A NOTE ON STARSHAPED HYPERSURFACES WITH ALMOST CONSTANT MEAN CURVATURE IN SPACE FORMS

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ABSTRACT. We show that closed starshaped hypersurfaces of space forms with almost constant mean curvature or almost constant higher order mean curvature are closed to geodesic spheres.

1. INTRODUCTION

Over the past years, the stability of many characterizations of geodesic spheres has been studied. One can cite for example, the stability of the Alexandrov theorem [10], the study of almost-Einstein [4, 13, 21], almost-umbilical [5, 14, 15], almost Weingarten [18] or almost stable hypersurfaces [16, 19], as well as for hypersurfaces that almost satisfy the limitting case of sharp inequalities (see for instance [1, 8, 12, 13, 16] and references therein).

The aim of this short note is to give a new result in this direction. Namely, we show that closed, connected and oriented starshaped hypersurfaces with almost constant mean curvature or almost constant higher order mean curvature in space forms are close to geodesic spheres for the Hausdorff distance. The setting for this problem is the followsing

Let (M^n, g) be a closed connected and oriented Riemannian manifold and $X : (M^n, g) \longrightarrow \mathbb{R}^{n+1}$ an isometric immersion of (M^n, g) into the Euclidean space $\mathbb{R}^{n+1}$. We consider ν a global unit normal vector field over M compatible with the orientation of M. We say that $X(M)$ is starshaped or simply M is starshaped if the function $\langle X, \nu \rangle$ has constant sign. It is a classical fact that if M is starshaped and has constant mean curvature or higher order mean curvature, then $X(M)$ is a geodesic sphere (see [9]). The proof of this result is a direct consequence of the classical Hsiung-Minkowski formulas. Moreover, Hsiung-Minkowski formulas have analogues in the half-sphere and the hyperbolic space which allows to prove the analogous characterization in these spaces. Namely, if M is a starshaped hypersurface of the half-sphere or the hyperbolic space and has constant mean curvature or higher order mean curvature, then $X(M)$ is a geodesic sphere.

We introduce the following notations before stating the main result of this note. The second fundamental form will be denoted B, the k-th mean curvature H_k and $\mathbb{M}^{n+1}(\delta)$ is the Euclidean space $\mathbb{R}^{n+1}$ if $\delta = 0$, the hyperbolic space $\mathbb{H}^{n+1}(\delta)$ if $\delta < 0$ and the upper half-sphere $\mathbb{S}_+^{n+1}(\delta)$ if $\delta > 0$. The main result of this note is the following stability result associated with the above characterization.

Theorem 1.1. *Let $n \geqslant 2$ and $r \in \{1, \cdots, n-1\}$ be two integers. Let M be a closed, connected and oriented hypersurface of $\mathbb{M}^{n+1}(\delta)$ contained in a ball of*

radius R and assume that $H_{r+1} > 0$ if $r > 1$. We denote by Z the position vector of M into $\mathbb{M}^{n+1}(\delta)$, assume that M is starshaped and set $R_0 = \min_M (|\langle Z, \nu \rangle|) > 0$. Let h be a positive real number. Then, there exist three constants γ, C and ε_1, with γ depending only on n; C and ε_1 depending on n, r, δ, h, $\min_M(H_{r+1;n,1})$, $\|B\|_\infty$, $V(\Sigma)$, R_0 and R so that if M has almost constant r-th mean curvature in the following sense

$$H_r = h + \varepsilon,$$

where ε is a smooth function satisfying $\|\varepsilon\|_\infty \leqslant \dfrac{h}{2}$ and $\|\varepsilon\|_1 \leqslant \varepsilon_1$, then

$$d_H(\Sigma, S_{\rho_0}) \leqslant C\|\varepsilon\|_1^\gamma,$$

where S_{ρ_0} is a geodesic sphere of a certain radius ρ_0 and d_H is the Hausdorff distance between compact sets into $\mathbb{M}^{n+1}(\delta)$.

