



HAL
open science

A topological tree of shapes

Nicolas Passat, Yukiko Kenmochi

► **To cite this version:**

Nicolas Passat, Yukiko Kenmochi. A topological tree of shapes. International Conference on Discrete Geometry and Mathematical Morphology (DGMM), Oct 2022, Strasbourg, France. pp.221-235, 10.1007/978-3-031-19897-7_18 . hal-03707130v2

HAL Id: hal-03707130

<https://hal.science/hal-03707130v2>

Submitted on 11 Oct 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A Topological Tree of Shapes

Nicolas Passat^{[0000-0002-0320-4581]1} and Yukiko Kenmochi^{[0000-0001-9648-326X]2}

¹ Université de Reims Champagne Ardenne, CReSTIC EA 3804, 51100 Reims, France

² Normandie Univ, UNICAEN, ENSICAEN, CNRS, GREYC, F-14050 Caen, France

Abstract. In this article, we enrich the framework of morphological hierarchies with new acyclic graphs and trees. These structures lie at the convergence of hierarchical models and topological descriptors. We define them in the context of digital grey-level imaging. We discuss their links with component-trees, trees of shapes and adjacency trees. This analysis leads to new notions, including a notion of *topological tree of shapes*.

1 Introduction

Many hierarchical, graph-based structures have been defined in the framework of mathematical morphology, especially for designing connected operators [25]. The most popular are trees (i.e. rooted, connected, acyclic graphs). They model finite sets of partitions organized with respect to the refinement order relation. These partitions can be partial. This is the case of the component-tree and its variants [24, 9], the level-line tree (a.k.a. tree of shapes) and its variants [11, 3]. These partitions can also be total. This is the case of the binary partition tree and its variants [23, 27, 19] and the hierarchical watershed [13, 26]. Other hierarchical structures are directed acyclic graphs (DAGs), e.g. the component-hypertree [15], the component-graph [17], the braid of partitions [8] and the directed component hierarchy [18].

The partitions modeled by these hierarchical structures are composed of connected sets defined with respect to a topology defined on a given space which is generally discrete (e.g. a part of \mathbb{Z}^n [22], a complex on / tessellation of \mathbb{R}^n). Hierarchical structures carry intrinsic, topological information. However, these information are often limited and generally not sufficient to perform high-level topological analysis of the modeled images / data. In particular, hierarchical structures are generally less informative than high-level topological invariants / descriptors, e.g. the homology groups / homology persistence [6] or the homotopy type.

In this article, we introduce a new family of hierarchical structures—DAGs and trees, including a new notion of *topological tree of shapes*—dedicated to the modeling of grey-level images. They aim to gather (i) connectedness / intensity information carried by component- (min- and max-) trees [24] and (ii) topological information carried by the adjacency tree, a classical topological invariant [21]. Basically, we will first build a DAG that is composed by the min-tree and max-tree of a grey-level image, and we will enrich the nodes of these two trees by the adjacency tree structure at each grey-level, leading to the notion of a graph of valued shapes. Then, we will establish that this graph of valued shape can be simplified in a lossless fashion as a tree structure by discarding some transitive, redundant edges. This will lead to a simpler tree structure called tree

of valued shapes. By factorizing some spatially equivalent nodes, we will then define a more compact structure, called the complete tree of shapes. We will establish that this complete tree of shapes can be reduced (in a lossy fashion) in two different ways, leading on the one hand to the usual notion of a tree of shapes and on the other hand to the new notion of a topological tree of shapes. (The chosen terminology of *topological trees of shapes* is justified by the way it is defined; however it will be shown to be different from another homonymous notion previously introduced in the literature.) We will finally evoke the links between these new structures (graph and tree of valued shapes, complete tree of shapes, topological tree of shapes) and usual morphological trees (component-tree, tree of shapes, adjacency tree).

This article is organized as follows. Section 2 provides definitions related to hierarchies and grey-level images. We introduce the notions of graph of valued shapes and tree of valued shapes in Sections 3 and 4, respectively. Section 5 derives from the tree of valued shapes the two essential notions of this work, namely the complete tree of shapes (that generalizes the tree of shapes) and the topological tree of shapes. In Section 6, we discuss on the links that exist between these new notions and well-known morphological hierarchies. We provide concluding remarks in Section 7.

2 Basics: Hierarchies and Images

Definition 1 (Hierarchical order) *Let X be a set and \leq be an order on X . We say that \leq is a hierarchical order if $\forall x \in X$ the subset $x^\uparrow = \{y \in X \mid x < y\}$ is totally ordered.*

Definition 2 (Hierarchical function) *Let X be a set and \leq a hierarchical order on X . The hierarchical function $\zeta_\leq : X \rightarrow X$ is defined by $\zeta_\leq(x) = \bigwedge_{\leq} x^\uparrow$. This function is defined everywhere on X except for the greatest elements of (X, \leq) .*

Remark 3 *Let \triangleleft be the Hasse relation obtained from \leq by reflexive-transitive reduction. $\forall x \in X$ we have*

$$x \triangleleft \zeta_\leq(x) \quad (1)$$

This formula induces an isomorphism between (X, \triangleleft) and (X, ζ_\leq) .

Definition 4 (Tree) *Let X be a set and \leq a hierarchical order on X such that (X, \leq) admits a maximum. The Hasse diagram (X, \triangleleft) is called a tree.*

Let \mathbb{U} be a discrete set endowed with a topological structure which provides the notions of adjacency and connectedness, and where the separation theorem (Jordan-Brouwer) holds.

Remark 5 *In this article we choose $\mathbb{U} = \mathbb{Z}^n$ ($n \geq 2$) and we consider the usual framework of digital topology on binary images [22], with the standard couples of $(2n, 3^n - 1)$ and $(3^n - 1, 2n)$ -adjacencies for the foreground and background.*

Let \mathbb{K} be a set of values endowed with a total order $\leq_{\mathbb{K}}$. Let $\mathcal{F} : \mathbb{U} \rightarrow \mathbb{K}$ be an application. We assume that there exist a finite, nonempty subset $\mathbb{S} \subset \mathbb{U}$ and two values $\perp <_{\mathbb{K}} \top \in \mathbb{K}$ such that for all $\mathbf{x} \in \mathbb{U}$

$$\begin{cases} \mathcal{F}(\mathbf{x}) = \perp & \text{if } \mathbf{x} \notin \mathbb{S} \\ \perp <_{\mathbb{K}} \mathcal{F}(\mathbf{x}) <_{\mathbb{K}} \top & \text{if } \mathbf{x} \in \mathbb{S} \end{cases} \quad (2)$$

We set $\mathbb{V} = \mathcal{F}(\mathbb{S}) \cup \{\perp, \top\}$. It is a finite set that we equip with the total order $\leq_{\mathbb{V}}$ induced by $\leq_{\mathbb{K}}$.

