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Tangential cover for 3D irregular noisy digital curves

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Abstract. This paper presents a discrete structure, named *adaptive tangential cover* (ATC), for studying 3D noisy digital curves. The structure relies mainly on the primitive of *blurred segment of width ν* and on the local noise estimator of *meaningful thickness*. More precisely, ATC is composed of maximal blurred segments of different widths deduced from the local noise values estimated at each point of the curve. Two applications of ATC for geometric estimators of 3D noisy digital curves are also presented in the paper. The experimental results demonstrate the efficiency of ATC for analyzing 3D irregular noisy curves.

Keywords: 3D digital curves, noise estimator, tangent and curvature estimators

1 Introduction

3D digital curves are often involved in many applications of 3D image processing and computer graphics. For instance, in [12], a system has been designed to model surfaces with collections of 3D curves. In [20], a sketch-based modeling method is proposed to reconstruct 3D curves and to reveal 3D shape information from typical design sketches. In applications of medical image processing [17], 3D digital curves are used for the analysis and can be obtained from a 3D curvilinear skeletonisation [4]. In those applications, the geometric characteristics of the curves play an important role for numerous purposes. In real context, the data present generally noise due to the acquisition process.

In digital geometry, new mathematical definitions of basic geometric objects are introduced to better fit the discrete/digital nature of data to process. In the context of 3D digital curve analysis, the notion of 3D *maximal digital straight segment* [2] has been used to describe the geometric properties of the curves. In particular, the sequence of all maximal segments along a digital curve C , called the *tangential cover*, has been shown to be an efficient tool to study digital curves. Indeed, it is involved in numerous geometric estimators of digital curves: length [2], tangent [18], curvature [3] . . .

However, the tangential covers based on maximal segments are not adapted to noisy or disconnected curves. For this, the notion of *blurred segment of width ν* [5, 16] was proposed to deal with 3D digital curves containing noise or other sources of imperfections from real data via the parameter ν . The sequence of maximal blurred segments of width ν along a digital curve is called a *ν -tangential cover*. This structure has been used in different contexts to study and to analyze the geometrical characteristics of noisy curves (e.g. [5, 15, 16]). Nevertheless, in these applications, the width ν needs to be manually adjusted to take into account the noise present on the curves. Furthermore, the structure is not well suited for noise which appears irregularly on the 3D curves.

Based on the works [13, 14], the present paper studies a discrete framework for 3D digital curves containing *irregular* noise, and proposes improvements for the 3D approach. More precisely, we present a discrete structure, called *adaptive tangential cover* (ATC), which relies on primitive of blurred segment of width ν . The particularity of ATC is that it contains a sequence of blurred segments with different widths varying in function of noise present along the digital curve. Such adaptive widths are computed thanks to the local noise estimator of *meaningful thickness* [9]. In particular, the ATC can be computed in quasi-linear time and the method is parameter-free. We then show two applications of ATC to geometric estimators: tangent based on λ -MST [18] and curvature based on osculating circle [3] of 3D digital noisy curves. The experimental results show that the proposed method, based on the structure of ATC, is an efficient tool for studying noisy digital curves.

2 Background notions

In the following, we define a digital curve $C = (C_i \in \mathbb{Z}^d)_{i=1..n}$, for $d = \{2, 3\}$, as a sequence of discrete points. We then denote $C_{i,j}$, with $1 \leq i \leq j \leq n$, the set of consecutive points from C_i to C_j of C .

Let us first recall some definitions in 2D and then the extensions to 3D since, in this work, the definitions in 3D are computed from the 2D projections.

2.1 2D blurred segment of width ν and noise detector

Definition 1 ([19]). A **2D digital line**, with direction vector $(b, a) \in \mathbb{Z}^2$, $\gcd(a, b) = 1$, shift $\mu \in \mathbb{Z}$ and thickness $\omega \in \mathbb{Z}_+$ is defined as the set of points $(x, y) \in \mathbb{Z}^2$ verifying

$$\mu \leq ax - by < \mu + \omega \quad (1)$$

Such a line is denoted by $\mathcal{D}_2(a, b, \mu, \omega)$.

A *2D digital segment* is a finite and bounded subset of a 2D digital line (see Fig.1 (a)). From the primitive of digital segment, the notion of 2D blurred segment of width ν is proposed and allows for more flexibility in handling noisy data via the width parameter.

Let us consider \mathcal{S}_2 a sequence of discrete points of \mathbb{Z}^2 .

Definition 2 ([6]). A digital line $\mathcal{D}_2(a, b, \mu, \omega)$ is said to be **bounding** for \mathcal{S}_2 if all points of \mathcal{S}_2 belong to \mathcal{D}_2 .

Definition 3 ([6]). A bounding digital line $\mathcal{D}_2(a, b, \mu, \omega)$ of \mathcal{S}_2 is said to be **optimal** if the value $\frac{\omega-1}{\max(|a|, |b|)}$ is minimal, i.e. if its vertical (or horizontal) distance is equal to the vertical (or horizontal) thickness of the convex hull of \mathcal{S}_2 .

