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► **To cite this version:**

T Alberti, Davide Faranda, V Lucarini, R V Donner, Berengere Dubrulle, et al.. Scale dependence of fractal dimension in deterministic and stochastic Lorenz-63 systems. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 2023, 10.1063/5.0106053 . hal-03707052v2

**HAL Id: hal-03707052**

**<https://hal.science/hal-03707052v2>**

Submitted on 13 Feb 2023

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# Scale dependence of fractal dimension in deterministic and stochastic Lorenz-63 systems

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(Dated: 13 February 2023)

Many natural systems show emergent phenomena at different scales, leading to scaling regimes with signatures of **deterministic** chaos at large scales and an apparently random behavior at small scales. These features are usually investigated quantitatively by studying the properties of the underlying attractor, the compact object asymptotically hosting the trajectories of the system with their invariant density in the phase-space. This multi-scale nature of natural systems makes it practically impossible to get a clear picture of the attracting set. Indeed, it spans over a wide range of spatial scales and may even change in time due to non-stationary forcing. Here we combine an adaptive decomposition method with extreme value theory to study the properties of the instantaneous scale-dependent dimension, which has been recently introduced to characterize such temporal and spatial scale-dependent attractors in turbulence and astrophysics. To provide a quantitative analysis of the properties of this metric, we test it on the well-known low-dimensional deterministic Lorenz-63 system perturbed with additive or multiplicative noise. We demonstrate that the properties of the invariant set depend on the scale we are focusing on and that the scale-dependent dimensions can discriminate between additive and multiplicative noise, despite the fact that the two cases **have exactly the same stationary invariant measure** at large scales. The proposed formalism can be generally helpful to investigate the role of multi-scale fluctuations within complex systems, allowing us to deal with the problem of characterizing the role of stochastic fluctuations across a wide range of physical systems.

**The collective dynamics of natural systems is the result of the dynamics of their individual components, often operating on multiple spatio-temporal scales and some-times related to intrinsic and extrinsic factors. These multiple components reflect in scaling laws, unpredictable vs. deterministic behavior, bifurcations between different regimes, and basin of attractions. Here, we propose a novel concept of fractal dimension in deterministic and stochastic Lorenz-63 systems to provide a more complete characterization of the geometric features of attractors at different scales.**

## I. INTRODUCTION

Since their first description by E. N. Lorenz in 1963<sup>1</sup>, the existence and properties of strange attractors have been fre-

quently discussed in the context of such diverse fields as the atmosphere<sup>2</sup>, climate<sup>3,4</sup>, biology<sup>5</sup>, and ecology<sup>6</sup>, to mention only a few examples. The concept of strange attractors is strictly related to that of dissipative dynamical systems with sensitive dependence on the initial conditions. Being revolutionary at the time of its invention, it has been attracting a lot of attention, especially in the context of developing measures to quantify the geometric and dynamical properties of attractors<sup>7</sup> and in revising some earlier concepts on the forecast horizon of physical systems<sup>8</sup>. A one-parametric family of measures, the so-called generalized fractal dimensions  $D_q$ , has been proposed based on a coarse-grained invariant measure linking the geometric properties of the phase-space trajectories to the statistics of the dynamical scaling properties<sup>9</sup>. These measures provided new insights not only in the field of dynamical system theory (where they have been developed)<sup>10</sup> but also into different more applied fields like fluid and magneto-hydrodynamic turbulence<sup>11,12</sup> and others<sup>13</sup>.

One of the peculiar aspects of physical systems is their variability over a wide range of scales, arising from both intrinsic interactions between characteristic variability components in one or several variables and external forcings, differently affecting the specific properties of the whole system at different scales<sup>4</sup>. Recently, Alberti *et al.*<sup>14</sup> proposed a method to investigate how scale-dependency affects the global phase-space properties and their statistical characteristics. This method requires to first identify scale-dependent components contributing to the observed dynamics of a given system as a whole, which can be achieved by applying time series decomposition techniques like empirical mode decomposition (EMD). Subsequently, quantitative scale-specific measures like generalized fractal dimensions are evaluated. The formalism resulting from the combination of those two approaches allows the introduction of *multi-scale measures* by computing the generalized fractal dimensions for each scale specific component and partial sums thereof<sup>14</sup>. The suitability of this approach has been demonstrated for several dynamical systems of different complexity, synthetic noisy signals, and real-world time series data<sup>14</sup>.

For systems exhibiting heterogeneous phase space structure or even non-stationarity, it would however be useful to track the instantaneous number of degrees of freedom, which are closely related to its associated recurrence characteristics<sup>15</sup>. Indeed, the spatial distribution of the instantaneous (i.e., local in phase-space) dimensions across the system's invariant set, as well as its geometric shape, provide us with more detailed information than established global (as in Hentschel and Procaccia<sup>9</sup>) and/or scale-dependent (as in Alberti *et al.*<sup>14</sup>) measures of complexity.

Accordingly, in this work we thoroughly extend the existing formalism of multi-scale measures<sup>14</sup> to characterize the instantaneous scale-dependent properties of strange attractors by combining time series decomposition methods with concepts from extreme value theory that are related to the instantaneous number of degrees of freedom of the observed dynamics. We then show the utility of our approach for the case of the well-known low-dimensional deterministically-chaotic Lorenz-63 system and two stochastic versions thereof<sup>16</sup>. We indeed show that the new formalism, based on instantaneous scale-dependent dimensions, allows us to discern two properties that are inaccessible by previous global or scale-dependent analysis, namely the existence of different scale-dependent source processes (as the presence of noise or a dominant scale) and the structural stability of fixed points.

## II. METHODS

In the following section, we start by introducing the decomposition procedure and the dynamical system metrics separately, before describing our proposed formalism. For a more general purpose, we assume to have a generic  $N$ -dimensional system, i.e., an  $N$ -dimensional phase-space, with  $N > 1$ . Thus, we describe our decomposition procedure in a general

multivariate framework. For univariate data (i.e.,  $N = 1$ ), we may proceed in a largely analogous way.

### A. Multivariate Empirical Mode Decomposition (MEMD)

Considering an  $N$ -dimensional system described via a multivariate time series signal  $\Theta_\mu(t) = [\Theta_1(t), \Theta_2(t), \dots, \Theta_N(t)]^\dagger$  (with  $\dagger$  indicating the transposition operator), the Multivariate Empirical Mode Decomposition (MEMD) decomposes the data into a finite number of multivariate oscillating patterns  $C_{\mu,k}(t)$ , referred to as Multivariate Intrinsic Mode Functions (MIMFs), and a monotonic residue  $R_\mu(t)$  as

$$\Theta_\mu(t) = \sum_{k=1}^{n_k} C_{\mu,k}(t) + R_\mu(t). \quad (1)$$

The decomposition basis, formed by the set of functions  $C_{\mu,k}(t)$ , is **empirically** derived via the so-called sifting process<sup>17</sup> modified for multivariate signals<sup>18</sup>. This sifting process consists of

1. identifying local extremes of  $\Theta_\mu(t)$ , i.e., where the  $N$ -variate derivative **vanishes**;
2. interpolating these points via cubic splines to derive the upper and lower envelopes  $\mathbf{u}(t)$  and  $\mathbf{l}(t)$ , respectively;
3. deriving the mean envelope  $\mathbf{m}(t)$  as  $\mathbf{m}(t) = \frac{\mathbf{u}(t) + \mathbf{l}(t)}{2}$ ;
4. evaluating the detail  $\mathbf{h}(t) = \mathbf{s}(t) - \mathbf{m}(t)$ .

