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Switching control design for LTI system with uncertain equilibrium : Application to parallel interconnection of DC/DC converters.

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Abstract

This paper addresses the problem of designing a robust control law for uncertain switched affine systems. In order to cope with model uncertainties, a dynamic controller is introduced. Local asymptotic stability of the closed-loop system as well as zero steady state error on the controlled output are proven. A constructive method based on LMI conditions is given. It delivers the control parameters and an estimation of the robust domain of attraction. The proposed strategy is applied on a parallel interconnection of m DC/DC converters. For this system, an extensive discussion is offered on specific difficulties about ill-conditioned matrix inequalities. Finally, by using a user-programmable Field-Programmable Gate Array (FPGA), experimental results are provided to illustrate the effectiveness of the control.

Key words: Robust control law, switched affine systems, LMI, DC/DC converters.

1 INTRODUCTION

The study of switched systems has attracted a lot of attention from the scientific community over the past few years [21]. They belong to the class of hybrid systems for which active subsystem is selected according to some switching rule [14]. The motivation for studying such systems comes from their practical applications for electromechanical and aerospace systems and more specifically power electronics [28]. In practice, however, there are few applications of this type of control law in power electronics, control synthesis based on average model using Pulse Width Modulation (PWM) is widely preferred due to the fact that analysis and design of switched affine systems are non-trivial tasks, from a theoretical point of view [21,6]. The control design procedure has to take into account complex phenomena such as limit cycles or sliding modes [30]. Various approaches have been proposed for the design of stabilizing switching rules in the literature under the assumption that the equilibrium point is perfectly known [9,16]. However, uncertainties prevent those results to be experimentally implemented.

Only few papers have investigated the case of uncertain equilibrium points [15,4,3]. In [15,4], the authors

focus on adaptive control where specific uncertainties are modeled, estimated and compensated by the control law. However, this strategy is expected to be fragile with respect to unmodeled uncertainties which prevent practical application. The methodology proposed in [3] for dealing with uncertainties in the model is based on the addition of an integral action. By dealing with weak Lyapunov function, a global stabilizing control law is designed. Zero steady state error is guaranteed, under the assumption that mismatch between the actual dynamics and its model is bounded. However, the limits of this mismatch are not explicitly defined. Besides, all of those solutions present a common characteristic: The stability proofs are based either on the fact that the open loop state matrix of each subsystem is Hurwitz or on the existence of a Hurwitz convex combination of them. This is in contrast with robust control of power converters modeled via averaging, which is the subject of a large number of publications, e.g. [27,20] to cite a few.

In this paper, we consider the parallel interconnection of m DC/DC converters. This system becomes more and more popular and offer several advantages such as better reliability due to redundancy, improved thermal management, etc [29,19]. As in any realistic context, not only the numerical value of the load but also the dynamics of parasitic elements are unknown. Therefore, we seek structural robustness. Unlike adaptive strategies,

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this property is offered by the robust output regulation paradigm, which intrinsically copes with unmodeled uncertainties [12].

Contribution of this paper is threefold. (i) Based on some arg min switching strategy, uncertain equilibria is treated with the introduction of a dynamic controller. Based on the existence of an exponentially globally stabilizing continuous state feedback controller as a reference to be emulated, local exponential stability is proved, thereby ensuring zero steady state error. Constructive methods proposed here allow to avoid the existence of a Hurwitz convex combination by restricting to local stabilization. Finally, insights on the construction of a subset included in the robust basin of attraction are provided. (ii) A controller gains design procedure is offered. It is formulated as a semi-definite program. A particular attention is devoted to numerical conditioning of this program. To this end, the main ingredient is the introduction of a new change of variable, which avoids poor conditioning induced by small magnitudes of inductances and capacitances of the converter. (iii) Finally, the proposed strategy is validated experimentally. A user-programmable Field-Programmable Gate Array (FPGA) is used. Large computing capacity of this device allows to cope with large switching frequency in steady state [21].

A preliminary version of this paper was presented in [24]. It concerns only the specific case of a buck converter associated with a scalar uncertainty. The more general case of (A, B, C) triple satisfying mild assumptions is treated in this paper.

This paper is organized as follows. Section II presents the problem formulation. Section III is devoted to the theoretical contribution consisting in a robust control law which guarantees local exponential stability. Section IV reports experimental results on parallel buck converters. The paper is ended with some concluding remarks.

Notations: The identity matrix of dimension $n \times n$ is denoted by \mathbf{I}_n . $\mathbf{0}_{m \times n}$ is the $m \times n$ matrix of zeros. $\mathbf{1}_m$ ($\mathbf{0}_m$) denotes a m dimensional column vector with 1 (0) entries. By $B(x, c_B)$, we denote the open ball centered on x with radius $c_B > 0$ i.e. $B(x, c_B) = \{y \in \mathbb{R}^n : |x - y| < c_B\}$. \mathcal{I}_N represents the set $\{1, 2, \dots, N\}$. We denote by $\mathbb{S}_+^n := \{P = P^\top \in \mathbb{R}^{n \times n} : \forall x \neq 0, x^\top P x > 0\}$. For a symmetric positive definite matrix $P \in \mathbb{S}_+^n$ and a positive scalar c , we denote by $\mathcal{E}(P, x_0, c)$ the open ellipsoid centered on x_0 with radius $c > 0$ i.e. $\{x \in \mathbb{R}^n : (x - x_0)^\top P (x - x_0) < c\}$. For a given set \mathcal{U} , $\text{conv}\{\mathcal{U}\}$ indicates the convex hull of the set and $\text{Int}\{\mathcal{U}\}$ its interior. The operator “diag $\{v\}$ ” builds a diagonal matrix of a column vector v .

2 Preliminaries and problem formulation

We consider the class of uncertain continuous time systems with switching control given by:

$$\begin{cases} \dot{x}(t) = A(\theta)x(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^p$ the controlled output and $u(t) \in \mathbb{R}^m$ is the control input vector which takes value in a finite set of constant vectors $\mathcal{U} = \{u_1, u_2, \dots, u_N\} \subset \mathbb{R}^m$, where N is a strictly positive integer. The uncertain terms are gathered in the vector $\theta = (\theta_1, \theta_2, \dots, \theta_{n_\theta}) \in \Theta$ and we assume that each uncertain parameter θ_i belongs to a bounded and known interval $\Theta_i := [\underline{\theta}_i, \bar{\theta}_i]$. The set Θ is then defined as $\Theta := \Theta_1 \times \Theta_2 \times \dots \times \Theta_{n_\theta}$. We consider that the matrix A depends on θ in an affine way i.e. there exist a matrix A_0 and a linear map $\theta \mapsto A_1(\theta)$ such that $A(\theta) = A_0 + A_1(\theta)$ for all $\theta \in \Theta$.

Our objective is to design a state-dependent switching law that guarantees robust regulation of the controlled output $y(t)$ to a constant reference y_{ref} . The challenge is to cancel steady state error due to parameter uncertainties.

Let $P_\Sigma(\theta, s)$ be the Rosenbrock matrix

$$P_\Sigma(\theta, s) := \begin{bmatrix} A(\theta) - s\mathbf{I}_n & B \\ C & \mathbf{0}_{p \times m} \end{bmatrix}.$$

For the problem to be well posed, following two assumptions are made:

Assumption 1 For all $\theta \in \Theta$, system $\{A(\theta), B, C, \mathbf{0}\}$ is robustly invertible, i.e. (i) $m = p$ holds and (ii) for all $\theta \in \Theta$, $\text{rank } P_\Sigma(\theta, s) = n + p$ holds for all but finitely many $s \in \mathbb{C}$.