Remark 1.2. (1) *The function $H_{r+1;n,1}$ appearing in the theorem is an extrinsic quantity defined from the second fundamental form. The precise definition is given by relation (11).*

(2) *If $r = 1$, that is, M has almost constant mean curvature, then the constants C and ε_1 do not depend on $\min_M(H_{2;n,1})$ since in the case $\min_M(H_{2;n,1})$ is just a dimensional constant.*

(3) *The assumption that $\|\varepsilon\|_\infty \leqslant \dfrac{h}{2}$ is here to ensure that the minimum of H_r over M is controlled by h. The choice of $\dfrac{h}{2}$ is arbitrary and can be replaced by $\|\varepsilon\|_\infty \leqslant \alpha h$ with $0 < \alpha < 1$. Therefore the constant C will also depends on α. Moreover, we could remove this assumption and therefore the constant C would depends on $\min_M(H_{r+1})$ instead of h by the Maclaurin inequality $H_r^{\frac{1}{r}} \geqslant H_{r+1}^{\frac{1}{r+1}}$.*

2. Preliminaries

Let (M^n, g) be an n-dimensional closed, connected and oriented Riemannian manifold isometrically immersed into the $(n+1)$-dimensional simply connected real space form $\mathbb{M}^{n+1}(\delta)$ of constant curvature δ. The (real-valued) second fundamental form B of the immersion is the bilinear symmetric form on $\Gamma(TM)$ defined for two vector fields X, Y by

$$B(X, Y) = -g\left(\overline{\nabla}_X \nu, Y\right),$$

where $\overline{\nabla}$ is the Riemannian connection on $\mathbb{M}^{n+1}(\delta)$ and ν a normal unit vector field on M.

From B, we can define the mean curvature,

$$H = \frac{1}{n}\operatorname{tr}(B).$$

Now, we recall the Gauss formula. For $X, Y, Z, W \in \Gamma(TM)$,

(1) $\quad R(X, Y, Z, W) = \overline{R}(X, Y, Z, W) + \langle SX, Z \rangle \langle SY, W \rangle - \langle SY, Z \rangle \langle SX, W \rangle$

where R and $\overline{R}$ are respectively the curvature tensor of M and $\mathbb{M}^{n+1}(\delta)$, and S is the Weingarten operator defined by $SX = -\overline{\nabla}_X \nu$.

By taking the trace and for $W = Y$, we get

$$(2) \qquad \mathrm{Ric}(Y) = \overline{\mathrm{Ric}}(Y) - \overline{R}(\nu, Y, \nu, Y) + nH \langle SY, Y \rangle - \langle S^2 Y, Y \rangle.$$

Since, the ambient space is of constant sectional curvature δ, by taking the trace a second time, we have

$$(3) \qquad \mathrm{Scal} = n(n-1)\delta + n^2 H^2 - \|S\|^2,$$

or equivalently

$$(4) \qquad \mathrm{Scal} = n(n-1)\left(H^2 + \delta\right) - \|\tau\|^2,$$

where $\tau = S - H\mathrm{Id}$ is the umbilicity tensor.

Now, we define the higher order mean curvatures, for $k \in \{1, \cdots, n\}$, by

$$H_k = \frac{1}{\binom{n}{k}} \sigma_k(\kappa_1, \cdots, \kappa_n),$$

where σ_k is the k-th elementary symmetric polynomial and $\kappa_1, \cdots, \kappa_n$ are the principal curvatures of the immersion.

From the definition, it is obvious that H_1 is the mean curvature H. We also remark from the Gauss formula (1) that

$$(5) \qquad H_2 = \frac{1}{n(n-1)}\mathrm{Scal} - \delta.$$

Hence, the equation (4) becomes $H^2 - H_2 = \dfrac{1}{n(n-1)}\|\tau\|^2$ and thus $H^2 \geqslant H_2$.