This definition is illustrated in Fig. 1 (b), it leads to the definition of 2D blurred segment.

Definition 4 ([6]). A sequence \mathcal{S}_2 is a **2D blurred segment of width ν** if its optimal bounding line has vertical/horizontal distance less than or equal to ν , i.e. $\frac{\omega-1}{\max(|a|, |b|)} \leq \nu$.

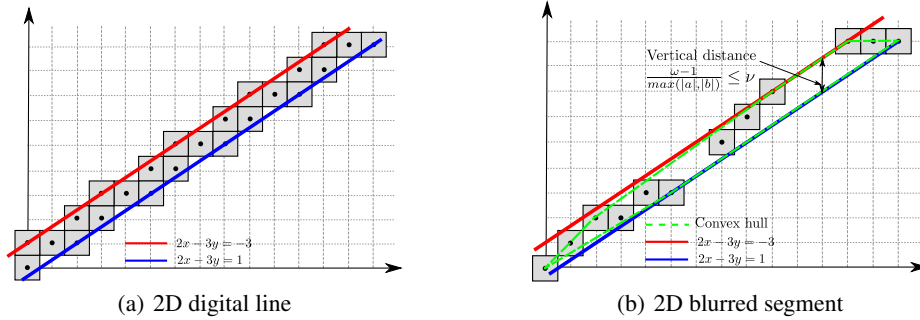


Fig. 1. Examples of (a) 2D digital line $\mathcal{D}_2(2, 3, -3, 5)$ and (b) 2D blurred segment of width $\nu = 1.5$ belongs to the optimal digital line $\mathcal{D}_2(2, 3, -3, 5)$.

From this definition, a linear algorithm of blurred segment recognition is proposed in [16]. The method is based on an incremental growth of convex hull of the points sequentially added to the segment, and the calculation of its vertical and horizontal thickness.

From the primitive of digital straight segment, the noise detector of *meaningful scale* (MS) [7, 8] is designed to locally estimate the best scale to analyze a digital curve. This detector is based on the study of the asymptotic properties of the discrete length L of maximal segments. In particular, it has been shown in [11] that the lengths of maximal segments covering a point p located on the boundary of a C^3 real object should be between $\Omega(1/h^{1/3})$ and $O(1/h^{1/2})$ if p is located on a strictly concave or convex part and near $O(1/h)$ elsewhere (where h represents the grid size). This theoretical property, defined on finer and finer grid sizes, was used by taking the opposite approach with the computation of the maximal segment lengths obtained with coarser and coarser grid sizes (from a subsampling process). From the graph of the maximal segment mean lengths \bar{L} , obtained at different scales, the noise estimator consists of recognizing the maximal scale for which the lengths follow the previous theoretical behavior.

The method has been extended to the detector of *meaningful thickness* (MT) [9] by using the blurred segment primitive with the scale definition given by the width of the blurred segment. Such a strategy presents the first advantage to be easier to implement without needing to apply different subsamplings and can be used for non-integer coordinate curves. In this work, we use the MT estimator since it allows to process disconnected curves which is suitable for our framework of studying digital noisy curves.

2.2 3D blurred segment of width ν

Definition 5 ([2]). A *3D digital line*, with main vector $(a, b, c) \in \mathbb{Z}^3$ such that $a \geq b \geq c > 0$ and $\gcd(a, b, c) = 1$, with shifts $\mu, \mu' \in \mathbb{Z}$, and thicknesses $\omega, \omega' \in \mathbb{Z}_+$ is defined as the set of points $(x, y, z) \in \mathbb{Z}^3$ verifying

$$\begin{cases} \mu' \leq bx - ay < \mu' + \omega' \\ \mu \leq cx - az < \mu + \omega \end{cases} \quad (2)$$

Such a line is denoted by $\mathcal{D}_3(a, b, c, \mu, \mu', \omega, \omega')$.

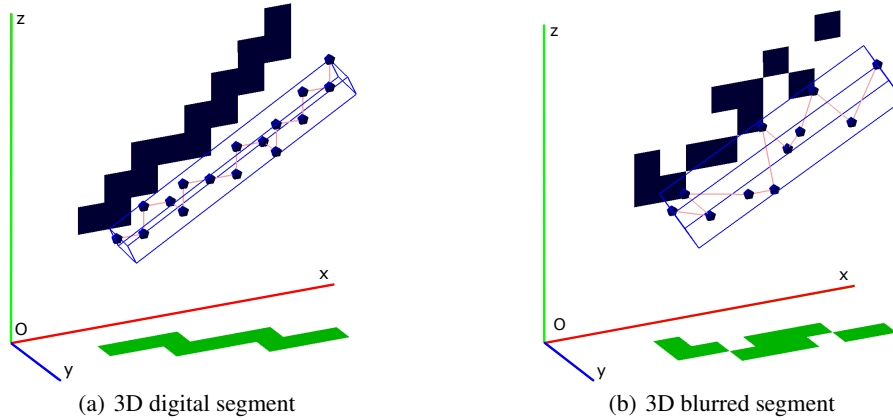


Fig. 2. Examples of (a) 3D digital segment of $\mathcal{D}_3(20, -25, 8, -10, -6, 8, 6)$ and (b) 3D blurred segment of width $\nu = 2.5$ belongs to the optimal digital line $\mathcal{D}_3(10, -15, 3, -10, -6, 22, 20)$.

