Chaos propagation in mean field networks of FitzHugh-Nagumo neurons

Laetitia Colombani, Pierre Le Bris

▶ **To cite this version:**

Laetitia Colombani, Pierre Le Bris. Chaos propagation in mean field networks of FitzHugh-Nagumo neurons. 2022. hal-03706842

HAL Id: hal-03706842
https://hal.science/hal-03706842

Preprint submitted on 28 Jun 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Chaos propagation in mean field networks of FitzHugh-Nagumo neurons

Laetitia Colombani[*], Pierre Le Bris[†]

Abstract

In this article, we are interested in the behavior of a fully connected network of N neurons, where N tends to infinity. We assume that neurons follow the stochastic FitzHugh-Nagumo model, whose specificity is the non-linearity with a cubic term. We prove a result of uniform in time propagation of chaos of this model in a mean-field framework. We also exhibit explicit bounds. We use a coupling method initially suggested by A. Eberle [Ebe16], and recently extended by [DEGZ20], known as the *reflection coupling*. We simultaneously construct a solution of the N particles system and N independent copies of the non-linear McKean-Vlasov limit such that, by considering an appropriate semimetrics that takes into account the various possible behaviors of the processes, the two solutions tend to get closer together as N increases, uniformly in time. The reflection coupling allows us to deal with the non-convexity of the underlying potential in the dynamics of the quantities defining our network, and show independence at the limit for the system in mean field interaction with sufficiently small Lipschitz continuous interactions.

1 Introduction

1.1 Understanding the model

Understanding the brain activity is both a complex and important challenge in current research. Of course, interests are plentiful: characterizing brain functions, understanding structures and links between them and figuring out some phenomena - such as cyclic heartbeat. A way of modeling this activity is by considering a very large number of individual neurons with interactions. Since the number of neurons in a human brain is around 10^{11}, and even "small" parts of the brain are thus constituted of very large number of them, such a strategy can be considered coherent.

The main quantity we study is the membrane potential of the nerve cells: it can "easily" be observed and its modification characterizes a synapse (an interaction between neurons). Neurons regulate their electrical potential. In general, without interaction, the potential evolves with time but has quite small changes. Incoming potentials from other neurons are usually what make the neuron fire, i.e. send potential to other neurons. We will here focus on an homogeneous network of neurons and consider mean-field interactions. This way, each neuron will interact with every other one, as it can be the case in small regions of the brain. The parameters of the model will be considered the same for each neuron.

A classical model was introduced by Hodgkin and Huxley [HH52] using experimental data of the activity of the giant squid axon. It describes the ion exchanges K^+, Na^+ and Cl^- through the membrane and their effects on the potential. A simplification of this model is the FitzHugh-Nagumo model, which reduces the dimension: from four-dimensional model (for one neuron) with Hodgkin-Huxley equations, we obtain a two-dimensional model. It's a compromise between the biological accuracy and the mathematical simplicity.

[*]Institut Mathématiques de Toulouse - Université de Toulouse. Email: laetitia.colombani[AT]math.univ-toulouse.fr

[†]Laboratoire Jacques-Louis Lions - Sorbonne Université. Email : pierre.lebris[AT]sorbonne-universite.fr

The deterministic FitzHugh-Nagumo model for one neuron (or one particle) is given by the following equations:

$$\begin{cases} dX_t = (X_t - (X_t)^3 - C_t - \alpha)dt \\ dC_t = (\gamma X_t - C_t + \beta)dt, \end{cases}$$

where X is the potential membrane and C is a recovery variable, called the adaptation variable. The parameters γ and β are positive constants that determine the duration of an excitation and the position of the equilibrium point of this system. Finally $\alpha \in \mathbb{R}$ is the magnitude of a stimulus current (an entrance current in the system). Note that the variable C isn't a physical quantity, and is used to allow X to mimic the behavior of the potential. This variable C has linear dynamics and provides a slower negative feedback.

This deterministic model has been largely studied. In Chapter 7 of [Thi13], Thieullen describes the behavior of the solution of one deterministic FitzHugh-Nagumo system. She also extends the result in the case of a stochastic FitzHugh-Nagumo system: she considers a noise on the dynamics of X.

In fact, noise can be introduced in both equations to model different types of randomness : when the noise is in the first equation (dynamics of X) with a standard deviation $\sigma_X > 0$, it models a noisy presynaptic current. When it is in the second equation (dynamics of C) with a standard deviation $\sigma_C > 0$, it describes a noisy conductance dynamic (a noise in the chemical behavior). In general, noise in this model is additive. Various mathematical questions can be studied. Some authors choose to focus on the properties of the natural macroscopic limit of the model as $N \to \infty$ when it is clearly defined (see system (1.2)), when others work on properties of the particles system for fixed N. These models can be quite complicated to study mathematically. The main objectives are to characterize the behavior of these models when the number of neurons N tends to $+\infty$ in a mean-field limit, and to prove whether or not there exists an equilibrium, a stationary behavior, when t tends to $+\infty$. The question of the synchronization of neurons can also be studied, since it is a phenomenon observed in different contexts, such as the generation of respiratory rhythm or complex neurological functionalities.

In [TRW03], the authors work on the determination of firing time. They consider a stochastic FitzHugh-Nagumo model for one neuron, with Brownian noise on X, obtain approximation of firing times and compare them with numerical simulations.

Even if the majority of authors consider a noise only on one equation of the model, some study stochastic models with two noises. Berglund and Landon describe the behavior of the deterministic FitzHugh-Nagumo model for one neuron in [BL12], and consider the stochastic model, with noise on both equations, to work on the behavior of the interspike interval and the distribution of oscillations of the solution.

In [TBSST13], Tatchim Bemmo, Siewe Siewe and Tchawoua focus on a quite different stochastic model by considering additive noise η on the dynamics of X, and multiplicative noise ξ on the dynamics of C, both defined as sinusoidal function of correlated Brownian motions. They choose to avoid Gaussian noises since it is an unbounded noise. They also consider a deterministic and periodic entrance signal in the first equation. They observe abrupt transitions of the potential membrane X when the intensity of the noise is gradually changed.

In general, a lot of authors focus on a noise on only one variable. In [LS18], León and Samson consider a FitzHugh-Nagumo model with a noise on C but not on X, i.e. $\sigma_X = 0$, and study the properties of the equations for one neuron. In particular, they focus on hypoellipticity of the model, the existence and uniqueness of an invariant probability and a mixing property by establishing a link between the model and the class of stochastic damping Hamiltonian systems. They also consider neuronal modeling questions and study the generation of spikes in function of parameters of the model. On the contrary, the article [Uda19] focuses on stochastic FitzHugh-Nagumo model with noise in the dynamics of X, and $\sigma_C = 0$. They study one neuron in a periodically forced regime. This study relies on the theory of Markovian Random Dynamical System. The model is driven by a cosinus signal, and Uda studies the spike rate and compares it with the probability of two-points motion of membrane potential.

As said above, we consider mean-field interactions. These interactions are described by two functions K_X and K_C, applied on the difference between two states $((X_t^i, C_t^i) - (X_t^j, C_t^j))$. In particular, this type of interaction

models electrical synapses, which are a sort of synapses.

In their article [BFFT12], Baladron, Fasoli, Faugeras and Touboul study FitzHugh-Nagumo and Hodgkin-Huxley models with mean-field interaction, only on X. They consider more general interactions, not only applied on the difference between two states, modeling chemical synapses and electrical synapses. For the FitzHugh-Nagumo model, they consider a noise on X, and prove propagation of chaos, i.e. the convergence of the law of k neurons towards the law of k independent solutions of the mean-field equations. This article is completed and clarified by the work of Bossy, Faugeras and Talay in [BFO19].
Mischler, Quininao and Touboul consider a FitzHugh-Nagumo model in [MQT16], with a linear interaction on X, and a noise only on X, i.e. $\sigma_C = 0$ and $K_X(z) = \lambda x$. The drift on X is not exactly the same as in the model above, but remains similar as it is a cubic function of X. They work on the properties of a solution of the McKean-Vlasov evolution PDE associated to this model and obtain the uniqueness of a global weak solution. Furthermore, they prove that there exists at least one stationary solution, and when the interaction is small, the stationary solution is unique and exponentially stable. They also exhibit numerical results with open problems, like attractive periodic solution in time. In a similar framework, Luçon and Poquet study the macroscopic limit of this mean-field model in [LP21], and in particular the periodicity of such a system. They analyze the influence of both noise and interaction on the emergence of periodic behavior, and prove the existence of periodic solution, exponentially attractive, when the parameters satisfy some assumptions and the drift is "small" enough with respect to interaction and noise. Their approach relies on a slow-fast analysis and Floquet theory.

This model can be complexified, by considering non mean-field interaction. In particular, Bayrak, Hövel and Vuksanović work on a stochastic FitzHugh-Nagumo model with a network interaction in [BHV21]. Their type of interaction take into account a connectivity coefficient between two neurons, and a propagation velocity.

Other authors choose to complexify the model by considering stochastic FitzHugh-Nagumo with a spatial model. A second spatial derivative of X is added in the dynamics of X. Various authors study the behavior of such a model, and explore the notion of random attractors [LW10], [Li21], [LL19] and [LX21] .

Various authors also study numerical schemes for the interacting particles system in the stochastic model. In [RS22], the authors adapt Euler-Maruyama scheme to approximate solution of the particles system.

1.2 Framework and results

Combining noise and interaction, we work specifically on the following equations, for $1 \leq i \leq N$, where N is the number of neurons:

$$\begin{cases} dX_t^{i,N} = (X_t^{i,N} - (X_t^{i,N})^3 - C_t^{i,N} - \alpha)dt + \frac{1}{N}\sum_{j=1}^N K_X(Z_t^{i,N} - Z_t^{j,N}) + \sigma_X dB_t^{i,X} \\ dC_t^{i,N} = (\gamma X_t^{i,N} - C_t^{i,N} + \beta)dt + \frac{1}{N}\sum_{j=1}^N K_C(Z_t^{i,N} - Z_t^{j,N}) + \sigma_C dB_t^{i,C}, \end{cases} \tag{1.1}$$

where we denote by Z_t^i the couple (X_t^i, C_t^i) to simplify the notations.

We assume $(B_t^{i,X})_i$ and $(B_t^{i,C})_i$ to be independent Brownian motions. Here, we consider two Brownian noises B^X and B^C, one on each equation, and thus assume that each neuron has its own independent noises, and that there is no environmental (or shared) noise.

We also assume K_X and K_C to be Lipschitz continuous and respectively denote their Lipschitz constants by L_X and L_C.

The goal of this article is to describe the behavior of this network as the number N of neurons tends to infinity.

To describe its behavior, we consider the $\mathbb{R}^2$-valued process $(\bar{Z}_t)_{t\geq 0} = (\bar{X}_t, \bar{C}_t)_{t\geq 0}$ evolving according to the following non-linear stochastic differential equation of *McKean-Vlasov* type

$$\begin{cases} d\bar{X}_t = (\bar{X}_t - (\bar{X}_t)^3 - \bar{C}_t - \alpha)dt + K_X * \bar{\mu}_t(\bar{Z}_t)dt + \sigma_x d\bar{B}_t^X \\ d\bar{C}_t = (\gamma\bar{X}_t - \bar{C}_t + \beta)dt + K_C * \bar{\mu}_t(\bar{Z}_t)dt + \sigma_c d\bar{B}_t^C, \end{cases} \tag{1.2}$$

where $\bar{\mu}_t = \text{Law}(\bar{Z}_t)$ is the law at time t of the process $(\bar{X}_t, \bar{C}_t)$, and $*$ denotes the operation of convolution, i.e.

$$K_X * \bar{\mu}_t(u) = \int K_X(u-v)\mu_t(dv).$$

To some extent, (1.1) can be seen as an approximation of (1.2) in which the operation of convolution is applied to the empirical measure $\mu_{t,\text{emp}} = \frac{1}{N}\sum_{i=1}^N \delta_{Z_t^i}$, and what We wish to prove is that, indeed, the law μ_t^N of the network (1.1) converges in some sense to $\bar{\mu}_t^{\otimes N}$ (i.e the law of the solution of (1.2) tensorized N times) as N tends to infinity. This phenomenon has been stated under the name *propagation of chaos* -an idea motivated by M. Kac [Kac56]- as it amounts to saying that, as the number of particle increases in the system, two particles will become "more and more" independent, converging towards a tensorized law. The notion of "propagation" refers to the fact that proving such convergence at time 0 is sufficient to prove it at a later time t.

In order to prove the convergence of μ_t^N to $\bar{\mu}_t^{\otimes N}$, we follow the coupling method described in a recent work by one of the authors in [GLM21], the result of which cannot be applied directly here. This method has been put forward by Eberle, following earlier works by Lindvall and Rogers [LR86]. Before recalling the method, let us also mention the recent work [Sch22], which uses a coupling approach adapted to a well-chosen distance.

We consider $r_t^i = |\bar{X}_t^i - X_t^{i,N}| + \delta|\bar{C}_t^i - C_t^{i,N}|$ with $\delta > 0$, a constant not yet specified.

A natural distance between probability measures is the Wasserstein distance, linked to the theory of optimal transport. For μ and ν two probability measures on $\mathbb{R}^d$, we denote

$$\mathcal{W}_p(\mu,\nu) = \inf_{X\sim\mu,\, Y\sim\nu} \mathbb{E}\left(||X-Y||_p^p\right)^{1/p}, \tag{1.3}$$

where $||\cdot||_p$ denotes the usual L^p distance on $\mathbb{R}^d$. It is thus defined as the minimum over all possible choices of a pair (X,Y), such that X is distributed according to μ and Y according to ν, of the expectation of the distance between X and Y. The basic idea behind a coupling method is then that an upper bound on the Wasserstein distance between μ and ν is given by the construction of any pair of random variables distributed according to these probability measures. Thus, instead of considering the minimum over all possible coupling, we construct simultaneously two solutions of (1.1) and (1.2) that will tend to get closer together as the number of neurons increases.

Have $\left(\bar{X}_t^i, \bar{C}_t^i\right)$, for i between 1 and N, be N independent copies of a solution of (1.2) driven by some independent Brownian motions $(\bar{B}_t^{i,X})_{t\geqslant 0}$ and $(\bar{B}_t^{i,C})_{t\geqslant 0}$. A coupling of $\left(\bar{X}_t^i, \bar{C}_t^i\right)$ and $\left(X_t^{i,N}, C_t^{i,N}\right)$ then follows from a coupling of the Brownian motions B and $\bar{B}$.

The first natural choice, popularized by Sznitman [Szn91], is the *synchronous* coupling and consists in choosing $B = \bar{B}$. By doing so, when considering the time evolution of $\bar{Z}_t^i - Z_t^{i,N} = \left(\bar{X}_t^i - X_t^{i,N}, \bar{C}_t^i - C_t^{i,N}\right)$, the noise cancels out. The contraction of a distance between the processes can then only be induced by the deterministic drift, as in [BGM10], and this usually only holds under rather restrictive conditions (in particular the drift should be strongly convex). Nevertheless, in our case, the calculation of the evolution of $\bar{X}_t^i - X_t^{i,N}$ and $\bar{C}_t^i - C_t^{i,N}$ (see later) shows that there is still some deterministic contraction when $\bar{X}_t^i - X_t^{i,N} = 0$. We can therefore use a synchronous coupling in the vicinity of this subspace.

Outside of this subspace, we use the noise to get the processes closer together. In the direction orthogonal to the contracting space we consider $B = -\bar{B}$, as this maximizes the variance of the noise. This yields the *reflection* coupling. Notice however at this stage that, because of the symmetry of the noise, there is *a priori* no reason r_t^i should decrease rather than increase. This invites us to consider $f(r_t^i)$, with f a concave function, so that a random decrease has more effect than a random increase of the same value. We will define the function f later.

Finally we construct a Lyapunov function H to take into account the trend of each process to come back to some compact set of $\mathbb{R}^2$. We are then led to the study of a suitable distance between the two processes, which will be of the form $\rho_t := \frac{1}{N}\sum_{i=1}^N f(r_t^i)(1 + \epsilon H(\bar{Z}_t^i) + \epsilon H(Z_t^{i,N}))$, where $\epsilon > 0$. This quantity controls the usual L^1 and L^2 distances between the two systems, and is interesting as, when r_t^i is small, $f(r_t^i)$ tends to decrease either because of the deterministic drift or the reflection coupling, and when r_t^i is big, the Lyapunov functions H will

tend to decrease. We thus show that $\mathbb{E}\rho_t$ decays exponentially fast. This leads to several constraints on δ, ϵ and on the parameters involved in the definition of f, and we have to prove that it is possible to meet all these conditions simultaneously. In reality, the quantity ρ_t considered will be a slight twist of the one given above (see (2.23)) so as to take into account the non linearity of the process.

As explained, this method requires some noise in the direction orthogonal to the naturally contracting subspace. This means, in the description of the method above, that one should have $\sigma_X > 0$ (so that we can use a reflection coupling to bring $\bar{X}_t^i$ and $X_t^{i,N}$ closer together). In the case $\sigma_X = 0$ and $\sigma_C > 0$, a modification of the calculations is necessary. We describe this case and the resulting modifications of the computations in Appendix B.

Assumption 1. *Let $L_{X,max}$ and $L_{C,max}$ be two universal constants*

$$\exists L_X \in [0, L_{X,max}], \forall z, z' \in \mathbb{R}^2 \ \ |K_X(z) - K_X(z')| \leq L_X(\|z - z'\|_1)$$
$$\exists L_C \in [0, L_{C,max}], \forall z, z' \in \mathbb{R}^2 \ \ |K_C(z) - K_C(z')| \leq L_C(\|z - z'\|_1)$$
$$K_X(0,0) = 0 \text{ and } K_C(0,0) = 0$$

These constants $L_{X,max}$ and $L_{C,max}$ might seem at first glance off putting as they are not given. When we prove non uniform in time propagation of chaos, these constants can be chosen to be $+\infty$. When we prove uniform in time propagation of chaos however, $L_{X,max}$ and $L_{C,max}$ are *a priori* bounds : Theorem 2 below will be true for L_X and L_C sufficiently small. The condition $L_X \leq L_{X,max}$ and $L_C \leq L_{C,max}$ are therefore not restrictive conditions, and are useful in proving some parameters are independent of L_X and L_C. Lemma 2.2 below shows that one can for instance consider $L_{X,max} = 4$ and $L_{C,max} = \frac{1}{5}$.

Before any result on propagation of chaos, we prove that both systems (1.1) and (1.2) have well-defined solutions:

Proposition 1.1. *Let K_X and K_C satisfy Assumptions 1, with $L_{X,max} = \infty$ and $L_{C,max} = \infty$. There exists a unique strong solution for system* (1.1) *and a unique strong solution for system* (1.2)*.*

We denote $\mathcal{W}_1$ and $\mathcal{W}_2$ the usual L^1 and L^2 Wasserstein distances defined in (1.3).

Theorem 1. *[Non uniform in time propagation of chaos] Let K_X and K_C satisfy Assumptions 1, with $L_{X,max} = \infty$ and $L_{C,max} = \infty$. There exist explicit $C_1, C_2 > 0$, such that for all probability measures μ_0 on $\mathbb{R}^2$ with finite second moment,*

$$\mathcal{W}_1\left(\mu_t^{k,N}, \bar{\mu}_t^{\otimes k}\right) \leq C_1 e^{C_2 t} \frac{k}{\sqrt{N}},$$

for all $k \in \mathbb{N}$, where $\mu_t^{k,N}$ is the marginal distribution at time t of the first k neurons $\left((X_t^1, C_t^1), ..., (X_t^k, C_t^k)\right)$ of an N particles system (1.1) *with initial distribution $(\mu_0)^{\otimes N}$, while $\bar{\mu}_t$ is a solution of* (1.2) *with initial distribution μ_0.*

This first theorem is in accordance with the theorem from [KNRS20], and explicits the dependency in t. Since its proof is rather quick, and provides a good entry point into coupling methods, we give it in Subsection 1.4.

Our main result consist in removing the time dependency in the previous upperbound. This uniform in time propagation of chaos however requires stronger assumptions on the interaction kernels.

Theorem 2. *[Uniform in time propagation of chaos] Let $\mathcal{C}_{init,exp} > 0$ and $\tilde{a} > 0$. There is an explicit $c^K > 0$ such that, for all K_X and K_C satisfying Assumptions 1 with $L_X, L_C < c^K$, there exist explicit $B_1, B_2 > 0$, such that for all probability measures μ_0 on $\mathbb{R}^2$ satisfying $\mathbb{E}_{\mu_0}\left(e^{\tilde{a}(|X|+|C|)}\right) \leq \mathcal{C}_{init,exp}$,*

$$\mathcal{W}_1\left(\mu_t^{k,N}, \bar{\mu}_t^{\otimes k}\right) \leq B_1 \frac{k}{\sqrt{N}}, \qquad \mathcal{W}_2^2\left(\mu_t^{k,N}, \bar{\mu}_t^{\otimes k}\right) \leq B_2 \frac{k}{\sqrt{N}},$$

for all $k \in \mathbb{N}$, where $\mu_t^{k,N}$ is the marginal distribution at time t of the first k neurons $\left((X_t^1, C_t^1), ..., (X_t^k, C_t^k)\right)$ of an N particles system (1.1) *with initial distribution $(\mu_0)^{\otimes N}$, while $\bar{\mu}_t$ is a solution of* (1.2) *with initial distribution μ_0.*

As previously mentioned, this work follows the method described in [GLM21]. Beyond the result of uniform in time propagation of chaos for the FitzHugh-Nagumo model, which is in itself an interesting result, the present work is also a testament to the robustness of the coupling method.

The reader will find **an index containing all notations**, constants and parameters for reference at the end of the document.

1.3 Existence of solutions

First of all, we prove Proposition 1.1, i.e existence of strong solutions of systems (1.1) and (1.2), under Assumption 1, and with begin with (1.1).
Let's denote, for $K \in \mathbb{R}^+$,

$$g_K(x) = \begin{cases} -K^3 & \text{if } x < -K \\ x^3 & \text{if } x \in [-K, K] \\ K^3 & \text{if } x > K. \end{cases}$$

g_K is locally Lipschitz and is bounded.

Thus, it's well known (see Chapter 3 [IW89]) that the following system (under Assumption 1)

$$\begin{cases} dX_t^{i,N} = (X_t^{i,N} - g_K(X_t^{i,N}) - C_t^{i,N} - \alpha)dt + \frac{1}{N}\sum_{j=1}^N K_X(Z_t^{i,N} - Z_t^{j,N}) + \sigma_X dB_t^{i,X} \\ dC_t^{i,N} = (\gamma X_t^{i,N} - C_t^{i,N} + \beta)dt + \frac{1}{N}\sum_{j=1}^N K_C(Z_t^{i,N} - Z_t^{j,N}) + \sigma_C dB_t^{i,C}, \end{cases} \tag{1.4}$$

for $1 \leq i \leq N$, have strong and unique solution.

In consequence, for a fixed $K \in \mathbb{R}^+$, there exists strong solution of system (1.1) until time $T_K = \sup\{t, \forall i, \forall s \leq t, X_t^i \leq K \text{ and } C_t^i \leq K\}$, and the solution coincide with the solution of the system with g_K.

We have the following Proposition:

Proposition 1.2. *If, for each $i \leq N$, $\mathbb{E}(|X_0^{i,N}|^2) < +\infty$ and $\mathbb{E}(|C_0^{i,N}|^2) < +\infty$, then for all $t \geq 0$ there exists $\mathcal{C}_t < \infty$ such that, for each $i \leq N$:*

$$\mathbb{E}\left(|X_t^{i,N}|^2 + |C_t^{i,N}|^2\right) \leq \mathcal{C}_t. \tag{1.5}$$

The proof relies on the Lyapunov function defined in the next Section, and is given in Appendix A.2.
Then, by denoting T_∞ the explosion time of a solution of system (1.1)

$$T_\infty = \inf\{t, \exists i, \forall A > 0, \exists \varepsilon > 0, \forall s \in (t-\varepsilon, t), X_s^{i,N} > A \text{ or } C_s^{i,N} > A\}$$

we deduce $\forall t \in \mathbb{R}^+, \mathbb{P}(T_\infty \leq t) = 0$ and $\mathbb{P}(\bar{T}_\infty \leq t) = 0$. Eventually, there exists unique and strong solution for system (1.1).

The existence and uniqueness of a solution of (1.2) is known from the Theorem 3.3 from [RST19]. We only have to prove that the Assumptions 3.2 [RST19] are verified. We define, for all $t \in \mathbb{R}^+$, $z = (x, c) \in \mathbb{R}^2$ and for all probability distribution ν with a finite variance:

$$b(t, z, \nu) = \begin{pmatrix} x - x^3 - c - \alpha + K_X * \nu(z) \\ \gamma x - c + \beta + K_C * \nu(z) \end{pmatrix} \quad \text{and} \quad \sigma(t, z, \nu) = \begin{pmatrix} \sigma_X \\ \sigma_C \end{pmatrix}.$$

σ is a constant function, so it clearly satisfies the various conditions.