Assumption 2 For all $\theta \in \Theta$, 0 is not an invariant zero of the system $\{A(\theta), B, C, \mathbf{0}\}$, i.e. $\text{rank } P_\Sigma(\theta, 0) = \max_{s \in \mathbb{C}} \text{rank } P_\Sigma(\theta, s)$ holds for all $\theta \in \Theta$.

Define the set Ψ_e of controlled equilibria of (1) for which $y = y_{\text{ref}}$ and $u(t)$ can take any value in \mathbb{R}^m

$$\Psi_e(\theta) = \{(x_\theta^*, u_\theta^*) \in \mathbb{R}^n \times \mathbb{R}^m : A(\theta)x_\theta^* + Bu_\theta^* = \mathbf{0}_n, Cx_\theta^* = y_{\text{ref}}\}. \quad (2)$$

Lemma 3 Let Assumptions 1 and 2 hold. Then, for all $\theta \in \Theta$ and for all $y_{\text{ref}} \in \mathbb{R}^p$, the controlled equilibria is unique, i.e. there exists (x_θ^*, u_θ^*) such that $\Psi_e(\theta) = \{(x_\theta^*, u_\theta^*)\}$.

PROOF. This result trivially follows from the observation that set $\Psi_e(\theta)$ defined in (2) may be re-expressed as follows:

$$\Psi_e(\theta) = \left\{ (x_\theta^*, u_\theta^*) : P_\Sigma(\theta, 0) \begin{bmatrix} x_\theta^* \\ u_\theta^* \end{bmatrix} = \begin{bmatrix} \mathbf{0}_n \\ y_{\text{ref}} \end{bmatrix} \right\}$$

and the fact that Assumptions 1 and 2 imply that $P_\Sigma(\theta, 0)$ is invertible for all $\theta \in \Theta$.

Assume also that inputs associated with controlled equilibria belongs to $\text{Int}\{\text{conv}(\mathcal{U})\}$:

Assumption 4 For all $\theta \in \Theta$, $\Psi_e(\theta)$ belongs to $\mathbb{R}^n \times \text{Int}\{\text{conv}(\mathcal{U})\}$.

This last hypothesis, together with Assumptions 1 and 2, imply that $\Psi_e(\theta) = \{(x_\theta^*, u_\theta^*)\}$ is such that $u_\theta^* \in \text{Int}\{\text{conv}(\mathcal{U})\}$. In this paper, we seek solutions to the following problem:

Problem 5 Given system (1) and the set \mathcal{U} . Assume that Assumptions 1, 2 and 4 hold. Design $n_z \in \mathbb{N}_{>0}$, vectors $\psi \in \mathbb{R}^m$ and $h_z \in \mathbb{R}^{n_z}$ and matrices $A_z \in \mathbb{R}^{n_z \times n_z}$, $B_z \in \mathbb{R}^{n_z \times n}$ and $\Gamma \in \mathbb{R}^{(n+n_z) \times m}$ characterizing dynamical state-feedback relay controller

$$\dot{z}(t) = A_z z(t) + B_z x(t) + h_z, \quad (3a)$$

$$u(t) \in \arg \min_{v \in \mathcal{U}} \left(\begin{bmatrix} x^\top(t) & z^\top(t) \end{bmatrix} \Gamma + \psi^\top \right) v, \quad (3b)$$

such that, for all $\theta \in \Theta$, resulting closed-loop admits a locally exponentially stable equilibrium (x_θ^*, z_θ^*) for which $y = y_{\text{ref}}$.

Remark 6 Since controller input $u(t)$ must belong to finite set \mathcal{U} , closed loop (1) with (3) is governed by a differential equation with discontinuous right-hand side (unless $u(t)$ is constant). In this paper, solutions are considered in the sense of Filippov, see e.g. [11] and [8].

3 Proposed robust control law

3.1 Augmented model

In order to cancel the error induced by the parameter uncertainties, an integral action is added in the control scheme. State z of the controller is governed by $\dot{z} = \int (y - y_{\text{ref}})$ which can be recast into (3a) by selecting $n_z = p$. Denoting $\zeta := \begin{bmatrix} x \\ z \end{bmatrix}$, the augmented model reads:

$$\dot{\zeta}(t) = \bar{A}(\theta)\zeta(t) + \bar{B}u(t) + h, \quad (4)$$

where $\zeta(t) \in \mathbb{R}^{(n+p)}$, with

$$\bar{A}(\theta) := \begin{bmatrix} A(\theta) & \mathbf{0}_{n \times p} \\ C & \mathbf{0}_{p \times p} \end{bmatrix}, \bar{B} := \begin{bmatrix} B \\ \mathbf{0}_{p \times p} \end{bmatrix}, h := \begin{bmatrix} \mathbf{0}_n \\ -y_{\text{ref}} \end{bmatrix}. \quad (5)$$

Now, note that by virtue of Lemma 3, there exists a unique couple (x_θ^*, u_θ^*) such that $\bar{A}(\theta)\zeta_\theta^* + \bar{B}u_\theta^* + h = \mathbf{0}_{n+p}$ with $\zeta_\theta^* := \begin{bmatrix} x_\theta^* \\ z_\theta^* \end{bmatrix} \in \mathbb{R}^{n+p}$ and $z_\theta^* \in \mathbb{R}^p$. Thereafter, we define by η the coordinates transformation:

$$\eta := \zeta - \zeta_\theta^*. \quad (6)$$

3.2 Main result

Definition 7 ([2]) $(A(\theta), B)$ is quadratically stabilizable via linear control if there exists K and $P \in \mathbb{S}_+^n$

such that, for all x_0 and for all $\theta \in \Theta$, the function $x \mapsto x^\top P x$ is strictly decreasing along trajectories of the system $\dot{x}(t) = A(\theta)x(t) + BKx(t)$, $x(0) = x_0$, i.e.

$$(A(\theta) + BK)^\top P + P(A(\theta) + BK) \prec \mathbf{0}_{n \times n}, \quad (7)$$

holds for all $\theta \in \Theta$.

System matrix $\bar{A}(\theta)$ depends affinely on the parameter θ which belongs to a convex polytope Θ . Therefore, $\bar{A}(\theta)$ belongs to a convex polytope of $N_s = 2^{n_\theta}$ vertices $\bar{A}_s^{[i]} := \bar{A}(\theta^{[i]})$, ($i \in \mathcal{I}_{N_s}$) where vectors $\theta^{[i]}$, ($i \in \mathcal{I}_{N_s}$) are vertices of Θ , see [7]:

$$\bar{A}(\theta) \in \text{conv}\{\bar{A}_s^{[1]}, \dots, \bar{A}_s^{[N_s]}\} := \left\{ \sum_{i=1}^{N_s} \gamma_i \bar{A}_s^{[i]} : 0 \leq \gamma_i \leq 1, \sum_{i=1}^{N_s} \gamma_i = 1 \right\}. \quad (8)$$

For the sequel, we will assume that:

Assumption 8 $(\bar{A}(\theta), \bar{B})$ is quadratically stabilizable via linear control.

Let us now state the main result of the paper.