For the 3D digital lines of coefficients ordered different from $a \geq b \geq c > 0$, they can be obtained by permuting x, y, z as well as their coefficients. From Def. 5, a 3D digital line is bijectively projected into two projection planes as two 2D digital lines. A *3D digital segment* is a finite, bounded and connected subset of a 3D digital line (see Fig. 2 (a)).

Definition 6 ([16]). Let S_3 be a sequence of points of \mathbb{Z}^3 . S_3 is a **3D blurred segment of width ν** , with main vector $(a, b, c) \in \mathbb{Z}^3$ such that $a \geq b \geq c > 0$, if it has an optimal digital line $\mathcal{D}_3(a, b, c, \mu, \mu', \omega, \omega')$ such that

- $\mathcal{D}_2(a, b, \mu', \omega')$ is **optimal** for the sequence of projections of points S_3 in the plane Oxy and $\frac{\omega'-1}{\max(|a|, |b|)} \leq \nu$;
- $\mathcal{D}_2(a, c, \mu, \omega)$ is **optimal** for the sequence of projections of points S_3 in the plane Oxz and $\frac{\omega-1}{\max(|a|, |c|)} \leq \nu$.

An illustration of 3D blurred segment is given in Fig. 2 (b).

Let $C \subset \mathbb{Z}^3$ be a 3D digital curve. Let $BS(i, j, \nu)$ denote the predicate “ $C_{i,j}$ is a blurred segment of width ν ”.

Definition 7 ([16]). $C_{i,j} \subset C$ is called a **maximal blurred segment of width ν** , noted $MBS(i, j, \nu)$, iff $BS(i, j, \nu)$, $\neg BS(i, j+1, \nu)$ and $\neg BS(i-1, j, \nu)$.

In [16], an incremental and linear algorithm is proposed to recognize a 3D maximal blurred segment (MBS) of width ν for a sequence of points. The main idea of the algorithm is to add simultaneously the 2D points in the projection planes as far as two of them are valid *i.e.*, we can add more point to the 2D MBS. Then, the 3D MBS is computed from the two corresponding 2D MBS of width ν projected onto the two basic planes. An algorithm for decomposing a digital curve C into 3D MBS of width ν is also presented in [16] with a complexity of $O(n \log^2 n)$ for n the number of points in C . The obtained structure is called (3D) ν -*tangential cover*. Some examples are given in Fig. 3.

Let $MBS_\nu(C) = \{MBS_i(B_i, E_i, \nu)_{i=1..m}\}$ denote the ν -tangential cover of C . By construction, we obtain the following property.

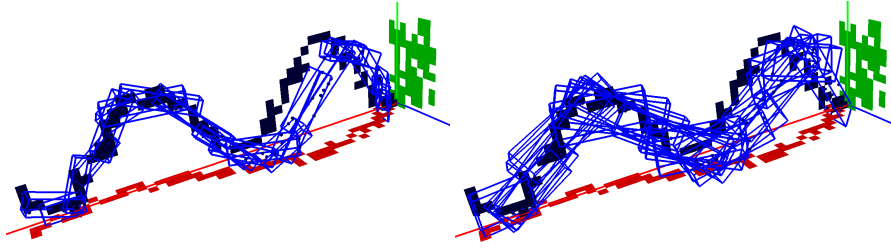


Fig. 3. Example of ν -tangential cover on a noisy digital curve: $\nu = 2$ (left) and $\nu = 4$ (right).

Property 1 ([16]) *Let $MBS_{\nu}(C)$ be the ν -tangential cover of C . Then, we have $MBS_{\nu}(C) = \{MBS(B_1, E_1, \nu), MBS(B_2, E_2, \nu), \dots, MBS(B_m, E_m, \nu)\}$ such that $B_1 < B_2 < \dots < B_m$ and $E_1 < E_2 < \dots < E_m$.*

3 Adaptive tangential cover

The notion of ν -tangential cover has been proved to be an efficient tool to study digital noisy curves [5, 15, 16]. However, the parameter ν needs to be manually adjusted by the user. In this section, we present a 3D discrete structure, named *adaptive tangential cover* (ATC). It is composed of 3D MBS of different widths according to the amount of noise present in the curve. Before detailing the 3D ATC and its construction, let us first describe the blurred segment recognition algorithm used for building the 3D ATC.

