TRANSVERSE INSTABILITY OF HIGH FREQUENCY WEAKLY STABLE QUASILINEAR BOUNDARY VALUE PROBLEMS

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Abstract. This work intends to prove that strong instabilities may appear for high order geometric optics expansions of weakly stable quasilinear hyperbolic boundary value problems, when the forcing boundary term is perturbed by a small amplitude oscillating function, with a transverse frequency. Since the boundary frequencies lie in the locus where the so-called Lopatinskii determinant is zero, the amplifications on the boundary give rise to a highly coupled system of equations for the profiles. A simplified model for this system is solved in an analytical framework using the Cauchy-Kovalevskaya theorem as well as a version of it ensuring analyticity in space and time for the solution. Then it is proven that, through resonances and amplification, a particular configuration for the phases may create an instability, in the sense that the small perturbation of the forcing term on the boundary interferes at the leading order in the asymptotic expansion of the solution. Finally we study the possibility for such a configuration of frequencies to happen for the isentropic Euler equations in space dimension three.

Contents

This work takes interest into the (in)-stability of multiphase geometric optics expansions for weakly stable quasilinear hyperbolic boundary value problems. The formal construction of such geometric optics expansions goes back to Majda, Artola, and Rosales, in [MR83, MR84, AM87, MA88]. In this paper, we prove that, for a simplified model, infinitely accurate approximate solutions can be unstable, in the sense that a small perturbation of the boundary forcing term interferes at the leading order in the asymptotic expansion.

For uniformly stable problems, the construction of a multiphase asymptotic expansion is performed, for Cauchy problems, notably in [HMR86] for the linear case, and in [JMR95] for the quasilinear one. In the case of boundary value problems, [Wil96] studies the semilinear case with multiple frequencies on the boundary, and the quasilinear case is treated in [CGW11] for one phase on the boundary. The case of multiple phases on the boundary is addressed in a previous work of the author, [Kil22]. In the weakly stable case, that is, when the weak Kreiss-Lopatinksii condition is satisfied, an amplification phenomenon occurs, as shown in works of Coulombel, Guès and Williams. Following the pioneering works by Majda and his collaborators, the first rigorous construction of a geometric optics expansion in the weakly stable case is performed in [CG10] for linear boundary value problems. Nonlinear problems are treated in [CGW14, CW14] for the semilinar case, and [CW17] for the quasilinear one. In [CGW14, CW14, CW17], the authors consider one phase on the boundary, and the present w work intends to address the extension of [CW17] to the multiphase case. Here, allowing multiple phases on the boundary permits us to consider a particular configuration of frequencies on the boundary, which, thanks to the amplification phenomenon, will lead to an instability for the asymptotic expansion. We show however that, fixing a locus of breaking of the Kreiss-Lopatinskii condition, this configuration of frequencies creating an instability cannot happen for the example of gas dynamics. This leaves hope to justify the validity of the geometric optics expansions with one amplification for the gas dynamics.

This work is divided in three main parts: (i) the derivation of the equations satisfied by the profiles in the multiphase case, following [CW17]; (ii) the proof of existence of solutions to the obtained system in an analytical framework; and (iii) the proof of instability for this system, namely, that there exists a perturbation of the boundary forcing term interfering at the leading order in the expansion. The general system being out of our reach for the moment, both for existence and for the instability mechanism, we deal with simplify models of the general system of equations for the profiles.

For the boundary value problem considered in this paper, the boundary condition is assumed to satisfy the weak Kreiss-Lopatinskii condition, namely that the Kreiss-Lopatinskii condition breaks on a certain locus of the frequency space. More precisely, we assume here that the locus where the Kreiss-Lopatinskii condition is not satisfied lies in the hyperbolic region (see [BGS07, Definition 2.1]). For boundary frequencies for which the Kreiss-Lopatinskii condition is not satisfied, an amplification phenomenon occurs on the boundary. The idea is to consider a particular configuration of frequencies on the boundary which will turn this amplification into a strong instability. For this purpose, we consider a boundary forcing term G oscillating at a frequency φ, belonging to the locus where the Kreiss-Lopatinskii condition is not satisfied, and we perturb this boundary forcing term G with a perturbation term H, oscillating at a *transverse* frequency ψ also belonging to the locus where the Kreiss-Lopatinskii condition is not satisfied, of small amplitude compared to the one of G. In [CW17], the boundary frequency φ is assumed to be non-resonant, in the sense that two interior frequencies lifted from φ cannot resonate with each other. We make the same assumption here, as well as for the boundary frequency ψ. We assume however that two well chosen resonance relations between φ and ψ exists, which will allow the perturbation H to ascend towards the leading order, through repeated amplification and resonances. We study in this article the possibility for such a configuration of frequencies to happen for the isentropic compressible Euler equations in space dimension 3. We prove that for a particular choice of locus where the Kreiss-Lopatinskii condition is not satisfied, the configuration of boundary frequencies considered here is (thankfully) impossible for the Euler system.

The derivation of equations for the amplitudes from the BKW cascade follows the one detailed in [CW17]. The main difference with the iterative process in the uniformly stable case, see e.g. [JMR95, Wil96, CGW11, Kil22], is that, for boundary frequencies lying in the locus where the Kreiss-Lopatinski condition is not satisfied, we cannot a priori determine a boundary condition for incoming profiles. Indeed, because of the weak Kreiss-Lopatinskii condition, for such boundary frequencies, the traces of incoming profiles are expressed through an unknown scalar function. For a given order, the evolution equations satisfied by these boundary terms are

derived using equations on profiles of the next order. This is where amplification occurs. The main difference with [CW17] is that, because of resonances, equations for each profile and for boundary terms are coupled with each other. Also, in comparison with [CW17], in equations for the boundary term of a given order, there is a term involving the trace of a profile of the next order, which was proven to be zero in [CW17], because resonances were absent in that work. This results into a highly coupled system. Nevertheless we discuss two points about this system, which are the existence of a solution to it and the creation of an instability.

The existence of a solution to the system of equations for the profiles is proven here in an analytical setting. The aim is to use the abstract Cauchy-Kovalevskaya theorem, whose proof can be found in [Nir72] and [Nis77]. We use in this work the formulation of [BG78]. The system of equations for the profiles is made of incoming and outgoing equations for interior profiles, whose traces of incoming profiles are expressed with boundary terms that in turn satisfy coupled evolution equations on the boundary. As already mentioned, the general system is difficult to treat, so we consider two simplified models of increasing difficulty for the study of existence. Both retain only a few profiles (which are the ones of interest), and remove some couplings between the equations. The first is only constituted by coupled equations on the boundary, and we make this first simplified model more complex into a second one by incorporating interior equations, whose traces on the boundary are given by the solutions to the equations on the boundary. For the first simplified model, containing only equations on the boundary, the formulation of [BG78] can be applied, using a chain of spaces quantifying analyticity by means of the Fourier transform. The only difficulty is to show that a certain bilinear operator appearing in the equations is semilinear in the considered spaces of functions, and this result is obtained adapting a result of [CW17]. For the second simplified model, incorporating interior equations, the aim is to apply the Cauchy-Kovalevskaya theorem to the interior equations, seen as propagation equations in the normal direction. Therefore, we need the boundary terms, which are solutions to boundary equations, to be analytical with respect to all their variables: both tangential space variables and *time*. However, if we apply the classical Cauchy-Kovalevskaya theorem to the boundary equations, we obtain a solution analytical only with respect to tangent space variables, and not with respect to time. We therefore need to adapt the Cauchy-Kovalevskaya theorem to obtain analyticity with respect to all variables for solutions to the boundary equations. This is done using the method of majoring series, and the phenomenon of regularization by integration in time introduced in [Uka01], see also [Mét09, Mor20]. We define for this purpose a chain of spaces of analytic functions, with a formulation adapted from [BG78] to the framework of majoring series, and prove the result using a fixed point theorem. We also define a chain of functional spaces suited to apply the Cauchy-Kovalevskaya theorem for interior equations, once we have constructed the analytic boundary terms. Applying the Cauchy-Kovalevskaya theorem to interior equations in this chain of functional spaces then presents no difficulty.

To prove that there is an instability, namely, that a small perturbation H of the boundary forcing term G interferes at a leading order, since the perturbation H is small compared to G, we consider the linearized version of the general system, around the particular solution of this system when the perturbation H is zero. We obtain a linearized system with a small boundary forcing term given by H, and we prove that there exists a boundary term H such that this system admits a solution whose first order profiles are not all zero. It shows that the small perturbation H interferes at the leading order for the linearized system, which constitutes an instability. The existence of H is proven by contradiction: we assume that for all boundary terms H, all leading profiles are zero, and we contradict a certain condition by constructing the second order correctors. As for the part about existence, we work here with simplified models, as the coupling of the general system of equations is too difficult to handle. The first simplified model allows us to construct explicitly the solution to the linearized system, solving the considered transport equations by the method of characteristics. For the second one, the coupling is more complex, preventing us to apply the latter method, and we use a perturbation method and solve equations with a fixed point theorem.

This article is organized as follows. First we state the problem that we study here, make structural assumptions about it, and specify some assumptions and preliminary results about

the oscillations at stake. Then, in a second part, the general system of equations for the profiles is derived, by detailing the iterative process for the leading profile and then the first corrector, and by writing down the general system satisfied by higher order correctors. We proceed in a third part with the proof of existence of a solution to simplified models of this general system. We start by detailing the obtaining of a first simplified model, then defining the functional framework which will be used, specifying the simplified model according to this functional framework, and applying the Cauchy-Kovalevskaya theorem for boundary equations. Then we detail how this first simplified model is made more complex into a second one, we define additional functional spaces and specify the second simplified model accordingly, and finally we show existence and analyticity of solutions to boundary equations by proving a new version of the Cauchy-Kovalevskaya theorem, leading to existence of solutions to interior equations, using a classical Cauchy-Kovalevskaya theorem. The fifth part is devoted to the proof of instability, first by deriving the linearization of the general system around the particular solution where the perturbation H is zero, and then by proving, for two different simplified models, that an instability is created. Finally, in a sixth part, the example of isentropic compressible Euler equations in space dimension 3 is studied.

In all the article the letter C denotes a positive constant that may vary during the analysis, possibly without any mention being made.

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1. Notation and assumptions

1.1. **Position of the problem.** Given a time $T > 0$ and an integer $d \geqslant 2$, let Ω_T be the domain $\Omega_T := (-\infty, T] \times \mathbb{R}^{d-1} \times \mathbb{R}_+$ and $\omega_T := (-\infty, T] \times \mathbb{R}^{d-1}$ its boundary. We denote as $t \in (-\infty, T]$ the time variable, $x = (y, x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}_+$ the space variable, with $y \in \mathbb{R}^{d-1}$ the tangential variable and $x_d \in \mathbb{R}_+$ the normal variable, and at last $z = (t, x) = (t, y, x_d)$. We also denote by $z' = (t, y) \in \omega_T$ the variable of the boundary $\{x_d = 0\}$. For $i = 1, \dots, d$, we denote by ∂_i the partial derivative operator with respect to x_i. Finally we denote as $\alpha \in \mathbb{R}^{d+1}$ and $\zeta \in \mathbb{R}^d$ the dual variables of $z \in \Omega_T$ and $z' \in \omega_T$. We consider the following problem

$$\begin{cases} L(u^\varepsilon, \partial_z)\, u^\varepsilon := \partial_t u^\varepsilon + \displaystyle\sum_{i=1}^{d} A_i(u^\varepsilon)\, \partial_i u^\varepsilon = 0 & \text{in } \Omega_T, \\ B\, u^\varepsilon_{|x_d=0} = \varepsilon^2\, g^\varepsilon + \varepsilon^M\, h^\varepsilon & \text{on } \omega_T, \\ u^\varepsilon_{|t\leqslant 0} = 0, \end{cases} \tag{1.1}$$

where the unknown u^ε is a function from Ω_T to an open set $\mathcal{O}$ of $\mathbb{R}^N$ containing zero, $N \geqslant 1$, the matrices A_j are smooth functions of $\mathcal{O}$ with values in $\mathcal{M}_N(\mathbb{R})$, the matrix B belongs to $\mathcal{M}_{\widetilde{p},N}(\mathbb{R})$ and is of maximal rank (integer $\widetilde{p} \geqslant 1$ will be made precise below).

The boundary term is a superposition of a reference forcing oscillating term $\varepsilon^2\, g^\varepsilon$ (of characteristic wavelength ε) and a smaller, transverse, oscillating term $\varepsilon^M\, h^\varepsilon$ with $M \geqslant 3$, namely, for $z' \in \omega_T$,

$$g^\varepsilon(z') = G\left(z', \frac{z' \cdot \varphi}{\varepsilon}\right), \tag{1.2a}$$

$$h^\varepsilon(z') = H\left(z', \frac{z' \cdot \psi}{\varepsilon}\right), \tag{1.2b}$$

where G, H are functions of the Sobolev space of infinite reguarity $H^\infty(\mathbb{R}^d \times \mathbb{T})$, are zero for negative time t, and with boundary frequencies φ, ψ given in $\mathbb{R}^d \setminus \{0\}$. Frequencies φ and ψ are taken linearly independent over $\mathbb{R}$, that is, $\psi \notin \mathbb{R}\varphi$. We denote by $\boldsymbol{\zeta}$ the couple $\boldsymbol{\zeta} := (\varphi, \psi)$. In this paper we wish to place ourselves in the framework of weakly nonlinear geometric optics. Usually to obtain this framework the amplitude of the boundary forcing term must be of order

$O(\varepsilon)$. Here, because we will assume that the Kreiss-Lopatinskii condition is not satisfied for φ (and ψ), an amplification phenomenon will happen at the boundary for this frequency, so a forcing term of amplitude of order $O(\varepsilon^2)$ should be chosen on the boundary. This scaling has been studied in [AM87, MR83, MR84, CW17]. Note that if we set $h^\varepsilon = 0$ in system (1.1), we obtain the system studied in [CW17].

To simplify the equations and computations we assume that the coefficients are affine maps, that is, for $j = 1, \dots, d$,

$$A_j(u) = A_j(0) + dA_j(0) \cdot u.$$

We make the following structural and classical assumption on the boundary.

Assumption 1 (noncharacteristic boundary)**.** *The boundary is noncharacteristic, that is, matrix $A_d(0)$ is invertible.*

To simplify the equations and the computations we will study here the case $M = 3$, but there is no apparent obstacle to generalize this analysis to any integer $M \geqslant 4$. For the same purpose we choose to work with the particular case of 3-dimensional vectors ($N = 3$) since it is sufficient in this analysis to create instabilities.

In this paper we study a geometric optics asymptotic expansion for system (1.1), namely, we look for an approximate solution to (1.1) in the form of a formal series

$$u^{\varepsilon,\mathrm{app}}(z) = \sum_{n\geqslant 1} \varepsilon^n U_n\Big(z, \frac{\Phi(z)}{\varepsilon}\Big), \tag{1.3}$$

where the collection of phases Φ will be made precise later. The approximate solution is expected to be of order $O(\varepsilon)$ because of the weakly nonlinear framework. The aim is to show that, with a well chosen configuration of frequencies, there is an instability in this asymptotic expansion, in the sense that, despite its small amplitude order $O\big(\varepsilon^3\big)$, perturbation $\varepsilon^3\, h^\varepsilon$ interferes at the leading order, i.e. in the construction of the leading profile U_1. In addition to this instability, we will study well-posedness for a simplified model associated with the equations for the profiles, and the possibility for such a frequency configuration to occur in the case of Euler equations in space dimension 3.

We start by making a series of structural assumptions on system (1.1) and detailing the configuration of frequencies considered here.

The following definition introduces the notion of characteristic frequency.

Definition 1.1. *For $\alpha = (\tau, \eta, \xi) \in \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}$, the symbol $L(0, \alpha)$ associated with $L(0, \partial_z)$ is defined as*

$$L(0, \alpha) := \tau I + \sum_{i=1}^{d-1} \eta_i A_i(0) + \xi A_d(0).$$

Then we define its characteristic polynomial as $p(\tau, \eta, \xi) := \det L\big(0, (\tau, \eta, \xi)\big)$. We say that $\alpha \in \mathbb{R}^{1+d}$ is a characteristic frequency *if it is a root of the polynomial p.*

The following assumption, called *strict hyperbolicity* (see [BGS07, Definition 1.2]), is made. Assumptions of hyperbolicity, whether strict or with constant multiplicity, are very usual, see e.g. [Wil96, CGW11, JMR95], and related to the structure of the problem. Assumption of hyperbolicity with constant multiplicity, which is more general than Assumption 2 of strict hyperbolicity below, is sometimes preferred like in [CGW11, JMR95]. We chose here to work with the latter for technical reasons. Recall that we placed ourselves in the particular case where the size of the system is $N = 3$.

Assumption 2 (strict hyperbolicity)**.** *There exist real functions $\tau_1 < \tau_2 < \tau_3$, analytic with respect to (η, ξ) in $\mathbb{R}^d \setminus \{0\}$, such that for all $(\eta, \xi) \in \mathbb{R}^d \setminus \{0\}$ and for all $\tau \in \mathbb{R}$, the following factorisation is verified*

$$p(\tau, \eta, \xi) = \det\Big(\tau I + \sum_{i=1}^{d-1} \eta_i A_i(0) + \xi A_d(0)\Big) = \prod_{k=1}^{3} \big(\tau - \tau_k(\eta, \xi)\big),$$

where the eigenvalues $-\tau_k(\eta,\xi)$ *of the matrix* $A(\eta,\xi) := \sum_{i=1}^{d-1} \eta_i A_i(0) + \xi A_d(0)$ *are therefore simple.*

1.2. **Weak Kreiss-Lopatinskii condition.** We define the following space of frequencies

$$\begin{aligned}
\Xi &:= \{\zeta = (\sigma = \tau - i\gamma, \eta) \in (\mathbb{C}\times\mathbb{R}^{d-1})\backslash\{0\} \mid \gamma \geqslant 0\},\\
\Sigma &:= \left\{\zeta \in \Xi \mid \tau^2 + \gamma^2 + |\eta|^2 = 1\right\},\\
\Xi_0 &:= \{\zeta \in \Xi \mid \gamma = 0\},\\
\Sigma_0 &:= \Xi_0 \cap \Sigma.
\end{aligned}$$

We also define the matrix valued symbol which we get when applying the Laplace-Fourier transform to the operator $L(0,\partial_z)$. For all $\zeta = (\sigma,\eta) \in \Xi$, let

$$\mathcal{A}(\zeta) := -i\, A_d(0)^{-1}\Big(\sigma I + \sum_{i=1}^{d-1} \eta_j\, A_j(0)\Big).$$

The Hersh lemma ([Her63]) ensures that for ζ in $\Xi\backslash\Xi_0$, the matrix $\mathcal{A}(\zeta) \in \mathcal{M}_3(\mathbb{C})$ has no eigenvalue of zero real part, and that the stable subspace associated with the eigenvalues of negative real part, denoted by $E_-(\zeta)$, is of constant dimension, denoted p. Furthermore, the integer p is obtained as the number of positive eigenvalues of the matrix $A_d(0)$. We denote by $E_+(\zeta)$ the unstable subspace $\mathcal{A}(\zeta)$ associated with eigenvalues of positive real part, which is of dimension $3-p$.

In [Kre70] (see also [CP82] and [BGS07]) it is shown that the stable and unstable subspaces $E_\pm$ extend continuously to the whole space Ξ in the strictly hyperbolic case (Assumption 2). We still denote by $E_\pm$ the extensions to Ξ. The *hyperbolic region*, denoted by $\mathcal{H}$, is defined as the set of frequencies ζ such that matrix $\mathcal{A}(\zeta)$ has only purely imaginary eigenvalues.

The following assumption is very structural to the problem, and is the one which allows amplification on the boundary, and thus instability.

Assumption 3 (weak Kreiss-Lopatinskii condition).

- *For all* $\zeta \in \Xi \setminus \Xi_0$, $\ker B \cap E_-(\zeta) = \{0\}$.
- *The set* $\Upsilon := \{\zeta \in \Sigma_0 \,|\, \ker B \cap E_-(\zeta) \neq \{0\}\}$ *is nonempty and included in the hyperbolic region* $\mathcal{H}$.
- *There exist a neighborhood* $\mathcal{V}$ *of* Υ *in* Σ, *a real valued* $\mathcal{C}^\infty$ *function* κ *defined on* $\mathcal{V}$, *a basis* $E_1(\zeta),\dots,E_p(\zeta)$ *of* $E_-(\zeta)$ *and a matrix* $P(\zeta) \in \mathrm{GL}_p(\mathbb{C})$ *which are of class* $\mathcal{C}^\infty$ *with respect to* $\zeta \in \mathcal{V}$ *such that, for all* ζ *in* $\mathcal{V}$,

$$B\big(E_1(\zeta)\cdots E_p(\zeta)\big) = P(\zeta)\ \mathrm{diag}\,\big(\gamma + i\kappa(\zeta), 1, \dots, 1\big).$$

Remark 1.2. First point of Assumption 3, requiring that $\ker B \cap E_-(\zeta) = \{0\}$ for all $\zeta \in \Xi \setminus \Xi_0$, implies in particular that $\widetilde{p}$, the rank of B, equals p, the dimension of $E_-(\zeta)$. These two equal integers will be denoted by p in the following. Assumption 5 below sets furthermore the integer $p = \widetilde{p}$ to be equal to 2.

The so-called *Kreiss-Lopatinskii condition* is the first point of Assumption 3 that stands in $\Xi \setminus \Xi_0$, and the next two points detail how this condition breaks on the boundary Ξ_0 of Ξ (for the *uniform Kreiss-Lopatinskii condition* to hold, equality $\ker B \cap E_-(\zeta) = \{0\}$ is assumed to be satisfied everywhere in Ξ, see [Kre70]). The second point asserts that the Kreiss-Lopatinskii condition breaks only in the hyperbolic region $\mathcal{H}$, and the third one ensures that when it breaks, the space $\ker B \cap E_-(\zeta)$ is of dimension 1, and that the default of injectivity of B on $E_-(\zeta)$ is parameterize by the $\mathcal{C}^\infty$ function κ. In particular, κ must be zero on Υ, and nonzero on $\Sigma_0 \setminus \Upsilon$.

Together with Assumptions 1 and 2, Assumption 3 ensures that for all $\varepsilon > 0$, system (1.1) is weakly well-posed locally in time (which depends on ε). A proof of a similar result, for characteristic free boundary problems can be found in [CS08]. Indeed, the three assumptions 1, 2 and 3 are stable under small perturbations around the equilibrium, see [BGS07, Section 8.3].

1.3. **Oscillations.** The notion of incoming, outgoing and glancing frequencies is now introduced.

Definition 1.3. *Let $\alpha = (\tau, \eta, \xi) \in \mathbb{R}^{d+1} \setminus \{0\}$ be a characteristic frequency, and k the integer between 1 and 3 such that $\tau = \tau_k(\eta, \xi)$. The group velocity $\mathbf{v}_\alpha \in \mathbb{R}^d$ associated with α is defined as*

$$\mathbf{v}_\alpha := \nabla_{\eta,\xi}\, \tau_k(\eta, \xi).$$

We shall say that α is glancing (resp. incoming, outgoing) if $\partial_\xi \tau_k(\eta, \xi)$ is zero (resp. negative, positive). Then the vector field X_α associated with α is defined as

$$X_\alpha := \frac{-1}{\partial_\xi \tau_k(\eta, \xi)} \Big(\partial_t - \mathbf{v}_\alpha \cdot \nabla_x \Big) = \frac{-1}{\partial_\xi \tau_k(\eta, \xi)} \Big(\partial_t - \nabla_\eta \tau_k(\eta, \xi) \cdot \nabla_y - \partial_\xi \tau_k(\eta, \xi)\, \partial_{x_d} \Big). \tag{1.4}$$

Lax lemma, see Lemma 1.10 below, ensures that these constant coefficients scalar transport operators X_α appear naturally in the equations satisfied by the profiles arising in weakly nonlinear asymptotic expansions (see [Rau12]).

We describe now a decomposition of the stable subspace $E_-(\zeta)$ for $\zeta \in \Xi_0$, that uses strict hyperbolicity (Assumption 2).

Proposition 1.4 ([Wil96], Proposition 3.4)**.** *Consider $\zeta = (\tau, \eta) \in \Xi_0$. We denote by $i\,\xi_j(\zeta)$ for $j = 1, \dots, \mathcal{M}(\zeta)$ the distinct complex eigenvalues of the matrix $\mathcal{A}(\zeta)$, and if $\xi_j(\zeta)$ is real, we shall denote by $\alpha_j(\zeta) := (\tau, \eta, \xi_j(\tau, \eta))$ the associated real characteristic frequency. If $\xi_j(\zeta)$ is real, we also denote by k_j the integer between 1 and 3 such that $\tau = \tau_{k_j}(\eta, \xi_j(\zeta))$. Then the set $\{1, 2, \dots, \mathcal{M}(\zeta)\}$ decomposes as the disjoint union*

$$\{1, 2, \dots, \mathcal{M}(\zeta)\} = \mathcal{G}(\zeta) \cup \mathcal{I}(\zeta) \cup \mathcal{P}(\zeta) \cup \mathcal{O}(\zeta) \cup \mathcal{N}(\zeta), \tag{1.5}$$

where the sets $\mathcal{G}(\zeta)$, $\mathcal{I}(\zeta)$, $\mathcal{P}(\zeta)$, $\mathcal{O}(\zeta)$ and $\mathcal{N}(\zeta)$ correspond to indexes j such that respectively $\alpha_j(\zeta)$ is glancing, $\alpha_j(\zeta)$ is incoming, $\mathrm{Im}(\xi_j(\zeta))$ is positive, $\alpha_j(\zeta)$ is outgoing and $\mathrm{Im}(\xi_j(\zeta))$ is negative.

Then the following decomposition of $E_-(\zeta)$ holds

$$E_-(\zeta) = \bigoplus_{j \in \mathcal{G}(\zeta)} E_-^j(\zeta) \oplus \bigoplus_{j \in \mathcal{R}(\zeta)} E_-^j(\zeta) \oplus \bigoplus_{j \in \mathcal{P}(\zeta)} E_-^j(\zeta), \tag{1.6}$$

where for each index j, the subspace $E_-^j(\zeta)$ is precisely described as follows.

- *i) If $j \in \mathcal{P}(\zeta)$, the space $E_-^j(\zeta)$ is the generalized eigenspace $\mathcal{A}(\zeta)$ associated with the eigenvalue $i\,\xi_j(\zeta)$.*
- *ii) If $j \in \mathcal{R}(\zeta)$, we have $E_-^j(\zeta) = \ker L\big(0, \alpha_j(\zeta)\big)$, which is of dimension 1.*
- *iii) If $j \in \mathcal{G}(\zeta)$, we denote by n_j the algebraic multiplicity of the imaginary eigenvalue $i\xi_j(\zeta)$. For small positive γ, the multiple eigenvalue $i\,\xi_j(\tau, \eta)$ splits into n_j simple eigenvalues, denoted by $i\,\xi_j^k(\tau - i\gamma, \eta)$, $k = 1, \dots, n_j$, all of nonzero real part. We denote by μ_j the number (independent of $\gamma > 0$) of the eigenvalues $i\,\xi_j^k(\tau - i\gamma, \eta)$ of negative real part. Then $E_-^j(\zeta)$ is of dimension μ_j and is generated by the vectors w satisfying $[\mathcal{A}(\zeta) - i\xi_j(\zeta)]^{\mu_j} w = 0$. Furthermore, if n_j is even, $\mu_j = n_j/2$ and if n_j is odd, μ_j is equal to $(n_j - 1)/2$ or $(n_j + 1)/2$.*

Likewise, the unstable subspace $E_+(\zeta)$ decomposes as

$$E_+(\zeta) = \bigoplus_{j \in \mathcal{G}(\zeta)} E_+^j(\zeta) \oplus \bigoplus_{j \in \mathcal{S}(\zeta)} E_+^j(\zeta) \oplus \bigoplus_{j \in \mathcal{N}(\zeta)} E_+^j(\zeta), \tag{1.7}$$

with similar description of the subspaces $E_+^j(\zeta)$. In particular, if the set $\mathcal{G}(\zeta)$ is empty, then

$$\mathbb{C}^3 = E_-(\zeta) \oplus E_+(\zeta).$$

For $\zeta \in \Xi_0$, we denote by $\mathcal{C}(\zeta)$ the set of indices such that $\alpha_j(\zeta)$ is real characteristic, that is

$$\mathcal{C}(\zeta) := \mathcal{I}(\zeta) \cup \mathcal{O}(\zeta) \cup \mathcal{G}(\zeta).$$

Definition 1.5. *A frequency ζ in Ξ_0 is said to be* glancing *if there exists $j = 1, \dots, \mathcal{M}(\zeta)$ such that $\alpha_j(\zeta)$ is glancing, i.e. if $\mathcal{G}(\zeta)$ is nonempty,* hyperbolic *if $\mathcal{A}(\zeta)$ has only purely imaginary eigenvalues, that is if $\mathcal{P}(\zeta) \cup \mathcal{N}(\zeta)$ is empty and* mixed *if $\mathcal{P}(\zeta) \cup \mathcal{N}(\zeta)$ is nonempty. We shall denote by $\mathcal{G}$ (resp. $\mathcal{H}$, $\mathcal{EH}$) the set of glancing (resp. hyperbolic, mixed) frequencies.*

Definition 1.6. *For $\zeta \in \Xi_0$ not glancing, according to Proposition 1.4, we have the following decomposition of $\mathbb{C}^3$:*

$$\mathbb{C}^3 = \bigoplus_{j \in \mathcal{O}(\zeta)} E_+^j(\zeta) \oplus \bigoplus_{j \in \mathcal{N}(\zeta)} E_+^j(\zeta) \oplus \bigoplus_{j \in \mathcal{I}(\zeta)} E_-^j(\zeta) \oplus \bigoplus_{j \in \mathcal{P}(\zeta)} E_-^j(\zeta).$$

In that case we denote by $\Pi^e(\zeta)$ the projection from $\mathbb{C}^3$ on the stable elliptic component $E_-^e(\zeta) := \oplus_{j \in \mathcal{P}(\zeta)} E_-^j(\zeta)$ according to this decomposition.

The following result is adapted from [CG10, Lemma 3.2] to the case of mixed frequencies.

Lemma 1.7. *For all $\zeta \in \Xi_0$ nonglancing, the following decompositions hold*

$$\mathbb{C}^3 = \bigoplus_{j \in \mathcal{C}(\zeta)} \ker L\big(0, \alpha_j(\zeta)\big) \oplus F_\zeta \tag{1.8a}$$

$$\mathbb{C}^3 = \bigoplus_{j \in \mathcal{C}(\zeta)} A_d(0) \ker L\big(0, \alpha_j(\zeta)\big) \oplus A_d(0) F_\zeta, \tag{1.8b}$$

where F_ζ is the generalized eigenspace of $\mathcal{A}(\zeta)$ associated with the eigenvalues of nonzero real part. Furthermore, if we denote by $P_j(\zeta)$ and P_{F_ζ} (resp. $Q_j(\zeta)$ and Q_{F_ζ}) the projectors associated with the decomposition (1.8a) (resp. (1.8b)), then we have

$$\operatorname{Im} L\big(0, \alpha_j(\zeta)\big) = \ker Q_j(\zeta), \tag{1.9}$$

for all j.

In [CG10], the result is proven only for frequencies ζ hyperbolic, and the proof is slightly simpler using directly the diagonalizability of matrix $\mathcal{A}(\zeta)$. Here the matrix is only block-diagonalizable, and we have to deal with eigenvalues of nonzero real part.

Proof. The two decompositions come from the block-diagonalizability of matrix $\mathcal{A}(\zeta)$, the fact that ζ is not glancing and the invertibility of matrix $A_d(0)$. Indeed, for any nonglancing frequency $\zeta \in \Xi_0$, there exists therefore an invertible matrix $T(\zeta)$ such that $T(\zeta)\,\mathcal{A}(\zeta)\,T(\zeta)^{-1}$ is the block diagonal matrix

$$T(\zeta)\,\mathcal{A}(\zeta)\,T(\zeta)^{-1} = \operatorname{diag}\big(i\xi_1(\zeta), \dots, i\xi_{m_\zeta}(\zeta), \mathcal{A}_\pm(\zeta)\big)$$

where the $\xi_j(\zeta)$ are real scalars, and the spectrum of the block $\mathcal{A}_\pm(\zeta)$ is contained in $\mathbb{C} \setminus i\mathbb{R}$. The proof decomposes in two main steps. First we construct a sequence of diagonalizable matrix converging toward $\mathcal{A}(\zeta)$, in order to be able to adapt the method used in [CG10]. Then using projectors defined for this sequence of matrix, analogous to $P_j(\zeta)$ and $Q_j(\zeta)$, we are able to prove relation (1.9), using diagonalizability.

Step 1. We consider a sequence $(\mathcal{A}_\pm^k(\zeta))_{k \geqslant 0}$ of diagonalizable matrices converging toward $\mathcal{A}_\pm(\zeta)$. For $k \geqslant 0$, we denote by $\widetilde{T}_k(\zeta)$ the invertible matrix such that

$$\widetilde{T}_k(\zeta)\,\mathcal{A}_\pm^k(\zeta)\,\widetilde{T}_k(\zeta)^{-1} = \operatorname{diag}(i\lambda_1, \dots, i\lambda_{3-m_\zeta}).$$

We also denote by $T_k(\zeta)$ the block diagonalizable matrix

$$T_k(\zeta) := \operatorname{diag}(I_{m_\zeta}, \widetilde{T}_k(\zeta)),$$

and we finally define, for $k \geqslant 0$ the matrix $\mathcal{A}^k(\zeta)$ as

$$\mathcal{A}^k(\zeta) := T(\zeta)\,T_k(\zeta)\,\operatorname{diag}\big(i\xi_1(\zeta), \dots, i\xi_{m_\zeta}(\zeta), i\lambda_1, \dots, i\lambda_{3-m_\zeta}\big)\,T_k(\zeta)^{-1}\,T(\zeta)^{-1}.$$

Note that the sequence $\big(\mathcal{A}^k(\zeta)\big)_{k \geqslant 0}$ is by definition a sequence of diagonalizable matrices which converges toward $\mathcal{A}(\zeta)$. Using this diagonalizability we get the two following decompositions of

$\mathbb{C}^3$, for $k \geqslant 0$:

$$\mathbb{C}^3 = \bigoplus_{j=1}^{m_\zeta} \ker\big(\mathcal{A}^k(\zeta) - i\xi_j(\zeta)I\big) \oplus \bigoplus_{j=1}^{3-m_\zeta} \ker\big(\mathcal{A}^k(\zeta) - i\lambda_j I\big) \tag{1.10a}$$

$$= \bigoplus_{j=1}^{m_\zeta} A_d(0)\ker\big(\mathcal{A}^k(\zeta) - i\xi_j(\zeta)I\big) \oplus \bigoplus_{j=1}^{3-m_\zeta} A_d(0)\ker\big(\mathcal{A}^k(\zeta) - i\lambda_j I\big). \tag{1.10b}$$

First we note that, by definition of the matrix $\mathcal{A}^k(\zeta)$, the eigenspace $\ker\big(\mathcal{A}^k(\zeta) - i\xi_j(\zeta)I\big)$ is equal to $\ker L\big(0, \alpha_j(\zeta)\big)$ and that

$$\bigoplus_{j=1}^{3-m_\zeta} \ker\big(\mathcal{A}^k(\zeta) - i\lambda_j I\big) = F_\zeta.$$

Thus we define the projectors $P_\pm^{k,j}(\zeta)$ (resp. $Q_\pm^{k,j}(\zeta)$) on $\ker\big(\mathcal{A}^k(\zeta) - i\lambda_j I\big)$ (resp. $A_d(0)$ $\ker\big(\mathcal{A}^k(\zeta) - i\lambda_j I\big)$) associated with the decomposition (1.10a) (resp. (1.10b)). According to the previous remark we then have

$$I = P_1(\zeta) + \cdots + P_{m_\zeta}(\zeta) + P_\pm^{k,1}(\zeta) + \cdots + P_\pm^{k,3-m_\zeta}(\zeta) \tag{1.11a}$$

$$= Q_1(\zeta) + \cdots + Q_{m_\zeta}(\zeta) + Q_\pm^{k,1}(\zeta) + \cdots + Q_\pm^{k,3-m_\zeta}(\zeta). \tag{1.11b}$$

Step 2. For j_0 between 1 and m_ζ, analogously to $L\big(0, \alpha_{j_0}(\zeta)\big)$, we define

$$L_k\big(0, \alpha_{j_0}(\zeta)\big) := iA_d(0)\big(\mathcal{A}_k(\zeta) - i\xi_{j_0}(\zeta)I\big).$$

By definition and since the following relation is satisfied

$$L\big(0, \alpha_{j_0}(\zeta)\big) = iA_d(0)\big(\mathcal{A}(\zeta) - i\xi_{j_0}(\zeta)I\big),$$

the sequence $\big(L_k\big(0, \alpha_{j_0}(\zeta)\big)\big)_{k\geqslant 0}$ converges to $L\big(0, \alpha_{j_0}(\zeta)\big)$. We consider $L_k\big(0, \alpha_{j_0}(\zeta)\big)\,X$ an element of $\operatorname{Im} L_k\big(0, \alpha_{j_0}(\zeta)\big)$ with $X \in \mathbb{C}^3$, and the aim is to prove that it belongs to $\ker Q_{j_0}(\zeta)$. The latter is a closed space, so, since the sequence $\big(L_k\big(0, \alpha_{j_0}(\zeta)\big)\,X\big)_{k\geqslant 0}$ converges to $L\big(0, \alpha_{j_0}(\zeta)\big)\,X$, it will follow that $\operatorname{Im} L\big(0, \alpha_{j_0}(\zeta)\big) \subset \ker Q_{j_0}(\zeta)$ and the conclusion then infers because of equality of dimension of the two spaces.

We have, by definition of the projectors $P_j(\zeta)$ and $P_\pm^{k,j}(\zeta)$ and because of the decomposition (1.11a),

$$\begin{aligned} L_k\big(\alpha_{j_0}(\zeta)\big)\,X &= iA_d(0)\big(\mathcal{A}_k(\zeta) - i\xi_{j_0}(\zeta)I\big)\Big\{\sum_{j=1}^{m_\zeta} P_j(\zeta)\,X + \sum_{j=1}^{3-m_\zeta} P_\pm^{k,j}(\zeta)\,X\Big\} \\ &= iA_d(0)\sum_{\substack{j=1\\ j\neq j_0}}^{m_\zeta} \big(i\xi_j(\zeta) - i\xi_{j_0}(\zeta)\big)P_j(\zeta)\,X + iA_d(0)\sum_{j=1}^{3-m_\zeta} \big(i\lambda_j(\zeta) - i\xi_{j_0}(\zeta)\big)P_\pm^{k,j}(\zeta)\,X, \end{aligned}$$

and the last term belongs to

$$\bigoplus_{\substack{j=1\\ j\neq j_0}}^{m_\zeta} A_d(0)\ker\big(\mathcal{A}^k(\zeta) - i\xi_j(\zeta)I\big) \oplus \bigoplus_{j=1}^{3-m_\zeta} A_d(0)\ker\big(\mathcal{A}^k(\zeta) - i\lambda_j I\big) = \ker Q_{j_0}(\zeta),$$

concluding the proof. □

The interest is now made on the frequencies created on the boundary and then lifted inside the domain. Recall that we considered a quasi-periodic boundary forcing term of frequencies φ/ε and ψ/ε, with $\varphi, \psi \in \mathbb{R}^d \setminus \{0\}$. In the following we will make restricting assumptions on φ and ψ in order to obtain a particular frequency configuration, eventually creating an instability.

By nonlinear interaction, frequencies φ and ψ on the boundary create the following lattice of frequencies on the boundary:

$$\mathcal{F}_b := \varphi\,\mathbb{Z} \oplus \psi\,\mathbb{Z}.$$

To avoid the complications induced by the glancing modes, we assume that there is no glancing frequency in $\mathcal{F}_b \setminus \{0\}$. This is a common assumption, see [CG10, CGW11].

Assumption 4. *We have*

$$\big(\mathcal{F}_b \setminus \{0\}\big) \cap \mathcal{G} = \emptyset.$$

To parameterize $\mathcal{F}_b$ we introduce the following subset of $\mathbb{Z}^2 \setminus \{0\}$:

$$\mathcal{B}_{\mathbb{Z}^2} := \left\{ (n_1, n_2) \in \mathbb{Z}^2 \setminus \{0\} \;\middle|\; \begin{array}{l} n_2 \wedge n_2 = 1, \\ n_1 > 0 \;\text{ or }\; n_1 = 0, n_2 > 0 \end{array} \right\},$$

of couples of coprime integers of which the first nonzero term is positive. Then, each frequency $\zeta := n_1\,\varphi + n_2\,\psi$ of $\mathcal{F}_b \setminus \{0\}$ is parameterized in a unique way by $\mathbf{n}_0 := (n_1^0, n_2^0) \in \mathcal{B}_{\mathbb{Z}^2}$ and $\lambda \in \mathbb{Z}^*$ such that $(n_1, n_2) = \lambda\,(n_1^0, n_2^0)$.

In the following, we will allow ourselves to alternate without mentioning it between the following representations of a frequency of $\zeta \in \mathcal{F}_b \setminus \{0\}$: $\mathbf{n} = (n_1, n_2)$ in $\mathbb{Z}^2 \setminus \{0\}$ such that $\zeta = n_1\,\varphi + n_2\,\psi$ and $\mathbf{n}_0 = (n_0^1, n_0^2)$ in $\mathcal{B}_{\mathbb{Z}^2}$ and λ in $\mathbb{Z}^*$ such that $\mathbf{n} = \lambda\,\mathbf{n}_0$.

Because of the hyperbolicity of the system, boundary frequencies ζ of $\mathcal{F}_b$ are lifted into frequencies (ζ, ξ) inside the domain, which must be characteristic frequencies due to polarization conditions. Furthermore, frequencies (ζ, ξ) with $\operatorname{Im}\xi < 0$ are excluded to obtain bounded solutions, and we have already discarded glancing frequencies by Assumption 4. Therefore, the set $\mathcal{F}$ of frequencies inside the domain is given by

$$\mathcal{F} := \{0\} \cup \big\{ \big(\zeta, \xi_j(\zeta)\big) \mid \zeta \in \mathcal{F}_b \setminus \{0\}\,, j \in \mathcal{C}(\zeta) \cup \mathcal{P}(\zeta) \big\}\,.$$

The following assumption details the configuration of frequencies which is assumed to hold in order to create an instability. It is a generalization to our case of [CW17, Assumptions 1.7 and 1.9], where the only frequency of the problem, φ, was supposed to be nonresonant, hyperbolic, and in Υ. In [CW17], the authors explain that allowing the boundary frequency φ to be resonant could lead to an over-determination of the system. Assumption 5 below requires in particular that frequencies φ and ψ are nonresonant[1], hyperbolic, and in Υ. We additionally assume two resonances between frequencies lifted from φ and ψ to hold, which will eventually allow us to create an instability.

Assumption 5. *There exists a frequency ν in $\mathcal{F}_b \setminus \{0\}$ defined by*

$$\lambda_\varphi\,\varphi + \lambda_\psi\,\psi + \nu = 0$$

with coprime integers $\lambda_\varphi, \lambda_\psi$ such that $(-\lambda_\varphi, -\lambda_\psi)$ is in $\mathcal{B}_{\mathbb{Z}^2}$, and such that the following conditions hold.

i.) Frequencies φ, ψ and ν are in the hyperbolic region $\mathcal{H}$.

ii.) Frequencies lifted from φ, ψ, ν, denoted by φ_j, ψ_j, ν_j, $j = 1, 2, 3$ are such that φ_j, ψ_j, ν_j, $j = 1, 3$ are incoming frequencies and φ_2, ψ_2, ν_2 are outgoing frequencies.

iii.) We have $\mathcal{F}_b \cap \Upsilon = \{\varphi, -\varphi, \psi, -\psi\}$ (so in particular we have $\varphi, \psi \in \Upsilon$ and $\nu \in \Xi_0 \setminus \Upsilon$).

iv.) The following two resonances hold:

$$\lambda_\varphi\,\varphi_1 + \lambda_\psi\,\psi_1 + \nu_2 = 0, \tag{1.12a}$$

$$\lambda_\varphi\,\varphi_3 + \lambda_\psi\,\psi_2 + \nu_2 = 0. \tag{1.12b}$$

v.) There is no other resonance between frequencies inside the domain. More precisely, if there exists a resonance relation of the form

$$\lambda_1\,\alpha_1 + \lambda_2\,\alpha_2 + \lambda_3\,\alpha_3 = 0,$$

[1] In the sense that two frequencies lifted from φ cannot resonate with each other, and the same for ψ.

with $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}^*$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{F} \setminus \{0\}$ noncolinear, then, there exists $\lambda \in \mathbb{Z}^*$, such that up to a renumbering, $\lambda_1 = \lambda\lambda_\varphi$, $\lambda_2 = \lambda\lambda_\psi$, $\lambda_3 = \lambda$ and ($\alpha_1 = \varphi_1$, $\alpha_2 = \psi_1$ and $\alpha_3 = \nu_2$) or ($\alpha_1 = \varphi_3$, $\alpha_2 = \psi_2$ and $\alpha_3 = \nu_2$).

Frequencies lifted inside from frequencies φ, ψ and ν are depicted in Figure 1. There is an amplification in the lifting of φ and ψ because these frequencies are in the region Υ where the Kreiss-Lopatinskii condition is not satisfied, in contrast to ν. Amplification arise since, for a frequency in Υ, there is an ascent of small amplitudes toward higher one, namely, a boundary source term of order $O(\varepsilon^{n+1})$ occurs in the equations for the profile of order $O(\varepsilon^n)$. Therefore, when amplification occurs, inside profiles lifted from boundary terms of order $O(\varepsilon^{n+1})$ are one order higher, namely $O(\varepsilon^n)$.

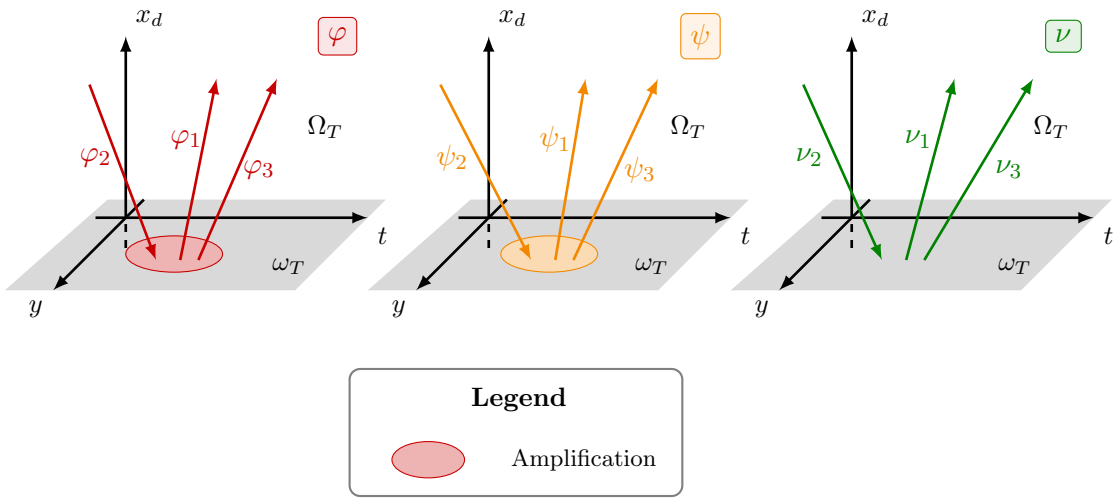

FIGURE 1. Frequencies lifted from φ, ψ and ν.

Remark 1.8.
- Point i.) of Assumption 5 asserts that each frequency φ, ψ and ν is lifted into three real characteristic frequencies inside the domain.
- Point ii.) of Assumption 5 implies in particular that the integer p, which is the rank of B and the dimension of the stable subspace $E_-(\zeta)$ for ζ in Ξ, is equal to 2.
- In relations (1.12), the numeration of the frequencies occurring in the resonances (1.12) is arbitrary. For the first resonance (1.12a), each of the two incoming frequencies lifted from φ and ψ can be chosen. It sets the numbering of the frequencies lifted from φ and ψ. Next, for the second resonance (1.12b), there is no choice, the incoming frequency lifted from φ which occurs in the resonance must be the one which did not occur in the first one, φ_3 in our fixed numbering, since we already required that $\lambda_\varphi \varphi_1 + \nu_2 = -\lambda_\psi \psi_1$, and ψ_2 is the only outgoing frequency associated with ψ.
- We choose a numbering of $\alpha_j(\zeta)$ for $\zeta = \varphi, \psi, \nu$ such that, for any $j = 1, 2, 3$, we have
$$\alpha_j(\zeta) = \zeta_j,$$
where the ζ_j are the hyperbolic frequencies defined in Assumption 5.
- The condition $(-\lambda_\varphi, -\lambda_\psi) \in \mathcal{B}_{\mathbb{Z}^2}$ is not restrictive and only relies on permuting the notation for φ and ψ or $-\varphi$ and φ. It is made to simplify notation in the following.

A useful notation is now introduced for the resonances.

Definition 1.9. *For $\zeta \in \{\varphi, \psi, \nu\}$, and $j = 1, 2, 3$, the set $\mathcal{R}(\zeta, j)$ is defined as the set of quadruples $(\zeta_1, \zeta_2, j_1, j_2)$ in $\{\varphi, \psi, \nu\}^2 \times \{1, 2, 3\}^2$ such that the following resonance holds*
$$\lambda_\zeta \, \alpha_j(\zeta) + \lambda_{\zeta_1} \, \alpha_{j_1}(\zeta_1) + \lambda_{\zeta_2} \, \alpha_{j_2}(\zeta_2) = 0,$$
where we have denoted $\lambda_\nu := 1$.

For example, we have, according to Assumption 5,

$$\mathcal{R}(\varphi,1)=\{(\psi,\nu,1,2),(\nu,\psi,2,1)\}\,,\quad \mathcal{R}(\varphi,2)=\emptyset,\quad \mathcal{R}(\varphi,3)=\{(\psi,\nu,3,2),(\nu,\psi,2,3)\}\,,\quad \text{and}$$

$$\mathcal{R}(\nu,2)=\{(\varphi,\psi,1,1),(\varphi,\psi,3,2),(\psi,\varphi,1,1),(\psi,\varphi,2,3)\}\,.$$

We conduct now a formal discussion on how the configuration of frequencies φ_j, ψ_j and ν_j, $j=1,2,3$ and the two resonances (1.12) are expected to create an instability, as represented in Figure 2. First, the boundary profiles $\varepsilon^2\, g^\varepsilon$ and $\varepsilon^3\, h^\varepsilon$ of frequencies φ and ψ in (1.1) create, because of the amplification due to the breaking of Kreiss-Lopatinskii condition for those frequencies, incoming interior profiles of frequencies φ_1, φ_3, and ψ_1, ψ_3 of orders respectively $O(\varepsilon)$ and $O(\varepsilon^2)$. Then because of the resonance relation (1.12a), the profiles associated with φ_1 and ψ_1 resonate to create a profile of outgoing frequency ν_2 and of order $O(\varepsilon^2)$[2]. This profile interacts, through resonance relation (1.12b), with the one of frequency φ_3 and of order $O(\varepsilon)$, which is lifted from the boundary forcing term $\varepsilon^2\, g^\varepsilon$. This resonance leads to a profile of frequency ψ_2 and amplitude $O(\varepsilon^2)$, which is an outgoing profile, so a reflection and thus an amplification occur. Indeed, it creates a boundary profile of frequency ψ and order $O(\varepsilon^2)$: we obtain instability. Indeed, this boundary profile creates, through amplification on the boundary for ψ, a profile of frequency ψ_1 and order $O(\varepsilon)$, which is one order higher than the profile of frequency ψ_1 we started with. Iterating this process leads to an explosion.

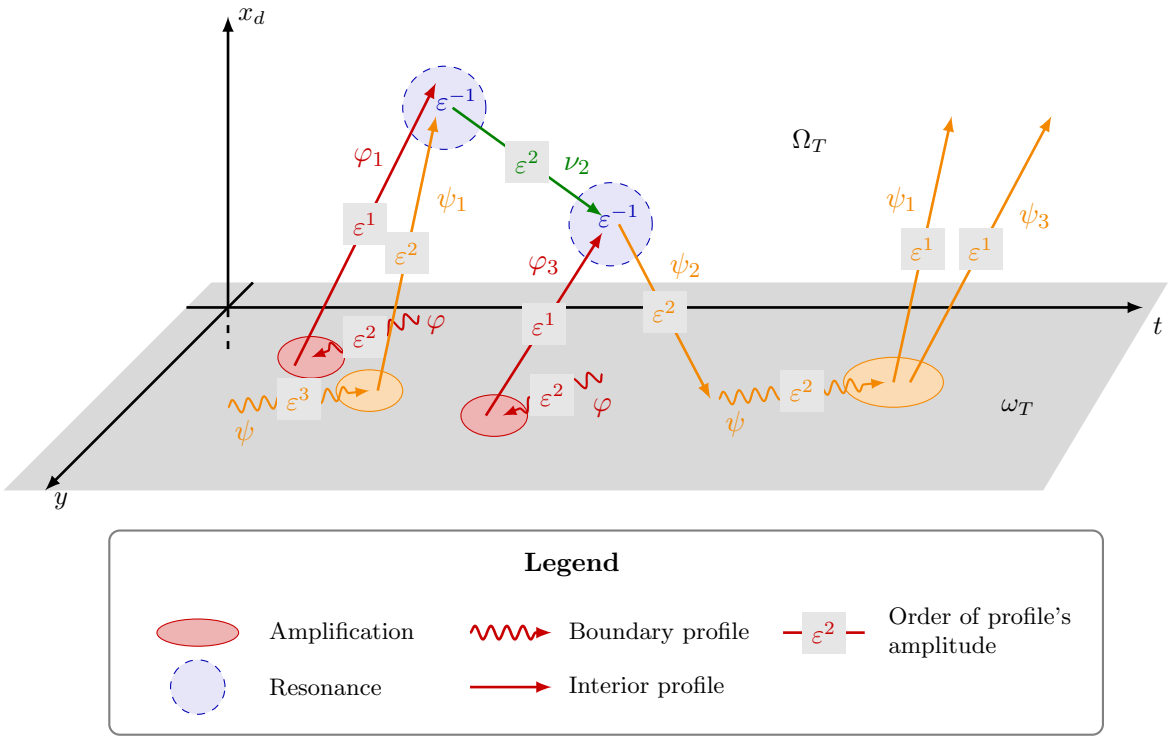

FIGURE 2. Creation of instability through amplification.