For $t \in \mathbb{R}^+$, z, z' in $\mathbb{R}^2$, and ν a probability measure:

$$\langle z - z', b(t, z, \nu) - b(t, z', \nu)\rangle = (x - x')\left((x - x') - (x^3 - x'^3) - (c - c') + K_X * \nu(z) - K_X * \nu(z')\right)$$

$$
\begin{aligned}
&+ (c-c')\left(\gamma(x-x') - (c-c') + K_C * \nu(z) - K_C * \nu(z')\right) \\
=&(x-x')^2 - (x-x')^2(x^2+xx'+x'^2) + (\gamma-1)(c-c')(x-x') - (c-c')^2 \\
&+ (K_X * \nu(z) - K_X * \nu(z')) + (K_C * \nu(z) - K_C * \nu(z')).
\end{aligned}
$$

Since $x^2 + xx' + x'^2 \geq 0$, the second term is non-positive. K_X and K_C are Lipschitz function, so the last line is clearly bounded by $\|z - z'\|_2^2$ up to a multiplicative constant. Then, there exists a constant L such that

$$\langle z - z', b(t,z,\nu) - b(t,z',\nu)\rangle \leq L\|z - z'\|_2^2.$$

Since K_X and K_C are Lipschitz function, we also have, for all probability distribution ν and ν' with a finite variance,

$$\|b(t,z,\nu) - b(t,z,\nu')\|_2 \leq L\mathcal{W}_2(\nu,\nu').$$

Eventually, since b is Locally Lipschitz with polynomial growth, each Assumption is satisfied and Theorem 3.3 [RST19] can be applied. Note that we could also apply Proposition 2.19 from [LS14] : assumptions are the same, and it gives a result for interaction depending on a spatial position.

To complete the Proposition 1.2, we also give the following

Proposition 1.3. *If $\mathbb{E}(|\bar{X}_0|^2) < +\infty$ and $\mathbb{E}(|\bar{C}_0|^2) < +\infty$, then there exists $C_{0,1}$ and $C_{0,2}$ such that:*

$$\mathbb{E}\left(|\bar{X}_t|^2 + |\bar{C}_t|^2\right) \leq C_{0,1}e^{C_{0,2}t}. \tag{1.6}$$

The proof is very similar with Proposition 1.2 and is in Appendix A.2.

1.4 Quick result : non uniform in time propagation of chaos

We start by proving Theorem 1, a non uniform in time propagation of chaos, as it highlights the basic strategy behind a coupling argument. Some of the following expressions will be used in the proof of Theorem 2, in Section 3.

We consider a synchronous coupling between $(Z_t^{i,N})_i$ and $(\bar{Z}_t^i)_i$, i.e. $\tilde{B}_t^{i,X} = B_t^{i,X}$ and $\tilde{B}_t^{i,C} = B_t^{i,C}$. We have

$$\begin{cases} dX_t^{i,N} = (X_t^{i,N} - (X_t^{i,N})^3 - C_t^{i,N} - \alpha)dt + \frac{1}{N}\sum_{j=1}^N K_X(Z_t^i - Z_t^j)dt + \sigma_x dB_t^{i,X} \\ dC_t^{i,N} = (\gamma X_t^{i,N} - C_t^{i,N} + \beta)dt + \frac{1}{N}\sum_{j=1}^N K_C(Z_t^i - Z_t^j)dt + \sigma_c dB_t^{i,C} \end{cases}$$

and

$$\begin{cases} d\bar{X}_t^i = (\bar{X}_t^i - (\bar{X}_t^i)^3 - \bar{C}_t^i - \alpha)dt + K_X * \bar{\mu}_t(\bar{Z}_t^i)dt + \sigma_x dB_t^{i,X} \\ d\bar{C}_t^i = (\gamma\bar{X}_t^i - \bar{C}_t^i + \beta)dt + K_C * \bar{\mu}_t(\bar{Z}_t^i)dt + \sigma_c dB_t^{i,C}, \end{cases}$$

with $\bar{\mu}_t$ the law of $\bar{Z}_t^1$. The method is the following :

- we compute the time evolution of $\mathbb{E}r_t^i = \mathbb{E}\left(|X_t^{i,N} - \bar{X}_t^i| + |C_t^{i,N} - \bar{C}_t^i|\right)$ using Ito's formula,
- we control the difference between the drifts $\frac{1}{N}\sum_{j\neq i} K(\bar{Z}_t^i - \bar{Z}_t^j)$ and $K * \bar{\mu}_t(\bar{Z}_t^i)$ using some form of law of large number. This is where the convergence rate $\sqrt{N}$ appears,
- and we conclude using Gronwall's lemma.

Time evolution : We have,

$$d(X_t^{i,N} - \bar{X}_t^i) = \left((X_t^{i,N} - \bar{X}_t^i) - \left((X_t^{i,N})^3 - (\bar{X}_t^i)^3\right) - (C_t^{i,N} - \bar{C}_t^i) + \frac{1}{N}\sum_{j=1}^N K_X(Z_t^i - Z_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right)dt.$$

We denote

$$\text{sign}(x) = \begin{cases} \frac{x}{|x|} \text{ if } x \neq 0, \\ 0 \text{ otherwise,} \end{cases}$$

and obtain, using Ito's formula,

$$\begin{aligned} d|X_t^{i,N} - \bar{X}_t^i| = \Big(&\text{sign}(X_t^{i,N} - \bar{X}_t^i)(X_t^{i,N} - \bar{X}_t^i) - \text{sign}(X_t^{i,N} - \bar{X}_t^i)\left((X_t^{i,N})^3 - (\bar{X}_t^i)^3\right) \\ &-\text{sign}(X_t^{i,N} - \bar{X}_t^i)(C_t^{i,N} - \bar{C}_t^i) + \text{sign}(X_t^{i,N} - \bar{X}_t^i)\frac{1}{N}\sum_{j=1}^N K_X(Z_t^i - Z_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\Big)\, dt \\ \leq &\left(|X_t^{i,N} - \bar{X}_t^i| - \left|(X_t^{i,N})^3 - (\bar{X}_t^i)^3\right| + \left|C_t^{i,N} - \bar{C}_t^i\right| + \left|\frac{1}{N}\sum_{j=1}^N K_X(Z_t^i - Z_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right|\right) dt. \end{aligned} \tag{1.7}$$

Similarly,

$$d(C_t^{i,N} - \bar{C}_t^i) = \left(\gamma(X_t^{i,N} - \bar{X}_t^i) - (C_t^{i,N} - \bar{C}_t^i) + \frac{1}{N}\sum_{j=1}^N K_C(Z_t^i - Z_t^j) - K_C * \bar{\mu}_t(\bar{Z}_t^i)\right) dt,$$

and we obtain

$$d|C_t^{i,N} - \bar{C}_t^i| \leq \left(\gamma\left|X_t^{i,N} - \bar{X}_t^i\right| - |C_t^{i,N} - \bar{C}_t^i| + \left|\frac{1}{N}\sum_{j=1}^N K_C(Z_t^i - Z_t^j) - K_C * \bar{\mu}_t(\bar{Z}_t^i)\right|\right) dt. \tag{1.8}$$

Thus, denoting $r_t^i = |X_t^{i,N} - \bar{X}_t^i| + |C_t^{i,N} - \bar{C}_t^i|$

$$\begin{aligned} dr_t^i \leq \Big(&(1+\gamma)|X_t^{i,N} - \bar{X}_t^i| - \left|(X_t^{i,N})^3 - (\bar{X}_t^i)^3\right| \\ &+ \left|\frac{1}{N}\sum_{j=1}^N K_X(Z_t^i - Z_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right| + \left|\frac{1}{N}\sum_{j=1}^N K_C(Z_t^i - Z_t^j) - K_C * \bar{\mu}_t(\bar{Z}_t^i)\right|\Big)\, dt. \end{aligned}$$

Difference of the drifts : Let us now consider these last two terms

$$\begin{aligned} &\left|\frac{1}{N}\sum_{j=1}^N K_X(Z_t^i - Z_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right| \\ &\qquad \leq \left|\frac{1}{N}\sum_{j=1}^N K_X(Z_t^i - Z_t^j) - \frac{1}{N}\sum_{j=1}^N K_X(\bar{Z}_t^i - \bar{Z}_t^j)\right| + \left|\frac{1}{N}\sum_{j=1}^N K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right|. \end{aligned}$$

The first sum can be decomposed, using Assumption 1,

$$\frac{1}{N}\left|\sum_{j=1}^N K_X(Z_t^i - Z_t^j) - K_X(\bar{Z}_t^i - \bar{Z}_t^j)\right| \leq \frac{L_X}{N}\sum_{j=1}^N \|Z_t^i - Z_t^j - (\bar{Z}_t^i - \bar{Z}_t^j)\|_1$$

$$\leq \frac{L_X}{N}\sum_{j=1}^{N}\left(\|Z_t^i-\bar{Z}_t^i\|_1+\|Z_t^j-\bar{Z}_t^j\|_1\right)$$
$$\leq L_X r_t^i+\frac{L_X}{N}\sum_{j=1}^{N} r_t^j.$$

Similarly, we obtain

$$\left|\frac{1}{N}\sum_{j=1}^{N}K_C(Z_t^i-Z_t^j)-K_C*\bar{\mu}_t(\bar{Z}_t^i)\right| \leq L_C r_t^i+\frac{L_C}{N}\sum_{j=1}^{N} r_t^j+\left|\frac{1}{N}\sum_{j=1}^{N}K_C(\bar{Z}_t^i-\bar{Z}_t^j)-K_C*\bar{\mu}_t(\bar{Z}_t^i)\right|.$$

Hence, we get

$$\begin{aligned} dr_t^i \leq & \left((1+\gamma)|X_t^{i,N}-\bar{X}_t^i|-\left|(X_t^{i,N})^3-(\bar{X}_t^i)^3\right|+(L_X+L_C)\left(r_t^i+\frac{1}{N}\sum_{j=1}^{N} r_t^j\right)\right.\\ & \left.+\left|\frac{1}{N}\sum_{j=1}^{N}K_X(\bar{Z}_t^i-\bar{Z}_t^j)-K_X*\bar{\mu}_t(\bar{Z}_t^i)\right|+\left|\frac{1}{N}\sum_{j=1}^{N}K_C(\bar{Z}_t^i-\bar{Z}_t^j)-K_C*\bar{\mu}_t(\bar{Z}_t^i)\right|\right)dt\\ \leq & \left((1+\gamma)r_t^i+(L_X+L_C)\left(r_t^i+\frac{1}{N}\sum_{j=1}^{N} r_t^j\right)\right.\\ & \left.+\left|\frac{1}{N}\sum_{j=1}^{N}K_X(\bar{Z}_t^i-\bar{Z}_t^j)-K_X*\bar{\mu}_t(\bar{Z}_t^i)\right|+\left|\frac{1}{N}\sum_{j=1}^{N}K_C(\bar{Z}_t^i-\bar{Z}_t^j)-K_C*\bar{\mu}_t(\bar{Z}_t^i)\right|\right)dt. \end{aligned}$$

By considering the expectation, since $\mathbb{E}(r_t^j)=\mathbb{E}(r_t^i)$ for each j, by exchangeability of the particles, we have

$$\begin{aligned} d\mathbb{E}(r_t^i) \leq & \left((1+\gamma+2L_X+2L_C)\mathbb{E}(r_t^i)+\mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^{N}K_X(\bar{Z}_t^i-\bar{Z}_t^j)-K_X*\bar{\mu}_t(\bar{Z}_t^i)\right|\right]\right.\\ & \left.+\mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^{N}K_C(\bar{Z}_t^i-\bar{Z}_t^j)-K_C*\bar{\mu}_t(\bar{Z}_t^i)\right|\right]\right)dt. \end{aligned}$$

Now, we bound the interaction part. We begin with K_X. By Cauchy-Schwarz, we can write

$$\mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^{N}K_X(\bar{Z}_t^i-\bar{Z}_t^j)-K_X*\bar{\mu}_t(\bar{Z}_t^i)\right|\right] \leq \mathbb{E}\left(\left|\frac{1}{N}\sum_{j=1}^{N}K_X(\bar{Z}_t^i-\bar{Z}_t^j)-K_X*\bar{\mu}_t(\bar{Z}_t^i)\right|^2\right)^{1/2}$$

We notice that $(\bar{Z}_t^j)_j$ are i.i.d with law $\bar{\mu}_t$. Let's denote $\bar{Z}_t$ a generic random variable of law $\bar{\mu}_t$ independent of $\bar{Z}_t^i$. What is more, $K_X*\bar{\mu}_t(\bar{Z}_t^i)=\int K_X(\bar{Z}_t^i-z)\bar{\mu}_t(dz)=\mathbb{E}[K_X(\bar{Z}_t^i-\bar{Z}_t)|\bar{Z}_t^i]$. Hence

$$\begin{aligned} \mathbb{E}\left(\mathbb{E}\left(\left|\frac{1}{N-1}\sum_{j\neq i}K_X(\bar{Z}_t^i-\bar{Z}_t^j)-K_X*\bar{\mu}_t(\bar{Z}_t^i)\right|^2\Big|\bar{Z}_t^i\right)\right) &= \mathbb{E}\left(\text{Var}\left(\frac{1}{N-1}\sum_{j\neq i}K_X(\bar{Z}_t^i-\bar{Z}_t^j)\Big|\bar{Z}_t^i\right)\right)\\ &= \frac{1}{N-1}\mathbb{E}\left(\text{Var}\left(K_X(\bar{Z}_t^i-\bar{Z}_t)\Big|\bar{Z}_t^i\right)\right) \end{aligned}$$

$$\leq \frac{L_X^2}{N-1}\mathbb{E}\left(\text{Var}\left(\|\bar{Z}_t^i - \bar{Z}_t\|_1 \middle| \bar{Z}_t^i\right)\right).$$

Since

$$\mathbb{E}\left[\text{Var}\left(\|\bar{Z}_t^i - \bar{Z}_t\|_1 \middle| \bar{Z}_t^i\right)\right] \leq \mathbb{E}\left[\mathbb{E}\left(\|\bar{Z}_t^i - \bar{Z}_t\|_1^2 \middle| \bar{Z}_t^i\right)\right] \leq \mathbb{E}\left[\mathbb{E}\left(2\|\bar{Z}_t^i\|_1^2 + 2\|\bar{Z}_t\|_1^2 \middle| \bar{Z}_t^i\right)\right] \leq 4\mathbb{E}(\|\bar{Z}_t\|_1^2),$$

we obtain

$$\mathbb{E}\left(\mathbb{E}\left(\left|\frac{1}{N-1}\sum_{j\neq i} K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right|^2 \middle| \bar{Z}_t^i\right)\right) \leq \frac{4L_X^2}{N-1}\mathbb{E}(\|\bar{Z}_t\|_1^2).$$

We now want to control $\mathbb{E}\left(\left|\frac{1}{N}\sum_{j=1}^N K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right|^2\right)$. We decompose it with

$$\begin{aligned}
&\mathbb{E}\left(\left|\frac{1}{N}\sum_{j=1}^N K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right|^2\right) \\
=&\mathbb{E}\left(\left|\frac{N-1}{N}\frac{1}{N-1}\sum_{j=1}^N K_X(\bar{Z}_t^i - \bar{Z}_t^j) - \left(\frac{N-1}{N} + \frac{1}{N}\right) K_X * \bar{\mu}_t(\bar{Z}_t^i)\right|^2\right) \\
\leq& 2\left(\frac{N-1}{N}\right)^2 \mathbb{E}\left(\left|\frac{1}{N-1}\sum_{j=1}^N K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right|^2\right) + \frac{2}{N^2}\mathbb{E}\left(|K_X * \bar{\mu}_t(\bar{Z}_t^i)|^2\right).
\end{aligned}$$

Since

$$\mathbb{E}\left(|K_X * \bar{\mu}_t(\bar{Z}_t^i)|^2\right) = \mathbb{E}\left(\left|\mathbb{E}\left(K_X(\bar{Z}_t^i - \bar{Z}_t)|\bar{Z}_t^i\right)\right|^2\right) \leq L_X^2\mathbb{E}\left(\mathbb{E}\left(\|\bar{Z}_t^i - \bar{Z}_t\|_1^2|\bar{Z}_t^i\right)\right) \leq 4L_X^2\mathbb{E}(\|\bar{Z}_t\|_1^2),$$

we obtain

$$\mathbb{E}\left(\left|\frac{1}{N}\sum_{j=1}^N K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right|^2\right) \leq \left(\frac{N-1}{N}\right)^2 \frac{4L_X^2}{N-1}\mathbb{E}(\|\bar{Z}_t\|_1^2) + \frac{4L_X^2}{N^2}\mathbb{E}(\|\bar{Z}_t\|_1^2) \leq \frac{8L_X^2}{N}\mathbb{E}(\|\bar{Z}_t\|_1^2), \tag{1.9}$$

and finally

$$\mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right|\right] \leq \left(\frac{8L_X^2}{N}\mathbb{E}(\|\bar{Z}_t\|_1^2)\right)^{1/2}.$$

Similarly, we have

$$\mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N K_C(\bar{Z}_t^i - \bar{Z}_t^j) - K_C * \bar{\mu}_t(\bar{Z}_t^i)\right|\right] \leq \left(\frac{8L_C^2}{N}\mathbb{E}(\|\bar{Z}_t\|_1^2)\right)^{1/2}.$$

Finally,

$$d\mathbb{E}(r_t^i) \leq \left((1+\gamma+2L_X+2L_C)\mathbb{E}(r_t^i) + \sqrt{8L_X^2+8L_C^2}\left(\frac{1}{N}\mathbb{E}(\|\bar{Z}_t\|_1^2)\right)^{1/2}\right)dt.$$

Then using Proposition 1.3, we obtain

$$d\mathbb{E}(r_t^i) \leq \left((1+\gamma+2L_X+2L_C)\mathbb{E}(r_t^i) + \frac{\sqrt{8L_X^2+8L_C^2}\sqrt{2C_{0,1}}}{\sqrt{N}} e^{\frac{1}{2}C_{0,2}t} \right) dt$$

Conclusion : We have thus obtained

$$\begin{aligned} &d\left(\mathbb{E}(r_t^i) + \sqrt{\frac{16(L_X^2+L_C^2)C_{0,1}}{N}} \frac{1}{1+\gamma+2L_X+2L_C-\frac{C_{0,2}}{2}} e^{\frac{1}{2}C_{0,2}t} \right) \\ &\quad \leq (1+\gamma+2L_X+2L_C)\left(\mathbb{E}(r_t^i)\sqrt{\frac{16(L_X^2+L_C^2)C_{0,1}}{N}} \frac{1}{1+\gamma+2L_X+2L_C-\frac{C_{0,2}}{2}} e^{\frac{1}{2}C_{0,2}t} \right) dt \end{aligned}$$

and Gronwall's lemma yields

$$\begin{aligned} &\mathbb{E}(r_t^i) + \sqrt{\frac{16(L_X^2+L_C^2)C_{0,1}}{N}} \frac{1}{1+\gamma+2L_X+2L_C-\frac{C_{0,2}}{2}} e^{\frac{1}{2}C_{0,2}t} \\ &\quad \leq e^{(1+\gamma+2L_X+2L_C)t}\left[\mathbb{E}(r_0^i) + \sqrt{\frac{16(L_X^2+L_C^2)C_{0,1}}{N}} \frac{1}{1+\gamma+2L_X+2L_C-\frac{C_{0,2}}{2}} \right], \end{aligned}$$

thus

$$\mathbb{E}(r_t^i) \leq C_1 e^{C_2 t} \frac{1}{\sqrt{N}}.$$

Let μ_0 a measure on $\mathbb{R}^2$, $\mu_t^{k,N}$ the marginal distribution at time t of the first k neurons $\left(Z_t^1, \ldots, Z_t^k\right)$ of an N particles system (1.1) with initial distribution $(\mu_0)^{\otimes N}$, and $\bar{\mu}_t$ is a solution of (1.2) with initial distribution μ_0.
We obtain for Wasserstein 1 distance

$$\begin{aligned} \mathcal{W}_1(\mu_t^{k,N}, \bar{\mu}_t^{\otimes k}) =& \inf\left\{ \mathbb{E}[\|Z^{(k)} - \bar{Z}^{(k)}\|_1], \mathbb{P}_{Z^{(k)}} = \mu_t^{k,N}, \mathbb{P}_{\bar{Z}^{(k)}} = \bar{\mu}_t^{\otimes k} \right\} \\ \leq& \inf\left\{ \mathbb{E}\left[\sum_{i=1}^k r_t^i \right], \mathbb{P}_{(Z_t^{i,N})_i} = \mu_t^{k,N}, \mathbb{P}_{(\bar{Z}_t^i)_i} = \bar{\mu}_t^{\otimes k} \right\} \\ \leq& k\mathbb{E}(r_t^1) \\ \leq& C_1 e^{C_2 t} \frac{k}{\sqrt{N}}. \end{aligned}$$

Hence Theorem 1.

2 Preliminaries

2.1 Notations

For $h : \mathbb{R}^{2N} \to \mathbb{R}$, for all $(z_i)_{1\leq i\leq N} = (x_i, c_i)_{1\leq i\leq N} \in \mathbb{R}^{2N}$, the generator of (1.1) is

$$\mathcal{L}^N h(z_1, .., z_N) = \sum_{i=1}^N \mathcal{L}^{i,N} h,$$

where

$$\mathcal{L}^{i,N}h(z_1,..,z_N) = \left(x_i - x_i^3 - c_i - \alpha + \frac{1}{N}\sum_{j=1}^{N} K_X(z_i - z_j)\right)\partial_{x_i}h + \left(\gamma x_i - c_i + \beta + \frac{1}{N}\sum_{j=1}^{N} K_C(z_i - z_j)\right)\partial_{c_i}h + \frac{\sigma_x^2}{2}\partial^2_{x_i,x_i}h + \frac{\sigma_c^2}{2}\partial^2_{c_i,c_i}h.$$

For $h : \mathbb{R}^2 \to \mathbb{R}$, for all $z = (x, y) \in \mathbb{R}^2$, the generator of (1.2) for a given distribution μ is

$$\mathcal{L}_\mu h(x,c) = \left(x - x^3 - c - \alpha + K_X * \mu(z)\right)\partial_x h + \left(\gamma x - c + \beta + K_C * \mu(z)\right)\partial_c h + \frac{\sigma_x^2}{2}\partial^2_{xx}h + \frac{\sigma_c^2}{2}\partial^2_{cc}h.$$

In particular, we notice that for fixed $(z_i)_{1\le i\le N} \in (\mathbb{R}^2)^N$, if we consider the empirical measure $\{\mu_{\text{emp}} = \frac{1}{N}\sum_j \delta_{z_j}\}$, we have for all $h : \mathbb{R}^2 \to \mathbb{R}$ and $\bar{z} \in \mathbb{R}^2$,

$$\begin{aligned}\mathcal{L}_{\mu_{\text{emp}}}h(\bar{z}) &= \left(\bar{x} - \bar{x}^3 - \bar{c} - \alpha + K_X * \mu_{\text{emp}}(\bar{z})\right)\partial_x h + \left(\gamma\bar{x} - \bar{c} + \beta + K_C * \mu_{\text{emp}}(\bar{z})\right)\partial_c h + \frac{\sigma_x^2}{2}\partial^2_{xx}h + \frac{\sigma_c^2}{2}\partial^2_{cc}h \\ &= \left(\bar{x} - \bar{x}^3 - \bar{c} - \alpha + \frac{1}{N}\sum_{j=1}^{N} K_X(\bar{z} - z_j)\right)\partial_x h + \left(\gamma\bar{x} - \bar{c} + \beta + \frac{1}{N}\sum_{j=1}^{N} K_C(\bar{z} - z_j)\right)\partial_c h \\ &\quad + \frac{\sigma_x^2}{2}\partial^2_{xx}h + \frac{\sigma_c^2}{2}\partial^2_{cc}h.\end{aligned}$$

In this case, if we consider $\bar{z} = z_i$ for a specific i and we denote $\bar{h}^i : (z_1,..,z_N) \to h(z_i)$, then

$$\begin{aligned}\mathcal{L}_{\mu_{\text{emp}}}h(z_i) &= \left(x_i - x_i^3 - c_i - \alpha + \frac{1}{N}\sum_{j=1}^{N} K_X(z_i - z_j)\right)\partial_x h \\ &\quad + \left(\gamma x_i - c_i + \beta + \frac{1}{N}\sum_{j=1}^{N} K_C(z_i - z_j)\right)\partial_c h + \frac{\sigma_x^2}{2}\partial^2_{xx}h + \frac{\sigma_c^2}{2}\partial^2_{cc}h \\ &= \mathcal{L}^{i,N}\bar{h}^i(z_1,\ldots,z_N).\end{aligned}$$

2.2 First Lyapunov function

Let $H : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$H(z) = H(x,c) = \frac{1}{2}\gamma x^2 + \beta x + \frac{1}{2}c^2 + \alpha c + H_0, \tag{2.1}$$

with

$$H_0 = \frac{\beta^2}{\gamma} + \alpha^2,$$

where γ, β and α are the parameters of the system (1.1).