More generally, we have the following classical inequalities between the the higher order mean curvatures H_r which are well-known. First, for any $r \in \{0, \cdots, n-2\}$, we have

$$(6) \qquad H_r H_{r+2} \leqslant H_{r+1}^2,$$

with equality at umbilical points, cf. [6, p. 104]. Moreover, if $H_{r+1} > 0$, then $H_s > 0$ for any $s \in \{0, \cdots, r\}$ [2] and we have the classical Maclaurin inequalities

$$(7) \qquad H_{r+1}^{\frac{1}{r+1}} \leqslant H_r^{\frac{1}{r}} \leqslant \cdots \leqslant H_2^{\frac{1}{2}} \leqslant H.$$

Finally, we recall the well-known Hsiung-Minkowski formula

$$(8) \qquad \int_M \left(H_{k+1} \langle Z, \nu \rangle + c_\delta(r) H_k \right) = 0,$$

where $r(x) = d(p_0, x)$ is the distance function to a base point p_0, Z is the position vector defined by $Z = s_\delta(r)\overline{\nabla}r$, and the functions c_δ and s_δ are defined by

$$c_\delta(t) = \begin{cases} \cos(\sqrt{\delta}t) & \text{if } \delta > 0 \\ 1 & \text{if } \delta = 0 \\ \cosh(\sqrt{-\delta}t) & \text{if } \delta < 0 \end{cases} \quad \text{and} \quad s_\delta(t) = \begin{cases} \frac{1}{\sqrt{\delta}}\sin(\sqrt{\delta}t) & \text{if } \delta > 0 \\ t & \text{if } \delta = 0 \\ \frac{1}{\sqrt{-\delta}}\sinh(\sqrt{-\delta}t) & \text{if } \delta < 0. \end{cases}$$

3. Proof of Theorem 1.1

The strategy of the proof consists in proving that M is almost umbilical. Precisely, we will show that the L^{n+1}-norm of τ is small (compared to ε) in order to apply the following result of [15] with $p = n + 1$ and where N^{n+1} is either the half-sphere or the hyperbolic space.

Theorem 3.1. *(Roth-Scheuer [15]) Let $D \subset \mathbb{R}^{n+1}$ be open and let $N^{n+1} = (D, h)$ be a conformally flat Riemannian manifold, i.e., $h = e^{2\varphi}\widetilde{h}$ where $\widetilde{h}$ is the Euclidean metric and $\varphi \in C^{\infty}(D)$. Let $\Sigma^n \hookrightarrow N^{n+1}$ be a closed, connected, oriented and isometrically immersed hypersurface. Let $p > n \geq 2$. Then there exist constants c and ε_0, depending on $n, p, V(\Sigma), \|B\|_p$ and $\|\varphi\|_{\infty}$, as well as a constant $\alpha = \alpha(n, p)$, such that whenever there holds*

$$\|\tau\|_p \leqslant \|\widetilde{H}\|_p \varepsilon_0,$$

there also holds

$$d_H(\Sigma, S_{\rho}) \leqslant \frac{c\rho}{\|\widetilde{H}\|_p^{\alpha}} \|\tau\|_p^{\alpha},$$

where S_{ρ} is the image of a Euclidean sphere considered as a hypersurface in N^{n+1} and the Hausdorff distance is also measured with respect to the metric h.

Remark 3.2. *We use the following convention for the L^p-norm*

$$\|f\|_p = \left(\frac{1}{V(M)} \int_M |f|^p dv_g \right)^{\frac{1}{p}}.$$

First, we have

$$\begin{aligned}
\|\tau\|_{n+1}^{2(n+1)} &= \left(\frac{1}{V(\Sigma)} \int_M \|\tau\|^{2(n+1)} dv_g \right)^2 \\
&\leqslant \frac{1}{V(\Sigma)^2} \left(\int_M \|\tau\|^{2n} dv_g \right) \left(\int_M \|\tau\|^2 dv_g \right)
\end{aligned}$$

by the Cauchy-Schwarz inequality. From this, we deduce immediately that

$$(9) \qquad \|\tau\|_{n+1}^{2(n+1)} \leqslant \frac{1}{V(\Sigma)} \|B\|_{\infty}^{2n} \left(\int_M \|\tau\|^2 dv_g \right).$$

Now, using the assumptions that M is starshaped and has almost constant r-th mean curvature, we estimate $\int_M \|\tau\|^2 dv_g$. First, we have the following lemma which bound $\|\tau\|$ from above by $HH_r - H_{r+1}$.