We make now a small divisors assumption, which is adapted from [CW17, Assumption 1.9]. This assumption is needed only for frequencies for which the uniform Kreiss-Lopatinskii condition is not satisfied, so, in our case, for φ and ψ. Analogously to [CW17, Assumption 1.9], it requires a polynomial control of the determinant of the symbol associated with combinations of incoming frequencies, using the fact that frequencies lifted from φ do not resonate, and the same for ψ. The formulation is simpler than the one of [CW17, Assumption 1.9] since in our case there is

[2]One of the quadratic term in the equations has a factor $1/\varepsilon$ in front of it, because the product $A_i(u^\varepsilon)\,\partial_i u^\varepsilon$ involves a derivative which counts as $1/\varepsilon$ for oscillating wave packets at frequency of order $1/\varepsilon$.

only two incoming frequencies, so the only possibility for a combination of it is $\lambda_1\varphi_1 + \lambda_3\varphi_3$, with $\lambda_1, \lambda_3 \in \mathbb{Z}^*$, and the same for ψ.

Assumption 6. *There exists a constant $C > 0$ and a real positive number m_0 such that, for $\zeta = \varphi, \psi$, for all $\lambda_1, \lambda_3 \in \mathbb{Z}^*$,*

$$\big|\det L\big(0, \lambda_1\,\zeta_1 + \lambda_3\,\zeta_3\big)\big| \geqslant C\big|(\lambda_1, \lambda_3)\big|^{-m_0}.$$

Finally, we define several vectors associated with the previously introduced eigenspaces. For ζ in $\mathcal{F}_b \setminus \{0\}$ and $j \in \mathcal{C}(\zeta)$, we denote by $r_{\zeta,j}$ a unit column vector of the one dimensional space $\ker L\big(0, \alpha_j(\zeta)\big)$, and $\ell_{\zeta,j}$ a row vector such that

$$\ell_{\zeta,j}\, L\big(0, \alpha_j(\zeta)\big) = 0 \tag{1.13}$$

with the following normalization: for all ζ in $\mathcal{F}_b \setminus \{0\}$ and for all j, j' in $\mathcal{C}(\zeta)$, we have

$$\ell_{\zeta,j'}\, A_d(0)\, r_{\zeta,j} = \delta^j_{j'}. \tag{1.14}$$

The projectors $P_j(\zeta)$, $Q_j(\zeta)$ (defined in Lemma 1.7) and the vectors $r_{\zeta,j}$ and $\ell_{\zeta,j}$ are chosen to be homogeneous of degree 0 with respect to ζ. Accordingly, we define the partial inverses $R_{\zeta,j}$, which satisfy, for ζ in $\mathcal{F}_b \setminus \{0\}$ represented by $\mathbf{n}_0, \lambda$ in $\mathcal{B}_{\mathbb{Z}^2} \times \mathbb{Z}^*$,

$$R_{\zeta,j}\, L\big(0, \alpha_j(\zeta)\big) = L\big(0, \alpha_j(\zeta)\big)\, R_{\zeta,j} = \lambda\,\big(I - P_{\zeta,j}\big). \tag{1.15}$$

Consider $\zeta \in \Upsilon$. Assumption 3 asserts that the space $\ker B \cap E_-(\zeta)$ is one dimensional, so we denote by e_ζ a unit vector in this space. Now, since, according to the same assumption, Υ is included in the hyperbolic region $\mathcal{H}$ and because of Proposition 1.4, we can decompose e_ζ as

$$e_\zeta = \sum_{j \in \mathcal{I}(\zeta)} e_{\zeta,j}, \tag{1.16}$$

with $e_{\zeta,j} \in \operatorname{Span} r_{\zeta,j}$ for $j \in \mathcal{I}(\zeta)$. We also denote by b_ζ a vector of $\mathbb{C}^2$ such that

$$B\, E_-(\zeta) = \big\{X \in \mathbb{C}^2 \,\big|\, b_\zeta \cdot X = 0\big\}, \tag{1.17}$$

that is, a nonzero vector of $\ker {}^tB_{|E_-(\zeta)}$, which is of dimension 1. Notation $b_\zeta \cdot X$ refers to the complex scalar product in $\mathbb{C}^2$.

Using vectors $r_{\zeta,j}$ and $\ell_{\zeta,j}$, we have the following lemma, analogous to the one of [Lax57]. The proof of this particular result can be found in [CGW11], and is recalled here for the sake of clarity.

Lemma 1.10 ([CGW11, Lemma 2.11])**.** *For $\zeta \in \mathcal{F}_b \setminus \{0\}$ and $j \in \mathcal{C}(\zeta)$, we have*

$$\ell_{\zeta,j}\, L(0, \partial_z)\, r_{\zeta,j} = X_{\alpha_j(\zeta)},$$

where $X_{\alpha_j(\zeta)}$ is the vector field defined in Definition 1.3.

Proof. Denote by k the integer between 1 and 3 such that, if $\alpha_j(\zeta) = (\tau, \eta, \xi)$, then $\tau = \tau_k(\eta, \xi)$. Since $\zeta \in \mathcal{F}_b \setminus \{0\}$, the frequency ζ is not glancing, so, according to definition 1.3, we have $\partial_\xi \tau_k(\eta, \xi) \neq 0$. Therefore, according to the implicit function theorem, the function $\zeta' \mapsto \xi_j(\zeta')$ is differentiable near ζ. Indeed, ξ_j is such that, if $\zeta' = (\tau', \eta')$,

$$\tau_k\big(\eta', \xi_j(\tau', \eta')\big) - \tau' = 0. \tag{1.18}$$

Therefore[3], seen as a function of ζ, the vector $r_{\zeta,j}$ is also differentiable with respect to ζ. Differentiating relation (1.18) even proves the following relations:

$$\partial_\tau \xi_j(\tau, \eta) = \frac{1}{\partial_\xi \tau_k(\eta, \xi)}, \qquad \partial_{\eta_p} \xi_j(\tau, \eta) = \frac{-\partial_{\eta_p} \tau_k(\eta, \xi)}{\partial_\xi \tau_k(\eta, \xi)}, \quad \forall p = 1, \dots, d-1. \tag{1.19}$$

Now, differentiating the relation

$$L\big(0, (\tau, \eta, \xi_j(\tau, \eta))\big)\, r_{\zeta,j} = 0$$

[3]Here we extend the definition of $r_{\zeta,j}$ to any frequency ζ in $\Xi \setminus \mathcal{G}$.

with respect to τ and η_p, $p = 1, \dots, d-1$, and multiplying on the left by $\ell_{\zeta,j}$, gives

$$\ell_{\zeta,j}\, r_{\zeta,j} + \partial_\tau \xi_j(\tau,\eta)\, \ell_{\zeta,j}\, A_d(0)\, r_{\zeta,j} = 0,$$

and, for $p = 1, \dots, d-1$,

$$\ell_{\zeta,j}\, A_p(0)\, r_{\zeta,j} + \partial_{\eta_p} \xi_j(\tau,\eta)\, \ell_{\zeta,j}\, A_d(0)\, r_{\zeta,j} = 0.$$

With relations (1.19), the result follows. □

The following result could be seen as an analogue to Lax lemma, for the boundary. Indeed, it asserts that a certain operator appearing in the boundary equations is actually a linear transport operator with constant velocity. The result is due to [CG10], and its technical proof is not recalled here.

Lemma 1.11 ([CG10, Proposition 3.5])**.** *Let $\zeta = \varphi, \psi$, and recall that κ is the scalar function of the weak Kreiss-Lopatinskii condition, see Assumption 3. Then, there exists a nonzero real scalar β_ζ such that*

$$b_\zeta \cdot B\left(R_{\zeta,1}\, L(0,\partial_z)\, e_{\zeta,1} + R_{\zeta,3}\, L(0,\partial_z)\, e_{\zeta,3}\right) = \beta_\zeta \left(\partial_\tau \kappa(\zeta)\, \partial_t + \sum_{j=1}^{d-1} \partial_{\eta_j} \kappa(\zeta)\, \partial_{x_j}\right).$$

Moreover, the coefficient $\partial_\tau \kappa(\zeta)$ is equal to 1.

Remark 1.12. In particular, the previous result ensures that the operator $b_\zeta \cdot B\left(R_{\zeta,1}\, L(0,\partial_z)\, e_{\zeta,1} + R_{\zeta,3}\, L(0,\partial_z)\, e_{\zeta,3}\right)$ is tangent to the boundary.

2. Derivation of the system

This section is devoted to the derivation of the general system studied in this article. We start by detailing the ansatz we choose here, and by displaying the WKB cascade associated with system (1.1). Then we proceed by trying to decouple this cascade for the profiles.

2.1. Ansatz and WKB cascade.

The ansatz for each amplitude U_n of (1.3) must allow to consider both oscillating modes (associated with characteristic frequencies $\alpha_j(\zeta)$ for $\zeta \in \mathcal{F}_b \setminus \{0\}$ and $j \in \mathcal{C}(\zeta)$) and evanescent modes (associated with evanescent frequencies $\alpha_j(\zeta)$ for $\zeta \in \mathcal{F}_b \setminus \{0\}$ and $j \in \mathcal{P}(\zeta)$). We define at this purpose the following spaces of profiles. We denote by $\mathbb{T}$ the one-dimensional torus.

Definition 2.1. *The space of evanescent profiles $\mathcal{P}_T^{\mathrm{ev}}$ is defined as the set of functions U^{ev} of $\mathcal{C}_b\big(\mathbb{R}^+_{\chi_d}, H^\infty(\Omega_T \times \mathbb{T}^2)\big)$ which converge to zero as χ_d goes to infinity.*

The space of oscillating profiles $\mathcal{P}_T^{\mathrm{osc}}$ is defined as the set of formal trigonometric functions in χ_d with values in the Sobolev space $H^\infty(\Omega_T \times \mathbb{T}^2)$, that is, formal series

$$U^{\mathrm{osc}}(z,\theta_1,\theta_2,\chi_d) = \sum_{\xi \in \mathbb{R}} U_\xi^{\mathrm{osc}}(z,\theta_1,\theta_2)\, e^{i\,\xi\,\chi_d},$$

with $U_\xi^{\mathrm{osc}} \in H^\infty(\Omega_T \times \mathbb{T}^2)$ for $\xi \in \mathbb{R}$.

Finally, $\mathcal{P}_T$ is defined as the direct sum

$$\mathcal{P}_T := \mathcal{P}_T^{\mathrm{osc}} \oplus \mathcal{P}_T^{\mathrm{ev}}.$$

The ansatz is the following: we look for an approximate solution

$$u^{\varepsilon,\mathrm{app}}(z) := v^\varepsilon\left(z, \frac{z'\cdot\varphi}{\varepsilon}, \frac{z'\cdot\psi}{\varepsilon}, \frac{x_d}{\varepsilon}\right)$$

where the formal series v^ε is given by

$$v^\varepsilon\big(z,\theta_1,\theta_2,\chi_d\big) := \sum_{n\geqslant 1} \varepsilon^n\, U_n\big(z,\theta_1,\theta_2,\chi_d\big), \tag{2.1}$$

where, for $n \geqslant 1$, U_n belongs to $\mathcal{P}_T$.

Formally plugging ansatz (2.1) in system (1.1), we obtain the following WKB cascade[4] (see [CW17]) for the profiles $(U_n)_{n\geqslant 1}$:

$$\mathcal{L}(\partial_\theta, \partial_{\chi_d})\, U_1 = 0 \tag{2.2a}$$

$$\mathcal{L}(\partial_\theta, \partial_{\chi_d})\, U_2 + L(0, \partial_z)\, U_1 + \mathcal{M}(U_1, U_1) = 0 \tag{2.2b}$$

$$\mathcal{L}(\partial_\theta, \partial_{\chi_d})\, U_{n+1} + L(0, \partial_z)\, U_n + \sum_{k=1}^{n} \mathcal{M}(U_{n-k+1}, U_k) + \sum_{k=1}^{n-1} \mathcal{N}(U_{n-k}, U_k) = 0, \tag{2.2c}$$

where (2.2c) should hold for any $n \geqslant 2$. In (2.2), the fast operator $\mathcal{L}(\partial_\theta, \partial_{\chi_d})$ and the quadratic operators $\mathcal{M}$ and $\mathcal{N}$ are defined by

$$\begin{aligned}
\mathcal{L}(\partial_\theta, \partial_{\chi_d}) &:= L(0, \varphi)\, \partial_{\theta_1} + L(0, \psi)\, \partial_{\theta_2} + A_d(0)\, \partial_{\chi_d}, \\
\mathcal{M}(u, v) &:= L_1(u, \varphi)\, \partial_{\theta_1} v + L_1(u, \psi)\, \partial_{\theta_2} v + dA_d(0)\, \cdot u\, \partial_{\chi_d} v \\
&= \sum_{k=1}^{d-1} dA_k(0) \cdot u \left(\varphi_k\, \partial_{\theta_1} + \psi_k\, \partial_{\theta_2}\right) v + dA_d(0) \cdot u\, \partial_{\chi_d} v, \\
\mathcal{N}(u, v) &:= L_1(u, \partial_z)\, v := \sum_{k=1}^{d} dA_k(0) \cdot u\, \partial_k v,
\end{aligned}$$

where we have denoted by, for X in $\mathbb{C}^3$ and $\zeta = (\zeta_0, \dots, \zeta_{d-1}) \in \mathbb{R}^d$,

$$L(0, \zeta) := L\big(0, (\zeta, 0)\big) = \sum_{k=1}^{d-1} \zeta_k A_k(0), \qquad L_1(X, \zeta) := \sum_{k=1}^{d-1} \zeta_k\, dA_k(0) \cdot X.$$

The boundary and initial conditions of (1.1) reads

$$B\left(U_1\right)_{|x_d, \chi_d = 0}(z', \theta_1, \theta_2) = 0, \qquad B\left(U_2\right)_{|x_d, \chi_d = 0}(z', \theta_1, \theta_2) = G(z', \theta_1) \tag{2.3a}$$

$$B\left(U_3\right)_{|x_d, \chi_d = 0}(z', \theta_1, \theta_2) = H(z', \theta_2), \qquad B\left(U_n\right)_{|x_d, \chi_d = 0}(z', \theta_1, \theta_2) = 0, \quad \forall n \geqslant 4, \tag{2.3b}$$

and

$$\left(U_n\right)_{|t\leqslant 0} = 0, \quad \forall n \geqslant 1. \tag{2.4}$$

The aim is now to decouple cascade (2.2). First we use polarization equation (2.2a) to obtain the form of the leading profile U_1, and proceed to show that the mean value U_1^* of U_1 is zero using evolution equation (2.2b). Then we need to determine the oscillating part of U_1. Equation (2.2b) leads to a transport equations for each mode. When the equation is outgoing (that is, when the frequency is outgoing), the transport equation can be solved with a source term eventually depending on other leading profiles, due to resonances. When it is incoming (i.e. when the frequency is incoming), we need to determine a boundary condition from first equation of (2.3a). Two cases may occur. If the associated boundary frequency ζ is not in Υ, that is, for $\zeta \neq \varphi, \psi$, we can write boundary condition (2.3a) for ζ as $B\,X = F$, where source term F depends on the trace of the outgoing leading profile for this boundary frequency ζ, and where X (containing traces of incoming leading profiles) belongs to $E_-(\zeta)$. Since B is invertible on $E_-(\zeta)$ according to Assumption 3 and the fact that $\zeta \notin \Upsilon$, this boundary condition $B\,X = F$ leads to a boundary condition for traces of incoming leading profiles. The second case is more complicated. If $\zeta \in \Upsilon$, that is, if $\zeta = \varphi, \psi$, matrix B is no longer invertible on $E_-(\zeta)$. Therefore, boundary condition $B\,X = F$ cannot be inverted, and leads to both a compatibility condition (which we shall see will over-determine the system), and expressions for traces of incoming leading profiles for φ and ψ depending on unknown scalar functions a^1_φ and a^1_ψ. Then, to determine these functions $a^1_\varphi$and a^1_ψ, we need investigate first corrector U_2.

According to equation (2.2b), the first corrector U_2 is not polarized. But this equation allows us to determine of formula for its nonpolarized part, depending on the leading profiles. We write

[4]We have used here the assumption that coefficients A_j are affine maps.

second equation of boundary condition (2.3a) with these expressions for the nonpolarized parts which leads to equations on the traces of the nonpolarized parts, and therefore, equations on the traces of leading profiles, namely a^1_φ and a^1_ψ. However, the system of equations obtained is still not closed, since equations on a^1_φ and a^1_ψ involve traces of the first corrector U_2.

The next step is to obtain equations on the polarized part of the first corrector U_2, which is achieved using equation (2.2c) for $n = 3$. Once again, depending on the frequency, we obtain incoming or outgoing transport equations with source term depending on leading profile and first corrector. For incoming equations, when the associated frequency ζ is not in Υ, boundary condition (2.3a) can be inverted to obtain a closed system. Otherwise, when ζ belongs to Υ, the same arguments as for the leading profiles leads to compatibility conditions (that are these time always satisfied by previous construction) and expressions for traces of incoming first corrector profiles for φ and ψ depending on unknown scalar functions a^2_φ and a^2_ψ.

Investigating the nonpolarized part of the second corrector U_3 leads, in its turn, to equations on a^2_φ and a^2_ψ, depending once again on trace of the second corrector U_2, preventing to close the system. This method applies recursively to any order.

2.2. **Rewriting the equations: leading profile and first corrector.** This subsection is devoted to the almost-decoupling of the cascade (2.2), (2.3) and (2.4). The computations are, for the most part of it, formal. Except for the leading profile, we will not detail the obtaining of formulas for the evanescent part, as it will not be interesting for the instability analysis, since all three frequencies φ, ψ and ν are hyperbolic.

2.2.1. *Leading profile.* We start by deriving the polarization condition for U_1 from (2.2a), recalling the analysis of [Les07]. If we write U_1 in $\mathcal{P}_T$ as

$$\begin{aligned} U_1(z,\theta,\chi_d) &= U_1^{\mathrm{osc}}(z,\theta,\chi_d) + U_1^{\mathrm{ev}}(z,\theta,\chi_d) \\ &= \sum_{\mathbf{n}\in\mathbb{Z}^m}\sum_{\xi\in\mathbb{R}} U^{1,\mathrm{osc}}_{\mathbf{n},\xi}(z)\, e^{i\,\mathbf{n}\cdot\theta}\, e^{i\,\xi\,\chi_d} + \sum_{\mathbf{n}\in\mathbb{Z}^m} U^{1,\mathrm{ev}}_{\mathbf{n}}(z,\chi_d)\, e^{i\,\mathbf{n}\cdot\theta}, \end{aligned}$$

equation (2.2a) reads

$$\begin{aligned} &\sum_{\mathbf{n}\in\mathbb{Z}^2}\sum_{\xi\in\mathbb{R}} iL\big(0,(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)\big)\, U^{1,\mathrm{osc}}_{\mathbf{n},\xi}(z)\, e^{i\,\mathbf{n}\cdot\theta}\, e^{i\,\xi\,\chi_d} \\ &\qquad + \sum_{\mathbf{n}\in\mathbb{Z}^2} \big\{ iL(0,\mathbf{n}\cdot\boldsymbol{\zeta}) + A_d(0)\,\partial_{\chi_d}\big\}\, U^{1,\mathrm{ev}}_{\mathbf{n}}(z,\chi_d)\, e^{i\,\mathbf{n}\cdot\theta} = 0. \end{aligned}$$

Therefore, on one hand, for the oscillating part, we get $L\big(0,(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)\big)\, U^{1,\mathrm{osc}}_{\mathbf{n},\xi} = 0$ for every $\mathbf{n}\in\mathbb{Z}^2\setminus\{0\}$ and $\xi\in\mathbb{R}$, so, if $(\mathbf{n}\cdot\boldsymbol{\zeta},\xi)$ is noncharacteristic, $U^{1,\mathrm{osc}}_{\mathbf{n},\xi} = 0$ and if $\xi = \xi_j(\mathbf{n}\cdot\boldsymbol{\zeta})$ for some $j\in\mathcal{C}(\mathbf{n}\cdot\boldsymbol{\zeta})$, we find that $U^{1,\mathrm{osc}}_{\mathbf{n},\xi}$ belongs to $\ker L\big(0,\alpha_j(\mathbf{n}\cdot\boldsymbol{\zeta})\big) = \operatorname{Span} r_{\mathbf{n}\cdot\boldsymbol{\zeta},j}$. Thus we write

$$U^{1,\mathrm{osc}}_{\mathbf{n},\xi} = \sigma^1_{\mathbf{n}_0,j,\lambda}\, r_{\mathbf{n}_0\cdot\boldsymbol{\zeta},j},$$

if $\mathbf{n} = \lambda\,\mathbf{n}_0$ with $\mathbf{n}_0\in\mathcal{B}_{\mathbb{Z}^2}$ and $\lambda\in\mathbb{Z}^*$, and where $\sigma^1_{\mathbf{n}_0,j,\lambda}$ is a scalar function of Ω_T. On the other hand, for the evanescent part, we get $U^{1,\mathrm{ev}}_0 = 0$, and, for $\mathbf{n}\in\mathbb{Z}^2\setminus\{0\}$, multiplying by $A_d(0)^{-1}$,

$$\partial_{\chi_d} U^{1,\mathrm{ev}}_{\mathbf{n}} - \mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})\, U^{1,\mathrm{ev}}_{\mathbf{n}} = 0.$$

Solving this differential equation in $\mathcal{P}^{\mathrm{ev}}_T$ leads to

$$U^{1,\mathrm{ev}}_{\mathbf{n}}(z,\chi_d) = e^{\chi_d\mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})}\,\Pi^e(\mathbf{n}\cdot\boldsymbol{\zeta})\, U^{1,\mathrm{ev}}_{\mathbf{n}}(z,0).$$

In short, polarization equation (2.2a) asserts that U_1 reads

$$\begin{aligned} U_1(z,\theta,\chi_d) = U_1^*(z) &+ \sum_{\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^2}}\sum_{j\in\mathcal{C}(\mathbf{n}\cdot\boldsymbol{\zeta})}\sum_{\lambda\in\mathbb{Z}^*} \sigma^1_{\mathbf{n},j,\lambda}(z)\, e^{i\,\lambda\,\mathbf{n}\cdot\theta}\, e^{i\,\lambda\,\xi_j(\mathbf{n}\cdot\boldsymbol{\zeta})\,\chi_d}\, r_{\mathbf{n}\cdot\boldsymbol{\zeta},j} \\ &+ \sum_{\mathbf{n}\in\mathbb{Z}^2\setminus\{0\}} e^{\chi_d\mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})}\,\Pi^e(\mathbf{n}\cdot\boldsymbol{\zeta})\, U^{1,\mathrm{ev}}_{\mathbf{n}}(z,0)\, e^{i\,\mathbf{n}\cdot\theta}. \end{aligned} \tag{2.5}$$

We start by showing that the mean value U_1^* is zero, using equation (2.2b). The oscillating part of $L(0,\partial_z)\,U_1+\mathcal{M}(U_1,U_1)$ is given by $L(0,\partial_z)\,U_1^{\rm osc}+\mathcal{M}(U_1^{\rm osc},U_1^{\rm osc})$ and according to (2.5) and the expression of the quadratic operator $\mathcal{M}$, the latter reads

$$L(0,\partial_z)\,U_1^* + \sum_{\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^2}}\sum_{j\in\mathcal{C}(\mathbf{n}\cdot\boldsymbol{\zeta})}\sum_{\lambda\in\mathbb{Z}^*} L(0,\partial_z)\,\sigma^1_{\mathbf{n},j,\lambda}\,e^{i\lambda\,\mathbf{n}\cdot\theta}\,e^{i\lambda\xi_j(\mathbf{n}\cdot\boldsymbol{\zeta})\chi_d}\,r_{\mathbf{n}\cdot\boldsymbol{\zeta},j} \tag{2.6a}$$

$$+\sum_{\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^2}}\sum_{j\in\mathcal{C}(\mathbf{n}\cdot\boldsymbol{\zeta})}\sum_{\lambda\in\mathbb{Z}^*} L_1\big(U_1^*,i\,\lambda\,\alpha_j(\mathbf{n}\cdot\boldsymbol{\zeta})\big)\,r_{\mathbf{n}\cdot\boldsymbol{\zeta},j}\,\sigma^1_{\mathbf{n},j,\lambda}\,e^{i\lambda\,\mathbf{n}\cdot\theta}\,e^{i\lambda\xi_j(\mathbf{n}\cdot\boldsymbol{\zeta})\chi_d} \tag{2.6b}$$

$$+\sum_{\mathbf{n}_1,\mathbf{n}_2\in\mathcal{B}_{\mathbb{Z}^2}}\sum_{\substack{j_1\in\mathcal{C}(\mathbf{n}_1\cdot\boldsymbol{\zeta})\\ j_2\in\mathcal{C}(\mathbf{n}_2\cdot\boldsymbol{\zeta})}}\sum_{\lambda_1,\lambda_2\in\mathbb{Z}^*} L_1\big(r_{\mathbf{n}_1\cdot\boldsymbol{\zeta},j_1},i\,\lambda_2\,\alpha_{j_2}(\mathbf{n}_2\cdot\boldsymbol{\zeta})\big)\,r_{\mathbf{n}_2\cdot\boldsymbol{\zeta},j_2} \tag{2.6c}$$
$$\sigma^1_{\mathbf{n}_1,j_1,\lambda_1}\,\sigma^1_{\mathbf{n}_2,j_2,\lambda_2}\,e^{i\,(\lambda_1\mathbf{n}_1+\lambda_2\mathbf{n}_2)\cdot\theta}\,e^{i\,(\lambda_1\xi_{j_1}(\mathbf{n}_1\cdot\boldsymbol{\zeta})+\lambda_2\xi_{j_2}(\mathbf{n}_2\cdot\boldsymbol{\zeta}))\,\chi_d},$$

where, for $\alpha=(\alpha_0,\alpha_1,\dots,\alpha_d)\in\mathbb{R}^{d+1}$ and $X\in\mathbb{C}^3$, we have denoted

$$L_1(X,\alpha):=\sum_{k=1}^{d}\alpha_k\,dA_k(0)\cdot X.$$

We now isolate the nonoscillating terms in (2.6), to obtain a system satisfied by the mean value U_1^*. In equation (2.6), the terms in the sums in (2.6a) and (2.6b) are always oscillating since $\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^2}$. As for them, the terms in the sum in (2.6c)are not oscillating if and only if $\lambda_1\mathbf{n}_1+\lambda_2\mathbf{n}_2=0$ and $\lambda_1\,\xi_{j_1}(\mathbf{n}_1\cdot\boldsymbol{\zeta})+\lambda_2\,\xi_{j_2}(\mathbf{n}_2\cdot\boldsymbol{\zeta})=0$, that is, if $\mathbf{n}_1=\mathbf{n}_2$, $\lambda_1=-\lambda_2$ and $j_1=j_2$. Therefore, we deduce from (2.6) that we have

$$L(0,\partial_z)\,U_1^*+\sum_{\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^2}}\sum_{j\in\mathcal{C}(\mathbf{n}\cdot\boldsymbol{\zeta})}\sum_{\lambda\in\mathbb{Z}^*} L_1\big(r_{\mathbf{n}\cdot\boldsymbol{\zeta},j},-i\,\lambda\,\alpha_j(\mathbf{n}\cdot\boldsymbol{\zeta})\big)\,r_{\mathbf{n}\cdot\boldsymbol{\zeta},j}\,\sigma^1_{\mathbf{n},j,\lambda}\,\sigma^1_{\mathbf{n},j,-\lambda}=0. \tag{2.7}$$

By a change of variable $\lambda=-\lambda$ we prove that the second term in the left-hand side of (2.7) is actually zero, so we have the following linear constant coefficient equation

$$L(0,\partial_z)\,U_1^*=0.$$

With the following boundary and initial conditions obtained from (2.3a) and (2.4),

$$B\big(U_1^*\big)_{|x_d,\chi_d=0}=0,\qquad \big(U_1^*\big)_{|t\leqslant 0}=0,$$

we get that the mean value U_1^* satisfies a system which is weakly well-posed, see [Cou05], with zero source term, boundary forcing term and initial term, so the mean value U_1^* is zero.

Since U_1^* is zero, equation (2.2b) now reads, for each nonzero characteristic mode $\lambda\,\alpha_j(\mathbf{n}\cdot\boldsymbol{\zeta})$, with $\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^2}$, $j\in\mathcal{C}(\mathbf{n}\cdot\boldsymbol{\zeta})$ and $\lambda\in\mathbb{Z}^*$,

$$i\,L\big(0,\alpha_j(\mathbf{n}\cdot\boldsymbol{\zeta})\big)\,U^{2,\rm osc}_{\mathbf{n},\xi_j(\mathbf{n}\cdot\boldsymbol{\zeta})}+L(0,\partial_z)\,\sigma^1_{\mathbf{n},j,\lambda}\,r_{\mathbf{n}\cdot\boldsymbol{\zeta},j} \tag{2.8}$$
$$+\sum_{(\mathbf{n}_1,\mathbf{n}_2,j_1,j_2,\lambda_1,\lambda_2)} L_1\big(r_{\mathbf{n}_1\cdot\boldsymbol{\zeta},j_1},i\,\lambda_2\,\alpha_{j_2}(\mathbf{n}_2\cdot\boldsymbol{\zeta})\big)\,r_{\mathbf{n}_2\cdot\boldsymbol{\zeta},j_2}\sigma^1_{\mathbf{n}_1,j_1,\lambda_1}\,\sigma^1_{\mathbf{n}_2,j_2,\lambda_2}=0,$$

where the sum is over the set of 6-tuples $(\mathbf{n}_1,\mathbf{n}_2,j_1,j_2,\lambda_1,\lambda_2)$ in $\big(\mathcal{B}_{\mathbb{Z}^2}\big)^2\times\mathcal{C}(\mathbf{n}_1\cdot\boldsymbol{\zeta})\times\mathcal{C}(\mathbf{n}_2\cdot\boldsymbol{\zeta})\times(\mathbb{Z}^*)^2$ such that $\lambda_1\alpha_{j_1}(\mathbf{n}_1\cdot\boldsymbol{\zeta})+\lambda_2\alpha_{j_2}(\mathbf{n}_2\cdot\boldsymbol{\zeta})=\lambda\alpha_j(\mathbf{n}\cdot\boldsymbol{\zeta})$. There are two possibilities for that to happen.

- Either frequencies $\lambda_1\,\alpha_{j_1}(\mathbf{n}_1\cdot\boldsymbol{\zeta})$ and $\lambda_2\,\alpha_{j_2}(\mathbf{n}_2\cdot\boldsymbol{\zeta})$ are colinear (therefore colinear to $\alpha_j(\mathbf{n}\cdot\boldsymbol{\zeta})$), that is to say $\mathbf{n}_1=\mathbf{n}_2=\mathbf{n}$ and $j_1=j_2=j$. This is called *self-interaction* of frequency $\alpha_j(\mathbf{n}\cdot\boldsymbol{\zeta})$ with itself. Note that the obtained frequency $\lambda_1\alpha_{j_1}(\mathbf{n}_1\cdot\boldsymbol{\zeta})+\lambda_2\alpha_{j_2}(\mathbf{n}_2\cdot\boldsymbol{\zeta})$ is then always real characteristic.
- Or frequencies $\lambda_1\,\alpha_{j_1}(\mathbf{n}_1\cdot\boldsymbol{\zeta})$ and $\lambda_2\,\alpha_{j_2}(\mathbf{n}_2\cdot\boldsymbol{\zeta})$ are noncolinear, in which case a true resonance in the sense of Assumption 5 occurs, namely (1.12a) or (1.12b). For example, if $\alpha_j(\mathbf{n}\cdot\boldsymbol{\zeta})=\psi_2$ (i.e. if $\mathbf{n}=(0,1)$ and $j=2$), then according to Assumption 5, it implies that $\lambda=k\lambda_\psi$ for some $k\in\mathbb{Z}^*$ and, up to a permutation, $\mathbf{n}_1=(1,0)$, $j_1=3$, $\lambda_1=-k\lambda_\psi$ and $\mathbf{n}_2=(-\lambda_\varphi,-\lambda_\psi)$, $j_2=2$, $\lambda_2=-k$.

Recall that, for a frequency $\lambda\zeta = \lambda\,\mathbf{n}\cdot\boldsymbol{\zeta} \in \mathcal{F}_b \setminus \{0\}$ with $\mathbf{n} \in \mathcal{B}_{\mathbb{Z}^2}$ and $\lambda \in \mathbb{Z}^*$, we alternate from the representations $\lambda\zeta$ and $(\lambda, \mathbf{n})$, so we shall denote

$$\sigma^1_{\zeta,j,\lambda} := \sigma^1_{\mathbf{n},j,\lambda}, \quad \forall j = 1,2,3, \forall \lambda \in \mathbb{Z}^*.$$

According to the previous analysis, we can now write the system satisfied by the leading profiles. For example, for ψ_2 which is involved in resonance (1.12b), multiplying equation (2.8) for $\mathbf{n} = (0,1)$ and $j = 2$ by the vector $\ell_{\psi,2}$ cancels the first term of (2.8), according to (1.13). Thus we obtain, for $\lambda \in \mathbb{Z}^*$,

$$\begin{aligned}\ell_{\psi,2}\,L(0,\partial_z)\,r_{\psi,2}\,\sigma^1_{\psi,2,\lambda} + \ell_{\psi,2}\,L_1\big(r_{\psi,2},\psi_2\big)\,r_{\psi,2}\sum_{\lambda_1+\lambda_2=\lambda} i\lambda_2\,\sigma^1_{\psi,2,\lambda_1}\,\sigma^1_{\psi,2,\lambda_2}\\ + \mathbb{1}_{\lambda=k\lambda_\psi}\ell_{\psi,2}\Big\{L_1\big(r_{\varphi,3},-\nu_2\big)\,r_{\nu,2} + L_1\big(r_{\nu,2},-\lambda_\varphi\varphi_3\big)\,r_{\varphi,3}\Big\}\,ik\,\sigma^1_{\varphi,3,-k\lambda_\varphi}\,\sigma^1_{\nu,2,-k} = 0.\end{aligned}$$

In the left-hand side of the previous equation, the first term, the transport one, corresponds to the second term of the left-hand side of (2.8), the second one, the Burgers type term corresponds to the self-interaction part of the third term of the left-hand side of (2.8), while the last one, the resonant term, corresponds to the resonance part of the third term of the left-hand side of (2.8). This splitting between transport, self-interaction and resonance terms can be generalized to any frequency. For $\zeta = \varphi, \psi, \nu$, $j = 1,2,3$ and $\lambda \in \mathbb{Z}^*$, we have

$$\begin{aligned}X_{\alpha_j(\zeta)}\,\sigma^1_{\zeta,j,\lambda} + D_{\zeta,j}\sum_{\lambda_1+\lambda_2=\lambda} i\lambda_2\,\sigma^1_{\zeta,j,\lambda_1}\,\sigma^1_{\zeta,j,\lambda_2}\\ + \mathbb{1}_{\lambda=k\lambda_\zeta}\sum_{\substack{(\zeta_1,\zeta_2,j_1,j_2)\\ \in\mathcal{R}(\zeta,j)}} J^{\zeta_1,j_1}_{\zeta_2,j_2}\,ik\,\sigma^1_{\zeta_1,j_1,-k\lambda_{\zeta_1}}\,\sigma^1_{\zeta_2,j_2,-k\lambda_{\zeta_2}} = 0,\end{aligned} \tag{2.9a}$$

and for other frequencies $\zeta \in \mathcal{F}_b \setminus \{0,\varphi,\psi,\nu\}$, $j \in \mathcal{C}(\zeta)$ and $\lambda \in \mathbb{Z}^*$,

$$X_{\alpha_j(\zeta)}\,\sigma^1_{\zeta,j,\lambda} + D_{\zeta,j}\sum_{\lambda_1+\lambda_2=\lambda} i\lambda_2\,\sigma^1_{\zeta,j,\lambda_1}\,\sigma^1_{\zeta,j,\lambda_2} = 0. \tag{2.9b}$$

We have denoted, for $\zeta \in \mathcal{F}_b$ and $j \in \mathcal{C}(\zeta)$, the vector field $X_{\alpha_j(\zeta)}$ and the self-interaction coefficient $D_{\zeta,j}$ as

$$X_{\alpha_j(\zeta)} := \ell_{\zeta,j}\,L(0,\partial_z)\,r_{\zeta,j}, \qquad D_{\zeta,j} := \ell_{\zeta,j}\,L_1\big(r_{\zeta,j},\alpha_j(\zeta)\big)\,r_{\zeta,j}, \tag{2.10a}$$

and, for $(\zeta_1,\zeta_2,j_1,j_2) \in \mathcal{R}(\zeta,j)$ (in the case $\zeta = \varphi,\psi,\nu$), the resonance coefficient $J^{\zeta_1,j_1}_{\zeta_2,j_2}$ as

$$J^{\zeta_1,j_1}_{\zeta_2,j_2} := \ell_{\zeta,j}\,L_1\big(r_{\zeta_1,j_1},\lambda_{\zeta_2}\,\alpha_{j_2}(\zeta_2)\big)\,r_{\zeta_2,j_2}. \tag{2.10b}$$

According to the Lax Lemma 1.10, the operator $\ell_{\zeta,j}\,L(0,\partial_z)\,r_{\zeta,j}$ of (2.10a) is equal to the vector field (1.4) which has already been denoted by $X_{\alpha_j(\zeta)}$, so the notation is coherent. In the following, for $\zeta \in \mathcal{F}_b \setminus \{0\}$ and $j \in \mathcal{C}(\zeta)$, we denote by $X_{\zeta,j}$ the vector field $X_{\alpha_j(\zeta)}$ and $\mathbf{v}_{\zeta,j} := \mathbf{v}_{\alpha_j(\zeta)}$ the velocity vector associated with it.

For all frequencies $\zeta = \mathbf{n}\cdot\boldsymbol{\zeta}$ with $\mathbf{n} \in \mathcal{B}_{\mathbb{Z}^2}$ except for $\zeta = \varphi,\psi,\nu$, and for all $j \in \mathcal{C}(\zeta)$, since the frequency $\alpha_j(\zeta)$ does not occur in any resonance, if we denote by $\sigma^1_{\zeta,j}$ the series

$$\sigma^1_{\zeta,j}(z,\Theta) := \sum_{\lambda\in\mathbb{Z}^*}\sigma^1_{\zeta,j,\lambda}(z)\,e^{i\lambda\Theta},$$

then, according to (2.9b), we have the Burgers type equation

$$X_{\zeta,j}\,\sigma^1_{\zeta,j} + D_{\zeta,j}\,\sigma^1_{\zeta,j}\partial_\Theta\sigma^1_{\zeta,j} = 0, \tag{2.11}$$

along with the initial condition $(\sigma^1_{\zeta,j})_{t=0} = 0$, which is a nonlinear scalar transport equation in the half space Ω_T. If the frequency $\alpha_j(\zeta)$ is outgoing, i.e. if the last component of $\mathbf{v}_{\zeta,j}$ is positive, there is no need for a boundary condition, so we deduce from (2.11) that $\sigma^1_{\zeta,j}$ is zero. Therefore, for $\lambda \in \mathbb{Z}^*$, we have $\sigma^1_{\zeta,j,\lambda} = 0$. The same arguments can be applied for the outgoing frequency φ_2 since there is no resonance for it.

If $\alpha_j(\zeta)$ is incoming we need in this case a boundary condition to determine the trace of $\sigma^1_{\zeta,k}(z,\Theta)$ at $x_d=0$. From boundary condition (2.3a) for the frequency $\lambda\zeta$, $\lambda\in\mathbb{Z}^*$, writing of U_1 (2.5) and the fact that all outgoing frequencies for ζ have been proved to be zero, we deduce that

$$B\sum_{j\in\mathcal{I}(\zeta)}\left(\sigma^1_{\zeta,j,\lambda}\right)_{|x_d=0}r_{\zeta,j}+B\,\Pi^e(\zeta)\left(U^{1,\mathrm{ev}}_{\lambda\zeta}\right)_{|x_d,\chi_d=0}=0. \tag{2.12}$$

Since $\lambda\zeta\in\mathcal{F}_b\setminus\Upsilon$ (because for now we consider $\zeta\neq\varphi,\psi$) and, according to decomposition (1.6) of the stable subspace $E_-(\zeta)$, the vector in (2.12) to which matrix B applies lies in $E_-(\lambda\zeta)$, on which B is invertible, according to Assumption 3. Therefore we deduce from the previous equation (2.12), using vectors $\ell_{\zeta,j}$, that

$$\left(\sigma^1_{\zeta,j}\right)_{|x_d=0}=0,\quad\forall j\in\mathcal{I}(\zeta). \tag{2.13}$$

Along with (2.11) we get $\sigma^1_{\zeta,j,\lambda}=0$ for all $\lambda\in\mathbb{Z}^*$. We have therefore proven that for all frequencies $\zeta=\mathbf{n}\cdot\boldsymbol{\zeta}$ with $\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^2}$ except for $\zeta=\varphi,\psi,\nu$, and for all $j\in\mathcal{C}(\zeta)$ and $\lambda\in\mathbb{Z}^*$, we have

$$\sigma^1_{\zeta,j,\lambda}=0.$$

In the same way we deduce from (2.12), using decomposition (1.6),

$$\left(U^{1,\mathrm{ev}}_{\lambda\zeta}\right)_{|x_d,\chi_d=0}=0,$$

so, according to (2.5), we can set the evanescent part $U^{1,\mathrm{ev}}$ of U^1 to be zero.

We now need to determine boundary conditions for the incoming frequencies φ_j, ψ_j and ν_j for $j=1,3$ as well. For the frequency ν we obtain, in the same fashion as before, since ν is in the hyperbolic region $\mathcal{H}$, for $\lambda\in\mathbb{Z}^*$,

$$B\left(\left(\sigma^1_{\nu,1,\lambda}\right)_{|x_d=0}r_{\nu,1}+\left(\sigma^1_{\nu,2,\lambda}\right)_{|x_d=0}r_{\nu,2}+\left(\sigma^1_{\nu,3,\lambda}\right)_{|x_d=0}r_{\nu,3}\right)=0,$$

so, for $j=1,3$, according to the normalization (1.14),

$$\left(\sigma^1_{\nu,j,\lambda}\right)_{|x_d=0}=-\ell_{\nu_j}\,A_d(0)\left(B_{|E_-(\nu)}\right)^{-1}B\left(\sigma^1_{\nu,2,\lambda}\right)_{|x_d=0}r_{\nu,2}.$$

We denote by $\mu_{\nu,j}$, for $j=1,3$, the coefficient

$$\mu_{\nu,j}:=-\ell_{\nu,j}\,A_d(0)\left(B_{|E_-(\nu)}\right)^{-1}B\,r_{\nu,2}, \tag{2.14}$$

so that, for $j=1,3$, we have

$$\left(\sigma^1_{\nu,j,\lambda}\right)_{|x_d=0}r_{\nu,j}=\mu_{\nu,j}\left(\sigma^1_{\nu,2,\lambda}\right)_{|x_d=0}r_{\nu,j}. \tag{2.15}$$

For φ we have, in a similar manner, since $\sigma^1_{\varphi,2,\lambda}$ is zero for $\lambda\in\mathbb{Z}^*$,

$$B\left(\left(\sigma^1_{\varphi,1,\lambda}\right)_{|x_d=0}r_{\varphi,1}+\left(\sigma^1_{\varphi,3,\lambda}\right)_{|x_d=0}r_{\varphi,3}\right)=0,$$

for every λ in $\mathbb{Z}^*$, so, according to (1.6), the vector in factor of B in the left-hand side belongs to $\ker B\cap E_-(\varphi)$. But since φ is in Υ, the latter space is of dimension 1 and reads $\operatorname{Span}e_\varphi$, so there exists a scalar function $a_{\varphi,\lambda}$ of ω_T such that,

$$\left(\sigma^1_{\varphi,1,\lambda}\right)_{|x_d=0}r_{\varphi,1}+\left(\sigma^1_{\varphi,3,\lambda}\right)_{|x_d=0}r_{\varphi,3}=a^1_{\varphi,\lambda}\,e_\varphi.$$

Therefore, according to decomposition (1.16) of e_ζ, for $j=1,3$,

$$\left(\sigma^1_{\varphi,j,\lambda}\right)_{|x_d=0}r_{\varphi,j}=a^1_{\varphi,\lambda}\,e_{\varphi,j}. \tag{2.16}$$

Finally the case of the phase ψ gather the two previous ones. We have, for $\lambda\in\mathbb{Z}^*$,

$$B\left(\left(\sigma^1_{\psi,1,\lambda}\right)_{|x_d=0}r_{\psi,1}+\left(\sigma^1_{\psi,2,\lambda}\right)_{|x_d=0}r_{\psi,2}+\left(\sigma^1_{\psi,3,\lambda}\right)_{|x_d=0}r_{\psi,3}\right)=0.$$

In particular, because of (1.6), it implies

$$B\big(\sigma^1_{\psi,2,\lambda}\big)_{|x_d=0}\, r_{\psi,2} \in \operatorname{Im} B_{|E_-(\psi)} = \Big(\ker {}^tB_{|E_-(\psi)}\Big)^\perp.$$

Therefore, according to the definition of b_ψ, the following necessary condition follows:

$$b_\psi \cdot B\big(\sigma^1_{\psi,2,\lambda}\big)_{|x_d=0}\, r_{\psi,2} = 0.$$

But since the scalar $b_\psi \cdot B\, r_{\psi,2}$ is not zero[5], we necessarily have

$$\big(\sigma^1_{\psi,2,\lambda}\big)_{|x_d=0} = 0. \tag{2.17}$$

When it is satisfied we can write, in the same way as for φ, for $j = 1, 3$,

$$\big(\sigma^1_{\psi,j,\lambda}\big)_{|x_d=0}\, r_{\psi,j} = a^1_{\psi,\lambda}\, e_{\psi,j}, \tag{2.18}$$

with $a_{\psi,\lambda}$ a scalar function of ω_T.

At this point we have obtained an constant coefficient equation for U_1^*, and transport equations (2.9a) and (2.11), associated with (when incoming) boundary conditions (2.13), (2.15), (2.16) and (2.18), but the last two ones are expressed through scalar functions $a^1_{\zeta,\lambda}$ which are still to be determined, so the system is not closed at this stage. Also note that condition (2.17) might raise an issue of over-determination of the system.

To determine the equations satisfied by coefficients $a^1_{\varphi,\lambda}$ and $a^1_{\psi,\lambda}$, we need to study the nonpolarized part of the first corrector U_2.

2.2.2. *Nonpolarized part of the first corrector.* For the first corrector we no longer have a polarization condition such as (2.2a), so noncharacteristic modes may appear through quadratic interaction of characteristic modes. Thus the first corrector U_2 reads

$$\begin{aligned} U_2(z,\theta,\chi_d) = U_2^*(z) &+ \sum_{\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^2}} \sum_{j\in\mathcal{C}(\mathbf{n}\cdot\boldsymbol{\zeta})} \sum_{\lambda\in\mathbb{Z}^*} U^{2,\mathrm{osc}}_{\mathbf{n},j,\lambda}(z)\, e^{i\,\lambda\,\mathbf{n}\cdot\theta}\, e^{i\,\lambda\,\xi_j(\mathbf{n}\cdot\boldsymbol{\zeta})\,\chi_d} \\ &+ \sum_{\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^2}} \sum_{\lambda\in\mathbb{Z}^*} U^{2,\mathrm{ev}}_{\mathbf{n},\lambda}(z,\chi_d)\, e^{i\,\lambda\,\mathbf{n}\cdot\theta} + U^{2,\mathrm{nc}}(z,\theta,\chi_d), \end{aligned}$$

where U_2^* is the mean value of U_2 and $U^{2,\mathrm{nc}}$ corresponds to the noncharacteristic modes. According to (2.2b), the noncharacteristic part $U^{2,\mathrm{nc}}$ satisfies (since there are only characteristic modes in $L(0,\partial_z)\,U_1$),

$$\begin{aligned} \mathcal{L}(\partial_\theta,\partial_{\chi_d})\, U^{2,\mathrm{nc}} = -\sum_{\substack{(\zeta_1,\zeta_2,j_1,j_2,\\ \lambda_1,\lambda_2)\in\mathcal{NR}}} & L_1(r_{\zeta_1,j_1},\alpha_{j_2}(\zeta_2))\, r_{\zeta_2,j_2}\, i\lambda_2\, \sigma^1_{\zeta_1,j_1,\lambda_1}\, \sigma^1_{\zeta_2,j_2,\lambda_2} \\ & e^{i(\lambda_1\mathbf{n}_1+\lambda_2\mathbf{n}_2)\cdot\theta}\, e^{i(\lambda_1\xi_{j_1}(\zeta_1)+\lambda_2\xi_{j_2}(\zeta_2))\chi_d}, \end{aligned} \tag{2.19}$$

where $\mathcal{NR}$ denotes the set of 6-tuples $(\zeta_1,\zeta_2,j_1,j_2,\lambda_1,\lambda_2)$ such that the frequency

$$\lambda_1\,\alpha_{j_1}(\zeta_1) + \lambda_2\,\alpha_{j_2}(\zeta_2)$$

is noncharacteristic (which is such that there is no resonance). Note that in (2.19), only occur the boundary frequencies φ, ψ and ν, since for all the others, the first profile is zero. Since all frequencies in $U^{2,\mathrm{nc}}$ are noncharacteristic, equation (2.19) determine $U^{2,\mathrm{nc}}$ totally. Indeed, for each mode of noncharacteristic frequency $\lambda_1\,\alpha_{j_1}(\zeta_1)+\lambda_2\,\alpha_{j_2}(\zeta_2)$ with $(\zeta_1,\zeta_2,j_1,j_2,\lambda_1,\lambda_2)\in\mathcal{NR}$, the operator $\mathcal{L}(\partial_\theta,\partial_{\chi_d})$ reads $i\,L\big(0,\lambda_1\alpha_{j_1}(\zeta_1)+\lambda_2\alpha_{j_2}(\zeta_2)\big)$, which is an invertible matrix.

Since for every boundary frequency $\zeta = \mathbf{n}\cdot\boldsymbol{\zeta} \in \mathcal{F}_b \setminus \{0,\varphi,\psi,\nu\}$, the oscillating profile $\sigma^1_{\zeta,j,\lambda}$ is zero for $j\in\mathcal{C}(\zeta)$ and $\lambda\in\mathbb{Z}^*$ and since there is no resonances generating theses frequencies, according to (2.2b), the profile $U^{2,\mathrm{osc}}_{\mathbf{n},j,\lambda}$ satisfies

$$L\big(0,\alpha_j(\zeta)\big)\, U^{2,\mathrm{osc}}_{\mathbf{n},j,\lambda} = 0,$$

[5]the linear form $b_\psi\cdot B$ is not uniformly zero and is already zero on two of the three vectors $r_{\psi,1}$, $r_{\psi,2}$ and $r_{\psi,3}$ constituting a basis of $\mathbb{C}^3$, so cannot be on the third one

so it is polarized, and we denote by $\sigma^2_{\zeta,j,\lambda}$ the scalar function of Ω_T such that

$$U^{2,\mathrm{osc}}_{\mathbf{n},j,\lambda} = \sigma^2_{\zeta,j,\lambda}\, r_{\zeta,j}.$$

With the same arguments as for the leading profile, we get the following polarization condition for the evanescent part: $U^{2,\mathrm{ev}}_0 = 0$, and, for $\mathbf{n} \in \mathbb{Z}^2 \setminus \{0\}$,

$$U^{2,\mathrm{ev}}_{\mathbf{n}}(z,\chi_d) = e^{\chi_d \mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})}\, \Pi^e(\mathbf{n}\cdot\boldsymbol{\zeta})\, U^{2,\mathrm{ev}}_{\mathbf{n}}(z,0).$$

Therefore, U_2 reads as the more precise following way,

$$\begin{aligned} U_2(z,\theta,\chi_d) = U_2^*(z) &+ \sum_{\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^2}} \sum_{j\in\mathcal{C}(\mathbf{n}\cdot\boldsymbol{\zeta})} \sum_{\lambda\in\mathbb{Z}^*} U^{2,\mathrm{osc}}_{\mathbf{n},j,\lambda}(z)\, e^{i\,\lambda\,\mathbf{n}\cdot\theta}\, e^{i\,\lambda\,\xi_j(\mathbf{n}\cdot\boldsymbol{\zeta})\,\chi_d} \\ &+ \sum_{\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^2}} \sum_{\lambda\in\mathbb{Z}^*} e^{\chi_d \mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})}\, \Pi^e(\mathbf{n}\cdot\boldsymbol{\zeta})\, U^{2,\mathrm{ev}}_{\mathbf{n}}(z,0)\, e^{i\,\lambda\,\mathbf{n}\cdot\theta} + U^{2,\mathrm{nc}}(z,\theta,\chi_d). \end{aligned} \tag{2.20}$$

Writing down boundary equation (2.3a) for U_2 will lead to equations on boundary terms a^1_φ and a^1_ψ. Thus we need to determine the nonpolarized part of the amplitudes associated with frequencies lifted from φ, ψ, ν. For $\zeta = \mathbf{n}\cdot\boldsymbol{\zeta} \in \{\varphi,\psi,\nu\}$, $j = 1,2,3$, and $\lambda \in \mathbb{Z}^*$, from polarization equation (2.2a), we get, (using the notation $U^{2,\mathrm{osc}}_{\zeta,j,\lambda} := U^{2,\mathrm{osc}}_{\mathbf{n},j,\lambda}$),

$$\begin{aligned} i\,L\big(0,\alpha_j(\zeta)\big)\, U^{2,\mathrm{osc}}_{\zeta,j,\lambda} = &- L(0,\partial_z)\,\sigma^1_{\zeta,j,\lambda}\, r_{\zeta,j} - L_1\big(r_{\zeta,j},\alpha_j(\zeta)\big)\, r_{\zeta,j} \sum_{\lambda_1+\lambda_2=\lambda} i\lambda_2\, \sigma^1_{\zeta,j,\lambda_1}\, \sigma^1_{\zeta,j,\lambda_2} \\ &- \mathbb{1}_{\lambda=k\lambda_\zeta} \sum_{\substack{(\zeta_1,\zeta_2,j_1,j_2)\\ \in\mathcal{R}(\zeta,j)}} ik \Big\{ L_1\big(r_{\zeta_1,j_1},\lambda_{\zeta_2}\,\alpha_{j_2}(\zeta_2)\big)\, r_{\zeta_2,j_2} \\ &\qquad + L_1\big(r_{\zeta_2,j_2},\lambda_{\zeta_1}\,\alpha_{j_1}(\zeta_1)\big)\, r_{\zeta_1,j_1} \Big\}\, \sigma^1_{\zeta_1,j_1,-k\lambda_{\zeta_1}}\, \sigma^1_{\zeta_2,j_2,-k\lambda_{\zeta_2}}. \end{aligned}$$

Then we multiply this equation on the left by the partial inverse $R_{\zeta,j}$ to obtain, according to relation (1.15),

$$\begin{aligned} i\lambda\,\big(I - P_{\zeta,j}\big)\, U^{2,\mathrm{osc}}_{\zeta,j,\lambda} = \\ &- R_{\zeta,j}\, L(0,\partial_z)\,\sigma^1_{\zeta,j,\lambda}\, r_{\zeta,j} - R_{\zeta,j}\, L_1\big(r_{\zeta,j},\alpha_j(\zeta)\big)\, r_{\zeta,j} \sum_{\lambda_1+\lambda_2=\lambda} i\lambda_2\, \sigma^1_{\zeta,j,\lambda_1}\, \sigma^1_{\zeta,j,\lambda_2} \\ &- R_{\zeta,j}\, \mathbb{1}_{\lambda=k\lambda_\zeta} \sum_{\substack{(\zeta_1,\zeta_2,j_1,j_2)\\ \in\mathcal{R}(\zeta,j)}} ik \Big\{ L_1\big(r_{\zeta_1,j_1},\lambda_{\zeta_2}\,\alpha_{j_2}(\zeta_2)\big)\, r_{\zeta_2,j_2} \\ &\qquad + L_1\big(r_{\zeta_2,j_2},\lambda_{\zeta_1}\,\alpha_{j_1}(\zeta_1)\big)\, r_{\zeta_1,j_1} \Big\}\, \sigma^1_{\zeta_1,j_1,-k\lambda_{\zeta_1}}\, \sigma^1_{\zeta_2,j_2,-k\lambda_{\zeta_2}}. \end{aligned} \tag{2.21}$$

We now write the boundary conditions for the first corrector U_2, for the frequencies φ and ψ.