Lemma 2.1. *We have*

(i) *For all $x, c \in \mathbb{R}$, we have $H(x,c) \geq \frac{\gamma}{4}x^2 + \frac{c^2}{4} \geq 0$,*

(ii) *For all $x, c \in \mathbb{R}$, we have $H(x,c) \geq \frac{1}{2\max(\gamma,1)}\left((\gamma x+\beta)^2 + (c+\alpha)^2\right)$,*

(iii) *For all $\delta > 0$ there is $\mathcal{C}_{r,H} > 0$ such that for all $x, x', c, c' \in \mathbb{R}$, we have*

$$\left(|x-x'| + \delta|c-c'|\right)^2 \leq \mathcal{C}_{r,H}(H(x,c) + H(x',c')),$$

(iv) *A direct consequence of the previous point is that for all $B \in \mathbb{R}$, $\lambda > 0$ and $\delta > 0$, there is $R \geq 0$ such that, for $x, x', c, c' \in \mathbb{R}$ satisfying $|x-x'| + \delta|c-c'| \geq R$, we have $H(x,c) + H(x',c') \geq \frac{80B}{\lambda}$. An explicit value of R is given by $R = \sqrt{\frac{1280(1+\delta^2)B}{\lambda\min(\gamma,1)}}$.*

The first two points are consequences of direct calculations. The last two points are proved in Appendix A.1.

Lemma 2.2 (Lyapunov's property of H)**.** *H, defined in* (2.1)*, is a Lyapunov function. In fact, let $\lambda \in \mathbb{R}$ such that*

$$\frac{L_X}{8} + L_C\left(2 + \frac{1}{8}\right) < 1 - \frac{\lambda}{2}, \tag{2.2}$$

then there exists $B > 0$ such that for all $(\bar{x}, \bar{c}) \in \mathbb{R}^2$, for all probability distribution μ on $\mathbb{R}^2$,

$$\mathcal{L}_\mu H(\bar{z}) \leq B + (\alpha_X L_X + \beta_X L_C)\left(\mathbb{E}_\mu(|X|)^2 - \bar{x}^2\right) + (\alpha_C L_X + \beta_C L_C)\left(\mathbb{E}_\mu(|C|)^2 - \bar{c}^2\right) - \lambda H(\bar{z}). \tag{2.3}$$

Moreover, for all $(z_i)_{1\leq i\leq N} \in \mathbb{R}^{2N}$, by denoting $H : (z_1, \ldots, z_N) \mapsto H(z_i)$,

$$\begin{aligned}\mathcal{L}^{i,N} H(z_1,\ldots,z_N) \leq & B + (\alpha_X L_X + \beta_X L_C)\left(\left(\frac{1}{N}\sum_{j=1}^N |x_j|\right)^2 - x_i^2\right) \\ & + (\alpha_C L_X + \beta_C L_C)\left(\left(\frac{1}{N}\sum_{j=1}^N |c_j|\right)^2 - c_i^2\right) - \lambda H(z_i),\end{aligned} \tag{2.4}$$

with

$$\alpha_X = \frac{\gamma}{2} + \frac{1}{2}, \quad \beta_X = \frac{17}{2}, \quad \alpha_C = \frac{1}{16}, \quad \beta_C = \frac{1}{2} + \frac{1}{32}$$

We refer to Appendix A.2 for the proof of this lemma and of the following Proposition.

Proposition 2.1. *We have*

$$\mathcal{L}^N\left(\frac{1}{N}\sum_{i=1}^N H\left(Z_t^{i,N}\right)\right) \leq B - \lambda\left(\frac{1}{N}\sum_{i=1}^N H\left(Z_t^{i,N}\right)\right), \tag{2.5}$$

A direct consequence of (2.3) is

$$\mathbb{E}H\left(\bar{Z}_t^i\right) \leq \mathbb{E}H\left(\bar{Z}_0^i\right) + \int_0^t \left(B - \lambda\mathbb{E}H\left(\bar{Z}_s^i\right)\right) ds, \tag{2.6}$$

and a consequence of (2.5) is

$$\left(\frac{1}{N}\sum_{i=1}^N \mathbb{E}H\left(Z_t^{i,N}\right)\right) \leq \left(\frac{1}{N}\sum_{i=1}^N \mathbb{E}H\left(Z_0^{i,N}\right)\right) + \int_0^t \left(B - \lambda\frac{1}{N}\sum_{i=1}^N \mathbb{E}H\left(Z_s^{i,N}\right)\right) ds. \tag{2.7}$$

From (2.7) we obtain bounds on the moments of $\left|X_t^{i,N}\right|^2$ and $\left|C_t^{i,N}\right|^2$, and from (2.6) Proposition 1.3 on the second moments of $\bar{X}_t^i$ and $\bar{C}_t^i$. The proof is given is Appendix A.2. It also yields the following result

Lemma 2.3. *Provided the interaction kernels satisfy*

$$\frac{L_X}{8} + L_C\left(2+\frac{1}{8}\right) < 1,$$

and that $\mathbb{E}(|\bar{X}_0|^2) < +\infty$ *and* $\mathbb{E}(|\bar{C}_0|^2) < +\infty$*, then there exists* $\mathcal{C}_{init,2}$ *such that for all* $t \geq 0$*:*

$$\mathbb{E}\left(|\bar{X}_t|^2 + |\bar{C}_t|^2\right) \leq \mathcal{C}_{init,2}.$$

From now on, we consider $\lambda > 0$ satisfying (2.2) (and use the *a priori* bounds $L_{X,max}$ and $L_{C,max}$ to ensure the existence of such a λ).

2.3 Modification of the function

Let $\mathcal{C}_{init,exp} > 0$, $\tilde{a} > 0$ and consider an initial measure μ_0 on $\mathbb{R}^2$ which satisfies $\mathbb{E}_{\mu_0}\left(e^{\tilde{a}(|X|+|C|)}\right) \leq \mathcal{C}_{init,exp}$.

For technical reasons, we need a greater restoring force by the Lyapunov function than the one given in Lemma 2.2. We thus modify it in order to obtain estimates such as (2.13) and (2.19) below.

Let $a > 0$, such that $a \leq \tilde{a}/\left(4\sqrt{2}\max\left(\sqrt{\gamma},1\right)\right)$. This choice of a is only necessary for further Propositions and Lemmas, in Section 3.

Let us consider for all $z \in \mathbb{R}^2$,

$$\tilde{H}(z) = \int_0^{H(z)} \exp\left(a\sqrt{u}\right) du = \frac{2}{a^2}\exp\left(a\sqrt{H(z)}\right)\left(a\sqrt{H(z)}-1\right) + \frac{2}{a^2}. \tag{2.8}$$

Direct calculations yield the following technical lemma.

Lemma 2.4. *We have, for all* $z \in \mathbb{R}^2$

$$H(z)\exp\left(a\sqrt{H(z)}\right) \geq \tilde{H}(z) \geq \exp\left(a\sqrt{H(z)}\right) - \frac{2}{a^2}\left(\exp\left(\frac{a^2}{2}\right)-1\right), \tag{2.9}$$

$$\frac{2}{a}\sqrt{H(z)}\exp\left(a\sqrt{H(z)}\right) \geq \tilde{H}(z) \geq \frac{1}{a}\sqrt{H(z)}\exp\left(a\sqrt{H(z)}\right) - \frac{1}{a^2}(e-2), \tag{2.10}$$

$$\tilde{H}(z) \geq H(z) \tag{2.11}$$

We may calculate, using Lemma 2.1 and Equation (2.3)

$$\begin{aligned}
\mathcal{L}_\mu\left(\tilde{H}\right) =& \exp\left(a\sqrt{H}\right)\mathcal{L}_\mu H + \frac{1}{2}\frac{a}{2\sqrt{H}}\exp\left(a\sqrt{H}\right)\left(|\sigma_x\partial_x H|^2 + |\sigma_c\partial_c H|^2\right)\\
=& \exp\left(a\sqrt{H}\right)\mathcal{L}_\mu H + \frac{a}{4\sqrt{H}}\exp\left(a\sqrt{H}\right)\left(\sigma_x^2(\gamma x+\beta)^2 + \sigma_c^2(c+\alpha)^2\right)\\
\leq& \exp\left(a\sqrt{H}\right)\left(B + (\alpha_X L_X + \beta_X L_C)\,\mathbb{E}_\mu(|X|)^2 + (\alpha_C L_X + \beta_C L_C)\,\mathbb{E}_\mu(|C|)^2 - \lambda H\right)\\
&+ \frac{1}{2}\max\left(\sigma_x^2,\sigma_c^2\right)\max(\gamma,1)\,a\sqrt{H}\exp\left(a\sqrt{H}\right)\\
\leq& \exp\left(a\sqrt{H}\right)\left(B + \frac{\left(\frac{1}{2}\max\left(\sigma_x^2,\sigma_c^2\right)\max(\gamma,1)\right)^2 a^2}{2\lambda} + (\alpha_X L_X + \beta_X L_C)\,\mathbb{E}_\mu(|X|)^2\right.\\
&\left. + (\alpha_C L_X + \beta_C L_C)\,\mathbb{E}_\mu(|C|)^2 - \frac{\lambda}{2}H\right),
\end{aligned} \tag{2.12}$$

where for this last inequality we used Young's inequality

$$\frac{1}{2}\max\left(\sigma_x^2,\sigma_c^2\right)\max(\gamma,1)\,a\sqrt{H} \leq \frac{\lambda}{2}H + \frac{\left(\frac{1}{2}\max\left(\sigma_x^2,\sigma_c^2\right)\max(\gamma,1)\right)^2 a^2}{2\lambda}.$$

Notice that (2.12) ensures that this new Lyapunov function also tends to bring back particle which ventured at infinity, and at an even greater rate. This new rate $H\exp(\sqrt{H})$ however comes at a cost : the initial condition must have a finite exponential moment, and no longer just have a finite second moment. First, by Lemma 2.3, $\mathbb{E}(\bar{X}_t)^2 + \mathbb{E}(\bar{C}_t)^2 \leq \mathcal{C}_{init,2}$. Furthermore, the function $h \mapsto \exp\left(a\sqrt{h}\right)\left(B - \frac{\lambda}{4}h\right)$ is bounded from above for $h \geq 0$. We therefore obtain from (2.12) the existence of $\tilde{B}$ such that

$$\mathcal{L}_{\bar{\mu}_t}\left(\tilde{H}\left(\bar{Z}_t^i\right)\right) \leq \tilde{B} - \frac{\lambda}{4}\left(H\left(\bar{Z}_t^i\right)\exp\left(a\sqrt{H\left(\bar{Z}_t^i\right)}\right)\right) \tag{2.13}$$

$$\frac{d}{dt}\mathbb{E}\tilde{H}\left(\bar{Z}_t^i\right) \leq \tilde{B} - \frac{\lambda}{4}\mathbb{E}\left(H\left(\bar{Z}_t^i\right)\exp\left(a\sqrt{H\left(\bar{Z}_t^i\right)}\right)\right) \tag{2.14}$$

$$\text{and} \quad \frac{d}{dt}\mathbb{E}\tilde{H}\left(\bar{Z}_t^i\right) \leq \tilde{B} - \frac{\lambda}{4}\mathbb{E}\tilde{H}\left(\bar{Z}_t^i\right), \tag{2.15}$$

where for this last inequality, we used (2.9). While (2.13) and (2.14) will be useful in ensuring a sufficient restoring force, Equation (2.15) give us a uniform in time bound on $\mathbb{E}\tilde{H}\left(\bar{Z}_t^i\right)$, provided we have an initial bound. These inequalities are to be understood in the sense of SDEs, where (2.15) should for instance be rigorously written

$$\mathbb{E}\tilde{H}\left(\bar{Z}_t^i\right) \leq \mathbb{E}\tilde{H}\left(\bar{Z}_0^i\right) + \int_0^t \left(\tilde{B} - \frac{\lambda}{4}\mathbb{E}\tilde{H}\left(\bar{Z}_s^i\right)\right) ds.$$

Now, for the system of particle, we have, using (2.12), $\forall i,\ \forall x_i, v_i \in \mathbb{R}^d$,

$$\begin{aligned}\mathcal{L}^N\tilde{H}\left(z_i\right) \leq & \exp\left(a\sqrt{H\left(z_i\right)}\right)\left(\tilde{B} + \left(\alpha_X L_X + \beta_X L_C\right)\left(\frac{1}{N}\sum_{j=1}^N |x_j|\right)^2\right. \\ & \left. + \left(\alpha_C L_X + \beta_C L_C\right)\left(\frac{1}{N}\sum_{j=1}^N |c_j|\right)^2 - \frac{\lambda}{2}H\left(z_i\right)\right).\end{aligned}$$

Summing over $i \in \{1, .., N\}$, we may calculate

$$\begin{aligned}&\left(\alpha_X L_X + \beta_X L_C\right)\sum_{j=1}^N\left(\frac{\sum_{j=1}^N |x_j|}{N}\right)^2\sum_{i=1}^N\frac{\exp\left(a\sqrt{H\left(z_i\right)}\right)}{N} - \frac{\lambda}{16}\sum_{i=1}^N\frac{H\left(z_i\right)\exp\left(a\sqrt{H\left(z_i\right)}\right)}{N} \\ \leq & \frac{\lambda}{16}\Big(\sum_{i,j=1}^N\frac{H\left(z_i\right)}{N}\frac{\exp\left(a\sqrt{H\left(z_j\right)}\right)}{N} - \sum_{i=1}^N\frac{H\left(z_i\right)\exp\left(a\sqrt{H\left(z_i\right)}\right)}{N}\Big) \\ \leq & 0.\end{aligned} \tag{2.16}$$

Here, we used Lemma 2.1, the fact that $\forall x, y \geq 0\ \ xe^{\sqrt{y}} + ye^{\sqrt{x}} - xe^{\sqrt{x}} - ye^{\sqrt{y}} = (e^{\sqrt{x}} - e^{\sqrt{y}})(y - x) \leq 0$ and assumed

$$\left(\alpha_X L_X + \beta_X L_C\right) \leq \frac{\gamma\lambda}{64}.$$

Likewise,

$$\left(\alpha_C L_X + \beta_C L_C\right)\sum_{j=1}^N\left(\frac{\sum_{j=1}^N |c_j|}{N}\right)^2\sum_{i=1}^N\frac{\exp\left(a\sqrt{H\left(z_i\right)}\right)}{N} - \frac{\lambda}{16}\sum_{i=1}^N\frac{H\left(z_i\right)\exp\left(a\sqrt{H\left(z_i\right)}\right)}{N} \leq 0, \tag{2.17}$$

provided

$$\left(\alpha_C L_X + \beta_C L_C\right) \leq \frac{\lambda}{64}.$$

There is therefore a constant, which for the sake of clarity we will also denote $\tilde{B}$ (as we may take the maximum of the previous constants), such that we get

$$\begin{aligned}\mathcal{L}^{i,N}\tilde{H}(Z_t^{i,N}) \leq & \tilde{B} + (\alpha_X L_X + \beta_X L_C)\left(\frac{\sum_{j=1}^N |X_t^{j,N}|}{N}\right)^2 \exp\left(a\sqrt{H\left(Z_t^{i,N}\right)}\right) \\ & + (\alpha_C L_X + \beta_C L_C)\left(\frac{\sum_{j=1}^N |C_t^{j,N}|}{N}\right)^2 \exp\left(a\sqrt{H\left(Z_t^{i,N}\right)}\right) \\ & - \frac{\lambda}{4} H\left(Z_t^{i,N}\right)\exp\left(a\sqrt{H\left(Z_t^{i,N}\right)}\right) \end{aligned} \tag{2.18}$$

$$\mathcal{L}^N\left(\frac{1}{N}\sum_{i=1}^N \tilde{H}(Z_t^{i,N})\right) \leq \tilde{B} - \frac{\lambda}{4}\left(\frac{1}{N}\sum_{i=1}^N H(Z_t^{i,N})\exp\left(a\sqrt{H\left(Z_t^{i,N}\right)}\right)\right) \tag{2.19}$$

and

$$\mathcal{L}^N\left(\frac{1}{N}\sum_{i=1}^N \tilde{H}(Z_t^{i,N})\right) \leq \tilde{B} - \frac{\lambda}{4}\left(\frac{1}{N}\sum_{i=1}^N \tilde{H}(Z_t^{i,N})\right) \tag{2.20}$$

Once again, (2.18) and (2.19) will be useful in ensuring a sufficient restoring force, and (2.20) yields a uniform in time bound on the expectation of $\tilde{H}(Z_t^{i,N})$, since $\mathbb{E}\left(\frac{1}{N}\sum_{j=1}^N \tilde{H}(Z_t^{j,N})\right) = \mathbb{E}\left(\tilde{H}(Z_t^{i,N})\right)$ by exchangeability of the particles.

2.4 Parameters

We start by fixing the values of some parameters. The somewhat intricate expressions in this section are dictated by the computations arising in the proofs later on. They are somewhat roughly chosen and far from optimal as we only wish to convey the fact that every constant is explicit.

Recall α_X, β_X, α_C and β_C given in Lemma 2.2. $a > 0$ is fixed from the last Subsection and the definition of $\tilde{H}$, and λ and $\tilde{B}$ are obtained from the same Subsection.

We consider the following parameters

Assumption 2. *Given any $\eta > 4$ and $\tilde{\delta} > 0$, consider the following set of parameters*

$$\begin{aligned}\delta =& (1+\tilde{\delta})\frac{1+L_{X,max}}{1-L_{C,max}},\ R_0 = \sqrt{\frac{1280\tilde{B}}{\lambda\min(\gamma,1)}},\ R = \sqrt{1+\delta^2}R_0, \\ \mathcal{C}_{f,1} =& 16\left(\frac{1}{a^2}\left(\gamma + a\left(\beta+\frac{\alpha}{\delta}\right)\sqrt{2\max(\gamma,1)}\right)\left(\exp\left(\frac{a^2}{2}\right)-1\right) + \sqrt{2\max(\gamma,1)}\left(\sqrt{\gamma}+\frac{1}{\delta}\right)(e-2)\right) \\ \mathcal{C}_{f,2} =& 4\left(\gamma + \left(a\left(\beta+\frac{\alpha}{\delta}\right) + 2a^2\left(\sqrt{\gamma}+\frac{1}{\delta}\right)\right)\sqrt{2\max(\gamma,1)}\right) \\ c =& \min\left\{\frac{2\tilde{B}}{\eta}, \frac{\lambda}{160}\frac{\eta-4}{\eta},\right. \\ & \left.\frac{1}{2(1+\eta)}\min\left(\frac{\sigma_x}{\sqrt{\pi}R}, 1-L_C-\frac{1+L_X}{\delta}\right)\exp\left(-\frac{1}{4\sigma_x^2}\left(1+\delta\gamma+L_{X,max}+\delta L_{C,max}+\left(\mathcal{C}_{f,1}+\mathcal{C}_{f,2}\right)\sigma_x^2\right)R^2\right)\right\}, \\ \epsilon =& \frac{\eta c}{2\tilde{B}},\ \phi_{min} = \exp\left(-\frac{1}{4\sigma_x^2}(1+\delta\gamma+L_{X,max}+\delta L_{C,max}+(\epsilon\mathcal{C}_{f,1}+\mathcal{C}_{f,2})\sigma_x^2)R^2\right) \\ \mathcal{C}_1 =& \frac{1}{\min(\delta,1)}\frac{2}{\phi_{min}}\max\left(\frac{16(1+\delta^2)}{\epsilon\min(\gamma,1)},1\right),\ \mathcal{C}_2 = \frac{1}{\min(\delta^2,1)}\frac{2}{\phi_{min}}\max\left(\frac{16(1+\delta^2)}{\epsilon\min(\gamma,1)},1\right)\end{aligned}$$

$$\mathcal{C}_z = \frac{2}{\phi_{min}} \max\left(1, \frac{4}{\epsilon}\max\left(\sqrt{\frac{1}{\gamma}}, 1\right)\right).$$

We define f as follows

$$\begin{aligned} f(r) &= \int_0^{r\wedge R} \phi(s)g(s)ds, \\ \phi(r) &= \exp\left(-\frac{1}{4\sigma_x^2}\left(1+\delta\gamma+L_X+\delta L_C+\left(\epsilon\mathcal{C}_{f,1}+\mathcal{C}_{f,2}\right)\sigma_x^2\right)r^2\right), \\ \Phi(s) &= \int_0^s \phi(u)du, \\ g(r) &= 1-\frac{c+2\epsilon\tilde{B}}{\sigma_x^2}\int_0^r \Phi(s)\phi(s)^{-1}ds. \end{aligned} \tag{2.21}$$

Assume furthermore that L_X and L_C, the Lipschitz constants, satisfy

$$\begin{aligned} L_X \leq \min\left(\frac{\lambda}{128\mathcal{C}_z}, \frac{\lambda a}{512\epsilon\mathcal{C}_z}, \frac{c}{2\mathcal{C}_1}\right) \quad \textit{and} \quad L_C \leq \min\left(\frac{\lambda}{128\delta\mathcal{C}_z}, \frac{\lambda a}{512\epsilon\delta\mathcal{C}_z}, \frac{c}{2\delta\mathcal{C}_1}\right) \\ \alpha_X L_X + \beta_X L_C \leq \frac{\gamma\lambda}{128} \quad \textit{and} \quad \alpha_C L_X + \beta_C L_C \leq \frac{\lambda}{128}, \\ \frac{L_X}{8} + L_C\left(2+\frac{1}{8}\right) < 1-\frac{\lambda}{2}. \end{aligned}$$

Notice how the bounds on L_X and L_C depend on c. This is one of the reasons why we use the *a priori* bounds $L_X \in [0, L_{X,max}]$ and $L_C \in [0, L_{C,max}]$ given in Assumption 1 : they allow us to bound c and δ independently of L_C and L_X. We are thus able to begin by choosing an acceptable values for those parameters, before then giving upper bounds on L_X and L_C.

We quickly mention that the constants $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_z$ above come from Lemma 2.6 later. We gather some properties required in the calculations of the proof of Theorem 2 in the following lemma. Again, these properties are the ones motivating the choice of parameters

Lemma 2.5. *The set of parameters given in Assumption 2 satisfy*

- *f is $\mathcal{C}^2$ on $(0,R)$ such that $f'_+(0)=1$ and $f'_-(R)>0$, and constant on $[R,\infty)$. Moreover, f is non-negative, non-decreasing and concave, and for all $s \geq 0$,*

$$\min(s,R) f'_-(R) \leq f(s) \leq \min(s, f(R)) \leq \min(s,R).$$

- *For all $r \in [0,R]$, $\phi(r) \geq \phi_{min}$ and $g(r) \geq \frac{1}{2}$.*

- *We have the conditions*

$$\begin{aligned} 2f'(R) &\geq \exp\left(-\frac{1}{4\sigma_x^2}\left(1+\delta\gamma+L_X+\delta L_C+\left(\epsilon\mathcal{C}_{f,1}+\mathcal{C}_{f,2}\right)\sigma_x^2\right)R^2\right), \\ 2c+4\epsilon\tilde{B} &\leq \left(1-L_C-\frac{1+L_X}{\delta}\right)\min_{r\in]0,R]}\frac{f'(r)r}{f(r)}, \\ c &\leq \frac{\lambda}{160}\frac{\frac{80\epsilon\tilde{B}}{\lambda}}{1+\frac{80\epsilon\tilde{B}}{\lambda}}, \qquad \frac{1+L_X}{1-L_C} < \delta \quad \textit{and} \quad \epsilon \leq 1. \end{aligned}$$

The proof of this lemma is done in Appendix A.3.

2.5 Control of the usual distances

As explained previously, we consider a modified semimetrics. For $z = (x, c) \in \mathbb{R}^2$ and $z' = (x', c') \in \mathbb{R}^2$, define

$$r(z, z') = r(x, c, x', c') = |x - x'| + \delta|c - c'|, \tag{2.22}$$

and let $\rho((z_j, z'_j)_{1\leq j\leq N})$ be defined as follows:

$$\rho\left((z_j, z'_j)_{1\leq j\leq N}\right) = \frac{1}{N}\sum_{i=1}^{N} f\left(r\left(z_i, z'_i\right)\right) G^i\left((z_j, z'_j)_j\right), \tag{2.23}$$

where for each $i \in \{1, ..., N\}$,

$$G^i\left((z_j, z'_j)_j\right) = 1 + \epsilon\tilde{H}\left(z_i\right) + \epsilon\tilde{H}\left(z'_i\right) + \frac{\epsilon}{N}\sum_{j=1}^{N}\tilde{H}\left(z_j\right) + \frac{\epsilon}{N}\sum_{j=1}^{N}\tilde{H}\left(z'_j\right). \tag{2.24}$$

An immediate corollary of the definition and properties of H is that ρ is a quantity on $\mathbb{R}^{4N}$ which controls the usual L^1 and L2 distances.