Remark 6 *The application \mathcal{F} is isomorphic to a grey-level image taking its values in an interval of \mathbb{Z} of size $|\mathbb{V}|$, e.g. $\llbracket 0, |\mathbb{V}| - 1 \rrbracket$. See Figure 1(a).*

3 Graph of Valued Shapes

3.1 (Valued) connected components

We set $\leq^{\circ} = \leq_{\mathbb{V}}$ and $\leq^{\bullet} = \geq_{\mathbb{V}}$. Let $v \in \mathbb{V}$. We define the threshold sets of \mathcal{F} at value $v \in \mathbb{V}$ (see Figure 1(b-i)) as

$$\begin{aligned} \Lambda_v^{\circ}(\mathcal{F}) &= \{\mathbf{x} \in \mathbb{U} \mid v \leq^{\circ} \mathcal{F}(\mathbf{x})\} \\ \Lambda_v^{\bullet}(\mathcal{F}) &= \{\mathbf{x} \in \mathbb{U} \mid v <^{\bullet} \mathcal{F}(\mathbf{x})\} \end{aligned} \quad (3)$$

Let $X \subseteq \mathbb{U}$. When X is nonempty, we note $\Pi[X] \subset 2^{\mathbb{U}}$ the partition gathering all the connected components of X . If X is empty, we set $\Pi[X] = \emptyset$.

Let $v \in \mathbb{V}$. We set

$$\begin{aligned} \mathcal{E}_v^{\circ} &= \Theta_v^{\circ} \times \{v\} & \Theta_v^{\circ} &= \Pi[\Lambda_v^{\circ}(\mathcal{F})] \\ \mathcal{E}_v^{\bullet} &= \Theta_v^{\bullet} \times \{v\} & \Theta_v^{\bullet} &= \Pi[\Lambda_v^{\bullet}(\mathcal{F})] \\ \mathcal{E}_v &= \Theta_v \times \{v\} & \Theta_v &= \Theta_v^{\circ} \cup \Theta_v^{\bullet} \end{aligned} \quad (4)$$

Remark 7 *We have $\mathcal{E}_{\top}^{\circ} = \mathcal{E}_{\perp}^{\bullet} = \emptyset$. For any $v \in \mathbb{V} \setminus \{\perp, \top\}$, we have $\mathcal{E}_v^{\circ} \neq \emptyset$ and $\mathcal{E}_v^{\bullet} \neq \emptyset$. In the sequel, (\mathbb{U}, \perp) and (\mathbb{U}, \top) are considered as a unique element noted ∞ . Then, we have $\mathcal{E}_{\perp}^{\circ} = \{(\mathbb{U}, \perp)\} = \{\infty\} = \{(\mathbb{U}, \top)\} = \mathcal{E}_{\top}^{\bullet}$ and $\mathcal{E}_{\perp}^{\bullet} = \mathcal{E}_{\top}^{\circ} = \{\infty\}$.*

We set

$$\begin{aligned} \mathcal{E}^{\circ} &= \bigcup_{v \in \mathbb{V}} \mathcal{E}_v^{\circ} & \Theta^{\circ} &= \bigcup_{v \in \mathbb{V}} \Theta_v^{\circ} \\ \mathcal{E}^{\bullet} &= \bigcup_{v \in \mathbb{V}} \mathcal{E}_v^{\bullet} & \Theta^{\bullet} &= \bigcup_{v \in \mathbb{V}} \Theta_v^{\bullet} \\ \mathcal{E} &= \mathcal{E}^{\circ} \cup \mathcal{E}^{\bullet} = \bigcup_{v \in \mathbb{V}} \mathcal{E}_v & \Theta &= \Theta^{\circ} \cup \Theta^{\bullet} = \bigcup_{v \in \mathbb{V}} \Theta_v \end{aligned} \quad (5)$$

3.2 Orders on valued connected components

We define the partial orders \sqsubseteq° on \mathcal{E}° and \sqsubseteq^{\bullet} on \mathcal{E}^{\bullet} as

$$\begin{aligned} ((X, v) \sqsubseteq^{\circ} (Y, w)) &\Leftrightarrow (X \subseteq Y \wedge w \leq^{\circ} v) \\ ((X, v) \sqsubseteq^{\bullet} (Y, w)) &\Leftrightarrow (X \subseteq Y \wedge w \leq^{\bullet} v) \end{aligned} \quad (6)$$

We define the order \sqsubseteq^{φ} as the union of \sqsubseteq° and \sqsubseteq^{\bullet} , i.e. $P \sqsubseteq^{\varphi} Q$ iff $P \sqsubseteq^{\circ} Q$ or $P \sqsubseteq^{\bullet} Q$.

Remark 8 *\sqsubseteq^{φ} , \sqsubseteq° and \sqsubseteq^{\bullet} are hierarchical orders. They admit ∞ as maximum.*

Let $v \in \mathbb{V}$. We define the order \sqsubseteq^v on \mathcal{E}_v as

$$((X, v) \sqsubseteq^v (Y, v)) \Leftrightarrow \tau(X) \subseteq \tau(Y) \quad (7)$$

where $\tau : 2^{\mathbb{U}} \rightarrow 2^{\mathbb{U}}$ is the hole closing application defined by $\tau(X) = X \cup \bigcup Z$ where $Z \subseteq \Pi[\bar{X}]$ is composed by the *finite* connected components of $\bar{X} = \mathbb{U} \setminus X$.

We define the order \sqsubseteq^{ψ} on \mathcal{E} as $\sqsubseteq^{\psi} = \bigcup_{v \in \mathbb{V}} \sqsubseteq^v$, i.e. $P \sqsubseteq^{\psi} Q$ iff $\exists v \in \mathbb{V}$ such that $P \sqsubseteq^v Q$.

Remark 9 \sqsubseteq^ψ , and \sqsubseteq^v ($v \in \mathbb{V}$) are hierarchical orders. Each ordered set $(\mathcal{E}_v, \sqsubseteq^v)$ ($v \in \mathbb{V}$) admits a maximum (\mathbb{U}_v, v) where $\mathbb{U}_v \subseteq \mathbb{U}$ is the unique element of Θ_v which is infinite.

We note \triangleleft^φ (resp. \triangleleft° , \triangleleft^\bullet) and \triangleleft^ψ (resp. \triangleleft^v) the Hasse relations associated to \sqsubseteq^φ (resp. \sqsubseteq° , \sqsubseteq^\bullet) and \sqsubseteq^ψ (resp. \sqsubseteq^v). The graph $(\mathcal{E}, \triangleleft^\varphi)$ is “similar” to the union of the max- and min-trees, whereas $(\mathcal{E}, \triangleleft^\psi)$ is “similar” to the union of the adjacency trees of each threshold set of \mathcal{F} . This will be more formally discussed in Section 6.

Remark 10 Let $P = (X, v), Q = (Y, w) \in \mathcal{E}$ such that $P \triangleleft^\varphi Q$. We have $P, Q \in \mathcal{E}^\star$ with $\star = \text{either } \circ \text{ or } \bullet$. In addition, we have $X \subseteq Y$ and $v = \zeta_{\leq \star}(w)$.

Remark 11 Let $P = (X, v), Q = (Y, w) \in \mathcal{E}$ such that $P \triangleleft^\psi Q$. We have $(P \in \mathcal{E}^\bullet \text{ and } Q \in \mathcal{E}^\circ)$ or $(P \in \mathcal{E}^\circ \text{ and } Q \in \mathcal{E}^\bullet)$. In addition, we have $v = w$ and $\tau(X) \in \Pi[\bar{Y}]$.