These steps are iterated until the detail  $\mathbf{h}(t)$  can be identified as a MIMF (also called multivariate empirical mode)<sup>18</sup>, i.e., it must **satisfy two properties**: it has the same number of local extremes and zeros (or both differing at most by one) and a zero-average mean envelope  $\mathbf{m}(t)$ <sup>17</sup>. The full sifting process stops when no more MIMFs  $C_{\mu,k}(t)$  can be filtered out from the data. Each  $C_{\mu,k}(t)$  represents a peculiar dynamical component intrinsic to the system that typically evolves on an average scale

$$\tau_k = \frac{1}{T} \int_0^T t' \langle C_{\mu,k}(t') \rangle dt', \quad (2)$$

where  $T$  is the length of data and  $\langle \bullet \rangle$  denotes an ensemble average over the  $N$ -dimensional space<sup>19</sup>. The MEMD allows us to interpret  $\Theta_\mu(t)$  as a collection of scale-dependent multivariate fluctuations contributing to the collective properties of the whole system. Indeed, each MIMF can be seen as representative of fluctuations at a typical scale that is the average of the instantaneous scales (i.e., the inverse instantaneous frequencies) derived from a given mode via the Hilbert transform<sup>19</sup>. The MEMD, due to its adaptive methodology, relieves some a priori mathematical constraints of fixed-basis decomposition methods and extracts a limited number of intrinsic components that can be visually inspected. **Other widely used** decomposition methods, like Fourier or **continuous** wavelet analysis, commonly return a large number of components

and/or need to project our data on a pre-defined decomposition basis. Moreover, at least classical Fourier transform methods also require that our data satisfy a stationarity condition. In this regard, we do not question the appropriateness of the aforementioned more traditional conventional analysis techniques, but rather acknowledge that they (as well as any other approaches) have intrinsic limitations in what we can learn from their application.

## B. Instantaneous dimension

Given the  $N$ -dimensional system described via the multivariate trajectory  $\Theta_\mu(t)$ , its dynamical properties can be investigated by combining the concept of recurrences in phase space and extreme value theory<sup>20</sup>. For some (arbitrary) state of interest  $\zeta$  in the associated phase-space, we first introduce the logarithmic return associated with each state on the trajectory (except for  $\zeta$  itself) as

$$G(\Theta_\mu(t), \zeta) = -\log [\text{dist}(\Theta_\mu(t), \zeta)] \quad (3)$$

where  $\text{dist}(\bullet)$  is a distance between two state vectors in phase-space, commonly the Euclidean one. By shortening the notation, we obtain a time series of logarithmic returns  $g(t) = G(\Theta_\mu(t), \zeta)$  that takes larger values whenever  $\Theta_\mu(t)$  is close to  $\zeta$ . If we now define a threshold  $s(q)$  as the  $q$ -th empirical quantile of  $g(t)$ , we can introduce the exceedances  $u(\zeta) \doteq \{t | g(t) > s(q)\}$ , i.e., the recurrences to the **neighborhood of the reference state** in the context first introduced by Poincaré **by exploiting a peaks-over-threshold like concept as widely used in extreme value theory**. According to the Freitas-Freitas-Todd theorem the cumulative probability distribution  $F(u, \zeta)$  then converges to the exponential member of the Generalised Pareto Distribution (GPD), i.e.,

$$F(u, \zeta) \simeq \exp \left[ -\frac{u(\zeta)}{\zeta(\zeta)} \right]. \quad (4)$$

The GPD parameter  $\zeta$  depends on the dynamical state  $\zeta$  and can be used to introduce the concept of an instantaneous dimension  $d$  **at the point in time where  $\zeta$  is attained, which is simply defined as  $d(\zeta) = \zeta(\zeta)^{-1}$** . Although it could merely be associated to a fitting parameter, it has a clear physical meaning:  $d$  is a proxy of the active number of degrees of freedom around each state  $\zeta$  in the phase-space. **Note, however, that from a practical perspective, this instantaneous dimension needs to be considered relative to the set of time series values available, and the interpretation of its values may be affected by nonstationarity or non-representative sampling of the presumed attractor at finer spatial scales.**

## C. Instantaneous scale-dependent dimension

The instantaneous dimension  $d$  introduced above provides a local **(in terms of phase-space) picture** of the properties of phase-space trajectories, i.e., allows **us to obtain** information for each **sampled point contributing to the global structure of the attractor under study**. Nevertheless, multi-scale

systems could have a scale-dependent phase-space structure<sup>14</sup> such that we can distinguish between features that emerge at different scales. To provide a scale-dependent instantaneous view of a given system we have to combine a decomposition method, **like the MEMD**, and the extreme value theory applied to inter-state distances in phase space.

Given again an  $N$ -dimensional system described via  $\Theta_\mu(t)$  with a multi-scale nature, i.e., being characterized by processes occurring over a wide range of scales, we can write

$$\Theta_\mu(t) = \langle \Theta_\mu(t) \rangle + \sum_{\tau} \delta \Theta_\mu^{(\tau)}(t) \quad (5)$$

where  $\langle \Theta_\mu(t) \rangle$  is a steady-state time-average value and  $\delta \Theta_\mu^{(\tau)}(t)$  is a component of the system operating at a mean scale  $\tau$ . It is easy to note the analogy between Eq. (5) and Eq. (1) via the correspondence  $\mathbf{C}_{\mu,k}(t) \leftrightarrow \delta \Theta_\mu^{(\tau)}(t)$  and  $\mathbf{R}_\mu(t) \leftrightarrow \langle \Theta_\mu(t) \rangle$ . This means that for each scale  $\tau$  we can identify the corresponding invariant set  $\mathbb{M}_\tau$  as the manifold obtained via the **partial sums** of MIMFs with scales  $\tau_* < \tau$ , i.e.,

$$\Theta_\mu^\tau(t) = \sum_{k^*=1}^k \mathbf{C}_{\mu,k^*}(t). \quad (6)$$

Then, for each scale  $\tau \in [\tau_1, \tau_{n_k}]$ , i.e., for each  $k \in [1, n_k]$ , given a trajectory  $\Theta_\mu^\tau(t)$  and a state of interest  $\zeta_\tau$ , the cumulative probability of logarithmic returns in the neighborhood of  $\zeta_\tau$  follows a GPD as

$$F(u_\tau, \zeta_\tau) \simeq \exp \left[ -\frac{u_\tau(\zeta_\tau)}{\zeta_\tau(\zeta_\tau)} \right]. \quad (7)$$

Thus, we can introduce a **quantity**  $D(t, \tau) = \zeta_\tau(\zeta_\tau)^{-1}$ , representing the number of active degrees of freedom **of fluctuations up to a maximum scale of  $\tau$**  around each state  $\zeta_\tau$ . In this way, we exploit the properties of MEMD in deriving **scale-dependent** components embedded into a given system and the instantaneous (in terms of time) properties of the extreme value theory based metric to derive the instantaneous scale-dependent metric  $D(t, \tau)$ .

Summarizing, our procedure consists of the following steps:

1. extract intrinsic components  $\mathbf{C}_{\mu,k}(t)$  and their mean scales  $\tau_k$  from  $\Theta_\mu(t)$  by using the MEMD;
2. evaluate partial sums of Eq. (1) at different scales

$$\Theta_\mu^\tau(t) = \sum_{k^*=1}^k \mathbf{C}_{\mu,k^*}(t) \quad (8)$$

with  $k^* = 1, \dots, n_k$  (by construction, MIMFs are ordered with increasing scales, i.e.,  $\tau_{k'} < \tau_{k''}$  if  $k' < k''$ );

3. for each scale  $\tau_k$  (i.e., for each  $k$ ) evaluate  $D(t, \tau_k)$ .

Our procedure is, by construction, complete, since when  $k \rightarrow n_k$  then  $D(t, \tau_k) = d(t)$ , with  $d(t)$  being the instantaneous fractal dimension of the full system<sup>2,15,20</sup>.

In the remainder of this work, we will discuss some examples to highlight the potential of our framework to disentangle distinct dynamical components of different origin in a multiscale complex system.