Lemma 2.6. *The constants $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_z > 0$, given in Assumption 2, are such that for all $z = (x, c) \in \mathbb{R}^2$ and $z' = (x', c') \in \mathbb{R}^2$*

(i) $\|z - z'\|_1 \leq \mathcal{C}_1 f\left(r\left(z, z'\right)\right)\left(1 + \epsilon\tilde{H}(z) + \epsilon\tilde{H}(z')\right)$

(ii) $\|z - z'\|_2^2 \leq \mathcal{C}_2 f\left(r\left(z, z'\right)\right)\left(1 + \epsilon\tilde{H}(z) + \epsilon\tilde{H}(z')\right)$

(iii) $\|z - z'\|_1 \leq \mathcal{C}_z f(r(z, z'))\left(1 + \epsilon\sqrt{H(z)} + \epsilon\sqrt{H(z')}\right)$

The proof of this lemma is postponed to Appendix A.4.

3 Proof of Theorem 2 in the case $\sigma_X > 0$

Let $\xi > 0$ be a parameter destined to vanish, and let $\varphi_{\text{sc}} : \mathbb{R}^+ \mapsto \mathbb{R}^+$ and $\varphi_{\text{rc}} : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be two Lipschitz continuous functions such that

$$\begin{aligned}\forall x,\ \ \varphi_{\text{sc}}^2(x) + \varphi_{\text{rc}}^2(x) =& 1\\ \varphi_{\text{rc}}(x) =& 1 \text{ if } \xi \leq x \leq R\\ \varphi_{\text{rc}}(x) =& 0 \text{ if } x \leq \frac{\xi}{2} \text{ or } x \geq R + \xi.\end{aligned}$$

Intuitively, φ_{rc} represents the region of space in which we consider a reflection coupling, and φ_{sc} the one in which we consider a synchronous coupling. We thus simultaneously construct the following solutions

$$\begin{cases} dX_t^{i,N} = (X_t^{i,N} - (X_t^{i,N})^3 - C_t^{i,N} - \alpha)dt + \frac{1}{N}\sum_{j=1}^{N} K_X(Z_t^i - Z_t^j)dt \\ \qquad\qquad + \sigma_x\varphi_{\text{sc}}\left(|X_t^{i,N} - \bar{X}_t^i|\right) dB_t^{i,sc,X} + \sigma_x\varphi_{\text{rc}}\left(|X_t^{i,N} - \bar{X}_t^i|\right) dB_t^{i,rc,X} \\ dC_t^{i,N} = (\gamma X_t^{i,N} - C_t^{i,N} + \beta)dt + \frac{1}{N}\sum_{j=1}^{N} K_C(Z_t^i - Z_t^j)dt + \sigma_c dB_t^{i,C}, \end{cases}$$

and

$$\begin{cases} d\bar{X}_t^i = (\bar{X}_t^i - (\bar{X}_t^i)^3 - \bar{C}_t^i - \alpha)dt + K_X * \bar{\mu}_t(\bar{Z}_t^i)dt \\ \qquad\qquad + \sigma_x\varphi_{\text{sc}}\left(|X_t^{i,N} - \bar{X}_t^i|\right) dB_t^{i,sc,X} - \sigma_x\varphi_{\text{rc}}\left(|X_t^{i,N} - \bar{X}_t^i|\right) dB_t^{i,rc,X} \\ d\bar{C}_t^i = (\gamma\bar{X}_t^i - \bar{C}_t^i + \beta)dt + K_C * \bar{\mu}_t(\bar{Z}_t^i)dt + \sigma_c dB_t^{i,C}. \end{cases}$$

Notice that we consider a symmetric coupling on the dynamics of C.

3.1 Main proof and results

Proposition 3.1. *We denote* $r_t^i = r(Z_t^{i,N}, \bar{Z}_t^i)$ *and* $G_t^i = G^i((Z_t^{j,N})_j, (\bar{Z}_t^j)_j)$. *For all* $c \in \mathbb{R}$, *for each* $i \in \{1, \dots, N\}$, *we have*

$$d(e^{ct} f(r_t^i) G_t^i) \leq e^{ct} K_t^i dt + dM_t^i, \tag{3.1}$$

where M_t^i *is a continuous local martingale and* K_t^i *can be written as*

$$K_t^i = \tilde{K}_t^i + I_t^{1,i} + I_t^{2,i} + I_t^{3,i}. \tag{3.2}$$

We define $\tilde{K}_t^i$, $I_t^{1,i}$, $I_t^{2,i}$, $I_t^{3,i}$ *and* $I_t^{4,i}$ *as followed:*

$$\begin{aligned}
\tilde{K}_t^i =& G_t^i \Big[2cf(r_t^i) + \frac{1}{2} f''(r_t^i) \left(2\sigma_x^2 \varphi_{rc} \left(|X_t^{i,N} - \bar{X}_t^i| \right)^2 \right) \\
&+ f'(r_t^i) \Big((1 + \gamma\delta + L_X + \delta L_C) |X_t^{i,N} - \bar{X}_t^i| - |(X_t^{i,N})^3 - (\bar{X}_t^i)^3| + (1 + L_X + \delta L_C - \delta) |C_t^{i,N} - \bar{C}_t^i| \\
&+ (\epsilon \mathcal{C}_{f,1} + \mathcal{C}_{f,2}) \sigma_x^2 \varphi_{rc} \left(|X_t^{i,N} - \bar{X}_t^i| \right)^2 r_t^i \Big) \Big] \\
&+ \epsilon f(r_t^i) \left(4\tilde{B} - \frac{\lambda}{16} \tilde{H}(\bar{Z}_t^i) - \frac{\lambda}{16} \tilde{H}(Z_t^{i,N}) - \frac{\lambda}{16N} \sum_{j=1}^N \tilde{H}(\bar{Z}_t^j) - \frac{\lambda}{16N} \sum_{j=1}^N \tilde{H}(Z_t^{j,N}) \right),
\end{aligned} \tag{3.3}$$

$$I_t^{1,i} = G_t^i f'(r_t^i) \left(\left| \frac{1}{N} \sum_{j=1}^N K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i) \right| \right) + \delta G_t^i f'(r_t^i) \left(\left| \frac{1}{N} \sum_{j=1}^N K_C(\bar{Z}_t^i - \bar{Z}_t^j) - K_C * \bar{\mu}_t(\bar{Z}_t^i) \right| \right), \tag{3.4}$$

$$\begin{aligned}
I_t^{2,i} =& G_t^i f'(r_t^i) \left(\frac{L_X}{N} \left(\sum_{j=1}^N \|Z_t^{j,N} - \bar{Z}_t^j\|_1 \right) \right) + \delta G_t^i f'(r_t^i) \left(\frac{L_C}{N} \left(\sum_{j=1}^N \|Z_t^{j,N} - \bar{Z}_t^j\|_1 \right) \right) \\
&- cf(r_t^i) G_t^i - \epsilon f(r_t^i) \left[\frac{\lambda}{16} H(\bar{Z}_t^i) \exp\left(a\sqrt{H(\bar{Z}_t^i)} \right) + \frac{\lambda}{16} H(Z_t^{i,N}) \exp\left(a\sqrt{H(Z_t^{i,N})} \right) \right] \\
&- \epsilon f(r_t^i) \left[\frac{\lambda}{16N} \sum_{j=1}^N H(\bar{Z}_t^j) \exp\left(a\sqrt{H(\bar{Z}_t^j)} \right) + \frac{\lambda}{16N} \sum_{j=1}^N H(Z_t^{j,N}) \exp\left(a\sqrt{H(Z_t^{j,N})} \right) \right],
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
I_t^{3,i} =& \epsilon f(r_t^i) \left((\alpha_X L_X + \beta_X L_C) \left(\frac{\sum_{j=1}^N |X_t^{j,N}|}{N} \right)^2 \exp\left(a\sqrt{H(Z_t^{i,N})} \right) \right. \\
&+ (\alpha_C L_X + \beta_C L_C) \left(\frac{\sum_{j=1}^N |C_t^{j,N}|}{N} \right)^2 \exp\left(a\sqrt{H(Z_t^{i,N})} \right) \\
&\left. - \frac{\lambda}{16} H(Z_t^{i,N}) \exp\left(a\sqrt{H(Z_t^{i,N})} \right) - \frac{\lambda}{16N} \sum_{j=1}^N H(Z_t^{j,N}) \exp\left(a\sqrt{H(Z_t^{j,N})} \right) \right).
\end{aligned} \tag{3.6}$$

We need a control on $\mathbb{E}(G_t^i)$, which is a consequence of Lyapunov's properties on $\tilde{H}$ and the initial assumption of the Theorem 2. A proof is given in Appendix A.5.

Lemma 3.1. *There exists* $\mathcal{C}_{G,1}$ *and* $\mathcal{C}_{G,2}$, *such that for each* $i \leq N$, *for all* $t > 0$, *we have*

$$\mathbb{E}(G_t^i) \leq \mathcal{C}_{G,1} \quad \textit{and} \quad \mathbb{E}[(G_t^i)^2] \leq \mathcal{C}_{G,2}.$$

Each of the terms given in Proposition 3.1 will be controlled in a different way. The following lemmas summarize it. The first term, $\tilde{K}_t^i$, contains the various behaviors we have previously identified : we deal with it either through a synchronous coupling (when the deterministic drift is contracting), or through a reflection coupling (notice the second derivative f" which will provide contraction provided f is sufficiently concave). Finally, notice the effect of Lyapunov function $\tilde{H}$ which yields a restoring force.

Lemma 3.2. *For each $i \leq N$, for all $t > 0$,*

$$\mathbb{E}\tilde{K}_t^i \leq \xi \left(2 + \delta\gamma + L_X + \delta L_C - L_C - \frac{1 + L_X}{\delta}\right) \mathbb{E}G_t^i. \tag{3.7}$$

The interaction term $\frac{1}{N}K_X(Z_t^{j,N} - Z_t^{i,N}) - K_X * \bar{\mu}_t(\bar{Z}_t^i)$ can be decomposed into $\frac{1}{N}K_X(\bar{Z}_t^j - \bar{Z}_t^i) - K_X * \bar{\mu}_t(\bar{Z}_t^i)$ and $\frac{1}{N}\sum \left[K_X(Z_t^{j,N} - Z_t^{i,N}) - K_X(\bar{Z}_t^j - \bar{Z}_t^i)\right]$. The first part, which is in $I_t^{1,i}$, is dealt with using some form of law of large number in a similar way as what has been done in the proof of Theorem 1.

Lemma 3.3. *For each $i \leq N$, for all $t > 0$,*

$$\mathbb{E}(I_t^{1,i}) \leq 4\sqrt{\frac{\mathcal{C}_{init,2}\mathcal{C}_{G,2}}{N}}(L_X + L_C), \tag{3.8}$$

where $\mathcal{C}_{G,2}$ is defined in Lemma 3.1 and $\mathcal{C}_{init,2}$ is defined in Lemma 2.3.

$I_t^{2,i}$ contains the leftovers of this decomposition and some of the additional terms of the Lyapunov function.

Lemma 3.4. *For all $t > 0$,*

$$\frac{1}{N}\sum_{i=1}^N I_t^{2,i} \leq 0. \tag{3.9}$$

Finally, $I_t^{3,i}$ deals with the non linearity appearing in the dynamics of the Lyapunov function, and will be non positive for values of L_X and L_C sufficiently small. It is also here we justify adding the last two terms in (2.24).

Lemma 3.5. *For each $i \leq N$, for all $t > 0$,*

$$I_t^{3,i} \leq 0. \tag{3.10}$$

Proof of Theorem 2. With these four Lemmas, we can calculate

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^N \mathbb{E}K_t^i =& \frac{1}{N}\sum_{i=1}^N \mathbb{E}\tilde{K}_t^i + \frac{1}{N}\sum_{i=1}^N \mathbb{E}I_t^{1,i} + \frac{1}{N}\sum_{i=1}^N \mathbb{E}I_t^{2,i} + \frac{1}{N}\sum_{i=1}^N \mathbb{E}I_t^{3,i} \\
\leq& \frac{1}{N}\sum_{i=1}^N \xi \left(2 + \delta\gamma + L_X + \delta L_C - L_C - \frac{1 + L_X}{\delta}\right) \mathbb{E}G_t^i + \frac{1}{N}\sum_{i=1}^N 4\sqrt{\frac{\mathcal{C}_{init,2}\mathcal{C}_{G,2}}{N}}(L_X + L_C) \\
\leq& \xi \left(2 + \delta\gamma + L_X + \delta L_C - L_C - \frac{1 + L_X}{\delta}\right) \frac{1}{N}\sum_{i=1}^N \mathbb{E}G_t^i + 4\sqrt{\frac{\mathcal{C}_{init,2}\mathcal{C}_{G,2}}{N}}(L_X + L_C)
\end{aligned}$$

Since by Lemma 3.1, we have $\frac{1}{N}\sum_{i=1}^N \mathbb{E}G_t^i \leq \mathcal{C}_{G,1}$,we obtain

$$\frac{1}{N}\sum_{i=1}^N \mathbb{E}K_t^i \leq \xi A + (L_X + L_C)\frac{B}{\sqrt{N}}$$

where A and B are constants.

For all initial couplings such that $\mathbb{E}\rho\left((Z_0^j,\bar{Z}_0^j)_{1\leq j\leq N}\right) < \infty$, by taking the expectation of (3.1) along a sequence of increasing localizing stopping times, we have thanks to Fatou's lemma

$$\begin{aligned}e^{ct}\mathbb{E}\left(\rho\left((Z_t^j,\bar{Z}_t^j)_{1\leq j\leq N}\right)\right) \leq& \mathbb{E}\left(\rho\left((Z_0^j,\bar{Z}_0^j)_{1\leq j\leq N}\right)\right) + \xi A\int_0^t e^{cs}ds + (L_X+L_C)\frac{B}{\sqrt{N}}\int_0^t e^{cs}ds\\ \leq& \mathbb{E}\left(\rho\left((Z_0^j,\bar{Z}_0^j)_{1\leq j\leq N}\right)\right) + \xi A\frac{e^{ct}-1}{c} + (L_X+L_C)\frac{B}{\sqrt{N}}\frac{e^{ct}-1}{c}.\end{aligned}$$

We obtain

$$\mathbb{E}\left(\rho\left((Z_t^j,\bar{Z}_t^j)_{1\leq j\leq N}\right)\right) \leq \mathbb{E}\left(\rho\left((Z_0^j,\bar{Z}_0^j)_{1\leq j\leq N}\right)\right)e^{-ct} + \frac{\xi A}{c}\left(1-e^{-ct}\right) + \frac{(L_X+L_C)B}{c}\frac{1}{\sqrt{N}}\left(1-e^{-ct}\right).$$

By using the exchangeability of the particles, we have $\mathbb{E}\left(\rho\left((Z_t^j,\bar{Z}_t^j)_{1\leq j\leq N}\right)\right) = \mathbb{E}\left(\frac{1}{N}\sum_{i=1}^N f(r_t^i)G_t^i\right) = \mathbb{E}\left(\frac{1}{k}\sum_{i=1}^k f(r_t^i)G_t^i\right)$ for all $k\in\mathbb{N}$. Then

$$\mathbb{E}\left(\sum_{i=1}^k f(r_t^i)G_t^i\right) = k\mathbb{E}\left(\rho\left((Z_t^j,\bar{Z}_t^j)_{1\leq j\leq N}\right)\right).$$

Let μ_0 a measure on $\mathbb{R}^2$, $\mu_t^{k,N}$ the marginal distribution at time t of the first k neurons $\left((X_t^1,C_t^1),....,(X_t^k,C_t^k)\right)$ of an N particles system (1.1) with initial distribution $(\mu_0)^{\otimes N}$, and $\bar{\mu}_t$ is a solution of (1.2) with initial distribution μ_0. This implies $\mathbb{E}\left(\rho\left((Z_0^j,\bar{Z}_0^j)_{1\leq j\leq N}\right)\right)=0$. By Lemma 2.6, we obtain for Wasserstein 1 distance

$$\begin{aligned}\mathcal{W}_1(\mu_t^{k,N},\bar{\mu}_t^{\otimes k}) =& \inf\left\{\mathbb{E}[\|Z^{(k)}-\bar{Z}^{(k)}\|_1], \mathbb{P}_{Z^{(k)}}=\mu_t^{k,N}, \mathbb{P}_{\bar{Z}^{(k)}}=\bar{\mu}_t^{\otimes k}\right\}\\ =& \inf\left\{\mathbb{E}\left[\sum_{i=1}^k \|Z_t^{i,N}-\bar{Z}_t^i\|_1\right], \mathbb{P}_{(Z_t^{i,N})_i}=\mu_t^{k,N}, \mathbb{P}_{(\bar{Z}_t^i)_i}=\bar{\mu}_t^{\otimes k}\right\}\\ \leq& \inf\left\{\mathcal{C}_1\mathbb{E}\left[\sum_{i=1}^k f(r_t^i)G_t^i\right], \mathbb{P}_{(Z_t^{i,N})_i}=\mu_t^{k,N}, \mathbb{P}_{(\bar{Z}_t^i)_i}=\bar{\mu}_t^{\otimes k}\right\}\\ \leq& \inf\left\{k\mathcal{C}_1\mathbb{E}\left(\rho\left((Z_t^j,\bar{Z}_t^j)_{1\leq j\leq N}\right)\right), \mathbb{P}_{(Z_t^{i,N})_i}=\mu_t^{k,N}, \mathbb{P}_{(\bar{Z}_t^i)_i}=\bar{\mu}_t^{\otimes k}\right\}\\ \leq& \frac{\xi Ak\mathcal{C}_1}{c}\left(1-e^{-ct}\right) + \frac{(L_X+L_C)B\mathcal{C}_1}{c}\frac{k}{\sqrt{N}}\left(1-e^{-ct}\right)\end{aligned}$$

By taking the limit as $\xi\to 0$ uniformly in time, we obtain the desired result. The same lemma and the same type of calculations yield the result for Wasserstein 2

$$\mathcal{W}_2(\mu_t^{k,N},\bar{\mu}_t^{\otimes k})^2 \leq \frac{k}{\sqrt{N}}\mathcal{C}_2\frac{(L_X+L_C)B}{c}.$$

□

3.2 Proof of the decomposition

Proof of Proposition 3.1. First, we need to calculate $d(e^{ct}f(r_t^i)G_t^i)$, where we recall

$$r_t^i = |X_t^{i,N}-\bar{X}_t^i| + \delta|C_t^{i,N}-\bar{C}_t^i|$$

and

$$G_t^i = 1 + \epsilon \tilde{H}(\bar{Z}_t^i) + \epsilon \tilde{H}(Z_t^{i,N}) + \frac{\epsilon}{N}\sum_{j=1}^N \tilde{H}(Z_t^{j,N}) + \frac{\epsilon}{N}\sum_{j=1}^N \tilde{H}(\bar{Z}_t^j).$$

We have already calculated $d(X_t^{i,N} - \bar{X}_t^i)$ and $d|X_t^{i,N} - \bar{X}_t^i|$ in the case of symmetric coupling in Subsection 1.4 in (1.7). Here, we need to use Ito's formula and usual convergence lemmas (see Lemma 7 of [DEGZ20]) to take care of the Brownian term. We obtain

$$d|X_t^{i,N} - \bar{X}_t^i| = A_t^X dt + 2\text{sign}(X_t^{i,N} - \bar{X}_t^i)\sigma_x \varphi_{\text{rc}}\left(|X_t^{i,N} - \bar{X}_t^i|\right) dB_t^{i,rc,X},$$

with

$$A_t^X \leq |X_t^{i,N} - \bar{X}_t^i| - \left|(X_t^{i,N})^3 - (\bar{X}_t^i)^3\right| + \left|C_t^{i,N} - \bar{C}_t^i\right| + \left|\frac{1}{N}\sum_{j=1}^N K_X(Z_t^i - Z_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right|.$$

Likewise, as it has already been calculated in (1.8) in Subsection 1.4,

$$d|C_t^{i,N} - \bar{C}_t^i| = A_t^C dt, \tag{3.11}$$

with

$$A_t^C \leq \gamma\left|X_t^{i,N} - \bar{X}_t^i\right| - |C_t^{i,N} - \bar{C}_t^i| + \left|\frac{1}{N}\sum_{j=1}^N K_C(Z_t^i - Z_t^j) - K_C * \bar{\mu}_t(\bar{Z}_t^i)\right|.$$

Now we have

$$dr_t^i = \left(A_t^X + \delta A_t^C\right) dt + 2\text{sign}(X_t^{i,N} - \bar{X}_t^i)\sigma_x \varphi_{\text{rc}}\left(|X_t^{i,N} - \bar{X}_t^i|\right) dB_t^{i,rc,X}$$

and we deduce with the Ito's formula

$$df(r_t^i) = f'(r_t^i)dr_t^i + \frac{1}{2}f''(r_t^i)\left(2\sigma_x\varphi_{\text{rc}}\left(|X_t^{i,N} - \bar{X}_t^i|\right)\right)^2 dt.$$

Finally, for $c > 0$,

$$d(e^{ct}f(r_t^i)) = ce^{ct}f(r_t^i)dt + e^{ct}df(r_t^i).$$

Then, by Ito's formula,

$$\begin{aligned}
\frac{1}{\epsilon}dG_t^i = &\left(\mathcal{L}_{\bar{\mu}_t}\tilde{H}(\bar{Z}_t^i) + \mathcal{L}^N\tilde{H}(Z_t^{i,N})\right)dt \\
&+ \sigma_x\varphi_{\text{rc}}\left(|X_t^{i,N} - \bar{X}_t^i|\right)\left(\partial_x\tilde{H}(Z_t^{i,N}) - \partial_x\tilde{H}(\bar{Z}_t^i)\right)dB_t^{i,rc,X} \\
&+ \sigma_x\varphi_{\text{sc}}\left(|X_t^{i,N} - \bar{X}_t^i|\right)\left(\partial_x\tilde{H}(Z_t^{i,N}) + \partial_x\tilde{H}(\bar{Z}_t^i)\right)dB_t^{i,sc,X} \\
&+ \sigma_c\left(\partial_c\tilde{H}(Z_t^{i,N}) + \partial_c\tilde{H}(\bar{Z}_t^i)\right)dB_t^{i,C} \\
&+ \frac{1}{N}\sum_{j=1}^N\left(\mathcal{L}_{\bar{\mu}_t}\tilde{H}(\bar{Z}_t^j) + \mathcal{L}^N\tilde{H}(Z_t^{j,N})\right)dt \\
&+ \frac{\sigma_x}{N}\sum_{j=1}^N\varphi_{\text{rc}}\left(|X_t^{j,N} - \bar{X}_t^j|\right)\left(\partial_x\tilde{H}(Z_t^{j,N}) - \partial_x\tilde{H}(\bar{Z}_t^j)\right)dB_t^{j,rc,X}
\end{aligned}$$

$$+\frac{\sigma_x}{N}\sum_{j=1}^{N}\varphi_{\text{sc}}\left(|X_t^{j,N}-\bar{X}_t^j|\right)\left(\partial_x\tilde{H}(Z_t^{j,N})+\partial_x\tilde{H}(\bar{Z}_t^j)\right)dB_t^{j,sc,X}$$
$$+\frac{\sigma_c}{N}\sum_{j=1}^{N}\left(\partial_c\tilde{H}(Z_t^{j,N})+\partial_c\tilde{H}(\bar{Z}_t^j)\right)dB_t^{j,C}.$$

We finally get

$$\begin{aligned}d(e^{ct}f(r_t^i)G_t^i) =&G_t^i d(e^{ct}f(r_t^i))+e^{ct}f(r_t^i)dG_t^i\\ &+2\epsilon\left(1+\frac{1}{N}\right)\sigma_x^2\varphi_{\text{rc}}\left(|X_t^{i,N}-\bar{X}_t^i|\right)^2\text{sign}(X_t^{i,N}-\bar{X}_t^i)\left(\partial_x\tilde{H}(Z_t^{i,N})-\partial_x\tilde{H}(\bar{Z}_t^i)\right)e^{ct}f'(r_t^i)dt.\end{aligned}$$

Now, we need to use the following Lemma, proven in Appendix A.5, to have a more tractable expression:

Lemma 3.6. *We have the upper bound*

$$\begin{aligned}2\epsilon\left(1+\frac{1}{N}\right)\sigma_x^2\varphi_{rc}\left(|X_t^{i,N}-\bar{X}_t^i|\right)^2 &sign(X_t^{i,N}-\bar{X}_t^i)\left(\partial_x\tilde{H}(Z_t^{i,N})-\partial_x\tilde{H}(\bar{Z}_t^i)\right)\\ &\leq(\epsilon\mathcal{C}_{f,1}+\mathcal{C}_{f,2})\,\sigma_x^2\varphi_{rc}\left(|X_t^{i,N}-\bar{X}_t^i|\right)^2 r_t^i G_t^i.\end{aligned}$$