Note that since φ is in the hyperbolic region, the stable elliptic component $E^e_-(\varphi)$ is zero, so $\Pi^e(\varphi)$ is also zero. Therefore, boundary condition (2.3a) for mode φ reads, according to equation (2.20),

$$\begin{aligned} B\,P_{\varphi,1}\,\big(U^2_{\varphi,1,\lambda}\big)_{|x_d,\chi_d=0} + B\,P_{\varphi,3}\,\big(U^2_{\varphi,3,\lambda}\big)_{|x_d,\chi_d=0} \\ + B\,(I-P_{\varphi,1})\,\big(U^2_{\varphi,1,\lambda}\big)_{|x_d,\chi_d=0} + B\,(I-P_{\varphi,3})\,\big(U^2_{\varphi,3,\lambda}\big)_{|x_d,\chi_d=0} \\ + B\,\big(U^2_{\varphi,2,\lambda}\big)_{|x_d,\chi_d=0} + B\,\big(U^{2,\mathrm{nc}}_{\varphi,\lambda}\big)_{|x_d,\chi_d=0} = G_\lambda, \end{aligned} \tag{2.22}$$

where we have expanded the source term G in Fourier series as

$$G(z',\Theta) = \sum_{\lambda\in\mathbb{Z}} G_\lambda(z')\, e^{i\lambda\Theta},$$

and where we have denoted by $U^{2,\mathrm{nc}}_{\varphi,\lambda}$ the sum of all the terms of $U^{2,\mathrm{nc}}$ of which the trace on the boundary of the associated frequency is equal to $\lambda\varphi$, namely,

$$U^{2,\mathrm{nc}}_{\varphi,\lambda} = - \sum_{\substack{(\zeta_1,\zeta_2,j_1,j_2,\\ \lambda_1,\lambda_2)\in\mathcal{NR}\\ \lambda_1\zeta_1+\lambda_2\zeta_2=\lambda\varphi}} L\big(0,\lambda_1\alpha_{j_1}(\zeta_1)+\lambda_2\alpha_{j_2}(\zeta_2)\big)^{-1} L_1(r_{\zeta_1,j_1},\alpha_{j_2}(\zeta_2))\, r_{\zeta_2,j_2}\, \lambda_2\, \sigma^1_{\zeta_1,j_1,\lambda_1}\, \sigma^1_{\zeta_2,j_2,\lambda_2}$$
$$e^{i(\lambda_1\mathfrak{n}_1+\lambda_2\mathfrak{n}_2)\cdot\theta}\, e^{i(\lambda_1\xi_{j_1}(\zeta_1)+\lambda_2\xi_{j_2}(\zeta_2))\chi_d}.$$

We investigate now which frequencies occur in this sum. If we denote by, for $i=1,2$, $\zeta_i = \mathfrak{m}_i\varphi+\mathfrak{n}_i\psi$ with $(\mathfrak{m}_i,\mathfrak{n}_i)\in\mathcal{B}_{\mathbb{Z}^2}$, since there are only frequencies lifted from φ,ψ,ν in $\mathcal{NR}$, we necessarily have ($\mathfrak{m}_i=1$ and $\mathfrak{n}_i=0$) or ($\mathfrak{m}_i=0$ and $\mathfrak{n}_i=1$) or ($\mathfrak{m}_i=\lambda_\varphi$ and $\mathfrak{n}_i=\lambda_\psi$). In this notation, the condition $\lambda_1\zeta_1+\lambda_2\zeta_2=\lambda\varphi$ is equivalent to

$$\begin{cases} \lambda_1\,\mathfrak{m}_1+\lambda_2\,\mathfrak{m}_2=\lambda\\ \lambda_1\,\mathfrak{n}_1+\lambda_2\,\mathfrak{n}_2=0, \end{cases}$$

and using that $\lambda_\varphi,\lambda_\psi$ are coprime integers, we find that this system admits the following solutions

$$\begin{cases} (\mathfrak{m}_1,\mathfrak{n}_1)=(1,0)\\ (\mathfrak{m}_2,\mathfrak{n}_2)=(1,0)\\ \lambda_1+\lambda_2=\lambda \end{cases}, \quad \begin{cases} (\mathfrak{m}_1,\mathfrak{n}_1)=(0,1)\\ (\mathfrak{m}_2,\mathfrak{n}_2)=(\lambda_\varphi,\lambda_\psi)\\ (\lambda,\lambda_1,\lambda_2)=k\,(\lambda_\varphi,-\lambda_\psi,1) \end{cases}, \text{ and } \begin{cases} (\mathfrak{m}_1,\mathfrak{n}_1)=(\lambda_\varphi,\lambda_\psi)\\ (\mathfrak{m}_2,\mathfrak{n}_2)=(0,1)\\ (\lambda,\lambda_1,\lambda_2)=k\,(\lambda_\varphi 1,-\lambda_\psi) \end{cases}.$$

Selecting only 6-tuples of $\mathcal{NR}$, we obtain that $U^{2,\mathrm{nc}}_{\varphi,\lambda}$ is equal to

$$\begin{aligned} U^{2,\mathrm{nc}}_{\varphi,\lambda} = \mathbb{1}_{\lambda=k\lambda_\varphi} &\sum_{\substack{j_1,j_2=1,2,3\\ (j_1,j_2)\neq(2,1),(2,2)}} L\big(0,k\nu_{j_1}-\lambda_\psi k\psi_{j_2}\big)^{-1}\big\{\lambda_\psi\, L_1(r_{\nu,j_1},\psi_{j_2})\, r_{\psi,j_2}\\ &- L_1(r_{\psi,j_2},\nu_{j_1})\, r_{\nu,j_1}\big\}\, k\,\sigma^1_{\nu,j_1,k}\,\sigma^1_{\psi,j_2,-\lambda_\psi k}\, e^{ik\lambda_\varphi\theta_1}\, e^{i(k\xi_{j_1}(\nu)-\lambda_\psi k\xi_{j_2}(\psi))\chi_d}\\ - \sum_{\lambda_1+\lambda_2=\lambda} &L\big(0,\lambda_1\varphi_1+\lambda_2\varphi_3\big)^{-1}\big\{\lambda_1\, L_1(r_{\varphi,3},\varphi_1)\, r_{\varphi,1}\\ &+\lambda_2\, L_1(r_{\varphi,1},\varphi_3)\, r_{\varphi,3}\big\}\,\sigma^1_{\varphi,1,\lambda_1}\,\sigma^1_{\varphi,3,\lambda_2}\, e^{i\lambda\,\theta_1}\, e^{i(\lambda_1\xi_1(\varphi)+\lambda_2\xi_3(\varphi))\chi_d}. \end{aligned} \tag{2.23}$$

Since the vectors $P_{\varphi,1}\left(U^2_{\varphi,1,\lambda}\right)_{|x_d,\chi_d=0}$ and $P_{\varphi,3}\left(U^2_{\varphi,3,\lambda}\right)_{|x_d,\chi_d=0}$ in (2.22) are respectively in $E^1_-(\varphi)$ and $E^3_-(\varphi)$, by definition (1.17) of b_φ, we have

$$b_\varphi\cdot B\, P_{\varphi,1}\left(U^2_{\varphi,1,\lambda}\right)_{|x_d,\chi_d=0} = b_\varphi\cdot B\, P_{\varphi,3}\left(U^2_{\varphi,3,\lambda}\right)_{|x_d,\chi_d=0}=0.$$

So if we take the scalar product of b_φ with equality (2.22) multiplied by $i\lambda$, using (2.21), (2.23) and the boundary conditions (2.16) for the leading profile associated with $\lambda\varphi$, we get the amplitude equation

$$\begin{aligned} &X^{\mathrm{Lop}}_\varphi\, a^1_{\varphi,\lambda} + D^{\mathrm{Lop}}_\varphi \sum_{\lambda_1+\lambda_2=\lambda} i\lambda_2\, a^1_{\varphi,\lambda_1}\, a^1_{\varphi,\lambda_2} + i\lambda \sum_{\lambda_1+\lambda_3=\lambda} \gamma_\varphi(\lambda_1,\lambda_3)\, a^1_{\varphi,\lambda_1}\, a^1_{\varphi,\lambda_3}\\ &\qquad + \mathbb{1}_{\lambda=k\lambda_\varphi}\,\Gamma^{\varphi,k}_1\, ik\, a^1_{\psi,-k\lambda_\psi}\left(\sigma^1_{\nu,2,-k}\right)_{|x_d=0} = i\lambda\, b_\varphi\cdot B\left(U^2_{\varphi,2,\lambda}\right)_{|x_d,\chi_d=0} - i\lambda\, b_\varphi\cdot G_\lambda, \end{aligned} \tag{2.24}$$

with

$$X^{\mathrm{Lop}}_\varphi := b_\varphi\cdot B\left(R_{\varphi,1}\, L(0,\partial_z)\, e_{\varphi,1} + R_{\varphi,3}\, L(0,\partial_z)\, e_{\varphi,3}\right), \tag{2.25a}$$

$$D^{\mathrm{Lop}}_\varphi := b_\varphi\cdot B\left(R_{\varphi,1}\, L_1\big(e_{\varphi,1},\varphi_1\big)\, e_{\varphi,1} + R_{\varphi,3}\, L_1\big(e_{\varphi,3},\varphi_3\big)\, e_{\varphi,3}\right), \tag{2.25b}$$

$$\gamma_\varphi(\lambda_1,\lambda_3) := b_\varphi\cdot B\, L\big(\lambda_1\varphi_1+\lambda_3\varphi_3\big)^{-1}\big\{\lambda_1\, L_1(e_{\varphi,3},\varphi_1)\, e_{\varphi,1} + \lambda_3\, L_1(e_{\varphi,1},\varphi_3)\, e_{\varphi,3}\big\}, \tag{2.25c}$$

$$\Gamma^\varphi := b_\varphi\cdot B\, R_{\varphi,1}\left(L_1(e_{\psi,1},-\nu_2)\, r_{\nu,2} + L_1(r_{\nu,2},-\lambda_\psi\psi_1)\, e_{\psi,1}\right) \tag{2.25d}$$

$$+\lambda_\varphi\, b_\varphi\cdot B\sum_{\substack{j_1=1,3,j_2=1,2,3\\(j_1,j_2)\neq(1,2)}}\mu_{\nu,j_2}\,L\big(\nu_{j_2}-\lambda_\psi\psi_{j_1}\big)^{-1}\Big\{L_1(r_{\psi,j_1},\nu_{j_2})\,r_{\nu,j_2}$$
$$-L_1(r_{\nu,j_2},\psi_{j_1})\,r_{\psi,j_1}\,\lambda_\psi\Big\}.$$

Equation (2.24) differs from the analogous one of [CW17, equation (2.19)] by the two terms $\mathbb{1}_{\lambda=k\lambda_\varphi}\,\Gamma^\varphi\,ik\,a^1_{\psi,-k\lambda_\psi}\,(\sigma^1_{\nu,2,-k})_{|x_d=0}$ and $i\lambda\,b_\varphi\cdot B\,(U^2_{\varphi,2,\lambda})_{|x_d,\chi_d=0}$. The first one appears here because of the resonances, and the second one wasn't in [CW17] because profiles $U^2_{\varphi,2,\lambda}$ were zero[6]. Computing $L\big(\lambda_1\varphi_1+\lambda_2\varphi_3\big)^{-1}$ on $r_{\varphi,2}$ leads to the following alternative expression for $\gamma_\varphi(\lambda_1,\lambda_3)$:

$$\gamma_\varphi(\lambda_1,\lambda_3)=b_\varphi\cdot B\,r_{\varphi,2}\,\frac{i\lambda_1\,\ell_{\varphi,2}\,E^\varphi_{3,1}+i\lambda_3\,\ell_{\varphi,2}\,E^\varphi_{1,3}}{\lambda_1\,\big(\xi_1(\varphi)-\xi_2(\varphi)\big)+\lambda_3\,\big(\xi_3(\varphi)-\xi_2(\varphi)\big)},$$

where we have denoted

$$E^\varphi_{3,1}:=L_1(e_{\varphi,3},\varphi_1)\,e_{\varphi,1},\qquad E^\varphi_{1,3}:=L_1(e_{\varphi,1},\varphi_3)\,e_{\varphi,3}.$$

This rewriting, which can be found in [CW17], will be useful in the following to study the bilinear operator associated with the symbol $\gamma_\varphi(\lambda_1,\lambda_2)$ of (2.25c). Finally, according to Lemma 1.11, the operator X^{Lop}_φ is actually equal to the tangential vector field

$$\beta_\varphi\Big(\partial_t+\nabla_\eta\kappa(\varphi)\cdot\nabla_y\Big),$$

which we still denote by X^{Lop}_φ.

Similarly, for ψ we have

$$\begin{aligned}(2.26)\qquad &B\,P_{\psi,1}\,(U^2_{\psi,1,\lambda})_{|x_d,\chi_d=0}+B\,P_{\psi,3}\,(U^2_{\psi,3,\lambda})_{|x_d,\chi_d=0}\\ &+B\,(I-P_{\psi,1})\,(U^2_{\psi,1,\lambda})_{|x_d,\chi_d=0}+B\,(I-P_{\psi,3})\,(U^2_{\psi,3,\lambda})_{|x_d,\chi_d=0}\\ &+B\,(U^2_{\psi,2,\lambda})_{|x_d,\chi_d=0}+B\,(U^{2,\mathrm{nc}}_{\psi,\lambda})_{|x_d,\chi_d=0}=0,\end{aligned}$$

where the vector $U^{2,\mathrm{nc}}_{\psi,\lambda}$ is given by

$$\begin{aligned}(2.27)\qquad U^{2,\mathrm{nc}}_{\psi,\lambda}=\mathbb{1}_{\lambda=k\lambda_\psi}&\sum_{\substack{j_1=1,3,j_2=1,2,3\\(j_1,j_2)\neq(1,2),(3,2)}}L\big(0,k\nu_{j_2}-\lambda_\varphi k\varphi_{j_1}\big)^{-1}\big\{\lambda_\varphi\,L_1(r_{\nu,j_2},\varphi_{j_1})\,r_{\varphi,j_1}\\ &-L_1(r_{\varphi,j_1},\nu_{j_2})\,r_{\nu,j_2}\big\}\,k\,\sigma^1_{\nu,j_2,k}\,\sigma^1_{\varphi,j_1,-\lambda_\varphi k}\,e^{ik\lambda_\varphi\theta_1}\,e^{i(k\xi_{j_2}(\nu)-\lambda_\varphi k\xi_{j_1}(\varphi))\chi_d}\\ -\sum_{\substack{j_1,j_2=1,2,3\\j_1\neq j_2}}&\sum_{\lambda_1+\lambda_2=\lambda}L\big(0,\lambda_1\psi_{j_1}+\lambda_2\psi_{j_2}\big)^{-1}\,L_1(r_{\psi,j_1},\psi_{j_2})\,r_{\psi,j_2}\,\lambda_2\\ &\sigma^1_{\psi,j_1,\lambda_1}\,\sigma^1_{\psi,j_2,\lambda_2}\,e^{i\lambda\,\theta_2}\,e^{i(\lambda_1\xi_{j_1}(\psi)+\lambda_2\xi_{j_2}(\psi))\chi_d}.\end{aligned}$$

If we take the scalar product of b_ψ with equality (2.26) multiplied by $i\lambda$, using (2.21), (2.23) and the boundary conditions (2.26) for the leading profile associated with $\lambda\psi$, we get the second amplitude equation

$$\begin{aligned}(2.28)\qquad &X^{\mathrm{Lop}}_\psi\,a^1_{\psi,\lambda}+v_\psi\sum_{\lambda_1+\lambda_2=\lambda}i\lambda_2\,a^1_{\psi,\lambda_1}\,a^1_{\psi,\lambda_2}+i\lambda\sum_{\lambda_1+\lambda_3=\lambda}\gamma_\psi(\lambda_1,\lambda_3)\,a^1_{\psi,\lambda_1}\,a^1_{\psi,\lambda_3}\\ &+\mathbb{1}_{\lambda=k\lambda_\psi}\,\Gamma^\psi\,ik\,(\sigma_{\nu,2,k})_{|x_d=0}\,a^1_{\varphi,-\lambda_\varphi k}=i\lambda\,b_\psi\cdot B\,(U^2_{\psi,2,\lambda})_{|x_d,\chi_d=0}.\end{aligned}$$

[6]Note that here, with a more precise analysis, we could also show that every profile $U^n_{\varphi,2,\lambda}$ is zero, since frequency φ_2 does not occur in any resonance. We choose however to not detail it, in order to simplify the whole analysis, and since $U^n_{\psi,2,\lambda}$ will not be zero, and proving that $U^n_{\varphi,2,\lambda}$ would not simplify the solving of the equations

with, using once again Lemma 1.11,

$$X_\psi^{\mathrm{Lop}} := b_\psi \cdot B\Big(R_{\psi,1}\, L(0,\partial_z)\, e_{\psi,1} + R_{\psi,3}\, L(0,\partial_z)\, e_{\psi,3}\Big)$$

$$= \beta_\psi\Big(\partial_t + \nabla_\eta \kappa(\psi)\cdot\nabla_y\Big), \tag{2.29a}$$

$$v_\psi := b_\psi \cdot B\Big(R_{\psi,1}\, L_1\big(e_{\psi,1},\psi_1\big)\, e_{\psi,1} + R_{\psi,3}\, L_1\big(e_{\psi,3},\psi_3\big)\, e_{\psi,3}\Big), \tag{2.29b}$$

$$\gamma_\psi(\lambda_1,\lambda_3) := b_\psi \cdot B\, L\big(\lambda_1\psi_1+\lambda_3\psi_3\big)^{-1}\big\{\lambda_1\, L_1(e_{\psi,3},\psi_1)\, e_{\psi,1} + \lambda_3\, L_1(e_{\psi,1},\psi_3)\, e_{\psi,3}\big\} \tag{2.29c}$$

$$\begin{aligned}\Gamma^\psi := {} & b_\psi \cdot B\, R_{\psi,1}\Big(L_1(e_{\varphi,1},-\nu_2)\, r_{\nu,2} + L_1(r_{\nu,2},-\lambda_\varphi\varphi_1)\, e_{\varphi,1}\Big)\\ & + \lambda_\psi\, b_\psi\cdot B \sum_{\substack{j_1=1,3, j_2=1,2,3\\ (j_1,j_2)\neq(1,2),(3,2)}} \mu_{\nu,j_2}\, L\big(\nu_{j_2}-\lambda_\varphi\varphi_{j_1}\big)^{-1}\big\{L_1(e_{\varphi,j_1},\nu_{j_2})\, r_{\nu,j_2}\\ & - \lambda_\varphi\, L_1(r_{\nu,j_2},\varphi_{j_1})\, e_{\varphi,j_1}\big\}.\end{aligned} \tag{2.29d}$$

Again, comparing to [CW17, equation (2.19)], equation (2.28) features two additional terms $\mathbb{1}_{\lambda=k\lambda_\psi}\,\Gamma^\psi\, ik\,\big(\sigma_{\nu,2,k}\big)_{|x_d=0}\, a^1_{\varphi,-\lambda_\varphi k}$ and $i\lambda\, b_\psi\cdot B\,\big(U^2_{\psi,2,\lambda}\big)_{|x_d,\chi_d=0}$ for the same reason. The is no boundary forcing term here because the one for ψ is of order $O(\varepsilon^3)$. In the same way as for φ we have

$$\gamma_\psi(\lambda_1,\lambda_3) = b_\psi\cdot B\, r_{\psi,2}\,\frac{i\lambda_1\,\ell_{\psi,2}\,E^\psi_{3,1} + i\lambda_3\,\ell_{\psi,2}\,E^\psi_{1,3}}{\lambda_1\big(\xi_1(\psi)-\xi_2(\psi)\big)+\lambda_3\big(\xi_3(\psi)-\xi_2(\psi)\big)},$$

where we have denoted

$$E^\psi_{3,1} := L_1(e_{\psi,3},\psi_1)\, e_{\psi,1}, \qquad E^\psi_{1,3} := L_1(e_{\psi,1},\psi_3)\, e_{\psi,3}.$$

In conclusion, in this paragraph, we have determined the nonpolarized part of the first corrector, with (2.21), and the evolution equations (2.24) and (2.28) satisfied by $a^1_{\varphi,\lambda}$ and $a^1_{\psi,\lambda}$. However, the obtained system of equation is still not closed since equations (2.24) and (2.28) involve traces of the first corrector U_2, of which the polarized part is still undetermined.

Therefore we proceed inductively, by deriving equations for the polarized part of the first corrector U_2, and then studying the nonpolarized part of the second corrector U_3, in order to obtain evolution equations on the boundary terms for the polarized part of U_2.

2.2.3. *Polarized part of the first corrector.* For $\zeta = \mathbf{n}\cdot\boldsymbol{\zeta} \in \mathcal{F}_b\setminus\{0\}$ with $\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^2}$, for $j\in\mathcal{C}(\zeta)$ and $\lambda\in\mathbb{Z}^*$, we decompose the profile $U^{2,\mathrm{osc}}_{\mathbf{n},j,\lambda}$ as $U^{2,\mathrm{osc}}_{\mathbf{n},j,\lambda} = P_{\zeta,j}\,U^{2,\mathrm{osc}}_{\mathbf{n},j,\lambda} + (I-P_{\zeta,j})\,U^{2,\mathrm{osc}}_{\mathbf{n},j,\lambda}$. Recall that the nonpolarized part $(I-P_{\zeta,j})\,U^{2,\mathrm{osc}}_{\mathbf{n},j,\lambda}$ is given by (2.21), so it remains to determine the polarized part, which is written as

$$P_{\zeta,j}\,U^{2,\mathrm{osc}}_{\mathbf{n},j,\lambda} = \sigma^2_{\zeta,j,\lambda}\, r_{\zeta,j},$$

with $\sigma^2_{\zeta,j,\lambda}$ a scalar function of Ω_T. We start by determining the mean value U_2^*, as in the general case of a corrector U_n, the mean value U_n^* appears in equations for the polarized components. According to equation (2.2b), U_2^* satisfies the equation

$$\begin{aligned}L(0,\partial_z)\,U_2^* + {} & \sum_{\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^2}}\sum_{j\in\mathcal{C}(\mathbf{n}\cdot\boldsymbol{\zeta})}\sum_{\lambda\in\mathbb{Z}^*} L_1\big(r_{\mathbf{n}\cdot\boldsymbol{\zeta},j}, -i\,\lambda\,\alpha_j(\mathbf{n}\cdot\boldsymbol{\zeta})\big)\, r_{\mathbf{n}\cdot\boldsymbol{\zeta},j}\Big(\sigma^1_{\mathbf{n},j,\lambda}\,\sigma^2_{\mathbf{n},j,-\lambda} + \sigma^2_{\mathbf{n},j,\lambda}\,\sigma^1_{\mathbf{n},j,-\lambda}\Big)\\ & + \sum_{\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^2}}\sum_{j\in\mathcal{C}(\mathbf{n}\cdot\boldsymbol{\zeta})}\sum_{\lambda\in\mathbb{Z}^*} L_1\big((I-P_{\mathbf{n}\cdot\boldsymbol{\zeta},j})\,U^{2,\mathrm{osc}}_{\mathbf{n},j,\lambda}, -i\,\lambda\,\alpha_j(\mathbf{n}\cdot\boldsymbol{\zeta})\big)\,\sigma^1_{\mathbf{n},j,-\lambda}\, r_{\mathbf{n}\cdot\boldsymbol{\zeta},j}\\ & + \sum_{\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^2}}\sum_{j\in\mathcal{C}(\mathbf{n}\cdot\boldsymbol{\zeta})}\sum_{\lambda\in\mathbb{Z}^*} L_1\big(\sigma^1_{\mathbf{n},j,\lambda}, -i\,\lambda\,\alpha_j(\mathbf{n}\cdot\boldsymbol{\zeta})\big)\,(I-P_{\mathbf{n}\cdot\boldsymbol{\zeta},j})\,U^{2,\mathrm{osc}}_{\mathbf{n},j,-\lambda}\, r_{\mathbf{n}\cdot\boldsymbol{\zeta},j}\\ & + \sum_{\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^2}}\sum_{j\in\mathcal{C}(\mathbf{n}\cdot\boldsymbol{\zeta})}\sum_{\lambda\in\mathbb{Z}^*} L_1\big(U^{1,\mathrm{osc}}_{\mathbf{n},j,\lambda},\partial_z\big)\,U^{1,\mathrm{osc}}_{\mathbf{n},j,\lambda} = 0.\end{aligned}$$

The change of variable $\lambda = -\lambda$ shows that the second term in first line of previous equation is zero. Boundary and initial condition (2.3a) and (2.4) as well as writing (2.20) for U_2 leads to the following boundary and initial condition for U_2^*:

$$B\left(U_2^*\right)_{|x_d=0} = G_0 - B\left(U_0^{2,\mathrm{nc}}\right)_{|x_d,\chi_d=0}, \qquad B\left(U_2^*\right)_{|t\leqslant 0} = 0,$$

where G_0 is the mean value of the forcing term G and $U_0^{2,\mathrm{nc}}$ is the sum of all terms of $U^{2,\mathrm{nc}}$ of which the trace on the boundary has zero frequency. According to expressions (2.21) of the nonpolarized parts and (2.19) of $U^{2,\mathrm{nc}}$, the mean value U_2^* satisfies an initial boundary value problem which is weakly well posed and of which the source and boundary terms depend only on the leading profile U_1 (with possibly a first order derivative applied to it).

We consider now modes $\lambda\,\alpha_j(\zeta)$ with $\zeta = \mathbf{n}\cdot\boldsymbol{\zeta} \in \mathcal{F}_b \setminus \{0,\varphi,\psi,\nu\}$, $\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^2}$, $j\in\mathcal{C}(\zeta)$ and $\lambda\in\mathbb{Z}^*$. Recall that it has been proven in the previous part that for these modes, the profile $U_{\mathbf{n},j}^{2,\mathrm{osc}}$ is polarized, that is

$$U_{\mathbf{n},j}^{2,\mathrm{osc}} = \sigma_{\zeta,j,\lambda}^2\, r_{\zeta,j}.$$

Since there is no resonance generating frequency $\alpha_j(\zeta)$, since the mean value U_1^* is zero and since profiles $\sigma_{\zeta,j,\lambda}^1$ are also zero, the terms $\mathcal{M}(U_1,U_2)$, $\mathcal{M}(U_2,U_1)$, and $\mathcal{N}(U_1,U_1)$ contain no term of frequency $\alpha_j(\zeta)$. Therefore, analogously as for the leading profile, multiplying equation (2.2c) for $n=3$ on the left by $\ell_{\zeta,j}$ leads to the following system of transport equations for the scalar functions $\sigma_{\zeta,j,\lambda}^2$:

$$X_{\alpha_j(\zeta)}\,\sigma_{\zeta,j,\lambda}^2 = 0, \qquad \left(\sigma_{\zeta,j,\lambda}^2\right)_{|t\leqslant 0} = 0. \tag{2.30a}$$

When frequency $\alpha_j(\zeta)$ is outgoing, transport equation (2.30a) leads to $\sigma_{\zeta,j,\lambda}^2 = 0$. When frequency $\alpha_j(\zeta)$ is incoming, according to boundary condition (2.3a) and decomposition (2.20) of U_2, since ζ is not in Υ, since G does not contain any oscillation in ζ and since the outgoing profile $\sigma_{\zeta,j,\lambda}^2$, $j\in\mathcal{O}(\zeta)$, is zero, we have

$$\left(\sigma_{\zeta,j,\lambda}^2\right)_{|x_d=0} = -\ell_{\zeta,j}\,A_d(0)\left(B_{|E_-(\zeta)}\right)^{-1} B\left(U_{\zeta,\lambda}^{2,\mathrm{nc}}\right)_{|x_d,\chi_d=0}, \tag{2.30b}$$

where $U_{\zeta,\lambda}^{2,\mathrm{nc}}$ is the sum of all terms of $U^{2,\mathrm{nc}}$ of which the trace on the boundary is $\lambda\,\zeta$. It is fully determined by (2.19), and thus depends only on $\left(\sigma_{\zeta,j,\lambda}^1\right)_{|x_d=0}$ for $\zeta=\varphi,\psi,\nu$. Therefore, if the traces $\left(\sigma_{\zeta,j,\lambda}^1\right)_{|x_d=0}$ for $\zeta=\varphi,\psi,\nu$ are determined, system (2.30) allows us to construct the profiles $\sigma_{\zeta,j,\lambda}^2$, for $\zeta\in\mathcal{F}_b\setminus\{0,\varphi,\psi,\nu\}$ and $j\in\mathcal{I}(\zeta)$.

We now take interest into modes φ_j, ψ_j and ν_j for $j=1,2,3$. Applying $\ell_{\zeta,j}$ to equation (2.2c) for $n=3$ leads to the following equation for $\sigma_{\zeta,j,\lambda}^2$,

(2.31)

$$\begin{aligned}
&X_{\zeta,j}\,\sigma_{\zeta,j,\lambda}^2 + D_{\zeta,j}\sum_{\lambda_1+\lambda_2=\lambda} i\lambda\,\sigma_{\zeta,j,\lambda_1}^2\,\sigma_{\zeta,j,\lambda_2}^1 + \mathbb{1}_{\lambda=k\lambda_\zeta}\sum_{\substack{(\zeta_1,\zeta_2,j_1,j_2)\\ \in\mathcal{R}(\zeta,j)}} J_{\zeta_1,j_1}^{\zeta_2,j_2}\,ik\,\sigma_{\zeta_1,j_1,-k\lambda_{\zeta_1}}^1\,\sigma_{\zeta_2,j_2,-k\lambda_{\zeta_2}}^2\\
&= -\ell_{\zeta,j}\,L(0,\partial_z)\left(I-P_{\zeta,j}\right)U_{\zeta,j,\lambda}^2 - \ell_{\zeta,j}\sum_{\lambda_1+\lambda_2=\lambda} L_1\left(\left(I-P_{\zeta,j}\right)U_{\zeta,j,\lambda_1}^2,\zeta_j\right) r_{\zeta,j}\,i\lambda_2\,\sigma_{\zeta,j,\lambda_2}^1\\
&\quad - \ell_{\zeta,j}\sum_{\lambda_1+\lambda_2=\lambda} L_1\left(r_{\zeta,j},\zeta_j\right)\left(I-P_{\zeta,j}\right)U_{\zeta,j,\lambda_2}^2\,i\lambda_2\,\sigma_{\zeta,j,\lambda_1}^1\\
&\quad - \mathbb{1}_{\lambda=k\lambda_\zeta}\sum_{\substack{(\zeta_1,\zeta_2,j_1,j_2)\\ \in\mathcal{R}(\zeta,j)}} \ell_{\zeta,j}\,ik\Big\{L_1\left(r_{\zeta_1,j_1},-\lambda_{\zeta_2}\alpha_{j_2}(\zeta_2)\right)\left(I-P_{\zeta_2,j_2}\right)U_{\zeta_2,j_2,-k\lambda_{\zeta_2}}^2\,\sigma_{\zeta_1,j_1,-k\lambda_{\zeta_1}}^1\\
&\qquad + L_1\left(\left(I-P_{\zeta_2,j_2}\right)U_{\zeta_2,j_2,-k\lambda_{\zeta_2}}^2,-\lambda_{\zeta_1}\alpha_{j_1}(\zeta_1)\right) r_{\zeta_1,j_1}\,\sigma_{\zeta_1,j_1,-k\lambda_{\zeta_1}}^1\Big\}\\
&\quad - \ell_{\zeta,j}\sum_{\lambda_1+\lambda_2=\lambda} L_1\left(\sigma_{\zeta,j,\lambda_1}^1\,r_{\zeta,j},\partial_z\right)\sigma_{\zeta,j,\lambda_2}^1\,r_{\zeta,j}
\end{aligned}$$

$$- \mathbb{1}_{\lambda=k\lambda_\zeta} \sum_{\substack{(\zeta_1,\zeta_2,j_1,j_2)\\ \in\mathcal{R}(\zeta,j)}} \ell_{\zeta,j}\, L_1\big(\sigma^1_{\zeta_1,j_1,-k\lambda_{\zeta_1}}\, r_{\zeta_1,j_1}, \partial_z\big)\, \sigma^1_{\zeta_2,j_2,-k\lambda_{\zeta_2}}\, r_{\zeta_2,j_2}.$$

Note that the source term on the right-hand side of equation (2.31) only depends on the leading profile U_1, according to formula (2.21) for nonpolarized parts, with possibly second order derivatives applied to it (since in the expression of the nonpolarized part, first order derivatives are applied to U_1).

For the incoming frequencies φ_1, φ_3 and so on, boundary conditions must be determined to solve the above transport equations (2.31). We have already seen that boundary equation (2.3a) for U_2 reads, for mode $\lambda\varphi$, as (2.22). From this boundary condition, according to decomposition (1.6) of $E_-(\zeta)$ and relation (1.17) defining the vector b_φ, we get the following necessary solvability condition

$$b_\varphi \cdot \Big(B\,(I-P_{\varphi,1})\,\big(U^2_{\varphi,1,\lambda}\big)_{|x_d,\chi_d=0} + B\,(I-P_{\varphi,3})\,\big(U^2_{\varphi,3,\lambda}\big)_{|x_d,\chi_d=0} \\ + B\,\big(U^2_{\varphi,2,\lambda}\big)_{|x_d,\chi_d=0} + B\,\big(U^{2,\mathrm{nc}}_{\varphi,\lambda}\big)_{|x_d,\chi_d=0}\Big) = b_\varphi \cdot G_\lambda,$$

which is satisfied as soon as the scalar functions $a_{\varphi,\lambda}$ satisfy the evolution equation (2.24), since these two equations are different writings of the same one. Thus we obtain, for $j = 1, 3$, in a similar manner than for the leading profile,

$$\big(\sigma^2_{\varphi,j,\lambda}\big)_{|x_d=0}\, r_{\varphi,j} = a^2_{\varphi,\lambda}\, e_{\varphi,j} + \widetilde{F}^2_{\varphi,j,\lambda}, \tag{2.32}$$

with $a^2_{\varphi,\lambda}$ a scalar function defined on ω_T and where, for $j = 1, 3$, we have denoted by $\widetilde{F}^2_{\varphi,j,\lambda}$ the function

$$\widetilde{F}^2_{\varphi,j,\lambda} := \ell_{\varphi,j}\cdot A_d(0)\,\big(B_{|E_-(\varphi)}\big)^{-1}\Big(G_\lambda - B\,(I-P_{\varphi,1})\,\big(U^2_{\varphi,1,\lambda}\big)_{|x_d,\chi_d=0} \\ - B\,(I-P_{\varphi,3})\,\big(U^2_{\varphi,3,\lambda}\big)_{|x_d,\chi_d=0} - B\,(I-P_{\varphi,2})\,\big(U^2_{\varphi,2,\lambda}\big)_{|x_d,\chi_d=0} \\ - B\,\big(\sigma^2_{\varphi,2,\lambda}\big)_{|x_d=0}\, r_{\varphi,2} - B\,\big(U^{2,\mathrm{nc}}_{\varphi,\lambda}\big)_{|x_d,\chi_d=0}\Big)\, r_{\varphi,j}.$$

In the same way, for ψ, from (2.26), the following condition must be satisfied:

$$b_\psi \cdot \Big(B\,(I-P_{\psi,1})\,\big(U^2_{\psi,1,\lambda}\big)_{|x_d,\chi_d=0} + B\,(I-P_{\psi,3})\,\big(U^2_{\psi,3,\lambda}\big)_{|x_d,\chi_d=0} \\ + B\,\big(U^2_{\psi,2,\lambda}\big)_{|x_d,\chi_d=0} + B\,\big(U^{2,\mathrm{nc}}_{\psi,\lambda}\big)_{|x_d,\chi_d=0}\Big) = 0,$$

and it is the case when $a_{\psi,\lambda}$ verify equation (2.28). Therefore, for $j = 1, 3$,

$$\big(\sigma^2_{\psi,j,\lambda}\big)_{|x_d=0}\, r_{\psi,j} = a^2_{\psi,\lambda}\, e_{\psi,j} + \widetilde{F}^2_{\psi,j,\lambda}, \tag{2.33}$$

with $a^2_{\psi,\lambda}$ a scalar function of ω_T, and where we have denoted by $\widetilde{F}^2_{\psi,j,\lambda}$ the function

$$\widetilde{F}^2_{\psi,j,\lambda} := -\ell_{\psi,j}\cdot A_d(0)\,\big(B_{|E_-(\psi)}\big)^{-1}\Big(B\,(I-P_{\psi,1})\,\big(U^2_{\psi,1,\lambda}\big)_{|x_d,\chi_d=0} \\ + B\,(I-P_{\psi,3})\,\big(U^2_{\psi,3,\lambda}\big)_{|x_d,\chi_d=0} + B\,(I-P_{\psi,2})\,\big(U^2_{\psi,2,\lambda}\big)_{|x_d,\chi_d=0} \\ - B\,\big(\sigma^2_{\psi,2,\lambda}\big)_{|x_d=0}\, r_{\psi,2} + B\,\big(U^{2,\mathrm{nc}}_{\psi,\lambda}\big)_{|x_d,\chi_d=0}\Big).$$

Note that expressions (2.32) and (2.33) of incoming traces $\big(\sigma^2_{\varphi,j,\lambda}\big)_{|x_d=0}$ and $\big(\sigma^2_{\psi,j,\lambda}\big)_{|x_d=0}$ for $j = 1, 3$ is respectively coupled to the outgoing trace $\big(\sigma^2_{\varphi,2,\lambda}\big)_{|x_d=0}$ and $\big(\sigma^2_{\psi,2,\lambda}\big)_{|x_d=0}$ through terms $\widetilde{F}^2_{\varphi,j,\lambda}$ and $\widetilde{F}^2_{\psi,j,\lambda}$.

Finally, for amplitudes associated with the boundary phase ν, we need to write a boundary condition for the first corrector. Boundary condition (2.3a) for U_2 reads

$$B\,P_{\nu,1}\,\big(U^2_{\nu,1,\lambda}\big)_{|x_d,\chi_d=0} + B\,P_{\nu,3}\,\big(U^2_{\nu,3,\lambda}\big)_{|x_d,\chi_d=0} \\ + B\,(I-P_{\nu,1})\,\big(U^2_{\nu,1,\lambda}\big)_{|x_d,\chi_d=0} + B\,(I-P_{\nu,3})\,\big(U^2_{\nu,3,\lambda}\big)_{|x_d,\chi_d=0}$$

$$+B\left(U^2_{\nu,2,\lambda}\right)_{|x_d,\chi_d=0}+B\left(U^{2,\mathrm{nc}}_{\nu,\lambda}\right)_{|x_d,\chi_d=0}=0,$$

where $U^{2,\mathrm{nc}}_{\nu,\lambda}$ is the sum of all the terms of $U^{2,\mathrm{nc}}$ of which the trace on the boundary of the associated frequency is equal to $\lambda\,\nu$, which is fully determined by (2.19). Therefore, since $\nu\in\mathcal{F}_b\setminus\Upsilon$, we get, for $j=1,3$,

$$\left(\sigma^2_{\nu,j,\lambda}\right)_{|x_d=0}r_{\nu,j}=\mu_{\nu,j}\left(\sigma^2_{\nu,2,\lambda}\right)_{|x_d=0}r_{\nu,j}+\widetilde{F}^2_{\nu,j,\lambda},\tag{2.34}$$

where $\mu_{\nu,j}$ has been defined in equation (2.14) and where we have denoted by $\widetilde{F}^2_{\nu,j,\lambda}$ the function

$$\begin{aligned}\widetilde{F}^2_{\nu,j,\lambda}:=-\ell_{\nu,j}\cdot A_d(0)\left(B_{|E_-(\nu)}\right)^{-1}\Big(&B\left(I-P_{\nu,1}\right)\left(U^2_{\nu,1,\lambda}\right)_{|x_d,\chi_d=0}\\&+B\left(I-P_{\nu,3}\right)\left(U^2_{\nu,3,\lambda}\right)_{|x_d,\chi_d=0}+B\left(I-P_{\nu,2}\right)\left(U^2_{\nu,2,\lambda}\right)_{|x_d,\chi_d=0}+B\left(U^{2,\mathrm{nc}}_{\nu,\lambda}\right)_{|x_d,\chi_d=0}\Big).\end{aligned}$$

In the same way as for the leading profile, we need to investigate the nonpolarized part of the second corrector to find equations on $a^2_{\varphi,\lambda}$ and $a^2_{\psi,\lambda}$.

2.2.4. *Nonpolarized part of the second corrector.* We follow the same analysis as for the first corrector. With similar arguments we get that the second corrector U_3 reads

$$\begin{aligned}U_3(z,\theta,\chi_d)=U_3^*(z)&+\sum_{\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^2}}\sum_{j\in\mathcal{C}(\mathbf{n}\cdot\boldsymbol{\zeta})}\sum_{\lambda\in\mathbb{Z}^*}U^{3,\mathrm{osc}}_{\mathbf{n},j,\lambda}(z)\,e^{i\,\lambda\,\mathbf{n}\cdot\theta}\,e^{i\,\lambda\,\xi_j(\mathbf{n}\cdot\boldsymbol{\zeta})\,\chi_d}\\&+\sum_{\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^2}}\sum_{\lambda\in\mathbb{Z}^*}e^{\chi_d\mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})}\,\Pi^e(\mathbf{n}\cdot\boldsymbol{\zeta})\,U^{3,\mathrm{ev}}_{\mathbf{n}}(z,0)\,e^{i\,\lambda\,\mathbf{n}\cdot\theta}+U^{3,\mathrm{nc}}(z,\theta,\chi_d),\end{aligned}\tag{2.35}$$

with U_3^* the mean value of U_3, $U^{3,\mathrm{nc}}$ the noncharacteristic terms, and where, for $\zeta=\mathbf{n}\cdot\boldsymbol{\zeta}\in\mathcal{F}_b\setminus\{0\}$, $\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^2}$, $j\in\mathcal{C}(\zeta)$ and $\lambda\in\mathbb{Z}^*$, profile $U^{3,\mathrm{osc}}_{\mathbf{n},j,\lambda}$ decomposes as

$$U^{3,\mathrm{osc}}_{\mathbf{n},j,\lambda}=\sigma^3_{\zeta\,j,\lambda}\,r_{\zeta,j}+\left(I-P_{\zeta,j}\right)U^{3,\mathrm{osc}}_{\mathbf{n},j,\lambda},$$

with $\sigma^3_{\zeta,j,\lambda}$ a scalar function of Ω_T. Furthermore, according to (2.2c) for $n=2$, the noncharacteristic part $U^{3,\mathrm{nc}}$ is given by

$$\begin{aligned}\mathcal{L}(\partial_\theta,\partial_{\chi_d})\,U^{3,\mathrm{nc}}=&\\&-\sum_{\substack{(\zeta_1,\zeta_2,j_1,j_2,\\ \lambda_1,\lambda_2)\in\mathcal{NR}}}\Big(L_1\big(U^{1,\mathrm{osc}}_{\zeta_1,j_1,\lambda_1},i\,\lambda_2\alpha_{j_2}(\zeta_2)\big)\,U^{2,\mathrm{osc}}_{\zeta_2,j_2,\lambda_2}+L_1\big(U^{2,\mathrm{osc}}_{\zeta_1,j_1,\lambda_1},i\,\lambda_2\alpha_{j_2}(\zeta_2)\big)\,U^{1,\mathrm{osc}}_{\zeta_2,j_2,\lambda_2}\\&+L_1\big(U^{1,\mathrm{osc}}_{\zeta_1,j_1,\lambda_1},\partial_z\big)\,U^{1,\mathrm{osc}}_{\zeta_2,j_2,\lambda_2}\Big)\,e^{i(\lambda_1\mathbf{n}_1+\lambda_2\mathbf{n}_2)\cdot\theta}\,e^{i(\lambda_1\xi_{j_1}(\zeta_1)+\lambda_2\xi_{j_2}(\zeta_2))\chi_d},\end{aligned}\tag{2.36}$$

where the set $\mathcal{NR}$ of nonresonant frequencies has already been defined. Since all frequencies in $U^{3,\mathrm{nc}}$ are noncharacteristic, equation (2.36) totally determines $U^{3,\mathrm{nc}}$. Note that opposite to what was done for the first corrector, this is no longer true that in $U^{3,\mathrm{nc}}$ there are only profiles of modes φ_j, ψ_j and ν_j, since now second order profiles $\sigma^2_{\zeta,j,\lambda}$ for $\zeta\in\mathcal{F}_b\setminus\{0,\varphi,\psi,\nu\}$ are possibly nonzero.

For the same reason, profiles $U^{3,\mathrm{osc}}_{\mathbf{n},j,\lambda}$ for $\mathbf{n}\cdot\boldsymbol{\zeta}\in\mathcal{F}_b\setminus\{0,\varphi,\psi,\nu\}$ are not necessarily polarized. Therefore we derive now the nonpolarized part for each frequency $\zeta=\mathbf{n}\cdot\boldsymbol{\zeta}\in\mathcal{F}_b\setminus\{0\}$, $\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^2}$ and $j\in\mathcal{C}(\zeta)$, $\lambda\in\mathbb{Z}^*$. Multiplying equation (2.2c) for $n=2$ and frequency $\lambda\,\alpha_j(\zeta)$ on the left by the partial inverse $R_{\zeta,j}$ leads to the relation

$$\begin{aligned}i\lambda\left(I-P_{\zeta,j}\right)U^{3,\mathrm{osc}}_{\mathbf{n},j,\lambda}=&\\&-R_{\zeta,j}\,L(0,\partial_z)\,\sigma^2_{\zeta,j,\lambda}\,r_{\zeta,j}-R_{\zeta,j}\,L_1\big(r_{\zeta,j},\alpha_j(\zeta)\big)\,r_{\zeta,j}\sum_{\lambda_1+\lambda_2=\lambda}i\lambda\,\sigma^1_{\zeta,j,\lambda_1}\,\sigma^2_{\zeta,j,\lambda_2}\\&-R_{\zeta,j}\,\mathbb{1}_{\lambda=k\lambda_\zeta}\sum_{\substack{(\zeta_1,\zeta_2,j_1,j_2)\\ \in\mathcal{R}(\zeta,j)}}L_1\big(r_{\zeta_1,j_1},-\lambda_{\zeta_2}\alpha_{j_2}(\zeta_2)\big)\,r_{\zeta_2,j_2}\,ik\,\sigma^1_{\zeta_1,j_1,-k\lambda_{\zeta_1}}\,\sigma^2_{\zeta_2,j_2,-k\lambda_{\zeta_2}}\\&-R_{\zeta,j}\,\partial_z\ \textit{terms in}\ \big(U_1,(I-P)\,U_2,U_2^*\big),\end{aligned}\tag{2.37}$$

where notation ∂_z *terms in* $\big(U_1,(I-P)\,U_2\big)$ refers to quadratic terms involving the leading profile U_1, the nonpolarized parts of the first corrector U_2 and the mean value U_2^*, with possibly first order derivative in front of it. The key point is that since the leading profile is polarized and since the mean value U_1^* is zero, all the terms involving a profile $\sigma^2_{\zeta',j',\lambda'}$ depend only on leading order polarized profiles $\sigma^1_{\zeta'',j'',\lambda''}$, and not on $(I-P)\,U_2$ or U_2^*. We now write the boundary conditions for the second corrector, for the frequencies φ and ψ, which will lead to equations on the amplitudes a^2_φ and a^2_ψ.

For φ we have, according to boundary condition (2.3b) and writing (2.35) of U_3, since the elliptic component $E^e_-(\varphi)$ of the stable subspace $E_-(\zeta)$ is zero,

$$
\begin{aligned}
(2.38)\qquad & B\,P_{\varphi,1}\,\big(U^3_{\varphi,1,\lambda}\big)_{|x_d,\chi_d=0}+B\,P_{\varphi,3}\,\big(U^3_{\varphi,3,\lambda}\big)_{|x_d,\chi_d=0}\\
&+B\,(I-P_{\varphi,1})\,\big(U^3_{\varphi,1,\lambda}\big)_{|x_d,\chi_d=0}+B\,(I-P_{\varphi,3})\,\big(U^3_{\varphi,3,\lambda}\big)_{|x_d,\chi_d=0}\\
&+B\,\big(U^{3,\mathrm{osc}}_{\varphi,2,\lambda}\big)_{|x_d,\chi_d=0}+B\,\big(U^{3,\mathrm{nc}}_{\varphi,\lambda}\big)_{|x_d,\chi_d=0}=0,
\end{aligned}
$$

where $U^{3,\mathrm{nc}}_{\varphi,\lambda}$ is the sum of all the terms of $U^{3,\mathrm{nc}}$ of which the trace on the boundary of the associated frequency is equal to $\lambda\,\varphi$, namely, according to expression (2.36) of $U^{3,\mathrm{nc}}$,

$$
\begin{aligned}
(2.39)\qquad U^{3,\mathrm{nc}}_{\varphi,\lambda}=\mathbb{1}_{\lambda=k\lambda_\varphi}&\sum_{\substack{j_1,j_2=1,2,3\\(j_1,j_2)\neq\\(2,1),(2,2)}}L\big(0,k\nu_{j_1}-\lambda_\psi k\psi_{j_2}\big)^{-1}\big\{\lambda_\psi L_1(r_{\nu,j_1},\psi_{j_2})\,r_{\psi,j_2}-L_1(r_{\psi,j_2},\nu_{j_1})\,r_{\nu,j_1}\big\}\\
&k\,\big\{\sigma^1_{\nu,j_1,k}\,\sigma^2_{\psi,j_2,-\lambda_\psi k}+\sigma^2_{\nu,j_1,k}\,\sigma^1_{\psi,j_2,-\lambda_\psi k}\big\}\,e^{ik\lambda_\varphi\theta_1}\,e^{i(k\xi_{j_1}(\nu)-\lambda_\psi k\xi_{j_2}(\psi))\chi_d}\\
-\sum_{\lambda_1+\lambda_2=\lambda}&L\big(0,\lambda_1\varphi_1+\lambda_2\varphi_3\big)^{-1}\big\{\lambda_1\,L_1(r_{\varphi,3},\varphi_1)\,r_{\varphi,1}+\lambda_2\,L_1(r_{\varphi,1},\varphi_3)\,r_{\varphi,3}\big\}\\
&\big\{\sigma^1_{\varphi,1,\lambda_1}\,\sigma^2_{\varphi,3,\lambda_2}+\sigma^2_{\varphi,1,\lambda_1}\,\sigma^1_{\varphi,3,\lambda_2}\big\}\,e^{i\lambda\,\theta_1}\,e^{i(\lambda_1\xi_1(\varphi)+\lambda_2\xi_3(\varphi))\chi_d}\\
+\,&\partial_{z,\theta}\ terms\ in\ \big(U_1,(I-P)\,U_2,(P\,U_2)_{\zeta\neq\varphi,\psi,\nu},U_2^*\big),
\end{aligned}
$$

where the notation ∂_z *terms in* $\big(U_1,(I-P)\,U_2,(P\,U_2)_{\zeta\neq\varphi,\psi,\nu},U_2^*\big)$ refers to quadratic terms in U_1, the nonpolarized part of U_2, the polarized part of U_2 of which the associated modes are different from $\lambda\varphi_j$, $\lambda\psi_j$ and $\lambda\nu_j$, and the mean value U_2^*, with possibly one derivative in front of it. Once again, the key point here is that since U_1 is polarized and of zero mean value, and since only the profiles $\sigma^1_{\zeta,j,\lambda}$ for $\zeta=\varphi,\psi,\nu$ are nonzero, in $U^{3,\mathrm{nc}}_{\varphi,\lambda}$, every term involving $\sigma^2_{\zeta,j,\lambda'}$ for $\zeta=\varphi,\psi,\nu$, is a quadratic term with a profile $\sigma^1_{\zeta',j',\lambda''}$ for $\zeta'=\varphi,\psi,\nu$.