Theorem 9 Assume that Assumptions 1, 2, 4 and 8 hold. Given any $\theta_n \in \Theta$. Denote $(x_{\theta_n}^*, u_{\theta_n}^*)$ the unique element of $\Psi_e(\theta_n)$ (see Lemma 3). Define

$$\zeta_{\theta_n}^* := \begin{bmatrix} x_{\theta_n}^* \\ \mathbf{0}_p \end{bmatrix}. \quad (9)$$

Then, there exists $P \in \mathbb{S}_+^{n+p}$ such that controller (3) with $n_z = p$, $A_z = \mathbf{0}_{p \times p}$, $B_z = C$, $h_z = -y_{\text{ref}}$, $\Gamma = P\bar{B}$ and $\psi^\top = -\zeta_{\theta_n}^{*\top} P\bar{B}$, solves Problem 5, i.e. controller (3) reads

$$\dot{z}(t) = Cx(t) - y_{\text{ref}}, z(0) = z_0 \in \mathbb{R}^p; \quad (10a)$$

$$u(t) = \arg \min_{v \in \mathcal{U}} (\zeta(t) - \zeta_{\theta_n}^*)^\top P \begin{bmatrix} B \\ \mathbf{0}_{p \times p} \end{bmatrix} v. \quad (10b)$$

Besides, there exists a strictly positive scalar ϵ such that ellipsoid

$$\Omega_0 := \mathcal{E}(P, \zeta_\theta^*, \epsilon) \subset \mathbb{R}^{n+p} \quad (11)$$

belongs to the closed-loop domain of attraction, i.e. for all $\theta \in \Theta$ and for all $\begin{bmatrix} x(0) \\ z(0) \end{bmatrix} \in \Omega_0$, trajectory of closed-loop (1) with (10b) converges exponentially to equilibrium of Problem 5.

PROOF. Select $\theta \in \Theta$ arbitrarily. Under Assumption 8, there exists $P \in \mathbb{S}_+^{n+p}$, $\delta > 0$ and $\lambda > 0$ such that

$$\frac{\partial V}{\partial \eta}(\eta) (\bar{A}(\theta)\eta + \bar{B}K\eta) < -2\delta V(\eta), \forall \eta \neq \mathbf{0}_{n+p}, \quad (12)$$

is satisfied for $V : \eta \mapsto \eta^T P \eta$ and

$$K = -\frac{\lambda}{2} \bar{B}^T P. \quad (13)$$

This proves that $\bar{A}(\theta) + \bar{B}K$ is Hurwitz.

According to Lemma 3, there exists a unique couple (x_θ^*, u_θ^*) in $\Psi_e(\theta)$. Those vectors allow to define the following control law:

$$u(\zeta) = K(\zeta - \zeta_{\theta_n}^*) + u_\theta^*, \quad (14)$$

which, together with system (4), gives rise to a closed-loop system described by:

$$\dot{\zeta} = \bar{A}(\theta)\zeta + \bar{B}(K(\zeta - \zeta_{\theta_n}^*) + u_\theta^*) + h. \quad (15)$$

This equation admits an equilibrium $\begin{bmatrix} \alpha \\ z_\theta^* \end{bmatrix}$ which is globally attractive (and hence necessarily unique) since $(\bar{A}(\theta) + \bar{B}K)$ is Hurwitz. Furthermore, this equilibrium is such that $\alpha = x_\theta^*$ and

$$K(\zeta_\theta^* - \zeta_{\theta_n}^*) = \mathbf{0}_m, \quad (16)$$

hold. Indeed, it holds

$$\begin{aligned} \bar{A}(\theta) \begin{bmatrix} \alpha \\ z_\theta^* \end{bmatrix} + \bar{B} \left(K \begin{bmatrix} \alpha \\ z_\theta^* \end{bmatrix} - \zeta_{\theta_n}^* + u_\theta^* \right) + h &= \mathbf{0}_{n+p}, \\ \Leftrightarrow \begin{bmatrix} A(\theta) & B \\ C & \mathbf{0}_{p \times p} \end{bmatrix} \begin{bmatrix} \alpha \\ K \begin{bmatrix} \alpha \\ z_\theta^* \end{bmatrix} - \zeta_{\theta_n}^* + u_\theta^* \end{bmatrix} &= \begin{bmatrix} \mathbf{0}_n \\ z_{\text{ref}} \end{bmatrix}, \\ \Leftrightarrow (\alpha, K \begin{bmatrix} \alpha \\ z_\theta^* \end{bmatrix} - \zeta_{\theta_n}^* + u_\theta^*) &\in \Psi_e(\theta). \end{aligned}$$

By virtue of Lemma 3, this ensures that

$$(\alpha, K \begin{bmatrix} \alpha \\ z_\theta^* \end{bmatrix} - \zeta_{\theta_n}^* + u_\theta^*) = (x_\theta^*, u_\theta^*),$$

so that $\alpha = x_\theta^*$ and (16) hold.

Let us now prove that, for all $\zeta(0) \in \Omega_0$, trajectory of closed-loop (4) with controller (10b) converges exponentially to $\zeta_\theta^* = \begin{bmatrix} x_\theta^* \\ z_\theta^* \end{bmatrix}$. To this end, define set $C_{\mathcal{U}}(\theta)$ as follows:

$$C_{\mathcal{U}}(\theta) := \{ \zeta : K(\zeta - \zeta_{\theta_n}^*) + u_\theta^* \in \text{conv}(\mathcal{U}) \}. \quad (17)$$

Under Assumption 4, one has $u_\theta^* \in \text{Int}\{\text{conv}(\mathcal{U})\}$. This equation is equivalent to $\mathbf{0}_m \in \text{Int}\{\text{conv}(\mathcal{U}) - u_\theta^*\}$, which guarantees the existence of $\gamma > 0$ such that $B(\mathbf{0}_m, \gamma) \subset \text{Int}\{\text{conv}(\mathcal{U}) - u_\theta^*\}$. Note that the function $k : \zeta \mapsto K(\zeta - \zeta_{\theta_n}^*)$ is continuous, so that

reciprocal image of the open set $B(\mathbf{0}_m, \gamma)$, namely $k^{-1}(B(\mathbf{0}_m, \gamma)) := \{ \zeta : k(\zeta) \in B(\mathbf{0}_m, \gamma) \}$, is also an open set. Clearly, this set contains ζ_θ^* (since from (16) $k(\zeta_\theta^*) = \mathbf{0}_m \in B(\mathbf{0}_m, \gamma)$) so that for all $Q \in \mathbb{S}_+^{n+p}$ there exists $c > 0$ defining the ellipsoid $\mathcal{E}(Q, \zeta_\theta^*, c)$ which is contained in $k^{-1}(B(\mathbf{0}_m, \gamma))$. Therefore, there exists $\epsilon > 0$ such that the ellipsoid $\Omega_0 := \mathcal{E}(P, \zeta_\theta^*, \epsilon)$, associated with P , satisfies $\Omega_0 \subset k^{-1}(B(\mathbf{0}_m, \gamma)) \subset C_{\mathcal{U}}(\theta)$. As a result, for all $\zeta \in \Omega_0$, there exists N scalars $\alpha_j(\zeta) \geq 0$, ($j \in \mathcal{I}_N$), with $\sum_{j=1}^N \alpha_j(\zeta) = 1$ such that:

$$K(\zeta - \zeta_{\theta_n}^*) + u_\theta^* = \sum_{j=1}^N \alpha_j(\zeta) u_j. \quad (18)$$