3.1 Blurred segment recognition

It must be recalled that, by definition, a 3D digital segment is bijectively projected into two projection planes as two 2D digital segments. From this property, the algorithm for 3D straight segment recognition [2] is performed by considering the projections in the basic planes, namely Oxy , Oxz and Oyz . In particular, a 3D digital segment is said *valid* if **at least two** of its projections are 2D digital segments and the points bijectively projected on the two planes, these planes are called *functional planes*. An algorithm for decomposing a digital curve into 3D straight segments is as well proposed. We refer the readers to [2, 5] for more details of the algorithm.

The recognition algorithm has been extended in [16] for the 3D blurred segment primitive by using the 2D blurred segment recognition in the projection planes. Similarly, a 3D blurred segment must be *valid* for at least two of its projections in the three basic planes together with the condition of two functional planes.

In this work, we are interested in 3D noisy digital curves, and in particular, the notion of 3D blurred segments of width ν for studying such curves. Although the 3D blurred segment recognition algorithm proposed in [16] works well for 3D noisy curves and its decomposition, we remarked various degeneracies that require us to modify the recognition algorithm. Firstly for the two functional planes, in case the amount of noise is important, this condition often fails, the algorithm therefore provides the short

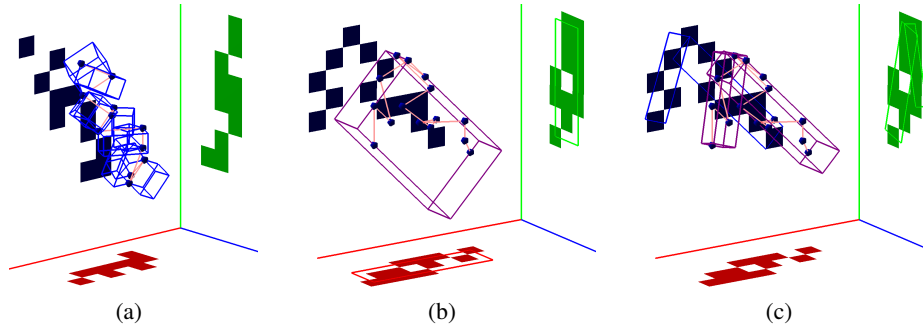


Fig. 4. Recognition of 3D blurred segment of width $\nu = 2$. (a) Result obtained when considering the bijectivity of the functional planes. (b) Result obtained when considering the two of the three projection are 2D blurred segments of width $\nu = 2$. (c) Result obtained when considering the three projection are 2D blurred segments of width $\nu = 2$ and relaxing the bijective condition.

segment as illustrated in Fig. 4 (a). This can bias the results of the geometric estimators on the curves. Secondly, a 3D blurred segment must have at least two of its projections being verified as 2D blurred segments. An example is given in Fig. 4 (b) for 3D blurred segment of width $\nu = 2$. One can observe that the 3D segment has its two projections on the planes Oxy and Oxz being two valid 2D blurred segments. However, the obtained 3D blurred segment is far to what we would like to have as a segment. More regular segments as in Fig. 4 (c) would be more preferable and relevant for analyzing this sequence of points. It must be mentioned that these issues are generally due to the nature of noisy data which makes discrete points irregularly distributed along the curve.

To overcome these limitations of the method, we propose some modifications in the recognition algorithm of 3D blurred segment. In particular, we **relax the constraints of valid and functional planes**. More precisely, for a given digital curve $C \subset \mathbb{Z}^3$, a sequence $C_{i,j}$ of C is a 3D blurred segment of width ν iff the three projections of $C_{i,j}$ on the basic planes are all 2D blurred segments of width ν . The characteristics of the optimal 3D digital line $\mathcal{D}_3(a, b, c, \mu, \mu', \omega, \omega')$ of $C_{i,j}$ is then computed from the two projections of 2D blurred segments having the longest lengths (in term of Euclidean distance). The rest of the algorithm stays the same as in [16].

3.2 Adaptive tangential cover construction

As previously mentioned, in the ν -tangential cover, the width parameter ν needs to be manually adjusted. Furthermore, the width ν is globally set for all MBS composing the ν -tangential cover. This approach works generally well when noise is uniformly distributed on the curve, but it is less relevant to local and irregular noisy curves.

In 2D, to overcome this issue of appropriate width ν , the structure of *adaptive tangential cover* (ATC) is introduced in [13, 14]. Such a structure is designed to capture the local noise on a digital curve by adjusting the width of 2D MBS in accordance with the amount of irregular noise present along the curve. In other words, the ATC is composed of MBS with different widths varying in function of detected noise present in the curve.

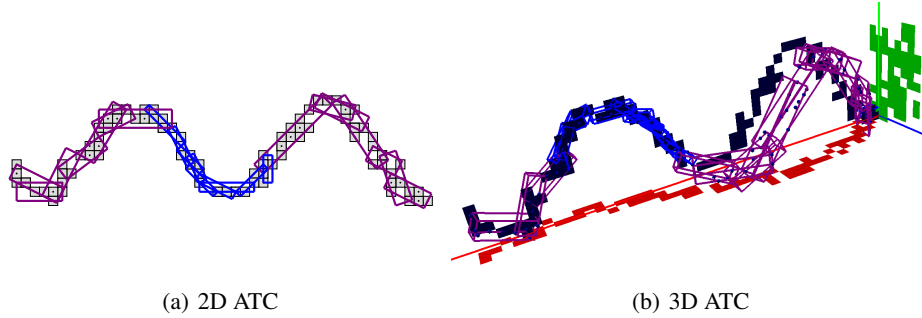


Fig. 5. Examples of ATC on 2D et 3D noisy digital curves.

