Infinite gain margin, contraction and optimality: an LMI-based design

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Mattia Giaccagli[1] and Vincent Andrieu[1] and Sophie Tarbouriech[2] and Daniele Astolfi[1]

***Abstract*— In this paper we focus on nonlinear systems composed by a linear term plus a nonlinearity satisfying a monotonic or a sector bound condition. We present sufficient conditions based on LMIs for the design of a feedback control law possessing an infinite gain margin property that makes the closed-loop system to define a contraction. Then, we analyze the connection between infinite gain margin contractive feedbacks and optimization problems by showing how such an infinite gain control design minimizes a given cost function. A practical case of study is given to illustrate our results.**

I. Introduction

The design of robust controllers is a central topic in system theory. In this article, we focus on controllers having an "infinite gain margin property", see, e.g. [1, Section 3]. Feedbacks possessing such a property are of practical importance because the closed-loop stability is preserved in presence of *static* and/or *unmodeled fast dynamic* plant uncertainties ([1, Section 3]). This robustness properties is guaranteed because infinite gain margin feedbacks stabilize a given plant always in the "right direction" and can be rescaled by any positive (large enough) parameter. This type of controllers are also optimal for a given cost function. In the case of linear systems, the construction of an infinite gain margin property boils down to the solution of the Algebraic Riccati Equation associated to a Linear Quadratic Regulator problem [1, Section 3].

In this work we focus on the notion of infinite gain margin in the context of incremental stability (see, e.g., [2]–[4]), similarly to what has been done in [5] for the case of observer design. In a few words, a system is globally incrementally exponentially stable when any two trajectories starting from different initial conditions asymptotically converge towards each other, uniformly in the difference of the initial conditions and with a decay that is exponential in time. This notion is also denoted as *contraction*. In the last decades, contractivity properties attracted more and more attention because of their potential applications in many control problems such as observers design (see, e.g., [5], [6]), output regulation (see, e.g., [7]–[10]) and multi-agent synchronization (see, e. g, [11]–[13]). To this end, the design of controllers that make the closed-loop system to define a contraction is gathering more and more importance, see, e.g., [10], [14], [15].

Among the main tools to study incremental stability, we recognize three (strictly related) different approaches. The first one relies on the use of Finsler-Lyapunov functions (see [4], [16]) which lift the classical notion of Lyapunov function to the tangent bundle (i.e. the tangent space). The second approach uses matrix measures (also called *logarithmic norms*) to directly characterize the properties of the Jacobian of a vector field ([17], [18]). Finally, the last approach uses a metric analysis and relies on the property that a system defines a contraction if there exists a Riemannian metric along which the vector-field's flow generates trajectories which are monotonically decreasing, forwarding in time, according to the distances associated to such a Riemannian metric, see, e.g., [2], [3], [18] and references therein. In this work we focus on this last approach.

Our objective is to study the design of infinite gain margin feedback laws providing contractive properties for the closed loop and to investigate its connection with optimal control. To this, we focus on the class of nonlinear systems which are defined by a linear dynamics coupled with a nonlinearity satisfying some incremental monotonic or sector bound condition. This a class of systems is of relevant practical interest since several real-life engineering systems can be modeled in a such way, see, e.g., [19]–[21] and references therein. To study the considered class of systems, we rely on a linear matrix inequality (LMI)-based approach in order to provide a set of sufficient conditions for the design of an infinite gain margin control feedback. Furthermore, differently from many other works employing similar LMIs techniques for the design of stabilizing feedbacks (see, e.g., [21]–[25]), here we also prove the contractivity properties of the closed-loop system. This extends the results in [12] in terms of robustness of the proposed control law. Then, we show that such a design can be associated with the solution to an optimization problem by means of the Hamilton-Jacobi-Bellman (HJB) equation.

The rest of the paper is structured as follows. In Section II we present some preliminaries and we formalize the problem we aim at solving. In Section III we provide our main result concerning the design of contractive feedback laws with infinite gain margin properties, while in Section IV we show that this control design can be seen as the solution to an optimization problem. In Section V we apply our results to a practical case study. Conclusions are in Section VI.