Eventually, by denoting the terms in $dB_t^{i,rc,X}$, $dB_t^{i,sc,X}$, $dB_t^{i,C}$, ... as the local martingale dM_t^i, we obtain

$$\begin{aligned}d(e^{ct}f(r_t^i)G_t^i) \leq&G_t^i ce^{ct}f(r_t^i)dt+e^{ct}G_t^i f'(r_t^i)\left(A_t^X+\delta A_t^C\right)dt\\ &+e^{ct}G_t^i\frac{1}{2}f''(r_t^i)\left(2\sigma_x\varphi_{\text{rc}}\left(|X_t^{i,N}-\bar{X}_t^i|\right)\right)^2dt\\ &+e^{ct}f(r_t^i)\left(\left(\mathcal{L}_{\bar{\mu}_t}\tilde{H}(\bar{Z}_t^i)+\mathcal{L}^N\tilde{H}(Z_t^{i,N})\right)dt+\frac{1}{N}\sum_{j=1}^{N}\left(\mathcal{L}_{\bar{\mu}_t}\tilde{H}(\bar{Z}_t^j)+\mathcal{L}^N\tilde{H}(Z_t^{j,N})\right)dt\right)\\ &+(\epsilon\mathcal{C}_{f,1}+\mathcal{C}_{f,2})\,\sigma_x^2\varphi_{\text{rc}}\left(|X_t^{i,N}-\bar{X}_t^i|\right)^2 r_t^i G_t^i e^{ct}f'(r_t^i)dt\\ &+dM_t^i.\end{aligned}$$

We use (2.13) to bound $\mathcal{L}_{\bar{\mu}_t}\tilde{H}(\bar{Z}_t^i)$ and (2.18) to bound $\mathcal{L}^N\tilde{H}(Z_t^{i,N})$. The interaction terms in A_t^X and A_t^C are decomposed and we define $I_t^{1,i}$ as follows

$$I_t^{1,i}=G_t^i f'(r_t^i)\left(\left|\frac{1}{N}\sum_{j=1}^{N}K_X(\bar{Z}_t^i-\bar{Z}_t^j)-K_X*\bar{\mu}_t(\bar{Z}_t^i)\right|\right)+\delta G_t^i f'(r_t^i)\left(\left|\frac{1}{N}\sum_{j=1}^{N}K_C(\bar{Z}_t^i-\bar{Z}_t^j)-K_C*\bar{\mu}_t(\bar{Z}_t^i)\right|\right).$$

The second part of the decomposition is grouped in $I_t^{2,i}$ with compensating terms, and in particular Lyapunov functions, which appears with the use of (2.13) and (2.18), to control the sum:

$$\begin{aligned}I_t^{2,i}=&G_t^i f'(r_t^i)\left(\frac{L_X}{N}\left(\sum_{j=1}^{N}|X_t^{j,N}-\bar{X}_t^j|+|C_t^{j,N}-\bar{C}_t^j|\right)\right)+\delta G_t^i f'(r_t^i)\left(\frac{L_C}{N}\left(\sum_{j=1}^{N}|X_t^{j,N}-\bar{X}_t^j|+|C_t^{j,N}-\bar{C}_t^j|\right)\right)\\ &-cf(r_t^i)G_t^i-\epsilon f(r_t^i)\left[\frac{\lambda}{16}H(\bar{Z}_t^i)\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)+\frac{\lambda}{16}H(Z_t^{i,N})\exp\left(a\sqrt{H(Z_t^{i,N})}\right)\right]\\ &-\epsilon f(r_t^i)\left[\frac{\lambda}{16N}\sum_{j=1}^{N}H(\bar{Z}_t^j)\exp\left(a\sqrt{H(\bar{Z}_t^j)}\right)+\frac{\lambda}{16N}\sum_{j=1}^{N}H(Z_t^{j,N})\exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right].\end{aligned}$$

We gather the expectations terms, obtained with (2.18), in $I_t^{3,i}$, and we keep a fraction of Lyapunov function to control it:

$$\begin{aligned} I_t^{3,i} = \epsilon f(r_t^i) \Bigg(& (\alpha_X L_X + \beta_X L_C) \left(\frac{\sum_{j=1}^N |X_t^{j,N}|}{N} \right)^2 \exp\left(a\sqrt{H(Z_t^{i,N})} \right) \\ & + (\alpha_C L_X + \beta_C L_C) \left(\frac{\sum_{j=1}^N |C_t^{j,N}|}{N} \right)^2 \exp\left(a\sqrt{H(Z_t^{i,N})} \right) \\ & - \frac{\lambda}{16} H(Z_t^{i,N}) \exp\left(a\sqrt{H(Z_t^{i,N})} \right) - \frac{\lambda}{16N} \sum_{j=1}^N H(Z_t^{j,N}) \exp\left(a\sqrt{H(Z_t^{j,N})} \right) \Bigg). \end{aligned}$$

Finally, we define $\tilde{K}_t^i$ with the leftovers. It will, in particular, give the constraints on f which explain its choice.

$$\begin{aligned} \tilde{K}_t^i = & G_t^i \Big[2cf(r_t^i) + \frac{1}{2} f''(r_t^i) \left(2\sigma_x^2 \varphi_{\text{rc}} \left(|X_t^{i,N} - \bar{X}_t^i| \right)^2 \right) \\ & + f'(r_t^i) \Big((1 + \gamma\delta + L_X + \delta L_C) |X_t^{i,N} - \bar{X}_t^i| - |(X_t^{i,N})^3 - (\bar{X}_t^i)^3| + (1 + L_X + \delta L_C - \delta) |C_t^{i,N} - \bar{C}_t^i| \\ & + (\epsilon \mathcal{C}_{f,1} + \mathcal{C}_{f,2}) \sigma_x^2 \varphi_{\text{rc}} \left(|X_t^{i,N} - \bar{X}_t^i| \right)^2 r_t^i \Big) \Big] \\ & + \epsilon f(r_t^i) \left(4\tilde{B} - \frac{\lambda}{16} \tilde{H}(\bar{Z}_t^i) - \frac{\lambda}{16} \tilde{H}(Z_t^{i,N}) - \frac{\lambda}{16N} \sum_{j=1}^N \tilde{H}(\bar{Z}_t^j) - \frac{\lambda}{16N} \sum_{j=1}^N \tilde{H}(Z_t^{j,N}) \right). \end{aligned}$$

□

3.3 Controls of $I_t^{1,i}$, $I_t^{2,i}$ and $I_t^{3,i}$

Proof of Lemma 3.5. Since we assume

$$\frac{4}{\gamma} (\alpha_X L_X + \beta_X L_C) \leq \frac{\lambda}{32} \quad \text{and} \quad 4 (\alpha_C L_X + \beta_C L_C) \leq \frac{\lambda}{32},$$

and since

$$H(Z_t^{j,N}) \exp\left(a\sqrt{H(Z_t^{i,N})} \right) \leq H(Z_t^{i,N}) \exp\left(a\sqrt{H(Z_t^{i,N})} \right) + H(Z_t^{j,N}) \exp\left(a\sqrt{H(Z_t^{j,N})} \right)$$

we obtain

$$\begin{aligned} & (\alpha_X L_X + \beta_X L_C) \left(\frac{\sum_{j=1}^N |X_t^{j,N}|}{N} \right)^2 \exp\left(a\sqrt{H(Z_t^{i,N})} \right) + (\alpha_C L_X + \beta_C L_C) \left(\frac{\sum_{j=1}^N |C_t^{j,N}|}{N} \right)^2 \exp\left(a\sqrt{H(Z_t^{i,N})} \right) \\ & \quad - \frac{\lambda}{16N} \left(N H(Z_t^{i,N}) \exp\left(a\sqrt{H(Z_t^{i,N})} \right) + \sum_{j=1}^N H(Z_t^{j,N}) \exp\left(a\sqrt{H(Z_t^{j,N})} \right) \right) \leq 0. \end{aligned}$$

Then, for each $i \leq N$, and for all $t > 0$, $I_t^{3,i} \leq 0$. □

Proof of Lemma 3.4. We prove the non-positivity of $\frac{1}{N} \sum_{i=1}^N I_t^{2,i}$. First, since $f'\left(r_t^i\right) \leq 1$, we have

$$\frac{1}{N} \sum_{i=1}^N \left(\frac{1}{N} f'\left(r_t^i\right) G_t^i \sum_{j=1}^N \|Z_t^{j,N} - \bar{Z}_t^j\|_1 \right) \leq \frac{1}{N^2} \sum_{i,j=1}^N \|Z_t^{j,N} - \bar{Z}_t^j\|_1 G_t^i$$

$$\leq \frac{1}{N}\sum_{i=1}^{N}\|Z_t^{i,N}-\bar{Z}_t^i\|_1+\frac{2\epsilon}{N^2}\sum_{i,j=1}^{N}\|Z_t^{i,N}-\bar{Z}_t^i\|_1\left(\tilde{H}\left(\bar{Z}_t^j\right)+\tilde{H}(Z_t^{j,N})\right)$$

and, using Lemma 2.6 (i)

$$\frac{1}{N}\sum_{i=1}^{N}\|Z_t^{i,N}-\bar{Z}_t^i\|_1 \leq \frac{\mathcal{C}_1}{N}\sum_{i=1}^{N} f(r_t^i)G_t^i$$

and with Lemma 2.6 (iii)

$$\begin{aligned}
&\sum_{i,j=1}^{N}\|Z_t^{i,N}-\bar{Z}_t^i\|_1\left(\tilde{H}\left(\bar{Z}_t^j\right)+\tilde{H}(Z_t^{j,N})\right)\\
&\quad\leq \mathcal{C}_z\sum_{i,j=1}^{N} f(r_t^i)\left(1+\epsilon\sqrt{H(Z_t^{i,N})}+\epsilon\sqrt{H(\bar{Z}_t^i)}\right)\left(\tilde{H}\left(\bar{Z}_t^j\right)+\tilde{H}(Z_t^{j,N})\right)\\
&\quad\leq \mathcal{C}_z\sum_{i,j=1}^{N} f(r_t^i)\left(\tilde{H}\left(\bar{Z}_t^j\right)+\tilde{H}(Z_t^{j,N})\right)+\epsilon\mathcal{C}_z\sum_{i,j=1}^{N} f(r_t^i)\left(\sqrt{H(Z_t^{i,N})}+\sqrt{H(\bar{Z}_t^i)}\right)\left(\tilde{H}\left(\bar{Z}_t^j\right)+\tilde{H}(Z_t^{j,N})\right).
\end{aligned}$$

Using (2.9) from Lemma 2.4, we obtain for the first sum:

$$\mathcal{C}_z\sum_{i,j=1}^{N} f(r_t^i)\left(\tilde{H}\left(\bar{Z}_t^j\right)+\tilde{H}(Z_t^{j,N})\right)\leq \mathcal{C}_z\sum_{i,j=1}^{N} f(r_t^i)\left(H(\bar{Z}_t^j)\exp\left(a\sqrt{H(\bar{Z}_t^j)}\right)+H(Z_t^{j,N})\exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right).$$

With (2.10) from the same Lemma, we obtain for the second sum:

$$\begin{aligned}
&\epsilon\mathcal{C}_z\sum_{i,j=1}^{N} f(r_t^i)\left(\sqrt{H(Z_t^{i,N})}+\sqrt{H(\bar{Z}_t^i)}\right)\left(\tilde{H}\left(\bar{Z}_t^j\right)+\tilde{H}(Z_t^{j,N})\right)\\
&\quad\leq\epsilon\mathcal{C}_z\frac{2}{a}\sum_{i,j=1}^{N} f(r_t^i)\left(\sqrt{H(Z_t^{i,N})}+\sqrt{H(\bar{Z}_t^i)}\right)\left(\sqrt{H\left(\bar{Z}_t^j\right)}\exp\left(a\sqrt{H\left(\bar{Z}_t^j\right)}\right)+\sqrt{H\left(Z_t^{j,N}\right)}\exp\left(a\sqrt{H\left(Z_t^{j,N}\right)}\right)\right).
\end{aligned}$$

Since for all $(y_1,y_2,y_3,y_4)\in(\mathbb{R}^+)^4$, we have

$$(y_1+y_2)\left(y_3e^{ay_3}+y_4e^{ay_4}\right)\leq 2\left(y_1^2e^{ay_1}+y_2^2e^{ay_2}+y_3^2e^{ay_3}+y_4^2e^{ay_4}\right),$$

we obtain for this last sum

$$\begin{aligned}
&\frac{2\epsilon\mathcal{C}_z}{a}\sum_{i,j=1}^{N} f(r_t^i)\left(\sqrt{H(Z_t^{i,N})}+\sqrt{H(\bar{Z}_t^i)}\right)\left(\sqrt{H\left(\bar{Z}_t^j\right)}\exp\left(a\sqrt{H\left(\bar{Z}_t^j\right)}\right)+\sqrt{H\left(Z_t^{j,N}\right)}\exp\left(a\sqrt{H\left(Z_t^{j,N}\right)}\right)\right)\\
&\quad\leq\frac{4\epsilon\mathcal{C}_z}{a}\sum_{i,j=1}^{N} f(r_t^i)\left(H(\bar{Z}_t^i)\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)+H(Z_t^{i,N})\exp\left(a\sqrt{H(Z_t^{i,N})}\right)\right)\\
&\qquad+\frac{4\epsilon\mathcal{C}_z}{a}\sum_{i,j=1}^{N} f(r_t^i)\left(H(\bar{Z}_t^j)\exp\left(a\sqrt{H(\bar{Z}_t^j)}\right)+H(Z_t^{j,N})\exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right)\\
&\quad\leq\frac{4\epsilon\mathcal{C}_z}{a}N\sum_{i=1}^{N} f(r_t^i)\left(H(\bar{Z}_t^i)\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)+H(Z_t^{i,N})\exp\left(a\sqrt{H(Z_t^{i,N})}\right)\right)\\
&\qquad+\frac{4\epsilon\mathcal{C}_z}{a}\sum_{i,j=1}^{N} f(r_t^i)\left(H(\bar{Z}_t^j)\exp\left(a\sqrt{H(\bar{Z}_t^j)}\right)+H(Z_t^{j,N})\exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right).
\end{aligned}$$

Then, by reconsidering the first expression:

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{N}f'\left(r_t^i\right)G_t^i\sum_{j=1}^{N}\|Z_t^{j,N}-\bar{Z}_t^j\|_1\right)\\
\leq&\frac{\mathcal{C}_1}{N}\sum_{i=1}^{N}f(r_t^i)G_t^i+\frac{2\epsilon}{N^2}\mathcal{C}_z\sum_{i,j=1}^{N}f(r_t^i)\left(H(\bar{Z}_t^j)\exp\left(a\sqrt{H(\bar{Z}_t^j)}\right)+H(Z_t^{j,N})\exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right)\\
&+\frac{2\epsilon}{N^2}\frac{4\epsilon\mathcal{C}_z}{a}N\sum_{i=1}^{N}f(r_t^i)\left(H(\bar{Z}_t^i)\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)+H(Z_t^{i,N})\exp\left(a\sqrt{H(Z_t^{i,N})}\right)\right)\\
&+\frac{2\epsilon}{N^2}\frac{4\epsilon\mathcal{C}_z}{a}\sum_{i,j=1}^{N}f(r_t^i)\left(H(\bar{Z}_t^j)\exp\left(a\sqrt{H(\bar{Z}_t^j)}\right)+H(Z_t^{j,N})\exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right)
\end{aligned}$$

This way, by Assumption 2 since

$$L_X\mathcal{C}_1\leq\frac{c}{2},\quad 2\mathcal{C}_zL_X\leq\frac{\lambda}{64}\quad\text{and}\quad L_X\epsilon\frac{8\mathcal{C}_z}{a}\leq\frac{\lambda}{64},$$

we get

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{N}f'\left(r_t^i\right)G_t^i\sum_{j=1}^{N}\|Z_t^{j,N}-\bar{Z}_t^j\|_1\right)\\
\leq&\frac{1}{N}\frac{c}{2L_X}\sum_{i=1}^{N}f(r_t^i)G_t^i+\frac{\epsilon}{N^2}\frac{\lambda}{64L_X}\sum_{i,j=1}^{N}f(r_t^i)\left(H(\bar{Z}_t^j)\exp\left(a\sqrt{H(\bar{Z}_t^j)}\right)+H(Z_t^{j,N})\exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right)\\
&+\frac{\epsilon}{N}\frac{\lambda}{64L_X}\sum_{i=1}^{N}f(r_t^i)\left(H(\bar{Z}_t^i)\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)+H(Z_t^{i,N})\exp\left(a\sqrt{H(Z_t^{i,N})}\right)\right)\\
&+\frac{\epsilon}{N^2}\frac{\lambda}{64L_X}\sum_{i,j=1}^{N}f(r_t^i)\left(H(\bar{Z}_t^j)\exp\left(a\sqrt{H(\bar{Z}_t^j)}\right)+H(Z_t^{j,N})\exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right)
\end{aligned}$$

and we finally obtain "half" the result

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}G_t^if'(r_t^i)\left(\frac{L_X}{N}\left(\sum_{j=1}^{N}\|Z_t^{j,N}-\bar{Z}_t^j\|_1\right)\right)\\
&-\frac{c}{2}\frac{1}{N}\sum_{i=1}^{N}f(r_t^i)G_t^i-\frac{\epsilon}{2}\frac{1}{N}\sum_{i=1}^{N}f(r_t^i)\left[\frac{\lambda}{16}H(\bar{Z}_t^i)\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)+\frac{\lambda}{16}H(Z_t^{i,N})\exp\left(a\sqrt{H(Z_t^{i,N})}\right)\right]\\
&-\frac{\epsilon}{2}\frac{1}{N}\sum_{i=1}^{N}f(r_t^i)\left[\frac{\lambda}{16N}\sum_{j=1}^{N}H(\bar{Z}_t^j)\exp\left(a\sqrt{H(\bar{Z}_t^j)}\right)+\frac{\lambda}{16N}\sum_{j=1}^{N}H(Z_t^{j,N})\exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right]\leq 0.
\end{aligned}$$

Likewise, by Assumption 2, since

$$\delta L_C\mathcal{C}_1\leq\frac{c}{2},\quad 2\mathcal{C}_z\delta L_C\leq\frac{\lambda}{64}\quad\text{and}\quad\delta L_C\epsilon\frac{8\mathcal{C}_z}{a}\leq\frac{\lambda}{64},$$

we obtain the second "half"

$$\frac{1}{N}\sum_{i=1}^{N}\delta G_t^i f'(r_t^i)\left(\frac{L_C}{N}\left(\sum_{j=1}^{N}\|Z_t^{j,N}-\bar{Z}_t^j\|_1\right)\right)$$
$$-\frac{c}{2}\frac{1}{N}\sum_{i=1}^{N}f(r_t^i)G_t^i-\frac{\epsilon}{2}\frac{1}{N}\sum_{i=1}^{N}f(r_t^i)\left[\frac{\lambda}{16}H(\bar{Z}_t^i)\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)+\frac{\lambda}{16}H(Z_t^{i,N})\exp\left(a\sqrt{H(Z_t^{i,N})}\right)\right]$$
$$-\frac{\epsilon}{2}\frac{1}{N}\sum_{i=1}^{N}f(r_t^i)\left[\frac{\lambda}{16N}\sum_{j=1}^{N}H(\bar{Z}_t^j)\exp\left(a\sqrt{H(\bar{Z}_t^j)}\right)+\frac{\lambda}{16N}\sum_{j=1}^{N}H(Z_t^{j,N})\exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right]\leq 0.$$

Eventually, we have proved $\sum_{i=1}^{N} I_t^{2,i}\leq 0$. □

Proof of Lemma 3.3. Since $f'(r)\leq 1$, we have by Cauchy-Schwarz

$$\mathbb{E}\left(G_t^i f'(r_t^i)\left(\left|\frac{1}{N}\sum_{j=1}^{N}K_X(\bar{Z}_t^i-\bar{Z}_t^j)-K_X*\bar{\mu}_t(\bar{Z}_t^i)\right|\right)\right)$$
$$\leq\mathbb{E}\left(|G_t^i|^2\right)^{1/2}\mathbb{E}\left(\left|\frac{1}{N}\sum_{j=1}^{N}K_X(\bar{Z}_t^i-\bar{Z}_t^j)-K_X*\bar{\mu}_t(\bar{Z}_t^i)\right|^2\right)^{1/2}.$$

By Lemma 3.1, we have for each $i\leq N$, for all $t\geq 0$, $\mathbb{E}[(G_t^i)^2]\leq\mathcal{C}_{G,2}$.

Moreover, we notice that $(\bar{Z}_t^j)_j$ are i.i.d with law $\bar{\mu}_t$. Let's denote $\bar{Z}_t$ a generic random variable of law $\bar{\mu}_t$ independent of $\bar{Z}_t^i$. The calculus of the right term of the product has already be done in Subsection 1.4, and we have (1.9):

$$\mathbb{E}\left(\left|\frac{1}{N}\sum_{j=1}^{N}K_X(\bar{Z}_t^i-\bar{Z}_t^j)-K_X*\bar{\mu}_t(\bar{Z}_t^i)\right|^2\right)\leq\frac{8L_X^2}{N}\mathbb{E}(\|\bar{Z}_t\|_1^2)\mathbb{E}(\|\bar{Z}_t\|_1^2).$$

A similar calculation yields

$$\mathbb{E}\left(\left|\frac{1}{N}\sum_{j=1}^{N}K_C(\bar{Z}_t^i-\bar{Z}_t^j)-K_C*\bar{\mu}_t(\bar{Z}_t^i)\right|^2\right)\leq\frac{8L_C^2}{N}\mathbb{E}(\|\bar{Z}_t\|_1^2).$$

By Lemma 2.3, $\mathbb{E}\left(|\bar{X}_t|^2+|\bar{C}_t|^2\right)\leq\mathcal{C}_{init,2}$. In particular,

$$\mathbb{E}\left(\|\bar{Z}_t\|_1^2\right)=\mathbb{E}\left(\left[|\bar{X}_t|+|\bar{C}_t|\right]^2\right)\leq 2\mathbb{E}\left(|\bar{X}_t|^2+|\bar{C}_t|^2\right)\leq 2\mathcal{C}_{init,2}.$$

Thus

$$\mathbb{E}\left(G_t^i f'(r_t^i)\left(\left|\frac{1}{N}\sum_{j=1}^{N}K_X(\bar{Z}_t^i-\bar{Z}_t^j)-K_X*\bar{\mu}_t(\bar{Z}_t^i)\right|\right)\right)\leq L_X\mathcal{C}_{G,2}^{1/2}\sqrt{2\mathcal{C}_{init,2}}\sqrt{\frac{8}{N}}$$

and likewise

$$\mathbb{E}\left(G_t^i f'(r_t^i)\left(\left|\frac{1}{N}\sum_{j=1}^{N}K_C(\bar{Z}_t^i-\bar{Z}_t^j)-K_C*\bar{\mu}_t(\bar{Z}_t^i)\right|\right)\right)\leq L_C\mathcal{C}_{G,2}^{1/2}\sqrt{2\mathcal{C}_{init,2}}\sqrt{\frac{8}{N}}.$$

□

3.4 Contraction in various regions of space

The goal of this section is to prove the Lemma 3.2 i.e show that for each $i \leq N$, for all $t > 0$, we have the following control

$$\mathbb{E}\tilde{K}_t^i \leq \xi \left(2 + \delta\gamma + L_X + \delta L_C - L_C - \frac{1+L_X}{\delta}\right) \mathbb{E}G_t^i.$$

Recall

$$\begin{aligned}
\tilde{K}_t^i =& G_t^i \Big[2cf(r_t^i) + \frac{1}{2} f''(r_t^i) \left(2\sigma_x^2 \varphi_{\text{rc}} \left(|X_t^{i,N} - \bar{X}_t^i| \right)^2 \right) \\
&+ f'(r_t^i) \Big((1 + \gamma\delta + L_X + \delta L_C) |X_t^{i,N} - \bar{X}_t^i| - |(X_t^{i,N})^3 - (\bar{X}_t^i)^3| + (1 + L_X + \delta L_C - \delta) |C_t^{i,N} - \bar{C}_t^i| \\
&+ (\epsilon \mathcal{C}_{f,1} + \mathcal{C}_{f,2}) \sigma_x^2 \varphi_{\text{rc}} \left(|X_t^{i,N} - \bar{X}_t^i| \right)^2 r_t^i \Big) \Big] \\
&+ \epsilon f(r_t^i) \left(4\tilde{B} - \frac{\lambda}{16} \tilde{H}(\bar{Z}_t^i) - \frac{\lambda}{16} \tilde{H}(Z_t^{i,N}) - \frac{\lambda}{16N} \sum_{j=1}^N \tilde{H}(\bar{Z}_t^j) - \frac{\lambda}{16N} \sum_{j=1}^N \tilde{H}(Z_t^{j,N}) \right),
\end{aligned}$$

which is a quantity that contains every term we have not yet dealt with. To prove Lemma 3.2, we divide for each $i \in \{1, ..., N\}$ the space into three regions

$$\begin{aligned}
\text{Reg}_1^i &= \left\{ (\bar{Z}_t^i, Z_t^{i,N}) \text{ s.t. } |\bar{X}_t^i - X_t^{i,N}| \geq \xi \text{ and } r_t^i \leq R \right\}, \\
\text{Reg}_2^i &= \left\{ (\bar{Z}_t^i, Z_t^{i,N}) \text{ s.t. } |\bar{X}_t^i - X_t^{i,N}| < \xi \text{ and } r_t^i \leq R_1 \right\}, \\
\text{Reg}_3^i &= \left\{ (\bar{Z}_t^i, Z_t^{i,N}) \text{ s.t. } r_t^i > R \right\},
\end{aligned}$$

where R was given in Lemma 2.1, and consider

$$\frac{1}{N} \sum_{i=1}^N \mathbb{E}\tilde{K}_t^i = \frac{1}{N} \sum_{i=1}^N \left(\mathbb{E}\left(\tilde{K}_t^i \mathbb{1}_{\text{Reg}_1^i} \right) + \mathbb{E}\left(\tilde{K}_t^i \mathbb{1}_{\text{Reg}_2^i} \right) + \mathbb{E}\left(\tilde{K}_t^i \mathbb{1}_{\text{Reg}_3^i} \right) \right).$$