Using this relation together with (16) imply that

$$K(\zeta - \zeta_\theta^*) = -u_\theta^* + \sum_{j=1}^N \alpha_j(\zeta) u_j, \quad \forall \zeta \in \Omega_0. \quad (19)$$

Now consider (6), system (15) can be re-written as follows:

$$\dot{\eta} = (\bar{A}(\theta) + \bar{B}K)\eta. \quad (20)$$

Using (19) and the equality $\frac{\partial V}{\partial \eta}(\eta) = 2(\zeta - \zeta_\theta^*)^T P$, (12) implies that

$$\begin{aligned} &(\zeta - \zeta_\theta^*)^T P \bar{A}(\theta)(\zeta - \zeta_\theta^*) + (\zeta - \zeta_{\theta_n}^*)^T P \bar{B} \left(\sum_{j=1}^N \alpha_j(\zeta) u_j - u_\theta^* \right) \\ &+ (\zeta_{\theta_n}^* - \zeta_\theta^*)^T P \bar{B} \left(\sum_{j=1}^N \alpha_j(\zeta) u_j - u_\theta^* \right) \\ &< -\delta V(\zeta - \zeta_\theta^*), \quad \forall \zeta \in \Omega_0 \setminus \{\zeta_\theta^*\}, \end{aligned} \quad (21)$$

holds. By using (16) and (13), one can conclude that $(\zeta_{\theta_n}^* - \zeta_\theta^*)^T P \bar{B} = 0$. Thus, (21) reads

$$\begin{aligned} &\sum_{j=1}^N \alpha_j(\zeta) \left((\zeta - \zeta_\theta^*)^T P \bar{A}(\theta)(\zeta - \zeta_\theta^*) + \right. \\ &\left. (\zeta - \zeta_{\theta_n}^*)^T P \bar{B} (u_j - u_\theta^*) \right) < -\delta V(\zeta - \zeta_\theta^*), \quad \forall \zeta \in \Omega_0 \setminus \{\zeta_\theta^*\}. \end{aligned} \quad (22)$$

Since $\alpha_j(\zeta) \geq 0$, ($j \in \mathcal{I}_N$), for all $\zeta \in \Omega_0 \setminus \{\zeta_\theta^*\}$, there must be at least one $j \in \mathcal{I}_N$ such that:

$$(\zeta - \zeta_\theta^*)^T P \bar{A}(\theta)(\zeta - \zeta_\theta^*) + (\zeta - \zeta_{\theta_n}^*)^T P \bar{B} (u_j - u_\theta^*) < -\delta V(\zeta - \zeta_\theta^*) \quad (23)$$

holds. As a result, controller (10b) ensures that, for all $\zeta(0) \in \Omega_0 \setminus \{\zeta_\theta^*\}$, (22) and, in turn, (12) are satisfied so that closed-loop trajectory converges exponentially to ζ_θ^* .

Remark 10 (On Ω_0) Set Ω_0 defined in (11) is an inner estimate of the basin of attraction. Then, a crucial question is the following: How to exploit this estimate since it depends on the unknown vector θ , via ζ_θ^* ? The idea is to make sure that the initial condition $(x(0), z(0))$ belongs to Ω_0 , for any $\theta \in \Theta$. This can be achieved via the following set, which is independent of θ :

$$\mathcal{G} := \cap_{\theta \in \Theta} \Omega_0. \quad (24)$$

As a result, if the initial condition $(x(0), z(0))$ belongs to \mathcal{G} , then $y(t)$ converges to y_{ref} . This suggests that one may have to adapt the value of $z(0)$ to the initial condition $x(0)$ of the system in order to ensure $(x(0), z(0)) \in \mathcal{G}$. Thus, instead of initialize the integrator at $z(0) = \mathbf{0}$, one should rather compute $z(0)$ in order to move $(x(0), z(0))$ away from the boundaries of \mathcal{G} . This is expected to enhance robustness with respect to $x(0)$.

Remark 11 Equation (10b) derives from continuous control law (14). The latter differs from classical controller

$$u(\zeta) = K\zeta, \quad (25)$$

usually implemented to control (4). Indeed, following steps of the proof of Theorem 9, following discontinuous controller can be derived from (25):

$$u(t) = \arg \min_{v \in \mathcal{U}} (\zeta(t) - \zeta_\theta^*)^T P \begin{bmatrix} B \\ \mathbf{0}_{p \times p} \end{bmatrix} v. \quad (26)$$

Dependency of this equation w.r.t. unknown vector θ via ζ_θ^* prevents practical implementation. Interestingly, for the same reason, (14) has no practical interest. This surprising interplay between those continuous controllers and their discontinuous counterparts can be summarized as: Unimplementable continuous law (14) leads to implementable discontinuous law (10b), whereas implementable continuous law (25) gives rise to unimplementable discontinuous law (26). This discussion suggests that Theorem 9 cannot be regarded as a direct adaptation to the strategy developed in [16].

3.3 Computation of the controller parameter P

This subsection provides guidelines to select numerical values of the matrix P . It is well known that (12) can be reformulated as a semidefinite programming (SDP) so that P can be computed for a given decay rate δ . Besides, matrix P , together with ϵ , also parametrizes ellipsoid Ω_0 included in the closed-loop basin of attraction. Therefore, computation of P aims (i) enlarging Ω_0 and, at the same time, (ii) being compatible with the largest possible value of δ . Those two objectives are known to be competing. For this reason, computation of P is recast into an optimization problem where (ii) is treated as a constraint whereas (i) is related to the criterion to be optimized. Specifically, size r of the ball $B(\zeta_\theta^*, r) \subset \Omega_0$

is maximized for a given value of δ . This strategy adapts the one of [16], which is inspired by the literature for systems with bounded controls [18,17]. A technical step toward this objective is the construction of the set:

$$\mathcal{V} := \cap_{\theta \in \Theta} (\text{conv}(\mathcal{U}) - u_\theta^*). \quad (27)$$

as an intersection of convex polytopes (which is convex [10, p. 8]) containing $\mathbf{0}$ in their interior (Assumption 4 ensures that $u_\theta^* \in \text{Int}\{\text{conv}(\mathcal{U})\}$), set \mathcal{V} can be characterized via $n_g \in \mathbb{N}$ and $g_j \in \mathbb{R}^m$, ($j \in \mathcal{I}_{n_g}$) [5, p. 87]:

$$\mathcal{V} = \{v \in \mathbb{R}^m : g_j^\top v \leq 1, (j \in \mathcal{I}_{n_g})\}. \quad (28)$$

Proposition 12 Consider the following SDP parametrized by strictly positive scalars δ and ϵ and vectors $g_j \in \mathbb{R}^m$:

$$\min_{Q \in \mathbb{S}_+^{n+p}, \lambda \in \mathbb{R}_{>0}, \gamma \in \mathbb{R}_{>0}} \gamma \text{ s.t.}$$

$$\bar{A}_s^{[i]} Q + Q \bar{A}_s^{[i]\top} - \lambda \bar{B} \bar{B}^\top \prec -2\delta Q, (i \in \mathcal{I}_{N_s}) \quad (29a)$$

$$\begin{bmatrix} 1 & \frac{\lambda}{2} g_j^\top \bar{B}^\top \epsilon \\ * & \epsilon Q \end{bmatrix} \succ \mathbf{0}, (j \in \mathcal{I}_{n_g}), \quad (29b)$$

$$\begin{bmatrix} \gamma I_{n+p} & I_{n+p} \\ * & \epsilon Q \end{bmatrix} \succ \mathbf{0}. \quad (29c)$$

Let $(Q^*, \lambda^*, \gamma^*)$ denotes numerical values of (Q, λ, γ) for which γ is minimized. Then, $P = (Q^*)^{-1}$ complies with statement of Theorem 9 and δ is a lower bound for robust decay rate of exponential convergence (Theorem 9). Furthermore, the size of Ω_0 is maximized in the sense that Ω_0 contains ball $B(\zeta_\theta^*, r)$ whose radius satisfies $r > 1/\gamma^*$.