We note $\varphi = \zeta_{\sqsubseteq^\varphi}$, $\varphi^\circ = \zeta_{\sqsubseteq^\circ}$, $\varphi^\bullet = \zeta_{\sqsubseteq^\bullet}$, $\psi = \zeta_{\sqsubseteq^\psi}$ and $\psi^v = \zeta_{\sqsubseteq^v}$ ($v \in \mathbb{V}$).

3.3 Definition of the graph of valued shapes

Let $\triangleleft_{\mathcal{E}}$ be the relation defined as the union of \triangleleft^φ and \triangleleft^ψ .

Definition 12 (Graph of valued shapes) The graph of valued shapes (or VS-graph, for brief) is the couple $\mathfrak{G}_{\text{VS}} = (\mathcal{E}, \triangleleft_{\mathcal{E}})$.

Remark 13 The intersection between \triangleleft^φ and \triangleleft^ψ is empty. We can then consider \mathfrak{G}_{VS} as $(\mathcal{E}, \triangleleft_{\mathcal{E}})$ or as $(\mathcal{E}, \triangleleft^\varphi, \triangleleft^\psi)$ and equivalently as $(\mathcal{E}, \varphi, \psi)$.

Property 14 $\mathfrak{G}_{\text{VS}} = (\mathcal{E}, \triangleleft_{\mathcal{E}})$ is a directed acyclic graph.

We define $\sqsubseteq_{\mathcal{E}}$ as the reflexive-transitive closure of $\triangleleft_{\mathcal{E}}$.

Remark 15 $(\mathcal{E}, \sqsubseteq_{\mathcal{E}})$ is an ordered set that admits ∞ as maximum.

4 Tree of Valued Shapes

4.1 Transitive reduction of the graph of valued shapes

Let $\blacktriangleleft_{\mathcal{E}}$ be the relation on \mathcal{E} defined as the transitive reduction of $\triangleleft_{\mathcal{E}}$.

Let $P \in \mathcal{E}$. Let us consider the following three equalities

$$\psi(P) = [\varphi \circ \psi \circ \varphi](P) \quad (8)$$

$$\varphi(P) = [\varphi \circ \psi \circ \psi](P) \quad (9)$$

$$\varphi(P) = [\varphi^{|\mathbb{V}|-2} \circ \psi](P) \quad (10)$$

Remark 16 If P satisfies Eq. (8), then we have $P \blacktriangleleft_{\mathcal{E}} \psi(P)$ and $P \blacktriangleleft_{\mathcal{E}} \psi(P)$. If P satisfies Eq. (9) or (10), then we have $P \blacktriangleleft_{\mathcal{E}} \varphi(P)$ and $P \blacktriangleleft_{\mathcal{E}} \varphi(P)$.

Proposition 17 Let $P \in \mathcal{E}$ be such that $\varphi(P)$ and $\psi(P)$ exist. One of Eqs. (8–10) is satisfied.

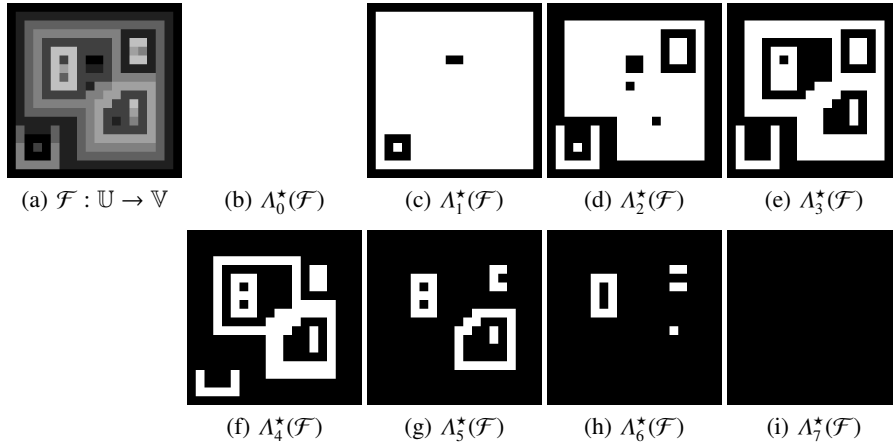


Fig. 1. (a) A grey-level image $\mathcal{F} : \mathbb{U} \rightarrow \mathbb{V}$ ($\mathbb{U} = \mathbb{Z}^2$ and $\mathbb{V} = \llbracket 0, 7 \rrbracket$). (b–i) The threshold sets $\Lambda_v^*(\mathcal{F})$ ($\Lambda_v^\circ(\mathcal{F})$ in white; $\Lambda_v^*(\mathcal{F})$ in black), for $v = 0$ (b) to 7 (i).

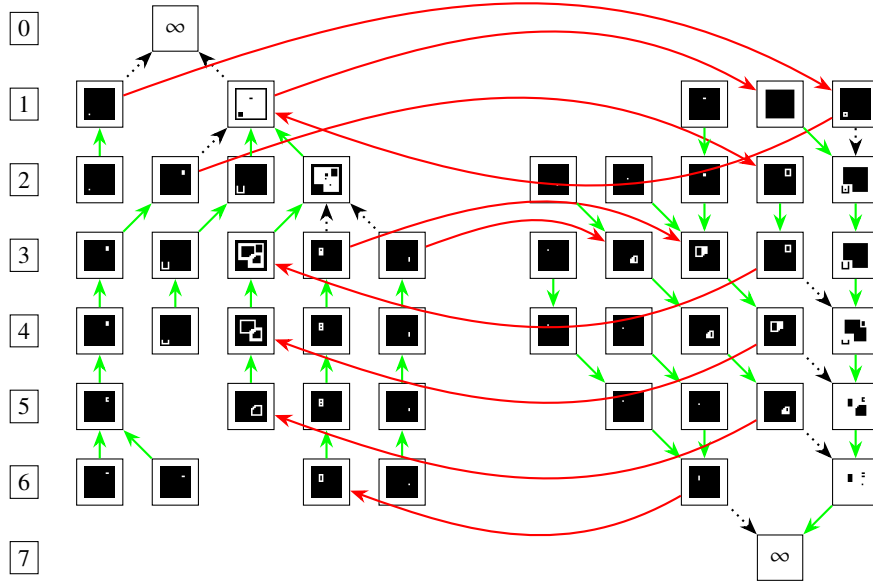


Fig. 2. Tree of valued shapes of the image \mathcal{F} (Figure 1(a)). The valued connected components are depicted by squares (\mathcal{E}° on the left side; \mathcal{E}^* on the right side) and are positioned with respect to the threshold value v (see on left), from 0 (top) to 7 (bottom). Red and green arrows correspond to the $\triangleleft_{\mathbb{E}}$ relation. Green and black dotted arrows correspond to the $\triangleleft_{\mathbb{V}}$ relation. Red arrows are a subset of the $\triangleleft_{\mathbb{V}}$ relation, not fully depicted for the sake of readability.