Lemma 3.3. *There exists a constant positive constant $K_1 = K_1(n, r, \min(H_{r+1;n,1}), h, \|B\|_{\infty})$ so that*

$$\|\tau\|^2 \leqslant K_1 \big(HH_r - H_{r+1} \big).$$

Remark 3.4. *If $r = 1$, then $\|\tau\|^2 = n(n-1)(H^2 - H_2)$ so that, in this case, K_1 is just a dimensional constant.*

Proof: First, as mention in the preliminaries section, we have the following inequalities, for any $k \in \{1, \cdots, n-1\}$,

$$H_k^2 - H_{k+1}H_{k-1} \geqslant 0.$$

Moreover, we have a more precise estimate of the positivity of this term. Namely,

$$(10) \qquad H_k^2 - H_{k+1}H_{k-1} \geqslant c_n \|\tau\|^2 H_{k+1;n,1}^2$$

where c_n is a constant depending only on n and where

$$(11) \qquad H_{l;i,j} = \frac{\partial H_l}{\partial \kappa_i \partial \kappa_j} = \frac{1}{\binom{n}{l}} \sum_{\substack{1 \leqslant i_1 < \cdots < i_{l-2} \leqslant n \\ i_1, \cdots, i_l \neq i, j}} \kappa_{i_1} \cdot \cdots \cdot \kappa_{i_{l-2}}.$$

One can find the proof in [20] for instance. Since we assume that $H_{r+1} > 0$, then all the functions H_k are also positive for $k \in \{1, \cdots, n-1\}$. Thus, dividing by $H_k H_{k-1}$, (10) becomes

$$(12) \qquad \frac{H_k}{H_{k-1}} - \frac{H_{k+1}}{H_k} \geqslant c_n \|\tau\|^2 \frac{H_{k+1;n,1}^2}{H_k H_{k-1}}.$$

Thus, by summing equation (12) for k from 1 to r, we get

$$(13) \qquad H - \frac{H_{r+1}}{H_r} \geqslant c_n \|\tau\|^2 \sum_{k=1}^{r} \frac{H_{k+1;n,1}^2}{H_k H_{k-1}},$$

and so

$$(14) \qquad HH_r - H_{r+1} \geqslant c_n \|\tau\|^2 \left(\sum_{k=1}^{r} \frac{H_{k+1;n,1}^2}{H_k H_{k-1}} \right) H_r.$$

Moreover, we have $H_k H_{k-1} \leqslant \|B\|_\infty^{2k-1}$. In addition, since H_{r+1} is positive, then all the function H_k are also positive and thus, as proved by Scheuer in [20], the functions $H_{k;n,1}$ are also positive. In addition, since they are the normalized symmetric polynomial evaluated for $\kappa_2, \cdots, \kappa_{n-1}$, they also satisfy the Maclaurin inequality, up to a normalization constant, that

$$(H_{k;n,1})^{\frac{1}{k-2}} \geqslant a_{n,k} (H_{k+1;n,1})^{\frac{1}{k-1}},$$

where $a_{n,k}$ is a positive constant depending only on n and k, and so

$$(H_{k;n,1})^{\frac{1}{k-2}} \geqslant b_{n,k,r} (H_{r+1;n,1})^{\frac{1}{r-1}},$$

where $b_{n,k,r}$ is a positive constant depending only on n, k and r. Note that the exponents come from the fact that $H_{k;n,1}$ is the symmetric polynomial of degree $k-2$. Thus (14) gives

$$(15) \qquad HH_r - H_{r+1} \geqslant c_n \|\tau\|^2 \left(\sum_{k=1}^{r} \frac{b_{n,k+1,r}^{2(k-1)} H_{r+1;n,1}^{\frac{2(k-1)}{r-1}}}{\|B\|_\infty^{2k-1}} \right) H_r \geqslant K_1 \|\tau\|^2,$$

where $K_1 = c_n \min_{1 \leqslant k \leqslant r} \left(b_{n,k+1,r}^{2(k-1)} \right) \frac{h}{2\|B\|_\infty} \sum_{k=1}^{r} \left(\frac{\min_M (H_{r+1;n,1})^{\frac{1}{r-1}}}{\|B\|_\infty} \right)^{2(k-1)}$.