PROOF. Condition (29a) implies the existence of K as defined in (13) and $P = Q^{-1}$ such that (12) is satisfied. Applying the Schur complement lemma, and using (13), condition (29b) leads to:

$$\epsilon < (g_j^\top K P^{-1} K^\top g_j)^{-1}, (j \in \mathcal{I}_{n_g}), \quad (30)$$

vectors g_j are defined as in (28). Since $\mathbf{0} \in \text{Int}\{\mathcal{V}\}$ holds, $\{\zeta : K(\zeta - \zeta_\theta^*) \in \mathcal{V}\}$ is a non-empty subset which, from (16), can be rewritten as $\{\zeta : K(\zeta - \zeta_\theta^*) \in \mathcal{V}\}$ and therefore contains ζ_θ^* . Recall that the minimum of $(\zeta - \zeta_\theta^*)^\top P(\zeta - \zeta_\theta^*)$ on the hyperplane $\{\zeta : g_j^\top K(\zeta - \zeta_\theta^*) = 1\}$ equals $(g_j^\top K P^{-1} K^\top g_j)^{-1}$ [17]. Therefore from (30), (16) and using the fact that Ω_0 is convex, we obtain that $\Omega_0 \subset \{\zeta \in \mathbb{R}^{n+p} : K(\zeta - \zeta_\theta^*) \in \mathcal{V}\}$ i.e. for all $\zeta \in \Omega_0$ and for all $\theta \in \Theta$ we have $K(\zeta - \zeta_\theta^*) + u_\theta^* \in \text{conv}(\mathcal{U})$. As explained in the proof of Theorem 9, this proves that Ω_0 is included in the robust basin of attraction. Condition (29c) is equivalent (by Schur complement) with

$$(\zeta - \zeta_\theta^*)^\top \frac{Q^{-1}}{\epsilon} (\zeta - \zeta_\theta^*) < \gamma (\zeta - \zeta_\theta^*)^\top (\zeta - \zeta_\theta^*), \forall \zeta \neq \zeta_\theta^*,$$

which guarantees that $B(\zeta_\theta^*, 1/\gamma) \subset \Omega_0$.

Remark 13 *Note that if the LMIs (29) are satisfied for some set of parameters $(Q_0, \lambda_0, \gamma_0)$ and for some given ϵ_0 , then they are also satisfied for $\epsilon = 1$ with $(Q_0\epsilon_0, \lambda_0\epsilon_0, \gamma_0)$. Therefore $\epsilon = 1$ can be imposed in Proposition 12 without loss of generality.*

3.4 Comparison with existing results

From the above discussion, proposed strategy somehow implements well-know strategy of robust output regulation consisting in (i) adding an internal model (the integrators in this specific case) before (ii) equipping the augmented model with a robust stabilizer, see e.g. [13]. Let us now clarify why existing results cannot be trivially employed to adapt this strategy to the context of this paper.

1) Relay controller design is often tackled in the framework of switched system for which general approaches have been proposed, see e.g. [9]. Augmented system (4) may be rewritten as the following switched system

$$\dot{\zeta}(t) = \bar{A}_{\sigma(t)}(\theta)\zeta(t) + \bar{B}_{\sigma(t)}$$

with $\bar{A}_{\sigma(t)}(\theta) = \bar{A}(\theta)$ and $\bar{B}_{\sigma(t)} \in \bar{B}\mathcal{U} + h$. In this context, controller design consists in defining switching law $t \mapsto \sigma(t)$ which selects instantaneous value of $\bar{B}_{\sigma(t)}$. Even in the nominal case where θ is known, this goal cannot be achieved via classical approaches, like e.g. [9], thought. This is due to the fact that, by construction, state matrix $\bar{A}(\theta)$ is not Hurwitz since it admits 0 as an eigenvalue, see (5). This feature makes the design problem out of the scope of [9]. However, based on a max-type Lyapunov function, [25] proposes a stabilizing switching law even in the case where $\bar{A}(\theta)$ is not Hurwitz, but no explicit design procedure is provided and uncertainties are not taken into account.

2) Besides, alternative strategies like [16] which are specifically devoted to relay controller and which emulates the behaviour of a stabilizing continuous controller cannot be directly applied in the robust context, see Remark 11.

Those two obstructions were motivation for authors of [3]. As in the present paper, they propose to add an integral action in order to asymptotically cancel steady-state error of the controlled output. However, proposed design procedure requires open-loop state matrix $A(\theta)$ to be Hurwitz for some $\theta \in \Theta$, which might not be the case. Besides, if robust basin of attraction \mathcal{R} is ensured to be non empty, neither \mathcal{R} nor explicit inner approximation of \mathcal{R} can be explicitly constructed. This is in contrast with Theorem 9 and Proposition 12. Therefore, as compared to [3], proposed control design (i) is applicable to any open-loop model (not necessarily having a Hurwitz state matrix) and (ii) provides insights on the construction of an explicit subset of the robust domain of attraction, see Remark 10.

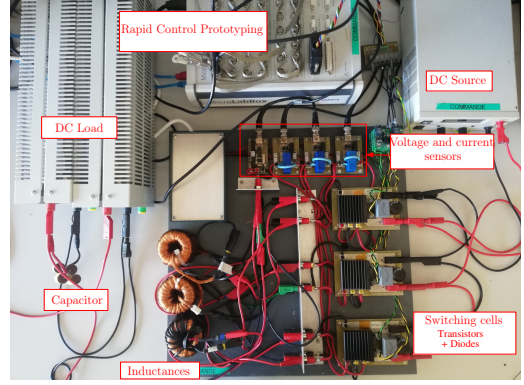


Fig. 1. Test bench

4 Experimental application : Parallel interconnection of DC/DC converters

The proposed approach is now implemented on parallel interconnection of m DC/DC converters sharing a single capacitor and connected to a common resistive load R . Here, we consider the case of unidirectional buck converter. An essential feature offered by this system is the possibility to distribute load current. Indeed, if regulation of output voltage imposes overall current, distribution of current among converters remains free. The most widespread strategy for dealing with this degree of freedom is the so-called balanced current sharing which uniformly distributes currents among converters [29]. The test bench represented on Fig. 1 for this application is composed of three heterogeneous buck converters ($m = 3$) in the sense that inductors, transistors and diodes are different. Each converter is equipped with an identical pair of transistors with anti-parallel diodes. For each converter, transistors are MOSFET whose references are respectively STP315N10F7, STP30NF10 and STP40NF10. For this application only one transistor per converter is driven while the second is used as a diode. The physical parameters values are $E_1 = E_2 = E_3 = 24V$, $C_0 = 40\mu F$, $L_1 = L_2 = 1.3mH$, $L_3 = 1.43mH$ and $R \in [5, 10]\Omega$. Voltage has to be regulated to $V_{ref} = 12V$. Note that the design of the test bench makes the uniform current distribution more challenging. This is to show