Notation: Given a $n \times n$ matrix A, we indicate with $\mathtt{He}\{A\} = A + A^\top$. If A is symmetric, write $A \succ 0$ (resp. $\succeq$, $\prec$, $\preceq$) if A is strictly positive (resp. semi positive, strictly negative, semi negative) defined. Given square matri-

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ces $A_1, \dots, A_N$ we indicate with $\texttt{blkdiag}\{A_1, \dots, A_N\}$ the block-diagonal matrix with $A_1, \dots, A_N$ in the main diagonal and 0 elsewhere. We indicate with I the identity matrix. We denote with $\mathbb{R}_{>0}$ the set $(0, \infty)$ while with $\mathbb{R}_{\geq 0}$ the set $[0, \infty)$. We say that a function f is C^1 if it is continuously differentiable with continuous first derivative. Given a vector field $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ and a 2-tensor $P : \mathbb{R}^n \mapsto \mathbb{R}^n \times \mathbb{R}^n$ both C^1, we indicate with $L_f P(x)$ the Lie derivative of the tensor P along f defined as $L_f P(x) = \lim_{h\to 0} \frac{(I+h\frac{\partial f}{\partial x}(x))^\top P(x+hf(x,t))(I+h\frac{\partial f}{\partial x}(x)) - P(x)}{}$, with coordinates $(L_f P(x))_{i,j} = \sum_k^h \left[2P_{ik}\frac{\partial f_k}{\partial x_j}(x) + \frac{\partial P_{ij}}{\partial x_k}(x) f_k(x)\right]$. In the case in which P is constant, the previous expression reduces to $L_f P(x) = P\frac{\partial f}{\partial x}(x) + \frac{\partial f}{\partial x}(x)^\top P$.

II. Preliminaries

A. Highlights on contraction theory

Consider the system

$$\dot{x} = f_0(x) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $f_0 : \mathbb{R}^n \mapsto \mathbb{R}^n$ is sufficiently smooth. Let $\mathcal{X}(x_0, t)$ denote the trajectory of system (1) at time t with initial condition x_0 at time 0. We have the following definition.

Definition 1. *We say that system* (1) *is incrementally globally exponentially stable (is a contraction) if there exist two strictly positive real numbers $\lambda, k > 0$ such that*

$$\|\mathcal{X}(x_1, t) - \mathcal{X}(x_2, t)\| \leq k \|x_1 - x_2\| \exp(-\lambda t) \tag{2}$$

for all $x_1, x_2 \in \mathbb{R}^n$ and for all $t \geq 0$.

A sufficient condition to claim that system (1) defines a contraction is the existence of a Riemannian metric along which the flow generates trajectories for which the distance associated with the metric is monotonically decreasing forward in time. This is recalled in the following result (see [2], [3]).

Theorem 1. *The system* (1) *is a contraction if there exist a C^1 matrix function $P : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$ taking symmetric and positive definite values and three strictly positive real numbers $\bar{p}, \underline{p}, q > 0$ such that*

$$\begin{aligned} &L_{f_0} P(x) \preceq -qI, \\ &\underline{p} I \preceq P(x) \preceq \bar{p} I \end{aligned} \tag{3}$$

for all $x \in \mathbb{R}^n$.

Consider now a system of the form

$$\dot{x} = f(x) + g(x)u \tag{4}$$

with $u \in \mathbb{R}^m$ being a control action, $f : \mathbb{R}^n \mapsto \mathbb{R}^n$, $g : \mathbb{R}^n \mapsto \mathbb{R}^{n\times m}$ and f, g to be sufficiently smooth.

Definition 2. *Consider system* (4)*. We say that the C^1 function $\alpha : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a contractive control law with infinite gain margin for system* (4) *if there exist $P, \underline{p}, \bar{p}, q$ such that, by letting*

$$f_\kappa(x) := f_0(x) + g(x)\kappa\alpha(x)\,,$$

the following holds

$$L_{f_\kappa} P(x) \preceq -qI, \qquad \underline{p} I \preceq P(x) \preceq \bar{p} I \tag{5}$$

for all $\kappa \geq 1$ and all $x \in \mathbb{R}^n$.

Note that Definition 2 is uniform with respect to P, that is, the metric is the same independently on the gain κ. Recall that for linear systems of the form $\dot{x} = Ax + Bu$, the typical design of a contractive feedback law is given by $K = \kappa B^\top P$ with P solution to the Algebraic Riccati Equation $PA + A^\top P - PBB^\top P \preceq -qI$, see, e.g. [1, Section 3.5.3].