Proof Let $P = (X, \nu) \in \mathcal{E}$ be such that $\varphi(P)$ and $\psi(P)$ exist. Case 1: $\varphi(P) = (\mathbb{U}, \perp) = \infty$ and $\psi(P) = (\mathbb{U}_\nu, \nu)$ (the unique element of $II[\Lambda_\nu^*(\mathcal{F})]$ which is infinite). It is plain that $\varphi^{|\mathbb{V}|-2}((\mathbb{U}_\nu, \nu)) = (\mathbb{U}, \top) = \infty = \varphi(P)$, and Eq. (10) then holds. Case 2: $\varphi(P) = (\mathbb{U}, \perp)$ and $\psi(P) \neq (\mathbb{U}_\nu, \nu)$. It is plain that $\psi^2(P)$ exists and $\varphi(\psi^2(P)) = (\mathbb{U}, \perp) = \varphi(P)$, thus Eq. (9) holds. Case 3: $\varphi(P) = (\mathbb{U}, \top)$. Since $\psi(P)$ exists (and is finite), it is plain that $\psi^2(P)$ also exists. But then $\varphi(\psi^2(P)) = (\mathbb{U}, \top) = \varphi(P)$, thus Eq. (9) holds. Case 4: $\varphi(P) \neq \infty$. If $\psi(P) = (\mathbb{U}_\nu, \nu)$, it is plain that $\psi(\varphi(P)) = (\mathbb{U}_w, w)$ with $(\mathbb{U}_\nu, \nu) = \varphi((\mathbb{U}_w, w))$ and Eq. (8) then holds. Let us now suppose that $\psi(P) \neq (\mathbb{U}_\nu, \nu)$ and that Eq. (8) does not hold, i.e. $[\varphi \circ \psi \circ \varphi](P) \neq \psi(P)$. Then we have $P \sqsubset^\psi \psi^2(P) \sqsubset^\psi [\varphi \circ \psi \circ \varphi](P)$. Now, let us suppose that $\varphi(\psi^2(P)) \neq \varphi(P)$. Then we have $\varphi(P) \sqsubset^\psi \varphi(\psi^2(P))$ and it comes $\varphi(P) \sqsubset^\psi \psi(\varphi(P)) \sqsubset^\psi \varphi(\psi^2(P))$ in contradiction with the Jordan theorem. Thus Eq. (9) holds. ■

It follows from Proposition 17 that for any $P \in \mathcal{E}$ such that both $\psi(P)$ and $\varphi(P)$ exist we have $P \not\triangleleft_{\mathcal{E}} \psi(P)$ or $P \not\triangleleft_{\mathcal{E}} \varphi(P)$. Since $(\mathcal{E}, \sqsubseteq_{\mathcal{E}})$ admits a maximum (namely ∞), for each $P \in \mathcal{E}$, $P \neq \infty$, we have either $P \triangleleft_{\mathcal{E}} \psi(P)$ or $P \triangleleft_{\mathcal{E}} \varphi(P)$. The following property derives from these facts.

Property 18 *Let $P \in \mathcal{E}$.*

- If $\varphi(P)$ is defined and $\psi(P)$ is not, then $P \triangleleft_{\mathcal{E}} \varphi(P)$.
- If $\psi(P)$ is defined and $\varphi(P)$ is not, then $P \triangleleft_{\mathcal{E}} \psi(P)$.
- If $\varphi(P)$ and $\psi(P)$ are defined, then either $P \triangleleft_{\mathcal{E}} \varphi(P)$ or $P \triangleleft_{\mathcal{E}} \psi(P)$.

Remark 19 *The transitive reduction from $\mathfrak{G}_{\text{VS}} = (\mathcal{E}, \triangleleft_{\mathcal{E}})$ to $(\mathcal{E}, \triangleleft_{\mathcal{E}})$ is a lossless compression. The graph $\mathfrak{G}_{\text{VS}} = (\mathcal{E}, \triangleleft_{\mathcal{E}})$ can be reconstructed from $(\mathcal{E}, \triangleleft_{\mathcal{E}})$.*

4.2 Definition of the tree of valued shapes

Property 20 $(\mathcal{E}, \triangleleft_{\mathcal{E}})$ is a tree. Equivalently, $\sqsubseteq_{\mathcal{E}}$ is a hierarchical order on \mathcal{E} .

Definition 21 (Tree of valued shapes) *The tree of valued shapes (or VS-tree, for brief) is the couple $\mathfrak{T}_{\text{VS}} = (\mathcal{E}, \triangleleft_{\mathcal{E}})$. See Figure 2.*

5 Complete Tree of Shapes and Topological Tree of Shapes

5.1 Spatial compression: From the tree of valued shapes to the complete tree of shapes

Let $\pi_{\theta} : \mathcal{E} \rightarrow \theta$ be the function defined by $\pi_{\theta}((X, \nu)) = X$. Let \sim_{θ} be the equivalence relation on \mathcal{E} defined by

$$P \sim_{\theta} Q \Leftrightarrow \pi_{\theta}(P) = \pi_{\theta}(Q) \tag{11}$$

Property 22 *The function $\tilde{\pi}_{\theta} : \mathcal{E}/\sim_{\theta} \rightarrow \theta$ defined by $\tilde{\pi}_{\theta}([P]_{\sim_{\theta}}) = \pi_{\theta}(P)$ is a bijection.*

Remark 23 *Based on the above property, we identify $\mathcal{E}/\sim_{\theta}$ and θ . More precisely, for any $P = (X, \nu) \in \mathcal{E}$, we identify $[P]_{\sim_{\theta}}$ and X . In particular, we have $[\infty]_{\sim_{\theta}} = \{\infty\} \in \mathcal{E}/\sim_{\theta}$ and it is identified to $\mathbb{U} \in \theta$.*

Property 24 Let $K \in \mathcal{E}/\sim_\theta$. Let \sqsubseteq_K be the order induced by $\sqsubseteq_{\mathcal{E}}$ on K . Then (K, \sqsubseteq_K) is a totally ordered set.

For any $K \in \mathcal{E}/\sim_\theta$, we note $\langle K \rangle_\theta = \bigwedge_{\mathcal{E}} K$ and $\langle\langle K \rangle\rangle_\theta = \bigvee_{\mathcal{E}} K$. We note $\rho_{\mathcal{E}} = \zeta_{\mathcal{E}}$.

Remark 25 From Property 24, it follows that $\forall p \in \llbracket 1, |K| - 1 \rrbracket$ we have $\rho_{\mathcal{E}}^p(\langle K \rangle_\theta) \in K$. In particular, we have $\rho_{\mathcal{E}}^{|\mathcal{E}|-1}(\langle K \rangle_\theta) = \langle\langle K \rangle\rangle_\theta$ whereas $\rho_{\mathcal{E}}^{|\mathcal{E}|}(\langle K \rangle_\theta) \notin K$.

Proposition 26 Let $K \in \mathcal{E}/\sim_\theta$, $K \neq \{\infty\}$. Let $P = \rho_{\mathcal{E}}^{|\mathcal{E}|}(\langle K \rangle_\theta)$. We have $P = \langle [P]_{\sim_\theta} \rangle_\theta$.