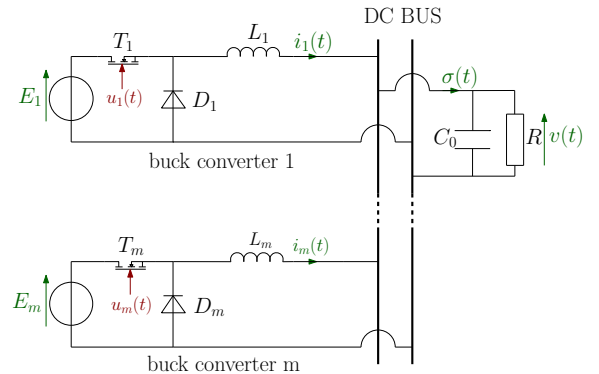


Fig. 2. Electrical schematic

the effectiveness of the control law. The controller hardware is Rapid Control Prototyping system dSpace MicroLabBox which is composed of a Real-Time Processor that communicates with a user-programmable FPGA (whose clock period $T_F = 1.10^{-8}\text{s}$) and several (digital and analog) inputs/outputs.

4.1 Loss free dynamical model

The electrical circuit of the system is represented on Fig. 2. By neglecting parasitic elements and by using Kirchhoff's laws, it can be modelled as follows:

$$L_k \frac{di_k(t)}{dt} = -v(t) + E_k u_k(t), (k \in \mathcal{I}_m), \quad (31a)$$

$$C_0 \frac{dv(t)}{dt} = \sigma(t) - \frac{v(t)}{R}, \quad (31b)$$

where $u_k(t) \in \{0, 1\}$ is the control input associated to k -th branch and

$$\sigma := \sum_{k \in \mathcal{I}_m} i_k, \quad (31c)$$

refers to the total current (see Fig. 2). Currents $i_k(t)$ as well as voltage $v(t)$ are measured and load R is unknown but belongs to a bounded and known interval $[R_{\min}, R_{\max}] \subset \mathbb{R}_{>0}$.

The two control objectives are: (i) regulate output voltage $v(t)$ to a fixed reference V_{ref} and (ii) uniformly distribute current over the m branches. This leads to the following definition of controlled output:

$$y = [i_1 - i_2 \ i_2 - i_3 \ \dots \ i_{m-1} - i_m \ v]^\top \in \mathbb{R}^p, \quad (32)$$

where $p = m$ (which is one of the requirements of Assumption 1), associated with reference y_{ref} given by:

$$y_{\text{ref}} := [\mathbf{0}_{m-1}^\top \ V_{\text{ref}}]^\top \in \mathbb{R}^m. \quad (33)$$

Let us denote the state vector $x = [i \ v]^\top$ with $i := [i_1, \dots, i_m]^\top$ and the control vector $u = [u_1, \dots, u_m]^\top$ with

$$u(t) \in \mathcal{U} = \{0, 1\}^m. \quad (34a)$$

Define parameter vectors $E := [E_1, \dots, E_m]^\top$, $L := [L_1, \dots, L_m]^\top$ and scalar $\theta = 1/R \in \Theta = [1/R_{\max}, 1/R_{\min}]$. In this case, (31) adopts state space representation (1) with:

$$A(\theta) := \begin{bmatrix} \mathbf{0}_{m \times m} & -\text{diag}\{L\}^{-1} \mathbf{1}_m \\ 1/C_0 \mathbf{1}_m^\top & -\theta/C_0 \end{bmatrix}, \quad (34b)$$

$$B := \begin{bmatrix} \text{diag}\{L\}^{-1} \text{diag}\{E\} \\ \mathbf{0}_m^\top \end{bmatrix}, \quad C := \begin{bmatrix} \Gamma^\top & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}, \quad (34c)$$

where $n = m + 1$ and $m = p$ holds and Γ reads

$$\Gamma := \begin{bmatrix} \mathbf{I}_{m-1} \\ \mathbf{0}_{m-1}^\top \end{bmatrix} - \begin{bmatrix} \mathbf{0}_{m-1}^\top \\ \mathbf{I}_{m-1} \end{bmatrix} \in \mathbb{R}^{m \times (m-1)}. \quad (34d)$$

Remark 14 For $m > 1$, the matrix $A(\theta)$ defined in (34b) admits 0 as eigenvalue therefore the example cannot be treated using the approaches proposed in [15, 4, 3, 9]. To the best of our knowledge, there is no solution in the switched systems framework able to deal with this problem.

Remark 15 In [22], a robust controller for the system considered here is designed, based on the combination of the concepts of integral-variable-structure and multiple-sliding-surface control. However, in comparison with this work, gains of our controller are computed automatically by solving an optimization problem.

Remark 16 There are many publications, generally based on PWM approaches, dealing with controlling a parallel dc-dc converter with different current sharing strategies, e.g. [1, 23]. With the approach proposed here, it is possible to impose an uneven distribution of current by modifying vector y defined in (32). However, this is beyond the scope of this paper.

4.2 Assumptions and control objectives

Physical parameters and voltage reference are such that following inequalities hold for all $k \in \mathcal{I}_m$

$$0 < V_{\text{ref}} < E_k, \ L_k > 0. \quad (34e)$$

Next lemma proves that previous assumptions are valid in this context.

Lemma 17 For model (1) and (34), Assumptions 1, 2 and 4 are satisfied. Furthermore, it holds

$$\Psi_e(\theta) = \left\{ \begin{bmatrix} \frac{\theta V_{\text{ref}}}{m} \mathbf{1}_m \\ V_{\text{ref}} \end{bmatrix}, V_{\text{ref}} \text{diag}\{E\}^{-1} \mathbf{1}_m \right\}. \quad (35)$$

PROOF. First observe that

$$\ker \Gamma^\top = \text{span } \mathbf{1}_m. \quad (36)$$

Hence,

$$\begin{aligned} \ker P_\Sigma(\theta, 0) &= \ker \begin{bmatrix} A(\theta) & B \end{bmatrix} \cap \ker \begin{bmatrix} C & \mathbf{0}_{p \times m} \end{bmatrix} \\ &= (\ker \mathbf{1}_m^\top \times \{0\} \times \{\mathbf{0}_m\}) \cap \\ &\quad (\text{span } \mathbf{1}_m \times \{0\} \times \text{span } \mathbf{I}_m) \\ &= \{\mathbf{0}_{n+m}\}, \end{aligned}$$

holds. Since $\text{rank } P_\Sigma(\theta, 0) = n + m - \dim \ker P_\Sigma(\theta, 0)$ is valid, one gets

$$n + m = \text{rank } P_\Sigma(\theta, 0) \leq \max_{s \in \mathbb{C}} \text{rank } P_\Sigma(\theta, s) \leq n + p.$$

From equality $m = p$, this proves that Assumptions 1 and 2 are satisfied. Expression of $\Psi_e(\theta)$ given in (35) follows from the definition of matrices in (34). Together with (34e), this proves that Assumption 4 is valid.

4.3 Controller design

4.3.1 Enhancing numerical conditioning

Controller (10) is parametrized by P whose computation is performed via Proposition 12. This involves inequality (29a) which is typically poorly conditioned for buck converters. This is due in general to difference of magnitude between inductances L_k and capacitor C_0 , which are involved in the definition of matrices defined in (34b) and (34c). To enhance numerical conditioning of optimization problem (29), following lemma provides an alternative formulation to model (34). It consists in changing time scale, state and input variables and is inspired from normalization procedure proposed in [26, p. 15] applied for the case of a *single* converter.