3.4.1 Region 1 : $\xi \leq |X_t^{i,N} - \bar{X}_t^i|$ and $r_t^i \leq R$.

In this region of space, since $\varphi_{\text{rc}}(|X_t^{i,N} - \bar{X}_t^i|) = 1$, we have

$$\begin{aligned}
\tilde{K}_t^i \mathbb{1}_{\text{Reg}_1^i} =& \mathbb{1}_{\text{Reg}_1^i} \Big(G_t^i \Big[2cf(r_t^i) + 2\sigma_x^2 f''(r_t^i) + f'(r_t^i) (\epsilon \mathcal{C}_{f,1} + \mathcal{C}_{f,2}) \sigma_x^2 r_t^i \\
&+ f'(r_t^i)(1 + \gamma\delta + L_X + \delta L_C) |X_t^{i,N} - \bar{X}_t^i| \Big] \\
&- G_t^i f'(r_t^i)(\delta - 1 - L_X - \delta L_C) |C_t^{i,N} - \bar{C}_t^i| - G_t^i f'(r_t^i) |(X_t^{i,N})^3 - (\bar{X}_t^i)^3| \\
&+ \epsilon f(r_t^i) 4\tilde{B} - \epsilon f(r_t^i) \left(\frac{\lambda}{16} \tilde{H}(\bar{Z}_t^i) + \frac{\lambda}{16} \tilde{H}(Z_t^{i,N}) + \frac{\lambda}{16N} \sum_{j=1}^N \tilde{H}(\bar{Z}_t^j) + \frac{\lambda}{16N} \sum_{j=1}^N \tilde{H}(Z_t^{j,N}) \right) \Big),
\end{aligned}$$

and since $\tilde{H}(z) \geq 0$, $|X_t^{i,N} - \bar{X}_t^i| \leq r_t^i$, $\delta > \frac{1+L_X}{1-L_C}$ (by Assumption 2(iii)) and $1 \leq G_t^i$ we have

$$\tilde{K}_t^i \mathbb{1}_{\text{Reg}_1^i} \leq \mathbb{1}_{\text{Reg}_1^i} G_t^i \left[(2c + 4\epsilon\tilde{B}) f(r_t^i) + 2\sigma_x^2 f''(r_t^i) + f'(r_t^i) \left(1 + \delta\gamma + L_X + \delta L_C + (\epsilon \mathcal{C}_{f,1} + \mathcal{C}_{f,2}) \sigma_x^2 \right) r_t^i \right].$$

Using the definition f given in (2.21) we get

$$2\sigma_x^2 f''(r_t^i) + f'(r_t^i) \left(1 + \delta\gamma + L_X + L_C + (\epsilon \mathcal{C}_{f,1} + \mathcal{C}_{f,2}) \sigma_x^2 \right) r_t^i$$

$$= 2\sigma_x^2\phi'(r_t^i)g(r_t^i) + 2\sigma_x^2\phi(r_t^i)g'(r_t^i) + \phi(r_t^i)g(r_t^i)\left(1+\delta\gamma + L_X + \delta L_C + (\epsilon\mathcal{C}_{f,1}+\mathcal{C}_{f,2})\,\sigma_x^2\right) r_t^i$$
$$= 2\sigma_x^2\phi(r_t^i)g'(r_t^i) = -(2c+4\epsilon\tilde{B})\Phi(r_t^i).$$

Thus

$$\begin{aligned}(2c+4\epsilon\tilde{B})f(r_t^i) + 2\sigma_x^2 f''(r_t^i) + f'(r_t^i)\left(1+\delta\gamma + L_X + \delta L_C + (\epsilon\mathcal{C}_{f,1}+\mathcal{C}_{f,2})\,\sigma_x^2\right) r_t^i \\ = (2c+4\epsilon\tilde{B})f(r_t^i) - (2c+4\epsilon\tilde{B})\Phi(r_t^i) \\ \leq 0.\end{aligned} \tag{3.12}$$

Eventually, in this region of space

$$\tilde{K}_t^i \mathbb{1}_{\text{Reg}_1^i} \leq 0.$$

3.4.2 Region 2 : $|X_t^{i,N} - \bar{X}_t^i| < \xi$ and $r_t^i \leq R$.

In this region, we can write $\tilde{K}_t^i$ as

$$\begin{aligned}\tilde{K}_t^i \mathbb{1}_{\text{Reg}_2^i} =& \mathbb{1}_{\text{Reg}_2^i} G_t^i \Big[2cf(r_t^i) + \varphi_{\text{rc}}\left(|X_t^{i,N} - \bar{X}_t^i|\right)^2 \left[2\sigma_x^2 f''(r_t^i) + (\epsilon\mathcal{C}_{f,1}+\mathcal{C}_{f,2})\,\sigma_x^2 r_t^i f'(r_t^i)\right] \\ &+ f'(r_t^i)\left((1+\gamma\delta + L_X + \delta L_C)|X_t^{i,N} - \bar{X}_t^i| - (\delta - 1 - L_X - \delta L_C)|C_t^{i,N} - \bar{C}_t^i|\right)\Big] \\ &- \mathbb{1}_{\text{Reg}_2^i} G_t^i f'(r_t^i)|(X_t^{i,N})^3 - (\bar{X}_t^i)^3| \\ &+ \mathbb{1}_{\text{Reg}_2^i}\epsilon f(r_t^i) 4\tilde{B} - \epsilon f(r_t^i)\mathbb{1}_{\text{Reg}_2^i}\left(\frac{\lambda}{16}\tilde{H}(\bar{Z}_t^i) + \frac{\lambda}{16}\tilde{H}(Z_t^{i,N}) + \frac{\lambda}{16N}\sum_{j=1}^N \tilde{H}(\bar{Z}_t^j) + \frac{\lambda}{16N}\sum_{j=1}^N \tilde{H}(Z_t^{j,N})\right).\end{aligned}$$

Since $r_t^i = |X_t^{i,N} - \bar{X}_t^i| + \delta|C_t^{i,N} - \bar{C}_t^i|$ and $|X_t^{i,N} - \bar{X}_t^i| < \xi$, we have $|C_t^{i,N} - \bar{C}_t^i| \geq (r_t^i - \xi)/\delta$. Since $\delta > \dfrac{1+L_X}{1-L_C}$, we obtain

$$\begin{aligned}\tilde{K}_t^i \mathbb{1}_{\text{Reg}_2^i} \leq& \mathbb{1}_{\text{Reg}_2^i} G_t^i \Big[2cf(r_t^i) + \varphi_{\text{rc}}\left(|X_t^{i,N} - \bar{X}_t^i|\right)^2 \left[2\sigma_x^2 f''(r_t^i) + (\epsilon\mathcal{C}_{f,1}+\mathcal{C}_{f,2})\,\sigma_x^2 r_t^i f'(r_t^i)\right] \\ &+ f'(r_t^i)\left((1+\gamma\delta + L_X + \delta L_C)\xi - (\delta - \delta L_C - 1 - L_X)\frac{r_t^i - \xi}{\delta}\right)\Big] \\ &+ \epsilon f(r_t^i)\mathbb{1}_{\text{Reg}_2^i} 4\tilde{B} \\ \leq& \varphi_{\text{rc}}\left(|X_t^{i,N} - \bar{X}_t^i|\right)^2 G_t^i \mathbb{1}_{\text{Reg}_2^i}\left[2\sigma_x^2 f''(r_t^i) + (\epsilon\mathcal{C}_{f,1}+\mathcal{C}_{f,2})\,\sigma_x^2 r_t^i f'(r_t^i)\right] \\ &+ \mathbb{1}_{\text{Reg}_2^i} G_t^i f'(r_t^i)\xi\left[1 + \gamma\delta + L_X + \delta L_C + 1 - L_C - \frac{1+L_X}{\delta}\right] \\ &+ \mathbb{1}_{\text{Reg}_2^i} G_t^i\left((2c+4\epsilon\tilde{B})f(r_t^i) - r_t^i f'(r_t^i)\left(1 - L_C - \frac{1+L_X}{\delta}\right)\right).\end{aligned}$$

By (3.12),

$$2\sigma_x^2 f''(r_t^i) + (\epsilon\mathcal{C}_{f,1}+\mathcal{C}_{f,2})\,\sigma_x^2 r_t^i f'(r_t^i) = -(2c+4\epsilon\tilde{B})\Phi(r_t^i) - f'(r_t^i)r_t^i\left(1+\delta\gamma + L_X + L_C\right) \leq 0,$$

and by Lemma 2.5

$$2c + 4\epsilon\tilde{B} \leq \left(1 - L_C - \frac{1+L_X}{\delta}\right)\min_{r\in]0,R]}\frac{f'(r)r}{f(r)},$$

we obtain

$$\tilde{K}_t^i \mathbb{1}_{\text{Reg}_2^i} \leq \mathbb{1}_{\text{Reg}_2^i} G_t^i f'(r_t^i) \xi \left[1 + \gamma\delta + L_X + \delta L_C + 1 - L_C - \frac{1+L_X}{\delta} \right].$$

Finally, since $f'(r) \leq 1$,

$$\mathbb{E}\tilde{K}_t^i \mathbb{1}_{\text{Reg}_2^i} \leq \xi \left(2 + \delta\gamma + L_X + \delta L_C - L_C - \frac{1+L_X}{\delta} \right) \mathbb{E} G_t^i.$$

3.4.3 Region 3 : $r_t^i \geq R$.

In this region of space $f' = f'' = 0$ and f is constant, and we therefore have

$$\tilde{K}_t^i \mathbb{1}_{\text{Reg}_3^i} = f(r_t^i) \mathbb{1}_{\text{Reg}_3^i} \left[2cG_t^i + 4\epsilon\tilde{B} - \frac{\lambda\epsilon}{16} \left(\tilde{H}(\bar{Z}_t^i) + \tilde{H}(Z_t^{i,N}) + \frac{1}{N}\sum_{j=1}^N \tilde{H}(\bar{Z}_t^j) + \frac{1}{N}\sum_{j=1}^N \tilde{H}(Z_t^{j,N}) \right) \right].$$

Since $G_t^i = 1 + \epsilon\tilde{H}(\bar{Z}_t^i) + \epsilon\tilde{H}(Z_t^{i,N}) + \frac{\epsilon}{N}\sum_{j=1}^N \tilde{H}(Z_t^{j,N}) + \frac{\epsilon}{N}\sum_{j=1}^N \tilde{H}(\bar{Z}_t^j)$ by definition (2.24), we can write

$$\tilde{K}_t^i \mathbb{1}_{\text{Reg}_3^i} = f(r_t^i) \mathbb{1}_{\text{Reg}_3^i} \left[2c + 4\epsilon\tilde{B} + \epsilon \left(2c - \frac{\lambda}{16} \right) \left(\tilde{H}(\bar{Z}_t^i) + \tilde{H}(Z_t^{i,N}) + \frac{1}{N}\sum_{j=1}^N \tilde{H}(\bar{Z}_t^j) + \frac{1}{N}\sum_{j=1}^N \tilde{H}(Z_t^{j,N}) \right) \right]$$

Since $c \leq \lambda/32$ from Assumption 2, we obtain

$$\tilde{K}_t^i \mathbb{1}_{\text{Reg}_3^i} \leq f(r_t^i) \mathbb{1}_{\text{Reg}_3^i} \left[2c + 4\epsilon\tilde{B} - \epsilon \left(\frac{\lambda}{16} - 2c \right) \left(H(\bar{Z}_t^i) + H(Z_t^{i,N}) \right) \right].$$

We have chosen R such that, for z, z' satisfying $r \geq R$, we have $H(z) + H(z') \geq 80\frac{\tilde{B}}{\lambda}$ by Lemma 2.1 (iv). Therefore

$$\begin{aligned}
\tilde{K}_t^i \mathbb{1}_{\text{Reg}_3^i} &\leq f(r_t^i) \mathbb{1}_{\text{Reg}_3^i} \left(2c + 4\epsilon\tilde{B} - \epsilon \left(\frac{\lambda}{16} - 2c \right) 80\frac{\tilde{B}}{\lambda} \right) \\
&= f(r_t^i) \mathbb{1}_{\text{Reg}_3^i} \left(2c \left(1 + 80\frac{\epsilon\tilde{B}}{\lambda} \right) - \epsilon\tilde{B} \right)
\end{aligned}$$

Lemma 2.5 and more specifically the inequality

$$c \leq \frac{1}{2} \frac{\epsilon\tilde{B}}{1 + 80\frac{\epsilon\tilde{B}}{\lambda}} = \frac{\lambda}{160} \frac{\frac{80\epsilon\tilde{B}}{\lambda}}{1 + \frac{80\epsilon\tilde{B}}{\lambda}}$$

yields the desired result: $\tilde{K}_t^i \mathbb{1}_{\text{Reg}_3^i} \leq 0$.

A Various technical lemmas

A.1 On Lemma 2.1

Lemma A.1. *For all $z = (x, c), z' = (x', c') \in \mathbb{R}^d$, denoting $r(z, z') = |x - x'| + \delta|c - c'|$*

$$r(z, z')^2 \leq \frac{16(1+\delta^2)}{\min(\gamma, 1)} \left(H(z) + H(z') \right), \tag{A.1}$$

so that, in particular, for any constant $B > 0$, if $r(z, z') \geq R = \sqrt{\frac{1280(1+\delta^2)B}{\lambda \min(\gamma, 1)}}$, then

$$\lambda H(z) + \lambda H(z') \geq 80B.$$

Proof. We have $H(z) \geq \frac{\gamma}{4}x^2 + \frac{c^2}{4} \geq \frac{1}{4}\min(\gamma, 1)\left(x^2 + c^2\right)$. Thus

$$\begin{aligned} r(z,z')^2 =& \left(|x - x'| + \delta|c - c'|\right)^2 \\ \leq& 2|x - x'|^2 + 2\delta^2|c - c'|^2 \\ \leq& 4x^2 + 4x'^2 + 4\delta^2 c^2 + 4\delta^2 c'^2 \\ \leq& 4(1+\delta^2)(x^2 + c^2) + 4(1+\delta^2)(x'^2 + c'^2) \\ \leq& 16\frac{(1+\delta^2)}{\min(\gamma, 1)}\left(H(z) + H(z')\right) \end{aligned}$$

□

A.2 Proof of Lyapunov's property of H and its consequences

Lyapunov's property

Proof of Lemma 2.2. We write the proof for (2.3), as it also yields (2.4) by considering μ to be the empirical measure. We notice

$$\partial_c H = c + \alpha \quad \text{and} \quad \partial_x H = \gamma x + \beta,$$

so

$$\begin{aligned} \mathcal{L}_\mu H(z) =& \partial_x H(z)(x - x^3) + \partial_x H(z) K_X * \mu(z) - c\partial_c H(z) + \partial_c H(z) K_C * \mu(z) + \frac{\sigma_x^2 \gamma}{2} + \frac{\sigma_c^2}{2} \\ =& (\gamma x + \beta)(x - x^3) - c(c + \alpha) + (\gamma x + \beta) K_X * \mu(z) + (c + \alpha) K_C * \mu(z) + \frac{\sigma_x^2 \gamma}{2} + \frac{\sigma_c^2}{2}. \end{aligned}$$

First, we focus on interaction terms. We have

$$\begin{aligned} |K_X * \mu(z)| \leq& \int_{\mathbb{R}^2} |K_X(z - z')| \mu(dz') \\ \leq& \int_{\mathbb{R}^2} L_X(\|z\|_1 + \|z'\|_1) \mu(dz'). \end{aligned}$$

Hence,

$$\begin{aligned} (\gamma x + \beta) K_X * \mu(z) \leq& L_X(\gamma|x| + \beta)(|x| + |c| + \mathbb{E}_\mu(|X|) + \mathbb{E}_\mu(|C|)) \\ \leq& L_X\left(\gamma|x|^2 + \gamma|x||c| + \gamma|x|\mathbb{E}_\mu(|X|) + \gamma|x|\mathbb{E}_\mu(|C|) + \beta|x| + \beta|c| + \beta\mathbb{E}_\mu(|X|) + \beta\mathbb{E}_\mu(|C|)\right), \end{aligned}$$

and using Young's inequality $ab \leq \frac{\alpha}{2}a^2 + \frac{1}{2\alpha}b^2$ ($\alpha = 16$ when we separate x term and c term, and $\alpha = 1$ otherwise on the various terms we get

$$\begin{aligned} (\gamma x + \beta) K_X * \mu(z) \leq& L_X\Big(\gamma|x|^2 + 8\gamma^2|x|^2 + \frac{|c|^2}{32} + \frac{\gamma}{2}|x|^2 + \frac{\gamma}{2}\mathbb{E}_\mu(|X|)^2 + 8\gamma^2|x|^2 + \frac{\mathbb{E}_\mu(|C|)^2}{32} + \frac{\beta^2}{2} \\ &+ \frac{|x|^2}{2} + 8\beta^2 + \frac{|c|^2}{32} + \frac{\beta^2}{2} + \frac{1}{2}\mathbb{E}_\mu(|X|)^2 + 8\beta^2 + \frac{\mathbb{E}_\mu(|C|)^2}{32}\Big) \end{aligned}$$

$$=L_X\left(17\beta^2+|x|^2\left(\frac{1}{2}+\frac{3}{2}\gamma+16\gamma^2\right)+\frac{|c|^2}{16}+\mathbb{E}_\mu(|X|)^2\left(\frac{\gamma}{2}+\frac{1}{2}\right)+\frac{\mathbb{E}_\mu(|C|)^2}{16}\right).$$

Likewise

$$(c+\alpha)K_C*\mu(z)\leq L_C\left(17\alpha^2+\frac{17}{2}|x|^2+|c|^2\left(\frac{3}{2}+\frac{3}{32}\right)+\mathbb{E}_\mu(|X|)^2\frac{17}{2}+\mathbb{E}_\mu(|C|)^2\left(\frac{1}{2}+\frac{1}{32}\right)\right).$$

The idea is to bound $\lambda H(z)+\mathcal{L}_\mu H(z)$, by distinguishing 3 types of terms: we isolate terms in $\mathbb{E}_\mu(|C|)^2-c^2$, $\mathbb{E}_\mu(|X|)^2-x^2$, and we group polynomials terms. Then, we notice the polynomial is upper bounded by a constant A. Thus

$$\begin{aligned}
&\lambda H(z)+\mathcal{L}_\mu H(z)-\frac{\sigma_x^2\gamma}{2}-\frac{\sigma_c^2}{2}\\
&\quad=\lambda\left(\frac{1}{2}\gamma x^2+\beta x+\frac{1}{2}c^2+\alpha c+H_0\right)+(\gamma x+\beta)(x-x^3)-c(c+\alpha)\\
&\qquad+(\gamma x+\beta)K_X*\mu(z)+(c+\alpha)K_C*\mu(z)\\
&\quad\leq\left(\lambda H_0+17\beta^2L_X+17\alpha^2L_C\right)-\gamma x^4-\beta x^3+(1+\lambda)\beta x\\
&\qquad+\left((1+\frac{\lambda}{2})\gamma+L_X\left(1+2\gamma+16\gamma^2\right)+17L_C\right)x^2\\
&\qquad+\left(\frac{L_C}{8}+L_C\left(2+\frac{1}{8}\right)-\left(1-\frac{\lambda}{2}\right)\right)c^2-(1-\lambda)\alpha c\\
&\qquad+\left(\frac{L_X}{16}+\frac{L_C}{2}+\frac{L_C}{32}\right)\left(\mathbb{E}_\mu(|C|)^2-c^2\right)\\
&\qquad+\left(\frac{\gamma}{2}L_X+\frac{1}{2}L_X+\frac{17}{2}L_C\right)\left(\mathbb{E}_\mu(|X|)^2-x^2\right)
\end{aligned}$$

Provided

$$\frac{L_X}{8}+L_C\left(2+\frac{1}{8}\right)<1-\frac{\lambda}{2},$$

there is $A\geq 0$ such that

$$\begin{aligned}
-\gamma x^4-\beta x^3+(1+\lambda)\beta x+\left((1+\frac{\lambda}{2})\gamma+L_X\left(1+2\gamma+16\gamma^2\right)+17L_C\right)x^2\\
+\left(\frac{L_C}{8}+L_C\left(2+\frac{1}{8}\right)-\left(1-\frac{\lambda}{2}\right)\right)c^2-(1-\lambda)\alpha c\leq A.
\end{aligned}$$

Hence the result:

$$\mathcal{L}_\mu H(\bar{z})\leq B+(\alpha_X L_X+\beta_X L_C)\left(\mathbb{E}_\mu(|X|)^2-\bar{x}^2\right)+(\alpha_C L_X+\beta_C L_C)\left(\mathbb{E}_\mu(|C|)^2-\bar{c}^2\right)-\lambda H(\bar{z}).$$

□

First consequences

Proof of Proposition 2.1. Inequality (2.5) simply relies on the sum of (2.4) for each i and the fact that $\mathcal{L}^{j,N}\left(H\left(Z_t^{i,N}\right)\right)=0$ for $i\neq j$:

$$\frac{1}{N}\sum_{i=1}^N\mathcal{L}^N\left(H\left(Z_t^{i,N}\right)\right)=\frac{1}{N}\sum_{i=1}^N\mathcal{L}^{i,N}\left(H\left(Z_t^{i,N}\right)\right)$$

$$
\begin{aligned}
\leq & \frac{1}{N}\sum_{i=1}^{N}\left[B+(\alpha_X L_X+\beta_X L_C)\left(\left(\frac{1}{N}\sum_{k=1}^{N}|X_t^{k,N}|\right)^2-(X_t^{i,N})^2\right)\right. \\
& \left.+(\alpha_C L_X+\beta_C L_C)\left(\left(\frac{1}{N}\sum_{k=1}^{N}|C_t^{k,N}|\right)^2-(C_t^{i,N})^2\right)-\lambda H\left(Z_t^{i,N}\right)\right] \\
\leq & B-\lambda\frac{1}{N}\sum_{i=1}^{N}H\left(Z_t^{i,N}\right).
\end{aligned}
$$

The last inequality uses the fact that $\left(\frac{1}{N}\sum_{i=1}^{N}|y_i|\right)^2-\frac{1}{N}\sum_{i=1}^{N}(y_i)^2\leq 0$ for all $(y_i)_{1\leq i\leq N}\in\mathbb{R}^N$. □

Bounds on the second moments of processes We can now prove the uniform in bounds on the second moments of $X_t^{i,N}$, $C_t^{i,N}$, $\bar{X}_t^i$ and $\bar{C}_t^i$ from (2.3) and (2.4).