Similarly as for the leading profile, taking the scalar product of b_φ with equation (2.38) multiplied by $i\lambda$, using (2.37), (2.39) and expression of traces (2.32), (2.33) and (2.34), we get

$$
\begin{aligned}
(2.40)\qquad & X^{\mathrm{Lop}}_\varphi\,a^2_{\varphi,\lambda}+D^{\mathrm{Lop}}_\varphi\,i\lambda\sum_{\lambda_1+\lambda_2=\lambda}a^1_{\varphi,\lambda_1}\,a^2_{\varphi,\lambda_2}+i\lambda\sum_{\lambda_1+\lambda_2=\lambda}\gamma_\varphi(\lambda_1,\lambda_2)\big(a^1_{\varphi,\lambda_1}\,a^2_{\varphi,\lambda_2}+a^2_{\varphi,\lambda_1}\,a^1_{\varphi,\lambda_2}\big)\\
&+\mathbb{1}_{\lambda=k\lambda_\varphi}\,\Gamma^\varphi\,ik\,\Big\{\big(\sigma^1_{\nu,2,-k}\big)_{|x_d=0}\,a^2_{\psi,-k\lambda_\psi}+\big(\sigma^2_{\nu,2,-k}\big)_{|x_d=0}\,a^1_{\psi,-k\lambda_\psi}\Big\}\\
&+\partial_{z,\theta}\ terms\ in\ \Big[\widetilde{F}^2_\varphi,\big(\widetilde{F}^2_\varphi,a^1_\varphi\big),\big(\widetilde{F}^2_\psi,(\sigma^1_{\nu,2})_{|x_d=0}\big),\big((\sigma^2_\varphi)_{|x_d=0},a^1_\varphi\big),\big((\sigma^2_\psi)_{|x_d=0},(\sigma^1_{\nu,2})_{|x_d=0}\big)\Big]\\
&=i\lambda\,b_\psi\cdot B\,\big(U^{3,\mathrm{osc}}_{\varphi,2,\lambda}\big)_{|x_d,\chi_d=0}+\partial_{z,\theta}\ terms\ in\ \big(U_1,(I-P)\,U_2,(P\,U_2)_{\zeta\neq\varphi,\psi,\nu},U_2^*\big)_{|x_d,\chi_d=0},
\end{aligned}
$$

where X^{Lop}_φ, D^{Lop}_φ, $\gamma_\varphi(\lambda_1,\lambda_2)$ and Γ^φ have already been defined in (2.25), and where

$$
\partial_{z,\theta}\ terms\ in\ \Big[\widetilde{F}^2_\varphi,\big(\widetilde{F}^2_\varphi,a^1_\varphi\big),\big(\widetilde{F}^2_\psi,(\sigma^1_{\nu,2})_{|x_d=0}\big),\big((\sigma^2_\varphi)_{|x_d=0},a^1_\varphi\big),\big((\sigma^2_\psi)_{|x_d=0},(\sigma^1_{\nu,2})_{|x_d=0}\big)\Big]
$$

refers to linear terms in $\widetilde{F}^2_{\varphi,j,\lambda}$ for $j=1,2$ and $\lambda\in\mathbb{Z}^*$, and quadratic terms in $\big(\widetilde{F}^2_{\varphi,j,\lambda},a^1_{\varphi,\lambda}\big)$, $\big(\widetilde{F}^2_{\psi,j,\lambda},(\sigma^1_{\nu,2,\lambda})_{|x_d=0}\big)$, $\big((\sigma^2_{\varphi,j,\lambda})_{|x_d=0},a^1_{\varphi,\lambda}\big)$ or $\big((\sigma^2_{\psi,j,\lambda})_{|x_d=0},(\sigma^1_{\nu,2,\lambda})_{|x_d=0}\big)$ for $j=1,2$ and $\lambda\in\mathbb{Z}^*$,

with, for all these terms, possibly one derivative in front of them. Recall that terms $\widetilde{F}^2_\varphi$ and $\widetilde{F}^2_\psi$ depend on the traces $(\sigma^2_{\varphi,2,\lambda})_{|x_d=0}$ and $(\sigma^2_{\psi,2,\lambda})_{|x_d=0}$.

For phase ψ we have the following boundary condition

$$
\begin{aligned}
(2.41)\qquad & B\,P_{\psi,1}\,\big(U^3_{\psi,1,\lambda}\big)_{|x_d,\chi_d=0} + B\,P_{\psi,3}\,\big(U^3_{\psi,3,\lambda}\big)_{|x_d,\chi_d=0} \\
& + B\,(I-P_{\psi,1})\,\big(U^3_{\psi,1,\lambda}\big)_{|x_d,\chi_d=0} + B\,(I-P_{\psi,3})\,\big(U^3_{\psi,3,\lambda}\big)_{|x_d,\chi_d=0} \\
& + B\,\big(U^3_{\psi,2,\lambda}\big)_{|x_d,\chi_d=0} + B\,\big(U^{3,\mathrm{nc}}_{\psi,\lambda}\big)_{|x_d,\chi_d=0} = H_\lambda,
\end{aligned}
$$

where we have expanded H in Fourier series as

$$
H(z',\Theta) = \sum_{\lambda\in\mathbb{Z}} H_\lambda(z')\,e^{i\lambda\Theta},
$$

and $U^{3,\mathrm{nc}}_{\psi,\lambda}$ is the sum of all the terms of $U^{3,\mathrm{nc}}$ of which the trace on the boundary of the associated frequency is equal to $\lambda\psi$, that is,

$$
\begin{aligned}
(2.42)\qquad U^{3,\mathrm{nc}}_{\psi,\lambda} = {} & \mathbb{1}_{\lambda=k\lambda_\psi} \sum_{\substack{j_1=1,3, j_2=1,2,3\\ (j_1,j_2)\\ \neq(1,2),(3,2)}} L\big(k\nu_{j_2}-\lambda_\varphi k\varphi_{j_1}\big)^{-1}\big\{\lambda_\varphi\,L_1(r_{\nu,j_2},\varphi_{j_1})\,r_{\varphi,j_1} - L_1(r_{\varphi,j_1},\nu_{j_2})\,r_{\nu,j_2}\big\} \\
& \qquad k\,\big\{\sigma^1_{\nu,j_2,k}\,\sigma^2_{\varphi,j_1,-\lambda_\varphi k} + \sigma^2_{\nu,j_2,k}\,\sigma^1_{\varphi,j_1,-\lambda_\varphi k}\big\}\,e^{ik\lambda_\varphi\theta_1}\,e^{i(k\xi_{j_2}(\nu)-\lambda_\varphi k\xi_{j_1}(\varphi))\chi_d} \\
& - \sum_{\substack{j_1,j_2=1,2,3\\ j_1\neq j_2}} \sum_{\lambda_1+\lambda_2=\lambda} L\big(\lambda_1\psi_{j_1}+\lambda_2\psi_{j_2}\big)^{-1}\,L_1(r_{\psi,j_1},\psi_{j_2})\,r_{\psi,j_2}\,\lambda_2\,\big\{\sigma^1_{\psi,j_1,\lambda_1}\,\sigma^2_{\psi,j_2,\lambda_2} \\
& \qquad + \sigma^2_{\psi,j_1,\lambda_1}\,\sigma^1_{\psi,j_2,\lambda_2}\big\}\,e^{i\lambda\,\theta_2}\,e^{i(\lambda_1\xi_{j_1}(\psi)+\lambda_2\xi_{j_2}(\psi))\chi_d} \\
& + \partial_{z,\theta}\ \textit{terms in}\ \big(U_1,(I-P)\,U_2,(P\,U_2)_{\zeta\neq\varphi,\psi,\nu},U_2^*\big).
\end{aligned}
$$

If we take the scalar product of b_ψ with the equality (2.41) multiplied by λ, using (2.37), (2.39) and expression of traces (2.32), (2.33) and (2.34), we get the amplitude equation

$$
\begin{aligned}
(2.43)\qquad & X^{\mathrm{Lop}}_\psi\,a^2_{\psi,\lambda} + D^{\mathrm{Lop}}_\psi\,i\lambda \sum_{\lambda_1+\lambda_2=\lambda} a^1_{\psi,\lambda_1}\,a^2_{\psi,\lambda_2} + i\lambda \sum_{\lambda_1+\lambda_2=\lambda} \gamma_\psi(\lambda_1,\lambda_2)\big(a^1_{\psi,\lambda_1}\,a^2_{\psi,\lambda_2} + a^2_{\psi,\lambda_1}\,a^1_{\psi,\lambda_2}\big) \\
& + \mathbb{1}_{\lambda=k\lambda_\psi}\,\Gamma^\psi\,ik\,\big\{\big(\sigma^1_{\nu,2,k}\big)_{|x_d=0}\,a^2_{\varphi,-\lambda_\varphi k} + \big(\sigma^2_{\nu,2,k}\big)_{|x_d=0}\,a^1_{\varphi,-\lambda_\varphi k}\big\} \\
& + \partial_{z,\theta}\ \textit{terms in}\ \Big[\widetilde{F}^2_\psi,\big(\widetilde{F}^2_\psi,a^1_\psi\big),\big(\widetilde{F}^2_\varphi,(\sigma^1_{\nu,2})_{|x_d=0}\big),\big((\sigma^2_\psi)_{|x_d=0},a^1_\psi\big),\big((\sigma^2_\varphi)_{|x_d=0},(\sigma^1_{\nu,2})_{|x_d=0}\big)\Big] \\
& = i\lambda\,b_\psi\cdot B\,\big(U^{3,\mathrm{osc}}_{\psi,2,\lambda}\big)_{|x_d,\chi_d=0} - i\lambda\,b_\psi\cdot H_\lambda \\
& + \partial_{z,\theta}\ \textit{terms in}\ \big(U_1,(I-P)\,U_2,(P\,U_2)_{\zeta\neq\varphi,\psi,\nu},U_2^*\big)_{|x_d,\chi_d=0},
\end{aligned}
$$

where X^{Lop}_ψ, D^{Lop}_ψ, $\gamma_\psi(\lambda_1,\lambda_2)$, and Γ^ψ have already been defined in (2.29), and where

$$
\partial_{z,\theta}\ \textit{terms in}\ \Big[\widetilde{F}^2_\psi,\big(\widetilde{F}^2_\psi,a^1_\psi\big),\big(\widetilde{F}^2_\varphi,(\sigma^1_{\nu,2})_{|x_d=0}\big),\big((\sigma^2_\psi)_{|x_d=0},a^1_\psi\big),\big((\sigma^2_\varphi)_{|x_d=0},(\sigma^1_{\nu,2})_{|x_d=0}\big)\Big]
$$

refers to linear terms in $\widetilde{F}^2_{\varphi,j,\lambda}$ for $j=1,2$ and $\lambda\in\mathbb{Z}^*$, and quadratic terms in $\big(\widetilde{F}^2_{\psi,j,\lambda},a^1_{\psi,\lambda}\big)$, $\big(\widetilde{F}^2_{\varphi,j,\lambda},(\sigma^1_{\nu,2,\lambda})_{|x_d=0}\big)$, $\big((\sigma^2_{\psi,j,\lambda})_{|x_d=0},a^1_{\psi,\lambda}\big)$ or $\big((\sigma^2_{\varphi,j,\lambda})_{|x_d=0},(\sigma^1_{\nu,2,\lambda})_{|x_d=0}\big)$ for $j=1,2$ and $\lambda\in\mathbb{Z}^*$, with, for all these terms, possibly one derivative in front of them.

Note that equations (2.40) and (2.43) can be seen as linearizations around the trace of the leading profile U_1 of equations (2.24) and (2.28). This is usual in weakly nonlinear geometric optics, where equations for the leading profile are nonlinear, and equations for higher order are linearizations of the former equations around the leading profile U_1. Again, the obtained system of equations is still not closed since traces of the second corrector U_3 appear in amplitude equations (2.40) and (2.43).

With the obtained equations, can have the intuition on how lower terms ascent toward higher order terms, eventually leading to an instability. In equation (2.43) for amplitudes $a^2_{\psi,\lambda}$, the boundary forcing term H occurs, and therefore this forcing term ascents to first corrector profiles $\sigma^2_{\psi,j,\lambda}$ for $j=1,3$ and $\lambda\in\mathbb{Z}^*$ through boundary conditions (2.33). Eventually, because of the resonances (1.12) leading to resonances terms in transport equations (2.31) for first corrector profiles, the boundary term H arises in profiles $\sigma^2_{\psi,2,\lambda}$ for $\lambda\in\mathbb{Z}^*$. In their turn, these profiles $\sigma^2_{\psi,2,\lambda}$ for $\lambda\in\mathbb{Z}^*$ interfere in amplitude equation (2.28) for $a^1_{\psi,\lambda}$, for $\lambda\in\mathbb{Z}^*$, because of the trace $\big(U^2_{\psi,2,\lambda}\big)_{|x_d,\xi_d=0}$. Then this reasoning can be applied recursively to obtain that the boundary forcing term H interferes in leading profiles $\sigma^1_{\zeta,j,\lambda}$, for $\zeta=\varphi,\psi,\nu$, $j=1,2,3$ and $\lambda\in\mathbb{Z}^*$.

2.3. **General system.** The above arguments can be extended recursively to any corrector U_n, $n\geqslant 3$. Doing so we get that the n-th profile U_n reads

$$\begin{aligned}(2.44)\quad U_n(z,\theta,\chi_d)=U_n^*(z)+\sum_{\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^2}}\sum_{j\in\mathcal{C}(\mathbf{n})}\sum_{\lambda\in\mathbb{Z}^*}U^{n,\mathrm{osc}}_{\mathbf{n},j,\lambda}\,e^{i\lambda\,\mathbf{n}\cdot\theta}\,e^{i\lambda\xi_j(\mathbf{n}\cdot\boldsymbol{\zeta})\chi_d}\\ +\sum_{\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^2}}\sum_{\lambda\in\mathbb{Z}^*}e^{\chi_d\mathcal{A}(\mathbf{n}\cdot\boldsymbol{\zeta})}\,\Pi^e(\mathbf{n}\cdot\boldsymbol{\zeta})\,U^{n,\mathrm{ev}}_{\mathbf{n}}(z,0)\,e^{i\,\lambda\,\mathbf{n}\cdot\theta}+U^{n,\mathrm{nc}}(z,\theta,\chi_d),\end{aligned}$$

with U_n^* the mean value of U_n, $U^{n,\mathrm{nc}}$ the noncharacteristic terms, and where, for $\zeta=\mathbf{n}\cdot\boldsymbol{\zeta}\in\mathcal{F}_b\setminus\{0\}$, $\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^2}$, $j\in\mathcal{C}(\zeta)$ and $\lambda\in\mathbb{Z}^*$, the oscillating profile $U^{n,\mathrm{osc}}_{\mathbf{n},j,\lambda}$ decomposes as

$$U^{n,\mathrm{osc}}_{\mathbf{n},j,\lambda}=\sigma^n_{\zeta\,j,\lambda}\,r_{\zeta,j}+\big(I-P_{\zeta,j}\big)\,U^{n,\mathrm{osc}}_{\mathbf{n},j,\lambda},$$

with $\sigma^n_{\zeta,j,\lambda}$ a scalar function defined on Ω_T. According to equation (2.2c) for $n-1$ and since U_1 is polarized and of zero mean value, $U^{n,\mathrm{nc}}$ is determined by the formula

$$\begin{aligned}(2.45)\qquad \mathcal{L}(\partial_\theta,\partial_{\chi_d})\,U^{n,\mathrm{nc}}=-\sum_{\substack{(\zeta_1,\zeta_2,j_1,j_2,\\ \lambda_1,\lambda_2)\in\mathcal{NR}}}L_1(r_{\zeta_1,j_1},\alpha_{j_2}(\zeta_2))\,r_{\zeta_2,j_2}\,i\lambda\,\sigma^1_{\zeta_1,j_1,\lambda_1}\,\sigma^{n-1}_{\zeta_2,j_2,\lambda_2}\\ e^{i(\lambda_1\mathbf{n}_1+\lambda_2\mathbf{n}_2)\cdot\theta}\,e^{i(\lambda_1\xi_{j_1}(\zeta_1)+\lambda_2\xi_{j_2}(\zeta_2))\chi_d}\\ +\partial_{z,\theta}\,terms\ in\ \big(U_1,\dots,U_{n-2},(I-P)\,U_{n-1},U^*_{n-1}\big),\end{aligned}$$

where notation $\partial_{z,\theta}\,terms\ in\ \big(U_1,\dots,U_{n-2},(I-P)\,U_{n-1},U^*_{n-1}\big)$ refers to quadratic terms involving the profiles $U_1,\dots,U_{n-2}$, the nonpolarized parts of the corrector U_{n-1} and the mean value U^*_{n-1}, with possibly first order derivatives in front of it. As for it, for $\zeta=\mathbf{n}\cdot\boldsymbol{\zeta}\in\mathcal{F}_b\setminus\{0\}$, $\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^2}$ and $j\in\mathcal{C}(\zeta)$, $\lambda\in\mathbb{Z}^*$, the nonpolarized part $\big(I-P_{\zeta,j}\big)\,U^{n,\mathrm{osc}}_{\zeta,j,\lambda}$ of $U^{n,\mathrm{osc}}_{\zeta,j,\lambda}$ is given by

$$\begin{aligned}(2.46)\quad i\lambda\,\big(I-P_{\zeta,j}\big)\,U^{n,\mathrm{osc}}_{\mathbf{n},j,\lambda}=\\ -R_{\zeta,j}\,L(0,\partial_z)\,\sigma^{n-1}_{\zeta,j,\lambda}\,r_{\zeta,j}-R_{\zeta,j}\,L_1\big(r_{\zeta,j},\alpha_j(\zeta)\big)\,r_{\zeta,j}\sum_{\lambda_1+\lambda_2=\lambda}i\lambda\,\sigma^1_{\zeta,j,\lambda_1}\,\sigma^{n-1}_{\zeta,j,\lambda_2}\\ -R_{\zeta,j}\,\mathbb{1}_{\lambda=k\lambda_\zeta}\sum_{\substack{(\zeta_1,\zeta_2,j_1,j_2)\\ \in\mathcal{R}(\zeta,j)}}L_1\big(r_{\zeta_1,j_1},-\lambda_{\zeta_2}\alpha_{j_2}(\zeta_2)\big)\,r_{\zeta_2,j_2}\,ik\,\sigma^1_{\zeta_1,j_1,-k\lambda_{\zeta_1}}\,\sigma^{n-1}_{\zeta_2,j_2,-k\lambda_{\zeta_2}}\\ -R_{\zeta,j}\,\partial_{z,\theta}\,terms\ in\ \big(U_1,\dots,U_{n-2},(I-P)\,U_{n-1},U^*_{n-1}\big).\end{aligned}$$

This formula is obtained by multiplying equation (2.2c) for $n-1$ by the partial inverse $R_{\zeta,j}$, using that U_1 is polarized and of zero mean value.

We specify now equations satisfied by the mean value U_n^* and the polarized components $\sigma^n_{\zeta,j,\lambda}$. Since U_1 is polarized and of zero mean value, equations (2.2c), (2.3b) for n and (2.4) lead to the following system for the mean value U_n^*:

$$(2.47)\qquad\begin{cases}L(0,\partial_z)\,U_n^*=\partial_{z,\theta}\,terms\ in\ \big(U_1,\dots,U_{n-1},(I-P)\,U_n\big)\\ B\,\big(U_n^*\big)_{|x_d=0}=\mathbb{1}_{n=3}\,H_0-B\,\big(U_0^{n,\mathrm{nc}}\big)_{|x_d,\chi_d=0}\\ B\,\big(U_n^*\big)_{|t\leqslant 0}=0,\end{cases}$$

where $U_0^{n,\mathrm{nc}}$ refers to the sum of all terms of $U^{n,\mathrm{nc}}$ of which the trace on the boundary is of zero frequency.

For a frequency $\zeta = \mathbf{n}\cdot\boldsymbol{\zeta} \in \mathcal{F}_b \setminus \{0,\varphi,\psi,\nu\}$, $\mathbf{n}\in\mathcal{B}_{\mathbb{Z}^2}$, $j\in\mathcal{C}(\zeta)$ and $\lambda\in\mathbb{Z}^*$, multiplying equation (2.2c) by $\ell_{\zeta,j}$ leads to, since all harmonics $\sigma^1_{\zeta,j,\lambda'}$, $\lambda\in\mathbb{Z}^*$ are zero and since U_1 is polarized and of zero mean value,

$$X_{\alpha_j(\zeta)}\,\sigma^n_{\zeta,j,\lambda} = \partial_{z,\theta}\ \textit{terms in}\ \big(U_1,\dots,U_{n-1},(I-P)\,U_n,U_n^*\big) \tag{2.48a}$$

$$\big(\sigma^n_{\zeta,j,\lambda}\big)_{|t\leqslant 0} = 0, \tag{2.48b}$$

with initial condition (2.4), and, if $j\in\mathcal{I}(\zeta)$, boundary condition (2.3b) gives, since $\zeta\in\mathcal{F}_b\setminus\Upsilon$,

$$\big(\sigma^n_{\zeta,j,\lambda}\big)_{|x_d=0} = -\ell_{\zeta,j}\,A_d(0)\,\big(E_-(\zeta)\big)^{-1}\,B\,\big(U^{n,\mathrm{nc}}_{\zeta,\lambda}\big)_{|x_d,\chi_d=0}. \tag{2.48c}$$

Finally, for $\zeta=\mathbf{n}\cdot\boldsymbol{\zeta}\in\{\varphi,\psi,\nu\}$, $j\in\mathcal{C}(\zeta)$ and $\lambda\in\mathbb{Z}^*$, multiplying equation (2.2c) by $\ell_{\zeta,j}$ gives, with the same arguments,

$$\begin{aligned} &X_{\zeta,j}\,\sigma^n_{\zeta,j,\lambda} + D_{\zeta,j}\sum_{\lambda_1+\lambda_2=\lambda} i\lambda\,\sigma^n_{\zeta,j,\lambda_1}\,\sigma^1_{\zeta,j,\lambda_2} + \mathbb{1}_{\lambda=k\lambda_\zeta}\sum_{\substack{(\zeta_1,\zeta_2,j_1,j_2)\\ \in\mathcal{R}(\zeta,j)}} J^{\zeta_2,j_2}_{\zeta_1,j_1}\,ik\,\sigma^1_{\zeta_1,j_1,-k\lambda_{\zeta_1}}\,\sigma^n_{\zeta_2,j_2,-k\lambda_{\zeta_2}} \\ &= \partial_{z,\theta}\ \textit{terms in}\ \big(U_1,\dots,U_{n-1},(I-P)\,U_n,U_n^*\big), \end{aligned} \tag{2.49a}$$

where the notation are defined by (2.10). Equation (2.49a) is coupled with the following initial condition,

$$\big(\sigma^n_{\zeta,j,\lambda}\big)_{|t\leqslant 0} = 0. \tag{2.49b}$$

It remains to determine the traces on the boundary of the corresponding incoming frequencies. The trace of the amplitudes associated with the boundary phase φ on the boundary is given by, for $j=1,3$,

$$\big(\sigma^n_{\varphi,j,\lambda}\big)_{|x_d=0}\,r_{\varphi,j} = a^n_{\varphi,\lambda}\,e_{\varphi,j} + \widetilde{F}^n_{\varphi,j,\lambda}, \tag{2.50a}$$

with $a^n_{\varphi,\lambda}$ a scalar function defined on ω_T and

$$\begin{aligned} \widetilde{F}^n_{\varphi,j,\lambda} := -\ell_{\varphi,j}\cdot A_d(0)\,\big(B_{|E_-(\varphi)}\big)^{-1}\Big(& B\,(I-P_{\varphi,1})\,\big(U^{n,\mathrm{osc}}_{\varphi,1,\lambda}\big)_{|x_d,\chi_d=0} \\ & + B\,(I-P_{\varphi,3})\,\big(U^{n,\mathrm{osc}}_{\varphi,3,\lambda}\big)_{|x_d,\chi_d=0} + B\,(I-P_{\varphi,2})\,\big(U^{n,\mathrm{osc}}_{\varphi,2,\lambda}\big)_{|x_d,\chi_d=0} \\ & + B\,\big(\sigma^n_{\varphi,2,\lambda}\big)_{|x_d=0}\,r_{\varphi,2} + B\,\big(U^{n,\mathrm{nc}}_{\varphi,\lambda}\big)_{|x_d,\chi_d=0}\Big)\,r_{\varphi,j}. \end{aligned}$$

For ψ we have, for $j=1,3$,

$$\big(\sigma^n_{\psi,j,\lambda}\big)_{|x_d=0}\,r_{\psi,j} = a^n_{\psi,\lambda}\,e_{\psi,j} + \widetilde{F}^n_{\psi,j,\lambda}, \tag{2.50b}$$

with $a^n_{\psi,\lambda}$ a scalar function of ω_T and

$$\begin{aligned} \widetilde{F}^n_{\psi,j,\lambda} := -\ell_{\psi,j}\cdot A_d(0)\,\big(B_{|E_-(\psi)}\big)^{-1}\Big(& -\mathbb{1}_{n=3}\,H_\lambda + B\,(I-P_{\psi,1})\,\big(U^{n,\mathrm{osc}}_{\psi,1,\lambda}\big)_{|x_d,\chi_d=0} \\ & + B\,(I-P_{\psi,3})\,\big(U^{n,\mathrm{osc}}_{\psi,3,\lambda}\big)_{|x_d,\chi_d=0} + B\,(I-P_{\psi,2})\,\big(U^{n,\mathrm{osc}}_{\psi,2,\lambda}\big)_{|x_d,\chi_d=0} \\ & + B\,\big(\sigma^n_{\psi,2,\lambda}\big)_{|x_d=0}\,r_{\psi,2} + B\,\big(U^{n,\mathrm{nc}}_{\psi,\lambda}\big)_{|x_d,\chi_d=0}\Big)\,r_{\psi,j}. \end{aligned}$$

Note that $\widetilde{F}^n_{\varphi,j,\lambda}$ and $\widetilde{F}^n_{\psi,j,\lambda}$, for $j=1,3$ and $\lambda\in\mathbb{Z}^*$ depends respectively on the traces $\big(\sigma^n_{\varphi,2,\lambda}\big)_{|x_d=0}$ and $\big(\sigma^n_{\psi,2,\lambda}\big)_{|x_d=0}$. Finally, for ν,we have for $j=1,3$,

$$\big(\sigma^n_{\nu,j,\lambda}\big)_{|x_d=0}\,r_{\nu,j} = \mu_{\nu,j}\,\big(\sigma^n_{\nu,2,\lambda}\big)_{|x_d=0}\,r_{\nu,j} + \widetilde{F}^n_{\nu,j,\lambda}, \tag{2.50c}$$

with

$$\widetilde{F}^n_{\nu,j,\lambda} := -\ell_{\nu,j}\cdot A_d(0)\,\big(B_{|E_-(\nu)}\big)^{-1}\Big(B\,(I-P_{\nu,1})\,\big(U^n_{\nu,1,\lambda}\big)_{|x_d,\chi_d=0}$$

$$+ B\left(I - P_{\nu,3}\right)\left(U^n_{\nu,3,\lambda}\right)_{|x_d,\chi_d=0} + B\left(I - P_{\nu,2}\right)\left(U^n_{\nu,2,\lambda}\right)_{|x_d,\chi_d=0} + B\left(U^{n,\mathrm{nc}}_{\nu,\lambda}\right)_{|x_d,\chi_d=0}\Big) r_{\nu,j}.$$

Coefficients $\mu_{\nu,j}$ have been introduced in (2.14).

Scalar functions $a^n_{\varphi,\lambda}$ and $a^n_{\psi,\lambda}$ satisfy the following equations, which are derived using boundary condition (2.3b) and formulas (2.45) and (2.46),

(2.51a)
$$\begin{aligned}
&X^{\mathrm{Lop}}_\varphi\, a^n_{\varphi,\lambda} + D^{\mathrm{Lop}}_\varphi\, i\lambda \sum_{\lambda_1+\lambda_2=\lambda} a^1_{\varphi,\lambda_1}\, a^n_{\varphi,\lambda_2} + i\lambda \sum_{\lambda_1+\lambda_2=\lambda} \gamma_\varphi(\lambda_1,\lambda_2)\big(a^1_{\varphi,\lambda_1}\, a^n_{\varphi,\lambda_2} + a^n_{\varphi,\lambda_1}\, a^1_{\varphi,\lambda_2}\big)\\
&+ \mathbb{1}_{\lambda=k\lambda_\varphi}\, \Gamma^\varphi\, ik\left\{\left(\sigma^1_{\nu,2,-k}\right)_{|x_d=0} a^n_{\psi,-k\lambda_\psi} + \left(\sigma^n_{\nu,2,-k}\right)_{|x_d=0} a^1_{\psi,-k\lambda_\psi}\right\}\\
&+ \partial_{z,\theta}\ terms\ in\left[\widetilde{F}^n_\varphi, \left(\widetilde{F}^n_\varphi, a^1_\varphi\right), \left(\widetilde{F}^n_\psi, (\sigma^1_{\nu,2})_{|x_d=0}\right), \left((\sigma^n_\varphi)_{|x_d=0}, a^1_\varphi\right), \left((\sigma^n_\psi)_{|x_d=0}, (\sigma^1_{\nu,2})_{|x_d=0}\right)\right]\\
&= i\lambda\, b_\varphi \cdot B\left(U^{n+1,\mathrm{osc}}_{\varphi,2,\lambda}\right)_{|x_d,\chi_d=0}\\
&+ \partial_{z,\theta}\ terms\ in\left(U_1,\dots,U_{n-1},(I-P)\,U_n,(P\,U_n)_{\zeta\neq\varphi,\psi,\nu},U^*_n\right)_{|x_d,\chi_d=0},
\end{aligned}$$

and

(2.51b)
$$\begin{aligned}
&X^{\mathrm{Lop}}_\psi\, a^n_{\psi,\lambda} + D^{\mathrm{Lop}}_\psi\, i\lambda \sum_{\lambda_1+\lambda_2=\lambda} a^1_{\psi,\lambda_1}\, a^n_{\psi,\lambda_2} + i\lambda \sum_{\lambda_1+\lambda_2=\lambda} \gamma_\psi(\lambda_1,\lambda_2)\big(a^1_{\psi,\lambda_1}\, a^n_{\psi,\lambda_2} + a^n_{\psi,\lambda_1}\, a^1_{\psi,\lambda_2}\big)\\
&+ \mathbb{1}_{\lambda=k\lambda_\psi}\, \Gamma^\psi\, ik\left\{\left(\sigma^1_{\nu,2,k}\right)_{|x_d=0} a^n_{\varphi,-\lambda_\varphi k} + \left(\sigma^n_{\nu,2,k}\right)_{|x_d=0} a^1_{\varphi,-\lambda_\varphi k}\right\}\\
&+ \partial_{z,\theta}\ terms\ in\left[\widetilde{F}^n_\psi, \left(\widetilde{F}^n_\psi, a^1_\psi\right), \left(\widetilde{F}^n_\varphi, (\sigma^1_{\nu,2})_{|x_d=0}\right), \left((\sigma^n_\psi)_{|x_d=0}, a^1_\psi\right), \left((\sigma^n_\varphi)_{|x_d=0}, (\sigma^1_{\nu,2})_{|x_d=0}\right)\right]\\
&= i\lambda\, b_\psi \cdot B\left(U^{n+1,\mathrm{osc}}_{\psi,2,\lambda}\right)_{|x_d,\chi_d=0} - \mathbb{1}_{n=3}\, b_\psi \cdot H_\lambda\\
&+ \partial_{z,\theta}\ terms\ in\left(U_1,\dots,U_{n-1},(I-P)\,U_n,(P\,U_n)_{\zeta\neq\varphi,\psi,\nu},U^*_n\right)_{|x_d,\chi_d=0},
\end{aligned}$$

where the notation have been defined in (2.25) and (2.29). These two equations come with the following initial conditions

(2.51c)
$$\left(a^n_{\varphi,\lambda}\right)_{|t\leqslant 0} = 0, \qquad \left(a^n_{\psi,\lambda}\right)_{|t\leqslant 0} = 0.$$

The system of equations (2.47), (2.48), (2.49), (2.50) and (2.51) is highly coupled. In all equations for the corrector of order n, there are terms depending on $U_1,\dots,U_{n-1},(I-P)\,U_n,(P\,U_n)_{\zeta\neq\varphi,\psi,\nu},U^*_n$, but this is not a big issue, since, if the lower order correctors $U_1,\dots,U_{n-1}$ are constructed, $(I-P)\,U_n,(P\,U_n)_{\zeta\neq\varphi,\psi,\nu},U^*_n$ can be determined with (2.46), (2.47) and (2.48). The terms inducing coupling which seem the most problematic are the terms $\lambda\, b_\psi \cdot B\left(U^{n+1,\mathrm{osc}}_{\varphi,2,\lambda}\right)_{|x_d,\chi_d=0}$ and $\lambda\, b_\psi \cdot B\left(U^{n+1,\mathrm{osc}}_{\psi,2,\lambda}\right)_{|x_d,\chi_d=0}$ in (2.51a) and (2.51b) which couple evolution equations for $a^n_{\varphi,\lambda}$ and $a^n_{\psi,\lambda}$ (and therefore evolution equations for the corrector U^n of order n), with the corrector of one order higher, U^{n+1}. In equations (2.51) there are also traces of profiles of order n, which prevents to solve this equations (having determined lower order correctors) before solving the evolution equations for U^n.

In addition to being highly coupled, system of equations (2.47), (2.48), (2.49), (2.50) and (2.51) seems also over-determined. Indeed, condition (2.17) imposing that the *outgoing* leading profile $\sigma^1_{\psi,2,\lambda}$ is of zero trace on the boundary gives one more boundary condition than the two boundary conditions prescribed by the structure of the problem. Therefore, this is not clear at all that the system of equations (2.47), (2.48), (2.49), (2.50) and (2.51) admits a solution satisfying the additional condition (2.17).

3. Existence of an analytic solution

In this section we focus on the well-posedness of (2.47), (2.48), (2.49), (2.50) and (2.51). Both because of the high coupling of the system, and the over-determination of it, we choose

to concentrate on a simplified version of the general system and try to prove well-posedness for it. This simplified model should focus on the profiles associated with frequencies φ_j, ψ_j and ν_j, because on one hand it greatly reduces the number of equations, and therefore complexity of the system and of the functional framework, and on the other hand because it seems that, due to amplification and resonances, equations on the profiles associated with φ_j, ψ_j and ν_j carry the main difficulties of system of equations (2.47), (2.48), (2.49) and (2.51). Indeed, we already pointed out that if profiles $\sigma^n_{\zeta,j,\lambda}$ for $n \geqslant 1$, $\zeta = \varphi, \psi, \nu$, $j = 1, 2, 3$ and $\lambda \in \mathbb{Z}^*$ are determined, system (2.47)-(2.48) becomes upper triangular, and could be studied in a rather classical way, see for example [Kil22]. Since we wish to study simplified versions of the system (2.47), (2.48), (2.49), (2.50) and (2.51) in an analytical setting, the initial conditions in (2.47), (2.48), (2.49) and (2.51), requiring that the profiles $\sigma_{\zeta,j,\lambda}$ and their boundary terms $a_{\zeta,\lambda}$ are zero for negative times t are not suited for analytic functions, since it would imply that these profiles and boundary terms are zero everywhere. Therefore, in the simplified models, we modify, in a non equivalent way, these boundary conditions into conditions requiring that the solutions are zero at $t = 0$, which are now adapted for analytic functions.

We start by describing a first simplified model, very simple, which concentrate on boundary equations, and detail the functional framework which will be used to prove well-posedness of it, and proceed with the proof. Then we describe a second simplified model, more elaborate, which incorporates interior (incoming) equations, introduce additional functional framework , make some specifications on the simplified model with regard to the functional framework, and state the main result, before proceeding by proving it.

3.1. **First simplified model.** For the first simplified model, we focus only on boundary equations (2.51). In these equations, terms

$$X^{\mathrm{Lop}}_\varphi\, a^n_{\varphi,\lambda}, \quad D^{\mathrm{Lop}}_\varphi\, i\lambda \sum_{\lambda_1+\lambda_2=\lambda} a^1_{\varphi,\lambda_1}\, a^n_{\varphi,\lambda_2} \quad \text{and} \quad i\lambda \sum_{\lambda_1+\lambda_2=\lambda} \gamma_\varphi(\lambda_1,\lambda_2)\big(a^1_{\varphi,\lambda_1}\, a^n_{\varphi,\lambda_2} + a^n_{\varphi,\lambda_1}\, a^1_{\varphi,\lambda_2}\big)$$

as well as the analogous ones for ψ appear in the chosen first simplified model, and the last one is rewritten as a semilinear term. On the contrary, terms

$$\mathbb{1}_{\lambda=k\lambda_\varphi}\, \Gamma^\varphi\, ik\Big\{\big(\sigma^1_{\nu,2,-k}\big)_{|x_d=0}\, a^n_{\psi,-k\lambda_\psi} + \big(\sigma^n_{\nu,2,-k}\big)_{|x_d=0}\, a^1_{\psi,-k\lambda_\psi}\Big\},$$

$$\partial_{z,\theta}\ terms\ in\ \Big[\widetilde{F}^n_\varphi, \big(\widetilde{F}^n_\varphi, a^1_\varphi\big), \big(\widetilde{F}^n_\psi, (\sigma^1_{\nu,2})_{|x_d=0}\big), \big((\sigma^n_\varphi)_{|x_d=0}, a^1_\varphi\big), \big((\sigma^n_\psi)_{|x_d=0}, (\sigma^1_{\nu,2})_{|x_d=0}\big)\Big]$$

and the analogous ones for ψ are removed in the simplified model, since they involved traces of outgoing interior profiles (recall that $\widetilde{F}^n_{\varphi,j,\lambda}$ and $\widetilde{F}^n_{\psi,j,\lambda}$ depend respectively on $\big(\sigma^n_{\varphi,j,\lambda}\big)_{|x_d=0}$ and $\big(\sigma^n_{\psi,j,\lambda}\big)_{|x_d=0}$). For the same reasons, terms

$$i\lambda\, b_\varphi \cdot B\,\big(U^{n+1,\mathrm{osc}}_{\varphi,2,\lambda}\big)_{|x_d,\chi_d=0} \quad \text{and} \quad i\lambda\, b_\psi \cdot B\,\big(U^{n+1,\mathrm{osc}}_{\psi,2,\lambda}\big)_{|x_d,\chi_d=0}$$

are not kept. Finally, source terms

$$\partial_{z,\theta}\ terms\ in\ \big(U_1, \dots, U_{n-1}, (I-P)\,U_n, (P\,U_n)_{\zeta\neq\varphi,\psi,\nu}, U^*_n\big)_{|x_d,\chi_d=0}$$

involves quadratic terms in the traces of profiles $\sigma^k_{\zeta,j,\lambda}$ for $1 \leqslant k \leqslant n$, $\zeta \in \mathcal{F}_b \setminus \{0\}$, $j \in \mathcal{C}(\zeta)$ and $\lambda \in \mathbb{Z}^*$, with possibly derivatives of order up to n in front of it. They are simplified in three ways: we keep only traces of profiles associated with boundary frequencies φ, ψ and ν, we express traces only through functions $a^k_{\zeta,\lambda}$ for $\zeta = \varphi, \psi$, and we choose only first order derivatives in Θ (but we shall see in the following that considering derivatives in y would present no additional difficulty). Finally, boundary terms G and H are represented by functions H^n_ζ, belonging to a space specified later on. Multiplying equations (2.51) by $e^{i\lambda\Theta}$ for $\Theta \in \mathbb{T}$ a periodic variable therefore leads to the following simplified model amplitude equations

$$X^{\mathrm{Lop}}_\varphi\, a^n_\varphi + D^{\mathrm{Lop}}_\varphi\, \partial_\Theta\big(a^1_\varphi\, a^n_\varphi\big) + \mathbf{w}_\varphi\, \mathbb{F}^{\mathrm{per}}_\varphi\big[\partial_\Theta\, a^1_\varphi, \partial_\Theta\, a^n_\varphi\big] = H^n_\varphi + K^{\mathrm{Lop}}_\varphi \sum_{k=1}^{n-1} \partial_\Theta\big(a^k_\varphi\, a^{n-k}_\psi\big), \tag{3.1a}$$

$$X^{\mathrm{Lop}}_{\psi}\, a^n_{\psi} + D^{\mathrm{Lop}}_{\psi}\, \partial_\Theta\big(a^1_{\psi}\, a^n_{\psi}\big) + \mathbf{w}_{\psi}\, \mathbb{F}^{\mathrm{per}}_{\psi}\big[\partial_\Theta\, a^1_{\psi}, \partial_\Theta\, a^n_{\psi}\big] = H^n_{\psi} + K^{\mathrm{Lop}}_{\psi} \sum_{k=1}^{n-1} \partial_\Theta\big(a^k_{\varphi}\, a^{n-k}_{\psi}\big), \tag{3.1b}$$

where, for $\zeta = \varphi, \psi$, we have denoted by a^n_ζ the function of $\omega_T \times \mathbb{T}$ defined as

$$a^n_\zeta(z, \Theta) := \sum_{\lambda \in \mathbb{Z}^*} a^n_{\zeta,\lambda}(z)\, e^{i\lambda\,\Theta},$$

where, for $\zeta = \varphi, \psi$, the bilinear operator $\mathbb{F}^{\mathrm{per}}_\zeta$ is defined as

$$\mathbb{F}^{\mathrm{per}}_\zeta\big[a, b\big] := \sum_{\lambda \in \mathbb{Z}^*} \sum_{\substack{\lambda_1 + \lambda_3 = \lambda \\ \lambda_1 \lambda_3 \neq 0}} \frac{i\, a_{\lambda_1}\, b_{\lambda_3}}{\lambda_1\, \delta^1_\zeta + \lambda_3\, \delta^3_\zeta}\, e^{i\lambda\,\Theta}, \tag{3.2}$$

with δ^1_ζ and δ^3_ζ scalars defined as

$$\delta^1_\zeta := \frac{\xi_1(\zeta) - \xi_2(\zeta)}{\xi_3(\zeta) - \xi_1(\zeta)}, \qquad \delta^3_\zeta := \frac{\xi_3(\zeta) - \xi_2(\zeta)}{\xi_3(\zeta) - \xi_1(\zeta)},$$

and where $\mathbf{w}_\varphi, \mathbf{w}_\psi, K^{\mathrm{Lop}}_\varphi, K^{\mathrm{Lop}}_\psi \in \mathbb{R}$. Here we have used analysis of [CW17, Section 3.1] to rewrite terms involving the $\gamma_\zeta(\lambda_1, \lambda_3)$ coefficients in (2.51) as $\mathbf{w}_\varphi\, \mathbb{F}^{\mathrm{per}}_\varphi\big[\partial_\Theta\, a^1_\varphi, \partial_\Theta\, a^n_\varphi\big]$, up to changing definition of the coefficients D^{Lop}_ζ. Note that since φ and ψ are nonresonant, the denominators in equation (3.2) defining $\mathbb{F}^{\mathrm{per}}_\zeta$ are nonzero. Up to changing all notation by a harmless multiplicative constant, we can assume that, for $\zeta = \varphi, \psi$, vector fields X^{Lop}_ζ read

$$X^{\mathrm{Lop}}_\zeta = \partial_t - \mathbf{v}^{\mathrm{Lop}}_\zeta \cdot \nabla_y, \tag{3.3}$$

with $\mathbf{v}^{\mathrm{Lop}}_\zeta \in \mathbb{R}^{d-1}$. Equations (3.1) are coupled with the initial conditions

$$\big(a^n_\varphi\big)_{|t=0} = 0, \qquad \big(a^n_\psi\big)_{|t=0} = 0. \tag{3.4}$$

Again, these initial conditions (3.4) are not the same as (2.51c), and are written in this (non equivalent) form to be suited for the analytical framework. Note that equations (3.1) are quasi-linear for a^1_φ, a^1_ψ when $n = 1$, and linear for a^n_φ, a^n_ψ when $n \geqslant 2$. As we will prove later that terms $\mathbb{F}^{\mathrm{per}}_\varphi\big[\partial_\Theta\, a^1_\varphi, \partial_\Theta\, a^n_\varphi\big]$ and $\mathbb{F}^{\mathrm{per}}_\psi\big[\partial_\Theta\, a^1_\psi, \partial_\Theta\, a^n_\psi\big]$ are semilinear, equations (3.1) are transport equations, with a Burgers type term (when $n = 1$), a semilinear term, a source term and a convolution type term. System of equations (3.1), (3.4) is a simplification of system (2.51) for which we propose to set up the analytical tools to solve it.

The aim is to solve system (3.1)-(3.4) with the following Cauchy-Kovalevskaya theorem. First proofs of this kind of result are due to [Nir72] and then [Nis77], and the proof of the following formulation goes back to [BG78].

Theorem 3.1 ([BG78]). *Let $(B_r)_{r_0 \leqslant r \leqslant r_1}$ be a decreasing sequence of Banach spaces (with $0 \leqslant r_0 \leqslant r_1 \leqslant 1$), i.e. such that, for $r_0 \leqslant r' < r \leqslant r_1$,*

$$B_r \subset B_{r'}, \quad \|.\|_{r'} \leqslant \|.\|_r\,.$$

Let T, R, C and M be positive real numbers, and consider a continuous function F from $[-T, T] \times \{u \in B_r \,|\, \|u\|_r < R\}$ to $B_{r'}$ for every $r_0 \leqslant r' < r \leqslant r_1$ which satisfies

$$\sup_{|t| \leqslant T} \|F(t, u) - F(t, v)\|_{r'} \leqslant \frac{C}{r - r'}\, \|u - v\|_r \tag{3.5}$$

for all $r_0 \leqslant r' < r \leqslant r_1$, $|t| < T$, and for all u, v in B_r such that $\|u\|_r \leqslant R$, $\|v\|_r \leqslant R$, and

$$\sup_{|t| \leqslant T} \|F(t, 0)\|_r \leqslant \frac{M}{r_1 - r}, \tag{3.6}$$

for every $r_0 \leqslant r < r_1$.

Then there exists a real number δ in $(0,T)$ and a unique function u, belonging to $\mathcal{C}^1\big((-\delta(r_1-r),\delta(r_1-r)),B_r\big)$ for every $r_0\leqslant r\leqslant r_1$, satisfying

$$\sup_{|t|<\delta(r_1-r)}\|u(t)\|_r<R,$$

and the system

$$\begin{cases}u'(t)=F\big(t,u(t)\big) & \text{for } |t|<\rho(r_1-r)\\ u(0)=0.\end{cases}$$

We therefore need to define a chain of Banach spaces of analytic functions adapted to our problem (3.1).

3.2. **Functional framework.**

3.2.1. *Functional spaces.* For a function u of $L^2(\mathbb{R}^{d-1})$, the symbol $\widehat{u}$ refers to the Fourier transform of u, with the following convention

$$\widehat{u}(\xi):=\int_{\mathbb{R}^{d-1}}u(y)\,e^{-i\,\xi\cdot y}\,dy,\quad\forall\xi\in\mathbb{R}^{d-1}.$$

For a complex vector X, notation $|X|$ refers to the norm $\sqrt{X\cdot X^*}$, and we denote by $\langle.\rangle$ the Japanese bracket, that is, for a complex vector X,

$$\langle X\rangle:=\big(1+|X|^2\big)^{1/2}.$$

We set d^* to be an integer such that $d^*>\widetilde{m}_0+2+(d+1)/2$, where $\widetilde{m}_0$ is the nonnegative real number of Lemma 3.7. The following definition quantifies analyticity by means of an exponential decay of the Fourier transform.

Definition 3.2. *For $s\in(0,1)$, the space Y_s is defined as the space of all functions u of $L^2(\mathbb{R}^{d-1}\times\mathbb{T})$ such that, if their Fourier series expansion in Θ reads*

$$u(y,\Theta)=\sum_{\lambda\in\mathbb{Z}}u_\lambda(y)\,e^{i\lambda\Theta},$$

then

$$\|u\|_{Y_s}^2:=\int_{\mathbb{R}^{d-1}}\sum_{\lambda\in\mathbb{Z}}e^{2s|(\lambda,\xi)|}\,\langle(\lambda,\xi)\rangle^{2d^*}\,\big|\widehat{u_\lambda}(\xi)\big|^2\,d\xi<+\infty.$$

The following results make precise how y and Θ derivatives act on Y_s, and assert that Y_s is a Banach algebra.

Lemma 3.3. *There exists $C>0$ such that, for $0\leqslant s'<s\leqslant 1$, for u in Y_s, functions $\nabla_y u$ and $\partial_\Theta u$ belong to $Y_{s'}$, and we have*

$$\|\nabla_y u\|_{Y_{s'}}\leqslant\frac{C}{s-s'}\,\|u\|_{Y_s}\quad\text{and}\quad\|\partial_\Theta u\|_{Y_{s'}}\leqslant\frac{C}{s-s'}\,\|u\|_{Y_s}\,. \tag{3.7}$$

Proof. We prove the estimate for $\nabla_y u$, the one for $\partial_\Theta u$ being similar. We have, by definition of the $Y_{s'}$-norm,

$$\begin{aligned}\|\nabla_y u\|_{Y_{s'}}^2&=\int_{\mathbb{R}^{d-1}}\sum_{\lambda\in\mathbb{Z}}e^{2s'|(\lambda,\xi)|}\,\langle(\lambda,\xi)\rangle^{2d^*}\,|\xi|^2\big|\widehat{u_\lambda}(\xi)\big|^2\,d\xi\\&\leqslant\frac{C^2}{(s-s')^2}\int_{\mathbb{R}^{d-1}}\sum_{\lambda\in\mathbb{Z}}e^{2s|(\lambda,\xi)|}\,\langle(\lambda,\xi)\rangle^{2d^*}\,\big|\widehat{u_\lambda}(\xi)\big|^2\,d\xi=\frac{C^2}{(s-s')^2}\,\|u\|_s^2\,,\end{aligned}$$

since $|\xi|^2\exp(2s'|\xi|)\leqslant C^2\exp(2s|\xi|)/(s-s')^2$ for ξ in $\mathbb{R}^d$, with $C>0$ independent of s,s' and ξ, which reads precisely $C=2e^{-1}$. □

Lemma 3.4. *For $s\in(0,1)$, the space Y_s is a Banach algebra, up to a positive constant, that is, there exists $C>0$ (independent of s), such that for u,v in Y_s, the function uv belongs to Y_s and we have*

$$\|uv\|_{Y_s}\leqslant C\,\|u\|_{Y_s}\,\|v\|_{Y_s}\,.$$

Proof. Let s be in $(0,1)$, and consider u, v in Y_s. We have,

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} \sum_{\lambda\in\mathbb{Z}} e^{2s|(\lambda,\xi)|} \langle(\lambda,\xi)\rangle^{2d^*} \left|\widehat{(uv)_\lambda}(\xi)\right|^2 d\xi \\
&= \int_{\mathbb{R}^{d-1}} \sum_{\lambda\in\mathbb{Z}} e^{2s|(\lambda,\xi)|} \langle(\lambda,\xi)\rangle^{2d^*} \left| \int_{\mathbb{R}^{d-1}} \sum_{\mu\in\mathbb{Z}} \widehat{u_\mu}(\eta)\widehat{v_{\lambda-\mu}}(\xi-\eta)\, d\eta \right|^2 d\xi \\
&\leqslant \int_{\mathbb{R}^{d-1}} \sum_{\lambda\in\mathbb{Z}} \left(\int_{\mathbb{R}^{d-1}} \sum_{\mu\in\mathbb{Z}} \frac{\langle(\lambda,\xi)\rangle^{2d^*}}{\langle(\mu,\eta)\rangle^{2d^*} \langle(\lambda-\mu,\xi-\eta)\rangle^{2d^*}}\, d\eta \right) \\
&\quad \int_{\mathbb{R}^{d-1}} \sum_{\mu\in\mathbb{Z}} e^{2s|(\mu,\eta)|} \langle(\mu,\eta)\rangle^{2d^*} \left|\widehat{u_\mu}(\eta)\right|^2 e^{2s|(\lambda-\mu,\xi-\eta)|} \langle(\lambda-\mu,\xi-\eta)\rangle^{2d^*} \left|\widehat{v_{\lambda-\mu}}(\xi-\eta)\right|^2 d\eta\, d\xi \\
&\leqslant C \|u\|_{Y_s}^2 \|v\|_{Y_s}^2 ,
\end{aligned}$$

by Cauchy-Schwarz inequality, if $\int_{\mathbb{R}^{d-1}} \sum_{\mu\in\mathbb{Z}} \frac{\langle(\lambda,\xi)\rangle^{2d^*}}{\langle(\mu,\eta)\rangle^{2d^*} \langle(\lambda-\mu,\xi-\eta)\rangle^{2d^*}}\, d\eta$ is bounded uniformly with respect to (λ,ξ). We conclude by making the proof of this latter result.

For $\mu, \lambda \in \mathbb{Z}$ and $\xi, \eta \in \mathbb{R}^{d-1}$, we have

$$\left|(\lambda,\xi)\right|^2 \leqslant 2\left|(\mu,\eta)\right|^2 + 2\left|(\lambda-\mu,\xi-\eta)\right|^2$$

so

$$\langle(\lambda,\xi)\rangle^2 \leqslant 2\langle(\mu,\eta)\rangle^2 + 2\langle(\lambda-\mu,\xi-\eta)\rangle^2 ,$$

and

$$\langle(\lambda,\xi)\rangle^{2d^*} \leqslant 2^{d^*+1}\langle(\mu,\eta)\rangle^{2d^*} + 2^{d^*+1}\langle(\lambda-\mu,\xi-\eta)\rangle^{2d^*}$$

Therefore, by a change of variables $(\eta,\mu) = (\xi-\eta, \lambda-\mu)$,

$$\int_{\mathbb{R}^{d-1}} \sum_{\mu\in\mathbb{Z}} \frac{\langle(\lambda,\xi)\rangle^{2d^*}}{\langle(\mu,\eta)\rangle^{2d^*} \langle(\lambda-\mu,\xi-\eta)\rangle^{2d^*}}\, d\eta \leqslant 2^{d^*+2} \int_{\mathbb{R}^{d-1}} \sum_{\mu\in\mathbb{Z}} \frac{1}{\langle(\mu,\eta)\rangle^{2d^*}}\, d\eta \leqslant C\, 2^{d^*+2}$$

with C depending only on d^*, since d^* is such that $d^* > d/2$. □

As we work with sequences of functions $(a_\varphi^n)_{n\geqslant 1}$ and $(a_\psi^n)_{n\geqslant 1}$, we define a functional space accordingly. We also specify the norm chosen on the product space, since we will work with couples of sequences.

Definition 3.5. *For $s \in (0,1)$, the space $\mathbf{Y}_s$ is defined as the set of sequences $\mathbf{a} = \left(a_n\right)_{n\in\mathbb{N}}$ of Y_s such that*

$$|||\mathbf{a}|||_{\mathbf{Y}_s}^2 := \sum_{n\geqslant 1} e^{2sn} \langle n\rangle^{2d^*} |||a_n|||_{Y_s}^2 < +\infty.$$

For $s \in (0,1)$, the norm on the product space $\mathbf{Y}_s^2$ is defined, for $(\mathbf{a},\mathbf{b}) \in \mathbf{Y}_s^2$, as

$$|||\left(\mathbf{a},\mathbf{b}\right)|||_{\mathbf{Y}_s^2}^2 := |||\mathbf{a}|||_{\mathbf{Y}_s}^2 + |||\mathbf{b}|||_{\mathbf{Y}_s}^2.$$

The space $\mathbf{Y}_s$ satisfies analogous properties as the ones of Lemmas 3.3 and 3.4 (with the convolution on sequences for product), but they will not be used directly.