B. Problem formulation

In this paper, we consider the class of nonlinear systems defined by the following equation

$$\dot{x} = Ax + G\varphi(y) + Bu, \qquad y = Hx \tag{6}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is a linear combination of the state, A, G, B and H are matrices of suitable dimensions and φ is a nonlinearity which satisfies the following assumption.

Standing Assumption 1. *One of the following holds.*

1) *The function $\varphi : \mathbb{R}^p \mapsto \mathbb{R}^p$ is C^1, has a symmetric Jacobian and there exists $\Gamma = \Gamma^\top \succ 0$ such that the following incremental **monotonic** condition holds*

$$0 \preceq \frac{1}{2}\frac{\partial \varphi}{\partial y}(y) \preceq \Gamma, \quad \forall y \in \mathbb{R}^p. \tag{7}$$

2) *The function $\varphi : \mathbb{R}^p \mapsto \mathbb{R}^q$ is C^1 and satisfies the following incremental **sector bound** condition*

$$\texttt{He}\left\{\frac{\partial \varphi}{\partial y}(y)^\top S \left[\frac{\partial \varphi}{\partial y}(y) + \Omega\right]\right\} \preceq 0, \quad \forall y \in \mathbb{R}^p, \tag{8}$$

for some matrices $\Omega \in \mathbb{R}^{q\times p}$ and $S \in \mathbb{R}^{q\times q}$, with $S = S^\top \succ 0$.

Figures 1 and 2 illustrate an example of a nonlinearity and its Jacobian when φ is a function from $\mathbb{R}$ to $\mathbb{R}$. Note that in such a scalar case, the monotonic and the sector bound assumptions coincide with a globally Lipschitz assumption.

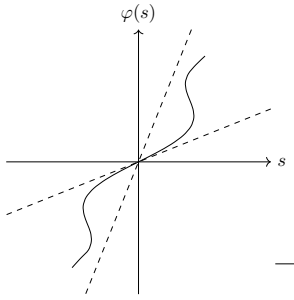

Fig. 1: Non-linearity

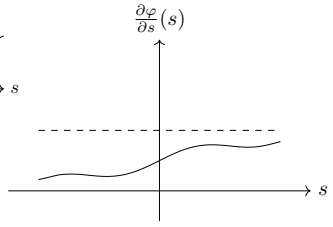

Fig. 2: Derivative term

The problem that we aim to solve is to design an infinite gain margin control law $\alpha(\cdot)$ that makes the closed-loop system to define a contraction according to Definition 2. In

this paper, motivated by the system structure in (6), we focus on state-feedback controllers of the form

$$\alpha(x) = Kx + N\varphi(y), \qquad y = Hx \tag{9}$$

for some matrices K, N of suitable dimensions. To achieve contractive properties for the closed-loop system (6), (9) we aim to apply Theorem 1. However, finding a metric P in the most general case, might be practically difficult as one has to solve a partial differential equation. Therefore, we restrict ourselves to the case in which P does not depend on x, i.e. P is a constant symmetric positive definite matrix $P = P^\top \succ 0$. This restricts the possible set of solutions. However, on the other hand, it allows to rely on tools which are easier to use in practical cases (LMIs) and simplify the design of the control law. In conclusion, the problem to which we aim to approach is formalized as follows.

Problem 1. *Consider system* (6) *with the function φ satisfying Standing Assumption 1. Find matrices K, N such that the feedback*

$$\alpha(x) := Kx + N\varphi(Hx) \tag{10}$$

is a contractive control law with infinite gain margin for system (6)*, namely, there exist $P = P^\top \succ 0$ and $q > 0$ such that, the following holds*

$$\mathtt{He}\left\{P\left[(A+\kappa BK) + (G+\kappa BK)\frac{\partial\varphi}{\partial y}(Hx)H\right]\right\} \preceq -qI \tag{11}$$

for all $x \in \mathbb{R}^n$ and for all $\kappa \geq 1$.

In the next sections, we establish a solution to the Problem 1 based on LMI-approaches.