Proof Let $P = \rho_{\mathcal{E}}^{|\mathcal{E}|}(\langle K \rangle_\theta) = \rho_{\mathcal{E}}(\langle\langle K \rangle\rangle_\theta)$, with $\langle\langle K \rangle\rangle_\theta = (X, v)$ and $P = (Y, w)$. In particular, we have $\langle\langle K \rangle\rangle_\theta \triangleleft_{\mathcal{E}} P$ and thus $\langle\langle K \rangle\rangle_\theta \triangleleft_{\mathcal{E}} P$. Let $Q = (Y, u) = \langle [P]_{\sim_\theta} \rangle_\theta$. Case 1: $\langle\langle K \rangle\rangle_\theta \triangleleft^\varphi P$. This implies $X \subseteq Y$ and $w \leq^\star v$ (with $\star =$ either \circ or \bullet). As $P \notin K$, we have $X \subset Y$, and it follows that $w <^\star v$. We have $w \leq^\star u$. If $u \neq w$, then we have $w <^\star u$ and it follows that $\langle\langle K \rangle\rangle_\theta \sqsubset^\varphi Q \sqsubset^\varphi P$, and thus $\langle\langle K \rangle\rangle_\theta \not\triangleleft^\psi P$: a contradiction. Then, we have $u = w$, and it follows that $P = \langle [P]_{\sim_\theta} \rangle_\theta$. Case 2: $\langle\langle K \rangle\rangle_\theta \triangleleft^\psi P$. We have $X \in$ either Θ_v° or Θ_v^\bullet (for instance, Θ_v° ; the same reasoning holds with Θ_v^\bullet) and $\varphi(\langle\langle K \rangle\rangle_\theta)$ exists. Let $R = (Z, t) = \varphi(\langle\langle K \rangle\rangle_\theta)$. We have $X \subseteq Z$, and since $R \notin K$, it comes $X \subset Z$. Let us suppose that $P \neq Q$. Then, we have $S = (Y, t) \in [P]_{\sim_\theta}$. It follows that $\tau(X) = \tau(Z)$. Consequently, we have $R \triangleleft^\psi S$. But then, we obtain $\langle\langle K \rangle\rangle_\theta \triangleleft^\varphi R \triangleleft^\psi S \triangleleft^\varphi P$, in contradiction with $\langle\langle K \rangle\rangle_\theta \triangleleft_{\mathcal{E}} P$. Then, we have $P = \langle [P]_{\sim_\theta} \rangle_\theta$. ■

For any $K \in \mathcal{E}/\sim_\theta$, we consider $\langle K \rangle_\theta \in \mathcal{E}$ as canonical element, and we identify $\langle K \rangle_\theta = (X, v)$ with $X \in \Theta$.

Let \sqsubseteq_θ be the order on $\{\langle K \rangle_\theta \mid K \in \mathcal{E}/\sim_\theta\} \subseteq \mathcal{E}$ —and equivalently on Θ —induced by $\sqsubseteq_{\mathcal{E}}$.

We note $\kappa_\theta = \tilde{\pi}_\theta^{-1}$. Let $\rho_\theta : \Theta \rightarrow \Theta$ be the function defined by

$$\rho_\theta(X) = \pi_\theta(\rho_{\mathcal{E}}^{|\mathcal{E}|}(\langle K \rangle_\theta)) \quad (12)$$

with $K = \kappa_\theta(X)$.

Remark 27 From the above property, we have $\rho_\theta = \zeta_{\mathcal{E}_\theta}$. We note \triangleleft_θ the relation on Θ associated to ρ_θ , namely the Hasse relation of \sqsubseteq_θ .

Definition 28 (Complete tree of shapes) The complete tree of shapes (or CS-tree, for brief) is the couple $\mathfrak{T}_{\text{CS}} = (\Theta, \triangleleft_\theta)$. See Figure 3 (left).

The following proposition directly derives from the above results.

Proposition 29 The equivalence relation \sim_θ induces a decreasing homeomorphism from $\mathfrak{T}_{\text{VS}} = (\mathcal{E}, \triangleleft_{\mathcal{E}})$ to $\mathfrak{T}_{\text{CS}} = (\Theta, \triangleleft_\theta)$.

Remark 30 The homeomorphism from $\mathfrak{T}_{\text{VS}} = (\mathcal{E}, \triangleleft_{\mathcal{E}})$ to $\mathfrak{T}_{\text{CS}} = (\Theta, \triangleleft_\theta)$ is a lossless compression. The tree $\mathfrak{T}_{\text{VS}} = (\mathcal{E}, \triangleleft_{\mathcal{E}})$ can be fully reconstructed from $\mathfrak{T}_{\text{CS}} = (\Theta, \triangleleft_\theta)$.

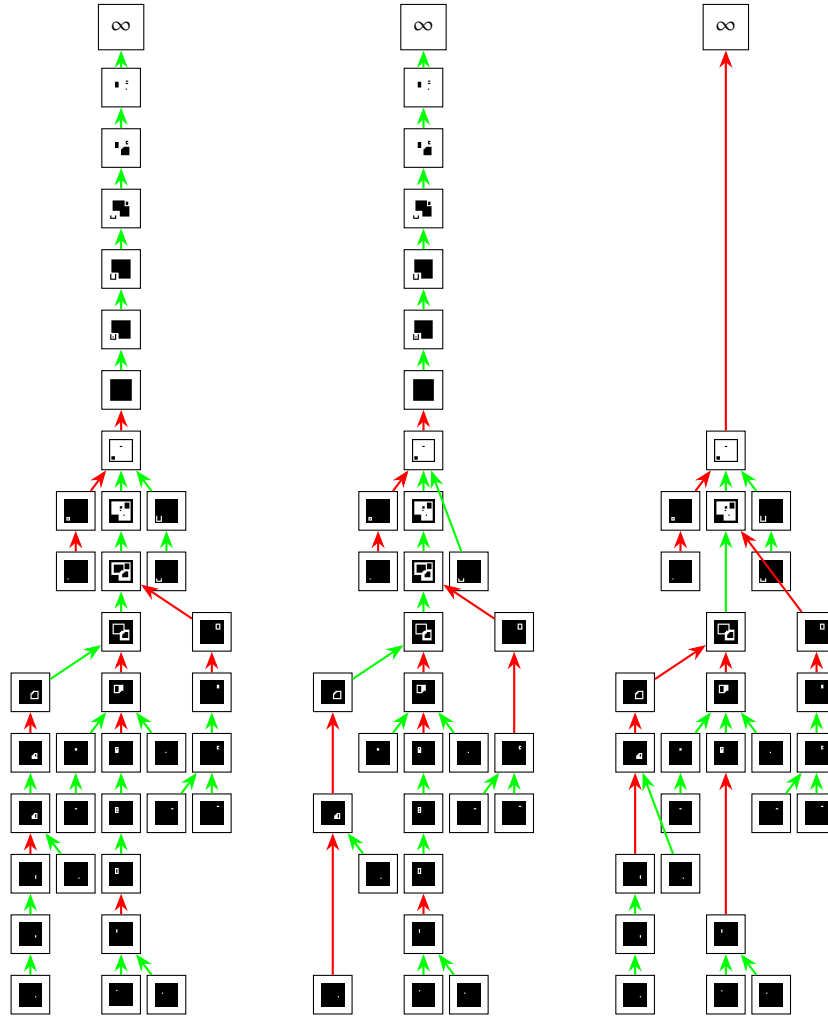


Fig. 3. From left to right: (1) the complete tree of shapes; (2) the topological tree of shapes; (3) the tree of shapes of the image \mathcal{F} of Figure 1. The complete tree of shapes (1) derives from the reduction of the tree of valued shapes of Figure 2. Green arrows are originated from the \prec^{φ} relation. Red arrows are originated from the \prec^{ψ} relation.