3.2.2. *Specifications on the simplified model.* We are now able to precise some properties of the study system (3.1), (3.4).

Boundary source terms H_φ^n, H_ψ^n for $n \geqslant 1$ are taken such that, defining $\mathbf{H} := \left(\mathbf{H}_\varphi, \mathbf{H}_\psi\right) := \left(H_\varphi^n, H_\psi^n\right)_{n\geqslant 1}$, function $\mathbf{H}$ is in $\mathcal{C}\left([-T,T], Y_1^2\right)$. In the statement 3.9 below of existence and uniqueness for system (3.1), (3.4), there will be an additional assumption on $\mathbf{H}$, requiring that there exists $M > 0$ such that, for $0 \leqslant s < 1$,

$$\sup_{|t|<T} |||\mathbf{H}|||_{\mathbf{Y}_s^2} \leqslant \frac{M}{1-s}.$$

This assumption on **H** is stronger than requiring H and G of (1.1) to be in $H^\infty(\mathbb{R}^d \times \mathbb{T})$, as it imposes analyticity with respect to space variables, but with bound on the norm increasing with regularity. We denote by $\gamma_0 > 0$ a positive constant such that, for $\zeta = \varphi, \psi$,

$$\left|\mathbf{v}_\zeta^{\mathrm{Lop}}\right| \leqslant \gamma_0, \qquad \left|D_\zeta^{\mathrm{Lop}}\right| \leqslant \gamma_0, \quad |\mathbf{w}_\zeta| \leqslant \gamma_0^{1/2} \quad \text{and} \quad \left|K_\zeta^{\mathrm{Lop}}\right| \leqslant \gamma_0, \tag{3.8a}$$

and, for $s \in (0,1)$, for u, v in Y_s, and for $\zeta = \varphi, \psi$,

$$\left\|\mathbb{F}_\zeta^{\mathrm{per}}\left[\partial_\Theta u, \partial_\Theta v\right]\right\|_{Y_s} \leqslant \gamma_0^{1/2} \, |||u|||_{Y_s} |||v|||_{Y_s}. \tag{3.8b}$$

All estimates relies on the fact that scalars D_ζ^{Lop}, $\mathbf{w}_\zeta$ and K_ζ^{Lop}, vectors $\mathbf{v}_\zeta^{\mathrm{Lop}}$ and operators $\mathbb{F}_\zeta^{\mathrm{per}}$ are indexed by finite sets. As for it, estimate (3.8b) asserting that the operator $\mathbb{F}_\zeta^{\mathrm{per}}$, composed with derivation in Θ, acts as a semilinear operator, is a result of [CW17, Theorem 3.1]. The proof in our case is a straightforward adaptation to our functional framework of the one of [CW17], which we detail here.

Proposition 3.6 ([CW17, Theorem 3.1])**.** *There exists a constant $C > 0$ such that for $s \in [0,1]$, for u, v in Y_s and $\zeta = \varphi, \psi$, we have*

$$\left\|\mathbb{F}_\zeta^{\mathrm{per}}\left[\partial_\Theta u, \partial_\Theta v\right]\right\|_{Y_s} \leqslant C \, |||u|||_{Y_s} |||v|||_{Y_s}. \tag{3.9}$$

Proof. The result relies on the following lemma, which constitutes a reformulation of the small divisors Assumption 6. Its proof is the same as the one in [CW17], and is recalled here for the sake of completeness.

Lemma 3.7 ([CW17, Lemma 3.2])**.** *There exists a constant $C > 0$ and a real number $\widetilde{m}_0$ such that, for $\lambda_1, \lambda_3 \in \mathbb{Z}^*$, and for $\zeta = \varphi, \psi$, we have*

$$\frac{1}{\left|\lambda_1 \, \delta_\zeta^1 + \lambda_3 \, \delta_\zeta^3\right|} \leqslant C \min\left(|\lambda_1|^{\widetilde{m}_0}, |\lambda_3|^{\widetilde{m}_0}\right). \tag{3.10}$$

Proof. Without loss of generality, we consider $\zeta = \varphi$. The aim is to use the bound of Assumption 6. Using equality $L(0, \varphi_2) \, r_{\varphi,2} = 0$, we get,

$$L\big(0, \lambda_1 \, \varphi_1 + \lambda_3 \, \varphi_3\big) \, r_{\varphi,2} = \Big[\lambda_1\big(\xi_1(\varphi) - \xi_2(\varphi)\big) + \lambda_3\big(\xi_3(\varphi) - \xi_2(\varphi)\big)\Big] A_d(0) \, r_{\varphi,2},$$

so the quantity $\lambda_1\big(\xi_1(\varphi) - \xi_2(\varphi)\big) + \lambda_3\big(\xi_3(\varphi) - \xi_2(\varphi)\big)$ is nonzero since otherwise $r_{\varphi,2}$ would be a nonzero vector in the kernel of $L\big(0, \lambda_1 \, \varphi_1 + \lambda_3 \, \varphi_3\big)$, contradicting Assumption 5 asserting that $\lambda_1 \, \varphi_1 + \lambda_3 \, \varphi_3$ is never characteristic for $\lambda_1, \lambda_3 \in \mathbb{Z}^*$. Therefore we have

$$\frac{1}{\Big|\lambda_1\big(\xi_1(\varphi) - \xi_2(\varphi)\big) + \lambda_3\big(\xi_3(\varphi) - \xi_2(\varphi)\big)\Big|} \leqslant C \left\|L\big(0, \lambda_1 \, \varphi_1 + \lambda_3 \, \varphi_3\big)^{-1}\right\|,$$

with a constant $C > 0$ independent on λ_1, λ_3. Using Assumption 6 and a polynomial bound on the transpose of the comatrix, we get that there exists a nonnegative real number $\widetilde{m}_0$ such that

$$\frac{1}{\Big|\lambda_1\big(\xi_1(\varphi) - \xi_2(\varphi)\big) + \lambda_3\big(\xi_3(\varphi) - \xi_2(\varphi)\big)\Big|} \leqslant C\big|(\lambda_1, \lambda_3)\big|^{\widetilde{m}_0},$$

with a new constant $C > 0$ independent on λ_1, λ_3. Up to changing constant $C > 0$, we obtain,

$$\frac{1}{\left|\lambda_1 \, \delta_\varphi^1 + \lambda_3 \, \delta_\varphi^3\right|} \leqslant C\big|(\lambda_1, \lambda_3)\big|^{\widetilde{m}_0}. \tag{3.11}$$

To get the formulation of (3.11) with a minimum, we see that two cases may occur. Either $\left|\lambda_1 \, \delta_\varphi^1 + \lambda_3 \, \delta_\varphi^3\right| > |\delta_\varphi^1|$, and in this case, with $C \geqslant 1/|\delta_\varphi^1|$, we have

$$\frac{1}{\left|\lambda_1 \, \delta_\varphi^1 + \lambda_3 \, \delta_\varphi^3\right|} \leqslant \frac{1}{|\delta_\varphi^1|} \leqslant C \leqslant C \, |\lambda_1|^{\widetilde{m}_0}$$

since $\widetilde{m}_0 \geqslant 0$. In the other case, if $\left|\lambda_1\,\delta^1_\varphi + \lambda_3\,\delta^3_\varphi\right| \leqslant |\delta^1_\varphi|$, we have

$$|\lambda_3| \leqslant \frac{1}{\left|\delta^3_\varphi\right|}\left|\lambda_1\,\delta^1_\varphi + \lambda_3\,\delta^3_\varphi\right| + \frac{1}{\left|\delta^3_\varphi\right|}\left|\lambda_1\,\delta^1_\varphi\right| \leqslant 2\frac{\left|\delta^1_\varphi\right|}{\left|\delta^3_\varphi\right|}|\lambda_1|,$$

so, up to changing constant C, estimate (3.11) rewrites

$$\frac{1}{\left|\lambda_1\,\delta^1_\varphi + \lambda_3\,\delta^3_\varphi\right|} \leqslant C\,|\lambda_1|^{\widetilde{m}_0}.$$

Applying the same arguments for λ_3 leads to the aimed estimate (3.10). □

The proof of Proposition 3.6 also relies on the following technical result, whose formulation is the one of [CW17]. Its proof is an immediate adaptation of a result of [RR82], and is not recalled here.

Lemma 3.8 ([RR82, Lemma 1.2.2],[CW17, Lemma 3.3])**.** *Let $\mathbb{K} : \mathbb{R}^{d-1}\times\mathbb{Z}\times\mathbb{R}^{d-1}\times\mathbb{Z} \to \mathbb{C}$ be a locally integrable measurable function such that, either*

$$\sup_{(\xi,\lambda)\in\mathbb{R}^{d-1}\times\mathbb{Z}} \int_{\mathbb{R}^{d-1}} \sum_{\mu\in\mathbb{Z}} \left|\mathbb{K}(\xi,\lambda,\eta,\mu)\right|^2 d\eta < +\infty,$$

or

$$\sup_{(\eta,\mu)\in\mathbb{R}^{d-1}\times\mathbb{Z}} \int_{\mathbb{R}^{d-1}} \sum_{\lambda\in\mathbb{Z}} \left|\mathbb{K}(\xi,\lambda,\eta,\mu)\right|^2 d\xi < +\infty.$$

Then the map

$$(f,g) \mapsto \int_{\mathbb{R}^{d-1}} \sum_{\mu\in\mathbb{Z}} \mathbb{K}(\xi,\lambda,\eta,\mu)\, f(\xi-\eta,\lambda-\mu)\, g(\eta,\mu)\, d\eta$$

is bounded on $L^2(\mathbb{R}^{d-1}\times\mathbb{Z})\times L^2(\mathbb{R}^{d-1}\times\mathbb{Z})$ with values in $L^2(\mathbb{R}^{d-1}\times\mathbb{Z})$.

We now proceed with the proof of Proposition 3.6, and we consider without loss of generality $\zeta = \varphi$. For $\xi \in \mathbb{R}^{d-1}$, $\lambda \in \mathbb{Z}$ and for u, v in Y_s, the Fourier transform of the λ-th term of the Fourier series expansion of $\mathbb{F}^{\mathrm{per}}_\zeta\left[\partial_\Theta u, \partial_\Theta v\right]$ is given by

$$\widehat{\mathbb{F}^{\mathrm{per}}_\zeta\left[\partial_\Theta u, \partial_\Theta v\right]}_\lambda(\xi) = -i \int_{\mathbb{R}^{d-1}} \sum_{\substack{\mu\in\mathbb{Z}\\ \mu\neq 0,\lambda}} \frac{\mu(\lambda-\mu)}{(\lambda-\mu)\delta^1_\varphi + \mu\delta^3_\varphi}\, \widehat{u_{\lambda-\mu}}(\xi-\eta)\, \widehat{v_\mu}(\eta)\, d\eta.$$

Therefore, to obtain inequality (3.9), we have to estimate the quantity

$$\int_{\mathbb{R}^{d-1}} \sum_{\lambda\in\mathbb{Z}} \left| \int_{\mathbb{R}^{d-1}} \sum_{\mu\in\mathbb{Z}} e^{s|(\lambda,\xi)|}\, \langle(\lambda,\xi)\rangle^{d^*}\, \mathbf{F}(\lambda,\mu)\, \widehat{u_{\lambda-\mu}}(\xi-\eta)\, \widehat{v_\mu}(\eta)\, d\eta \right|^2 d\xi,$$

where we have denoted, for $\lambda, \mu \in \mathbb{Z}$,

$$\mathbf{F}(\lambda,\mu) := \begin{cases} \frac{\mu(\lambda-\mu)}{(\lambda-\mu)\delta^1_\varphi + \mu\delta^3_\varphi} & \text{if } \mu \neq 0, \lambda \\ 0 & \text{otherwise,} \end{cases}$$

and we will do it using Lemma 3.8. We consider two nonnegative functions χ_1, χ_2 on $\mathbb{R}^d \times \mathbb{R}^d$ such that $\chi_1 + \chi_2 \equiv 1$ and

$$\chi_1(\xi,\lambda,\eta,\mu) = 0 \quad \text{if } \langle(\eta,\mu)\rangle \geqslant (2/3)\,\langle(\xi,\lambda)\rangle$$
$$\chi_2(\xi,\lambda,\eta,\mu) = 0 \quad \text{if } \langle(\eta,\mu)\rangle \leqslant (1/3)\,\langle(\xi,\lambda)\rangle\,.$$

We first consider the quantity

$$\int_{\mathbb{R}^{d-1}} \sum_{\mu\in\mathbb{Z}} \chi_1(\xi,\lambda,\eta,\mu)\, e^{s|(\lambda,\xi)|}\, \langle(\lambda,\xi)\rangle^{d^*}\, \mathbf{F}(\lambda,\mu)\, \widehat{u_{\lambda-\mu}}(\xi-\eta)\, \widehat{v_\mu}(\eta)\, d\eta, \tag{3.12a}$$

rewritten as

$$\int_{\mathbb{R}^{d-1}} \sum_{\mu\in\mathbb{Z}} \frac{\chi_1(\xi,\lambda,\eta,\mu)\, e^{s|(\lambda,\xi)|}\, \langle(\lambda,\xi)\rangle^{d^*}\, \mathbf{F}(\lambda,\mu)}{e^{s|(\lambda-\mu,\xi-\eta)|}\, \langle(\lambda-\mu,\xi-\eta)\rangle^{d^*}\, e^{s|(\mu,\eta)|}\, \langle(\mu,\eta)\rangle^{d^*}} \times \Big(e^{s|(\lambda-\mu,\xi-\eta)|}\, \langle(\lambda-\mu,\xi-\eta)\rangle^{d^*}\, \widehat{u_{\lambda-\mu}}(\xi-\eta)\Big)\Big(e^{s|(\mu,\eta)|}\, \langle(\mu,\eta)\rangle^{d^*}\, \widehat{v_\mu}(\eta)\Big)\, d\eta. \tag{3.12b}$$

We have

$$e^{2s|(\lambda,\xi)|} \leqslant e^{2s|(\mu,\eta)|} e^{2s|(\lambda-\mu,\xi-\eta)|},$$

and, on the support of χ_1,

$$\langle(\lambda-\mu,\xi-\eta)\rangle \geqslant \langle(\lambda,\xi)\rangle - \langle(\mu,\eta)\rangle \geqslant \frac{1}{3}\langle(\lambda,\xi)\rangle,$$

so

$$\int_{\mathbb{R}^{d-1}} \sum_{\mu\in\mathbb{Z}} \left| \frac{\chi_1(\xi,\lambda,\eta,\mu)\, e^{s|(\lambda,\xi)|}\, \langle(\lambda,\xi)\rangle^{d^*}\, \mathbf{F}(\lambda,\mu)}{e^{s|(\lambda-\mu,\xi-\eta)|}\, \langle(\lambda-\mu,\xi-\eta)\rangle^{d^*}\, e^{s|(\mu,\eta)|}\, \langle(\mu,\eta)\rangle^{d^*}} \right|^2 d\eta \leqslant C \int_{\mathbb{R}^{d-1}} \sum_{\mu\in\mathbb{Z}} \left| \frac{\mathbf{F}(\lambda,\mu)}{\langle(\mu,\eta)\rangle^{d^*}} \right|^2 d\eta.$$

Using Lemma 3.7 we get

$$\left| \frac{\mu(\lambda-\mu)}{(\lambda-\mu)\delta_\varphi^1 + \mu\delta_\varphi^3} \right| = \frac{1}{|\delta_\varphi^1|} \left| \mu - \frac{\delta_\varphi^3 \mu^2}{(\lambda-\mu)\delta_\varphi^1 + \mu\delta_\varphi^3} \right| \leqslant C|\mu|^{\widetilde{m}_0+2},$$

so

$$\sup_{(\xi,\lambda)\in\mathbb{R}^{d-1}\times\mathbb{Z}} \int_{\mathbb{R}^{d-1}} \sum_{\mu\in\mathbb{Z}} \left| \frac{\chi_1(\xi,\lambda,\eta,\mu)\, e^{s|(\lambda,\xi)|}\, \langle(\lambda,\xi)\rangle^{d^*}\, \mathbf{F}(\lambda,\mu)}{e^{s|(\lambda-\mu,\xi-\eta)|}\, \langle(\lambda-\mu,\xi-\eta)\rangle^{d^*}\, e^{s|(\mu,\eta)|}\, \langle(\mu,\eta)\rangle^{d^*}} \right|^2 d\eta \leqslant C \int_{\mathbb{R}^{d-1}} \sum_{\mu\in\mathbb{Z}} \frac{|\mu|^{2(\widetilde{m}_0+2)}}{\langle(\mu,\eta)\rangle^{2d^*}}\, d\eta < +\infty,$$

since we chose $d^* > \widetilde{m}_0 + 2 + (d+1)/2$. Applying Lemma 3.8 to the quantity in (3.12) we obtain

$$\int_{\mathbb{R}^{d-1}} \sum_{\lambda\in\mathbb{Z}} \left| \int_{\mathbb{R}^{d-1}} \sum_{\mu\in\mathbb{Z}} \chi_1(\xi,\lambda,\eta,\mu)\, e^{s|(\lambda,\xi)|}\, \langle(\lambda,\xi)\rangle^{d^*}\, \mathbf{F}(\lambda,\mu)\, \widehat{u_{\lambda-\mu}}(\xi-\eta)\, \widehat{v_\mu}(\eta)\, d\eta \right|^2 d\xi$$

$$\leqslant C\, \|u\|_{Y_s}^2\, \|v\|_{Y_s}^2.$$

Applying similar arguments for χ_2 leads to the analogous estimate for χ_2, and combining estimates for χ_1 and χ_2 gives the sought one (3.9), concluding the proof. □

We are now able to prove well-posedness for system (3.1), (3.4), using Theorem 3.1, with the above properties.

3.3. **A Cauchy-Kovalevskaya theorem for boundary equations.** System (3.1), (3.4) reads

$$\begin{cases} \partial_t \mathbf{a} = \mathbf{F}(t,\mathbf{a}) := L\,\mathbf{a} - \partial_\Theta\, \mathbf{D}^{\mathrm{Lop}}(\mathbf{a},\mathbf{a}) - \mathbb{F}^{\mathrm{per}}(\partial_\Theta\,\mathbf{a}, \partial_\Theta\,\mathbf{a}) + \mathbf{H} + \mathbf{K}^{\mathrm{Lop}}(\mathbf{a},\mathbf{a}), \\ \mathbf{a}(0) = 0, \end{cases} \tag{3.13}$$

where $\mathbf{a} := (\mathbf{a}_\varphi, \mathbf{a}_\psi) := \big(a_\varphi^n, a_\psi^n\big)_{n\geqslant 1}$, and, if $\mathbf{c} := \big(c_\varphi^n, c_\psi^n\big)_{n\geqslant 1}$,

$$L\,\mathbf{a} := \big(\mathbf{v}_\varphi^{\mathrm{Lop}} \cdot \nabla_y\, a_\varphi^n, \mathbf{v}_\varphi^{\mathrm{Lop}} \cdot \nabla_y\, a_\psi^n\big)_{n\geqslant 1},$$

$$\mathbf{D}^{\mathrm{Lop}}(\mathbf{a},\mathbf{c}) := \big(D_\varphi^{\mathrm{Lop}}\, a_\varphi^1\, c_\varphi^n, D_\psi^{\mathrm{Lop}}\, a_\psi^1\, c_\psi^n\big),$$

$$\mathbb{F}^{\mathrm{per}}(\mathbf{a},\mathbf{c}) := \big(\mathbf{w}_\varphi\, \mathbb{F}_\varphi^{\mathrm{per}}[a_\varphi^1, c_\varphi^n], \mathbf{w}_\psi \mathbb{F}_\psi^{\mathrm{per}}[a_\psi^1, c_\psi^n]\big)_{n\geqslant 1},$$

$$\mathbf{K}^{\mathrm{Lop}}(\mathbf{a},\mathbf{c}) := \Big(K_\varphi^{\mathrm{Lop}} \sum_{k=1}^{n-1} \partial_\Theta\big(a_\varphi^k\, c_\psi^{n-k}\big), K_\psi^{\mathrm{Lop}} \sum_{k=1}^{n-1} \partial_\Theta\big(a_\varphi^k\, c_\psi^{n-k}\big)\Big)_{n\geqslant 1}.$$

System (3.13) is now in the right shape to apply Theorem 3.1, and we prove the following result.

Proposition 3.9. *For $M_0 > 0$, there exists $\delta \in (0,T)$ such that for all $\mathbf{H}$ in $\mathcal{C}\big([-T,T],\mathbf{Y}_1^2\big)$ satisfying, for $0 < s < 1$,*

$$\sup_{|t|<T} |||\mathbf{H}(t)|||_{Y_s^2} \leqslant \frac{M_0}{1-s}, \tag{3.14}$$

system (3.13) admits a unique solution $\mathbf{a}$ in $\mathcal{C}^1\big((-\delta(1-s),\delta(1-s)),\mathbf{Y}_s^2\big)$ for every $0 < s \leqslant 1$.

The key estimates to prove this result are the following. These are classical, and their proof is recalled here for the sake of completeness.

Lemma 3.10. *There exists a constant $C > 0$ such that for $0 < s' < s \leqslant 1$, for $\mathbf{b},\mathbf{c}$ in $\mathbf{Y}_s^2$, the following estimates hold*

$$|||L\,\mathbf{b}|||_{\mathbf{Y}_{s'}^2} \leqslant \frac{C\,\gamma_0\,|||\mathbf{b}|||_{\mathbf{Y}_s^2}}{s-s'}, \tag{3.15a}$$

$$\Big|\Big|\Big|\partial_\Theta\,\mathbf{D}^{\mathrm{Lop}}\big(\mathbf{b},\mathbf{c}\big)\Big|\Big|\Big|_{\mathbf{Y}_{s'}^2} \leqslant \frac{C\,\gamma_0\,|||\mathbf{b}|||_{\mathbf{Y}_s^2}|||\mathbf{c}|||_{\mathbf{Y}_s^2}}{s-s'}, \tag{3.15b}$$

$$\big|\big|\big|\mathbb{F}^{\mathrm{per}}\big(\partial_\Theta\mathbf{b},\partial_\Theta\mathbf{c}\big)\big|\big|\big|_{\mathbf{Y}_s^2} \leqslant C\,\gamma_0\,|||\mathbf{b}|||_{\mathbf{Y}_s^2}|||\mathbf{c}|||_{\mathbf{Y}_s^2}, \tag{3.15c}$$

$$\Big|\Big|\Big|\mathbf{K}^{\mathrm{Lop}}\big(\mathbf{b},\mathbf{c}\big)\Big|\Big|\Big|_{\mathbf{Y}_{s'}^2} \leqslant \frac{C\,\gamma_0\,|||\mathbf{b}|||_{\mathbf{Y}_s^2}|||\mathbf{c}|||_{\mathbf{Y}_s^2}}{s-s'}. \tag{3.15d}$$

Proof. First, estimate (3.15a) follows directly from estimate (3.8a) and Lemma 3.3 since L is a linear combination of a bounded vector and a first order derivative. For the second one (3.15b), we have, according to Lemma 3.4, for $\zeta = \varphi,\psi$ and $n \geqslant 1$,

$$\Big\|D_\zeta^{\mathrm{Lop}}\,b_\zeta^1\,c_\zeta^n\Big\|_{Y_s} \leqslant C\,\gamma_0\,\Big\|b_\zeta^1\Big\|_{Y_s}\,\Big\|c_\zeta^n\Big\|_{Y_s} \leqslant C\,\gamma_0\,\|\mathbf{b}\|_{\mathbf{Y}_s^2}\,\Big\|c_\zeta^n\Big\|_{Y_s}.$$

Therefore, according to Lemma 3.3,

$$\Big\|\partial_\Theta\,D_\zeta^{\mathrm{Lop}}\,b_\zeta^1\,c_\zeta^n\Big\|_{Y_{s'}^2} \leqslant \frac{C}{s-s'}\,\Big\|D_\zeta^{\mathrm{Lop}}\,b_\zeta^1\,c_\zeta^n\Big\|_{Y_s} \leqslant \frac{C\,\gamma_0}{s-s'}\,\|\mathbf{b}\|_{\mathbf{Y}_s^2}\,\Big\|c_\zeta^n\Big\|_{Y_s}.$$

Multiplying by $e^{2sn}\,\langle n\rangle^{2d^*}$ and summing over $n \geqslant 1$ and $\zeta = \varphi,\psi$ gives the estimate (3.15b) for the $\mathbf{Y}_s^2$-norm. With (3.8b), the proof of (3.15c) is analogous but simpler since the operator is semilinear. Finally, for (3.15d), according to Lemmas 3.3 and 3.4, we have, for $n \geqslant 1$,

$$\Big\|\sum_{k=1}^{n-1}\partial_\Theta\big(b_\varphi^k\,c_\psi^{n-k}\big)\Big\|_{Y_{s'}} \leqslant \frac{C}{s-s'}\sum_{k=1}^{n-1}\Big\|b_\varphi^k\Big\|_{Y_s}\,\Big\|c_\psi^{n-k}\Big\|_{Y_s}.$$

Thus, by Cauchy-Schwarz inequality,

$$\begin{aligned}
&2\sum_{n\geqslant 1}e^{2s'n}\,\langle n\rangle^{2d^*}\,\Big\|\sum_{k=1}^{n-1}\partial_\Theta\big(b_\varphi^k\,c_\psi^{n-k}\big)\Big\|_{Y_{s'}}^2 \\
&\quad\leqslant \frac{C^2}{(s-s')^2}\sum_{n\geqslant 1}e^{2sn}\,\langle n\rangle^{2d^*}\,\Big(\sum_{k=1}^{n-1}\Big\|b_\varphi^k\Big\|_{Y_s}\,\Big\|c_\psi^{n-k}\Big\|_{Y_s}\Big)^2 \\
&\quad\leqslant \frac{C^2}{(s-s')^2}\sum_{n\geqslant 1}\Big(\sum_{k=1}^{n-1}\frac{\langle n\rangle^{2d^*}}{\langle k\rangle^{2d^*}\,\langle n-k\rangle^{2d^*}}\Big) \\
&\qquad\times\sum_{k=1}^{n-1}e^{2sk}\,\langle k\rangle^{2d^*}\,\Big\|b_\varphi^k\Big\|_{Y_s}^2\,e^{2s(n-k)}\,\langle n-k\rangle^{2d^*}\,\Big\|c_\psi^{n-k}\Big\|_{Y_s}^2
\end{aligned}$$

$$\leqslant \frac{C^2}{(s-s')^2} |||\mathbf{b}|||^2_{\mathbf{Y}^2_s} |||\mathbf{c}|||^2_{\mathbf{Y}^2_s},$$

since $\sum_{k=1}^{n-1} \frac{\langle n\rangle^{2d^*}}{\langle k\rangle^{2d^*}\langle n-k\rangle^{2d^*}}$ is bounded uniformly with respect to $n \geqslant 1$, and the result follows. □

We proceed with proof of Proposition 3.9, which essentially amounts to verify assumptions of Theorem 3.1.

Proof (Proposition 3.9). We apply Theorem 3.1 to system (3.31) with the scale of Banach spaces $(\mathbf{Y}^2_s)_{0<s\leqslant 1}$. First note that assumption (3.6) is satisfied as soon as assumption (3.14) for $\mathbf{H}$ is verified. Next we take interest into continuity assumption for $\mathbf{F}$ and assumption (3.5). For $0 < s' < s \leqslant 1$, and for $t, t' \in (-T, T)$ and $\mathbf{b}, \mathbf{c}$ in $\mathbf{Y}^2_s$, we have,

$$\begin{aligned}\mathbf{F}(t,\mathbf{b}) - \mathbf{F}(t',\mathbf{c}) = {}& \mathbf{L}(\mathbf{b}-\mathbf{c}) - \partial_\Theta \mathbf{D}^{\mathrm{Lop}}(\mathbf{b}, \mathbf{b}-\mathbf{c}) - \partial_\Theta \mathbf{D}^{\mathrm{Lop}}(\mathbf{b}-\mathbf{c}, \mathbf{c}) \\ & - \mathbb{F}^{\mathrm{per}}(\partial_\Theta \mathbf{b}, \partial_\Theta(\mathbf{b}-\mathbf{c})) - \mathbb{F}^{\mathrm{per}}(\partial_\Theta(\mathbf{b}-\mathbf{c}), \partial_\Theta \mathbf{c}) + \mathbf{H}(t) - \mathbf{H}(t') \\ & + \mathbf{K}^{\mathrm{Lop}}(\mathbf{b}, \mathbf{b}-\mathbf{c}) + \mathbf{K}^{\mathrm{Lop}}(\mathbf{b}-\mathbf{c}, \mathbf{c})\end{aligned}$$

so, according to estimates of Lemma 3.10,

$$\begin{aligned}(3.16)\quad & |||\mathbf{F}(t,\mathbf{b}) - \mathbf{F}(t',\mathbf{c})|||_{\mathbf{Y}^2_{s'}} \\ & \leqslant |||\mathbf{H}(t) - \mathbf{H}(t')|||_{\mathbf{Y}^2_s} + C\,\gamma_0\Big(1 + 3|||\mathbf{b}|||_{\mathbf{Y}^2_s} + 3|||\mathbf{c}|||_{\mathbf{Y}^2_s}\Big)\frac{|||\mathbf{b}-\mathbf{c}|||_{\mathbf{Y}^2_s}}{s-s'}.\end{aligned}$$

Therefore, since $\mathbf{H}$ is continuous from $[-T,T]$ to $\mathbf{Y}^2_1$, if we set $R > 0$ (which is therefore arbitrary), we both get, from (3.16), continuity of $\mathbf{F}$ from $[-T,T] \times \big\{\mathbf{b} \in \mathbf{Y}^2_s \mid |||\mathbf{b}|||_{\mathbf{Y}^2_s} < R\big\}$ to $\mathbf{Y}^2_{s'}$, and (setting $t' = t$) estimate (3.5), with constant C given by $C\,\gamma_0(1+6R)$. Theorem 3.1 therefore applies here and gives the sought result. □

Here we used that system (3.1), (3.4) presents quadratic nonlinearities, but, using the same arguments, other types of nonlinearities could also be treated.

3.4. **Second simplified model.** We now refine the previous simplified model by incorporating interior equations in it. According to remarks from the introduction of this section, in the new chosen simplified model, we remove the coupling with profiles of frequencies different from φ_j, ψ_j and ν_j, which were appearing in (2.49a) in terms $\partial_{z,\theta}$ *terms in* $(U_1, \dots, U_{n-1}, (I-P)U_n, U_n^*)$. The latter terms are also simplified since they carry derivatives of order higher than one[7], and we keep only first order derivatives in Θ (once again, considering derivatives in y presents no additional difficulty). They are therefore represented through terms of the form

$$\sum_{\zeta_1,\zeta_2=\varphi,\psi,\nu} \sum_{j_1,j_2=1,3} \sum_{k=1}^{n-1} \partial_{y,\Theta}(\sigma^k_{\zeta_1,j_1}\,\sigma^{n-k}_{\zeta_2,j_2}).$$

We also remove couplings with outgoing frequencies φ_2, ψ_2 and ν_2, as incoming equations will be solved seen as propagation in the normal variable equations, a form which is not suited to solve outgoing equations. Finally, we multiply equations (2.49) by $e^{i\lambda\Theta}$ for $\Theta \in \mathbb{T}$ a periodic variable. It leads to the following study interior evolution equations, for $n \geqslant 1$, $\zeta = \varphi, \psi, \nu$ and $j = 1, 3$,

$$\begin{aligned}(3.17)\quad & X_{\zeta,j}\,\sigma^n_{\zeta,j} + D_{\zeta,j}\,\partial_\Theta(\sigma^n_{\zeta,j}\,\sigma^1_{\zeta,j}) + \sum_{\substack{\zeta_1,\zeta_2\{\varphi,\psi,\nu\}\\ j_1,j_2\in\{1,3\}}} \partial_\Theta\,\mathbb{J}^{\zeta_2,j_2}_{\zeta_1,j_1}[\sigma^1_{\zeta_1,j_1}, \sigma^n_{\zeta_2,j_2}] \\ & \qquad = K_{\zeta,j} \sum_{\substack{\zeta_1,\zeta_2\{\varphi,\psi,\nu\}\\ j_1,j_2\in\{1,3\}}} \sum_{k=1}^{n-1} \partial_\Theta(\sigma^k_{\zeta_1,j_1}\,\sigma^{n-k}_{\zeta_2,j_2}),\end{aligned}$$

[7] In expression (2.46) of nonpolarized parts, there are already derivatives.

where we have defined $\sigma^n_{\zeta,j}$, a function of $\Omega_T \times \mathbb{T}$, as

$$\sigma^n_{\zeta,j}(z,\Theta) := \sum_{\lambda\in\mathbb{Z}^*} \sigma^n_{\zeta,j,\lambda}(z)\, e^{i\lambda\,\Theta},$$

and where $K_{\zeta,j} \in \mathbb{R}$ for $\zeta = \varphi,\psi,\nu$ and $j = 1,3$. For $(\zeta_1,\zeta_2,j_1,j_2) \in \{\varphi,\psi,\nu\}^2 \times \{1,3\}^2$, the bilinear operator $\mathbb{J}^{\zeta_2,j_2}_{\zeta_1,j_1}$ is defined as

$$\mathbb{J}^{\zeta_2,j_2}_{\zeta_1,j_1}[\sigma,\tau] = J^{\zeta_2,j_2}_{\zeta_1,j_1} \sum_{\lambda\in\mathbb{Z}^*} \sigma_\lambda \tau_\lambda\, e^{i\lambda\Theta}, \tag{3.18}$$

with some coefficients $J^{\zeta_2,j_2}_{\zeta_1,j_1}$. Similarly as for the boundary equations, up to changing all notation by a harmless multiplicative constant, according to expression (1.4) of vector field $X_{\zeta,j}$, it can be assumed to read

$$X_{\zeta,j} = \partial_t - \mathbf{v}_{\zeta,j}\cdot\nabla_x,$$

where vector $\mathbf{v}_{\zeta,j}$ has been defined in Definition 1.3. Recall last component of each vector $\mathbf{v}_{\zeta,j}$ is positive. Equation (3.17) is not provided with an initial condition, as we will see it as a propagation in the normal variable equation.

For boundary conditions for profiles $\sigma^n_{\zeta,j}$, $j = 1,3$, the coupling terms in $\big(\sigma^n_{\zeta,2}\big)_{|x_d=0}$ (appearing in terms in $\widetilde{F}^n_{\zeta,j,\lambda}$ for boundary conditions for profiles associated with φ and ψ) are not kept, since it would require trace estimates to solve interior equations, and we do not have such estimates in our possession. Terms in $\widetilde{F}^n_{\zeta,j,\lambda}$ also convey first order derivatives of lower order terms $a^1_\zeta,\dots,a^{n-1}_\zeta$. For the functional framework chosen later, these derivatives are an issue, and since coupling with lower order terms $a^1_\zeta,\dots,a^{n-1}_\zeta$ will be expressed in evolution equations for a^n_ζ, terms $\widetilde{F}^n_{\zeta,j,\lambda}$ are only represented in the study equations by boundary terms $g^n_{\zeta,j}$, belonging to one of the spaces defined later on. This lead to the following study boundary conditions, for $j = 1,3$,

$$\big(\sigma^n_{\varphi,j}\big)_{|x_d=0} = (e_{\varphi,j}\cdot r_{\varphi,j})\, a^n_\varphi + g^n_{\varphi,j}, \qquad \big(\sigma^n_{\psi,j}\big)_{|x_d=0} = (e_{\psi,j}\cdot r_{\psi,j})\, a^n_\psi + g^n_{\psi,j}, \tag{3.19a}$$

$$\big(\sigma^n_{\nu,j}\big)_{|x_d=0} = g^n_{\nu,j}, \tag{3.19b}$$

where, for $\zeta = \varphi,\psi$, we have denoted by a^n_ζ the function of $\Omega_T\times\mathbb{T}$ defined as

$$a^n_\zeta(z,\Theta) := \sum_{\lambda\in\mathbb{Z}^*} a^n_{\zeta,\lambda}(z)\, e^{i\lambda\,\Theta}.$$

Finally, equations for boundary terms a^n_φ and a^n_ψ are the same as for the first simplified model, namely,

$$X^{\mathrm{Lop}}_\varphi a^n_\varphi + D^{\mathrm{Lop}}_\varphi \partial_\Theta\big(a^1_\varphi a^n_\varphi\big) + \mathbf{w}_\varphi \mathbb{F}^{\mathrm{per}}_\varphi\big[\partial_\Theta a^1_\varphi, \partial_\Theta a^n_\varphi\big] = H^n_\varphi + K^{\mathrm{Lop}}_\varphi \sum_{k=1}^{n-1} \partial_\Theta\big(a^k_\varphi a^{n-k}_\psi\big), \tag{3.20a}$$

$$X^{\mathrm{Lop}}_\psi a^n_\psi + D^{\mathrm{Lop}}_\psi \partial_\Theta\big(a^1_\psi a^n_\psi\big) + \mathbf{w}_\psi \mathbb{F}^{\mathrm{per}}_\psi\big[\partial_\Theta a^1_\psi, \partial_\Theta a^n_\psi\big] = H^n_\psi + K^{\mathrm{Lop}}_\psi \sum_{k=1}^{n-1} \partial_\Theta\big(a^k_\varphi a^{n-k}_\psi\big), \tag{3.20b}$$

coupled with the initial conditions

$$\big(a^n_\varphi\big)_{|t=0} = 0, \qquad \big(a^n_\psi\big)_{|t=0} = 0. \tag{3.21}$$

The strategy to solve the above system of equations (3.17), (3.19), (3.20) and (3.21) is to apply a Cauchy-Kovalevskaya theorem such as Theorem 3.1 to interior system (3.17), (3.19), seen as a propagation equation in the normal variable. In order to do that, we need the boundary terms in (3.19) to be analytical with respect to all variables (even with respect to time). For the first simplified model, in Proposition 3.9, we obtained only continuity with respect to time. Therefore we need to refine this result to obtain analyticity with respect to all variables. In the next part we define functional spaces which will be used for this purpose.

3.5. Additional functional framework.

3.5.1. *Functional spaces.* We define two different types of spaces, which all are spaces of functions defined on $\omega_T \times \mathbb{T}$, analytical with respect to all variables (t, y, Θ). The first ones, which will be denoted by E_ρ and $\mathbf{E}_\rho$, will be used to solve boundary equations (3.20)-(3.21), which will be viewed as a fixed point problem in $\mathbf{E}_\rho$. The second one, denoted by X_r, $\mathbf{X}_r$, are the one fitted for interior system (3.17)-(3.19), where equation (3.17) will be seen as a differential equation with values in $\mathbf{X}_r$. Features and relations of this spaces are summarized in Figure 3. In addition to defining the functional spaces, we have to describe action of differentiation on it, and to prove that every function of $\mathbf{E}_\rho$ is in $\mathbf{X}_r$.

Previously introduced spaces Y_s, $s \in (0,1)$, are used to defined spaces E_ρ, $\rho \in (0,1)$. If $I \subset \mathbb{R}$ is an interval and E a Banach space, we denote by $\mathcal{C}^\omega(I, E)$ the space of analytic functions from I to E.

Definition 3.11. *For $\rho \in (0,1)$, the space $\widetilde{E}_\rho$ is defined as*

$$\widetilde{E}_\rho := \bigcap_{s\in(0,1)} \mathcal{C}^\omega\Big(\big(-\rho(1-s), \rho(1-s)\big), Y_s\Big).$$

In the next definition we use the *Catalan numbers* (see [Com74]), defined by, for $n \geqslant 0$,

$$\mathfrak{C}_n := \frac{1}{n+1}\binom{2n}{n}.$$

They satisfy, for $n \geqslant 0$,

$$\sum_{i=0}^{n} \mathfrak{C}_i\, \mathfrak{C}_{n-i} = \mathfrak{C}_{n+1}. \tag{3.22}$$

The Catalan numbers appear in the power series expansion of $x \mapsto (1-x)^{-1/2}$:

$$\frac{1}{\sqrt{1-x}} = \sum_{n\geqslant 0} \frac{1}{n!} \frac{(n+1)!\,\mathfrak{C}_n}{4^n}\, x^n, \quad \forall |x| < 1. \tag{3.23}$$

Next definition takes inspiration from the method of majoring series, see for example [Joh91, Chapter II], since, in this formalism, it requires for a function to admit a dilatation of $x \mapsto (1-x)^{-1/2}$ as a majoring series.

Definition 3.12. *For $\rho \in (0,1)$, the space E_ρ is defined as the set of functions a of $\widetilde{E}_\rho$ such that there exists $M > 0$ such that for all $s \in (0,1)$ and $\nu \in \mathbb{N}$,*

$$\|\partial_t^\nu a(0)\|_{Y_s} \leqslant \frac{M}{(1-s)^{\nu+1}} \frac{(\nu+1)!\,\mathfrak{C}_\nu}{(4\rho)^\nu}. \tag{3.24}$$

The infimum of all M satisfying condition (3.24) is denoted by $|||a|||_{E_\rho}$.

If a is in E_ρ for some $\rho \in (0,1)$, then, for $s \in (0,1)$, for $|t| < \rho(1-s)$, by expanding a in power series with respect to t at 0, using estimate (3.24) and the power series expansion of $x \mapsto (1-x)^{-1/2}$, we get

$$\|a(t)\|_{Y_s} \leqslant \frac{|||a|||_{E_\rho}}{1-s} \left(1 - \frac{|t|}{\rho(1-s)}\right)^{-1/2}.$$

We find here the formulation of [BG78].

Since we work with couples of sequences of functions, we define a space accordingly, and we specify the norm used on the product space.

Definition 3.13. *For $\rho \in (0,1)$, the space $\mathbf{E}_\rho$ is defined as the set of sequences $\mathbf{a} = \big(a_n\big)_{n\in\mathbb{N}}$ of E_ρ such that*

$$|||\mathbf{a}|||^2_{\mathbf{E}_\rho} := \sum_{n\geqslant 1} e^{2\rho n} \langle n\rangle^{2d^*} |||a_n|||^2_{E_\rho} < +\infty.$$

For $\rho \in (0,1)$, the norm on the product space $\mathbf{E}_\rho^2$ is defined, for $(\mathbf{a},\mathbf{b}) \in \mathbf{E}_\rho^2$, as

$$\left|\!\left|\!\left| (\mathbf{a},\mathbf{b}) \right|\!\right|\!\right|_{\mathbf{E}_\rho^2}^2 := \left|\!\left|\!\left| \mathbf{a} \right|\!\right|\!\right|_{\mathbf{E}_\rho}^2 + \left|\!\left|\!\left| \mathbf{b} \right|\!\right|\!\right|_{\mathbf{E}_\rho}^2.$$

Spaces E_ρ and $\mathbf{E}_\rho$ are not normed algebras, and neither do they satisfy a derivation property such as (3.7). Indeed, for a function a of E_ρ with $\rho \in (0,1)$, we have, by Lemma 3.3, for $0 < s' < s < 1$,

$$\|\partial_t^\nu \partial_\Theta\, a(0)\|_{Y_{s'}} \leqslant \frac{1}{s-s'} \|\partial_t^\nu a(0)\|_{Y_s} \leqslant \frac{1}{s-s'} \frac{\left|\!\left|\!\left| a \right|\!\right|\!\right|_{E_\rho}}{(1-s)^{\nu+1}} \frac{(\nu+1)!\,\mathfrak{C}_\nu}{(4\rho)^\nu}.$$

To obtain an estimate for (3.24), it seems that we should have the existence of $C > 0$ such that for all $0 < s' < s < 1$ and $\nu \geqslant 0$,

$$\frac{1}{(s-s')(1-s)^{\nu+1}} \leqslant \frac{C}{(1-s')^{\nu+1}},$$

which is false. However, as we shall see later, estimating $t \mapsto \int_0^t \partial_\Theta\, a(s)\, ds$ instead of $\partial_\Theta\, a$ could solve the problem. This is what is referred to as *regularization by integration in time*, see [Uka01, Mor20]. These spaces seem well suited to prove existence of solutions to boundary system (3.20), analytical with respect to all variables, but the absence of above mentioned properties prevents to apply a Cauchy-Kovalevskaya theorem with these spaces for interior system. This is why we need to define other, more appropriate spaces.

Spaces for interior equations are spaces in (t, y, Θ) variables since interior equations will be seen as propagation equation in x_d, valued in these spaces. In the following, H^{d^*} denotes the Sobolev space $H^{d^*}(\mathbb{R}_y^{d-1} \times \mathbb{T}_\Theta)$ of regularity d^*. Recall that d^* has been chosen such that $d^* > \widetilde{m}_0 + 2 + (d+1)/2$, where $\widetilde{m}_0$ is the real nonnegative number of Lemma 3.7. The next definition is based on the classical way to characterize analytic functions. For a $(d+1)$-tuple $\alpha = (\alpha_0, \dots, \alpha_d) \in \mathbb{N}^{d+1}$, notation $\alpha!$ refers to $\alpha! := \alpha_0! \cdots \alpha_d!$.

Definition 3.14. *Consider $\rho \in (0,1)$. For $r \in (0,1)$, the space X_r is defined as the set of smooth functions a of $(t, y, \Theta) \in [-\rho/2, \rho/2] \times \mathbb{R}^{d-1} \times \mathbb{T}$ with values in $\mathbb{C}$ such that there exists $M > 0$ such that for every α in $\mathbb{N}^{d+1}$,*

$$\left\| \partial_{t,y,\Theta}^\alpha a(0,.,.) \right\|_{H^{d^*}} \leqslant \frac{M \alpha!}{r^{|\alpha|}\left(|\alpha|^{2d+1}+1\right)}.$$

The infimum of such $M > 0$ is denoted by $\left|\!\left|\!\left| a \right|\!\right|\!\right|_{X_r}$.

Note that in the previous definition, the space X_r depends on the fixed constant $\rho \in (0,1)$, but we chose not to include this dependence in the notation since in the following ρ will be fixed. The time interval of the form $[-\rho/2, \rho/2]$ is required because, in the following, functions of X_r will come from functions of E_ρ, which are defined on time intervals $\big(-\rho(1-s), \rho(1-s)\big)$ for $s \in (0,1)$, so we choose arbitrarily $s = 1/2$. Analogously as for E_ρ, we define a space for sequences of X_r.

Definition 3.15. *For $r \in (0,1)$, the space $\mathbf{X}_r$ is defined as the set of sequences $\mathbf{a} = \big(a_n\big)_{n\in\mathbb{N}}$ of X_r such that*

$$\left|\!\left|\!\left| \mathbf{a} \right|\!\right|\!\right|_{\mathbf{X}_r}^2 := \sum_{n \geqslant 1} e^{2\,r\,n} \langle n \rangle^{2d^*} \left|\!\left|\!\left| a_n \right|\!\right|\!\right|_{X_r}^2 < +\infty.$$

For $r \in (0,1)$, the norm on the product space $\mathbf{X}_r^6$ is defined, for $\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_6) \in \mathbf{X}_r^6$, as

$$\left|\!\left|\!\left| \mathbf{a} \right|\!\right|\!\right|_{\mathbf{X}_r^6}^2 := \left|\!\left|\!\left| \mathbf{a}_1 \right|\!\right|\!\right|_{\mathbf{X}_r}^2 + \cdots + \left|\!\left|\!\left| \mathbf{a}_6 \right|\!\right|\!\right|_{\mathbf{X}_r}^2.$$

The following result asserts that, for every $\rho \in (0,1)$, there exists $r \in (0,1)$ such that space $\mathbf{E}_\rho$ is continuously injected in $\mathbf{X}_r$, with a constant independent of ρ. The proof is recalled here for the sake of clarity.

Lemma 3.16. *There exists $C > 0$, such that, for $\rho \in (0,1)$, if $\mathbf{a}$ is in $\mathbf{E}_\rho$ then there exists $r \in (0,1)$ such that $\mathbf{a}$ belongs to $\mathbf{X}_r$ (for the same ρ) and, furthermore,*

$$\left|\!\left|\!\left| \mathbf{a} \right|\!\right|\!\right|_{\mathbf{X}_r} \leqslant C \left|\!\left|\!\left| \mathbf{a} \right|\!\right|\!\right|_{\mathbf{E}_\rho}.$$

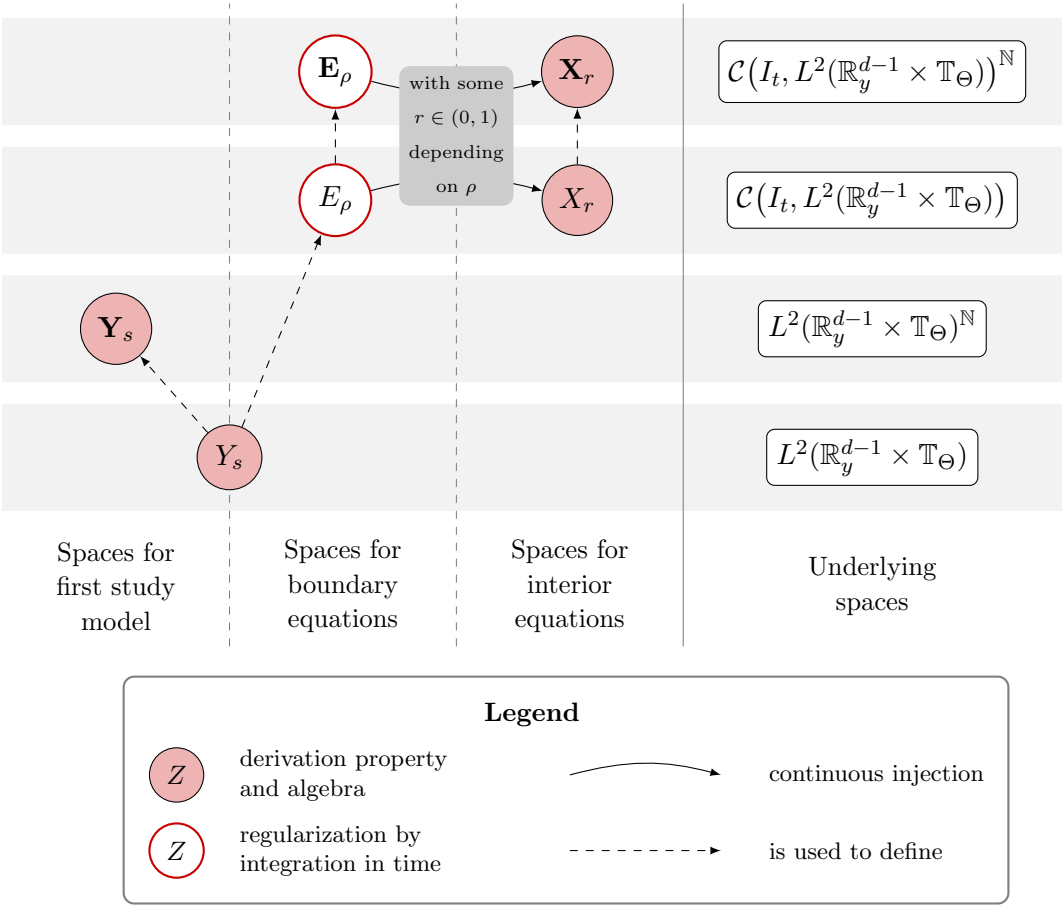

FIGURE 3. Features of functional spaces and links between them

Proof. Let $\mathbf{a} = (a^n)_{n\geqslant 1}$ be in $\mathbf{E}_\rho$ for some $\rho \in (0,1)$. For $\alpha = (\alpha_0, \alpha', \beta) = (\alpha_0, \alpha_1, \dots, \alpha_{d-1}, \beta)$ in $\mathbb{N} \times \mathbb{N}^{d-1} \times \mathbb{N}$, we have, for $s \in (0,1)$ and $n \geqslant 1$,

$$\begin{aligned}
\left\|\partial^\alpha_{t,y,\Theta} a^n(0,.,.)\right\|^2_{H^{d^*}} &= \int_{\mathbb{R}^{d-1}} \sum_{\lambda\in\mathbb{Z}} \xi_1^{2\alpha_1}\cdots\xi_{d-1}^{2\alpha_{d-1}}\lambda^{2\beta}\left\langle(\xi,\lambda)\right\rangle^{2d^*}\left|\partial_t^{\alpha_0}\widehat{a^n_\lambda}(0,\xi)\right|^2 d\xi \\
&\leqslant \int_{\mathbb{R}^{d-1}} \sum_{\lambda\in\mathbb{Z}} \frac{\alpha'!^2\beta!^2}{s^{2(|\alpha'|+\beta)}} e^{2s|(\xi,\lambda)|}\left\langle(\xi,\lambda)\right\rangle^{2d^*}\left|\partial_t^{\alpha_0}\widehat{a^n_\lambda}(0,\xi)\right|^2 d\xi \\
&= \frac{\alpha'!^2\beta!^2}{s^{2(|\alpha'|+\beta)}}\left\|\partial_t^{\alpha_0}a^n(0)\right\|^2_{Y_s},
\end{aligned}$$

using the inequality

$$\frac{(s\xi)^{\alpha'}(s\lambda)^\beta}{\alpha'!\beta!} \leqslant e^{s|(\xi,\lambda)|}.$$

Since a^n is in E_ρ (because $\mathbf{a}$ is in $\mathbf{E}_\rho$), we therefore have

$$\left\|\partial^\alpha_{t,y,\Theta} a^n(0,.,.)\right\|_{H^{d^*}} \leqslant \frac{\alpha'!\beta!}{s^{|\alpha'|+\beta}}\frac{|||a^n|||_{E_\rho}(\alpha_0+1)!\,\mathfrak{C}_{\alpha_0}}{(1-s)^{\alpha_0+1}(4\rho)^{\alpha_0}} \leqslant \frac{\alpha'!\beta!}{s^{|\alpha'|+\beta}}\frac{C\,|||a^n|||_{E_\rho}\,\alpha_0!}{(1-s)^{\alpha_0+1}(3\rho)^{\alpha_0}}$$

using $(\alpha_0+1)\,\mathfrak{C}_{|\alpha_0|}/(4\rho)^{\alpha_0} \leqslant C/(3\rho)^{\alpha_0}$. Finally, we have, if $s \leqslant \min(\rho, 2/3)$,

$$\left\|\partial^\alpha_{t,y,\Theta} a^n(0,.,.)\right\|_{H^{d^*}} \leqslant \frac{\alpha!}{s^{|\alpha|}}\frac{C\,|||a^n|||_{\mathbf{E}_\rho}}{3} \leqslant \frac{C\,|||a^n|||_{\mathbf{E}_\rho}\alpha!}{s'^{|\alpha|}\,(|\alpha|^{2d+1}+1)},$$

with $s' < s$, because $s'^{|\alpha|}\,(|\alpha|^{2d+1}+1) \leqslant Cs^{|\alpha|}$. Therefore,

$$|||a^n|||_{X_{s'}} \leqslant C|||a^n|||_{E_\rho}, \tag{3.25}$$

with a constant C which does not depend on $n \geqslant 1$. Therefore, multiplying inequality (3.25) by $e^{2s'n} \langle n \rangle^{2d^*}$ and summing over $n \geqslant 1$ leads to

$$|||\mathbf{a}|||_{\mathbf{X}_{s'}} \leqslant C |||\mathbf{a}|||_{\mathbf{E}_\rho},$$

so $\mathbf{a}$ belongs to $\mathbf{X}_r$ with $r < \min(\rho, 2/3)$ which depends only on ρ, concluding the proof. □

Following results state that partial derivatives with respect to t, y, Θ act on $\mathbf{X}_r$ in the same way as partial derivatives with respect to y, Θ act on Y_s, and that spaces $\mathbf{X}_r$ satisfy an algebra property. For the sake of completeness, we recall the proof of these classical results.