In particular, the MBS has bigger widths at noisy zones, and smaller widths in zones with less or no noise (see Fig. 5 (a)).

Definition 8 ([13]). Let $C = (C_i)_{1 \leq i \leq n}$ be a digital curve. Let $\eta = (\eta_i)_{1 \leq i \leq n}$ be the vector of noise level associated to each C_i of C . Let $MBS(C) = \{MBS_{v_k}(C)\}$ be the set of v_k -tangential covers for $v_k \in \eta$. An **adaptive tangential cover** of C , noted $ATC(C)$, is the sequence of maximal blurred segments: $ATC(C) = \{MBS_j = MBS(B_j, E_j, v) \in MBS(C) \mid v = \max\{\eta_i \mid t \in \llbracket B_j, E_j \rrbracket\}\}$ such that $MBS_j \not\subseteq MBS_i$ for $i \neq j$, where $\llbracket a, b \rrbracket$ is the integer interval between a and b , including both.

Still in [13], an algorithm is proposed to build the ATC of a 2D digital curve C . The algorithm is composed of two steps: (1) labeling the points with the noise level vector η and (2) building the $ATC(C)$ with MBS of width from the obtained labels.

It should be mentioned that the definition of ATC is given in 2D, but still valid for 3D. A primary extension in 3D of ATC is also presented in [13]. The method is however restricted to noisy curves $C \subset \mathbb{Z}^3$ having **two functional planes**.

Regarding the noise vector η for each point of C , the method in [13] performs separately the noise detection – with MT estimator – on the projected 2D curves in the two functional planes, and then chooses the maximum noise level to assign to each corresponding 3D point. Although this strategy works well for 3D noisy curves containing few noise (see Fig. 5 (b)), it becomes less performance for important noisy cases. One common problem is that the high noise level is quickly propagated along the constructed ATC. As a consequence, it creates the segments of big widths encompassing the significant details of the input curve (see Fig. 6 (a)). Furthermore, in general, the 3D curves do not always have only two functional planes. We refer the reader to [13] for more details. In this work, we keep the idea of the ATC algorithm in [13]. We will, however, make some changes in the method for a more efficient construction of 3D ATC, and allow to handle general curves – without the constraint of two functional planes. More precisely, the noise estimator is performed on the three projected curves in the basic planes. For a better local fitting of the input curve C , **the median value is used, instead of the maximum** as in [13], for the noise associated to each 3D point of C . Indeed, it allows to prevent the strong increment of the width of blurred segments when the noise becomes important. A comparison of the two strategies on a noisy curve is given in

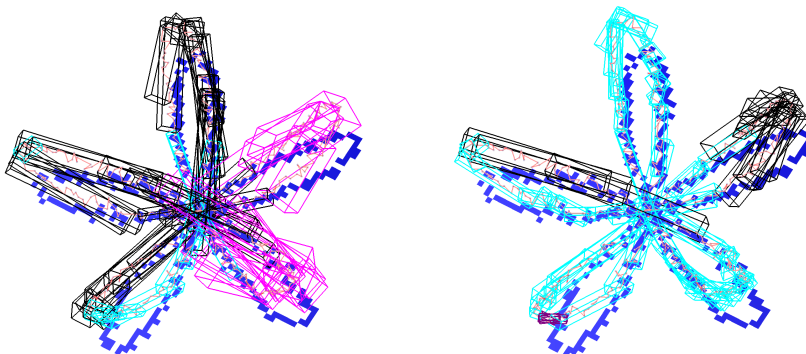


Fig. 6. Illustration of 3D ATC with the max (left) and median (right) value for combing noise.

Fig. 6. We can observe that the 3D ATC with maximal values contains the segments of big widths where the noise is important. It causes the lost of the curve characteristic at certain places and may bias the geometric estimators of the curve. On the other hand, the 3D ATC with median allows a closer approximation of the curve.

In the following, we use the blurred segment recognition described in Sec. 3.1. The 3D ATC construction is given in *Algo. 1* with modifications explained previously. Examples of 3D ATC with the method are given in Fig. 7. Note that, by definition (Def. 8) and construction, the 3D ATC satisfies also Property 1.

In [16], it is shown that the ν -TC of a 3D digital curve C can be computed in $O(n \log^2 n)$ for n the number of points in C . The construction of ATC is based on the ν -TC for different widths ν obtained for the MT estimator. The number ν -TC to be computed is equal to the size of ϑ – the number of noise levels present in C – and $|\vartheta| \ll n$. In other words, the complexity of *Algo. 1* for computing a 3D ATC is also $O(n \log^2 n)$.