Proof of Proposition 1.2. K_X and K_C are Lipschitz with constants L_X and L_C respectively. We do not assume any bound on these constants. We assume for each $i\leq N$, $\mathbb{E}(|X_0^{i,N}|^2)<+\infty$ and $\mathbb{E}(|C_0^{i,N}|^2)<+\infty$. We have

$$
d\left(\frac{e^{\lambda t}}{N}\sum_{i=1}^{N}H\left(Z_t^{i,N}\right)\right)=\lambda\frac{e^{\lambda t}}{N}\sum_{i=1}^{N}H\left(Z_t^{i,N}\right)dt+e^{\lambda t}\mathcal{L}^N\left(\frac{1}{N}\sum_{i=1}^{N}H\left(Z_t^{i,N}\right)\right)dt+dM_t,
$$

where M_t is a local martingale. Using (2.4)

$$
d\left(\frac{e^{\lambda t}}{N}\sum_{i=1}^{N}H\left(Z_t^{i,N}\right)\right)=A_t dt+dM_t,
$$

where $A_t\leq Be^{\lambda t}$. Let τ_n be an increasing sequence of localizing stopping times converging to ∞ for M_t

$$
\begin{aligned}
\mathbb{E}\left(\frac{e^{\lambda t\wedge\tau_n}}{N}\sum_{i=1}^{N}H\left(Z_{t\wedge\tau_n}^{i,N}\right)\right)\leq & \mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}H\left(Z_0^{i,N}\right)\right)+\mathbb{E}\left(\int_0^{t\wedge\tau_n}Be^{\lambda s}ds\right) \\
\leq & \mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}H\left(Z_0^{i,N}\right)\right)+B\frac{\mathbb{E}\left(e^{\lambda t\wedge\tau_n}\right)-1}{\lambda} \\
\leq & \mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}H\left(Z_0^{i,N}\right)\right)+B\max\left(\frac{e^{\lambda t}-1}{\lambda},\frac{1}{|\lambda|}\right),
\end{aligned}
$$

where the maximum on this last inequality depends on the sign of λ. By Fatou's lemma, we obtain

$$
\begin{aligned}
e^{\lambda t}\mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}H\left(Z_t^{i,N}\right)\right)= & \mathbb{E}\left(\liminf_{n\to\infty}\frac{e^{\lambda t\wedge\tau_n}}{N}\sum_{i=1}^{N}H\left(Z_{t\wedge\tau_n}^{i,N}\right)\right) \\
\leq & \liminf_{n\to\infty}\mathbb{E}\left(\frac{e^{\lambda t\wedge\tau_n}}{N}\sum_{i=1}^{N}H\left(Z_{t\wedge\tau_n}^{i,N}\right)\right) \\
\leq & \mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}H\left(Z_0^{i,N}\right)\right)+B\max\left(\frac{e^{\lambda t}-1}{\lambda},\frac{1}{|\lambda|}\right).
\end{aligned}
$$

Hence the various bounds on $\mathbb{E}\left(|X_t^{i,N}|^2\right)$ and $\mathbb{E}\left(|C_t^{i,N}|^2\right)$, since by Lemma 2.1 (i) we have

$$
\mathbb{E}H\left(Z_t^{i,N}\right)\geq\frac{\gamma}{4}\mathbb{E}\left(|X_t^{i,N}|^2\right)+\frac{1}{4}\mathbb{E}\left(|C_t^{i,N}|^2\right)\quad\text{and}\quad\mathbb{E}H\left(Z_0^{i,N}\right)\leq\gamma\mathbb{E}\left(|X_0^{i,N}|^2\right)+\mathbb{E}\left(|C_0^{i,N}|^2\right)+\frac{3}{2}H_0.
$$

These bounds are uniform in time provided $\lambda > 0$, i.e $\frac{L_X}{8} + L_C\left(2 + \frac{3}{32}\right) < 1$. □

Proof of Proposition 1.3 and Lemma 2.3. The proof are done in exactly the same fashion as the proof of Proposition 1.2 above using (2.3).

□

A.3 Proof of Lemma 2.5

We now prove that there are constants c, ϵ and δ such that

$$c + 2\epsilon\tilde{B} \leq \frac{\sigma_x^2}{2}\left(\int_0^R \Phi(s)\phi(s)^{-1}ds\right)^{-1} \tag{A.2}$$

$$2c + 4\epsilon\tilde{B} \leq \left(1 - L_C - \frac{1 + L_X}{\delta}\right)\min_{r\in]0,R]}\frac{f'(r)r}{f(r)} \tag{A.3}$$

$$c \leq \frac{\lambda}{160}\frac{\frac{80\epsilon\tilde{B}}{\lambda}}{1 + \frac{80\epsilon\tilde{B}}{\lambda}} \tag{A.4}$$

$$\delta > \frac{1 + L_X}{1 - L_C} \tag{A.5}$$

- Since for all $u \geq 0$, $0 < \phi(u) \leq 1$, we have $0 < \Phi(s) = \int_0^s \phi(u)\,du \leq s$, i.e $s/\Phi(s) \geq 1$. Therefore

$$\inf_{r\in]0,R]}\frac{r\phi(r)}{\Phi(r)} \geq \inf_{r\in]0,R]}\phi(r) = \phi(R).$$

  It is thus sufficient for (A.3) to have

$$2c + 4\epsilon\tilde{B} \leq \frac{1}{2}\left(1 - L_C - \frac{1 + L_X}{\delta}\right)\phi(R).$$

- We have

$$\phi(r) \leq \exp\left(-\frac{1}{4\sigma_x^2}r^2\right).$$

  So

$$\Phi(r) \leq \int_0^\infty \exp\left(-\frac{r^2}{4\sigma_x^2}\right)dr = \sigma_x\sqrt{\pi}.$$

  Then

$$\int_0^R \frac{\Phi(r)}{\phi(r)}dr \leq \sigma_x\sqrt{\pi}R\frac{1}{\phi(R)}.$$

  It is thus sufficient for (A.2) that

$$c + 2\epsilon\tilde{B} \leq \frac{\sigma_x}{2\sqrt{\pi}}\frac{\phi(R)}{R}.$$

- The various conditions involving c invite us to consider $2\epsilon\tilde{B} = \eta c$. Then

$$\begin{aligned} c \leq \frac{\lambda}{160}\frac{\frac{80\epsilon\tilde{B}}{\lambda}}{1 + \frac{80\epsilon\tilde{B}}{\lambda}} &\iff c \leq \frac{\lambda}{160}\frac{40\eta c}{\lambda + 40\eta c} \\ &\iff 1 \leq \lambda\frac{\eta}{4\lambda + 160\eta c} \text{ (since } c \geq 0) \\ &\iff c \leq \frac{\lambda}{160}\frac{\eta - 4}{\eta}. \end{aligned}$$

- We choose to write
$$\delta = (1+\tilde{\delta})\frac{1+L_{X,max}}{1-L_{C,max}} > \frac{1+L_X}{1-L_C}$$
- Let us assume, for simplicity, that $\epsilon \leq 1$. It is sufficient for this later condition to have
$$c \leq \frac{2\tilde{B}}{\eta}.$$
- The appearance of $\phi(R)$ suggests we should try to minimize it. We recall
$$\begin{aligned}\phi(r) =& \exp\left(-\frac{1}{4\sigma_x^2}\left(1+\delta\gamma+L_X+\delta L_C+(\epsilon\mathcal{C}_{f,1}+\mathcal{C}_{f,2})\,\sigma_x^2\right)r^2\right)\\ \geq& \exp\left(-\frac{1}{4\sigma_x^2}\left(1+\delta\gamma+L_X+\delta L_C+(\mathcal{C}_{f,1}+\mathcal{C}_{f,2})\,\sigma_x^2\right)r^2\right).\end{aligned}$$
It is therefore sufficient for (A.2) to have
$$c \leq \frac{1}{1+\eta}\frac{\sigma_x}{2\sqrt{\pi}}\frac{1}{R}\exp\left(-\frac{1}{4\sigma_x^2}\left(1+\delta\gamma+L_X+\delta L_C+(\mathcal{C}_{f,1}+\mathcal{C}_{f,2})\,\sigma_x^2\right)R^2\right),$$
and for (A.3) to have
$$c \leq \frac{1}{2(1+\eta)}\left(1-L_C-\frac{1+L_X}{\delta}\right)\exp\left(-\frac{1}{4\sigma_x^2}\left(1+\delta\gamma+L_X+\delta L_C+(\mathcal{C}_{f,1}+\mathcal{C}_{f,2})\,\sigma_x^2\right)R^2\right).$$
- Finally, we bound L_X and L_C by either 0 or $L_{X,max}$ and $L_{C,max}$, to obtain bounds on c independent of L_X and L_C.

A.4 Proof of Lemma 2.6

Let $z, z' \in \mathbb{R}^2$.

Proof of control of the L^1 distance : We have
$$\|z-z'\|_1 = |x-x'|+|c-c'| \leq \frac{1}{\min(\delta,1)}\left(|x-x'|+\delta|c-c'|\right) = \frac{1}{\min(\delta,1)}r(z,z').$$
If $r(z,z') \leq 1 \leq R$, we have, using Lemma 2.5
$$r(z,z') \leq \frac{f(r)}{f'_-(R)} \leq \frac{f(r)}{\phi(R)g(R)}\left(1+\epsilon\tilde{H}(z)+\epsilon\tilde{H}(z')\right).$$
If $r(z,z') \geq 1$, we have, using (A.1)
$$\begin{aligned}r(z,z') \leq& r(z,z')^2\\ \leq& \frac{16(1+\delta^2)}{\epsilon\min(\gamma,1)}\left(\epsilon H(z)+\epsilon H(z')\right)\\ \leq& \frac{16(1+\delta^2)}{\epsilon\min(\gamma,1)}\frac{f(r)}{f(1)}\left(1+\epsilon H(z)+\epsilon H(z')\right)\\ \leq& \frac{16(1+\delta^2)}{\epsilon\min(\gamma,1)}\frac{f(r)}{\phi(R)g(R)}\left(1+\epsilon\tilde{H}(z)+\epsilon\tilde{H}(z')\right).\end{aligned}$$
Thus
$$\|z-z'\|_1 \leq \frac{1}{\min(\delta,1)}\frac{1}{\phi(R)g(R)}\max\left(\frac{16(1+\delta^2)}{\epsilon\min(\gamma,1)},1\right)f(r(z,z'))\left(1+\epsilon\tilde{H}(z)+\epsilon\tilde{H}(z')\right).$$

Proof of control of the L2 distance : We have

$$r(z,z')^2 = \left(|x-x'| + \delta|c-c'|\right)^2 \geq |x-x'|^2 + \delta^2|c-c'|^2 \geq \min\left(1,\delta^2\right)\left(|x-x'|^2 + |c-c'|^2\right).$$

If $r(z,z') \geq 1$, we have, using (A.1)

$$\begin{aligned} r(z,z')^2 \leq & \frac{16(1+\delta^2)}{\epsilon \min(\gamma,1)}\left(\epsilon H(z) + \epsilon H(z')\right) \\ \leq & \frac{16(1+\delta^2)}{\epsilon \min(\gamma,1)}\frac{f(r)}{f(1)}\left(1 + \epsilon H(z) + \epsilon H(z')\right) \\ \leq & \frac{16(1+\delta^2)}{\epsilon \min(\gamma,1)}\frac{f(r)}{\phi(R)g(R)}\left(1 + \epsilon \tilde{H}(z) + \epsilon \tilde{H}(z')\right). \end{aligned}$$

If $r(z,z') \leq 1 \leq R$, we have, using Lemma 2.5

$$r(z,z')^2 \leq r(z,z') \leq \frac{f(r)}{f'_-(R)} \leq \frac{f(r)}{\phi(R)g(R)}\left(1 + \epsilon \tilde{H}(z) + \epsilon \tilde{H}(z')\right).$$

Thus

$$\|z-z'\|_2^2 \leq \frac{1}{\min(\delta^2,1)}\frac{1}{\phi(R)g(R)}\max\left(\frac{16(1+\delta^2)}{\epsilon\min(\gamma,1)},1\right) f(r(z,z'))\left(1 + \epsilon \tilde{H}(z) + \epsilon \tilde{H}(z')\right).$$

Proof of the second control of the L^1 distance : We have, if $r(z,z') \leq 1 \leq R$

$$r(z,z') \leq \frac{f(r)}{f'_-(R)} \leq \frac{f(r)}{\phi(R)g(R)}\left(1 + \epsilon\sqrt{H(z)} + \epsilon\sqrt{H(z')}\right).$$

and, if $r(z,z') \geq 1$, recall Lemma 2.1.

$$\begin{aligned} \|z-z'\|_1 \leq & \sqrt{\frac{4}{\gamma}H(z)} + \sqrt{\frac{4}{\gamma}H(z')} + \sqrt{4H(z)} + \sqrt{4H(z')} \\ \leq & 4\max\left(\sqrt{\frac{1}{\gamma}},1\right)\left(\sqrt{H(z)} + \sqrt{H(z')}\right) \\ \leq & \frac{4}{\epsilon}\max\left(\sqrt{\frac{1}{\gamma}},1\right)\frac{f(r)}{\phi(R)g(R)}\left(1 + \epsilon\sqrt{H(z)} + \epsilon\sqrt{H(z')}\right), \end{aligned}$$

and thus

$$\|z-z'\|_1 \leq \frac{1}{\phi(R)g(R)}\max\left(1, \frac{4}{\epsilon}\max\left(\sqrt{\frac{1}{\gamma}},1\right)\right) f(r(z,z'))\left(1 + \epsilon\sqrt{H(z)} + \epsilon\sqrt{H(z')}\right).$$

Independence with respect to L_X and L_C The *a priori* bounds $L_X \in [0, L_{X,max}]$ and $L_C \in [0, L_{C,max}]$ allow us to bound $\phi(R)$ independently of L_C and L_X by $\phi_{\min}$ (and we also use $g(R) \geq \frac{1}{2}$), thus giving us constant $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_z$ independent of L_C and L_X.

A.5 Proof of Lemmas 3.1 and 3.6

Proof of Lemma 3.1. Let's prove there exists an uniform in time bound on $\mathbb{E}(G_t^i)$ and $\mathbb{E}[(G_t^i)^2]$. First, let's recall the definition of G from(2.24):

$$G_t^i = 1 + \epsilon\tilde{H}(\bar{Z}_t^i) + \epsilon\tilde{H}(Z_t^{i,N}) + \frac{\epsilon}{N}\sum_{j=1}^N \tilde{H}(Z_t^{j,N}) + \frac{\epsilon}{N}\sum_{j=1}^N \tilde{H}(\bar{Z}_t^j).$$

The idea is to bound the different expectation in terms of the expectation at time $t = 0$. Since $\mathbb{E}(e^{\tilde{a}(|X_0|+|C_0|)})$ is finite, we know that for each $k \in \mathbb{N}$, $\mathbb{E}(|X_0|^k)$ and $\mathbb{E}(|C_0|^k)$ are also finite. We deduce that for each $k \in \mathbb{N}$, for each $j \leq N$, $\mathbb{E}[H(\bar{Z}_0^j)^k]$ and $\mathbb{E}[H(Z_0^{j,N})^k]$ are finite.

In fact, to bound uniformly in time the first moment, we only have to bound $\mathbb{E}(\tilde{H}(Z_t^{j,N}))$ and $\mathbb{E}(\tilde{H}(\bar{Z}_t^j))$ for each $j \leq N$. Let's begin with $\bar{Z}^j$. By (2.15), we have

$$\frac{d}{dt}\mathbb{E}\left[\tilde{H}\left(\bar{Z}_t^j\right)\right] \leq \tilde{B} - \frac{\lambda}{4}\mathbb{E}\left[\tilde{H}\left(\bar{Z}_t^j\right)\right].$$

By using Itô's formula on $e^{\lambda t/4}\tilde{H}\left(\bar{Z}_t^j\right)$ and the bound above, we obtain

$$\begin{aligned}\mathbb{E}\left[\tilde{H}\left(\bar{Z}_t^j\right)\right] \leq& \frac{4\tilde{B}}{\lambda} + e^{-\frac{\lambda}{4}t}\left(\mathbb{E}\left[\tilde{H}\left(\bar{Z}_0^j\right)\right] - \frac{4\tilde{B}}{\lambda}\right)\\ \leq& \max\left(\mathbb{E}\left[\tilde{H}\left(\bar{Z}_0^j\right)\right], \frac{4\tilde{B}}{\lambda}\right).\end{aligned}$$

By (2.9), in Lemma 2.4, we deduce the following inequality and we apply Cauchy-Schwarz

$$\begin{aligned}\mathbb{E}\left[\tilde{H}\left(\bar{Z}_0^j\right)\right] \leq& \mathbb{E}\left[H\left(\bar{Z}_0^j\right)\exp\left(a\sqrt{H\left(\bar{Z}_0^j\right)}\right)\right]\\ \leq& \mathbb{E}\left[H\left(\bar{Z}_0^j\right)^2\right]^{1/2}\mathbb{E}\left[\exp\left(2a\sqrt{H\left(\bar{Z}_0^j\right)}\right)\right]^{1/2}\end{aligned} \tag{A.6}$$

We already know $\mathbb{E}\left[H\left(\bar{Z}_0^j\right)^2\right]$ is bounded. Now, it is enough to prove that there exist C such that for all $z \in \mathbb{R}^2$

$$\exp\left(2a\sqrt{H(z)}\right) \leq C \times e^{\tilde{a}(|x|+|c|)}.$$

In fact, from the definition of H in (2.1), we have

$$\begin{aligned}2\sqrt{H(z)} =& \sqrt{2}\sqrt{\gamma\left(x + \frac{\beta}{\gamma}\right)^2 + (c+\alpha)^2 + H_0}\\ \leq& \sqrt{2\gamma}\left|x + \frac{\beta}{\gamma}\right| + \sqrt{2}\,|c+\alpha| + \sqrt{H_0}\\ \leq& \sqrt{2\gamma}|x| + \sqrt{2}|c| + \frac{1}{a}\ln C,\end{aligned}$$

where C is a constant independent of z. Finally, since $\max\left(a\sqrt{2\gamma}, a\sqrt{2}\right) \leq \tilde{a}$, we have

$$\exp\left(2a\sqrt{H(z)}\right) \leq C \times e^{\tilde{a}(|x|+|c|)}.$$

Then, $\mathbb{E}\left[\exp\left(2a\sqrt{H\left(\bar{Z}_0^j\right)}\right)\right]$ is bounded and we deduce $\mathbb{E}(\tilde{H}(\bar{Z}_t^j))$ is bounded for each $j \leq N$ and all $t \geq 0$.

The same calculations can be done for $Z_t^{j,N}$. By (2.20), we have

$$\mathcal{L}^N\left(\frac{1}{N}\sum_{i=1}^N \tilde{H}(Z_t^{i,N})\right) \leq \tilde{B} - \frac{\lambda}{4}\left(\frac{1}{N}\sum_{i=1}^N \tilde{H}(Z_t^{i,N})\right).$$

In particular,

$$\frac{d}{dt}\left[\mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{H}(Z_t^{i,N})\right)\right] \leq \mathbb{E}\left[\mathcal{L}^N\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{H}(Z_t^{i,N})\right)\right] \leq \tilde{B} - \frac{\lambda}{4}\mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{H}(Z_t^{i,N})\right),$$

and we can use the same method as above.

Finally, we have proved that for each $j \leq N$, $\mathbb{E}(\tilde{H}(Z_t^{j,N}))$ and $\mathbb{E}(\tilde{H}(\bar{Z}_t^j))$ are bounded uniformly in time . Thus, $\mathbb{E}(G_t^i)$ is bounded uniformly in time (and in N).

To bound the second moment of G_t^i, we have to bound each type of the following expectations: $\mathbb{E}[\tilde{H}(Z_t^{j_1,N})\tilde{H}(Z_t^{j_2,N})]$, $\mathbb{E}[\tilde{H}(Z_t^{j_1,N})\tilde{H}(\bar{Z}_t^{j_2})]$, $\mathbb{E}[\tilde{H}(\bar{Z}_t^{j_1})\tilde{H}(\bar{Z}_t^{j_2})]$, $\mathbb{E}[\tilde{H}(Z_t^{j,N})^2]$ and $\mathbb{E}[\tilde{H}(\bar{Z}_t^{j})^2]$. By Cauchy-Schwarz, it is in fact enough to bound $\mathbb{E}[\tilde{H}(Z_t^{j,N})^2]$ and $\mathbb{E}[\tilde{H}(\bar{Z}_t^{j})^2]$.

First, by the definition of $\tilde{H}$ in (2.8),

$$\begin{aligned}
\tilde{H}(z)^2 =& \left(\frac{2}{a^2}\exp\left(a\sqrt{H(z)}\right)\left(a\sqrt{H(z)}-1\right)+\frac{2}{a^2}\right)^2\\
\leq& 2\frac{2^2}{a^4}\exp\left(2a\sqrt{H(z)}\right)\left(a\sqrt{H(z)}-1\right)^2+2\frac{2^2}{a^4}\\
\leq& \frac{8}{a^4}\exp\left(2a\sqrt{H(z)}\right)\left(2a^2H(z)+2\right)+\frac{8}{a^4}.
\end{aligned}$$

As for the first moment, the study of $Z_t^{j,N}$ is very similar to the one of $\bar{Z}_t^j$. Here, we only focus on the second one.

To bound $\mathbb{E}[\tilde{H}(\bar{Z}_t^j)^2]$, by Cauchy-Schwarz, it is sufficient to bound $\mathbb{E}\left[\exp\left(4a\sqrt{H(\bar{Z}_t^j)}\right)\right]$ and $\mathbb{E}\left[H(\bar{Z}_t^j)^2\right]$. The latter has already been bounded uniformly in time, and the former can be obtained by the same calculations as previously, replacing a by $4a$ (and thus assuming $\tilde{a} \geq 4\sqrt{2}a\max(\sqrt{\gamma},1)$, which we do).

Finally, we deduce $\mathbb{E}\left((G_t^i)^2\right)$ is bounded uniformly in time. □

Proof of Lemma 3.6. Using $\partial_x H(z) = \gamma x + \beta$, we have

$$\begin{aligned}
\left|\partial_x\tilde{H}(Z_t^{i,N}) - \partial_x\tilde{H}(\bar{Z}_t^i)\right| =& \left|\left(\gamma X_t^{i,N}+\beta\right)\exp\left(a\sqrt{H(Z_t^{i,N})}\right) - \left(\gamma\bar{X}_t^i+\beta\right)\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right|\\
\leq& \left|\gamma X_t^{i,N}-\gamma\bar{X}_t^i\right|\left(\exp\left(a\sqrt{H(Z_t^{i,N})}\right)+\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right)\\
&+\left|\gamma\bar{X}_t^i+\beta\right|\left|\exp\left(a\sqrt{H(Z_t^{i,N})}\right)-\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right|.
\end{aligned}$$

Since $\left|X_t^{i,N}-\bar{X}_t^i\right| \leq r_t^i$,

$$\left|\gamma X_t^{i,N}-\gamma\bar{X}_t^i\right|\left(\exp\left(a\sqrt{H(Z_t^{i,N})}\right)+\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right) \leq \gamma r_t^i\left(\exp\left(a\sqrt{H(Z_t^{i,N})}\right)+\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right).$$

By Lemma 2.1 (ii), we have $H(z) \geq \frac{1}{2}\min\left(\frac{1}{\gamma},1\right)(\gamma x+\beta)^2$. By the mean value theorem, for all $y_1 \leq y_2$ in $\mathbb{R}$, there exists $y_3 \in [y_1,y_2]$ such that: $e^{ay_1}-e^{ay_2} = a(y_1-y_2)e^{ay_3}$. In particular, $|e^{ay_1}-e^{ay_2}| \leq a|y_1-y_2|(e^{ay_1}+e^{ay_2})$. Thus

$$\left|\gamma\bar{X}_t^i+\beta\right|\left|\exp\left(a\sqrt{H(Z_t^{i,N})}\right)-\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right|$$

$$
\begin{aligned}
&\leq a\sqrt{\frac{2H(\bar{Z}_t^i)}{\min\left(\frac{1}{\gamma},1\right)}}\left|\sqrt{H(Z_t^{i,N})}-\sqrt{H(\bar{Z}_t^i)}\right|\left(\exp\left(a\sqrt{H(Z_t^{i,N})}\right)+\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right)\\
&\leq a\sqrt{2\max(\gamma,1)}\left|H(Z_t^{i,N})-H(\bar{Z}_t^i)\right|\left(\exp\left(a\sqrt{H(Z_t^{i,N})}\right)+\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right).
\end{aligned}
$$

Then by the definition of H we get

$$
\begin{aligned}
\left|H(Z_t^{i,N})-H(\bar{Z}_t^i)\right| &= \left|\frac{1}{2}\gamma\left((X_t^{i,N})^2-(\bar{X}_t^i)^2\right)+\beta(X_t^{i,N}-\bar{X}_t^i)+\frac{1}{2}\left((C_t^{i,N})^2-(\bar{C}_t^i)^2\right)+\alpha(C_t^{i,N}-\bar{C}_t^i)\right|\\
&\leq \frac{1}{2}\gamma\left|X_t^{i,N}-\bar{X}_t^i\right|\left|X_t^{i,N}+\bar{X}_t^i\right|+\beta\left|X_t^{i,N}-\bar{X}_t^i\right|+\frac{1}{2}\left|C_t^{i,N}-\bar{C}_t^i\right|\left|C_t^{i,N}+\bar{C}_t^i\right|+\alpha\left|C_t^{i,N}-\bar{C}_t^i\right|.
\end{aligned}
$$

Now, by Lemma 2.1 (i), we have $H(z)\geq\frac{\gamma}{4}x^2+\frac{1}{4}c^2$ and since $\left|X_t^{i,N}-\bar{X}_t^i\right|\leq r_t^i$ and $\left|C_t^{i,N}-\bar{C}_t^i\right|\leq r_t^i/\delta$, we get

$$
\left|X_t^{i,N}-\bar{X}_t^i\right|\left(\frac{1}{2}\gamma\left|X_t^{i,N}+\bar{X}_t^i\right|+\beta\right)\leq r_t^i\left(\sqrt{\gamma}\left(\sqrt{H(Z_t^{i,N})}+\sqrt{H(\bar{Z}_t^i)}\right)+\beta\right)
$$

and

$$
\left|C_t^{i,N}-\bar{C}_t^i\right|\left(\frac{1}{2}\left|C_t^{i,N}+\bar{C}_t^i\right|+\alpha\right)\leq\frac{r_t^i}{\delta}\left(\sqrt{H(Z_t^{i,N})}+\sqrt{H(\bar{Z}_t^i)}+\alpha\right).
$$

Thus

$$
\left|H(Z_t^{i,N})-H(\bar{Z}_t^i)\right|\leq\left(\beta+\frac{\alpha}{\delta}\right)r_t^i+\left(\sqrt{\gamma}+\frac{1}{\delta}\right)r_t^i\left(\sqrt{H(Z_t^{i,N})}+\sqrt{H(\bar{Z}_t^i)}\right).
$$