This concludes the proof of the lemma since K_1 depends only on n, r, $\min(H_{r+1;n,1})$, h and $\|B\|_\infty$. We have used here that $\min_M (H_r) \geqslant \frac{h}{2}$ from the assumption $H_r = h + \varepsilon$ with $\|\varepsilon\|_\infty \leqslant \frac{h}{2}$. $\qquad \square$

Remark 3.5. *Note that at the end of the proof, one can remove the assumption $\|\varepsilon\|_\infty \leqslant \dfrac{h}{2}$ and, since $H_r^{\frac{1}{r}} \geqslant H_{r+1}^{\frac{1}{r+1}}$, replace the dependence on h by a dependence on $\min_M(H_{r+1})$. In this case, the constant K_1 is replaced by the constant K_1' given by*

$$K_1' = c_n \min_{1 \leqslant k \leqslant r} \left(b_{n,k+1,r}^{2(k-1)} \right) \frac{\left(\min_M (H_{r+1}) \right)^{\frac{r}{r+1}}}{\|B\|_\infty} \sum_{k=1}^{r} \left(\frac{\min_M (H_{r+1;n,1})^{\frac{1}{r-1}}}{\|B\|_\infty} \right)^{2(k-1)}.$$

The conclusion of this lemma is obtained independentely of any condition of starshapedness or on H_r. Now, we will use the assumptions of starshapedness and almost constant r-th mean curvtaure to estimate the term $HH_r - H_{r+1}$. The key point here is the use of the Hsiung-Minkowski formulas. Form the assumption that $\langle Z, \nu \rangle$ have fixed sign and the definition of $R_0 = \min_M (|\langle Z, \nu \rangle|)$, we have

$$(16) \qquad \int_M \|\tau\|^2 dv_g \ \leqslant \ \frac{1}{R_0} \left| \int_M \|\tau\|^2 \langle Z, \nu \rangle dv_g \right|.$$

Now, using Lemma 3.3, we get

$$(17) \qquad \int_M \|\tau\|^2 dv_g \ \leqslant \ \frac{K_1}{R_0} \left| \int_M (HH_r - H_{r+1}) \langle Z, \nu \rangle dv_g \right|.$$

Note that we used again the fact that $\langle Z, \nu \rangle$ has constant sign to obtain this last inequality.

Now, using the assumption that $H_r = h + \varepsilon$, we estimate

$$\int_M (HH_r - H_{r+1}) \langle Z, \nu \rangle dv_g.$$

Namely, we have

$$\begin{aligned}
\int_M (HH_r - H_{r+1}) \langle Z, \nu \rangle dv_g &= \int_M (H(h+\varepsilon) - H_{r+1}) \langle Z, \nu \rangle dv_g \\
&= h \int_M H \langle Z, \nu \rangle dv_g + \int_M H\varepsilon \langle Z, \nu \rangle dv_g - \int_M H_{r+1} \langle Z, \nu \rangle dv_g \\
(18) \qquad &= -\int_M hc_\delta(\rho) dv_g + \int_M H\varepsilon \langle Z, \nu \rangle dv_g + \int_M H_r c_\delta(\rho) dv_g,
\end{aligned}$$

where we have used Hsiung-Minkowski formulas for the first and third terms of the right-hand side. Using again the assumption $H_r = h + \varepsilon$, we obtain

$$\begin{aligned}
\int_M (HH_r - H_{r+1}) \langle Z, \nu \rangle dv_g &= -\int_M (H_r - \varepsilon)c_\delta(\rho) dv_g + \int_M H\varepsilon \langle Z, \nu \rangle dv_g + \int_M H_r c_\delta(\rho) dv_g \\
(19) \qquad &= \int_M \varepsilon c_\delta(\rho) dv_g + \int_M H\varepsilon \langle Z, \nu \rangle dv_g.
\end{aligned}$$