### III. THE LORENZ-63 MODEL AND ITS STOCHASTIC VERSIONS

The Lorenz-63 system<sup>1</sup>, originally developed as a simplified model for atmospheric convection, is one of the most famous and widely studied paradigmatic dissipative-chaotic dynamical systems<sup>21</sup>, which can be written as

$$dx = s(-x + y) dt \quad (9)$$

$$dy = (rx - y - xz) dt \quad (10)$$

$$dz = (xy - bz) dt \quad (11)$$

with the parameters  $(s, r, b)$  related to the Prandtl number, the Rayleigh number, and the geometry of the atmospheric convective layer. With the **canonical** set of parameters  $(s, r, b) = (10, 28, 8/3)$  the system admits chaotic **behavior** with all initial points (**except for a set of measure zero**) tending towards an invariant set **with fractal structure**, usually termed the Lorenz attractor. It is a strange attractor whose Hausdorff dimension (and all its generalizations  $D_q$ ) take a value of  $2.05 \pm 0.02$ <sup>7,9,22</sup>.

A simple way to investigate the role of hidden fast dynamical components is to couple deterministic equations to "noise" mimicking the action of unknown fast variables. This can be also easily done for the Lorenz-63 system by rewriting the original system in terms of a set of coupled stochastic differential equations as

$$dx = s(-x + y) dt + \sigma dW_t \quad (12)$$

$$dy = (rx - y - xz) dt + \sigma dW_t \quad (13)$$

$$dz = (xy - bz) dt + \sigma dW_t \quad (14)$$

In nonlinear deterministic systems, such additive noise can lead to non-trivial effects<sup>23</sup>, including transitions between existing states or attractors, shifting bifurcations, or acting as an external forcing to the intrinsic variability of the system<sup>11,24</sup>, also observed for the Lorenz-63 system<sup>16</sup>.

More recently, another stochastic version of the Lorenz-63 system has been proposed by Chekroun, Simonnet, and Ghil<sup>16</sup>, considering a linearly multiplicative noise term to the original system as

$$dx = s(-x + y) dt + \sigma x dW_t \quad (15)$$

$$dy = (rx - y - xz) dt + \sigma y dW_t \quad (16)$$

$$dz = (xy - bz) dt + \sigma z dW_t \quad (17)$$

This system provided a first example for the existence of random attractors, extending the concept of a strange attractor still supporting nontrivial sample measures from deterministic to stochastic dynamics<sup>1,16</sup> that have been shown to be random Sinai–Ruelle–Bowen measures<sup>25</sup>. Note that in the weak noise limit, response theory allows one to compute explicitly

the change in the expectation value of the measurable observables when perturbing an underlying chaotic dynamics with stochastic terms of rather general nature<sup>26</sup>.

In the following, we apply our formalism to the three different versions of the Lorenz-63 system **described above**. In the case of the stochastic models featuring multiplicative noise, we use the Itô convention for the stochastic integration,  $dW_t$  is a Wiener process obtained by sampling at each time step a random variable with **the same Gaussian** density ( $W_t \sim \mathcal{N}(0, t)$ ) and intensity  $\sigma$ . The numerical simulation of Eqs. (9)-(17) is obtained by using the Euler-Maruyama method with a time resolution  $dt = 5 \times 10^{-3}$  over  $N = 10^7$  time steps, using the classical set of parameters  $(s, r, b) = (10, 28, 8/3)$  and  $\sigma = 0.4$  as in Chekroun, Simonnet, and Ghil<sup>16</sup>.

## IV. RESULTS

### A. Full system attractor

Figures 1-3 report the trajectories (left panels) of the three different Lorenz-63 systems (deterministic, Fig. 1; additive noise, Fig. 2; multiplicative noise, Fig. 3) and their corresponding attractors in the 3-D phase-space (right panels). As expected, a breakdown of the symmetric shape of the Lorenz attractor is observed when the classical Lorenz-63 system is subject to either additive or multiplicative noise. Furthermore, intermittency appears to be reduced, thus moving from a deterministic strange attractor towards a random stochastic attractor<sup>16</sup>. **Nevertheless, by only looking at the full system attractor we are not able to identify any significant difference in the geometric shape between the additive and the multiplicative model. Furthermore, both random attractors are characterized by the same dimension, equal to the full dimension of the phase-space, due to the corresponding property of the invariant measure of an elliptic diffusion process.**

### B. Average dimensions of scale-dependent attractors

**To further inspect and characterize the role of the noise versus the deterministic dynamics of the Lorenz system, we apply our formalism to derive  $D(t, \tau)$  for the three different systems. We first decompose every multivariate trajectory  $\Theta_\mu(t)$  via the MEMD through which we obtained a set of 15, 20, and 24 MIMFs, respectively, whose ranges of timescales are  $\tau \in [1.64, 2.5 \times 10^5]$ ,  $\tau \in [9.7 \times 10^{-2}, 2.5 \times 10^5]$ , and  $\tau \in [8.9 \times 10^{-2}, 2.5 \times 10^5]$ , respectively. Then, using Eqs. (6)-(7) we derive the instantaneous scale-dependent metric  $D(\tau, t)$  for the three systems.**

As a first step, we inspect the behavior of the average instantaneous scale-dependent dimension  $\langle D(\tau, t) \rangle_t$  as a function of the scale  $\tau$  as reported in Figure 4. **This is equivalent to the method proposed by Alberti *et al.*<sup>14</sup> where local (in terms of time-scale) and time-independent (i.e., averaged) multi-scale measures have been introduced.**

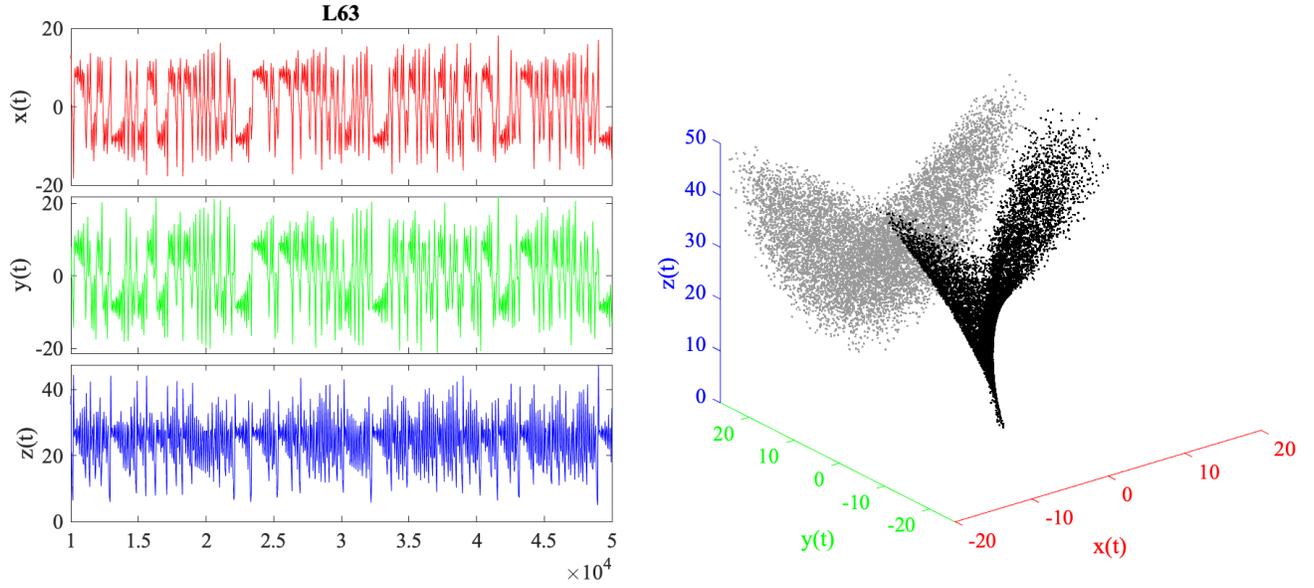


FIG. 1. (Left) Zoom of the trajectory components of the deterministic Lorenz-63 system as in Eqs. (9)-(11) (L63). (Right) Corresponding attractor in the 3-D phase-space (black points) and its projection in the  $x-z$  plane (gray points).