4 Applications: tangent and curvature estimators

Geometric properties of curves are important characteristics to be exploited in geometry processing. In particular, tangent and curvature are among the important properties to describe a curve. In this section, we present two applications of ATC for geometric estimators at each point of 3D noisy digital curves. The ATC can be used to improve the precision of existing tangent and curvature estimators for irregular noisy curves.

To evaluate the proposed estimators, we consider two following curves: Flower and Astroid. The first curve is a *smooth* curve with nearly the same mean-error along the curve, while the second is a *sharp 4-pointed* curve. These two curves are defined, respectively, by the following parametric equations:

$$\begin{cases} x = 2 \cos(t) \cos(5t) \\ y = 2 \sin(t) \cos(5t) \\ z = \cos^2(5t) \end{cases} \quad (3) \quad \text{and} \quad \begin{cases} x = \cos^3(t) \\ y = \sin^3(t) \\ z = t \end{cases} \quad (4)$$

Algorithm 1: Calculation of 3D adaptive tangential cover.

```

Input   :  $C = (C_i)_{i=1\dots n}$  input 3D digital curve,
           :  $\eta = (\eta_i)_{i=1\dots n}$  noise level of each  $C_i$  with median MT of the projected planes
           :  $\vartheta = \{v_k \mid v_k \in \eta\}$  ordered set of  $\eta$ , and
           :  $MBS(C) = \{MBS_{v_k}(C)\}_{k=1\dots m}$  sets of MBS of  $C$  for each width value  $v_k \in \vartheta$ 
Output :  $ATC(C)$  the 3D adaptive tangential cover of  $C$ 
Variable :  $\gamma = (\gamma_i)_{i=1\dots n}$  vector of labels to each  $C_i$ 
1 begin
2    $ATC(C) = \emptyset$ ;  $\gamma_i = \eta_i$  for  $i \in \llbracket 0, n-1 \rrbracket$ 
   // Step 1: Label each point  $C_i$  of  $C$ 
3   foreach  $v_k \in \vartheta$  do
4     foreach  $MBS(B_i, E_i, v_k) \in MBS_{v_k}(C)$  do
5        $\alpha = \max\{\eta_i \mid i \in \llbracket B_i, E_i \rrbracket\}$ 
6       if  $\alpha = v_k$  then  $\gamma_i = v_k$  for  $i \in \llbracket B_i, E_i \rrbracket$ 
   // Step 2: Calculate the ATC of  $C$  with MBS of width from  $\gamma$ 
7   foreach  $v_k \in \vartheta$  do
8     foreach  $MBS(B_i, E_i, v_k) \in MBS_{v_k}(C)$  do
9       if  $\exists \gamma_i$ , for  $i \in \llbracket B_i, E_i \rrbracket$ , such that  $\gamma_i = v_k$  then
10       $ATC(C) = ATC(C) \cup \{MBS(B_i, E_i, v_k)\}$ 

```

These curves are discretized using the class `NaiveParametricCurveDigitizer3D` of `DGtal` [1] to obtain 3D connected curve C . To generate noise for C , a random process with a uniform distribution is considered to change one coordinate of points of C one unit, ± 1 . Thirty curves with random noise are generated for the error measurements between the theoretical tangent/curvature and the estimated ones on these curves.

4.1 Discrete tangent

In [18], a discrete tangent estimator, called *λ -maximal segment tangent* (λ -MST), has been proposed for *regular* digital curves. It is based on the tangential cover with maximal straight segments. The method is an extension of the algorithm presented in [11] for 2D curves. In particular, the estimator is a simple parameter-free method and has multi-grid convergent properties [10, 11].

The λ -MST can be easily applied to ATC with blurred segment primitive, and thus allows to handle noise. More precisely, the tangent at $C_k \in C$ is obtained via the definition of a *pencil of maximal blurred segments* $\mathcal{P}(C_k) = \{MBS_i = MBS(B_i, E_i, \cdot), \text{ with } B_i \leq k \leq E_i\}$. From this, a notion of eccentricity was introduced in order to distribute weights on all the segments covering C_k . More formally, the eccentricity is defined as

$$e_i(C_k) = \begin{cases} \|C_k - C_{E_i}\|_2^2 / L_i & \text{if } MBS_i \in \mathcal{P}(C_k) \\ 0 & \text{otherwise} \end{cases}, \text{ with } L_i = \|C_{E_i} - C_{B_i}\|_2^2 \quad (5)$$

Then, the tangent direction $\mathbf{t}(C_k)$ at C_k is computed as a weighted combination of the direction vectors \mathbf{t}_i of the maximal blurred segments $MBS_i \in \mathcal{P}(C_k)$ covering C_k by

$$\mathbf{t}(C_k) = \frac{\sum_{MBS_i \in \mathcal{P}(C_k)} \lambda(e_i(C_k)) \frac{\mathbf{t}_i}{\|\mathbf{t}_i\|_2}}{\sum_{MBS_i \in \mathcal{P}(C_k)} \lambda(e_i(C_k))} \quad (6)$$

where λ is a mapping function defined from $[0, 1]$ to \mathbb{R}^+ such that $\lambda(0) = \lambda(1) = 0$ and $\lambda > 0$ elsewhere and λ needs to satisfy convexity/concavity property [11]. In this paper, the C^2 function $\lambda(x) = 64(-x^6 + 3x^5 - 3x^4 + x^3)$ is used. Furthermore, thanks to *Prop. 1*, the pencil $\mathcal{P}(C_k)$, for $C_k \in C$, can be easily computed from $ATC(C)$.