Reporting this into (17), we get

$$\begin{aligned}
\int_M \|\tau\|^2 dv_g \ &\leqslant \ \frac{K_1}{R_0} \left| \int_M \varepsilon c_\delta(\rho) dv_g + \int_M H\varepsilon \langle Z, \nu \rangle dv_g \right| \\
(20) \qquad &\leqslant \ \frac{K_1}{R_0} \left(\max_M(c_\delta(\rho)) + \|B\|_\infty \max_M(s_\delta(\rho)) \right) \int_M |\varepsilon| dv_g.
\end{aligned}$$

Now, we set

$$K_2 = \begin{cases} \dfrac{K_1}{R_0} \left(1 + \dfrac{\|B\|_\infty}{\sqrt{\delta}} \right) & \text{if } \delta > 0 \\[3mm] \dfrac{K_1}{R_0} \left(1 + \|B\|_\infty R \right) & \text{if } \delta = 0, \\[3mm] \dfrac{K_1}{R_0} \left(c_\delta(R) + \|B\|_\infty s_\delta(R) \right) & \text{if } \delta < 0, \end{cases}$$

where R is the radius of a ball $B(p, R)$ containing M. Thus, we have

$$(21) \qquad \int_M \|\tau\|^2 dv_g \;\; \leqslant \;\; K_2 \int_M |\varepsilon| dv_g,$$

with K_2 depending on n, r, δ, h, $\min\limits_M (H_{r+1;n,1})$, $\|B\|_\infty$, R_0 and R.

In order to apply Theorem 3.1, we need to compare the L^{n+1}-norms of τ and the mean curvature $\widetilde{H}$ of M viewed as a hypersurface of the Euclidean space after the conformal change of metric $h = e^{2\varphi}\widetilde{h}$.

As a first step to prove this, we have the following lemma:

Lemma 3.6. *There exists a constant depending only on n and φ so that*

$$1 \leqslant c_{n,\varphi}^2 V(M)^{\frac{2(n+1)}{n}} \|\widetilde{H}\|_{n+1}^{2(n+1)}.$$

Proof: For this, we first recall the extrinsic Sobolev inequality of Michael and Simon. If (Σ, g_0) is a closed connected and oriented hypersurface of the Euclidean space, for any $\mathcal{C}^1$ function f on M, the following inequality holds

$$(22) \qquad \left(\int_\Sigma f^{\frac{n}{n-1}} dv_g \right)^{\frac{n-1}{n}} \leqslant K(n) \int_\Sigma \left(|\nabla f| + |Hf| \right) dv_{g_0},$$

where $K(n)$ is a constant that depends only on n. Applying this inequality for the function $f \equiv 1$, we get

$$(23) \qquad V(\Sigma)^{\frac{n-1}{n}} \leqslant K(n) \int_\Sigma |H| dv_{g_0}.$$

Now, since $\mathbb{M}^{n+1}(\delta)$ is conformally flat, we have $\mathbb{M}^{n+1}(\delta)$ can be viewed as a Euclidean domain D endowed with the metric $h = e^{2\varphi}\widetilde{h}$ where $\widetilde{h}$ is the Euclidean metric and $\varphi \in C^\infty(D)$. Applying (23) for $(\Sigma, g_0) = (M, \widetilde{g})$, we have

$$(24) \qquad \widetilde{V}(M)^{\frac{n-1}{n}} \leqslant K(n) \int_M |\widetilde{H}| dv_{\widetilde{g}},$$

where $\widetilde{V}(M)$ is the volume of $(M, \widetilde{g})$ and $\widetilde{H}$ is the mean curvature of the isometric immersion $(M, \widetilde{g}) \hookrightarrow (D, \widetilde{h})$. Thus, we deduce immediately that

$$(25) \qquad V(M)^{\frac{n-1}{n}} \leqslant c_{n,\varphi} \int_M |\widetilde{H}| dv_{\widetilde{g}},$$

where $c_{n,\varphi}$ is a constant depending on n and φ. Note that here, $V(M)$ is the volume of M with the metric g which explain the dependence of the constant $c_{n,\varphi}$ on the conformal factor φ. Thus, we deduce immediately that

$$(26) \qquad V(M)^{-\frac{1}{n}} \leqslant c_{n,\varphi} \|\widetilde{H}\|_1$$

and so

$$(27) \qquad V(M)^{-\frac{n+1}{n}} \leqslant c_{n,\varphi} \|\widetilde{H}\|_{n+1}^{n+1}.$$