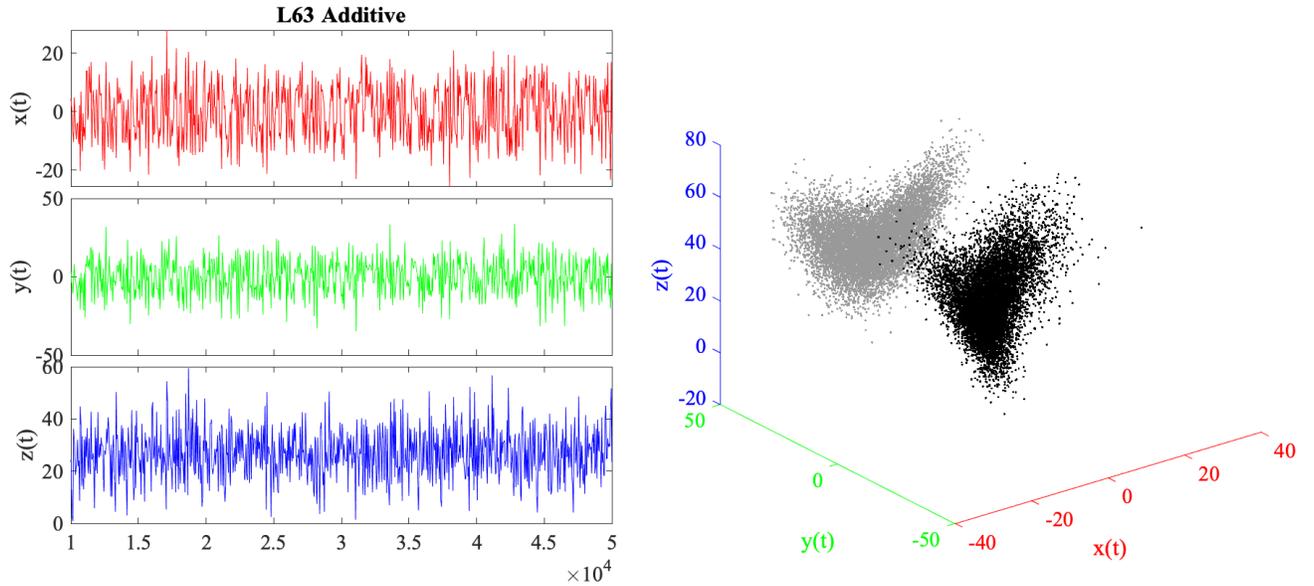


FIG. 2. (Left) Zoom of the trajectory components of the additive noise model as in Eqs. (12)-(14) (L63 Additive). (Right) Corresponding attractor in the 3-D phase-space (black points) and its projection in the  $x-z$  plane (gray points). The stochastic noise term has an amplitude  $\sigma = 0.4^{16}$ .

369 **Our corresponding analysis evidences the absence of**<sup>378</sup>  
 370 **time scales  $\tau < 1.64$  for the deterministic Lorenz-63 sys-**<sup>379</sup>  
 371 **tem as opposed to its stochastic versions. This is clearly a**<sup>380</sup>  
 372 **reflection of the absence of stochastic terms in the classical**<sup>381</sup>  
 373 **deterministic system, which are responsible for the ver-**<sup>382</sup>  
 374 **fast fluctuations in the stochastic cases.**<sup>383</sup>

375 **Furthermore, we generally observe larger average scale-**<sup>384</sup>  
 376 **dependent dimensions for the multiplicative noise case than**<sup>385</sup>  
 377 **for the deterministic Lorenz-63 system and the additive noise**<sup>386</sup>

model. This reflects the effect of the stochastic term on the dynamical features of the Lorenz-63 system: it does not only act at short scales, exciting variability at additional scales with respect to the classical Lorenz-63 system, but also affects the attractor geometry and, hence, the time-averaged number of active degrees of freedom as reflected by the scale-dependent fractal dimension metric at larger timescales. The latter property can be linked to the fact that the Lorenz-63 system with non-degenerate noise has an invariant mea-

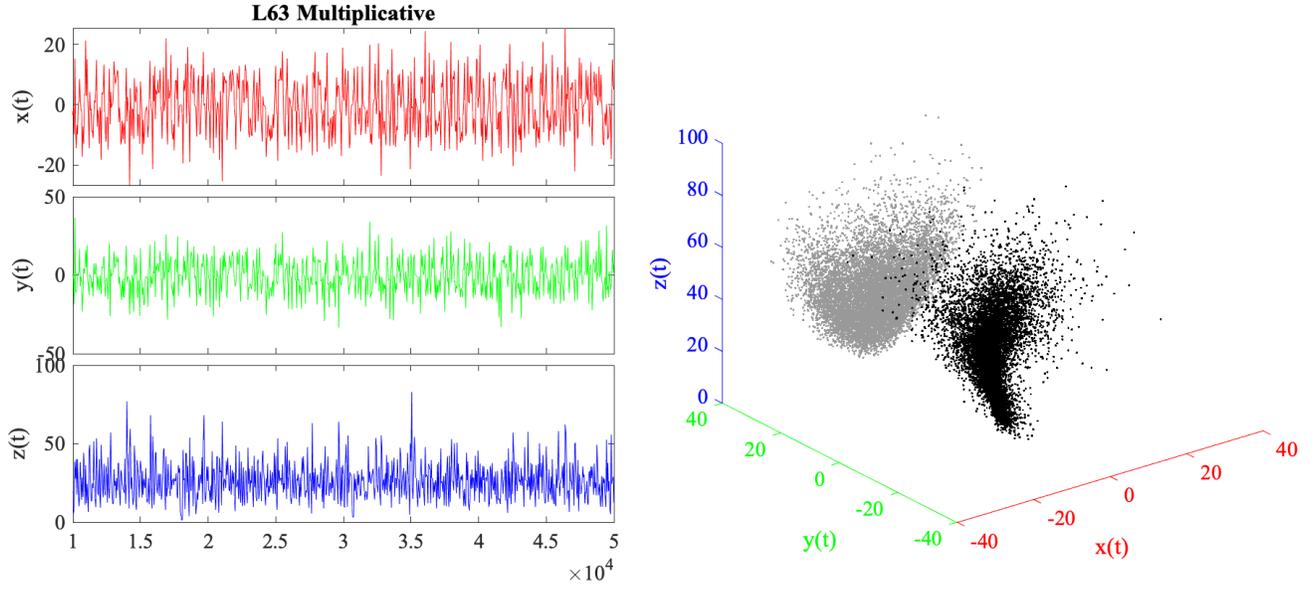


FIG. 3. (Left) Zoom of the trajectory components of the multiplicative noise model as in Eqs. (15)-(17) (L63 Multiplicative). (Right) Corresponding attractor in the 3-D phase-space (black points) and its projection in the  $x - z$  plane (gray points). The stochastic noise term has an amplitude  $\sigma = 0.4^{16}$ .

