). J Comp.Phys : hal-03445455
PART IV

NUMERICAL RESULTS

a. Uncertainty quantification (Prior)

b. Data assimilation (Posterior)
UNCERTAINTY QUANTIFICATION (PRIOR)
Known initial conditions \( b(t = 0) \)

\[ db(t) = H(b(t)) \, dt + \alpha(d\beta_t) \, b(t) \]

Metrics choice
- \( b_i(t) \) VS reference
- Error metrics

Test cases

**Full-order reference**
- Reynolds number (Re) = 100 / 2D
  (full-order simulation has \( 10^4 \) dof)
- Reynolds number (Re) = 300 3D
  (full-order simulation has \( 10^7 \) dof)

**Reduced-order reference**
PCA-projection of the full-order simulation
(Optimal from 8-dof linear decomposition)

- Wind Q-criterion
- Vorticity
- (round) wind turbine blade
- Q-criterion

From \( 10^7 \) to 8 degrees of freedom
No data assimilation
Known initial conditions \( b(t = 0) \)

Resseguier et al. (2021). SIAM-ASA J Uncertain. hal-03169957
UNCERTAINTY QUANTIFICATION (PRIOR)

\( b_i(t) \) VS reference

From \( 10^7 \) to 8 degrees of freedom
No data assimilation
Known initial conditions \( b(t = 0) \)

Reference (full-order simulation)

State of the art

Red. LUM realization
Red. LUM mean
Red. LUM confidence interval

Resseguier et al. (2021). SIAM-ASA J Uncertain. hal- 03169957
UNCERTAINTY QUANTIFICATION (PRIOR)
Error on the reduced solution $w$

$$v = w + v'$$

Resolved fluid velocity:
$$w = \sum_{i=0}^{n} b_i \phi_i$$

Unresolved fluid velocity:
$$v'$$

Reynolds number (Re) = $100 / 2D$  (full-order simulation has $10^4$ dof)

From $10^7$ to 8 degrees of freedom
No data assimilation
Known initial conditions $b(t = 0)$

Resseguier et al. (2021). SIAM-ASA J Uncertain. hal-03169957
**UNCERTAINTY QUANTIFICATION (PRIOR)**

Error on the reduced solution $w$

\[ v = w + v' \]

Resolved fluid velocity:
\[ w = \sum_{i=0}^{n} b_i \phi_i \]

Unresolved fluid velocity:
\[ v' \]

Reynolds number (Re) = 100 / 2D
(full-order simulation has $10^4$ dof)

The Reference remains always close to the Red. LUM ensemble

From $10^7$ to 8 degrees of freedom
No data assimilation
Known initial conditions $b(t = 0)$

Resseguier et al. (2021). SIAM-ASA J Uncertain. hal- 03169957
DATA ASSIMILATION (POSTERIOR)
On-line estimation of the solution

**Reference**
PCA-projection of the DNS (Optimal from 8-dof linear decomposition)

**Our method**
POD-Galerkin with Navier-Stokes under location uncertainty (LUM)

**State-of-the-art**
POD-Galerkin with Navier-Stokes + optimally tuned eddy viscosity & additive noise

From $10^7$ to 8 degrees of freedom
- Single measurement point (blurred & noisy velocity)

Resseguier et al. (2022). *J Comp Phys*. hal-03445455
DATA ASSIMILATION (POSTERIOR)
Error on the solution estimation

\[ v = w + v' \]

Resolved fluid velocity:
\[ w = \sum_{i=0}^{n} b_i \phi_i \]

Unresolved fluid velocity:
\[ v' \]

Reynolds number (Re) = 100 / 2D
(full-order simulation has \(10^4\) dof)

Reynolds number (Re) = 300 3D
(full-order simulation has \(10^7\) dof)

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Resseguier et al. (2022). J Comp. Phys. hal-03445455
CONCLUSION
CONCLUSION

- Reduced order model (ROM) : for very fast and robust CFD ($10^7 \rightarrow 8$ degrees of freedom.)
  - Combine data & physics (built off-line)
  - Closure problem handled by LUM
  - Efficient estimator for the multiplicative noise
  - Efficient generation of prior / Model error quantification

- Data assimilation (Bayesian inverse problem) : to correct the fast simulation on-line by incomplete/noisy measurements

- First results
  - Optimal \textit{unsteady} flow estimation/prediction in the whole spatial domain (large-scale structures)
  - Robust far outside the training set

NEXT STEPS

- Real measurements
- Better stochastic closure
- Parametric ROM (unknown inflow)

- Increasing Reynolds (ROM of (non-polynomial) turbulence models)