Lemma 3.17. *There exists $C > 0$ such that, for $0 \leqslant r' < r \leqslant 1$, for $\mathbf{a}$ in $\mathbf{X}_r$ and for e_j in $\mathbb{N}^{d+1}$ with $|e_j| = 1$, function $\partial^{e_j}_{t,y,\Theta}\mathbf{a}$ belongs to $\mathbf{X}_{r'}$ and satisfies*

$$\left|\left|\left| \partial^{e_j}_{t,y,\Theta}\mathbf{a} \right|\right|\right|_{\mathbf{X}_{r'}} \leqslant \frac{C}{r - r'} |||\mathbf{a}|||_{\mathbf{X}_r}.$$

Proof. In the same way as for the previous Lemma 3.16, proving the estimate for the space X_r leads to the one for $\mathbf{X}_r$ and the associate result, by multiplying by $e^{2s'n} \langle n \rangle^{2d^*}$ and summing over $n \geqslant 1$. Let a be in X_r. Without loss of generality, we make the proof for $e_j = e_0 = (1, 0, \dots, 0)$. For $\alpha = (\alpha_0, \alpha', \beta)$ in $\mathbb{N} \times \mathbb{N}^{d-1} \times \mathbb{N}$, we have, by definition of X_r-norm,

$$\begin{aligned}\left\| \partial^{\alpha}_{t,y,\Theta} \partial^{e_0}_{t,y,\Theta} a(0, ., .) \right\|_{H^{d^*}} &\leqslant |||a|||_{X_r} \frac{(\alpha + e_0)!}{r^{|\alpha|+1} \left(|\alpha + e|^{2d+1} + 1\right)} \\ &= |||a|||_{X_r} \frac{\alpha!}{(r')^{|\alpha|} \left(|\alpha|^{2d+1} + 1\right)} \frac{\left(|\alpha|^{2d+1} + 1\right)}{\left(|\alpha + e|^{2d+1} + 1\right)} (\alpha_0 + 1) \frac{(r')^{|\alpha|}}{r^{|\alpha|+1}}.\end{aligned}$$

Since $(|\alpha|^{2d+1} + 1)/(|\alpha + e|^{2d+1} + 1)$ is bounded uniformly with respect to $\alpha \in \mathbb{N}^{d+1}$ and since

$$(\alpha_0 + 1) \frac{(r')^{|\alpha|}}{r^{|\alpha|+1}} \leqslant \left(|\alpha| + 1\right) \frac{(r')^{|\alpha|}}{r^{|\alpha|+1}} \leqslant \frac{1}{r - r'},$$

the result follows. □

Lemma 3.18. *For $r \in (0, 1)$, spaces X_r and $\mathbf{X}_r$ are Banach algebras (the latter for the convolution on sequences), up to a positive constant, that is, there exists $C > 0$ (independent of $r \in (0, 1)$), such that for $\mathbf{a}, \mathbf{b}$ in $\mathbf{X}_r$, the function $\mathbf{ab}$ belongs to $\mathbf{X}_r$ and we have*

$$|||\mathbf{ab}|||_{\mathbf{X}_r} \leqslant C |||\mathbf{a}|||_{\mathbf{X}_r} |||\mathbf{b}|||_{\mathbf{X}_r},$$

and the analogous estimate for X_r.

Proof. We make the proof for spaces X_r, and the result for $\mathbf{X}_r$ follows using the same arguments as in the proof of Lemma 3.4. Let r be in $(0, 1)$ and consider a, b in X_r. We need to show that there exists $C > 0$ such that for all $\alpha \in \mathbb{N}^{d+1}$, we have

$$\left\| \partial^{\alpha}_{t,y,\Theta} (ab)(0, ., .) \right\|_{H^{d^*}} \leqslant \frac{C |||a|||_{X_r} |||b|||_{X_r} \alpha!}{r^{|\alpha|} \left(|\alpha|^{2d+1} + 1\right)}$$

So consider $\alpha \in \mathbb{N}^{d+1}$. We have, since $d^* > d/2$, so H^{d^*} is an algebra,

$$\begin{aligned}\left\| \partial^{\alpha}_{t,y,\Theta} (ab)(0, ., .) \right\|_{H^{d^*}} &\leqslant C \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \left\| \partial^{\beta}_{t,y,\Theta} a(0, ., .) \right\|_{H^{d^*}} \left\| \partial^{\alpha-\beta}_{t,y,\Theta} b(0, ., .) \right\|_{H^{d^*}} \\ &\leqslant C \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \frac{|||a|||_{X_r} \beta!}{r^{|\beta|} \left(|\beta|^{2d+1} + 1\right)} \frac{|||b|||_{X_r} (\alpha - \beta)!}{r^{|\alpha|-|\beta|} \left(|\alpha - \beta|^{2d+1} + 1\right)} \\ &= \frac{C \, |||a|||_{X_r} |||b|||_{X_r} \alpha!}{r^{|\alpha|} \left(|\alpha|^{2d+1} + 1\right)} \sum_{\beta \leqslant \alpha} \frac{\left(|\alpha|^{2d+1} + 1\right)}{\left(|\beta|^{2d+1} + 1\right) \left(|\alpha - \beta|^{2d+1} + 1\right)},\end{aligned}$$

and the result follows since $\sum_{\beta \leqslant \alpha} \frac{(|\alpha|^{2d+1}+1)}{(|\beta|^{2d+1}+1)\,(|\alpha-\beta|^{2d+1}+1)}$ is bounded uniformly with respect to $\alpha \in \mathbb{N}^{d+1}$. □

We summarize the main features of the functional spaces introduced in this section in Figure 3. The concept of *regularization by integration in time* is detailed below, in Lemma 3.21.

3.5.2. *Specifications on the simplified model and main result.* In view of the functional spaces defined above, we are able to make precise the study system (3.17)-(3.19)-(3.20)-(3.21).

Boundary terms $g^n_{\zeta,j}$ appearing in (3.19) are taken such that, if we define $\mathbf{g}_{\zeta,j} := \big(g^n_{\zeta,j}\big)_{n\in\mathbb{N}}$, then function $\mathbf{g}_{\zeta,j}$ is in $\mathbf{X}_1$ for $\zeta = \varphi, \psi, \nu$ and $j = 1, 3$. Analogously, source terms H^n_ζ of boundary equations (3.20) are taken such that, defining $\mathbf{H}_\zeta := \big(H^n_\zeta\big)_{n\geqslant 1}$, sequence $\mathbf{H}_\zeta$ is in $\mathbf{E}_1$ for $\zeta = \varphi, \psi$. Once again, assumption on $\mathbf{H}_\zeta$ imposing it to be analytical with respect to all its variables is stronger than requiring H and G to be in $H^\infty(\mathbb{R}^d \times \mathbb{T})$. We also denote by $\gamma_0 > 0$ a positive constant such that, for $\zeta = \varphi, \psi, \nu$ and $j = 1, 3$,

$$|\mathbf{v}_{\zeta,j}| \leqslant \gamma_0, \quad \big|D_{\zeta,j}\big| \leqslant \gamma_0 \quad \text{and} \quad \big|K_{\zeta,j}\big| \leqslant \gamma_0, \tag{3.26a}$$

for $\zeta = \varphi, \psi$,

$$\big|\mathbf{v}^{\mathrm{Lop}}_\zeta\big| \leqslant \gamma_0, \qquad \big|D^{\mathrm{Lop}}_\zeta\big| \leqslant \gamma_0, \quad |\mathbf{w}_\zeta| \leqslant \gamma_0^{1/2}, \quad \text{and} \quad \big|K^{\mathrm{Lop}}_\zeta\big| \leqslant \gamma_0, \tag{3.26b}$$

for $r \in (0,1)$ and for σ, τ in X_r, for $\zeta_1, \zeta_2 = \varphi, \psi, \nu$ and $j_1, j_2 = 1, 3$,

$$\Big|\Big|\Big|\mathbb{J}^{\zeta_2,j_2}_{\zeta_1,j_1}[\sigma,\tau]\Big|\Big|\Big|_{X_r} \leqslant \gamma_0 \,|||\sigma|||_{X_r} |||\tau|||_{X_r}, \tag{3.26c}$$

and, for $s \in (0,1)$, for u, v in Y_s, and for $\zeta = \varphi, \psi$,

$$\Big\|\mathbb{F}^{\mathrm{per}}_\zeta\big[\partial_\Theta u, \partial_\Theta v\big]\Big\|_{Y_s} \leqslant \gamma_0^{1/2} \,\|u\|_{Y_s} \|v\|_{Y_s}\,. \tag{3.26d}$$

All estimates relies on the fact that scalars $D_{\zeta,j}$, $K_{\zeta,j}$, D^{Lop}_ζ, $\mathbf{w}_\zeta$ and K^{Lop}_ζ, vectors $\mathbf{v}_{\zeta,j}$ and $\mathbf{v}^{\mathrm{Lop}}_\zeta$ and operators $\mathbb{J}^{\zeta_1,j_1}_{\zeta_2,j_2}$ and $\mathbb{F}^{\mathrm{per}}_\zeta$ are indexed by finite sets. Estimate (3.26d) is the result of Proposition 3.6, and (3.26c) is the result of the following lemma.

Lemma 3.19. *There exists a constant $C > 0$ such that, for $r \in (0,1)$, for σ, τ in X_r, ζ_1, ζ_2 in$\{\varphi, \psi\}$ and j_1, j_2 in $\{1, 3\}$, we have*

$$\Big|\Big|\Big|\mathbb{J}^{\zeta_2,j_2}_{\zeta_1,j_1}[\sigma,\tau]\Big|\Big|\Big|_{X_r} \leqslant C \,|||\sigma|||_{X_r} |||\tau|||_{X_r}. \tag{3.27}$$

Proof. Without loss of generality, we make the proof for $\zeta_1 = \zeta_2 = \varphi$ and $j_1 = j_2 = 1$. Recall that $\mathbb{J}^{\varphi,1}_{\varphi,1}$ is defined by (3.18) as

$$\mathbb{J}^{\varphi,1}_{\varphi,1}[\sigma,\tau] = J^{\varphi,1}_{\varphi,1} \sum_{\lambda\in\mathbb{Z}^*} \sigma_\lambda \tau_\lambda \, e^{i\lambda\Theta}.$$

For $\alpha = (\alpha', \alpha_d)$ in $\mathbb{N}^d \times \mathbb{N}$, we want to estimate in H^{d^*} the following function

$$\partial^\alpha_{t,y,\Theta} \mathbb{J}^{\varphi,1}_{\varphi,1}[\sigma,\tau](0,.,.) = J^{\varphi,1}_{\varphi,1} \sum_{\lambda\in\mathbb{Z}^*} \sum_{\beta'\leqslant\alpha'} \binom{\alpha'}{\beta'} (i\lambda)^{\alpha_d} \partial^{\beta'}_{t,y} \sigma_\lambda(0,.)\, \partial^{\alpha'-\beta'}_{t,y} \tau_\lambda(0,.)\, e^{i\lambda\Theta}. \tag{3.28}$$

We prove now an intermediate result. For every function u, v defined on $\mathbb{R}^{d-1} \times \mathbb{T}$, whose Fourier series expansions read

$$u(y,\Theta) := \sum_{\lambda\in\mathbb{Z}} u_\lambda(y)\, e^{i\lambda\Theta}, \qquad v(y,\Theta) := \sum_{\lambda\in\mathbb{Z}} v_\lambda(y)\, e^{i\lambda\Theta},$$

we have,

$$\begin{aligned}
&\Big\|(y,\Theta) \mapsto \sum_{\lambda\in\mathbb{Z}} u_\lambda(y) v_\lambda(y)\, e^{i\lambda\Theta}\Big\|^2_{H^{d^*}} \\
&\qquad = \sum_{k\in\mathbb{Z}^*} \int_{\mathbb{R}^{d-1}} \langle(\lambda,\xi)\rangle^{2d^*} \left|\int_{\mathbb{R}^{d-1}} \widehat{u_\lambda}(\eta)\, \widehat{v_\lambda}(\xi-\eta)\, d\eta\right|^2 d\xi
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \sum_{k\in\mathbb{Z}^*}\int_{\mathbb{R}^{d-1}}\left(\int_{\mathbb{R}^{d-1}}\frac{\langle(\lambda,\xi)\rangle^{2d^*}}{\langle(\lambda,\eta)\rangle^{2d^*}\,\langle(\lambda,\xi-\eta)\rangle^{2d^*}}\,d\eta\right)\\
&\quad\times\int_{\mathbb{R}^{d-1}}\langle(\lambda,\eta)\rangle^{2d^*}\left|\widehat{u_\lambda}(\eta)\right|^2\,\langle(\lambda,\xi-\eta)\rangle^{2d^*}\left|\widehat{v_\lambda}(\xi-\eta)\right|^2 d\eta\,d\xi\\
&\leqslant C\left\|(y,\Theta)\mapsto\sum_{\lambda\in\mathbb{Z}}u_\lambda(y)\,e^{i\lambda\Theta}\right\|^2_{H^{d^*}}\left\|(y,\Theta)\mapsto\sum_{\lambda\in\mathbb{Z}}v_\lambda(y)\,e^{i\lambda\Theta}\right\|^2_{H^{d^*}},
\end{aligned}
\tag{3.29}
$$

with a constant $C>0$ independent on u and v, since $\int_{\mathbb{R}^{d-1}}\frac{\langle(\lambda,\xi)\rangle^{2d^*}}{\langle(\lambda,\eta)\rangle^{2d^*}\langle(\lambda,\xi-\eta)\rangle^{2d^*}}\,d\eta$ is bounded uniformly with respect to (λ,ξ). We have, according to (3.28),

$$
\begin{aligned}
&\left\|\partial^\alpha_{t,y,\Theta}\mathbb{J}^{\varphi,1}_{\varphi,1}[\sigma,\tau](0,.,.)\right\|^2_{H^{d^*}}\\
&\qquad\leqslant\left(J^{\varphi,1}_{\varphi,1}\right)^2\sum_{\beta'\leqslant\alpha'}\binom{\alpha'}{\beta'}\left\|(y,\Theta)\mapsto\sum_{\lambda\in\mathbb{Z}^*}(i\lambda)^{\alpha_d}\partial^{\beta'}_{t,y}\sigma_\lambda(0,.)\,\partial^{\alpha'-\beta'}_{t,y}\tau_\lambda(0,.)\,e^{i\lambda\Theta}\right\|,
\end{aligned}
\tag{3.30}
$$

so, applying inequality (3.29) to quantity (3.30) we get

$$
\begin{aligned}
&\left\|\partial^\alpha_{t,y,\Theta}\mathbb{J}^{\varphi,1}_{\varphi,1}[\sigma,\tau](0,.,.)\right\|^2_{H^{d^*}}\\
&\quad\leqslant C\left(J^{\varphi,1}_{\varphi,1}\right)^2\sum_{\beta'\leqslant\alpha'}\binom{\alpha'}{\beta'}\left\|(y,\Theta)\mapsto\sum_{\lambda\in\mathbb{Z}}(i\lambda)^{\alpha_d}\partial^{\beta'}_{t,y}\sigma_\lambda(0,y)\,e^{i\lambda\Theta}\right\|^2_{H^{d^*}}\\
&\qquad\times\left\|(y,\Theta)\mapsto\sum_{\lambda\in\mathbb{Z}}\partial^{\alpha'-\beta'}_{t,y}\tau_\lambda(0,y)\,e^{i\lambda\Theta}\right\|^2_{H^{d^*}}\\
&\quad=C\left(J^{\varphi,1}_{\varphi,1}\right)^2\sum_{\beta'\leqslant\alpha'}\binom{\alpha'}{\beta'}\left\|\partial^{(\beta',\alpha_d)}_{t,y,\Theta}\sigma(0,.,.)\right\|^2_{H^{d^*}}\left\|\partial^{(\alpha'-\beta',0)}_{t,y,\Theta}\tau(0,.,.)\right\|^2_{H^{d^*}}.
\end{aligned}
$$

Therefore, by definition of the X_r-norm,

$$
\begin{aligned}
&\left\|\partial^\alpha_{t,y,\Theta}\mathbb{J}^{\varphi,1}_{\varphi,1}[\sigma,\tau](0,.,.)\right\|^2_{H^{d^*}}\\
&\quad\leqslant C\sum_{\beta'\leqslant\alpha'}\binom{\alpha'}{\beta'}\frac{|||\sigma|||_{X_r}\,\beta'!\alpha_d!}{r^{|\beta'|+\alpha_d}\left(|(\beta,\alpha_d)|^{2d+1}+1\right)}\frac{|||b|||_{X_r}\,(\alpha'-\beta')!}{r^{|\alpha'|-|\beta'|}\left(|\alpha'-\beta'|^{2d+1}+1\right)}\\
&\quad=\frac{C\,|||a|||_{X_r}|||b|||_{X_r}\alpha!}{r^{|\alpha|}\left(|\alpha|^{2d+1}+1\right)}\sum_{\beta'\leqslant\alpha'}\frac{\left(|\alpha|^{2d+1}+1\right)}{\left(|(\beta',\alpha_d)|^{2d+1}+1\right)\left(|\alpha'-\beta'|^{2d+1}+1\right)},
\end{aligned}
$$

and the result follows since $\sum_{\beta'\leqslant\alpha'}\frac{\left(|\alpha|^{2d+1}+1\right)}{\left(|(\beta',\alpha_d)|^{2d+1}+1\right)\left(|\alpha'-\beta'|^{2d+1}+1\right)}$ is bounded uniformly with respect to $\alpha\in\mathbb{N}^{d+1}$. □

The rest of the section is devoted to the proof of the following existence and uniqueness result.

Theorem 3.20. *For every $M_0,M_1>0$, there exist $0<r_1<1$ and $\delta>0$ such that for every $\mathbf{g}$ in $\mathbf{X}^6_{r_1}$ and $\mathbf{H}$ in $\mathbf{E}^2_1$ satisfying respectively $|||\mathbf{H}|||_{\mathbf{E}^2_1}<M_0$ and $|||\mathbf{g}|||_{\mathbf{X}^6_{r_1}}<M_1$, system of equations (3.17), (3.19), (3.20) and (3.21) admits a unique solution given by $\boldsymbol{\sigma}$ in $\mathcal{C}^1\left((-\delta(r_1-r),\delta(r_1-r)),\mathbf{X}^6_r\right)$ for each $0<r<r_1$ and $\mathbf{a}$ in $\mathbf{X}^2_{r_1}$, where we have denoted $\boldsymbol{\sigma}:=\left(\sigma^n_{\varphi,1},\sigma^n_{\varphi,3},\sigma^n_{\psi,1},\sigma^n_{\psi,3},\sigma^n_{\nu,1},\sigma^n_{\nu,3}\right)_{n\geqslant1}$, $\mathbf{g}:=\left(\mathbf{g}_{\varphi,1},\mathbf{g}_{\varphi,3},\mathbf{g}_{\psi,1},\mathbf{g}_{\psi,3},\mathbf{g}_{\nu,1},\mathbf{g}_{\nu,3}\right)$ and $\mathbf{H}:=\left(\mathbf{H}_\varphi,\mathbf{H}_\psi\right)$.*

To prove this result, we will start by proving existence for boundary system (3.20)-(3.21), with the Banach fixed point theorem applied in a closed ball of $\mathbf{E}^2_\rho$ for some $\rho\in(0,1)$. We will therefore have a solution $\mathbf{a}$ of (3.20)-(3.21) analytical both with respect to space and time. The strategy is to write equations (3.20) as a fixed point, by the change of variables $\mathbf{b}:=\partial_t\,\mathbf{a}$,

to obtain a problem like the one of [BG78]. This will allow us to prove that the operator at stake is a contraction, using the phenomenon of regularization by integration in time. Then we proceed with the existence of solution for interior system (3.17)-(3.19), by applying a classical Cauchy-Kovalevskaya result, in $\mathbf{X}^6_r$ for some $r \in (0,1)$. For this purpose, equations (3.17) will be seen as propagation equations in the normal variable. Verifying assumptions of Theorem 3.1 for interior equations presents no difficulty. Finally Lemma 3.16 will be used to assert that the obtained solution $\mathbf{a}$ of (3.20)-(3.21) in $\mathbf{E}^2_\rho$ is actually in $\mathbf{X}^2_r$ for some $r \in (0,1)$.

3.6. Time analyticity on the boundary and Cauchy-Kovalevskaya theorem for incoming equations.

3.6.1. *Existence and time analyticity for boundary equations.* This part is devoted to solving boundary system (3.20)-(3.21). The goal is to obtain solutions which are analytical not only with respect to (y,Θ) but also with respect to time. In the same way as for the first simplified model, system (3.20)-(3.21) can be displayed in the form

$$\begin{cases} \partial_t \mathbf{a} = L\,\mathbf{a} - \partial_\Theta \mathbf{D}^{\mathrm{Lop}}(\mathbf{a},\mathbf{a}) - \mathbb{F}^{\mathrm{per}}(\partial_\Theta \mathbf{a}, \partial_\Theta \mathbf{a}) + \mathbf{H} + \mathbf{K}^{\mathrm{Lop}}(\mathbf{a},\mathbf{a}), \\ \mathbf{a}(0) = 0, \end{cases} \tag{3.31}$$

where $\mathbf{a} := (\mathbf{a}_\varphi, \mathbf{a}_\psi) := (a^n_\varphi, a^n_\psi)_{n\geqslant 1}$, $\mathbf{H} := (\mathbf{H}_\varphi, \mathbf{H}_\psi)$, and, if $\mathbf{c} := (c^n_\varphi, c^n_\psi)_{n\geqslant 1}$,

$$L\,\mathbf{a} := (\mathbf{v}^{\mathrm{Lop}}_\varphi \cdot \nabla_y a^n_\varphi, \mathbf{v}^{\mathrm{Lop}}_\varphi \cdot \nabla_y a^n_\psi)_{n\geqslant 1},$$

$$\mathbf{D}^{\mathrm{Lop}}(\mathbf{a},\mathbf{c}) := (D^{\mathrm{Lop}}_\varphi a^1_\varphi\, a^n_\varphi, D^{\mathrm{Lop}}_\psi a^1_\psi\, a^n_\psi)_{n\geqslant 1},$$

$$\mathbb{F}^{\mathrm{per}}(\mathbf{a},\mathbf{c}) := (\mathbf{w}_\varphi \mathbb{F}^{\mathrm{per}}_\varphi[a^1_\varphi, c^n_\varphi], \mathbf{w}_\psi \mathbb{F}^{\mathrm{per}}_\psi[a^1_\psi, c^n_\psi])_{n\geqslant 1},$$

$$\mathbf{K}^{\mathrm{Lop}}(\mathbf{a},\mathbf{c}) := \Big(K^{\mathrm{Lop}}_\varphi \sum_{k=1}^{n-1} \partial_\Theta(a^k_\varphi\, c^{n-k}_\psi), K^{\mathrm{Lop}}_\psi \sum_{k=1}^{n-1} \partial_\Theta(a^k_\varphi\, c^{n-k}_\psi)\Big)_{n\geqslant 1}.$$

Setting $\mathbf{b} := \partial_t \mathbf{a}$, system (3.31) is equivalent to

$$\begin{aligned} \mathbf{b}(t) = {} & L\int_0^t \mathbf{b}(\sigma)\,d\sigma - \partial_\Theta \mathbf{D}^{\mathrm{Lop}}\Big(\int_0^t \mathbf{b}(\sigma)\,d\sigma, \int_0^t \mathbf{b}(\sigma)\,d\sigma\Big) \\ & - \mathbb{F}^{\mathrm{per}}\Big(\partial_\Theta \int_0^t \mathbf{b}(\sigma)\,d\sigma, \partial_\Theta \int_0^t \mathbf{b}(\sigma)\,d\sigma\Big) + \mathbf{K}^{\mathrm{Lop}}\Big(\int_0^t \mathbf{b}(\sigma)\,d\sigma, \int_0^t \mathbf{b}(\sigma)\,d\sigma\Big) + \mathbf{H}(t), \end{aligned} \tag{3.32}$$

with $\mathbf{a}(t) = \int_0^t \mathbf{b}(\sigma)\,d\sigma$. The aim is to solve equation (3.32) with a fixed point theorem in $\mathbf{E}^2_\rho$, so we start by proving the following key estimates, which will allow us to prove contraction for the operator at stake.

Estimates (3.33a), (3.33b) and (3.33d) below constitute what we call *regularization by integration in time*, where composing derivation in (y,Θ) with integration in time leads to no loss of regularity. This phenomenon was introduced by [Uka01], and can also be found in [Mét09, Mor20].

Lemma 3.21. *There exists $C>0$ such that for $\rho \in (0,1)$, for $\mathbf{b},\mathbf{c}$ in $\mathbf{E}^2_\rho$, the following estimates hold*

$$\Big|\!\Big|\!\Big| t \mapsto L\int_0^t \mathbf{b}(\sigma)\,d\sigma \Big|\!\Big|\!\Big|_{\mathbf{E}^2_\rho} \leqslant C\,\rho\,\gamma_0\,|\!|\!|\mathbf{b}|\!|\!|_{\mathbf{E}^2_\rho}, \tag{3.33a}$$

$$\Big|\!\Big|\!\Big| t \mapsto \partial_\Theta \mathbf{D}^{\mathrm{Lop}}\Big(\int_0^t \mathbf{b}(\sigma)\,d\sigma, \int_0^t \mathbf{c}(\sigma)\,d\sigma\Big) \Big|\!\Big|\!\Big|_{\mathbf{E}^2_\rho} \leqslant C\,\rho^2\,\gamma_0\,|\!|\!|\mathbf{b}|\!|\!|_{\mathbf{E}^2_\rho} |\!|\!|\mathbf{c}|\!|\!|_{\mathbf{E}^2_\rho}, \tag{3.33b}$$

$$\Big|\!\Big|\!\Big| t \mapsto \mathbb{F}^{\mathrm{per}}\Big(\partial_\Theta \int_0^t \mathbf{b}(\sigma)\,d\sigma, \partial_\Theta \int_0^t \mathbf{c}(\sigma)\,d\sigma\Big) \Big|\!\Big|\!\Big|_{\mathbf{E}^2_\rho} \leqslant C\,\rho^2\,\gamma_0\,|\!|\!|\mathbf{b}|\!|\!|_{\mathbf{E}^2_\rho} |\!|\!|\mathbf{c}|\!|\!|_{\mathbf{E}^2_\rho}, \tag{3.33c}$$

$$\Big|\!\Big|\!\Big| t \mapsto \mathbf{K}^{\mathrm{Lop}}\Big(\int_0^t \mathbf{b}(\sigma)\,d\sigma, \int_0^t \mathbf{c}(\sigma)\,d\sigma\Big) \Big|\!\Big|\!\Big|_{\mathbf{E}^2_\rho} \leqslant C\,\rho^2\,\gamma_0\,|\!|\!|\mathbf{b}|\!|\!|_{\mathbf{E}^2_\rho} |\!|\!|\mathbf{c}|\!|\!|_{\mathbf{E}^2_\rho}. \tag{3.33d}$$

Proof. First note that since $\mathbf{b}, \mathbf{c}$ are in $\mathbf{E}^2_\rho$, functions which we wish to estimate are in $\big(\widetilde{E}^{\mathbb{N}^*}_\rho\big)^2$. In all this proof, we denote

$$\mathbf{b} := (\mathbf{b}_\varphi, \mathbf{b}_\psi) := \big(b^n_\varphi, b^n_\psi\big)_{n\geqslant 1}, \qquad \mathbf{c} := (\mathbf{c}_\varphi, \mathbf{c}_\psi) := \big(c^n_\varphi, c^n_\psi\big)_{n\geqslant 1}.$$

For $\nu \geqslant 1$ and $s \in (0,1)$, we define $s_\nu := s + \frac{1}{\nu+1}(1-s)$ which is such that $s_\nu > s$ and satisfy, for $\nu \geqslant 1$,

$$\frac{1-s}{s_\nu - s} = \nu + 1 \qquad \text{and} \qquad \frac{1-s}{1-s_\nu} = 1 + \frac{1}{\nu}. \tag{3.34}$$

We proceed with the proof of estimate (3.33a) dealing with function $\mathbf{V} := L\int_0^t \mathbf{b}(\sigma)\,d\sigma$, and we denote $\mathbf{V} := (\mathbf{V}_\varphi, \mathbf{V}_\psi) := \big(V^n_\varphi, V^n_\psi\big)_{n\geqslant 1}$. According to Definition 3.12, the aim is to estimate, for $s \in (0,1)$, $\nu \geqslant 0$, $\zeta = \varphi, \psi$ and $n \geqslant 1$, the Y_s-norm of $\partial_t^\nu V^n_\zeta(0)$. Fix $s \in (0,1)$, $\zeta = \varphi, \psi$ and $n \geqslant 1$, and recall that

$$V^n_\zeta(t) = \mathbf{v}^{\mathrm{Lop}}_\zeta \cdot \nabla_y \int_0^t b^n_\zeta(\sigma)\,d\sigma.$$

Therefore we have $V^n_\zeta(0) = 0$ and, for $\nu \geqslant 1$,

$$\partial_t^\nu V^n_\zeta(0) = \mathbf{v}^{\mathrm{Lop}}_\zeta \cdot \nabla_y\, \partial_t^{\nu-1} b^n_\zeta(0),$$

so, for $s \in (0,1)$, using (3.7) with $0 < s < s_\nu \leqslant 1$, estimate (3.26a) and definition (3.24) of E_ρ-norm,

$$\begin{aligned}\Big\|\partial_t^\nu V^n_\zeta(0)\Big\|_{Y_s} &\leqslant \gamma_0 \Big\|\nabla_y\, \partial_t^{\nu-1} b^n_\zeta(0)\Big\|_{Y_s} \leqslant \frac{C\,\gamma_0}{s_\nu - s}\Big\|\partial_t^{\nu-1} b^n_\zeta(0)\Big\|_{Y_{s_\nu}} \\ &\leqslant \frac{C\,\gamma_0}{s_\nu - s}\frac{1}{(1-s_\nu)^\nu}\frac{\nu!\,\mathfrak{C}_{\nu-1}}{(4\rho)^{\nu-1}}\Big|\!\Big|\!\Big|b^n_\zeta\Big|\!\Big|\!\Big|_{E_\rho}.\end{aligned}$$

Therefore, using relations (3.34),

$$\begin{aligned}\Big\|\partial_t^\nu V^n_\zeta(0)\Big\|_{Y_s} &\leqslant C\,\gamma_0 \Big|\!\Big|\!\Big|b^n_\zeta\Big|\!\Big|\!\Big|_{E_\rho} \frac{1}{(1-s)^{\nu+1}}\frac{(\nu+1)\,\nu!\,\mathfrak{C}_{\nu-1}}{(4\rho)^{\nu-1}}\Big(1+\frac{1}{\nu}\Big)^\nu \\ &\leqslant C\,\rho\,\gamma_0 \Big|\!\Big|\!\Big|b^n_\zeta\Big|\!\Big|\!\Big|_{E_\rho} \frac{(\nu+1)!\,\mathfrak{C}_\nu}{(1-s)^{\nu+1}\,(4\rho)^\nu},\end{aligned}$$

so that $\Big|\!\Big|\!\Big|V^n_\zeta\Big|\!\Big|\!\Big|_{E_\rho} \leqslant C\,\rho\,\gamma_0 \Big|\!\Big|\!\Big|b^n_\zeta\Big|\!\Big|\!\Big|_{E_\rho}$. Since this estimate is independent of n, ζ, multiplying it by $e^{2\rho n}\langle n\rangle^{2d^*}$ and summing over $n \geqslant 1$ leads to the analogous one for $\mathbf{E}^2_\rho$, which reads as (3.33a).

For $\mathbf{V} := \partial_\Theta \mathbf{D}^{\mathrm{Lop}}\Big(\int_0^t \mathbf{b}(\sigma)\,d\sigma, \int_0^t \mathbf{c}(\sigma)\,d\sigma\Big)$ which we decompose in $\big(\widetilde{E}^{\mathbb{N}^*}_\rho\big)^2$ as $\mathbf{V} =: (\mathbf{V}_\varphi, \mathbf{V}_\psi) =: \big(V^n_\varphi, V^n_\psi\big)_{n\geqslant 1}$, with, for $n \geqslant 1$ and $\zeta = \varphi, \psi$,

$$V^n_\zeta := \partial_\Theta\, D^{\mathrm{Lop}}_\zeta \int_0^t b^1_\zeta(\sigma)\,d\sigma \cdot \int_0^t c^n_\zeta(\sigma)\,d\sigma,$$

for $n \geqslant 1$ and $\zeta = \varphi, \psi$, we compute $V^n_\zeta(0) = \partial_t V^n_\zeta(0) = 0$, and for $\nu \geqslant 2$,

$$\partial_t^\nu V^n_\zeta(0) = D^{\mathrm{Lop}}_\zeta\, \partial_\Theta \sum_{\mu=1}^{\nu-1}\binom{\nu}{\mu}\partial_t^{\mu-1} b^1_\zeta(0)\,\partial_t^{\nu-\mu-1} c^n_\zeta(0).$$

Therefore, for $s \in (0,1)$, using (3.7) with $0 < s < s_\nu \leqslant 1$ and estimates (3.26a) and (3.24), we have

$$\begin{aligned}\Big\|\partial_t^\nu V^n_\zeta(0)\Big\|_{Y_s} &\leqslant \frac{C\,\gamma_0}{s_\nu - s}\sum_{\mu=1}^{\nu-1}\binom{\nu}{\mu}\Big\|\partial_t^{\mu-1} b^1_\zeta(0)\Big\|_{Y_{s_\nu}}\Big\|\partial_t^{\nu-\mu-1} c^n_\zeta(0)\Big\|_{Y_{s_\nu}} \\ &\leqslant \frac{C\,\gamma_0}{s_\nu - s}\frac{\Big|\!\Big|\!\Big|b^1_\zeta\Big|\!\Big|\!\Big|_{E_\rho}\Big|\!\Big|\!\Big|c^n_\zeta\Big|\!\Big|\!\Big|_{E_\rho}}{(4\rho)^{\nu-2}}\sum_{\mu=1}^{\nu-1}\binom{\nu}{\mu}\frac{\mu!\,\mathfrak{C}_{\mu-1}}{(1-s_\nu)^\mu}\frac{(\nu-\mu)!\,\mathfrak{C}_{\nu-\mu-1}}{(1-s_\nu)^{\nu-\mu}}.\end{aligned}$$

Again with relations (3.34) we get

$$\begin{aligned}\left\|\partial_t^\nu V_\zeta^n(0)\right\|_{Y_s} &\leqslant \frac{C\,\gamma_0\,(\nu+1)}{1-s}\,\frac{\left|\!\left|\!\left|b_\zeta^1\right|\!\right|\!\right|_{E_\rho}\left|\!\left|\!\left|c_\zeta^n\right|\!\right|\!\right|_{E_\rho}}{(4\rho)^{\nu-2}}\,\frac{\nu!}{(1-s)^\nu}\sum_{\mu=1}^{\nu-1}\mathfrak{C}_{\mu-1}\,\mathfrak{C}_{\nu-\mu-1}\left(1+\frac{1}{\nu}\right)^\nu\\ &\leqslant C\,\rho^2\,\gamma_0\,e\,\left|\!\left|\!\left|b_\zeta^1\right|\!\right|\!\right|_{E_\rho}\left|\!\left|\!\left|c_\zeta^n\right|\!\right|\!\right|_{E_\rho}\frac{(\nu+1)!}{(1-s)^{\nu+1}\,(4\rho)^\nu}\sum_{\mu=1}^{\nu-1}\mathfrak{C}_{\mu-1}\,\mathfrak{C}_{\nu-\mu-1}.\end{aligned}$$

We can conclude using (3.22):

$$\sum_{\mu=1}^{\nu-1}\mathfrak{C}_{\mu-1}\,\mathfrak{C}_{\nu-\mu-1}=\sum_{\mu=0}^{\nu-2}\mathfrak{C}_\mu\,\mathfrak{C}_{\nu-2-\mu}=\mathfrak{C}_{\nu-1}\leqslant\mathfrak{C}_\nu.$$

Last inequality follows from relation (3.22) and $\mathfrak{C}_0=1$.

The proof of estimate (3.33c) for $\mathbf{V}:=\mathbb{F}^{\rm per}\Big(\partial_\Theta\int_0^t\mathbf{b}(\sigma)\,d\sigma,\partial_\Theta\int_0^t\mathbf{c}(\sigma)\,d\sigma\Big)$ follows the same argument as the previous one, but it is simpler since, according to (3.26d), $\mathbb{F}_\zeta^{\rm per}(\partial_\Theta\,\mathbf{a},\partial_\Theta\,\mathbf{a})$ acts as a semilinear term in $\mathbf{a}$. It is therefore omitted.

Finally, we take interest into $\mathbf{V}:=\mathbf{K}^{\rm Lop}\Big(\int_0^t\mathbf{b}(\sigma)\,d\sigma,\int_0^t\mathbf{c}(\sigma)\,d\sigma\Big)$ which we decompose in $\big(\widetilde{E}_\rho^{\mathbb{N}^*}\big)^2$ as $\mathbf{V}=:(\mathbf{V}_\varphi,\mathbf{V}_\psi)=:\big(V_\varphi^n,V_\psi^n\big)_{n\geqslant1}$, with, for $n\geqslant1$ and $\zeta=\varphi,\psi$,

$$V_\zeta^n:=K_\zeta^{\rm Lop}\sum_{k=1}^{n-1}\partial_\Theta\big(b_\varphi^k,c_\psi^{n-k}\big)$$

For $n\geqslant1$ and $\zeta=\varphi,\psi$, we compute $V_\zeta^n(0)=\partial_tV_\zeta^n(0)=0$, and for $\nu\geqslant2$,

$$\partial_t^\nu V_\zeta^n(0)=K_\zeta^{\rm Lop}\sum_{k=1}^{n-1}\partial_\Theta\sum_{\mu=1}^{\nu-1}\binom{\nu}{\mu}\partial_t^{\mu-1}b_\varphi^k(0)\,\partial_t^{\nu-\mu-1}c_\psi^{n-k}(0),$$

so, for $s\in(0,1)$, using (3.26b) and (3.7) with $0<s<s_\nu\leqslant1$ and then (3.24),

$$\begin{aligned}\left\|\partial_t^\nu V_\zeta^n(0)\right\|_{Y_s}&\leqslant\frac{C\,\gamma_0}{s_\nu-s}\sum_{k=1}^{n-1}\sum_{\mu=1}^{\nu-1}\binom{\nu}{\mu}\left\|\partial_t^{\mu-1}b_\varphi^k(0)\right\|_{Y_{s_\nu}}\left\|\partial_t^{\nu-\mu-1}c_\psi^{n-k}(0)\right\|_{Y_{s_\nu}}\\ &\leqslant\frac{C\,\gamma_0}{s_\nu-s}\,\frac{1}{(4\rho)^{\nu-2}}\sum_{k=1}^{n-1}\left|\!\left|\!\left|b_\varphi^k\right|\!\right|\!\right|_{E_\rho}\left|\!\left|\!\left|c_\psi^{n-k}\right|\!\right|\!\right|_{E_\rho}\\ &\quad\sum_{\mu=1}^{\nu-1}\binom{\nu}{\mu}\frac{\mu!\,\mathfrak{C}_{\mu-1}}{(1-s_\nu)^\mu}\,\frac{(\nu-\mu)!\,\mathfrak{C}_{\nu-\mu-1}}{(1-s_\nu)^{\nu-\mu}}\\ &\leqslant C\,\rho^2\,\gamma_0\,\frac{(\nu+1)!\,\mathfrak{C}_\nu}{(1-s)^{\nu+1}(4\rho)^\nu}\sum_{k=1}^{n-1}\left|\!\left|\!\left|b_\varphi^k\right|\!\right|\!\right|_{E_\rho}\left|\!\left|\!\left|c_\psi^{n-k}\right|\!\right|\!\right|_{E_\rho},\end{aligned}$$

using (3.22) as before. Therefore we get

$$\left|\!\left|\!\left|V_\zeta^n\right|\!\right|\!\right|_{E_\rho}\leqslant C\,\rho^2\,\gamma_0\sum_{k=1}^{n-1}\left|\!\left|\!\left|b_\varphi^k\right|\!\right|\!\right|_{E_\rho}\left|\!\left|\!\left|c_\psi^{n-k}\right|\!\right|\!\right|_{E_\rho},$$

so that, multiplying by $e^{2\,\rho\,n}\,\langle n\rangle^{2d^*}$ and summing over $n\geqslant1$, we obtain

$$\begin{aligned}&\sum_{\zeta=\varphi,\psi}\sum_{n\geqslant1}e^{2\,\rho\,n}\,\langle n\rangle^{2d^*}\left|\!\left|\!\left|V_\zeta^n\right|\!\right|\!\right|_{E_\rho}^2\\ &\qquad\leqslant C^2\,\rho^4\,\gamma_0^2\sum_{\zeta=\varphi,\psi}\sum_{n\geqslant1}e^{2\,\rho\,n}\,\langle n\rangle^{2d^*}\left(\sum_{k=1}^{n-1}\left|\!\left|\!\left|b_\varphi^k\right|\!\right|\!\right|_{E_\rho}\left|\!\left|\!\left|c_\psi^{n-k}\right|\!\right|\!\right|_{E_\rho}\right)^2\end{aligned}$$

$$\leqslant C^2\,\rho^4\,\gamma_0^2 \sum_{n\geqslant 1}\left(\sum_{k=1}^{n-1}\frac{\langle n\rangle^{2d^*}}{\langle k\rangle^{2d^*}\,\langle n-k\rangle^{2d^*}}\right)$$
$$\sum_{k=1}^{n-1} e^{2\,\rho\,k}\,\langle k\rangle^{2d^*}\,\left|\!\left|\!\left|b_\varphi^k\right|\!\right|\!\right|_{E_\rho}^2 e^{2\,\rho\,(n-k)}\,\langle n-k\rangle^{2d^*}\,\left|\!\left|\!\left|c_\psi^{n-k}\right|\!\right|\!\right|_{E_\rho}^2,$$

using Cauchy-Schwarz inequality. Therefore, since $\sum_{k=1}^{n-1}\frac{\langle n\rangle^{2d^*}}{\langle k\rangle^{2d^*}\langle n-k\rangle^{2d^*}}$ is bounded uniformly with respect to $n\geqslant 1$, we get

$$|||\mathbf{V}|||_{\mathbf{E}_\rho^2}^2=\sum_{\zeta=\varphi,\psi}\sum_{n\geqslant 1} e^{2\,\rho\,n}\,\langle n\rangle^{2d^*}\,\left|\!\left|\!\left|V_\zeta^n\right|\!\right|\!\right|_{E_\rho}^2\leqslant C^2\,\rho^4\,\gamma_0^2\,|||\mathbf{b}|||_{\mathbf{E}_\rho^2}^2|||\mathbf{c}|||_{\mathbf{E}_\rho^2}^2,$$

which is the sought inequality (3.33d). □

We are now in place to prove existence for (3.32) (which is equivalent to (3.31)) in the space $\mathbf{E}_\rho^2$, of analytic functions with respect to (t,y,Θ). We follow here the method of [Uka01, Mor20].

Proposition 3.22. *For every $M_0>0$, there exists $\rho\in(0,1)$ such that for every $\mathbf{H}$ in $\mathbf{E}_\rho^2$ satisfying $|||\mathbf{H}|||_{\mathbf{E}_\rho^2}<M_0$, equation (3.32) admits a unique solution $\mathbf{b}$ in $\mathbf{E}_\rho^2$.*

Proof. In all this proof, for $R>0$, $B_\rho(0,R)$ denotes the closed ball of $\mathbf{E}_\rho^2$ centered at 0 and of radius R. For $\mathbf{b}$ and $\mathbf{H}$ in $\mathbf{E}_\rho^2$, we denote

$$F\big(\mathbf{H},\mathbf{b}\big):=L\int_0^t\mathbf{b}(\sigma)\,d\sigma-\partial_\Theta\,\mathbf{D}^{\mathrm{Lop}}\Big(\int_0^t\mathbf{b}(\sigma)\,d\sigma,\int_0^t\mathbf{b}(\sigma)\,d\sigma\Big)$$
$$-\mathbb{F}^{\mathrm{per}}\Big(\partial_\Theta\int_0^t\mathbf{b}(\sigma)\,d\sigma,\partial_\Theta\int_0^t\mathbf{b}(\sigma)\,d\sigma\Big)+\mathbf{K}^{\mathrm{Lop}}\Big(\int_0^t\mathbf{b}(\sigma)\,d\sigma,\int_0^t\mathbf{b}(\sigma)\,d\sigma\Big)+\mathbf{H},$$

so that solving (3.32) amounts to find a fixed point of $F\big(\mathbf{H},.\big):\mathbf{b}\mapsto F\big(\mathbf{H},\mathbf{b}\big)$. Therefore, we will prove that there exist $R>0$ and $\rho\in(0,1)$ such that for every $\mathbf{H}$ in $\mathbf{E}_\rho^2$ satisfying $|||\mathbf{H}|||_{\mathbf{E}_\rho^2}<M_0$, the map $F\big(\mathbf{H},.\big)$ is a contraction from the complete space $B_\rho(0,R)$ to itself.

Consider $M_0>0$ and $\mathbf{H}$ in $\mathbf{E}_\rho^2$ such that $|||\mathbf{H}|||_{\mathbf{E}_\rho^2}<M_0$. First we need to show that there exist $\rho\in(0,1)$ and $R>0$ such that $F\big(\mathbf{H},.\big)$ maps $B_\rho(0,R)$ to itself. Lemma 3.21 asserts that $F\big(\mathbf{H},.\big)$ is well defined from $\mathbf{E}_\rho^2$ to itself and that it satisfies, for $\mathbf{b}$ in $\mathbf{E}_\rho^2$,

$$\left|\!\left|\!\left|F\big(\mathbf{H},\mathbf{b}\big)\right|\!\right|\!\right|_{\mathbf{E}_\rho^2}\leqslant C\,\rho\,\gamma_0\,|||\mathbf{b}|||_{\mathbf{E}_\rho^2}+C\,\rho^2\,\gamma_0\,|||\mathbf{b}|||_{\mathbf{E}_\rho^2}^2+M_0,$$

for a new positive constant $C>0$ independent on ρ, $\mathbf{H}$, M_0 and $\mathbf{b}$. Therefore, setting $R:=2M_0$, for $0<\rho<C(\gamma_0,M_0)$, with

$$C(\gamma_0,M_0):=\min\Big(\big[4\,C\,\gamma_0\big]^{-1},\big[8\,C\,\gamma_0\,M_0\big]^{-1/2}\Big),$$

the application $F\big(\mathbf{H},.\big)$ maps the ball $B_\rho(0,R)$ to itself.

Now we need to show that this map is a contraction, for $\rho<C(M_0,\gamma_0)$ small enough. We compute, for $0<\rho<C(M_0,\gamma_0)$ and for $\mathbf{b},\mathbf{c}$ in $B_\rho(0,R)$,

$$F\big(\mathbf{H},\mathbf{b}\big)-F\big(\mathbf{H},\mathbf{c}\big)=$$
$$L\int_0^t\big(\mathbf{b}-\mathbf{c}\big)(\sigma)\,d\sigma-\partial_\Theta\,\mathbf{D}^{\mathrm{Lop}}\Big(\int_0^t\mathbf{b}(\sigma)\,d\sigma,\int_0^t\big(\mathbf{b}-\mathbf{c}\big)(\sigma)\,d\sigma\Big)$$
$$-\partial_\Theta\,\mathbf{D}^{\mathrm{Lop}}\Big(\int_0^t\big(\mathbf{b}-\mathbf{c}\big)(\sigma)\,d\sigma,\int_0^t\mathbf{c}(\sigma)\,d\sigma\Big)-\mathbb{F}^{\mathrm{per}}\Big(\partial_\Theta\int_0^t\mathbf{b}(\sigma)\,d\sigma,\partial_\Theta\int_0^t\big(\mathbf{b}-\mathbf{c}\big)(\sigma)\,d\sigma\Big)$$
$$-\mathbb{F}^{\mathrm{per}}\Big(\partial_\Theta\int_0^t\big(\mathbf{b}-\mathbf{c}\big)(\sigma)\,d\sigma,\partial_\Theta\int_0^t\mathbf{c}(\sigma)\,d\sigma\Big)+\mathbf{K}^{\mathrm{Lop}}\Big(\int_0^t\mathbf{b}(\sigma)\,d\sigma,\int_0^t\big(\mathbf{b}-\mathbf{c}\big)(\sigma)\,d\sigma\Big)$$
$$+\mathbf{K}^{\mathrm{Lop}}\Big(\int_0^t\big(\mathbf{b}-\mathbf{c}\big)(\sigma)\,d\sigma,\int_0^t\mathbf{c}(\sigma)\,d\sigma\Big).$$

Therefore using estimates of Lemma 3.21 and the fact that $\mathbf{b}, \mathbf{c}$ are in $B_\rho(0,R)$, we get,

$$\begin{aligned}\big|\!\big|\!\big| F(\mathbf{H},\mathbf{b}) - F(\mathbf{H},\mathbf{c}) \big|\!\big|\!\big|_{\mathbf{E}^2_\rho} &\leqslant C\,\rho\,\gamma_0\Big(1+\rho\,\big(|\!|\!|\mathbf{b}|\!|\!|_{\mathbf{E}^2_\rho} + |\!|\!|\mathbf{c}|\!|\!|_{\mathbf{E}^2_\rho}\big)\Big)\,|\!|\!|\mathbf{b}-\mathbf{c}|\!|\!|_{\mathbf{E}^2_\rho}\\ &\leqslant C\,\rho\,\gamma_0\,\big(1+\rho\,R\big)\,|\!|\!|\mathbf{b}-\mathbf{c}|\!|\!|_{\mathbf{E}^2_\rho},\end{aligned}$$

up to changing the constant $C>0$ in every line. Thus, for $\rho < \widetilde{C}(\gamma_0, M_0)$, with

$$\widetilde{C}(\gamma_0,M_0) := \min\Big(\big[4\,\gamma_0\,M_0\big]^{-1}, \big[8\,\gamma_0\,M_0^2\big]^{-1/2}, \big[2\,C\,\gamma_0\,\big(1+2\,M_0\big)\big]^{-1}\Big),$$

the map $F(\mathbf{H},.)$ is a contraction from $B_\rho(0,R)$ to itself.

Since $B_\rho(0,R)$ is a closed subspace of the Banach space $\mathbf{E}^2_\rho$, the Banach fixed-point theorem gives a unique solution to (3.32). □

3.6.2. *A Cauchy-Kovalevskaya theorem for incoming interior equations.* The aim is now to prove existence of solution to (3.17)-(3.19) with the Cauchy-Kovalevskaya type Theorem 3.1 using the chain of Banach spaces $\big(\mathbf{X}_r\big)_{r\in(0,1)}$.

We start by writing system (3.17)-(3.19) in a form suited to apply Theorem 3.1. Up to multiplying (3.17) by a nonzero constant (which is the x_d-component of $-\mathbf{v}_{\zeta,j}$), system (3.17)-(3.19) can be written as

$$\begin{cases}\partial_{x_d}\boldsymbol{\sigma} = L\,\boldsymbol{\sigma} - \partial_\Theta\,\mathbf{D}\big(\boldsymbol{\sigma},\boldsymbol{\sigma}\big) - \partial_\Theta\,\mathbb{J}\big(\boldsymbol{\sigma},\boldsymbol{\sigma}\big) + \partial_\Theta\,\mathbf{K}\big(\boldsymbol{\sigma},\boldsymbol{\sigma}\big)\\ \boldsymbol{\sigma}_{|x_d=0} = \widetilde{\mathbf{a}} + \mathbf{g},\end{cases} \tag{3.35}$$

where $\boldsymbol{\sigma} := \big(\boldsymbol{\sigma}_{\varphi,1},\boldsymbol{\sigma}_{\varphi,3},\boldsymbol{\sigma}_{\psi,1},\boldsymbol{\sigma}_{\psi,3},\boldsymbol{\sigma}_{\nu,1},\boldsymbol{\sigma}_{\nu,3}\big) := \big(\sigma^n_{\varphi,1},\sigma^n_{\varphi,3},\sigma^n_{\psi,1},\sigma^n_{\psi,3},\sigma^n_{\nu,1},\sigma^n_{\nu,3}\big)_{n\geqslant1}$, function $\mathbf{a}$ is the solution to (3.20)-(3.21), function $\widetilde{\mathbf{a}}$ is defined from $\mathbf{a}$ by $\widetilde{\mathbf{a}} := \big((e_{\varphi,1}\cdot r_{\varphi,1})\,\mathbf{a}_\varphi, (e_{\varphi,3}\cdot r_{\varphi,3})\,\mathbf{a}_\varphi, (e_{\psi,1}\cdot r_{\psi,1})\,\mathbf{a}_\psi, (e_{\psi,3}\cdot r_{\psi,3})\,\mathbf{a}_\psi, 0, 0\big)$, boundary term $\mathbf{g}$ is defined as $\mathbf{g} := \big(\mathbf{g}_{\varphi,1},\dots,\mathbf{g}_{\nu,3}\big)$, and, if $\boldsymbol{\tau} := \big(\tau^n_{\varphi,1},\dots,\tau^n_{\nu,3}\big)_{n\geqslant1}$,

$$\begin{aligned}L\,\boldsymbol{\sigma} &:= \big(\mathbf{v}_{\varphi,1}\cdot\nabla_{t,y}\,\boldsymbol{\sigma}_{\varphi,1},\dots,\mathbf{v}_{\nu,3}\cdot\nabla_{t,y}\,\boldsymbol{\sigma}_{\nu,3}\big),\\ \mathbf{D}\big(\boldsymbol{\sigma},\boldsymbol{\tau}\big) &:= \big(D_{\varphi,1}\,\sigma^n_{\varphi,1}\,\tau^1_{\varphi,1},\dots,D_{\nu,3}\,\sigma^n_{\nu,3}\,\tau^1_{\nu,3}\big)_{n\geqslant1},\\ \mathbb{J}\big(\boldsymbol{\sigma},\boldsymbol{\tau}\big) &:= \left(\sum_{\zeta_1,\zeta_2\in\{\varphi,\psi,\nu\}}\ \sum_{j_1,j_2\in\{1,3\}} \mathbb{J}^{\zeta_2,j_2}_{\zeta_1,j_1}\big[\sigma^1_{\zeta_1,j_1},\tau^n_{\zeta_2,j_2}\big]\right)_{\substack{\zeta=\varphi,\psi,\nu\\ j=1,3\\ n\geqslant1}},\\ \mathbf{K}\big(\boldsymbol{\sigma},\boldsymbol{\tau}\big) &:= \left(K_{\zeta,j}\sum_{\zeta_1,\zeta_2\in\{\varphi,\psi,\nu\}}\ \sum_{j_1,j_2\in\{1,3\}}\ \sum_{k=1}^{n-1}\sigma^k_{\zeta_1,j_1}\,\sigma^{n-k}_{\zeta_2,j_2}\right)_{\substack{\zeta=\varphi,\psi,\nu\\ j=1,3\\ n\geqslant1}},\end{aligned}$$

with new[8] $\mathbf{v}_{\zeta,j}$, $D_{\zeta,j}$, $K_{\zeta,j}$ and $\mathbb{J}^{\zeta_2,j_2}_{\zeta_1,j_1}$, satisfying the same assumptions as the old ones (3.26a) and (3.26c). Note that there exists a constant $C>0$ such that for any $\mathbf{a}$ in $\mathbf{X}^2_r$, we have $|\!|\!|\widetilde{\mathbf{a}}|\!|\!|_{\mathbf{X}^6_r} \leqslant C|\!|\!|\mathbf{a}|\!|\!|_{\mathbf{X}^2_r}$.