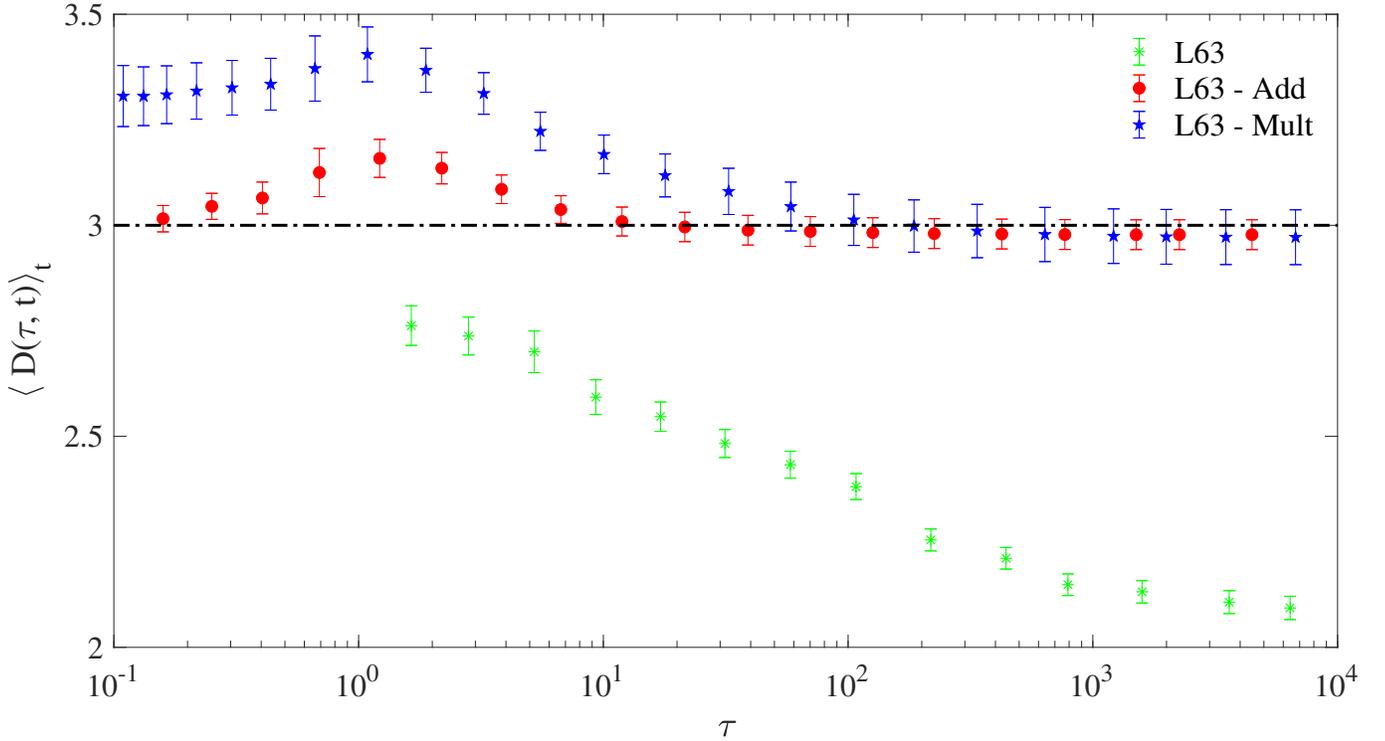


FIG. 4. Average instantaneous scale-dependent dimension  $\langle D(\tau, t) \rangle_t$  as a function of the scale  $\tau$ . Green asterisks refer to the Lorenz '63 system, red circles to the additive noise model, and blue stars to the multiplicative noise case. Error bars are obtained as the standard deviations of  $D(t, \tau)$  along the respective trajectory.

sure that is absolutely continuous with respect to Lebesgue such that when introducing a stochastic term the dimension must converge to 3, as observed at large timescales for both stochastic models.

The most interesting feature emerging for the average dimensions is that the largest value  $D(t, \tau)$  for both the additive and the multiplicative case is obtained for  $\tau$  of the order of the fundamental period (i.e., the Lyapunov time scale  $\tau_L \approx 1.12$  time units) of the dominating unstable periodic orbit of the deterministic system<sup>27,28</sup>; see ref.<sup>29</sup> for a discussion of how unstable periodic orbits are responsible for resonant behaviour in forced systems, and ref.<sup>30</sup> for evidence of the resonant response of the Lorenz-63 system.

Another interesting feature is the quantitative difference of  $\langle D(\tau, t) \rangle$  in the range of scales dominated by the stochastic contribution (below  $\tau_L$ ). While for the additive model the average dimensions converge, as expected, to 3 at the same timescales, thus suggesting that the dynamics at these scales behave as a forcing-like contribution. Conversely, by looking at the full system attractor, i.e., when considering the whole timescales, we are not able to identify any difference in the dimensions between the additive and the multiplicative model. Indeed, as expected, the average dimensions tend to saturate to those expected for the full dynamics when  $\tau \rightarrow \tau_{N_k}$ , being  $\langle D(\tau, t) \rangle = 2.05 \pm 0.02$  for the deterministic Lorenz-63 system and  $\langle D(\tau, t) \rangle = 2.98 \pm 0.04$  for its stochastic versions, because the invariant measure of an elliptic diffusion process has full dimension. The results obtained for the deterministic Lorenz-63 system are in agreement with previous findings by Alberti *et al.*<sup>14</sup>, while our corresponding findings on the stochastic models have been reported here for the first time.

### C. Instantaneous scale-dependent dimensions

As a second step of our analysis, which is also the main novelty introduced in this work, we investigate the behavior of the instantaneous scale-dependent dimension  $D(t, \tau)$  for the three different systems as reported in Fig. 5.

The deterministic Lorenz-63 system is characterized by instantaneous dimension values close to  $D_F \simeq 2.05$  at timescales larger than  $10^2$ , as expected due to the monofractal nature of the system with all generalized fractal dimensions  $D_q$  taking the same value for the full system. Conversely, larger values are found for the range  $\tau \in [10^0, 10^2]$ , the reason for which will be discussed below.

When a stochastic term is considered, we observe values  $D(t, \tau) \gtrsim 3$  at short timescales, extending towards larger scales when localized (in time) intermittent bursts in the trajectory take place. Typically, dimensions larger than 3 imply the presence of external forcing components, increasing the number of active degrees of freedom. By further inspecting the behavior of the trajectory in the phase-space at large scales (see Fig. 6) this excess over the topological dimension of the phase-space appears to be related to situations associated with approaches of the unstable fixed points in the

centers of each of the two lobes and subsequent fast escapes from the neighborhoods of those points along their unstable manifolds. This indicates that the increase in the number of active degrees of freedom at short timescales, not observed for the deterministic model, is related to the stochastic component. Indeed, the imposed noise term acts as an additional forcing to the autonomous dynamics. This means that the noise introduces additional degrees of freedom in the dynamics because it adds energy to the system: the attractor can deform through scales by increasing/decreasing its dimensions depending on the instantaneous concurrent effect between the noise forcing term and the intrinsic dynamics of the Lorenz-63 system. The main differences between the two stochastic versions emerge at short timescales ( $\tau < 10^0$ ), where larger dimensions are found for the multiplicative noise case as compared to the additive one. This could be explained by invoking the fact that in the multiplicative case, the amplitude of the stochastic term depends on the state variables of the system.

### D. A scale-dependent instantaneous view of the attractor

As a final step and to better highlight the scale-dependent instantaneous properties of the attractor, Fig. 6 reports three views of the attractor at different timescales color-coded with respect to the instantaneous dimensions. The shapes of the different scale-dependent attractors are obtained by summing up empirical modes in a certain range of scales as in Eq. (8). Due to the associated properties of MIMFs they have a zero-average envelope, thus they fluctuate around zero such that the attractors occupy only a small region (especially, at short scales) of the phase-space of scale-dependent fluctuations.

At large timescales (right panels in Fig. 6), roughly corresponding to 100 times the Lyapunov time of the deterministic Lorenz-63 system at the considered parameter values, the geometric shapes and the spatial distribution of dimensions across both stochastic attractors are qualitatively similar and clearly distinct from the chaotic attractor of the deterministic model. This means that we can visually distinguish between the chaotic and the two stochastic attractors, while a clear distinction cannot be made qualitatively and quantitatively between the two random invariant sets.