Finally,

$$
\begin{aligned}
&|\partial_x\tilde{H}(Z_t^{i,N})-\partial_x\tilde{H}(\bar{Z}_t^i)|\\
&\leq\gamma r_t^i\left(\exp\left(a\sqrt{H(Z_t^{i,N})}\right)+\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right)\\
&\quad+a\sqrt{2\max(\gamma,1)}\left(\beta+\frac{\alpha}{\delta}\right)r_t^i\left(\exp\left(a\sqrt{H(Z_t^{i,N})}\right)+\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right)\\
&\quad+a\sqrt{2\max(\gamma,1)}\left(\sqrt{\gamma}+\frac{1}{\delta}\right)r_t^i\left(\sqrt{H(Z_t^{i,N})}+\sqrt{H(\bar{Z}_t^i)}\right)\left(\exp\left(a\sqrt{H(Z_t^{i,N})}\right)+\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right)\\
&\leq r_t^i\left(\gamma+a\sqrt{2\max(\gamma,1)}\left(\beta+\frac{\alpha}{\delta}\right)\right)\left(\exp\left(a\sqrt{H(Z_t^{i,N})}\right)+\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right)\\
&\quad+ar_t^i\sqrt{2\max(\gamma,1)}\left(\sqrt{\gamma}+\frac{1}{\delta}\right)\left(2\sqrt{H(Z_t^{i,N})}\exp\left(a\sqrt{H(Z_t^{i,N})}\right)+2\sqrt{H(\bar{Z}_t^i)}\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right)\right).
\end{aligned}
$$

Now, we can finally use Lemma 2.4, and more precisely (2.9) and (2.10), we obtain

$$
\begin{aligned}
&|\partial_x\tilde{H}(Z_t^{i,N})-\partial_x\tilde{H}(\bar{Z}_t^i)|\\
&\leq r_t^i\left(\gamma+a\sqrt{2\max(\gamma,1)}\left(\beta+\frac{\alpha}{\delta}\right)\right)\left(\tilde{H}(Z_t^{i,N})+\tilde{H}(\bar{Z}_t^i)+\frac{4}{a^2}\left(\exp\left(\frac{a^2}{2}\right)-1\right)\right)\\
&\quad+ar_t^i\sqrt{2\max(\gamma,1)}\left(\sqrt{\gamma}+\frac{1}{\delta}\right)\left(2a\tilde{H}(Z_t^{i,N})+\frac{2}{a}(e-2)+2a\tilde{H}(\bar{Z}_t^i)+\frac{2}{a}(e-2)\right)\\
&\leq r_t^i\left(\tilde{H}(Z_t^{i,N})+\tilde{H}(\bar{Z}_t^i)\right)\left[\gamma+a\sqrt{2\max(\gamma,1)}\left(\beta+\frac{\alpha}{\delta}\right)+2a^2\sqrt{2\max(\gamma,1)}\left(\sqrt{\gamma}+\frac{1}{\delta}\right)\right]
\end{aligned}
$$

$$+ r_t^i \left[\left(\gamma + a\sqrt{2\max(\gamma,1)} \left(\beta + \frac{\alpha}{\delta} \right) \right) \frac{4}{a^2} \left(\exp\left(\frac{a^2}{2} \right) - 1 \right) + 4\sqrt{2\max(\gamma,1)} \left(\sqrt{\gamma} + \frac{1}{\delta} \right) (e-2) \right].$$

We denote by $\mathcal{C}_{f,1}$ and $\mathcal{C}_{f,2}$ (given in Lemma 2.5) the following constants

$$\mathcal{C}_{f,1} = 4 \left[\left(\gamma + a\sqrt{2\max(\gamma,1)} \left(\beta + \frac{\alpha}{\delta} \right) \right) \frac{4}{a^2} \left(\exp\left(\frac{a^2}{2} \right) - 1 \right) + 4\sqrt{2\max(\gamma,1)} \left(\sqrt{\gamma} + \frac{1}{\delta} \right) (e-2) \right]$$
$$\mathcal{C}_{f,2} = 4 \left[\gamma + a\sqrt{2\max(\gamma,1)} \left(\beta + \frac{\alpha}{\delta} \right) + 2a^2\sqrt{2\max(\gamma,1)} \left(\sqrt{\gamma} + \frac{1}{\delta} \right) \right]$$

By the definition of G_t^i and since $G_t^i \geq 1$, we obtain

$$|\partial_x \tilde{H}(Z_t^{i,N}) - \partial_x \tilde{H}(\bar{Z}_t^i)| \leq r_t^i \frac{G_t^i}{\epsilon} \frac{\mathcal{C}_{f,2}}{4} + r_t^i G_t^i \frac{\mathcal{C}_{f,1}}{4},$$

and eventually

$$2\epsilon \left(1 + \frac{1}{N} \right) \sigma_x^2 \varphi_{\mathrm{rc}} \left(|X_t^{i,N} - \bar{X}_t^i| \right)^2 \left| \partial_x \tilde{H}(Z_t^{i,N}) - \partial_x \tilde{H}(\bar{Z}_t^i) \right|$$
$$\leq \left(\epsilon \mathcal{C}_{f,1} + \mathcal{C}_{f,2} \right) \sigma_x^2 \varphi_{\mathrm{rc}} \left(|X_t^{i,N} - \bar{X}_t^i| \right)^2 r_t^i G_t^i.$$

□

B Proof of Theorem 2 in the case $\sigma_X = 0$ and $\sigma_C > 0$

We quickly explain in this section how we may also deal with the case $\sigma_X = 0$ and $\sigma_C > 0$. Recall how the choice of the coupling method was motivated by the observation in (1.8) that the difference of potentials $\left| C_t^{i,N} - \bar{C}_t^i \right|$ was naturally contracting when $\left| X_t^{i,N} - \bar{X}_t^i \right|$ was close to 0. This lead us to using a reflection coupling on the Brownian motions acting on the potential X, to bring the difference close to 0, and it was thus necessary for σ_X to be positive (σ_C however did not hold any importance). In the case $\sigma_X = 0$, we then have to assume $\sigma_C > 0$, and we do a change of variable, motivated by the following observation. We have, when $\sigma_X = 0$

$$d(X_t^i - \bar{X}_t^i) = \left((X_t^i - \bar{X}_t^i) - ((X_t^i)^3 - (\bar{X}_t^i)^3) - (C_t^i - \bar{C}_t^i) \right) dt + \left(\frac{1}{N} \sum_{j=1}^N K_X(Z_t^i - Z_t^j) - K_X * \bar{\rho}(\bar{Z}_t^i) \right) dt,$$
$$= \left(2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i) - (X_t^i - \bar{X}_t^i) - ((X_t^i)^3 - (\bar{X}_t^i)^3) \right) dt$$
$$+ \left(\frac{1}{N} \sum_{j=1}^N K_X(Z_t^i - Z_t^j) - K_X * \bar{\rho}(\bar{Z}_t^i) \right) dt.$$

Thus

$$d|X_t^i - \bar{X}_t^i| = \left(\mathrm{sign}(X_t^i - \bar{X}_t^i) \left(2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i) \right) - |(X_t^i)^3 - (\bar{X}_t^i)^3| - |X_t^i - \bar{X}_t^i| \right) dt$$
$$+ \mathrm{sign}(X_t^i - \bar{X}_t^i) \left(\frac{1}{N} \sum_{j=1}^N K_X(Z_t^i - Z_t^j) - K_X * \bar{\rho}(\bar{Z}_t^i) \right) dt.$$

The quantity $|X_t^i - \bar{X}_t^i|$ is therefore naturally contracting when $|2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i)|$ is close to 0. Thanks to the presence of a Brownian motion in the stochastic differential equations defining the potential C, we can now

use a reflection coupling to have $|2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i)|$ go to 0. Consider the following coupling

$$\begin{cases} dX_t^i = (X_t^i - (X_t^i)^3 - C_t^i - \alpha)dt + \frac{1}{N}\sum_{j=1}^N K_X(Z_t^i - Z_t^j)dt \\ dC_t^i = (\gamma X_t^i - C_t^i + \beta)dt + \frac{1}{N}\sum_{j=1}^N K_C(Z_t^i - Z_t^j)dt + \sigma_c\varphi_{\text{sc}}\left(|2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i)|\right) dB_t^{i,sc,C} \\ \qquad\qquad +\sigma_c\varphi_{\text{rc}}\left(|2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i)|\right) dB_t^{i,rc,C} \end{cases} \tag{B.1}$$

and

$$\begin{cases} d\bar{X}_t^i = (\bar{X}_t^i - (\bar{X}_t^i)^3 - \bar{C}_t^i - \alpha)dt + K_X * \bar{\rho}(\bar{Z}_t^i)dt \\ d\bar{C}_t^i = (\gamma\bar{X}_t^i - \bar{C}_t^i + \beta)dt + K_C * \bar{\rho}(\bar{Z}_t^i)dt + \sigma_c\varphi_{\text{sc}}\left(|2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i)|\right) dB_t^{i,sc,C} \\ \qquad\qquad -\sigma_c\varphi_{\text{rc}}\left(|2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i)|\right) dB_t^{i,rc,C} \\ \bar{\rho} = \mathcal{L}((\bar{X}_t^1, \bar{C}_t^1)) \end{cases}, \tag{B.2}$$

and for $\delta > 0$, the following modified distance $r_t^i = \delta|X_t^i - \bar{X}_t^i| + |2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i)|$. Like previously, we consider a modified semimetric of the form $\frac{1}{N}\sum f(r_t^i)G_t^i$, and similar calculations yield

$$d(e^{ct}f(r_t^i)G_t^i) \leq e^{ct}K_t^i dt + dM_t^i,$$

where M_t^i is a continuous local martingale and

$$K_t^i = \tilde{K}_t^i + I_t^{1,i} + I_t^{2,i} + I_t^{3,i}.$$

We define $\tilde{K}_t^i$, $I_t^{1,i}$, $I_t^{2,i}$, $I_t^{3,i}$ and $I_t^{4,i}$ as follows:

$$\begin{aligned}
\tilde{K}_t^i =& G_t^i\Big[2cf(r_t^i) + 2f''(r_t^i)\sigma_c^2\varphi_{\text{rc}}\left(|2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i)|\right)^2 \\
&+f'(r_t^i)\left(\left|2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i)\right|(\delta+1) - |(X_t^i)^3 - (\bar{X}_t^i)^3|(\delta-2)\right. \\
&+|X_t^i - \bar{X}_t^i|\left(-\delta + \gamma + L_X(\delta+2) + L_C\right) + |C_t^i - \bar{C}_t^i|(L_X(\delta+2) + L_C) \\
&\left.+\sigma_c^2\varphi_{\text{rc}}\left(|2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i)|\right)^2(\epsilon\mathcal{C}_{f,1} + \mathcal{C}_{f,2})\, r_t^i\right)\Big] \\
&+\epsilon f(r_t^i)\left(4\tilde{B} - \frac{\lambda}{8}\tilde{H}(\bar{Z}_t^i) - \frac{\lambda}{8}\tilde{H}(Z_t^{i,N}) - \frac{\lambda}{8N}\sum_{j=1}^N \tilde{H}(\bar{Z}_t^j) - \frac{\lambda}{32N}\sum_{j=1}^N \tilde{H}(Z_t^{j,N})\right), \\
I_t^{1,i} =& G_t^i f'(r_t^i)\left[(\delta+2)\left(\left|\frac{1}{N}\sum_{j=1}^N K_X(\bar{Z}_t^i - \bar{Z}_t^j) - K_X * \bar{\mu}_t(\bar{Z}_t^i)\right|\right) + \left(\left|\frac{1}{N}\sum_{j=1}^N K_C(\bar{Z}_t^i - \bar{Z}_t^j) - K_C * \bar{\mu}_t(\bar{Z}_t^i)\right|\right)\right], \\
I_t^{2,i} =& G_t^i f'(r_t^i)\left[(\delta+2)\left(\frac{L_X}{N}\left(\sum_{j=1}^N |X_t^{j,N} - \bar{X}_t^j| + |C_t^{j,N} - \bar{C}_t^j|\right)\right) + \left(\frac{L_C}{N}\left(\sum_{j=1}^N |X_t^{j,N} - \bar{X}_t^j| + |C_t^{j,N} - \bar{C}_t^j|\right)\right)\right] \\
&- cf(r_t^i)G_t^i - \epsilon f(r_t^i)\left[\frac{\lambda}{32}H(\bar{Z}_t^i)\exp\left(a\sqrt{H(\bar{Z}_t^i)}\right) + \frac{\lambda}{32}H(Z_t^{i,N})\exp\left(a\sqrt{H(Z_t^{i,N})}\right)\right] \\
&- \epsilon f(r_t^i)\left[\frac{\lambda}{32N}\sum_{j=1}^N H(\bar{Z}_t^j)\exp\left(a\sqrt{H(\bar{Z}_t^j)}\right) + \frac{\lambda}{32N}\sum_{j=1}^N H(Z_t^{j,N})\exp\left(a\sqrt{H(Z_t^{j,N})}\right)\right], \\
I_t^{3,i} =& \epsilon f(r_t^i)\left((\alpha_X L_X + \beta_X L_C)\left(\frac{\sum_{j=1}^N |X_t^{j,N}|}{N}\right)^2\exp\left(a\sqrt{H(Z_t^{i,N})}\right)\right. \\
&\left.+(\alpha_C L_X + \beta_C L_C)\left(\frac{\sum_{j=1}^N |C_t^{j,N}|}{N}\right)^2\exp\left(a\sqrt{H(Z_t^{i,N})}\right)\right.
\end{aligned}$$

$$-\frac{\lambda}{16}H(Z_t^{i,N})\exp\left(a\sqrt{H(Z_t^{i,N})}\right)-\frac{\lambda}{16N}\sum_{j=1}^{N}H(Z_t^{j,N})\exp\left(a\sqrt{H(Z_t^{j,N})}\right)\Bigg).$$

We then have the additional constraint of $\delta > 2$ (so that the coefficient appearing in front of $|(X_t^i)^3 - (\bar{X}_t^i)^3|$ in the expression of $\tilde{K}_t^i$ is non positive). Otherwise, we deal with the various terms exactly as previously, through the choice of a sufficiently concave function f and a law of large numbers, and by considering the regions of space

$$\begin{aligned}
\text{Reg}_1^i &= \left\{(\bar{Z}_t^i, Z_t^{i,N}) \text{ s.t. } |2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i)| \geq \xi \text{ and } r_t^i \leq R\right\},\\
\text{Reg}_2^i &= \left\{(\bar{Z}_t^i, Z_t^{i,N}) \text{ s.t. } |2(X_t^i - \bar{X}_t^i) - (C_t^i - \bar{C}_t^i)| < \xi \text{ and } r_t^i \leq R_1\right\},\\
\text{Reg}_3^i &= \left\{(\bar{Z}_t^i, Z_t^{i,N}) \text{ s.t. } r_t^i > R\right\}.
\end{aligned}$$

Acknowledgements

Laetitia Colombani is a PhD student under the supervision of Patrick Cattiaux and Manon Costa, and Pierre Le Bris is a PhD student under the supervision of Arnaud Guillin and Pierre Monmarché. The authors would like to thank them, as well as Samir Salem, for their help throughout the redaction of the present article.

This work has been (partially) supported by the Project EFI ANR-17-CE40-0030 of the French National Research Agency.

Index

Throughout this article, we define many parameters and constants. For the sake of clarity, we list the main ones here so as to give the reader an index to refer to.

- X, C, Z : X and C are the processes we consider (see (1.1) and (1.2)) and we often refer to $Z = (X, C)$,
- $\bar{\mu}_t = \text{Law}(\bar{Z}_t)$: the density of the non linear limit (see (1.2)),
- α, β, γ, σ_X, σ_C : parameters of the problem (see (1.1)),
- $K_X, K_C, L_X, L_C, L_{X,max}, L_{C,max}$: K_X (resp. K_C) is an Lipschitz continuous interaction kernel, with Lipschitz constant $L_X \in [0, L_{X,max}]$ (resp. $L_C \in [0, L_{C,max}]$), as given in Assumption 1. In the case of uniform in time propagation of chaos, the inequalities L_X and L_C must satisfy are listed in Assumption 2,
- $\mathcal{W}_p$: the usual Wasserstein distance associated to the L^p distance (see (1.3)),
- a, $\tilde{a}$, $\mathcal{C}_{init,exp}$: constants used to give an exponential initial moment to the problem (see the assumptions of Theorem 2 and Section 2.3),
- $\lambda, B, \tilde{B}, H, \tilde{H}, \alpha_X, \alpha_C, \beta_X, \beta_C$: H (resp. $\tilde{H}$) is a Lyapunov functions given in (2.1) (resp. (2.8)). Its main property involves parameters λ and B (resp. λ and $\tilde{B}$), as can for instance be seen in (2.3) (resp. (2.13)). $\alpha_X, \alpha_C, \beta_X$ and β_C are intermediate constants given in Lemma 2.2,
- c : a contraction rate (see Assumption 2),
- $r, f, g, \phi, \Phi, G, \rho, \delta, R, \epsilon, \mathcal{C}_{f,1}, \mathcal{C}_{f,2}$: f (see (2.21)) is a concave function, the definition of which involves g, ϕ, Φ (see Assumption 2). Function G (see (2.24)) is then used to define ρ (see (2.23)), the semimetrics we consider in the end. All those notations thus refer to the modified distance we consider. These functions will be applied to r a modification of the usual L^1 distance (see equation (2.22)). Then, parameters δ, R, ϵ, $\mathcal{C}_{f,1}$, and $\mathcal{C}_{f,2}$ are used to define such functions (see Assumption 2 for some explicit values),

- R_0, ϕ_{min} : intermediate constants (see Assumption 2),
- $\mathcal{C}_{init,2}$: uniform in time bound on the second moment of the processes (see Lemma 2.3),
- $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_z$: constants used to quantify the control our modify distance has over the usual L^1 and L^2 distance (see Lemma 2.6 for the control and Assumption 2 for explicit values),
- $\phi_{rc}, \phi_{sc}, \xi$: ϕ_{rc} and ϕ_{sc} are two Lipschitz continuous functions used to define the coupling method, and their definitions involve a parameter ξ which converges to 0 in the end (see the beginning of Section 3),
- $\mathcal{C}_{r,H}$: used to explicit the control of the Lyapunov function H over the distance r (see Lemma 2.1),

References

[BFFT12] Javier Baladron, Diego Fasoli, Olivier Faugeras, and Jonathan Touboul. Mean-field description and propagation of chaos in networks of hodgkin-huxley and fitzhugh-nagumo neurons. *The Journal of Mathematical Neuroscience*, 2(1):10, May 2012.

[BFO19] Mireille Bossy, Joaquin Fontbona, and Hector Olivero. Synchronization of stochastic mean field networks of hodgkin-huxley neurons with noisy channels. *Journal of Mathematical Biology*, 78(6):1771–1820, May 2019. arXiv: 1803.04379.

[BGM10] François Bolley, Arnaud Guillin, and Florent Malrieu. Trend to equilibrium and particle approximation for a weakly selfconsistent Vlasov-Fokker-Planck equation. *ESAIM, Math. Model. Numer. Anal.*, 44(5):867–884, 2010.

[BHV21] Şeyma Bayrak, Philipp Hövel, and Vesna Vuksanović. Modeling functional connectivity on empirical and randomized structural brain networks. *Differential Equations and Dynamical Systems*, 29(4):789–805, 2021.

[BL12] Nils Berglund and Damien Landon. Mixed-mode oscillations and interspike interval statistics in the stochastic fitzhugh-nagumo model. *Nonlinearity*, 25(8):2303–2335, 2012.

[DEGZ20] Alain Durmus, Andreas Eberle, Arnaud Guillin, and Raphael Zimmer. An elementary approach to uniform in time propagation of chaos. *Proc. Amer. Math. Soc.*, 148(12):5387–5398, 2020.

[Ebe16] Andreas Eberle. Reflection couplings and contraction rates for diffusions. *Probab. Theory Relat. Fields*, 166(3-4):851–886, 2016.

[GLM21] Arnaud Guillin, Pierre Le Bris, and Pierre Monmarché. Convergence rates for the vlasov-fokker-planck equation and uniform in time propagation of chaos in non convex cases, 2021. arXiv: 2105.09070.

[HH52] Alan L. Hodgkin and Andrew F. Huxley. A quantitative description of membrane current and its application to conduction and excitation in nerve. *The Journal of Physiology*, 117(4):500–544, Aug 1952.

[IW89] Nobuyuki Ikeda and Shinzo Watanabe. *Stochastic Differential Equations and Diffusion Processes*. Number 24 in North-Holland Mathematical Library. North-Holland [u.a.], 2. ed edition, 1989.

[Kac56] Mark Kac. Foundations of kinetic theory. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, Volume 3: Contributions to Astronomy and Physics*, pages 171–197, Berkeley, Calif., 1956. University of California Press.

[KNRS20] Chaman Kumar, Neelima, Christoph Reisinger, and Wolfgang Stockinger. Well-posedness and tamed schemes for mckean-vlasov equations with common noise. *arXiv:2006.00463 [cs, math]*, May 2020. arXiv: 2006.00463.

[Li21] Fuzhi Li. Dynamics for stochastic fitzhugh-nagumo systems with general multiplicative noise on thin domains. *Mathematical Methods in the Applied Sciences*, 44(6):5050–5078, 2021.

[LL19] Yangrong Li and Fuzhi Li. Limiting dynamics for stochastic fitzhugh-nagumo equations on large domains. *Stochastics and Dynamics*, 19(5):25, 2019.

[LP21] Eric Luçon and Christophe Poquet. Periodicity induced by noise and interaction in the kinetic mean-field fitzhugh–nagumo model. *The Annals of Applied Probability*, 31(2):561–593, Apr 2021.

[LR86] Torgny Lindvall and L. C. G. Rogers. Coupling of multidimensional diffusions by reflection. *The Annals of Probability*, 14(3):860–872, Jul 1986.

[LS14] Eric Luçon and Wilhelm Stannat. Mean field limit for disordered diffusions with singular interactions. *The Annals of Applied Probability*, 24(5), 2014.

[LS18] José R. León and Adeline Samson. Hypoelliptic stochastic fitzhugh-nagumo neuronal model: mixing, up-crossing and estimation of the spike rate. *The Annals of Applied Probability*, 28(4):2243–2274, 2018.

[LW10] Yan Lv and Wei Wang. Limit dynamics for the stochastic fitzhugh-nagumo system. *Nonlinear Analysis. Real World Applications*, 11(4):3091–3105, 2010.

[LX21] Fuzhi Li and Dongmei Xu. Regular dynamics for stochastic fitzhugh-nagumo systems with additive noise on thin domains. *Discrete and Continuous Dynamical Systems. Series B*, 26(7):3517–3542, 2021.

[MQT16] Stéphane Mischler, Cristóbal Quiñinao, and Jonathan Touboul. On a kinetic fitzhugh-nagumo model of neuronal network. *Communications in Mathematical Physics*, 342(3):1001–1042, Mar 2016. arXiv: 1503.00492.

[RS22] Christoph Reisinger and Wolfgang Stockinger. An adaptive euler-maruyama scheme for mckean-vlasov sdes with super-linear growth and application to the mean-field fitzhugh-nagumo model. *Journal of Computational and Applied Mathematics*, 400:23, 2022.

[RST19] Goncalo Dos Reis, William Salkeld, and Julian Tugaut. Freidlin-wentzell ldp in path space for mckean-vlasov equations and the functional iterated logarithm law. *The Annals of Applied Probability*, 29(3), Jun 2019. arXiv: 1708.04961.

[Sch22] Katharina Schuh. Global contractivity for langevin dynamics with distribution-dependent forces and uniform in time propagation of chaos, 2022. arXiv:2206.03082.

[Szn91] Alain-Sol Sznitman. Topics in propagation of chaos. In Paul-Louis Hennequin, editor, *Ecole d'Eté de Probabilités de Saint-Flour XIX — 1989*, pages 165–251, Berlin, Heidelberg, 1991. Springer Berlin Heidelberg.

[TBSST13] D. Tatchim Bemmo, M. Siewe Siewe, and C. Tchawoua. Combined effects of correlated bounded noises and weak periodic signal input in the modified fitzhugh–nagumo neural model. *Communications in Nonlinear Science and Numerical Simulation*, 18(5):1275–1287, May 2013.

[Thi13] Michèle Thieullen. *Deterministic and Stochastic FitzHugh–Nagumo Systems*, volume 2058 of *Lecture Notes in Mathematics*, page 175–186. Springer Berlin Heidelberg, 2013.

[TRW03] Henry C. Tuckwell, Roger Rodriguez, and Frederic Y. M. Wan. Determination of firing times for the stochastic fitzhugh-nagumo neuronal model. *Neural Computation*, 15(1):143–159, 2003.

[Uda19] Kenneth Uda. Ergodicity and spike rate for stochastic fitzhugh-nagumo neural model with periodic forcing. *Chaos, Solitons and Fractals*, 123:383–399, 2019.