For $N_0>0$, denote by $\widetilde{F}$ the function of $[-N_0,N_0]\times\mathbf{X}^6_r$ defined by, for $|x_d|\leqslant N_0$ and $\boldsymbol{\sigma}\in\mathbf{X}^6_r$,

$$\widetilde{F}(x_d,\boldsymbol{\sigma}) := L\,\boldsymbol{\sigma} - \partial_\Theta\,\mathbf{D}\big(\boldsymbol{\sigma},\boldsymbol{\sigma}\big) - \partial_\Theta\,\mathbb{J}\big(\boldsymbol{\sigma},\boldsymbol{\sigma}\big) + \partial_\Theta\,\mathbf{K}\big(\boldsymbol{\sigma},\boldsymbol{\sigma}\big),$$

and set $\boldsymbol{\sigma}^0 := \widetilde{\mathbf{a}} + \mathbf{g}$. Now system (3.35) is equivalent to the following one, with $\boldsymbol{\tau} := \boldsymbol{\sigma} - \boldsymbol{\sigma}^0$,

$$\begin{cases}\boldsymbol{\tau}'(x_d) = \widetilde{F}\big(x_d,\boldsymbol{\tau}(x_d)+\boldsymbol{\sigma}^0\big)\\ \boldsymbol{\tau}(0) = 0,\end{cases} \tag{3.36}$$

which is, with $F\big(x_d,\boldsymbol{\tau}(x_d)\big) := \widetilde{F}\big(x_d,\boldsymbol{\tau}(x_d)+\boldsymbol{\sigma}^0\big)$ for $|x_d|<N_0$, in the right form to apply Theorem 3.1. Note that the operator F actually does not depend on x_d, so all suprema in x_d

[8] Due to the fact that we multiplied equation (3.17) by a nonzero coefficient to obtain a propagation equation in the normal variable.

in assumptions of Theorem 3.1 may be removed below when verifying the assumptions of this theorem on our particular problem (3.36). We will therefore omit to indicate the dependency in x_d of F and simply write $F(\boldsymbol{\tau})$. It remains to check the assumption of Theorem 3.1 to obtain existence of solutions to (3.36). The key estimates to do so are the following ones.

Lemma 3.23. *There exists $C>0$ such that for $0\leqslant r'<r\leqslant 1$, for $\boldsymbol{\sigma},\boldsymbol{\tau}$ in $\mathbf{X}^6_r$, the following estimates hold*

$$|||L\,\boldsymbol{\sigma}|||_{\mathbf{X}^6_{r'}}\leqslant\frac{C\,\gamma_0}{r-r'}\,|||\boldsymbol{\sigma}|||_{\mathbf{X}^6_r} \tag{3.37a}$$

$$|||\mathbf{D}(\boldsymbol{\sigma},\boldsymbol{\tau})|||_{\mathbf{X}^6_r}\leqslant C\gamma_0|||\boldsymbol{\sigma}|||_{\mathbf{X}^6_r}|||\boldsymbol{\tau}|||_{\mathbf{X}^6_r} \tag{3.37b}$$

$$|||\mathbb{J}(\boldsymbol{\sigma},\boldsymbol{\tau})|||_{\mathbf{X}^6_r}\leqslant C\gamma_0|||\boldsymbol{\sigma}|||_{\mathbf{X}^6_r}|||\boldsymbol{\tau}|||_{\mathbf{X}^6_r} \tag{3.37c}$$

$$|||\mathbf{K}(\boldsymbol{\sigma},\boldsymbol{\tau})|||_{\mathbf{X}^6_r}\leqslant C\gamma_0|||\boldsymbol{\sigma}|||_{\mathbf{X}^6_r}|||\boldsymbol{\tau}|||_{\mathbf{X}^6_r}. \tag{3.37d}$$

Proof. Estimate (3.37a) follows directly from Lemma 3.17 and assumption (3.26a) on $\mathbf{v}_{\zeta,j}$, for $\zeta=\varphi,\psi,\nu$, and $j=1,3$. As for them, estimates (3.37b) and (3.37c) rely on the algebra property of X_r and assumptions (3.26a) and (3.26c) on $D_{\zeta,j}$ and $\mathbb{J}^{\zeta_2,j_2}_{\zeta_1,j_1}$. Finally, estimate (3.37d) is proven using assumption (3.26a) on $K_{\zeta,j}$ for $\zeta=\varphi,\psi,\nu$, and $j=1,3$, and the same arguments used to prove algebra property of $\mathbf{X}_r$. □

The main result of this part is the following one, which, along with Proposition 3.22, will prove Theorem 3.20.

Proposition 3.24. *Consider $0<r_1<1$, and $\mathbf{a}$ a solution to system (3.20)-(3.21) given in $\mathbf{X}^2_{r_1}$. Then, for every $M_1>0$, the following existence and uniqueness result holds: there exists $\delta>0$ such that for every $\mathbf{g}$ in $\mathbf{X}^6_{r_1}$ satisfying $|||\mathbf{g}|||_{\mathbf{X}^6_{r_1}}<M_1$, system (3.36) admits a unique solution in $\mathcal{C}^1\big((-\delta(r_1-r),\delta(r_1-r)),\mathbf{X}^6_r\big)$ for each $r\in(0,r_1)$.*

Proof. The aim is to apply Theorem 3.1 with the scale of Banach spaces $\big(\mathbf{X}^6_r\big)_{0<r\leqslant r_1}$. Fix now a constant $M_1>0$ as well as[9] $R>0$, and consider $\mathbf{g}$ in $\mathbf{X}^6_{r_1}$ such that $|||\mathbf{g}|||_{\mathbf{X}^6_{r_1}}<M_1$. We will now verify assumptions (3.5) and (3.6). We compute that for $\boldsymbol{\tau}$ in $\mathbf{X}^6_{r_1}$, we have

$$\begin{aligned}F(\boldsymbol{\tau})=&L\,\boldsymbol{\tau}+L\,\boldsymbol{\sigma^0}-\partial_\Theta\,\mathbf{D}(\boldsymbol{\tau},\boldsymbol{\tau})-\partial_\Theta\,\mathbf{D}(\boldsymbol{\sigma^0},\boldsymbol{\tau})-\partial_\Theta\,\mathbf{D}(\boldsymbol{\tau},\boldsymbol{\sigma^0})-\partial_\Theta\,\mathbf{D}(\boldsymbol{\sigma^0},\boldsymbol{\sigma^0})\\&-\partial_\Theta\,\mathbb{J}(\boldsymbol{\tau},\boldsymbol{\tau})-\partial_\Theta\,\mathbb{J}(\boldsymbol{\tau},\boldsymbol{\sigma^0})-\partial_\Theta\,\mathbb{J}(\boldsymbol{\sigma^0},\boldsymbol{\tau})-\partial_\Theta\,\mathbb{J}(\boldsymbol{\sigma^0},\boldsymbol{\sigma^0})\\&+\partial_\Theta\,\mathbf{K}(\boldsymbol{\tau},\boldsymbol{\tau})+\partial_\Theta\,\mathbf{K}(\boldsymbol{\tau},\boldsymbol{\sigma^0})+\partial_\Theta\,\mathbf{K}(\boldsymbol{\sigma^0},\boldsymbol{\tau})+\partial_\Theta\,\mathbf{K}(\boldsymbol{\sigma^0},\boldsymbol{\sigma^0}).\end{aligned}$$

Therefore,

$$F(0)=L\,\boldsymbol{\sigma^0}-\partial_\Theta\,\mathbf{D}(\boldsymbol{\sigma^0},\boldsymbol{\sigma^0})-\partial_\Theta\,\mathbb{J}(\boldsymbol{\sigma^0},\boldsymbol{\sigma^0})+\partial_\Theta\,\mathbf{K}(\boldsymbol{\sigma^0},\boldsymbol{\sigma^0}),$$

so, using Lemmas 3.17 and 3.23, we get that there exists a constant $C>0$ such that for all $0<r'<r<r_1$,

$$|||F(0)|||_{\mathbf{X}^6_{r'}}\leqslant\frac{C\,\gamma_0}{r-r'}\Big(|||\boldsymbol{\sigma^0}|||_{\mathbf{X}^6_r}+|||\boldsymbol{\sigma^0}|||^2_{\mathbf{X}^6_r}\Big)\leqslant\frac{C\,\gamma_0}{r-r'}\Big(|||\boldsymbol{\sigma^0}|||_{\mathbf{X}^6_r}+|||\boldsymbol{\sigma^0}|||^2_{\mathbf{X}^6_r}\Big),$$

and then, using the fact that $|||\widetilde{\mathbf{a}}|||_{\mathbf{X}^6_r}\leqslant C|||\mathbf{a}|||_{\mathbf{X}^2_r}$ and $|||\mathbf{g}|||_{\mathbf{X}^6_{r_1}}<M_1$,

$$|||F(0)|||_{\mathbf{X}^6_{r'}}\leqslant\frac{C\,\gamma_0}{r-r'}\Big(|||\mathbf{a}|||_{\mathbf{X}^2_r}+|||\mathbf{a}|||^2_{\mathbf{X}^2_r}+M_1+M_1^2\Big),$$

so assumption (3.6) is satisfied with $M:=C\,\gamma_0\big(|||a|||_{\mathbf{X}^2_r}+|||a|||^2_{\mathbf{X}^2_r}+M_1+M_1^2\big)$.

On the other hand, we have, for $\boldsymbol{\tau},\boldsymbol{\omega}$ in $\mathbf{X}^6_r$,

$$\begin{aligned}F(\boldsymbol{\tau})-F(\boldsymbol{\omega})=&\,L\,(\boldsymbol{\tau}-\boldsymbol{\omega})-\partial_\Theta\,\mathbf{D}(\boldsymbol{\tau}-\boldsymbol{\omega},\boldsymbol{\tau})-\partial_\Theta\,\mathbf{D}(\boldsymbol{\omega},\boldsymbol{\tau}-\boldsymbol{\omega})-\partial_\Theta\,\mathbf{D}(\boldsymbol{\sigma^0},\boldsymbol{\tau}-\boldsymbol{\omega})\\&-\partial_\Theta\,\mathbf{D}(\boldsymbol{\tau}-\boldsymbol{\omega},\boldsymbol{\sigma^0})-\partial_\Theta\,\mathbb{J}(\boldsymbol{\tau}-\boldsymbol{\omega},\boldsymbol{\tau})-\partial_\Theta\,\mathbb{J}(\boldsymbol{\omega},\boldsymbol{\tau}-\boldsymbol{\omega})-\partial_\Theta\,\mathbb{J}(\boldsymbol{\sigma^0},\boldsymbol{\tau}-\boldsymbol{\omega})\\&-\partial_\Theta\,\mathbb{J}(\boldsymbol{\tau}-\boldsymbol{\omega},\boldsymbol{\sigma^0})+\partial_\Theta\,\mathbf{K}(\boldsymbol{\tau}-\boldsymbol{\omega},\boldsymbol{\tau})+\partial_\Theta\,\mathbf{K}(\boldsymbol{\omega},\boldsymbol{\tau}-\boldsymbol{\omega})+\partial_\Theta\,\mathbf{K}(\boldsymbol{\sigma^0},\boldsymbol{\tau}-\boldsymbol{\omega})\end{aligned}$$

[9]Constant R takes part only in the proof, and can be chosen arbitrarily large.

$$+\,\partial_\Theta \mathbf{K}\big(\boldsymbol{\tau}-\boldsymbol{\omega},\boldsymbol{\sigma}^{\mathbf{0}}\big).$$

Therefore, using Lemmas 3.17 and 3.23, we get that there exists $C>0$ such that for all $0<r'<r<r_1$, for each $\boldsymbol{\tau},\boldsymbol{\omega}$ in $\Big\{\boldsymbol{\sigma}\in\mathbf{X}^6_r\Big|\,|||\boldsymbol{\sigma}|||_{\mathbf{X}^6_r}<R\Big\}$,

$$\begin{aligned}|||F(\boldsymbol{\tau})-F(\boldsymbol{\omega})|||_{\mathbf{X}^6_{r'}} &\leqslant C\,\gamma_0\Big(1+|||\boldsymbol{\tau}|||_{\mathbf{X}^6_r}+|||\boldsymbol{\omega}|||_{\mathbf{X}^6_r}+|||\boldsymbol{\sigma}^0|||_{\mathbf{X}^6_r}\Big)\frac{|||\boldsymbol{\tau}-\boldsymbol{\omega}|||_{\mathbf{X}^6_r}}{r-r'}\\ &\leqslant \frac{C\,\gamma_0\,\big(1+R+|||\mathbf{a}|||_{\mathbf{X}^2_r}+M_1\big)}{r-r'}\,|||\boldsymbol{\tau}-\boldsymbol{\omega}|||_{\mathbf{X}^6_r}.\end{aligned}$$

This estimate asserts that both the continuity property of F and assumption (3.5) with $C:=C\,\gamma_0\,\big(1+R+|||\mathbf{a}|||_{\mathbf{X}^2_r}+M_1\big)$ are satisfied. We can therefore apply Theorem 3.1 which gives the sought result. □

Proof of Theorem 3.20 is now straightforward. Fix two positive constants M_0 and M_1. Proposition 3.22 asserts the existence of $\rho_1\in(0,1)$ such that for all $\mathbf{H}$ in $\mathbf{E}^2_1$ satisfying $|||\mathbf{H}|||_{\mathbf{E}^2_1}<M_0$, there exists a solution $\mathbf{a}$ in $\mathbf{E}^2_{\rho_1}$ to system (3.20)-(3.21). Then, Lemma 3.16 ensures that there exists r_1 (depending only on ρ_1), such that the solution $\mathbf{a}$ to (3.20)-(3.21) is in $\mathbf{X}_{r_1}$. Proposition 3.24 gives the existence of $\delta>0$ such that for every $\mathbf{g}$ in $\mathbf{X}^6_{r_1}$ satisfying $|||\mathbf{g}|||_{\mathbf{X}^6_{r_1}}<M_1$, there exists a solution $\boldsymbol{\sigma}$ in $\mathcal{C}^1\big((-\delta(r_1-r),\delta(r_1-r)),\mathbf{X}^6_r\big)$, for each $r\in(0,r_1)$, to system (3.17)-(3.19). This is precisely the statement of Theorem 3.20.

To make the simplified model (3.17), (3.19), (3.20) and (3.21) more complicated and closer to the general system (2.47), (2.48), (2.49), (2.50) and (2.51), several aspects could be incorporated in the former one. Outgoing equations associated with boundary frequencies φ, ψ and ν could be integrated in interior equations (3.17). It raises mainly an issue of functional framework, as we solved incoming equations (3.17) as propagation equations in the normal variable, which is not a framework suited for outgoing equations. Then it would be possible to incorporate traces of outgoing profiles in boundary conditions (3.19) and boundary evolution equations (3.20). For that we would need trace estimates for the chosen functional framework. In a more distant perspective, we could integrate profiles associated with boundary frequencies different from φ, ψ and ν in interior equations (3.17) and boundary equations (3.20), which would require a total change of the functional framework, since we would have to work with almost-periodic functions. We could also consider derivatives of order higher than one in source terms of theses equations (3.17) and (3.20)

4. Instability

This section is devoted to the proof of instability. More precisely, the aim is to show that the perturbation H in (1.1) interferes at a leading order in the asymptotic expansion (2.1). This is not the case in general, where the perturbation $\varepsilon^3\,h^\varepsilon$ only interferes at order ε^2 and higher, see [MA88]. As the perturbation $\varepsilon^3\,h^\varepsilon$ of $\varepsilon^2\,g^\varepsilon$ in (1.1) is small, we will work with the linearized system of system (2.47), (2.48), (2.49), (2.50) and (2.51), around the particular solution when the perturbation is zero. To simplify even more the computations we will prove instability on simplified models of the linearized system. The first part of the section focuses on deriving the linearized system for the profiles.

4.1. Linearization around a particular solution.

If the perturbation H is uniformly zero in (1.1), then we are brought back to the case of [CW17], and the solution obtained in the mentioned work is thus a solution to our cascade of equations (2.47), (2.48), (2.49), (2.50) and (2.51) in this particular case. Therefore, according to [CW17], we have the following result.

Proposition 4.1 ([CW17, Theorem 1.10]). *Let $T_0>0$, and consider G in $\mathcal{C}^\infty\big((-\infty,T_0],H^\infty(\mathbb{R}^{d-1}\times\mathbb{T})\big)$, zero for negative times t, and $H\equiv 0$. Then there exists $T\in(0,T_0]$ and unique sequences of functions $\big(\overline{U}^*_n\big)_{n\geqslant 0}$, and $\big(\overline{\sigma}^n_{\zeta,j,\lambda}\big)_{n\geqslant 0}$ for $\zeta=\varphi,\psi,\nu$ and $j=1,2,3$ in*

$\mathcal{C}^\infty\big((-\infty,T_0],H^\infty(\mathbb{R}^{d-1}\times\mathbb{R}_+\times\mathbb{T})\big)$ *and sequences* $\big(\overline{a}^n_{\zeta,\lambda}\big)_{n\geqslant 1}$ *for* $\zeta=\varphi,\psi$ *in* $\mathcal{C}^\infty\big((-\infty,T_0],H^\infty(\mathbb{R}^{d-1}\times\mathbb{T})\big)$*, solution of the cascade of equations* (2.47)*,* (2.48)*,* (2.49)*,* (2.50) *and* (2.51)*.*

Note that Theorem 3.20 constitute a version of Proposition 4.1 in the case where H is possibly nonzero, but only on a simplified model of system (2.47), (2.48), (2.49), (2.50) and (2.51), and with a different functional framework. Note also that since H is zero, we have, for $n\geqslant 1$, $\lambda\in\mathbb{Z}^*$, for the solution of Proposition 4.1,

$$\overline{\sigma}^n_{\zeta,j,\lambda}=0 \quad \text{for } \zeta\neq\varphi \text{ and } j\in\mathcal{C}(\zeta), \qquad \text{and} \qquad \overline{a}^n_{\psi,\lambda}=0. \tag{4.1}$$

The aim of this part is to derive the linearization of system (2.47), (2.48), (2.49), (2.50) and (2.51) around the particular solution of Proposition 4.1. Schematically, in order to study the general problem of the form $\mathcal{F}(u)=(G,H,0,\dots)$, we linearize this problem around the particular solution $\overline{u}$ of $\mathcal{F}(\overline{u})=(G,0,0,\dots)$ to obtain the linearized problem $d\mathcal{F}(\overline{u})\cdot u=(0,H,0,\dots)$. We will also simplify the linearized system during its derivation, since for some profiles it is easy to show that they are zero.

We only detail the linearized equations for the order we are interested in, which are first and second orders, and only for profiles of interest, that is, $\sigma^n_{\psi,1,\lambda}$, $\sigma^n_{\psi,2,\lambda}$, $\sigma^n_{\nu,2,\lambda}$ and $a^n_{\psi,\lambda}$, for $\lambda\in\mathbb{Z}^*$ and $n=1,2$. Here, opposite to the formulation of (2.49), we write down each equation separately, as they are now different since $\overline{\sigma}^n_{\zeta,j,\lambda}$ is zero for $\zeta\neq\varphi$. We also adopt a new color code for these equations.

For the leading profile, starting from equations (2.9), we get, for the phases ψ_1, ψ_2 and ν_2, and for $\lambda\in\mathbb{Z}^*$,

$$X_{\psi,1}\,\sigma^1_{\psi,1,\lambda}+\mathbb{1}_{\lambda=k\lambda_\psi}\,J^{\varphi,1}_{\nu,2}\,ik\,\overline{\sigma}^1_{\varphi,1,-k\lambda_\varphi}\,\sigma^1_{\nu,2,-k}=0, \tag{4.2a}$$

$$X_{\psi,2}\,\sigma^1_{\psi,2,\lambda}+\mathbb{1}_{\lambda=k\lambda_\psi}\,J^{\varphi,3}_{\nu,2}\,ik\,\overline{\sigma}^1_{\varphi,3,-k\lambda_\varphi}\,\sigma^1_{\nu,2,-k}=0, \tag{4.2b}$$

$$X_{\nu,2}\,\sigma^1_{\nu,2,\lambda}+J^{\varphi,1}_{\psi,1}\,i\lambda\,\overline{\sigma}^1_{\varphi,1,-\lambda\lambda_\varphi}\,\sigma^1_{\psi,1,-\lambda\lambda_\psi}+J^{\varphi,3}_{\psi,2}\,i\lambda\,\overline{\sigma}^1_{\varphi,3,-\lambda\lambda_\varphi}\,\sigma^1_{\psi,2,-\lambda\lambda_\psi}=0. \tag{4.2c}$$

In equations (4.2a) and (4.2b), if $\lambda\notin\lambda_\psi\mathbb{Z}$, no resonance happens, but if $\lambda=k\lambda_\psi$ for some $k\in\mathbb{Z}^*$, then, for example for the phase ψ_1, the resonance $k\lambda_\varphi\,\varphi_1+k\lambda_\psi\,\psi_1+k\,\nu_2=0$ occurs. This explains the presence of factors $\mathbb{1}_{\lambda=k\lambda_\psi}$ in equations (4.2a) and (4.2b). We also have, for $j=1,3$ and $\lambda\in\mathbb{Z}^*$, the transport equation

$$X_{\varphi,j}\,\sigma^1_{\varphi,j,\lambda}+D_{\varphi,j}\sum_{\lambda_1+\lambda_2=\lambda}i\lambda_2\,\overline{\sigma}^1_{\varphi,j,\lambda_1}\,\sigma^1_{\varphi,j,\lambda_2}=0. \tag{4.3}$$

As for them, the linearized equations for boundary terms a^1_φ and a^1_ψ read

$$X^{\mathrm{Lop}}_\varphi\,a^1_{\varphi,\lambda}+v_\varphi\sum_{\lambda_1+\lambda_2=\lambda}i\lambda_2\,\overline{a}^1_{\varphi,\lambda_1}\,a^1_{\varphi,\lambda_2}+\lambda\sum_{\lambda_1+\lambda_2=\lambda}\gamma_\varphi(\lambda_1,\lambda_2)\,\big(\overline{a}^1_{\varphi,\lambda_1}\,a^1_{\varphi,\lambda_2}+a^1_{\varphi,\lambda_1}\,\overline{a}^1_{\varphi,\lambda_2}\big)=0, \tag{4.4}$$

and

$$X^{\mathrm{Lop}}_\psi\,a^1_{\psi,\lambda}+\mathbb{1}_{\lambda=k\lambda_\psi}\,\Gamma^\psi\,ik\,\big(\sigma^1_{\nu,2,k}\big)_{|x_d=0}\,\overline{a}^1_{\varphi,-\lambda_\varphi k}=i\lambda\,b_\psi\cdot B\,\big(U^2_{\psi,2,\lambda}\big)_{|x_d,\chi_d=0}. \tag{4.5}$$

There is no term in G in equation (4.4) since we linearized around the solution given by Proposition 4.1 corresponding to the source term $H=0$ and we study the influence of a small source term H on the leading amplitudes σ^1.

Equations for the boundary phase φ are decoupled from the others, they can therefore be solved, using for example [CW17, Theorem 1.10]. From (4.4), along with the initial condition $\big(a^1_{\varphi,\lambda}\big)_{|t\leqslant 0}=0$, we obtain $a^1_{\varphi,\lambda}=0$ for $\lambda\in\mathbb{Z}^*$. Using boundary condition (2.16) as well as initial condition $\big(\sigma^1_{\varphi,j,\lambda}\big)_{|t\leqslant 0}=0$, we also get $\sigma^1_{\varphi,j,\lambda}=0$ for $j=1,3$ and $\lambda\in\mathbb{Z}^*$. Summing up, we have

$$\sigma^1_{\varphi,\lambda,j}=0, \quad a^1_{\varphi,\lambda}=0, \quad \forall\lambda\in\mathbb{Z}^*,\forall j=1,3. \tag{4.6}$$

The other frequencies ψ_1, ψ_2 and ν_2 are totally coupled through equations (4.2) and (4.5), and we need to determine the function a^1_ψ on the boundary and thus the outgoing amplitude $\sigma^2_{\psi,2,\lambda}$.

In the same way as for the leading profile, for the first corrector, starting from (2.31), we get, for the phases ψ_1, ψ_2 and ν_2,

$$X_{\psi,1}\,\sigma^2_{\psi,1,\lambda} + \mathbb{1}_{\lambda=k\lambda_\psi}\,J^{\varphi,1}_{\nu,2}\,ik\left(\overline{\sigma}^1_{\varphi,1,-k\lambda_\varphi}\,\sigma^2_{\nu,2,-k} + \overline{\sigma}^2_{\varphi,1,-k\lambda_\varphi}\,\sigma^1_{\nu,2,-k}\right) = \textit{terms in}\left(U_1,(I-P)\,U_2\right), \tag{4.7a}$$

$$X_{\psi,2}\,\sigma^2_{\psi,2,\lambda} + \mathbb{1}_{\lambda=k\lambda_\psi}J^{\varphi,3}_{\nu,2}\,ik\left(\overline{\sigma}^1_{\varphi,3,-k\lambda_\varphi}\,\sigma^2_{\nu,2,-k} + \overline{\sigma}^2_{\varphi,3,-k\lambda_\varphi}\,\sigma^1_{\nu,2,-k}\right) = \textit{terms in}\left(U_1,(I-P)\,U_2\right), \tag{4.7b}$$

$$\begin{aligned} &X_{\nu,2}\,\sigma^2_{\nu,2,\lambda} + J^{\varphi,1}_{\psi,1}\,i\lambda\left(\overline{\sigma}^1_{\varphi,1,-\lambda\lambda_\varphi}\,\sigma^2_{\psi,1,-\lambda\lambda_\psi} + \overline{\sigma}^2_{\varphi,1,-\lambda\lambda_\varphi}\,\sigma^1_{\psi,1,-\lambda\lambda_\psi}\right)\\ &\quad + J^{\varphi,3}_{\psi,2}\,i\lambda\left(\overline{\sigma}^1_{\varphi,3,-\lambda\lambda_\varphi}\,\sigma^2_{\psi,2,-\lambda\lambda_\psi} + \overline{\sigma}^2_{\varphi,3,-\lambda\lambda_\varphi}\,\sigma^1_{\psi,2,-\lambda\lambda_\psi}\right) = \textit{terms in}\left(U_1,(I-P)\,U_2\right), \end{aligned} \tag{4.7c}$$

where *terms in* $\left(U_1,(I-P)\,U_2\right)$ refer to quadratic terms in U_1 or the nonpolarized parts of U_2, both of them for frequencies ζ_j, with $\zeta=\varphi,\psi,\nu$ and $j=1,3$, terms which therefore will be zero if the corresponding profiles are zero. Equations on other profiles $\sigma^2_{\zeta,j,\lambda}$, for $\zeta\in\mathcal{F}_b\setminus\{\psi,\nu\}$ and $(\zeta,j)=(\psi,3),(\nu,1),(\nu,3)$, are not of interest so we do not write them.

For the boundary term a^2_ψ, we have, according to (2.43),

$$\begin{aligned} &X^{\mathrm{Lop}}_\psi\,a^2_{\psi,\lambda} + \widetilde{X}^{\mathrm{Lop}}_\psi\left(\sigma^2_{\psi,2,\lambda}\right)_{|x_d=0}\\ &\quad + \mathbb{1}_{\lambda=k\lambda_\psi}\,\Gamma^\psi\,ik\left\{\left(\sigma^1_{\nu,2,k}\right)_{|x_d=0}\overline{a}^2_{\varphi,-\lambda_\varphi k} + \left(\sigma^2_{\nu,2,k}\right)_{|x_d=0}\overline{a}^1_{\varphi,-\lambda_\varphi k}\right\}\\ &= i\lambda\,b_\psi\cdot B\left(U^{3,\mathrm{osc}}_{\psi,2,\lambda}\right)_{|x_d,\chi_d=0} - i\lambda\,b_\psi\cdot H_\lambda\\ &\quad + \partial_{z,\theta}\,\textit{terms in}\left(U_1,(I-P)\,U_2,(P\,U_2)_{\zeta\neq\varphi,\psi,\nu},U^*_2\right)_{|x_d,\chi_d=0}. \end{aligned} \tag{4.8}$$

Again, here, equations for ψ_1, ψ_2 and ν_2 are coupled. As the coupling is difficult to handle, especially with the term $i\lambda\,b_\psi\cdot B\left(U^{3,\mathrm{osc}}_{\psi,2,\lambda}\right)_{|x_d,\chi_d=0}$, we will simplify equations (4.7) and (4.8) to reduce the coupling, in order to study instability.

We have obtained the system (4.2), (4.5), (4.7) and (4.8), which is the linearization of system of equations (2.49) and (2.51), around the particular solution of (2.47), (2.48), (2.49), (2.50) and (2.51) of Proposition 4.1 for which the boundary term H is zero.

4.2. **Instability on simplified models.** The aim of this section is to show that the system (1.1) considered in this article is unstable, namely that a small perturbation H in the boundary term may interfere up to the leading order. More precisely we prove that there exists a boundary term H such that, for simplified models of the linearized system (4.2), (4.5), (4.7) and (4.8), the leading perturbations $\sigma^1_{\psi,j,\lambda}$ and $\sigma^1_{\nu,j,\lambda}$ are not all zero. For this purpose, we argue by contradiction and assume that for every boundary term H, all amplitudes $\sigma^1_{\psi,j,\lambda}$ and $\sigma^1_{\nu,j,\lambda}$, for $j=1,3$ and $\lambda\in\mathbb{Z}^*$ are zero. Then we seek for a contradiction. In particular, according to (2.18), it implies that $a^1_{\psi,\lambda}=0$ for all $\lambda\in\mathbb{Z}^*$.

Recall that we have shown above that for the linearized system, profiles $\sigma^1_{\varphi,j,\lambda}$ for $j=1,3$ and $\lambda\in\mathbb{Z}^*$ are zero. Therefore, all leading profiles of frequencies ζ_j, $\zeta=\varphi,\psi,\nu$ and $j=1,3$ are zero. Furthermore, according to formula (2.21) giving the nonpolarized parts of the first corrector, the nonpolarized parts of U_2 for frequencies ζ_j, $\zeta=\varphi,\psi,\nu$ and $j=1,3$ are consequently also zero. We can also show in a similar manner, that the mean value U^*_2 and the polarized parts of frequencies different from ζ_j, $\zeta=\varphi,\psi,\nu$ and $j=1,3$, are also zero.

Therefore, equation (4.5) now reads

$$\left(\sigma^2_{\psi,2,\lambda}\right)_{|x_d=0}=0, \tag{4.9}$$

since $U^2_{\psi,2,\lambda}$ is polarized, $a^1_{\psi,\lambda}$ is zero for $\lambda\in\mathbb{Z}^*$ and the scalar $b_\psi\cdot B\,r_{\psi,2}$ is nonzero. Equation (4.9) is the condition which we wish to contradict.

The general linearized equations (4.2), (4.5), (4.7) and (4.8) being too difficult to handle at this stage of comprehension, two simplified models are investigated.

4.2.1. *First simplified model.* We focus first on a very simple simplified model, for which computations can be easily followed through the end, and reflect the general idea of the instability mechanism. In equations (4.7), most of the resonant terms, which couple the equations, are removed. We also use that both leading profiles and nonpolarized parts of U_2 of frequencies ζ_j for $\zeta = \varphi, \psi, \nu$ and $j = 1, 3$ are zero. We retain at the end, for phases ψ_1, ψ_2 and ν_2, the incoming evolution equation

$$X_{\psi,1}\, \sigma^2_{\psi,1,\lambda} = 0, \tag{4.10a}$$

and the two outgoing evolution equations with resonance terms

$$X_{\psi,2}\, \sigma^2_{\psi,2,\lambda} + \mathbb{1}_{\lambda = k\lambda_\psi}\, J^{\varphi,3}_{\nu,2}\, ik\, \overline{\sigma}^1_{\varphi,3,-k\lambda_\varphi}\, \sigma^2_{\nu,2,-k} = 0, \tag{4.10b}$$

$$X_{\nu,2}\, \sigma^2_{\nu,2,\lambda} + J^{\varphi,1}_{\psi,1}\, i\lambda\, \overline{\sigma}^1_{\varphi,1,-\lambda\lambda_\varphi}\, \sigma^2_{\psi,1,-\lambda\lambda_\psi} = 0. \tag{4.10c}$$

As for the boundary amplitudes $a^2_{\psi,\lambda}$ for $\lambda \in \mathbb{Z}^*$, we remove all traces of first or second profile, and, as usual, $i\lambda\, b_\psi \cdot B\, (U^{3,\mathrm{osc}}_{\psi,2,\lambda})_{|x_d, \chi_d = 0}$, and we use that terms of $\partial_{z,\theta}$ *terms in* $\big(U_1, (I - P)\, U_2, (P\, U_2)_{\zeta \neq \varphi,\psi,\nu}, U_2^*\big)_{|x_d,\chi_d=0}$ are zero to retain the simple forced transport equation

$$X^{\mathrm{Lop}}_\psi\, a^2_{\psi,\lambda} = -i\lambda\, b_\psi \cdot H_\lambda. \tag{4.11}$$

According to above remarks, boundary condition (2.33) for the incoming amplitude $\sigma^2_{\psi,1,\lambda}$ now reads

$$\big(\sigma^2_{\psi,1,\lambda}\big)_{|x_d=0}\, r_{\psi,1} = a^2_{\psi,\lambda}\, e_{\psi,1}. \tag{4.12}$$

Although system (4.10), (4.11) and (4.12) is coupled, it is in an upper triangular form, so it can be solved using explicit formulas, since we are in presence of transport equations with constant coefficients. This is made precise now, with the proof of the following result.

Theorem 4.2. *There exists a boundary term H in $L^2\big((-\infty, T]_t \times \mathbb{R}^{d-1}_y \times \mathbb{T}_{\theta_2}\big)$ such that, if the sequence $(\sigma^2_{\psi,1,\lambda}, \sigma^2_{\psi,2,\lambda}, \sigma^2_{\nu,2,\lambda})_{\lambda\in\mathbb{Z}^*}$ of tuples of $\mathcal{C}\big(\mathbb{R}^+_{x_d}, L^2\big((-\infty, T]_t \times \mathbb{R}^{d-1}_y\big)\big)$ and the sequence $(a^2_{\psi,\lambda})_{\lambda\in\mathbb{Z}^*}$ of $L^2\big((-\infty, T]_t \times \mathbb{R}^{d-1}_y\big)$ are solutions to the system (4.10), (4.11) and (4.12), then the trace $\big(\sigma^2_{\psi,2,\lambda_\psi}\big)_{|x_d=0}$ is nonzero.*

Proof. We consider any boundary term H in $L^2\big((-\infty, T]_t \times \mathbb{R}^{d-1}_y \times \mathbb{T}_{\theta_2}\big)$, and we look for an expression of the trace $\big(\sigma^2_{\psi,2,\lambda}\big)_{|x_d=0}$ of the associated solution of (4.10), (4.11) and (4.12).

First of all, the transport equation (4.11) on the boundary $\{x_d = 0\}$ can be solved to find

$$a^2_{\psi,\lambda}(t, y) = \int_0^t -i\lambda\, b_\psi \cdot H_\lambda\big(s, y - \mathbf{v}^{\mathrm{Lop}}_\psi\, (t - s)\big)\, ds,$$

recalling the notation[10]

$$X^{\mathrm{Lop}}_\psi = \partial_t + \mathbf{v}^{\mathrm{Lop}}_\psi \cdot \nabla_y.$$

According to boundary condition (4.12), it follows, with notation[11]

$$X_{\psi,1} = \partial_t - \mathbf{v}_{\psi,1} \cdot \nabla_x =: \partial_t - \mathbf{v}'_{\psi,1} \cdot \nabla_y + \partial_{x_d},$$

using the incoming transport equation (4.10a),

$$\sigma^2_{\psi,1,\lambda}(t, y, x_d) = -\mathbb{1}_{x_d \leqslant t} \int_0^{t - x_d} i\lambda\, p_{\psi,1}\, b_\psi \cdot H_\lambda\Big(s, y + x_d\, \big(\mathbf{v}^{\mathrm{Lop}}_\psi + \mathbf{v}'_{\psi,1}\big) - \mathbf{v}^{\mathrm{Lop}}_\psi\, (t - s)\Big)\, ds,$$

with a coefficient $p_{\psi,1} \in \mathbb{R}$ such that $e_{\psi,1} = p_{\psi,1}\, r_{\psi,1}$. To simplify notation, the coefficient $p_{\psi,1}$ will be omitted in the following. Thus, with notation[12]

$$X_{\nu,2} = \partial_t - \mathbf{v}_{\nu,2} \cdot \nabla_x =: \partial_t - \mathbf{v}'_{\nu,2} \cdot \nabla_y - \partial_{x_d},$$

[10] Without lost of generality, we have set $\beta_\psi = 1$ in Lemma 1.11, to simplify the equations.

[11] We assumed here that, with notation of Definition 1.3, $-1/\partial_\xi \tau_k(\eta, \xi)$ is equal to 1.

[12] See footnote 11.

according to the outgoing transport equation (4.10c), we have

$$\begin{aligned}\sigma^2_{\nu,2,\lambda}(t,y,x_d) = &-\int_0^t J^{\varphi,1}_{\psi,1}\, i\lambda\, \overline{\sigma}^1_{\varphi,1,-\lambda\lambda_\varphi}(s, x+\mathbf{v}_{\nu,2}(t-s))\, \mathbb{1}_{x_d\leqslant 2s-t}\\ &\times\int_0^{2s-x_d-t} i\lambda\lambda_\psi\, b_\psi\cdot H_{-\lambda\lambda_\psi}\Big(\tau, y+\mathbf{v}'_{\nu,2}\,(t-s)\\ &+(x_d+t-s)\left(\mathbf{v}^{\mathrm{Lop}}_\psi+\mathbf{v}'_{\psi,1}\right)-\mathbf{v}^{\mathrm{Lop}}_\psi\,(s-\tau)\Big)\,d\tau\,ds.\end{aligned}$$

In the same way, with similar notation, the outgoing transport equation (4.10b) leads to

$$\begin{aligned}\sigma^2_{\psi,2,\lambda}(t,y,x_d) = &\int_0^t \mathbb{1}_{\lambda=k\lambda_\psi}\, J^{\varphi,3}_{\nu,2}\, ik\,\overline{\sigma}^1_{\varphi,3,-k\lambda_\varphi}\big(s, x+\mathbf{v}_{\psi,2}(t-s)\big)\\ &\times\int_0^s J^{\varphi,1}_{\psi,1}\, ik\,\overline{\sigma}^1_{\varphi,1,k\lambda_\varphi}\big(\tau, x+\mathbf{v}_{\psi,2}(t-s)+\mathbf{v}_{\nu,2}(s-\tau)\big)\,\mathbb{1}_{x_d\leqslant 2\tau-t}\\ &\times\int_0^{2\tau-x_d-t} ik\lambda_\psi\, b_\psi\cdot H_{k\lambda_\psi}\Big(\sigma, y+\mathbf{v}'_{\nu,2}\,(s-\tau)+\mathbf{v}'_{\psi,2}\,(t-s)\\ &+(x_d+t-\tau)\left(\mathbf{v}^{\mathrm{Lop}}_\psi+\mathbf{v}'_{\psi,1}\right)-\mathbf{v}^{\mathrm{Lop}}_\psi\,(\tau-\sigma)\Big)\,d\sigma\,d\tau\,ds.\end{aligned}$$

The trace of $\sigma^2_{\psi,2,\lambda}$ on the boundary $\{x_d=0\}$ is therefore given by

$$\begin{aligned}\sigma^2_{\psi,2,\lambda}(t,y,0) = &-i\int_0^t\int_0^s\int_0^{2\tau-t} \mathbb{1}_{\lambda=k\lambda_\psi}\, J^{\varphi,3}_{\nu,2}\, J^{\varphi,1}_{\psi,1}\, k^3\,\lambda_\psi\,\overline{\sigma}^1_{\varphi,3,-k\lambda_\varphi}\big(s, y+\mathbf{v}_{\psi,2}(t-s)\big)\\ &\times\overline{\sigma}^1_{\varphi,1,k\lambda_\varphi}\big(\tau, y+\mathbf{v}_{\psi,2}(t-s)+\mathbf{v}_{\nu,2}(s-\tau)\big)\\ &\times b_\psi\cdot H_{k\lambda_\psi}\Big(\sigma, y+\mathbf{v}'_{\nu,2}\,(s-\tau)+\mathbf{v}'_{\psi,2}(t-s)\\ &+(t-\tau)\left(\mathbf{v}^{\mathrm{Lop}}_\psi+\mathbf{v}'_{\psi,1}\right)-\mathbf{v}^{\mathrm{Lop}}_\psi\,(\tau-\sigma)\Big)\,d\sigma\,d\tau\,ds.\end{aligned}\tag{4.13}$$

We justify now why there is a choice of a boundary term H such that this trace is nonzero. We take interest into the trace $\left(\sigma^2_{\psi,2,\lambda_\psi}\right)_{|x_d=0}$, which is given by formula (4.13) with $\lambda=\lambda_\psi$ and therefore $k=1$, namely,

$$\begin{aligned}\sigma^2_{\psi,2,\lambda_\psi}(t,y,0) = &-i\int_0^t\int_0^s\int_0^{2\tau-t} J^{\varphi,3}_{\nu,2}\, J^{\varphi,1}_{\psi,1}\,\lambda_\psi\,\overline{\sigma}^1_{\varphi,3,-\lambda_\varphi}\big(s, y+\mathbf{v}_{\psi,2}(t-s)\big)\\ &\times\overline{\sigma}^1_{\varphi,1,\lambda_\varphi}\big(\tau, y+\mathbf{v}_{\psi,2}(t-s)+\mathbf{v}_{\nu,2}(s-\tau)\big)\\ &\times b_\psi\cdot H_{\lambda_\psi}\Big(\sigma, y+\mathbf{v}'_{\nu,2}\,(s-\tau)+\mathbf{v}'_{\psi,2}(t-s)\\ &+(t-\tau)\left(\mathbf{v}^{\mathrm{Lop}}_\psi+\mathbf{v}'_{\psi,1}\right)-\mathbf{v}^{\mathrm{Lop}}_\psi\,(\tau-\sigma)\Big)\,d\sigma\,d\tau\,ds.\end{aligned}\tag{4.14}$$

We start by constructing $\overline{\sigma}^1_{\varphi,1,\lambda_\varphi}$ and $\overline{\sigma}^1_{\varphi,3,-\lambda_\varphi}$ suited for our purpose. It is proven in [CW17, section 2.2] that $\overline{a}^1_{\varphi,\lambda}$, solution to equation (2.51a) in the particular case where H is zero, is the solution to the following equation, for $\lambda\in\mathbb{Z}^*$

$$X^{\mathrm{Lop}}_\varphi\,\overline{a}^1_{\varphi,\lambda}+D^{\mathrm{Lop}}_\varphi\sum_{\lambda_1+\lambda_2=\lambda} i\lambda_2\,\overline{a}^1_{\varphi,\lambda_1}\,\overline{a}^1_{\varphi,\lambda_2}+i\lambda\sum_{\lambda_1+\lambda_3=\lambda}\gamma_\varphi(\lambda_1,\lambda_3)\,\overline{a}^1_{\varphi,\lambda_1}\,\overline{a}^1_{\varphi,\lambda_3}=-i\lambda\, b_\varphi\cdot G_\lambda.\tag{4.15}$$

We set $G_\lambda=0$ for $\lambda\in\mathbb{Z}\setminus\{\lambda_\varphi,-\lambda_\varphi\}$ and G_{λ_φ}, $G_{-\lambda_\varphi}$ real, non-negative, and equal to one on the set $[1/2,2]_t\times[-(h+2)\mathbf{V},(h+2)\mathbf{V}]^{d-1}_y$, where we have denoted $\mathbf{V}:=\left|\mathbf{v}^{\mathrm{Lop}}_\varphi\right|$ and with $h\geqslant 1$. Solving the transport equation (4.15), we get $\overline{a}^1_{\varphi,\lambda}=0$ for $\lambda\in\mathbb{Z}\setminus\{\lambda_\varphi,-\lambda_\varphi\}$ and $\overline{a}^1_{\varphi,\lambda_\varphi}$, $\overline{a}^1_{\varphi,-\lambda_\varphi}$ real, non-negative, and greater than $1/2$ on the set $[1,2]_t\times[-(h+1)\mathbf{V},(h+1)\mathbf{V}]^{d-1}_y$. Now we know that, according to the condition (4.1) on profiles $\overline{\sigma}^1_{\zeta,j,\lambda}$ for $\zeta=\psi,\nu$, $j=1,2,3$ and $\lambda\in\mathbb{Z}^*$, there are no resonance terms in the evolution equation (2.9a) for $\overline{\sigma}^1_{\varphi,j,\lambda}$ for $j=1,3$ and $\lambda\in\mathbb{Z}^*$,

so theses profiles $\overline{\sigma}^1_{\varphi,j,\lambda}$ for $j=1,3$ and $\lambda\in\mathbb{Z}^*$ satisfy the following incoming transport equation

$$X_{\varphi,j}\,\overline{\sigma}^1_{\varphi,j,\lambda}+D_{\varphi,j}\sum_{\lambda_1+\lambda_2=\lambda} i\lambda_2\,\overline{\sigma}^1_{\varphi,j,\lambda_1}\,\overline{\sigma}^1_{\varphi,j,\lambda_2}=0, \tag{4.16a}$$

with the following boundary condition (2.16)

$$\big(\overline{\sigma}^1_{\varphi,j,\lambda}\big)_{|x_d=0}\,r_{\varphi,j}=\overline{a}^1_{\varphi,\lambda}\,e_{\varphi,j}. \tag{4.16b}$$

Solving system (4.16) seen as a transport propagation equation in the normal direction, with notation[13]

$$X_{\varphi,j}=:\partial_t-\mathbf{v}'_{\varphi,j}\cdot\nabla_y+\partial_{x_d},$$

we get $\overline{\sigma}^1_{\varphi,j,\lambda}=0$ for $j=1,3$ and $\lambda\in\mathbb{Z}\setminus\{\lambda_\varphi,-\lambda_\varphi\}$, and that $\overline{\sigma}^1_{\varphi,1,\lambda_\varphi}$, $\overline{\sigma}^1_{\varphi,3,-\lambda_\varphi}$ are real, non-negative, and larger than $A/2$ on the set $[1+\mathbf{V}/(2\mathbf{w}),2+\mathbf{V}/(2\mathbf{W})]_t\times[-h\mathbf{V},h\mathbf{V}]_y\times[0,\mathbf{V}/\mathbf{W}]_{x_d}$, where $A:=\min\big(|e_{\varphi,1}\cdot r_{\varphi,1}|,|e_{\varphi,3}\cdot r_{\varphi,3}|\big)$, $\mathbf{w}:=\min\big(|\mathbf{v}'_{\varphi,1}|,|\mathbf{v}'_{\varphi,3}|\big)$ and $\mathbf{W}:=\max\big(|\mathbf{v}'_{\varphi,1}|,|\mathbf{v}'_{\varphi,3}|\big)$.

Now that $\overline{\sigma}^1_{\varphi,1,\lambda_\varphi}$ and $\overline{\sigma}^1_{\varphi,3,-\lambda_\varphi}$ have been constructed appropriately, we make precise a suitable choice of boundary term H. We denote $\mathbf{t}:=1+\mathbf{V}/(2\mathbf{w})$ and $\mathbf{T}:=2+\mathbf{V}/(2\mathbf{W})$, and we set the integer h such that

$$h\geqslant\frac{8(\mathbf{T}-\mathbf{t})}{\mathbf{V}}\max\Big(\big|\mathbf{v}'_{\psi,2}\big|,\big|\mathbf{v}'_{\nu,2}\big|,\big|\mathbf{v}'_{\psi,1}\big|,\big|\mathbf{v}^{\mathrm{Lop}}_{\psi}\big|\Big).$$

We also take the boundary term H such that $b_\psi\cdot H_{\lambda_\psi}$ is pure imaginary, and such that its imaginary part is of the sign of $J^{\varphi,3}_{\nu,2}\,J^{\varphi,1}_{\psi,1}\,\lambda_\psi$ and of modulus one on $[\mathbf{t},\mathbf{T}]_t\times[-h\mathbf{V},h\mathbf{V}]_y$, namely,

$$b_\psi\cdot H_{\lambda_\psi}=i\,\mathrm{sign}\big(J^{\varphi,3}_{\nu,2}\,J^{\varphi,1}_{\psi,1}\,\lambda_\psi\big)\quad\text{on}\quad[\mathbf{t},\mathbf{T}]_t\times[-h\mathbf{V},h\mathbf{V}]_y.$$

Then we note that, according to (4.14), the trace $\big(\sigma^2_{\psi,2,\lambda_\psi}\big)_{|x_d=0}$ satisfies, for $t\in[\mathbf{t},\mathbf{T}]$ and $y\in[-h\mathbf{V}/2,h\mathbf{V}/2]$,

$$\begin{aligned}\sigma^2_{\psi,2,\lambda_\psi}(t,y,0)\geqslant -i\int_{\mathbf{t}}^{t}\int_{\mathbf{t}}^{s}\int_{\mathbf{t}}^{2\tau-t} & J^{\varphi,3}_{\nu,2}\,J^{\varphi,1}_{\psi,1}\,\lambda_\psi\,\overline{\sigma}^1_{\varphi,3,-\lambda_\varphi}\big(s,y+\mathbf{v}_{\psi,2}(t-s)\big)\\
&\times\overline{\sigma}^1_{\varphi,1,\lambda_\varphi}\big(\tau,y+\mathbf{v}_{\psi,2}(t-s)+\mathbf{v}_{\nu,2}(s-\tau)\big)\\
&\times b_\psi\cdot H_{\lambda_\psi}\Big(\sigma,y+\mathbf{v}'_{\nu,2}\,(s-\tau)+\mathbf{v}'_{\psi,2}(t-s)\\
&\quad+(t-\tau)\,\big(\mathbf{v}^{\mathrm{Lop}}_{\psi}+\mathbf{v}'_{\psi,1}\big)-\mathbf{v}^{\mathrm{Lop}}_{\psi}\,(\tau-\sigma)\Big)\,d\sigma\,d\tau\,ds\\
\geqslant\big|J^{\varphi,3}_{\nu,2}\,J^{\varphi,1}_{\psi,1}\,\lambda_\psi\big|&\frac{A^2}{4},\end{aligned}$$

since, for $\mathbf{t}\leqslant\sigma\leqslant 2\tau-t$, $\mathbf{t}\leqslant\tau\leqslant s\leqslant t\leqslant\mathbf{T}$ and $y\in[-h\mathbf{V}/2,h\mathbf{V}/2]$, according to the assumption on h, we have $\sigma,\tau,s\in[\mathbf{t},\mathbf{T}]$ and

$$\begin{aligned}y+\mathbf{v}_{\psi,2}(t-s)&\in[-h\mathbf{V},h\mathbf{V}]_y\times[0,\mathbf{V}/\mathbf{W}]_{x_d},\\
y+\mathbf{v}_{\psi,2}(t-s)+\mathbf{v}_{\nu,2}(s-\tau)&\in[-h\mathbf{V},h\mathbf{V}]_y\times[0,\mathbf{V}/\mathbf{W}]_{x_d},\\
y+\mathbf{v}'_{\nu,2}\,(s-\tau)+\mathbf{v}'_{\psi,2}(t-s)&\\
+(t-\tau)\,\big(\mathbf{v}^{\mathrm{Lop}}_{\psi}+\mathbf{v}'_{\psi,1}\big)-\mathbf{v}^{\mathrm{Lop}}_{\psi}\,(\tau-\sigma)&\in[-h\mathbf{V},h\mathbf{V}]_y,\end{aligned}$$

because $\mathbf{T}-\mathbf{t}\leqslant\mathbf{V}/2\mathbf{W}$, up to shrinking $\mathbf{w}$.