By further inspecting the spatial distributions of instantaneous dimensions of all three attractors, we clearly observe that larger dimensions are found at the edges of the attractors and close to the origin as compared to lower dimensions observed within the two lobes. While this feature has been also previously highlighted for the deterministic chaotic attractor (see, e.g., Faranda, Messori, and Yiou<sup>31</sup>), it is the first time that the spatial distribution of dimensions is inspected for both random attractors. In particular, our analysis reveals that forcing-like mechanisms leading to  $D(\tau, t) > 3$  are operating at the edges of the attractor and close to the unstable fixed point at the origin, reflecting the repelling nature of the fixed points of the Lorenz-63 system. However, the observation of  $D(\tau, t) > 3$  suggests

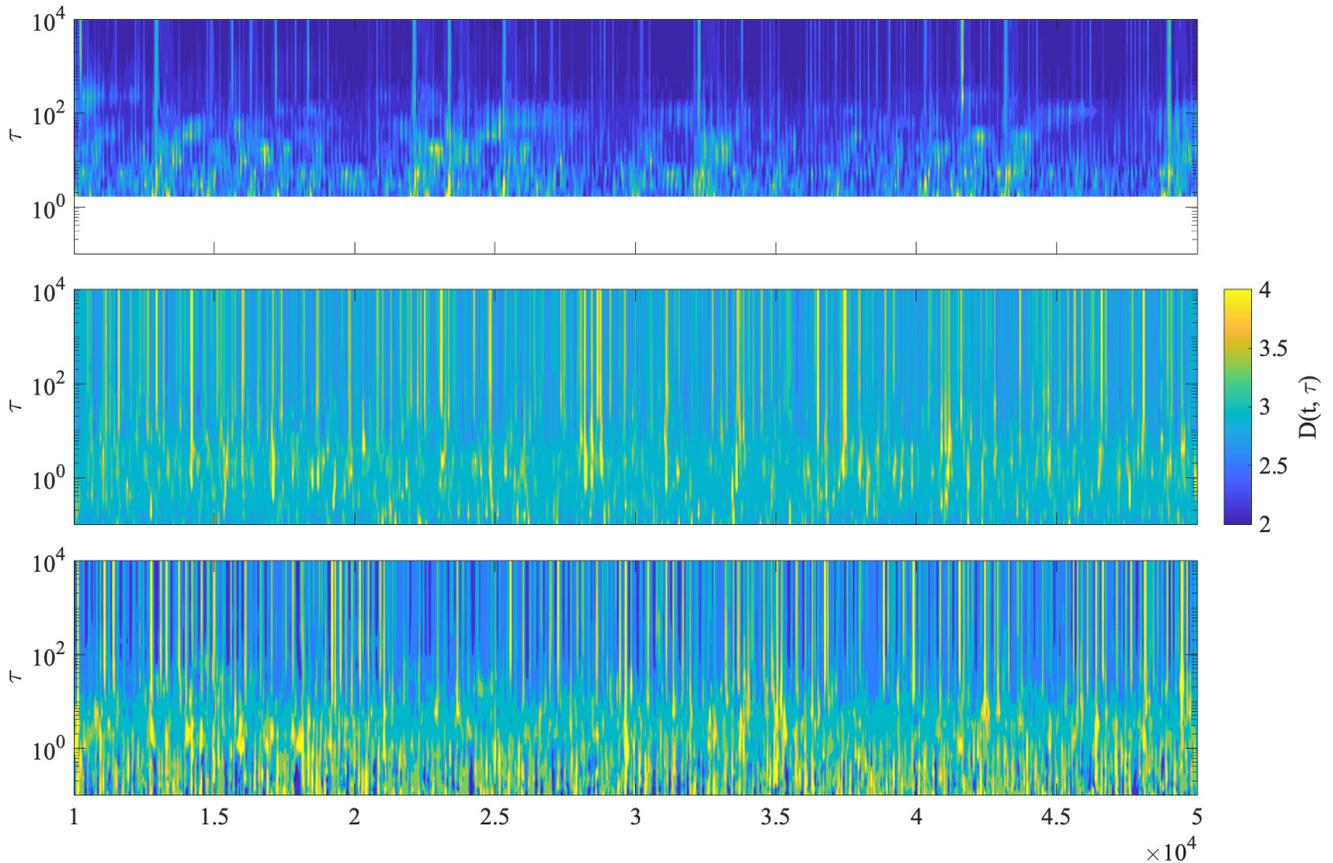


FIG. 5. Behavior of the instantaneous scale-dependent dimension  $D(t, \tau)$  for the deterministic Lorenz-63 system (top), its version with additive noise (middle), and the multiplicative noise case (bottom). The colormap for  $D(t, \tau)$  has been saturated between 2 and 4 for a better visualization. In all three cases, an excerpt comprising 40,000 time units is shown.

498 that the number of active degrees of freedom near the ori-521  
 499 gin is increased with respect to the deterministic model,522  
 500 likely related to the action of the noise term.523

501 To further investigate the last point on the role of the524  
 502 stochastic fluctuations in increasing the number of de-525  
 503 grees of freedom near the origin, we investigate the spatial526  
 504 distribution of dimensions across the trajectory at short527  
 505 timescales, below the Lyapunov time (left panels in Fig. 6)528  
 506 Clearly, we do not have any dynamical component below529  
 507 the Lyapunov time for the deterministic Lorenz-63 system,530  
 508 indicating that in both noisy systems, variability at those531  
 509 fast timescales is intimately related to the stochastic forcings532  
 510 However, a completely different spatial distributions of di-533  
 511 mensions across the trajectory is observed between the ad-534  
 512 ditive and the multiplicative case. While the former has 2835  
 513 more homogeneous spatial distribution of dimensions with36  
 514 the most probable value close to 3 and small fluctuations37  
 515 around it, the latter is characterized by a saddle point-like38  
 516 dynamics  $D(\tau, t) > 3$  in a ring-like configuration lying in39  
 517 the  $x-y$  plane and  $D(\tau, t) < 3$  elongated in the  $z$  direction.40  
 518 This is due to the different structure of the noise terms,41  
 519 being a "pure" noise term in the additive case, reflecting42  
 520 into  $D(\tau, t) = 3 + \varepsilon$ , with  $\varepsilon \ll 1$ , while acting as a "forcing"43

for the multiplicative one, providing  $D(\tau, t)$  values larger  
 than the system's dimension<sup>15,20</sup>.

When approaching the Lyapunov scale  $\tau_L$  (middle pan-  
 els in Fig. 6), a different spatial distribution of the dimen-  
 sions is again observed, together with a different coverage  
 of the available phase-space when comparing the chaotic  
 attractor with the two stochastic ones. The latter are char-  
 acterized by regions with low dimension surrounded by  
 higher-dimensional ones, markedly differing from the de-  
 terministic Lorenz-63 system. We hypothesize that the  
 regions with low instantaneous dimensions could indicate  
 the location of weakly repulsive low-period unstable peri-  
 odic orbits<sup>32,33</sup>.

Overall, our results indicate, for both the determinis-  
 tic and the two stochastic models, a clear different spatial  
 distribution of the dimensions across the phase-space at  
 large timescales. In particular, the two stochastic attrac-  
 tors are characterized by a region with dimensions larger  
 than the topological dimension of the system close to the  
 origin  $O = (0, 0, 0)$ , suggesting the existence of forcing-like  
 mechanisms altering the structure of this fixed point of  
 the deterministic model. Since the stochastic term mainly op-  
 erates at short timescales (below the Lyapunov scale  $\tau_L$ )