Lemma 18 Define following matrices:

$$\begin{aligned} T_x^{-1} &:= \begin{bmatrix} H^{-1} & \mathbf{0}_{n-1} \\ \mathbf{0}_{n-1}^\top & 1 \end{bmatrix}, \quad T_y^{-1} := \begin{bmatrix} F & \mathbf{0}_{m-1} \\ \mathbf{0}_{m-1}^\top & 1 \end{bmatrix}, \\ T_u^{-1} &:= \begin{bmatrix} \frac{L_{eq}}{L_M} \Gamma^\top \\ L_{eq} \mathbf{1}_m^\top \text{diag}\{L\}^{-1} \end{bmatrix} \text{diag}\{E\}, \end{aligned}$$

with:

$$H^{-1} := \sqrt{\frac{L_{eq}}{C_0}} \begin{bmatrix} \frac{1}{L_M} \Gamma^\top \text{diag}\{L\} \\ \mathbf{1}_m^\top \end{bmatrix}, \quad (38a)$$

$$F := \frac{1}{L_m} \sqrt{\frac{L_{eq}}{C_0}} (\Gamma^\top \text{diag}\{L\} \Gamma) (\Gamma^\top \Gamma)^{-1}, \quad (38b)$$

$$\frac{1}{L_{eq}} := \sum_{k \in \mathcal{I}_m} \frac{1}{L_k}, \quad L_M = \max_k L_k. \quad (38c)$$

Define also change of coordinates via $x_a = T_x^{-1}x$, $u_a = T_u^{-1}u$, $y_a = T_y^{-1}y$ and $\tau = \frac{t}{\sqrt{L_{eq}C_0}}$ so that system (1) becomes:

$$\begin{cases} \frac{dx_a(\tau)}{d\tau} = A_a(\theta)x_a(\tau) + B_a u_a(\tau), \\ y_a(\tau) = C_a x_a(\tau), \end{cases} \quad (39)$$

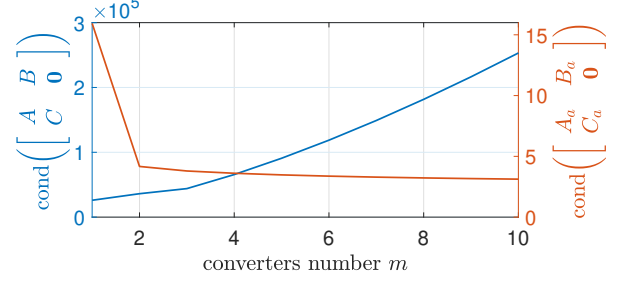


Fig. 3. Impact of the numerical conditioning for m converters

where

$$\begin{aligned} A_a(\theta) &= \begin{bmatrix} \mathbf{0}_{m \times m} & \begin{bmatrix} \mathbf{0}_{m-1} \\ -1 \end{bmatrix} \\ \begin{bmatrix} \mathbf{0}_{m-1}^\top & 1 \end{bmatrix} & -\frac{\theta}{Q_c} \end{bmatrix}, \quad B_a = \begin{bmatrix} I_m \\ \mathbf{0}_m^\top \end{bmatrix}, \\ C_a &= \begin{bmatrix} I_{m-1} & \frac{-1}{mL_M} \Gamma^\top L & \mathbf{0}_{m-1} \\ \mathbf{0}_{m-1}^\top & 0 & 1 \end{bmatrix}, \end{aligned} \quad (40)$$

with $Q_c := \sqrt{\frac{C_0}{L_{eq}}}$. In this case, $u_a(t)$ belongs to $\mathcal{U}_a := T_u^{-1}\mathcal{U}$ for all t . See Appendix A for the proof.

To illustrate conditioning enhance, the Matlab function **cond** which returns the 2-norm condition number of a matrix is used: It computes the ratio of the largest singular value of a matrix to the smallest. Large condition number indicates ill-conditioned matrix. We plot on Fig. 3 the condition number of the original model (34) and the normalized model (40) for different values of m . It shows that the normalized model is much better conditioned even for large values of m .

4.3.2 Control design

The control structure Following methodology proposed in Section 3 and relying on the changes of variable defined earlier and in particular the fact $C_a = T_y^{-1}CT_x$ holds, equality $y = y_{\text{ref}}$ is proved to be equivalent to $C_a x_a = T_y^{-1}y_{\text{ref}}$. This is the motivation for considering the following equation, in place of (10a),

$$\frac{dz_a(\tau)}{d\tau} = C_a x_a(\tau) - T_y^{-1}y_{\text{ref}}. \quad (41)$$

State of the resulting augmented model equals $\zeta_a := \begin{bmatrix} x_a \\ z_a \end{bmatrix} = T_\zeta^{-1}\zeta$ with

$$T_\zeta^{-1} := \begin{bmatrix} T_x^{-1} & \mathbf{0} \\ \mathbf{0} & T_y^{-1} \frac{1}{\sqrt{L_{eq}C_0}} \end{bmatrix}. \quad (42)$$

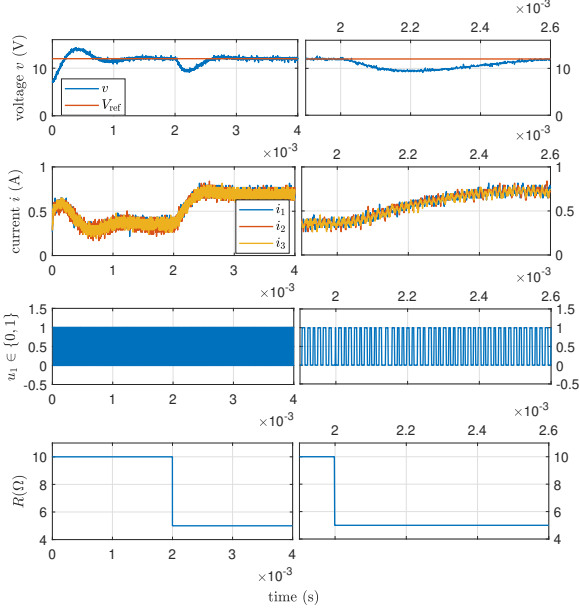


Fig. 4. Experimental results: chronograph and its zoom (depicted on the right) of the voltage, the current and the load for $\delta_a = 0.22$.

Normalized controller parameters The strategy first consists in considering augmented model based on state space (39) in Proposition 12. From physical parameters valued defined earlier and (35), u_θ^* does not depend on θ and equals to $0.5 \times \mathbf{1}_3$. In this case, \mathcal{V} equals $[-0.5, 0.5]^3$. Alternatively, it can be characterized as in (28) with $n_g = 6$ and $g_1 = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix}^\top$, $g_2 = -g_1$, $g_3 = \begin{bmatrix} 0 & 2 & 0 \end{bmatrix}^\top$, $g_4 = -g_3$, $g_5 = \begin{bmatrix} 0 & 0 & 2 \end{bmatrix}^\top$ and $g_6 = -g_5$. Normalized version of this set reads: $\mathcal{V}_a := \{v \in \mathbb{R}^m : g_{j_a}^\top v \leq 1, (j \in \mathcal{I}_6)\}$ with $g_{j_a} := T_u^\top g_j$. Without loss of generality we consider $\epsilon = 1$ to solve (29) (see Remark 13). Now using the normalized vectors g_{j_a} , ($j \in \mathcal{I}_6$) and choosing $\delta_a = 0.22$, (29) leads to :

$$\lambda_a^* = 96, \gamma_a^* = 0.0114, \det(P_a^*)^{-1/2} = 1.47 \cdot 10^9, \quad (43)$$

so that Theorem 9 applies and gives a solution to Problem 5 for the normalized model.