Fig. 7 shows a visual result of the tangent estimator λ -MST with ATC on two noisy curves. Further results are shown in Tab. 1 in which we compare the λ -MST estimator using ATC and ν -tangential covers with $\nu = 1 \dots 5$ (as there are 5 noise levels detected in the thirty input curves). Different error measures are considered: mean and maximal error and standard deviation. The result shows that the combination of λ -MST with ATC improves globally tangent estimation on digital noisy curves; it has the best mean error measures, and the other measures are every closed to the best ones as well.

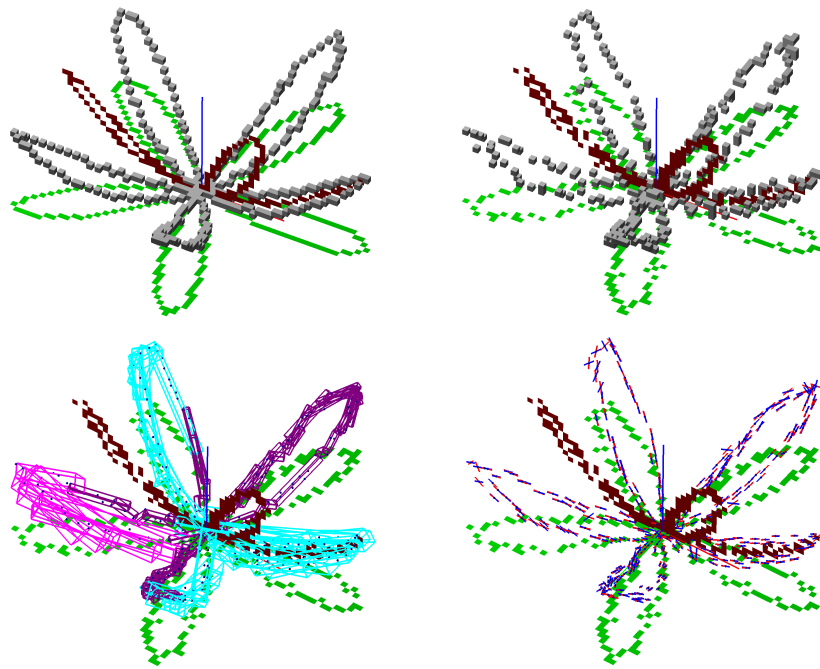
Table 1. Error measures of tangent estimator λ -MST on the digitized curves of Flower (Eq. 3) and Astroid (Eq. 4) with random noise added.

Curves		Error of estimated tangents					
		ATC_{MT}	1-TC	2-TC	3-TC	4-TC	5-TC
Flower	Mean Error	0.220012	0.369261	0.222374	0.283572	0.320565	0.343145
	Std. Dev.	0.214158	0.250986	0.193687	0.211761	0.225935	0.23631
	Max Error	1.26362	1.26657	1.25397	1.29662	1.29617	1.30408
Astroid	Mean error	0.293318	0.379116	0.295132	0.302137	0.309408	0.344594
	Std. Dev.	0.216065	0.285028	0.234184	0.222423	0.231388	0.24023
	Max error	1.06834	1.22489	1.13035	1.06979	1.04051	1.05925