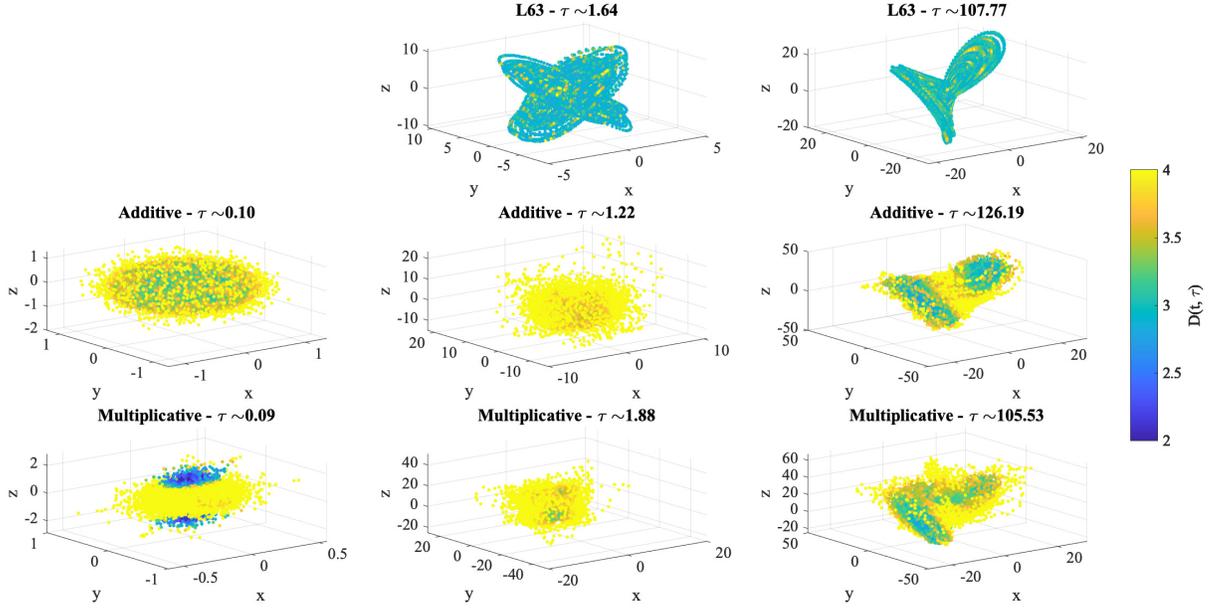


FIG. 6. Three views of the Lorenz attractor at different timescales color-coded with respect the instantaneous dimensions: (top) deterministic system, (middle) additive noise and (bottom) multiplicative noise.

544 the source of this difference must be searched in this range 572  
 545 of scales whose corresponding attractors for the additive 573  
 546 and the multiplicative models are structurally different, 574  
 547 both in terms of the geometric shape and in the spatial 575  
 548 distribution of the dimension values. In this regard, the 576  
 549 stochastic term is able to change the stability of the ori- 577  
 550 gin, revealing a new structure of attractors whose prop- 578  
 551 erties (i.e., fractal dimensions) evolve in time and across 579  
 552 scales. This difference disappears when reaching larger 580  
 553 and larger timescales due to the existence of an invariant 581  
 554 measure for the Lorenz-63 system that is absolutely con- 582  
 555 tinuous with respect to Lebesgue, such that when intro- 583  
 556 ducing a stochastic term the dimension must converge to 584  
 557 3, as observed at large timescales. 585

558 Finally, it is important to underline that in the deter- 586  
 559 ministic Lorenz-63 system with standard parameters (i.e. 587  
 560  $(s, r, b) = (10, 28, 8/3)$ ), the origin  $O$  is an unstable sad- 588  
 561 dle point whose structure is preserved when considering 589  
 562 a multiplicative stochastic term. Conversely, this nature 590  
 563 seems to be modified when considering an additive noise 591  
 564 altering the structural stability of the unstable point to- 592  
 565 wards a different nature. However, a linear stability anal- 593  
 566 ysis of the stochastic models is beyond the scope of the 594  
 567 present work and is left for a future devoted study. 595

## 568 V. CONCLUSIONS

569 We have presented a formalism to study the behavior of 600  
 570 chaotic or stochastic attractors as a function of the timescale, 601  
 571 indicating that when considering different timescales the con- 602

cept of a single universal attractor should be revised. Specif-  
 ically, using the famous Lorenz-63 system in its standard de-  
 terministic as well as two stochastically forced versions, we  
 have demonstrated that the attractor of this system is scale de-  
 pendent.

To reach this conclusion, we have extended an approach  
 recently introduced by Alberti *et al.*<sup>14</sup> to investigate the in-  
 stantaneous scale-dependent properties of attractors by combin-  
 ing concepts from time series decomposition methods and  
 extreme value theory applied to recurrences in phase space.  
 More specifically, we have used the Multivariate Empirical  
 Mode Decomposition (MEMD) to derive intrinsic compo-  
 nents of a given system at different timescales. Based on this  
 decomposition, we have estimated the instantaneous scale-  
 dependent dimensions of the system's attractor at different  
 scales. We have shown that a new structure of attractors,  
 whose properties evolve in time, space and scale, is discov-  
 ered by looking for fixed points and following their evolution  
 from small to large scale and vice versa. Thus, the geometric  
 structure of the attractor is gradually deformed and depends  
 on the scale at which we are investigating the respective sys-  
 tem.

The main novelty introduced in this study is a powerful  
 method to identify the existence of processes of different  
 origin by looking at the spatial distribution of fractal dimen-  
 sions across the full phase-space trajectories at differ-  
 ent timescales. Concerning the systems considered in this  
 work our formalism allowed us to clearly distinguish be-  
 tween a purely noise-like contribution at short timescales  
 for the additive noise model, being characterized by an er-  
 godic coverage of the available phase-space with dimen-

sions fluctuating around 3 (as expected), as compared to a more forcing-like contribution for the multiplicative noise model at the same timescales, where the dimensions are larger than 3 and differently distributed across the attractor. Conversely, by looking at the full system attractor, i.e., when considering the whole range of contributing timescales, we are not able to identify any clear difference both in the attractor shape and in the spatial pattern of the instantaneous (i.e., local in time and phase-space) dimensions, between the additive and the multiplicative model since the invariant measure of an elliptic diffusion process has full dimension. Thus, our method allows us to evidence where the difference between the two stochastic models resides only by looking at the spatial distribution (see Fig. 6) of instantaneous fractal dimensions at different timescales.

Our formalism can be easily modified by using any alternative time series decomposition technique (like wavelet decomposition, singular spectrum analysis, or others). Our choice of the MEMD has been motivated by its empirical and adaptive nature, reducing *a priori* constraints and possible artifacts of fixed-frequency/fixed-basis decomposition methods. Furthermore, the instantaneous nature (i.e., time-dependency) of the intrinsic components derived via the MEMD allows us to perform a more detailed investigation of the dynamical evolution (in time) of a system variable, better suited for evaluating instantaneous dynamical system metrics (as the dimension) than fixed-basis methods as Fourier transforms.

We are confident that the proposed formalism provides a novel way to investigate the underlying geometric (fractal) properties of physical systems at different scales during their time evolution. The concept of a scale-dependent attractor could tackle the problem of defining a more useful concept for the analysis of multiscale systems like in the case of the climate or for turbulence, which has largely remained unsolved despite numerous efforts reported in the last four decades. In a companion paper<sup>34</sup>, where our formalism has been applied to laboratory experiments on fluids, we observe the emergence of an intrinsic timescale, solely determined by nonlinear interactions, controlling the geometric and topological properties of phase-space trajectories.

In this first study, we focused only on the geometric properties of attractors, in order to show that the universal concept of attractor can be insufficient for fully describing multiscale systems, in presence or in absence of noise. The counterpart of our geometric view of each point in phase-space is the instantaneous, i.e., time behavior, of the scale-dependent dimension. Indeed, this interesting aspect can be used for further studying some crucial aspects of physical systems, e.g., bifurcations, tipping points, small vs. large-scale forcing and/or driving mechanisms<sup>19</sup>. The corresponding prospects call for further studies to investigate these aspects in more detail, which is beyond the scope of the present paper and will be the subject of future work.

## ACKNOWLEDGMENTS

We wish to acknowledge the support by ANR TILT grant agreement no. ANR-20-CE30-0035. VL acknowledges the support received from the Horizon 2020 project TiPES (grant no. 820970) and from the EPSRC project EP/T018178/1. RVD has received funding by the German Federal Ministry for Education and Research via the JPI Climate/JPI Oceans project ROADMAP (grant no. 01LP2002B). The authors thank two anonymous reviewers for fruitful suggestions.