5.2 Topological compression: From the tree of valued shapes to the topological tree of shapes

Let $X, Y \in \mathbb{U}$, with $Y \subset X$. We aim to characterize the preservation of topological properties by a decreasing transformation from X to Y . A frequent strategy is to consider the notion of homotopic transformation. In particular, if there exists a (decreasing) homotopic transformation from X to Y , then X and Y have the same homotopy type. However, this is hardly tractable in 3D [10] and in higher dimensions. Then we consider a

weaker topological invariant induced by the notion of strongly deletable set [20]. More precisely, if $X \searrow Y$ is strongly deletable, then the inclusion relation induces a bijection between the (foreground and background) connected components of X and those of Y .

Remark 31 *If $\mathbb{U} = \mathbb{Z}^2$, the notion of strongly deletable set is equivalent to the notion of simple set [14]. This implies that if $X \searrow Y$ is strongly deletable, then X and Y have the same homotopy type and Y is obtained from X by a decreasing homotopic transformation defined as the iterative removal of a sequence of simple points [4].*

Let $P, Q \in \mathcal{E}$ be such that $\rho_{\mathcal{E}}(P) = Q$. If $\rho_{\mathcal{E}}^{-1}(\{Q\}) = \{P\}$ and $\pi_{\theta}(Q) \setminus \pi_{\theta}(P)$ is a strongly deletable set, then we note $Q \searrow P$.

Remark 32 *If $Q \searrow P$, then we have $\rho_{\mathcal{E}}(P) = \varphi(P)$.*

Proposition 33 *Let $P \in \mathcal{E}$. Let $A = \varphi^{-1}(\{\varphi(P)\}) \cup \psi^{-1}(\{\varphi(P)\})$. Let $B = \{\varphi(P)\} \cup \psi^{-1}(\{P\})$. We have $\varphi(P) \searrow P$ iff the restriction $\varphi|_A : A \rightarrow B$ is a bijection.*

Proof Let $X = \pi_{\theta}(P)$. By definition, X is connected, i.e. $\Pi[X] = \{X\}$. The set $\Pi[\bar{X}]$ is composed by one infinite set $X_0 = \mathbb{U} \setminus \tau(X)$ and $k \geq 0$ sets X_i ($1 \leq i \leq k$) such that $\{X_i\}_{i=1}^k = \tau(\pi_{\theta}(\psi^{-1}(\{P\})))$. Let $Y = \pi_{\theta}(\varphi(P))$. By definition, Y is connected, i.e. $\Pi[Y] = \{Y\}$. The set $\Pi[\bar{Y}]$ is composed by one infinite set $Y_0 = \mathbb{U} \setminus \tau(Y)$ and $l \geq 0$ sets Y_j ($1 \leq j \leq l$) such that $\{Y_j\}_{j=1}^l = \tau(\pi_{\theta}(\psi^{-1}(\{\varphi(P)\})))$. Let $D = Y \setminus X$. Let us suppose that $\varphi(P) \searrow P$. Then, we have $\varphi^{-1}(\{\varphi(P)\}) = \{P\}$, i.e. φ is bijective between $\varphi^{-1}(\{\varphi(P)\})$ and $\{P\}$. Since D is deletable we have $k = l$ and (up to reindexing), for any $i \in \llbracket 0, k \rrbracket$, $Y_i \subseteq X_i$. For each $i \in \llbracket 0, k \rrbracket$, there exist $\widehat{P}_i = (\widehat{X}_i, v) \in \psi^{-1}(\{P\})$ such that $X_i = \tau(\widehat{X}_i)$ and $\widehat{Q}_i = (\widehat{Y}_i, w) \in \psi^{-1}(\{\varphi(P)\})$ such that $Y_i = \tau(\widehat{Y}_i)$. We have $Y_i \subseteq X_i$ and then $\tau(\widehat{Y}_i) \subseteq \tau(\widehat{X}_i)$. We set $D_i = D \cap \widehat{X}_i$. We have $\tau(\widehat{X}_i \setminus D_i) = \tau(\widehat{X}_i) \setminus D_i = \tau(\widehat{Y}_i)$. It follows that $\widehat{Y}_i \subseteq \widehat{X}_i$, and φ is then bijective between $\psi^{-1}(\{\varphi(P)\})$ and $\psi^{-1}(\{P\})$. Let us suppose that φ is bijective between $\varphi^{-1}(\{\varphi(P)\})$ and $\{\varphi(P)\}$. Then both $P = \varphi(P) \setminus D$ and $\varphi(P)$ are connected and $P \subset \varphi(P)$. Let us suppose that φ is bijective between $\psi^{-1}(\{\varphi(P)\})$ and $\psi^{-1}(\{P\})$. The function $\tau \circ \pi_{\theta}$ is a bijection between $\psi^{-1}(\{P\})$ (resp. $\psi^{-1}(\{\varphi(P)\})$) and $\{X_i\}_{i=1}^k$ (resp. $\{Y_j\}_{j=1}^l$). It follows that $\varphi(P) \searrow P$. ■

Let \sim_H be the equivalence relation on \mathcal{E} defined as the reflexive-transitive-symmetric closure of \searrow .

Remark 34 *We have $[\infty]_{\sim_H} = \{\infty\} \in \mathcal{E}/\sim_H$.*

Property 35 *Let $K \in \mathcal{E}/\sim_H$. Let \sqsubseteq_K be the order induced by $\sqsubseteq_{\mathcal{E}}$ on K . (K, \sqsubseteq_K) is a totally ordered set.*

For any $K \in \mathcal{E}/\sim_H$, we note $\langle K \rangle_H = \bigwedge_{\sqsubseteq_{\mathcal{E}}} K$ and $\langle\langle K \rangle\rangle_H = \bigvee_{\sqsubseteq_{\mathcal{E}}} K$.

Remark 36 *From these results, it follows that $\forall p \in \llbracket 1, |K| - 1 \rrbracket$ we have $\rho_{\mathcal{E}}^p(\langle K \rangle_H) \in K$. In particular, we have $\rho_{\mathcal{E}}^{|K|-1}(\langle K \rangle_H) = \langle\langle K \rangle\rangle_H$ whereas $\rho_{\mathcal{E}}^{|K|}(\langle K \rangle_H) \notin K$.*

Property 37 *Let $K \in \mathcal{E}/\sim_H$, $K \neq \{\infty\}$. Let $P = \rho_{\mathcal{E}}^{|K|}(\langle K \rangle_H)$. We have $P = \langle [P]_{\sim_H} \rangle_H$.*

For any $K \in \mathcal{E}/\sim_H$, we consider $\langle K \rangle_H \in \mathcal{E}$ as canonical element, and we identify $\langle K \rangle_H = (X, v)$ with $X \in \mathcal{O}$. We set $H = \mathcal{E}/\sim_H$.