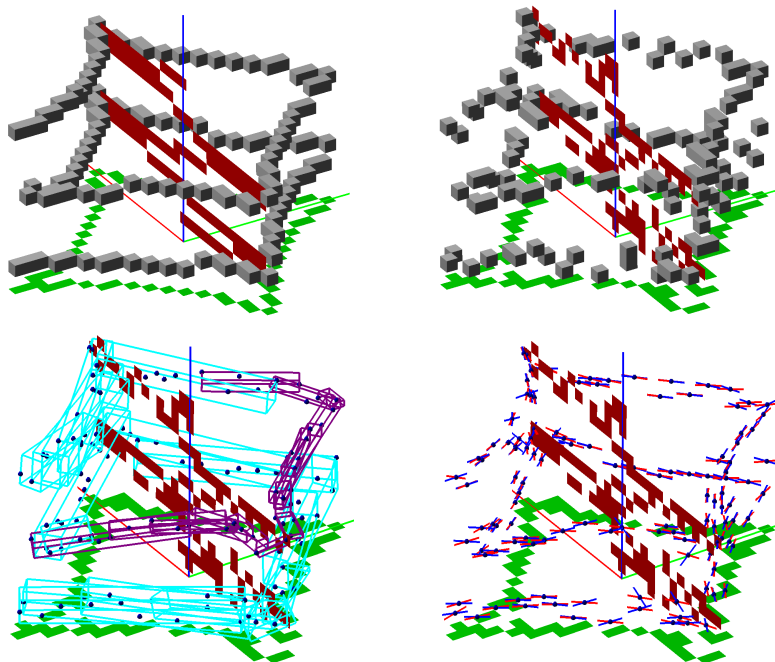
4.2 Discrete curvature

In the study of geometric characteristics of 3D digital curves, Coeurjolly and Svensson have proposed in [3] the calculation of discrete curvature based on the osculating circle with maximal straight segments. Inspired by this idea, Nguyen and Debled-Rennesson have presented in [15] the discrete curvature estimator using 3D maximal blurred segments. The proposed method can be naturally applied to the structure of 3D ATC. More precisely, from the 3D ATC of an input curve $C = (C_i)_{i=1 \dots n}$, we compute the pencil $\mathcal{P}(C_k)$ for each point $C_k \in C$. Let C_l (resp. C_r) be the left (resp. right) end point of $\mathcal{P}(C_k)$. The discrete curvature at C_k is estimated using the radius $R_\nu(C_k)$ of the osculating circle of the triangle formed by three points C_l , C_k and C_r

$$R_\nu(C_k) = \frac{s_1 s_2 s_3}{\sqrt{(s_1 + s_2 + s_3)(s_1 + s_2 - s_3)(s_1 + s_2 - s_3)(s_2 + s_3 - s_1)}} \quad (7)$$



(a) Noisy digital Flower curve (Eq. 3)



(b) Noisy digital Astroid curve (Eq. 4)

Fig. 7. Tangent estimator λ -MST with ATC on noisy digital curves. In each figure (a) and (b), the ATC of the input curve is on the left, the result of estimated tangents is on the right: red segments are expected theoretical tangents and blue segments are estimated tangents.

with $s_1 = \|\overrightarrow{C_k C_r}\|$, $s_2 = \|\overrightarrow{C_k C_l}\|$ et $s_3 = \|\overrightarrow{C_l C_r}\|$. The curvature of C_k is then calculated by

$$c_\nu(C_k) = \frac{s}{R_\nu(C_k)} \quad (8)$$

where $s = \text{signe}(\det(\overrightarrow{C_k C_r}, \overrightarrow{C_k C_l}))$ indicating the concavity or convexity of C at C_k .

Tab. 2 presents the results of the proposed curvature estimator using 3D ATC and ν -tangential covers with $\nu = 1 \dots 5$ on the error measures of mean and maximal error and standard deviation. The estimator with ATC does not always give the best result but we obtain fairly accurate curvature estimation comparing to the ν -tangential covers.

Table 2. Error measures of curvature estimator on the digitized curves of Flower (Eq. 3) and Astroid (Eq. 4) with random noise added.

Curves		Error of estimated curvatures					
		ATC _{MT}	1-TC	2-TC	3-TC	4-TC	5-TC
Flower	Mean Error	0.042239	0.192657	0.043795	0.041935	0.045646	0.048014
	Std. Dev.	0.118326	0.173881	0.108671	0.125055	0.130469	0.131886
	Max Error	1.05404	1.906708	0.977172	1.08214	1.11167	1.11805
Astroid	Mean Error	0.026844	0.202735	0.043616	0.024496	0.024548	0.028414
	Std. Dev.	0.015149	0.176366	0.058435	0.016437	0.016609	0.017491
	Max Error	0.439742	0.828724	0.349773	0.08767	0.078456	0.078618

5 Conclusion

From the studies in [13, 14], this paper presents different improvements of the structure of *adaptive tangential cover* (ATC) for 3D noisy digital curves. ATC is a discrete structure composed of a sequence of maximal blurred segment of width ν adjusted according to the amount of noise present on the curve. Due to the nature of noisy data, 3D blurred segment recognition is also modified by considering all three projection planes. From this, a parameter-free and quasi-linear algorithm is proposed to compute the structure of ATC.

Two applications of ATC are also presented for tangent and curvature estimators of noisy curves. The proposed structure opens numerous perspectives for studying and analyzing 3D noisy curves such as 3D curve segmentation or approximation, convex and concave detection, dominant point detection in 3D ... The source code of the proposed method for constructing 3D ATC, based on DGtal library [1], is available at the *GitHub* repository: https://github.com/ngophuc/ATC_3D.

In further works, we also would like to study the aspect of multi-grid convergent estimators of 3D noisy digital curves using the proposed structure of ATC. Furthermore, we would like to elaborate a more efficient computation of ATC without calculation of all ν -tangential covers. Other perspective consists of studying a 3D noise estimator for ATC instead of using the combination of MT estimators on the three projection planes.

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