Finally we deduce immediately that

$$(28) \qquad 1 \leqslant c_{n,\varphi}^2 V(M)^{\frac{2(n+1)}{n}} \|\widetilde{H}\|_{n+1}^{2(n+1)}.$$

$$\square$$

Now inequality (9) together with (21) and Lemma 3.6 gives

$$
\begin{aligned}
\|\tau\|_{n+1}^{2(n+1)} &\leqslant \|B\|_{\infty}^{2n}\|\tau\|_2^2 \\
&\leqslant \|B\|_{\infty}^{2n}K_2\|\varepsilon\|_1 \\
&\leqslant K_2 c_{n,\varphi}^2 V(\Sigma)^{\frac{2n+2}{n}}\|\widetilde{H}\|_{n+1}^{2(n+1)}\|\varepsilon\|_1 \\
(29) \qquad &= K_3\|\widetilde{H}\|_{n+1}^{2(n+1)}\|\varepsilon\|_1,
\end{aligned}
$$

where K_3 is a constant depending on n, r, δ, h, $\min\limits_{M}(H_{r+1;n,1})$, $\|B\|_{\infty}$, $V(\Sigma)$, R_0 and R. Note that K_3 depends also on $\|\varphi\|_{\infty,\Omega}$ due to (25), but since φ is the conformal change of metric between $\mathbb{R}^{n+1}$ and $\mathbb{H}^{n+1}$ or $\mathbb{S}_+^{n+1}$, this dependence can be replaced by a dependence on δ and R.

Now, if $\|\varepsilon\|_1$ is supposed to be smaller than $\varepsilon_1 = \dfrac{\varepsilon_0^{2(n+1)}}{K_3}$, where ε_0 is the constant of Theorem 3.1, then we have

$$\|\tau\|_{n+1} \leqslant \|\widetilde{H}\|_{n+1}\varepsilon_0,$$

so that we can apply Theorem 3.1. Note that ε_1 is a positive constant depending on n, r, δ, h, $\min\limits_{\Sigma}(H_{r+1;n,1})$, $\|B\|_{\infty}$, $V(\Sigma)$, R_0 and R. Thus, there exists $\rho_0 > 0$ so that

$$(30) \qquad d_H(\Sigma, S_{\rho_0}) \leqslant \frac{c\rho_0}{\|\widetilde{H}\|_{n+1}^{\alpha}}\|\tau\|_{n+1}^{\alpha}.$$

Using (29) once again, we get

$$(31) \qquad d_H(\Sigma, S_{\rho_0}) \leqslant c\rho_0 K_3^{\frac{\alpha}{2(n+1)}}\|\varepsilon\|_1^{\frac{\alpha}{2(n+1)}} = C\|\varepsilon\|_1^{\gamma},$$

where $C = c\rho_0 K_3^{\frac{\alpha}{2(n+1)}}$ is a positive constant depending on n, r, δ, h, $\min\limits_{M}(H_{r+1;n,1})$, $\|B\|_{\infty}$, $V(\Sigma)$, R_0 and R and γ is a positive constant depending only on n. This concludes the proof of Theorem 1.1. $\square$

Remark 3.7. *Here again, the dependence on h can be replaced by a dependence on $\min\limits_{M}(H_{r+1})$ if we use Lemma 3.3 with the constant K_1' given instead of K_1, where K_1' is the constant given in Remark 3.5.*

Remark 3.8. *One can also obtain an anisotropic version of Theorem 1.1 as it is done for almost Weingarten hypersurfaces in [18]. This generalizes for higher order anisotropic mean curvatures the result obtain in [17]. The proof is similar and the conclusion is obtained by using the result of de Rosa and Gioffrè for nearly umbilical anisotropic hypersurfaces [3]. For a sake of briefness, we do not state this immediate adaptation here.*

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