Let \sqsubseteq_H be the order on $\{\langle K \rangle_H \mid K \in H\} \subseteq \mathcal{E}$ —and equivalently on H — induced by $\sqsubseteq_{\mathcal{E}}$.

Let $\rho_H : H \rightarrow H$ be the function defined by

$$\rho_H(K) = [\rho_{\mathcal{E}}^{|\langle K \rangle|}(\langle K \rangle)]_{\sim_H} \quad (13)$$

We define the relation \blacktriangleleft_H on H , induced by the relation $\blacktriangleleft_{\mathcal{E}}$ on \mathcal{E} by $K \blacktriangleleft_H \rho_H(K)$.

Definition 38 (Topological tree of shapes) *The topological tree of shapes (or TS-tree, for brief) is the couple $\mathfrak{T}_{\text{TS}} = (H, \blacktriangleleft_H)$. See Figure 3 (centre).*

The following proposition directly derives from the two above properties.

Proposition 39 *The equivalence relation \sim_H induces a decreasing homeomorphism from $\mathfrak{T}_{\text{VS}} = (\mathcal{E}, \blacktriangleleft_{\mathcal{E}})$ to $\mathfrak{T}_{\text{TS}} = (H, \blacktriangleleft_H)$.*

Remark 40 *The homeomorphism from $\mathfrak{T}_{\text{VS}} = (\mathcal{E}, \blacktriangleleft_{\mathcal{E}})$ to $\mathfrak{T}_{\text{TS}} = (H, \blacktriangleleft_H)$ is a topologically lossless but a geometrically lossy compression. The structure of the tree $\mathfrak{T}_{\text{VS}} = (\mathcal{E}, \blacktriangleleft_{\mathcal{E}})$ can be fully reconstructed from $\mathfrak{T}_{\text{TS}} = (\mathcal{O}, \blacktriangleleft_{\mathcal{O}})$, but not the shapes of its nodes.*

6 Links With Other Trees

The graph of valued shapes \mathbb{G}_{VS} presents a DAG structure, similarly to other morphological hierarchies, e.g. the component-graph [17], the directed component hierarchy [18] or the braid of partitions [8]. \mathbb{G}_{VS} is also organized via two kinds of relations, similarly to the component-hypertree [15] and the directed component hierarchy [18] (where the initial order can be split into two distinct orders).

But, contrary to these morphological hierarchies, \mathbb{G}_{VS} can be modeled as a tree structure, namely \mathfrak{T}_{VS} . This should open the way to efficient construction strategies, compared, e.g. to the component-hypertree [12], the component-graph [16] or the braid of partitions [29], the construction of which remains complex and/or costly.

Beyond these considerations, the graph of valued shapes and the induced trees (trees of valued shapes, complete tree of shapes) also allow to unify various morphological trees.

With the notations introduced in Section 3, the max-tree (resp. min-tree) [24] of \mathcal{F} is defined as $\mathfrak{T}_{\text{max}} = (\mathcal{O}_{\text{max}}, \blacktriangleleft_{\text{max}})$ (resp. $\mathfrak{T}_{\text{min}} = (\mathcal{O}_{\text{min}}, \blacktriangleleft_{\text{min}})$) with $\mathcal{O}_{\text{max}} = \mathcal{O}^\circ$ (resp. $\mathcal{O}_{\text{min}} = \mathcal{O}^\bullet$) and $\blacktriangleleft_{\text{max}}$ (resp. $\blacktriangleleft_{\text{min}}$) is the Hasse relation induced by the restriction of \sqsubseteq on \mathcal{O}_{max} (resp. \mathcal{O}_{min}).

Proposition 41 *There is a decreasing homeomorphism from the subgraph $(\mathcal{E}^\circ, \blacktriangleleft^\circ)$ (resp. $(\mathcal{E}^\bullet, \blacktriangleleft^\bullet)$) of \mathbb{G}_{VS} to the max-tree $\mathfrak{T}_{\text{max}} = (\mathcal{O}_{\text{max}}, \blacktriangleleft_{\text{max}})$ (resp. the min-tree $\mathfrak{T}_{\text{min}} = (\mathcal{O}_{\text{min}}, \blacktriangleleft_{\text{min}})$).*

Proof The proof is similar to that of Proposition 29 (Properties 22, 24, 26) by considering \mathcal{E}° and Θ° (resp. \mathcal{E}^\bullet and Θ^\bullet) instead of \mathcal{E} and Θ . ■

The adjacency tree [21] of a binary set $X \subset \mathbb{U}$ is defined as $\mathfrak{T}_{\text{adj}} = (\Theta_{\text{adj}}(X), \blacktriangleleft_{\text{adj}})$ where $\Theta_{\text{adj}}(X) = \Pi[X] \cup \Pi[\bar{X}]$ and $\blacktriangleleft_{\text{adj}}$ is the Hasse relation induced by the ‘‘surrounding’’ order relation on $\Theta_{\text{adj}}(X)$.

Proposition 42 *Let $v \in \mathbb{V}$. The subgraph $(\Theta_v, \blacktriangleleft^v)$ of $\mathbb{G}_{\mathbb{V}\mathbb{S}}$ is isomorphic to the adjacency-tree $\mathfrak{T}_{\text{adj}} = (\Theta_{\text{adj}}(\Lambda_v^\circ(\mathcal{F})), \blacktriangleleft_{\text{adj}})$.*

Proof This proposition directly derives from the equivalence of the definitions. ■

The tree of shapes [11] of \mathcal{F} is defined as $\mathfrak{T}_{\text{shape}} = (\Theta_{\text{shape}}, \blacktriangleleft_{\text{shape}})$ where $\Theta_{\text{shape}} = \tau(\Theta)$ and $\blacktriangleleft_{\text{shape}}$ is the Hasse relation induced by \subseteq on Θ_{shape} . See Figure 3 (right).

Proposition 43 *There is a decreasing homeomorphism from the tree $(\Theta, \blacktriangleleft_\Theta)$ to the tree of shapes $\mathfrak{T}_{\text{shape}} = (\Theta_{\text{shape}}, \blacktriangleleft_{\text{shape}})$.*

Proof The proof is similar to that of Proposition 29 (Properties 22, 24 and Proposition 26) by considering Θ instead of \mathcal{E} and the equivalence relation on Θ defined by $X \sim_S Y \Leftrightarrow \tau(X) = \tau(Y)$. ■

Remark 44 *In [28], the notion of a topological monotonic tree was introduced, where ‘‘monotonic tree’’ has the same meaning as ‘‘tree of shapes’’. However, the topological monotonic tree of [28] is indeed different from our topological tree of shapes. Unformally, the difference between both structures lies in the fact that our topological tree of shapes relies on a topological equivalence relation between the nodes of the complete tree of shapes, whereas the topological monotonic as defined in [28] relies on a similar equivalence relation between the external border of the nodes. From [28], there is a decreasing homeomorphism from the tree of shapes to the topological monotonic tree. It can be proved that there is also a decreasing homeomorphism from our topological tree of shapes to the topological monotonic tree. These links are not formalized here by lack of room; they will be more deeply investigated in our further works.*

The following result derives from the above propositions and properties.