Here, due to the large dimension of Q_a^* (7×7), only the numerical value of $\det(P_a^*)^{-1/2}$ with $P_a^* = (Q_a^*)^{-1}$ is provided. This quantity is the volume of the inner approximation of the domain of attraction, see Proposition 12 and Theorem 9.

Denormalization In order to recover original physical units, the actual gains values have to be "de-normalized" before implementation. From the change of variables defined previously, this is achieved by implementing the following relation : $u = T_u u_a$.

4.4 Experimental results

Now let us apply the proposed methodology. We fix the initial condition $x(0)$ to $[0.24 \ 0.24 \ 0.24 \ 7.2]^\top$, z_0 to $\mathbf{0}_3$ and we choose $\theta_n = 1/10$. The latter leads to

$$\zeta_{\theta_n}^* = \begin{bmatrix} 0.4 & 0.4 & 0.4 & 12 & \mathbf{0}_3^\top \end{bmatrix}^\top.$$

From Remark 10 and relying on the change of variables defined earlier, one need to make sure that the following relation holds in order to apply the proposed methodology:

$$T_\zeta^{-1} \zeta(0) \in \cap_{\theta \in \Theta} \mathcal{E}(P_a, T_\zeta^{-1} \zeta_\theta^*, 1). \quad (44)$$

One can show that it is the case for the numerical values obtained in (43).

The experimental results are represented on Fig. 4 for which the initial value of the load is, first, fixed to its nominal value equals to 10Ω . Then a load step is applied, which changes this value to 5Ω . For the two load values, one observes that after a short transient, the desired steady state is achieved: Voltage converges to V_{ref} and overall current is uniformly distributed. This proves the effectiveness of our approach.

Note that exponential stability is achieved under the assumption of possibly infinite switching frequency as well as ideal electrical components, including the ones required to implement the control law. Those assumptions are obviously not valid in practice. This induces very fast switching and chattering around the desired equilibrium as shown on Fig 4. Yet, magnitude of this chattering is fully acceptable from the application point of view, which validates relevance of the previous assumptions for the control design purpose.

Remark 19 *The transistors switching induce transient phenomena on states to be measured. For this reason, the ADC sampling needs to be triggered with the maximum switching frequency so that measurements are interspersed between two switches [24]. Knowing that a ADC measurement are available every $1.10^{-6}s$, we have imposed a minimum time between two switches $T_s = 5.10^{-6}s$. The choice of T_s ensures that the controller dynamics is much faster than system dynamics. These latter are set by L_j/R , $j \in \mathcal{I}_3$ whose minimum is equal to $260.10^{-6}s$.*

5 CONCLUSION

A switching controller for LTI system with uncertain equilibrium was proposed in this paper. The main contribution was the design of a robust switching control law in the case of uncertain and non Hurwitz open loop state matrix. The proposed approach is applied on a parallel interconnection of m DC/DC converters where a new change of variables is presented to avoid ill-conditioned matrix. Finally, experimental results show the effectiveness of the method. Generalizing this approach for a

larger class of system is the topic of future works. Other interesting points are (i) to limit the switching frequency at the steady state, (ii) to get rid of Assumption 1 in the stability proofs and (iii) to provide a precise framework for a comparison with PWM approach.

A APPENDIX: Proof of Lemma 18

From (36), we can show that H reads

$$H = \sqrt{\frac{C_0}{L_{eq}}} \left[L_M \Gamma (\Gamma^\top \text{diag}\{L\} \Gamma)^{-1} L_{eq} \text{diag}\{L\}^{-1} \mathbf{1}_m \right]. \quad (\text{A.1})$$

Now, notice that from the definitions of matrices (38) and (A.1), it holds:

$$\mathbf{1}_m^\top H = \sqrt{\frac{C_0}{L_{eq}}} \left[\mathbf{0}_{m-1}^\top \ 1 \right], \quad (\text{A.2a})$$

$$H^{-1} \text{diag}\{L\}^{-1} \mathbf{1}_m = \sqrt{\frac{L_{eq}}{C_0}} \begin{bmatrix} \mathbf{0}_{m-1} \\ 1/L_{eq} \end{bmatrix}. \quad (\text{A.2b})$$

By using the change of variables $x_a = T_x^{-1} x$, one has:

$$T_x^{-1} A(\theta) T_x = \begin{bmatrix} \mathbf{0}_{m \times m} & -H^{-1} \text{diag}\{L\}^{-1} \mathbf{1}_m \\ \frac{1}{C_0} \mathbf{1}_m^\top H & -\frac{\theta}{C_0} \end{bmatrix}, \quad (\text{A.2})$$

$$\stackrel{(\text{A.2})}{=} \sqrt{\frac{1}{L_{eq} C_0}} A_a(\theta),$$

and

$$T_x^{-1} B u = \begin{bmatrix} H^{-1} \text{diag}\{L\}^{-1} \text{diag}\{E\} \\ \mathbf{0}_m^\top \end{bmatrix} u \stackrel{(37)}{=} \sqrt{\frac{1}{L_{eq} C_0}} B_a u_a.$$

Finally, by using the expression of τ in (37), we end up with matrices $A_a(\theta)$ and B_a given in (40).

Let us compute expression of C_a :

$$C_a = T_y^{-1} C T_x = \begin{bmatrix} F \Gamma^\top H & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \stackrel{(\text{A.1})}{=} \begin{bmatrix} \mathbf{I}_{m-1} & Z & \mathbf{0} \\ \mathbf{0} & 0 & 1 \end{bmatrix},$$

where

$$Z := \frac{L_{eq}}{L_M} \Gamma^\top \text{diag}\{L\} W \text{diag}\{L\}^{-1} \mathbf{1}_m, \quad (\text{A.3})$$

$$W := \Gamma (\Gamma^\top \Gamma)^{-1} \Gamma^\top = \mathbf{I}_m - \frac{1}{m} \mathbf{1}_{m \times m}. \quad (\text{A.4})$$

Thus, it remains to prove that

$$Z = \frac{-1}{m L_M} \Gamma^\top L \quad (\text{A.5})$$

holds. Indeed, right multiplication by invertible matrix $\begin{bmatrix} \Gamma & \mathbf{1}_m \end{bmatrix}$ of either W and right hand side of (A.4) gives the same matrix $\begin{bmatrix} \Gamma & \mathbf{0}_m \end{bmatrix}$. Bearing (A.4) in mind and noticing that L_{eq}^{-1} equals $\mathbf{1}_m^\top \text{diag}\{L\}^{-1} \mathbf{1}_m$, it follows that

$$Z = \frac{1}{L_M} \Gamma^\top (L_{eq} \mathbf{1}_m - \frac{1}{m} \text{diag}\{L\} \mathbf{1}_m),$$

which immediately leads to (A.5). Note that for identical numerical values of inductances, $Z = \mathbf{0}_{m-1}$.

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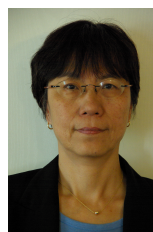
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