It concludes the proof of Theorem 4.2 since we proved that there exists a choice of H such that the trace $\big(\sigma^2_{\psi,2,\lambda_\psi}\big)_{|x_d=0}$ is nonzero. □

Theorem 4.2, stating that the trace $\big(\sigma^2_{\psi,2,\lambda_\psi}\big)_{|x_d=0}$ is nonzero, contradicts the condition (4.9), so instability is proven. Indeed, we have assumed that for all boundary terms H, the associated amplitudes $\sigma^1_{\psi,j,\lambda}$ and $\sigma^1_{\nu,j,\lambda}$ for $j=1,3$ and $\lambda\in\mathbb{Z}^*$ are zero, and, for the simplified model equations (4.10), (4.11) and (4.12), we found a contradiction. Therefore, for this simplified model, there exists a boundary term H such that the leading profiles $\sigma^1_{\psi,j,\lambda}$ and $\sigma^1_{\nu,j,\lambda}$ for $j=1,3$

[13] See footnote 11.

and $\lambda \in \mathbb{Z}^*$ are not all zero. It proves that the boundary term H may interfere at the leading order, which constitutes an instability.

4.2.2. *Second simplified model.* The second simplified model that we shall consider features additional resonance coupling terms which add difficulties. This time we keep the formulation of simplified models of section 3 and multiply equations (4.7) and (4.8) by $e^{i\lambda\Theta}$, to obtain, for the phases ψ_1, ψ_2 and ν_2, the incoming evolution equation with a resonance term

$$X_{\psi,1}\,\sigma^2_{\psi,1} + \partial_\Theta\,\mathbb{J}^{\varphi,1}_{\nu,2}\big(\overline{\sigma}^1_{\varphi,1}, \sigma^2_{\nu,2}\big) = 0, \tag{4.17a}$$

and the two outgoing evolution equations with resonance terms

$$X_{\psi,2}\,\sigma^2_{\psi,2} + \partial_\Theta\,\mathbb{J}^{\varphi,3}_{\nu,2}\big(\overline{\sigma}^1_{\varphi,3}, \sigma^2_{\nu,2}\big) = 0, \tag{4.17b}$$

$$X_{\nu,2}\,\sigma^2_{\nu,2} + \partial_\Theta\,\mathbb{J}^{\varphi,1}_{\psi,1}\big(\overline{\sigma}^1_{\varphi,1}, \sigma^2_{\psi,1}\big) + \partial_\Theta\,\mathbb{J}^{\varphi,3}_{\psi,2}\big(\overline{\sigma}^1_{\varphi,3}, \sigma^2_{\psi,2}\big) = 0, \tag{4.17c}$$

where operators $\mathbb{J}^{\zeta_1,j_1}_{\zeta_2,j_2}$ have been defined in (3.18), and equations on the boundary are kept as in the first simplified model (4.11) and (4.12), namely,

$$X^{\mathrm{Lop}}_\psi\, a^2_\psi = -b_\psi \cdot \partial_\Theta\, H, \tag{4.18}$$

$$\big(\sigma^2_{\psi,1}\big)_{|x_d=0}\, r_{\psi,1} = a^2_\psi\, e_{\psi,1}. \tag{4.19}$$

Note that this time, the simplified model features all resonance terms of the general equations.

The obtained system is no longer triangular since additional resonance terms in (4.17) relatively to (4.10) couple each equation with the others, so we cannot solve it as a sequence of transport equations as before. We use a perturbation method and solve equations (4.17) with a fixed point theorem. We start by solving (4.18) as a transport equation, and then we deduce, using the incoming transport equations (4.17a) and boundary condition (4.19), an expression of $\sigma^2_{\psi,1}$ depending on $\sigma^2_{\nu,2}$. We use this expression in (4.17c) to obtain an equation in $\sigma^2_{\nu,2}$ with a source term depending only on $\sigma^2_{\psi,2}$. This equation is solved with a fixed point method, using that the source term depending on $\sigma^2_{\nu,2}$ is "small", in a convenient topology, comparing to the transport term, and we get an expression of $\sigma^2_{\nu,2}$ depending on $\sigma^2_{\psi,2}$. This expression is finally used in (4.17b) which is solved with the same fixed point method. The result is the following.

Theorem 4.3. *There exists a boundary term H in $\mathcal{C}\big([0,T]_t, H^\infty(\mathbb{R}^{d-1}_y \times \mathbb{T}_{\theta_2})\big)$ such that, if $\sigma^2_{\psi,1}, \sigma^2_{\psi,2}, \sigma^2_{\nu,2}$ in $\mathcal{C}^1\big([0,T]_t, H^\infty(\mathbb{R}^{d-1}_y \times \mathbb{R}^+_{x_d} \times \mathbb{T}_\Theta)\big)$ and a^2_ψ in $\mathcal{C}^1\big([0,T]_t, H^\infty(\mathbb{R}^{d-1}_y \times \mathbb{T}_\Theta)\big)$ are solutions to the system (4.17), (4.18) and (4.19), then the trace $\big(\sigma^2_{\psi,2}\big)_{|x_d=0}$ is nonzero.*

Proof. Similarly as for the first simplified model, from equation (4.18), reusing previous notation, we get

$$a^2_\psi(t,y) = -\int_0^t b_\psi \cdot \partial_\Theta\, H_\lambda\big(s, y - \mathbf{v}^{\mathrm{Lop}}_\psi\,(t-s)\big)\,ds.$$

Then system (4.17a), (4.19) seen as a transport equation with a source term depending on $\sigma^2_{\nu,2}$, leads to

$$\begin{aligned}\sigma^2_{\psi,1}(t,y,x_d) = -\mathbb{1}_{x_d \leqslant t} \int_0^{t-x_d} b_\psi \cdot \partial_\Theta\, H\Big(s, y + x_d\,\big(\mathbf{v}^{\mathrm{Lop}}_\psi + \mathbf{v}'_{\psi,1}\big) - \mathbf{v}^{\mathrm{Lop}}_\psi\,(t-s)\Big)\,ds \\ + \int_{\max(0,t-x_d)}^t \partial_\Theta\,\mathbb{J}^{\varphi,1}_{\nu,2}\big(\overline{\sigma}^1_{\varphi,1}, \sigma^2_{\nu,2}\big)\big(s, x + \mathbf{v}_{\psi,1}\,(t-s)\big)\,ds.\end{aligned}$$

Therefore, equation (4.17c) now reads

$$X_{\nu,2}\,\sigma^2_{\nu,2} + \partial_\Theta\,\mathbb{J}^{\varphi,3}_{\psi,2}\big(\overline{\sigma}^1_{\varphi,3}, \sigma^2_{\psi,2}\big) \tag{4.20a}$$

$$- \partial_\Theta\,\mathbb{J}^{\varphi,1}_{\psi,1}\left[\overline{\sigma}^1_{\varphi,1}, \mathbb{1}_{x_d \leqslant t} \int_0^{t-x_d} b_\psi \cdot \partial_\Theta\, H\Big(s, y + x_d\,\big(\mathbf{v}^{\mathrm{Lop}}_\psi + \mathbf{v}'_{\psi,1}\big) - \mathbf{v}^{\mathrm{Lop}}_\psi\,(t-s)\Big)\,ds\right] \tag{4.20b}$$

$$+ \partial_\Theta\,\mathbb{J}^{\varphi,1}_{\psi,1}\left[\overline{\sigma}^1_{\varphi,1}, \int_{\max(0,t-x_d)}^t \partial_\Theta\,\mathbb{J}^{\varphi,1}_{\nu,2}\big(\overline{\sigma}^1_{\varphi,1}, \sigma^2_{\nu,2}\big)\big(s, x + \mathbf{v}_{\psi,1}\,(t-s)\big)\,ds\right] = 0. \tag{4.20c}$$

This is a transport equation but with a perturbation term (4.20c) depending on the unknowns $\sigma^2_{\nu,2,\lambda}$. It is solved using the following result. For $s \in [0,+\infty)$, we denote by H^s the Sobolev space $H^s(\mathbb{R}^{d-1}_y \times \mathbb{R}^+_{x_d} \times \mathbb{T}_\Theta)$ of regularity s, and $H^\infty := \bigcap_{s\geqslant 0} H^s$.

Lemma 4.4. *There exists $T > 0$ such that, for any function f in $\mathcal{C}([0,T], H^\infty)$, the transport equation*

$$X_{\nu,2}\,\sigma^2_{\nu,2} + \partial_\Theta\, \mathbb{J}^{\varphi,1}_{\psi,1}\Big[\overline{\sigma}^1_{\varphi,1}, \int_{\max(0,t-x_d)}^t \partial_\Theta\, \mathbb{J}^{\varphi,1}_{\nu,2}\big(\overline{\sigma}^1_{\varphi,1}, \sigma^2_{\nu,2}\big)\big(s, x + \mathbf{v}_{\psi,1}\,(t-s)\big)\, ds\Big] = f(t,x), \tag{4.21a}$$

$$\big(\sigma^2_{\nu,2}\big)_{|t=0} = 0, \tag{4.21b}$$

admits a unique solution $\sigma^2_{\nu,2}$ in $\mathcal{C}^1([0,T], H^\infty)$.

If, for f in $\mathcal{C}([0,T], H^\infty)$ we denote by Ψf the solution $\sigma^2_{\nu,2}$ of (4.21) in $\mathcal{C}^1([0,T], H^\infty)$, then, for any $s \geqslant 0$, there exists $C_s > 0$ such that for f in $\mathcal{C}([0,T], H^s)$, we have

$$\|\Psi f\|_{\mathcal{C}([0,T_s],H^s)} \leqslant C_s\, T\, \|f\|_{\mathcal{C}([0,T_s],H^s)}\,. \tag{4.22}$$

Before proving Lemma 4.4, we prove the following preliminary result, asserting that the operators $u \mapsto \partial_\Theta\, \mathbb{J}^{\varphi,j}_{\zeta,k}\big[\overline{\sigma}^1_{\varphi,j}, u\big]$ for $(j,\zeta,k) \in \{(1,\psi,1),(1,\nu,2),(3,\nu,2),(3,\psi,2)\}$ are bounded from $\mathcal{C}([0,T], H^s)$ to itself, for $s \in [0,+\infty)$.

Lemma 4.5. *For $s \in [0,+\infty)$ and $\overline{\sigma}^1_{\varphi,1} \in \mathcal{C}([0,T], H^\infty)$, there exists $C_s > 0$ such that for u in $\mathcal{C}([0,T], H^s)$, functions $\partial_\Theta\, \mathbb{J}^{\varphi,j}_{\zeta,k}\big[\overline{\sigma}^1_{\varphi,j}, u\big]$ for $(j,\zeta,k) \in \{(1,\psi,1),(1,\nu,2),(3,\nu,2),(3,\psi,2)\}$ belong to $\mathcal{C}([0,T], H^s)$ and satisfy*

$$\Big\|\partial_\Theta\, \mathbb{J}^{\varphi,j}_{\zeta,k}\big[\overline{\sigma}^1_{\varphi,1}, u\big]\Big\|_{\mathcal{C}([0,T],H^s)} \leqslant C_s\, \|u\|_{\mathcal{C}([0,T],H^s)}\,. \tag{4.23}$$

Proof. We make the proof for the operator $u \mapsto \partial_\Theta\, \mathbb{J}^{\varphi,1}_{\psi,1}\big[\overline{\sigma}^1_{\varphi,1}, u\big]$, namely, $(j,\zeta,k) = (1,\psi,1)$. According to the expression (3.18) of the operator $\mathbb{J}^{\varphi,1}_{\psi,1}$, we want to estimate, for $t \in [0,T]$, the following quantity:

$$\Big\|\partial_\Theta\, \mathbb{J}^{\varphi,1}_{\psi,1}\big[\overline{\sigma}^1_{\varphi,1}, u\big](t)\Big\|_{H^s} = \Big\|(x,\Theta) \mapsto J^{\zeta_2,j_2}_{\zeta_1,j_1} \sum_{\lambda\in\mathbb{Z}^*} \lambda\, \overline{\sigma}^1_{\varphi,1,\lambda}(t,x)\, u_\lambda(t,x)\, e^{i\lambda\Theta}\Big\|_{H^s}\,.$$

Using the same proof as the one of estimate (3.29), we get

$$\begin{aligned}
&\Big\|\partial_\Theta\, \mathbb{J}^{\varphi,1}_{\psi,1}\big[\overline{\sigma}^1_{\varphi,1}, u\big](t)\Big\|_{H^s} \\
&\quad\leqslant \Big\|(x,\Theta) \mapsto J^{\zeta_2,j_2}_{\zeta_1,j_1} \sum_{\lambda\in\mathbb{Z}^*} \lambda\, \overline{\sigma}^1_{\varphi,1,\lambda}(t,x)\, e^{i\lambda\Theta}\Big\|_{H^s} \Big\|(x,\Theta) \mapsto \sum_{\lambda\in\mathbb{Z}^*} u_\lambda(t,x)\, e^{i\lambda\Theta}\Big\|_{H^s} \\
&\quad\leqslant C\, \big\|\overline{\sigma}^1_{\varphi,1}(t)\big\|_{H^{s+1}}\, \|u(t)\|_{H^s}\,,
\end{aligned}$$

which, taking the supremum in $t \in [0,T]$, leads to the sought estimate (4.23), since $\overline{\sigma}^1_{\varphi,1}$ is in $\mathcal{C}([0,T], H^{s+1})$ according to Proposition 4.1. This is sufficient since we do not seek here for a tame estimate. □

Proof (Lemma 4.4). From now on we fix an integer $s \geqslant 0$, and the aim is to use the Banach fixed point theorem in the Banach space $\mathcal{C}([0,T], H^s)$. For v in $\mathcal{C}([0,T], H^s)$, we denote by $\Phi\, v$ the solution in $\mathcal{C}([0,T], H^s)$ of

$$\begin{cases} X_{\nu,2}\, u + v = 0, \\ u_{|t=0} = 0, \end{cases}$$

which is therefore given by

$$\Phi\, v(t,x,\Theta) = -\int_0^t v\big(s, x + \mathbf{v}_{\nu,2}\,(t-s), \Theta\big)\, ds.$$

Note that Φ is continuous from $\mathcal{C}([0,T],H^s)$ to itself, and satisfies

$$\|\Phi\, v\|_{\mathcal{C}([0,T],H^s)} \leqslant T\, \|v\|_{\mathcal{C}([0,T],H^s)}\,. \tag{4.24}$$

Now, if $\sigma^2_{\nu,2}$ is a solution to (4.21), it is in this notation a fixed point of the map $\mathbf{F}$, where, for w in $\mathcal{C}([0,T],H^s)$, $\mathbf{F}(w)$ is defined by

$$\mathbf{F}(w) : (t,x,\Theta) \mapsto \Phi\, \partial_\Theta\, \mathbb{J}^{\varphi,1}_{\psi,1}\Big[\overline{\sigma}^1_{\varphi,1}, \int_{\max(0,t-x_d)}^t \partial_\Theta\, \mathbb{J}^{\varphi,1}_{\nu,2}\big(\overline{\sigma}^1_{\varphi,1}, w\big)\big(s, x+\mathbf{v}_{\psi,1}\,(t-s),\Theta\big)\, ds\Big](t,x,\Theta) \\ -\Phi\, f(t,x,\Theta).$$

We derive now an estimate on the difference $\mathbf{F}(w)-\mathbf{F}(w')$ for w,w' in $\mathcal{C}([0,T],H^s)$. By linearity of the operators, we have

$$\mathbf{F}(w)-\mathbf{F}(w') = \Phi\, \partial_\Theta\, \mathbb{J}^{\varphi,1}_{\psi,1}\Big[\overline{\sigma}^1_{\varphi,1}, \int_{\max(0,t-x_d)}^t \partial_\Theta\, \mathbb{J}^{\varphi,1}_{\nu,2}\big(\overline{\sigma}^1_{\varphi,1}, w-w'\big)\big(s, x+\mathbf{v}_{\psi,1}\,(t-s),\Theta\big)\, ds\Big].$$

Therefore, according to estimates (4.24) and (4.23), we have

$$\begin{aligned}
&\big\|\mathbf{F}(w)-\mathbf{F}(w')\big\|_{\mathcal{C}([0,T],H^s)} \\
&\quad\leqslant C_s\, T\, \Big\|\int_{\max(0,t-x_d)}^t \partial_\Theta\, \mathbb{J}^{\varphi,1}_{\nu,2}\big(\overline{\sigma}^1_{\varphi,1}, w-w'\big)\big(s, x+\mathbf{v}_{\psi,1}\,(t-s),\Theta\big)\, ds\Big\|_{\mathcal{C}([0,T],H^s)} \\
&\quad\leqslant C_s\, T^2\, \Big\|\partial_\Theta\, \mathbb{J}^{\varphi,1}_{\nu,2}\big(\overline{\sigma}^1_{\varphi,1}, w-w'\big)\Big\|_{\mathcal{C}([0,T],H^s)} \\
&\quad\leqslant C_s^2\, T^2\, \big\|w-w'\big\|_{\mathcal{C}([0,T],H^s)}\,.
\end{aligned}$$

Therefore, for $T>0$ small enough, $\mathbf{F}$ is a contraction of $\mathcal{C}([0,T],H^s)$, and the Banach fixed point theorem therefore gives a unique solution Ψf in $\mathcal{C}([0,T],H^s)$. By linearity of system (4.21), the solution Ψf may be extended to any time interval, so the time of existence T does not depend on the regularity $s\in[0,+\infty)$. Finally, using equation (4.21a), we obtain

$$\begin{aligned}
\partial_t\, \Psi f(t,x,\Theta) &= f(t,x,\Theta) + \mathbf{v}_{\nu,2}\cdot\nabla_x\, \Psi f(t,x,\Theta) \\
&\quad - \partial_\Theta\, \mathbb{J}^{\varphi,1}_{\psi,1}\Big[\overline{\sigma}^1_{\varphi,1}, \int_{\max(0,t-x_d)}^t \partial_\Theta\, \mathbb{J}^{\varphi,1}_{\nu,2}\big(\overline{\sigma}^1_{\varphi,1}, \Psi f\big)\big(s, x+\mathbf{v}_{\psi,1}\,(t-s),\Theta\big)\, ds\Big](t,x,\Theta)
\end{aligned} \tag{4.25}$$

so $\partial_t \Psi f$ belongs to $\mathcal{C}([0,T],H^\infty)$ and therefore Ψf is actually in $\mathcal{C}^1([0,T],H^\infty)$. We have proven the first part of Lemma 4.4.

The interest is now made on the boundedness of Ψ. We have, since $\Psi f = \mathbf{F}(\Psi f)$,

$$\Psi f(t,x,\Theta) = \Phi\, \partial_\Theta\, \mathbb{J}^{\varphi,1}_{\psi,1}\Big[\overline{\sigma}^1_{\varphi,1}, \int_{\max(0,t-x_d)}^t \partial_\Theta\, \mathbb{J}^{\varphi,1}_{\nu,2}\big(\overline{\sigma}^1_{\varphi,1}, \Psi f\big)\big(s, x+\mathbf{v}_{\psi,1}\,(t-s),\Theta\big)\, ds\Big](t,x,\Theta) \\ -\Phi\, f(t,x,\Theta),$$

and therefore, using estimates (4.24) and (4.23), we have, for $s\in[0,+\infty)$,

$$\begin{aligned}
\|\Psi f\|_{\mathcal{C}([0,T],H^s)} &\leqslant C_s\, T\, \Big\|\int_{\max(0,t-x_d)}^t \partial_\Theta\, \mathbb{J}^{\varphi,1}_{\nu,2}\big(\overline{\sigma}^1_{\varphi,1}, \Psi f\big)\big(s, x+\mathbf{v}_{\psi,1}\,(t-s),\Theta\big)\, ds\Big\|_{\mathcal{C}([0,T],H^s)} \\
&\quad + T\, \|f\|_{\mathcal{C}([0,T],H^s)} \\
&\leqslant C_s^2\, T^2\, \|\Psi f\|_{\mathcal{C}([0,T],H^s)} + T\, \|f\|_{\mathcal{C}([0,T],H^s)}\,.
\end{aligned}$$

Thus, for T small enough (depending on $s\in[0,+\infty)$), we have

$$\|\Psi f\|_{\mathcal{C}([0,T],H^s)} \leqslant C\, T\, \|f\|_{\mathcal{C}([0,T],H^s)}\,. \tag{4.26}$$

Once again, by linearity of system (4.21), the estimate (4.26) is propagated to the whole interval $[0,T]$, which concludes the proof. □

Returning to (4.20) and using Lemma 4.4, by linearity, the solution $\sigma^2_{\nu,2}$ of equation (4.20) reads

(4.27)
$$\sigma^2_{\nu,2} = \Psi\,\partial_\Theta\,\mathbb{J}^{\varphi,1}_{\psi,1}\left(\overline{\sigma}^1_{\varphi,1}, \mathbb{1}_{x_d\leqslant t}\int_0^{t-x_d} b_\psi\cdot\partial_\Theta\, H\Big(s, y + x_d\left(\mathbf{v}^{\mathrm{Lop}}_\psi + \mathbf{v}'_{\psi,1}\right) - \mathbf{v}^{\mathrm{Lop}}_\psi\,(t-s), \Theta\Big)\,ds\right) - \Psi\,\partial_\Theta\,\mathbb{J}^{\varphi,3}_{\psi,2}\big(\overline{\sigma}^1_{\varphi,3}, \sigma^2_{\psi,2}\big).$$

We proceed now with equation (4.17b) which now reads, according to the expression (4.27) of $\sigma^2_{\nu,2}$,

(4.28)
$$\begin{aligned} &X_{\psi,2}\,\sigma^2_{\psi,2} - \partial_\Theta\,\mathbb{J}^{\varphi,3}_{\nu,2}\Big[\overline{\sigma}^1_{\varphi,3}, \Psi\,\partial_\Theta\,\mathbb{J}^{\varphi,3}_{\psi,2}\big(\overline{\sigma}^1_{\varphi,3}, \sigma^2_{\psi,2}\big)\Big] \\ &= -\partial_\Theta\,\mathbb{J}^{\varphi,3}_{\nu,2}\Big[\overline{\sigma}^1_{\varphi,3}, \Psi\,\partial_\Theta\,\mathbb{J}^{\varphi,1}_{\psi,1}\Big(\overline{\sigma}^1_{\varphi,1}, \\ &\qquad \mathbb{1}_{x_d\leqslant t}\int_0^{t-x_d} b_\psi\cdot\partial_\Theta\, H\Big(s, y + x_d\left(\mathbf{v}^{\mathrm{Lop}}_\psi + \mathbf{v}'_{\psi,1}\right) - \mathbf{v}^{\mathrm{Lop}}_\psi\,(t-s)\Big)\,ds\Big)\Big]. \end{aligned}$$

This equation is solved using the same method as the one of Lemma 4.4. For v in $\mathcal{C}([0,T], H^\infty)$, we still denote by $\Phi\,v$ the solution in $\mathcal{C}([0,T], H^\infty)$ of

$$\begin{cases} X_{\psi,2}\,u + v = 0, \\ u_{|t=0} = 0, \end{cases}$$

and we recall that it satisfies, for $s\in[0,+\infty)$,

$$\|\Phi\,v\|_{\mathcal{C}([0,T],H^s)} \leqslant T\,\|v\|_{\mathcal{C}([0,T],H^s)}. \tag{4.29}$$

Now, $\sigma^2_{\psi,2}$ is a solution to (4.28) if and only if it is a fixed point of the map $\mathbf{F}$ of $\mathcal{C}([0,T], H^\infty)$ given by

$$\begin{aligned} \mathbf{F} : w \mapsto &-\Phi\,\partial_\Theta\,\mathbb{J}^{\varphi,3}_{\nu,2}\Big[\overline{\sigma}^1_{\varphi,3}, \Psi\,\partial_\Theta\,\mathbb{J}^{\varphi,3}_{\psi,2}\big(\overline{\sigma}^1_{\varphi,3}, w\big)\Big] + \Phi\,\partial_\Theta\,\mathbb{J}^{\varphi,3}_{\nu,2}\Big[\overline{\sigma}^1_{\varphi,3}, \Psi\,\partial_\Theta\,\mathbb{J}^{\varphi,1}_{\psi,1}\Big(\overline{\sigma}^1_{\varphi,1}, \\ &\mathbb{1}_{x_d\leqslant t}\int_0^{t-x_d} b_\psi\cdot\partial_\Theta\, H\Big(s, y + x_d\left(\mathbf{v}^{\mathrm{Lop}}_\psi + \mathbf{v}'_{\psi,1}\right) - \mathbf{v}^{\mathrm{Lop}}_\psi\,(t-s), \Theta\Big)\,ds\Big)\Big]. \end{aligned}$$

For w, w' in $\mathcal{C}([0,T], H^\infty)$, the difference $\mathbf{F}(w) - \mathbf{F}(w')$ is given by

$$\mathbf{F}(w) - \mathbf{F}(w') = \Phi\,\partial_\Theta\,\mathbb{J}^{\varphi,3}_{\nu,2}\Big[\overline{\sigma}^1_{\varphi,3}, \Psi\,\partial_\Theta\,\mathbb{J}^{\varphi,3}_{\psi,2}\big(\overline{\sigma}^1_{\varphi,3}, w' - w\big)\Big].$$

Therefore, for $s\in[0,+\infty)$, according to estimates (4.23) and (4.22), we have

$$\begin{aligned} \big\|\mathbf{F}(w) - \mathbf{F}(w')\big\|_{\mathcal{C}([0,T],H^s)} &\leqslant C_s\,T\,\Big\|\Psi\,\partial_\Theta\,\mathbb{J}^{\varphi,3}_{\psi,2}\big(\overline{\sigma}^1_{\varphi,3}, w' - w\big)\Big\|_{\mathcal{C}([0,T],H^s)} \\ &\leqslant C_s\,T^2\,\big\|w - w'\big\|_{\mathcal{C}([0,T],H^s)}, \end{aligned}$$

so, for $T>0$ small enough, $\mathbf{F}$ is a contraction of $\mathcal{C}([0,T], H^s)$. The Banach fixed point theorem therefore gives a unique solution $\sigma^2_{\psi,2}$ to (4.28) in $\mathcal{C}([0,T], H^s)$ for $s\in[0,+\infty)$, that reads,

$$\begin{aligned} \sigma^2_{\psi,2}(t,x) = &\,\Phi\,\mathbb{J}^{\varphi,3}_{\nu,2}\Big[\overline{\sigma}^1_{\varphi,3}, \Psi\,\partial_\Theta\,\mathbb{J}^{\varphi,3}_{\psi,2}\big(\overline{\sigma}^1_{\varphi,3}, \sigma^2_{\psi,2}\big)\Big] \\ &- \Phi\,\mathbb{J}^{\varphi,3}_{\nu,2}\Big[\overline{\sigma}^1_{\varphi,3}, \Psi\,\partial_\Theta\,\mathbb{J}^{\varphi,1}_{\psi,1}\Big(\overline{\sigma}^1_{\varphi,1}, \\ &\qquad \mathbb{1}_{x_d\leqslant t}\int_0^{t-x_d} b_\psi\cdot\partial_\Theta\, H\Big(s, y + x_d\left(\mathbf{v}^{\mathrm{Lop}}_\psi + \mathbf{v}'_{\psi,1}\right) - \mathbf{v}^{\mathrm{Lop}}_\psi\,(t-s), \Theta\Big)\,ds\Big)\Big]. \end{aligned}$$

Therefore, according to the expression of Φ and taking the trace in $x_d = 0$, we have

$$\big(\sigma^2_{\psi,2}\big)_{|x_d=0}(t,y) - \Phi\,\mathbb{J}^{\varphi,3}_{\nu,2}\Big[\overline{\sigma}^1_{\varphi,3}, \Psi\,\partial_\Theta\,\mathbb{J}^{\varphi,3}_{\psi,2}\big(\overline{\sigma}^1_{\varphi,3}, \sigma^2_{\psi,2}\big)\Big]_{|x_d=0}(t,y) \tag{4.30}$$

$$= \int_0^t \partial_\Theta \, \mathbb{J}^{\varphi,3}_{\nu,2}\Big[\overline{\sigma}^1_{\varphi,3}, \Psi\, \partial_\Theta \, \mathbb{J}^{\varphi,1}_{\psi,1}\Big(\overline{\sigma}^1_{\varphi,1}, \int_0^t b_\psi \cdot \partial_\Theta \, H\Big(s, y - \mathbf{v}^{\mathrm{Lop}}_\psi\,(t-s)\Big)\, ds\Big)\Big]$$
$$\big(s, y + \mathbf{v}_{\psi,2}\,(t-s)\big)\, ds.$$

Using similar arguments as the one of Theorem 4.2, we can construct profiles $\overline{\sigma}^1_{\varphi,1}$ and $\overline{\sigma}^1_{\varphi,3}$ and choose a boundary term H such that the right-hand side of equation (4.30) is nonzero. Therefore, we obtain

$$\mathbf{C} := \Big\| \big(\sigma^2_{\psi,2}\big)_{|x_d=0} - \Phi\, \mathbb{J}^{\varphi,3}_{\nu,2}\Big[\overline{\sigma}^1_{\varphi,3}, \Psi\, \partial_\Theta\, \mathbb{J}^{\varphi,3}_{\psi,2}\big(\overline{\sigma}^1_{\varphi,3}, \sigma^2_{\psi,2}\big)\Big]_{|x_d=0} \Big\|_{\mathcal{C}([0,T],L^2(\mathbb{R}^{d-1}\times\mathbb{T}))} > 0. \tag{4.31}$$

Now note that, using the exact same arguments as the one used to prove estimates (4.22), (4.23) and (4.29), we can prove that theses estimates (4.22), (4.23) and (4.29) still hold for the traces, in $\mathcal{C}([0,T], L^2(\mathbb{R}^{d-1}\times\mathbb{T}))$. Therefore, we get

$$\begin{aligned}
\mathbf{C} &\leqslant \Big\| \big(\sigma^2_{\psi,2}\big)_{|x_d=0} \Big\|_{\mathcal{C}([0,T],L^2)} + \Big\| \Phi\, \mathbb{J}^{\varphi,3}_{\nu,2}\Big[\overline{\sigma}^1_{\varphi,3}, \Psi\, \partial_\Theta\, \mathbb{J}^{\varphi,3}_{\psi,2}\big(\overline{\sigma}^1_{\varphi,3}, \sigma^2_{\psi,2}\big)\Big]_{|x_d=0} \Big\|_{\mathcal{C}([0,T],L^2)} \\
&\leqslant \Big\| \big(\sigma^2_{\psi,2}\big)_{|x_d=0} \Big\|_{\mathcal{C}([0,T],L^2)} + C\,T \Big\| \Psi\, \partial_\Theta\, \mathbb{J}^{\varphi,3}_{\psi,2}\big(\overline{\sigma}^1_{\varphi,3}, \sigma^2_{\psi,2}\big)_{|x_d=0} \Big\|_{\mathcal{C}([0,T],L^2)} \\
&\leqslant \Big\| \big(\sigma^2_{\psi,2}\big)_{|x_d=0} \Big\|_{\mathcal{C}([0,T],L^2)} + C\,T^2 \Big\| \big(\sigma^2_{\psi,2}\big)_{|x_d=0} \Big\|_{\mathcal{C}([0,T],L^2)} \\
&= \big(1 + C\,T^2\big) \Big\| \big(\sigma^2_{\psi,2}\big)_{|x_d=0} \Big\|_{\mathcal{C}([0,T],L^2)}.
\end{aligned}$$

In conclusion, we obtain that the norm $\Big\| \big(\sigma^2_{\psi,2}\big)_{|x_d=0} \Big\|_{\mathcal{C}([0,T],L^2(\mathbb{R}^{d-1})\times\mathbb{T}))}$ is positive, so, for T sufficiently small, the trace $\big(\sigma^2_{\psi,2}\big)_{|x_d=0}$ is nonzero, which is the sought result, concluding the proof. □

In the same manner as for the first simplified model, Theorem 4.3 proves that an instability is created for the simplified model (4.17), (4.18) and (4.19).

Once again, as in section 3, it is conceivable to consider a more complex simplified model than (4.17), (4.18) and (4.19), by integrating, in the equations (4.17), coupling terms with profiles $\sigma_{\zeta,j}$ with $\zeta \neq \varphi, \psi, \nu$. What seems to be a further step is to add, in equation (4.18), terms involving the traces of interior profiles. Among these terms, the one that seems to raise the most difficult issue is $i\lambda\, b_\psi \cdot B\, \big(U^{3,\mathrm{osc}}_{\psi,2,\lambda}\big)_{|x_d,\chi_d=0}$, since it couples equation (4.18) on the first corrector with the second corrector U^3.

5. The example of gas dynamics

We study here the example of three dimensional compressible isentropic Euler equations. The aim is to determine whether or not the configuration of frequencies considered in this work can happen for this system. For $\mathcal{C}^1$ solutions, away from vacuum, the equations read

$$\begin{cases}
\partial_t V^\varepsilon + A_1(V^\varepsilon)\, \partial_1 V^\varepsilon + A_2(V^\varepsilon)\, \partial_2 V^\varepsilon + A_3(V^\varepsilon)\, \partial_3 V^\varepsilon = 0 & \text{in } \Omega_T, \\
B\, V^\varepsilon_{|x_3=0} = \varepsilon^2\, g^\varepsilon + \varepsilon^M\, h^\varepsilon & \text{on } \omega_T, \\
V^\varepsilon_{|t=0} = 0,
\end{cases} \tag{5.1}$$

with $V^\varepsilon = (v^\varepsilon, \mathbf{u}^\varepsilon) \in \mathbb{R}^*_+ \times \mathbb{R}^3$, where $v^\varepsilon \in \mathbb{R}^*_+$ represents the fluid volume, and $\mathbf{u}^\varepsilon \in \mathbb{R}^3$ its velocity, and where the functions A_j, $j = 1, 2, 3$ are defined on $\mathbb{R}^*_+ \times \mathbb{R}^3$ as

$$A_j(V) := \begin{pmatrix} \mathbf{u}_j & -v\, {}^t e_j \\ -c(v)^2/v\, e_j & \mathbf{u}_j\, I_3 \end{pmatrix} \in \mathcal{M}_4(\mathbb{R}), \tag{5.2}$$

where e_j is the j-th vector of the canonical basis, with $c(v) > 0$ representing the sound velocity in the fluid, depending on its volume v. We study here a perturbation of this system around the equilibrium $V_0 := (v_0, 0, 0, u_0)$ with $v_0 > 0$ a fixed volume and $(0, 0, u_0)$ an incoming subsonic velocity, that is, such that $0 < u_0 < c(v_0)$. We denote by $c_0 := c(v_0)$ the sound velocity in a fluid of the fixed volume v_0.

In order to study the possibility of existence of a configuration of frequencies satisfying Assumption 5, we need to determine a matrix B satisfying Assumption 3. For which we need to know the dimension of the stable subspace $E_-(\zeta)$, and construct a basis of it.

Although it will not be used in this part, we derive the expression of various quantities related to hyperbolicity of the Euler system. For $(\eta, \xi) \in \mathbb{R}^2 \times \mathbb{R}$, the matrix $A(\eta, \xi) := \eta_1 A_1(V_0) + \eta_2 A_2(V_0) + \xi A_3(V_0)$ associated with the system (5.1) is given by

$$A(\eta, \xi) = \begin{pmatrix} u_0\,\xi & -v_0\,{}^t\eta & -v_0\,\xi \\ -c_0^2/v_0\,\eta & u_0\,\xi\,I_2 & 0 \\ -c_0^2/v_0\,\xi & 0 & u_0\,\xi \end{pmatrix},$$

so the polynomial p defined as $p(\tau, \eta, \xi) := \det\big(\tau\, I_4 + A(\eta, \xi)\big)$ reads

$$p(\tau, \eta, \xi) = (\tau + \xi\, u_0)^2\big((\tau + \xi\, u_0)^2 - c_0^2(|\eta|^2 + \xi^2)\big).$$

Thus the matrix $A(\eta, \xi)$ admits a double eigenvalue $-\tau_2(\eta, \xi)$ and two simple eigenvalues $-\tau_1(\eta, \xi)$ and $-\tau_3(\eta, \xi)$ given by

$$\tau_1(\eta, \xi) = -\xi\, u_0 - c_0\sqrt{|\eta|^2 + \xi^2}, \quad \tau_2(\eta, \xi) = -\xi\, u_0, \quad \tau_3(\eta, \xi) = -\xi\, u_0 + c_0\sqrt{|\eta|^2 + \xi^2}.$$

Note that since $-\tau_2(\eta, \xi)$ is a double eigenvalue, the Euler system is not strictly hyperbolic, but hyperbolic with constant multiplicity. Despite this difference with Assumption 2, we study this system since Assumption 2 seems to be a technical assumption.

Now to determine the expression of the stable subspace $E_-(\zeta)$, we need to study the eigenvalues of the matrix

$$\mathcal{A}(\tau, \eta) := -i\, A_3(V_0)^{-1}\Big(\tau I + \eta_1\, A_1(V_0) + \eta_2\, A_2(V_0)\Big).$$

We determine that in this case, the hyperbolic region[14] is given by

$$\mathcal{H} := \left\{ (\tau, \eta) \;\middle|\; |\tau| > \sqrt{c_0^2 - u_0^2}\,|\eta| \right\}.$$

Then, for (τ, η) in $\mathcal{H}$, the eigenvalues of $\mathcal{A}(\tau, \eta)$ are given by

$$i\,\xi_1(\tau, \eta) := i\,\frac{\tau\, u_0 - \operatorname{sign}(\tau)\, c_0\,\sqrt{\tau^2 - |\eta|^2\,(c_0^2 - u_0^2)}}{c_0^2 - u_0^2}, \tag{5.3a}$$

$$i\,\xi_2(\tau, \eta) := i\,\frac{\tau\, u_0 + \operatorname{sign}(\tau)\, c_0\,\sqrt{\tau^2 - |\eta|^2\,(c_0^2 - u_0^2)}}{c_0^2 - u_0^2}, \tag{5.3b}$$

$$i\,\xi_3(\tau, \eta) := i\,\frac{-\tau}{u_0}, \tag{5.3c}$$

where $\operatorname{sign}(x) := x/|x|$ for $x \neq 0$. The eigenvalue $i\,\xi_3$ is double, when the two others are simple. We determine that, if we denote $\alpha_j(\tau, \eta) := \big(\tau, \eta, \xi_j(\tau, \eta)\big)$, the frequency $\alpha_2(\tau, \eta)$ is outgoing when frequencies $\alpha_1(\tau, \eta)$ and $\alpha_3(\tau, \eta)$ are incoming. Since $i\,\xi_3(\tau, \eta)$ is a double eigenvalue, the dimension p of the stable subspace $E_-(\zeta)$ is therefore equal to 3. This could also have been determined by the number of positive eigenvalues of $A_3(V_0)$.

[14] That is, the region where $\mathcal{A}(\tau, \eta)$ has only pure imaginary eigenvalues and is diagonalizable.

The interest is now made on a basis for $E_-(\zeta)$, which, according to (1.6), can be constructed with eigenvectors of $\mathcal{A}(\zeta)$ associated with incoming frequencies. We determine that the eigenvectors associated with eigenvalues $i\,\xi_1(\zeta)$ and $i\,\xi_3(\zeta)$ are respectively given by

$$\lambda \begin{pmatrix} |(\eta, \xi_2(\tau,\eta))|\, v_0 \\ c_0\,\eta \\ c_0\,\xi_2(\tau,\eta) \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \mathbf{a} \end{pmatrix},$$

with $\lambda \in \mathbb{R}$ and where $\mathbf{a}$ is any vector satisfying $\mathbf{a} \cdot \big(\eta, \xi_3(\tau,\eta)\big) = 0$. For $\mathbf{a}$ we can choose for example the two linearly independent vectors $(\tau\,\eta, |\eta|^2\, u_0)$ and $(c_0\,\eta_2, -c_0\,\eta_1, 0)$ to obtain the following basis of the stable subspace $E_-(\zeta)$:

$$r_1(\zeta) := \begin{pmatrix} |(\eta, \xi_2(\tau,\eta))|\, v_0 \\ c_0\,\eta \\ c_0\,\xi_2(\tau,\eta) \end{pmatrix}, \quad r_3^1(\zeta) := \begin{pmatrix} 0 \\ \tau\,\eta \\ |\eta|^2\, u_0 \end{pmatrix}, \quad r_3^2(\zeta) := \begin{pmatrix} 0 \\ c_0\,\eta_2 \\ -c_0\,\eta_1 \\ 0 \end{pmatrix},$$

We look now for a matrix B, of size 3×4, satisfying the weak Kreiss-Lopatinskii condition 3. More precisely, we want a matrix B such that $\ker B \cap E_-(\zeta)$ is nonzero on the specific frequency $\tau = c_0\,|\eta|$. Note that here we make a restrictive choice, about the locus where $\ker B \cap E_-(\zeta)$ should be nontrivial. This choice is made since it makes the following computations easier. Since every quantity is homogeneous of degree 1, we can make the computations for $|\eta| = 1$. For $\tau = c_0\,|\eta|$ we have $\xi_2(\tau,\eta) = 0$, so, denoting $\eta = (\cos\theta, \sin\theta)$, basis $\big\{r_1(\zeta), r_3^1(\zeta), r_3^2(\zeta)\big\}$ reads

$$r_1(\zeta) = \begin{pmatrix} v_0 \\ c_0\,\cos\theta \\ c_0\,\sin\theta \\ 0 \end{pmatrix}, \quad r_3^1(\zeta) = \begin{pmatrix} 0 \\ c_0\,\cos\theta \\ c_0\,\sin\theta \\ u_0 \end{pmatrix}, \quad r_3^2(\zeta) = \begin{pmatrix} 0 \\ c_0\,\sin\theta \\ -c_0\,\cos\theta \\ 0 \end{pmatrix}.$$

The condition that $\ker B \cap E_-(\zeta)$ is trivial is equivalent to the three vectors $B\,r_1(\zeta), B\,r_3^1(\zeta), B\,r_3^2(\zeta)$ being linearly dependent. To study this condition, we write B in column as

$$B = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \end{pmatrix},$$

and, since B has to be of rank 3, we can assume that column b_4 is a linear combination of the three linearly independent vectors b_1, b_2, b_3 which we chose to be the canonical basis of $\mathbb{R}^3$. We write $b_4 = \mu_1\,b_1 + \mu_2\,b_2 + \mu_3\,b_3$, with $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$. In this notation, the linear dependence of $B\,r_1(\zeta), B\,r_3^1(\zeta), B\,r_3^2(\zeta)$ is equivalent to

$$v_0\,c_0^2 = \mu_1\,u_0\,c_0^2 \qquad \text{and} \qquad \mu_2\,v_0\,c_0\,\cos\theta = \mu_3\,v_0\,c_0\,\sin\theta, \quad \forall\theta \in \mathbb{T},$$

so $\mu_1 = v_0/u_0$ and $\mu_2 = \mu_3 = 0$. Multiplying B by a nonzero constant we obtain

$$B = \begin{pmatrix} u_0 & 0 & 0 & v_0 \\ 0 & u_0 & 0 & 0 \\ 0 & 0 & u_0 & 0 \end{pmatrix},$$

which gives an example of a matrix B for which $\ker B \cap E_-(\zeta)$ is nonzero, and actually of dimension 1, on $\tau = c_0\,|\eta|$.

We investigate now if $\ker B \cap E_-(\zeta)$ is nontrivial only on $\tau = c_0\,|\eta|$. At this purpose we introduce a practical tool, the Lopatinskii determinant (see [BGS07, section 4.2.2]), denoted by $\Delta(\sigma,\eta)$ for $(\sigma,\eta) \in \Xi$. It is a scalar function such that its zeros are exactly the frequencies for

which $\ker B \cap E_-(\zeta)$ is nontrivial. Its construction can be found in [BGS07, section 4.6.1]. If we write $E_-(\sigma,\eta)$ as[15]

$$E_-(\sigma,\eta) = \ell(\sigma,\eta)^\perp,$$

the Lopatinskii determinant is given by the following block determinant:

$$\Delta(\sigma,\eta) = \begin{vmatrix} B \\ \ell(\sigma,\eta) \end{vmatrix}.$$

Calculations made in [BGS07, section 14.3.1] show that, for $(\sigma,\eta) \in \Xi$, we can choose

$$\ell(\sigma,\eta) := \big(a, -v_0\, u_0\, {}^t\eta, v_0\, \sigma\big)$$

with

$$a := u_0\, \sigma - \xi_-\, (c_0^2 - u_0^2),$$

ξ_- being the root of negative real part of the following dispersion relation

$$(\sigma + u_0\, \xi)^2 - c_0^2\, (\xi^2 + |\eta|^2) = 0.$$

Thus the Lopatinskii determinant is given by

$$\Delta(\sigma,\eta) := \begin{vmatrix} u_0 & 0 & 0 & v_0 \\ 0 & v_0 & 0 & 0 \\ 0 & 0 & v_0 & 0 \\ a & -v_0\, u_0\, \eta_1 & -v_0\, u_0\, \eta_2 & v_0\, \sigma \end{vmatrix} = v_0^3 \big[u_0\, \sigma - a\big] = v_0^3\, \xi_-(c_0^2 - u_0^2).$$

It is zero if and only if ξ_- is zero, and this is the case only when σ is real (i.e. for $(\sigma,\eta) = (\tau,\eta) \in \Xi_0$) and $\tau = c_0\, |\eta|$. Therefore $\ker B \cap E_-(\zeta)$ is nontrivial only on $\tau = c_0\, |\eta|$, and thus matrix B satisfies Assumption 3, with

$$\Upsilon := \{(\tau,\eta)\,|\,\tau = c_0\, |\eta|\}\,.$$

Now that we have determined a boundary condition B suited for our problem, we take interest into oscillations. Thus we consider two hyperbolic frequencies φ and ψ on the boundary which will satisfy our assumptions. First, according to Assumption 5, frequencies φ and ψ must be zeros of the Lopatinskii determinant, thus satisfy $\tau = c_0|\eta|$. If we still take $|\eta| = 1$, it leads to consider

$$\varphi := (c_0, \cos\theta_\varphi, \sin\theta_\varphi) \quad \text{and} \quad \psi := (c_0, \cos\theta_\psi, \sin\theta_\psi),$$

with $\theta_\varphi, \theta_\psi \in [0, 2\pi)$. An immediate computation then gives

$$\xi_1(\varphi) = \xi_1(\psi) = 0, \quad \xi_2(\varphi) = \xi_2(\psi) = \frac{2\,M}{1-M^2}, \quad \xi_3(\varphi) = \xi_3(\psi) = -\frac{1}{M},$$

with $M := u_0/c_0 \in (0,1)$ being the Mach number. Therefore, in order to have no resonances between frequencies lifted from φ and no resonances between frequencies lifted from ψ, it is sufficient to assume M^2 irrational.

We now look for a boundary frequency $\nu := -\lambda_\varphi\, \varphi - \lambda_\psi\, \psi$ with $\lambda_\varphi, \lambda_\psi \in \mathbb{Z}^*$, which satisfies Assumption 5. Frequency ν reads

$$\nu = \big(-c_0(\lambda_\varphi + \lambda_\psi), -\lambda_\varphi \cos\theta_\varphi - \lambda_\psi \cos\theta_\psi, -\lambda_\varphi \sin\theta_\varphi - \lambda_\psi \sin\theta_\psi\big).$$

First we determine in which case ν is not in Υ. If we denote $\nu = (\tau,\eta)$, we have

$$\tau^2 = c_0^2(\lambda_\varphi + \lambda_\psi)^2 \quad \text{and} \quad c_0^2\, |\eta|^2 = c_0^2(\lambda_\varphi^2 + \lambda_\psi^2) + 2c_0^2\lambda_\varphi\lambda_\psi \cos(\theta_\varphi - \theta_\psi),$$

so, according to the description of Υ, frequency ν is not in Υ if and only if $\theta_\varphi \neq \theta_\psi$. Generalizing this to any frequency $\zeta = \lambda_1\, \varphi + \lambda_2\, \psi \in \mathcal{F}_b \setminus \{0\}$ asserts that $\mathcal{F}_b \cap \Upsilon = \{\varphi, -\varphi, \psi, -\psi\}$ as required

[15] Notation $^\perp$ refers to the orthogonal complement relatively to the complex scalar product.

by Assumption 5. This assumption also demands ν to be in the hyperbolic region. We have, if we still denote $\nu = (\tau, \eta)$,

$$\tau^2 - |\eta|^2(c_0^2 - u_0^2) = u_0^2(\lambda_\varphi + \lambda_\psi)^2 + 2\lambda_\varphi\lambda_\psi(c_0^2 - u_0^2)\big[1 - \cos(\theta_\varphi - \theta_\psi))\big],$$

so the hyperbolicity condition $\tau^2 - |\eta|^2(c_0^2 - u_0^2) > 0$ reads

$$M^2(\lambda_\varphi + \lambda_\psi)^2 + 2\lambda_\varphi\lambda_\psi(1 - M^2)\big[1 - \cos(\theta_\varphi - \theta_\psi))\big] > 0, \tag{5.4}$$

which is satisfied for example when λ_φ and λ_ψ are positive.

We take interest now in resonance assumptions (1.12a) and (1.12b). We compute

$$\xi_1(\nu) = \frac{-M(\lambda_\varphi + \lambda_\psi)}{1 - M^2} + \frac{\operatorname{sign}(\lambda_\varphi + \lambda_\psi)\sqrt{2\lambda_\varphi\lambda_\psi\big[1 - \cos(\theta_\varphi - \theta_\psi)\big] + M^2\big[\lambda_\varphi^2 + \lambda_\psi^2 + 2\lambda_\varphi\lambda_\psi\cos(\theta_\varphi - \theta_\psi)\big]}}{1 - M^2}$$

$$\xi_2(\nu) = \frac{-M(\lambda_\varphi + \lambda_\psi)}{1 - M^2} - \frac{\operatorname{sign}(\lambda_\varphi + \lambda_\psi)\sqrt{2\lambda_\varphi\lambda_\psi\big[1 - \cos(\theta_\varphi - \theta_\psi)\big] + M^2\big[\lambda_\varphi^2 + \lambda_\psi^2 + 2\lambda_\varphi\lambda_\psi\cos(\theta_\varphi - \theta_\psi)\big]}}{1 - M^2}$$

$$\xi_3(\nu) = \frac{\lambda_\varphi + \lambda_\psi}{M}.$$

Recalling Remark 1.8 about the numbering of frequencies, we need to check the four possibilities for the couple of resonance, namely,

$$\lambda_\varphi\,\varphi_1 + \lambda_\psi\,\psi_1 + \nu_2 = 0 \quad \text{and} \quad \lambda_\varphi\,\varphi_3 + \lambda_\psi\,\psi_2 + \nu_2 = 0, \tag{5.5a}$$

$$\lambda_\varphi\,\varphi_1 + \lambda_\psi\,\psi_3 + \nu_2 = 0 \quad \text{and} \quad \lambda_\varphi\,\varphi_3 + \lambda_\psi\,\psi_2 + \nu_2 = 0, \tag{5.5b}$$

$$\lambda_\varphi\,\varphi_3 + \lambda_\psi\,\psi_1 + \nu_2 = 0 \quad \text{and} \quad \lambda_\varphi\,\varphi_1 + \lambda_\psi\,\psi_2 + \nu_2 = 0, \tag{5.5c}$$

$$\lambda_\varphi\,\varphi_3 + \lambda_\psi\,\psi_1 + \nu_2 = 0 \quad \text{and} \quad \lambda_\varphi\,\varphi_1 + \lambda_\psi\,\psi_2 + \nu_2 = 0. \tag{5.5d}$$

- Since $\xi_1(\varphi) = \xi_1(\psi) = 0$, relation $\lambda_\varphi\,\varphi_1 + \lambda_\psi\,\psi_1 + \nu_2 = 0$ implies that $\xi_2(\nu) = 0$, and therefore $\lambda_\varphi + \lambda_\psi = 0$ which is impossible, since it contradicts condition (5.4).
- We determine that $\lambda_\varphi\,\varphi_1 + \lambda_\psi\,\psi_3 + \nu_2 = 0$ is equivalent to

  $$2M^2\lambda_\varphi\lambda_\psi\big[2 - \cos(\theta_\varphi - \theta_\psi)\big] + M^4\big[\lambda_\psi^2 + 2\lambda_\varphi\lambda_\psi\cos(\theta_\varphi - \theta_\psi)\big] - \lambda_\psi^2 = 0$$

  and

  $$(\lambda_\varphi + \lambda_\psi)(\lambda_\varphi M^2 - \lambda_\psi) \geqslant 0. \tag{5.6}$$

  The corresponding second resonance is $\lambda_\varphi\,\varphi_3 + \lambda_\psi\,\psi_2 + \nu_2 = 0$, which is equivalent to

  $$2M^2\lambda_\varphi\lambda_\psi\big[2 - \cos(\theta_\varphi - \theta_\psi)\big] + M^4\big[\lambda_\varphi^2 + 2\lambda_\varphi\lambda_\psi\cos(\theta_\varphi - \theta_\psi)\big] - \lambda_\varphi^2 = 0$$

  and

  $$(\lambda_\varphi + \lambda_\psi)(\lambda_\psi M^2 - \lambda_\varphi) \geqslant 0. \tag{5.7}$$

  Now conditions (5.6) and (5.7) are incompatible, so the configuration of resonances (5.5a) is impossible.
- The case of (5.5c) is analogous, and is not detailed here.
- Finally, if for the first resonance we have $\lambda_\varphi\,\varphi_3 + \lambda_\psi\,\psi_3 + \nu_2 = 0$, then the second one must be $\lambda_\varphi\,\varphi_1 + \lambda_\psi\,\psi_2 + \nu_2 = 0$, which is equivalent to

  $$\big[1 - \cos(\theta_\varphi - \theta_\psi)\big] + M^2\big[1 + \cos(\theta_\varphi - \theta_\psi)\big] = 0 \quad \text{and} \quad (\lambda_\varphi + \lambda_\psi)(\lambda_\psi - \lambda_\varphi) \geqslant 0.$$

  First equation rewrites

  $$\cos(\theta_\varphi - \theta_\psi) = \frac{1 + M^2}{1 - M^2},$$

  which cannot be satisfied by $M^2 \in (0, 1)$. Thus the fourth possibility (5.5d) is also impossible.

Therefore, in this case, a situation like the one described in Assumption 5 **cannot happen**.

To conclude as for the Euler system, we need to have a discussion about where we have made a choice which puts us in a particular case. The above analysis about frequencies φ, ψ and ν does not depend on B, but only on the location of cancellation of the Lopatinskii determinant. Thus the only restrictive choice we made is to choose this location as $\tau = c_0 |\eta|$. Therefore, for the compressible isentropic Euler equations in space dimension 3, in this particular case, the configuration of frequencies considered in this work which leads to an instability cannot happen.

We have considered here the Euler system in space dimension 3, since, in space dimension 2, the condition $\tau = c|\eta|$ leads to $\tau = \pm c\eta$, preventing to obtain a transverse oscillation. We could also consider the shock problem for the Euler equations, which is the original problem of Majda and Rosales in [MR83, MR84].

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