- <sup>1</sup>E. N. Lorenz, “Deterministic Nonperiodic Flow.” *Journal of Atmospheric Sciences* **20**, 130–148 (1963).
- <sup>2</sup>D. Faranda, G. Messori, and S. Vannitsem, “Attractor dimension of time-averaged climate observables: insights from a low-order ocean-atmosphere model,” *Tellus A: Dynamic Meteorology and Oceanography* **71**, 1554413 (2019), <https://doi.org/10.1080/16000870.2018.1554413>.
- <sup>3</sup>C. Nicolis and G. Nicolis, “Is there a climatic attractor?” *Nature* **311**, 529–532 (1984).
- <sup>4</sup>M. Ghil and V. Lucarini, “The physics of climate variability and climate change,” *Reviews of Modern Physics* **92**, 035002 (2020), arXiv:1910.00583 [physics.ao-ph].
- <sup>5</sup>S. Nikolov, O. Wolkenhauer, and J. Vera, “Tumors as chaotic attractors,” *Mol. BioSyst.* **10**, 172–179 (2014).
- <sup>6</sup>W. M. Schaffer and M. Kot, “Do strange attractors govern ecological systems?” *BioScience* **35**, 342–350 (1985).
- <sup>7</sup>P. Grassberger and I. Procaccia, “Characterization of strange attractors,” *Phys. Rev. Lett.* **50**, 346–349 (1983).
- <sup>8</sup>A. N. Kolmogorov, “Entropy per unit time as a metric invariant of automorphism,” *Doklady of Russian Academy of Sciences* **124**, 754–755 (1959).
- <sup>9</sup>H. G. E. Hentschel and I. Procaccia, “The infinite number of generalized dimensions of fractals and strange attractors,” *Physica D Nonlinear Phenomena* **8**, 435–444 (1983).
- <sup>10</sup>J. L. Kaplan and J. A. Yorke, “Chaotic behavior of multidimensional differential equations,” in *Functional Differential Equations and Approximation of Fixed Points*, edited by H.-O. Peitgen and H.-O. Walthers (Springer Berlin Heidelberg, Berlin, Heidelberg, 1979) pp. 204–227.
- <sup>11</sup>R. Benzi, G. Paladin, A. Vulpiani, and G. Parisi, “On the multifractal nature of fully developed turbulence and chaotic systems,” *Journal of Physics A Mathematical General* **17**, 3521–3531 (1984).
- <sup>12</sup>W. M. Macek, R. Bruno, and G. Consolini, “Generalized dimensions for fluctuations in the solar wind,” *Physical Review E* **72**, 017202 (2005).
- <sup>13</sup>M. Cencini and F. Ginelli, “Lyapunov analysis: from dynamical systems theory to applications,” *Journal of Physics A Mathematical General* **46**, 250301 (2013).
- <sup>14</sup>T. Alberti, G. Consolini, P. D. Ditlevsen, R. V. Donner, and V. Quattrocchi, “Multiscale measures of phase-space trajectories,” *Chaos* **30**, 123116 (2020).
- <sup>15</sup>D. Faranda, V. Lucarini, G. Turchetti, and S. Vaienti, “Generalized Extreme Value Distribution Parameters as Dynamical Indicators of Stability,” *International Journal of Bifurcation and Chaos* **22**, 1250276 (2012), arXiv:1107.5972 [math.DS].
- <sup>16</sup>M. D. Chekroun, E. Simonnet, and M. Ghil, “Stochastic climate dynamics: Random attractors and time-dependent invariant measures,” *Physica D Nonlinear Phenomena* **240**, 1685–1700 (2011).
- <sup>17</sup>N. E. Huang, Z. Shen, S. R. Long, M. C. Wu, H. H. Shih, Q. Zheng, N. C. Yen, C. C. Tung, and H. H. Liu, “The empirical mode decomposition and the Hilbert spectrum for nonlinear and non-stationary time series analysis,” *Proceedings of the Royal Society of London Series A* **454**, 903–998 (1998).
- <sup>18</sup>N. Rehman and D. P. Mandic, “Multivariate empirical mode decomposition,” *Proceedings of the Royal Society of London Series A* **466**, 1291–1302 (2010).
- <sup>19</sup>T. Alberti, R. V. Donner, and S. Vannitsem, “Multiscale fractal dimension analysis of a reduced order model of coupled ocean-atmosphere dynamics,” *Earth System Dynamics* **12**, 837–855 (2021).
- <sup>20</sup>V. Lucarini, D. Faranda, G. Turchetti, and S. Vaienti, “Extreme value theory for singular measures,” *Chaos* **22**, 023135 (2012).
- <sup>21</sup>E. Ott, *Chaos in Dynamical Systems - 2nd Edition* (2002).

- 723 <sup>22</sup>P. Grassberger and I. Procaccia, “Measuring the strangeness of strange at-741  
724 tractors,” *Physica D Nonlinear Phenomena* **9**, 189–208 (1983). 742
- 725 <sup>23</sup>L. Schimansky-Geier, A. V. Tolstopjatenko, and W. Ebelin, “Noise induced743  
726 transitions due to external additive noise,” *Physics Letters A* **108**, 329–332744  
727 (1985). 745
- 728 <sup>24</sup>L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, “Stochastic reso-746  
729 nance,” *Reviews of Modern Physics* **70**, 223–287 (1998). 747
- 730 <sup>25</sup>J. P. Eckmann and D. Ruelle, “Ergodic theory of chaos and strange attrac-748  
731 tors,” *Rev. Mod. Phys.* **57**, 617–656 (1985). 749
- 732 <sup>26</sup>V. Lucarini, “Stochastic perturbations to dynamical systems: A response750  
733 theory approach,” *Journal of Statistical Physics* **146**, 774–786 (2012). 751
- 734 <sup>27</sup>B. Eckhardt and G. Ott, “Periodic orbit analysis of the lorenz attractor,”752  
735 *Zeitschrift für Physik B Condensed Matter* **93**, 259–266 (1994). 753
- 736 <sup>28</sup>C. C. Maiocchi, V. Lucarini, and A. Gritsun, “Decomposing the dynamics754  
737 of the lorenz 1963 model using unstable periodic orbits: Averages, tran-755  
738 sitions, and quasi-invariant sets,” *Chaos: An Interdisciplinary Journal of*756  
739 *Nonlinear Science* **32**, 033129 (2022), <https://doi.org/10.1063/5.0067673>.
- 740 <sup>29</sup>A. Gritsun and V. Lucarini, “Fluctuations, response, and resonances in a  
simple atmospheric model,” *Physica D: Nonlinear Phenomena* **349**, 62–76  
(2017).
- <sup>30</sup>V. Lucarini, “Evidence of dispersion relations for the nonlinear response of  
the lorenz 63 system,” *Journal of Statistical Physics* **134**, 381–400 (2009).
- <sup>31</sup>D. Faranda, G. Messori, and P. Yiou, “Dynamical proxies of North Atlantic  
predictability and extremes,” *Scientific Reports* **7**, 41278 (2017).
- <sup>32</sup>R. V. Donner, Y. Zou, J. F. Donges, N. Marwan, and J. Kurths, “Recur-  
rence networks—a novel paradigm for nonlinear time series analysis,” *New  
Journal of Physics* **12**, 033025 (2010).
- <sup>33</sup>R. V. Donner, M. Small, J. F. Donges, N. Marwan, Y. Zou, R. Xiang, and  
J. Kurths, “Recurrence-based time series analysis by means of complex net-  
work methods,” *International Journal of Bifurcation and Chaos* **21**, 1019–  
1046 (2011).
- <sup>34</sup>T. Alberti, F. Daviaud, R. V. Donner, B. Dubrulle, D. Faranda, and V. Lu-  
carini, “Chameleon attractors in turbulent flows,” *Chaos, Solitons & Frac-  
tals* (2022), 10.2139/ssrn.4191328.