Property 45 *We have*

$$|H| \leq |\Theta| \leq |\mathcal{E}| \leq (|\mathbb{S}| + 1) \cdot |\mathbb{V}| \quad (14)$$

$$|\Theta_{\text{shape}}| \leq |\Theta| = |\Theta_{\text{max}}| + |\Theta_{\text{min}}| - 1 \leq 2 \cdot |\mathbb{S}| \quad (15)$$

7 Concluding Remarks and Perspectives

This article gathers some preliminary results about the notions of graph / tree of valued shapes and complete / topological tree of shapes. These notions shed a new light on well

known morphological hierarchies, namely the component-tree and the tree of shapes. In particular they allow to unify and extend these notions and to link them to topological invariants related to the adjacency tree, the deletable sets and—under favourable hypotheses—the homotopy type. We believe that these structures constitute a promising subject of research in the framework of morphological hierarchies. At this stage, our introductory study focused only on the structural side of these notions.

Our perspective works will also consider the algorithmic aspects, in particular the way to build these structures efficiently. Due to their strong links with the component-tree, it is possible to propose first, naive strategies to build the graph and then the tree of valued shapes from the min- and max-trees [2], and then to derive the complete and topological trees of shapes. It is also possible to start from the construction of the tree of shapes [7] to derive the same structures. However, such approaches, although tractable, are not optimal, and seeking dedicated construction algorithms makes sense.

We initially designed the graph of valued shapes by “mixing” the min- / max-trees and adjacency trees with a precise idea in mind. Our purpose was to develop conceptual tools that would allow one to carry out the topological analysis of objects in non-binary paradigms (e.g. for grey-level images or fuzzy modeling). In this regard, our next step will be to investigate the links that exist between these new structures and frameworks developed in topological analysis, especially with respect to grey-scale topology [5] and to homology persistence and Morse theory [1].

More generally, we also believe that these new structures could be useful for developing approaches dedicated e.g. to homotopic morphology, topological compression or topological comparison.

Acknowledgements

This work was supported by the *Centre National de la Recherche Scientifique* (IEA Project DiTopAM, INS2I Acc-CH-EC TopIA) and by the French *Agence Nationale de la Recherche* (Grants ANR-18-CE23-0025, ANR-20-CE23-0022).

References

1. Boutry, N., Géraud, T., Najman, L.: An equivalence relation between morphological dynamics and persistent homology in n -D. In: DGMM, Procs. pp. 525–537 (2021)
2. Carlinet, E., Géraud, T.: A comparative review of component tree computation algorithms. *IEEE T Image Process* **23**, 3885–3895 (2014)
3. Carlinet, E., Géraud, T.: MToS: A tree of shapes for multivariate images. *IEEE T Image Process* **24**, 5330–5342 (2015)
4. Couprie, M., Bertrand, G.: New characterizations of simple points in 2D, 3D, and 4D discrete spaces. *IEEE T Pattern Anal* **31**, 637–648 (2009)
5. Couprie, M., Nivando Bezerra, F., Bertrand, G.: Topological operators for grayscale image processing. *J Electron Imaging* **10**, 1003–1015 (2001)
6. Edelsbrunner, H., Letscher, D., Zomorodian, A.: Topological persistence and simplification. *Discrete Comput Geom* **28**, 511–533 (2002)
7. Géraud, T., Carlinet, E., Crozet, S., Najman, L.: A quasi-linear algorithm to compute the tree of shapes of n D images. In: ISMM, Procs. pp. 98–110 (2013)

8. Kiran, B., Serra, J.: Braids of partitions. In: ISMM, Procs. pp. 217–228 (2015)
9. Kurtz, C., Naegel, B., Passat, N.: Connected filtering based on multivalued component-trees. *IEEE T Image Process* **23**, 5152–5164 (2014)
10. Malgouyres, R., Francés, A.R.: Determining whether a simplicial 3-complex collapses to a 1-complex is NP-complete. In: DGCI, Procs. pp. 177–188 (2008)
11. Monasse, P., Guichard, F.: Fast computation of a contrast-invariant image representation. *IEEE T Image Process* **9**, 860–872 (2000)
12. Morimitsu, A., Passat, N., Luz Alves, W.A., Hashimoto, R.F.: Efficient component-hypertree construction based on hierarchy of partitions. *Pattern Recogn Lett* **135**, 30–37 (2020)
13. Najman, L., Schmitt, M.: Geodesic saliency of watershed contours and hierarchical segmentation. *IEEE T Pattern Anal* **18**, 1163–1173 (1996)
14. Passat, N., Mazo, L.: An introduction to simple sets. *Pattern Recogn Lett* **30**, 1366–1377 (2009)
15. Passat, N., Naegel, B.: Component-hypertrees for image segmentation. In: ISMM, Procs. pp. 284–295 (2011)
16. Passat, N., Naegel, B., Kurtz, C.: Component-graph construction. *J Math Imaging Vis* **61**, 798–823 (2019)
17. Passat, N., Naegel, N.: Component-trees and multivalued images: Structural properties. *J Math Imaging Vis* **49**, 37–50 (2014)
18. Perret, B., Cousty, J., Tankyevych, O., Talbot, H., Passat, N.: Directed connected operators: Asymmetric hierarchies for image filtering and segmentation. *IEEE T Pattern Anal* **37**, 1162–1176 (2015)
19. Randrianasoa, J.F., Kurtz, C., Desjardin, E., Passat, N.: Binary partition tree construction from multiple features for image segmentation. *Pattern Recogn* **84**, 237–250 (2018)
20. Ronse, C.: A topological characterization of thinning. *Theor Comput Sci* **43**, 31–41 (1986)
21. Rosenfeld, A.: Adjacency in digital pictures. *Inform Control* **26**, 24–33 (1974)
22. Rosenfeld, A.: Digital topology. *Am Math Mon* **86**, 621–630 (1979)
23. Salembier, P., Garrido, L.: Binary partition tree as an efficient representation for image processing, segmentation, and information retrieval. *IEEE T Image Process* **9**, 561–576 (2000)
24. Salembier, P., Oliveras, A., Garrido, L.: Anti-extensive connected operators for image and sequence processing. *IEEE T Image Process* **7**, 555–570 (1998)
25. Salembier, P., Serra, J.: Flat zones filtering, connected operators, and filters by reconstruction. *IEEE T Image Process* **4**(8), 1153–1160 (1995)
26. Santana Maia, D., Cousty, J., Najman, L., Perret, B.: Characterization of graph-based hierarchical watersheds: Theory and algorithms. *J Math Imaging Vis* **62**, 627–658 (2020)
27. Soille, P.: Constrained connectivity for hierarchical image decomposition and simplification. *IEEE T Pattern Anal* **30**, 1132–1145 (2008)
28. Song, Y., Zhang, A.: Monotonic tree. In: DGCI, Procs. pp. 114–123 (2002)
29. Tochon, G., Dalla Mura, M., Veganzones, M.A., Géraud, T., Chanussot, J.: Braids of partitions for the hierarchical representation and segmentation of multimodal images. *Pattern Recogn* **95**, 162–172 (2019)