III. Feedback design

To find a state-feedback stabiliser of the form (9) which is of infinite gain margin, we need (11) to hold for all $\kappa \geq 1$. This is trivially satisfied if the autonomous (i.e. with $u = 0$) system already defines a contraction and the control action is chosen such that it provides negativity always in the correct directions. By considering the case for the nonlinearity satisfying the incremental monotonic condition (7) and having in mind the results in [12], the following result can be stated.

Proposition 1. *Consider system* (6) *and assume that φ satisfies the monotonic condition* (7) *for some matrix $\Gamma = \Gamma^\top \succ 0$. If there exist a matrix $W \in \mathbb{R}^{n\times n}$, $W = W^\top \succ 0$, two matrices $Z \in \mathbb{R}^{m\times n}$, $N \in \mathbb{R}^{m\times q}$ and a real number $\epsilon > 0$ such that the following LMIs hold*

$$\mathtt{He}\left\{\begin{bmatrix} AW+BZ & 0 & 0 \\ HW+(G+BN)^\top & -2\Gamma^{-1} & 0 \\ W & 0 & -\frac{\epsilon}{2}I \end{bmatrix}\right\} \preceq 0, \tag{12}$$

$$\mathtt{He}\left\{\begin{bmatrix} BZ & 0 \\ HW+(BN)^\top & -2\Gamma^{-1} \end{bmatrix}\right\} \preceq 0, \tag{13}$$

then Problem 1 is solved with $K = ZW^{-1}$ and such a N.

Proof. The closed-loop system can be written in the form

$$\begin{aligned} \dot{x} &= (A+\kappa BK)x + (G+\kappa BN)\varphi(Hx) \\ &= (A+BK)x + (G+BN)\varphi(Hx) \\ &\quad + (\kappa-1)\left[BKx + BN\varphi(Hx)\right] \end{aligned} \tag{14}$$

Let

$$\begin{aligned} \xi(x) &= (A+BK)x + (G+BN)\varphi(Hx) \\ \zeta(x,\kappa) &= (\kappa-1)[BKx + BN\varphi(Hx)] \end{aligned} \tag{15}$$

By [12, Theorem 2], if (12) admits a solution, then $\xi(x)$ defines a contraction with respect to the metric $P = W^{-1}$ for $K = ZW^{-1}$ and N. To make the proof more self-contained, we recall the main steps of such a result. Let us define

$$\mathcal{L} := \mathtt{He}\left\{P\left[\overline{A} + \overline{G}\frac{\partial\varphi}{\partial y}(y)H\right]\right\} \tag{16}$$

for all $y \in \mathbb{R}^p$, where $\overline{A} = A + BK$ and $\overline{G} = (G + BN)$. The function $\mathcal{L}$ defined in (16) can be rewritten as:

$$\mathcal{L} := P\mathcal{A} + \mathcal{A}^\top P + w^\top \frac{\partial\varphi}{\partial y}(y)v + v^\top \frac{\partial\varphi}{\partial y}(y)^\top w,$$

with the notation $v = H$ and $w = \overline{G}^\top P$. The latter gives

$$\begin{aligned} \mathcal{L} = \mathtt{He}\{P\mathcal{A}\} &- \frac{1}{4}(v-w)^\top \mathtt{He}\left\{\frac{\partial\varphi}{\partial y}(y)\right\}(v-w) \\ &+ \frac{1}{4}(v+w)^\top \mathtt{He}\left\{\frac{\partial\varphi}{\partial y}(y)\right\}(v+w)\ . \end{aligned}$$

With the monotonicity property (7) in Standing Assumption 1, this implies

$$\mathcal{L} \leq \mathtt{He}\{P\mathcal{A}\} + \frac{1}{4}(v+w)^\top\Gamma(v+w)\ ,$$

or equivalently

$$\mathcal{L} \leq -\eta I + \mathtt{He}\{P\mathcal{A}\} + \eta I + \frac{1}{4}(v+w)^\top\Gamma(v+w)\ . \tag{17}$$

for some $\eta > 0$ to be defined. By applying the Schur complement to (12), and by pre- and post-multiplying the resulting one by $\mathtt{blkdiag}\{P, I\}$ with $P = W^{-1}$ and $K = ZW^{-1}$, we obtain

$$\begin{bmatrix} P\mathcal{A} + \mathcal{A}^\top P + \nu^{-1}I & P\overline{G} + H^\top \\ (P\overline{G} + H^\top)^\top & -4\Gamma^{-1} \end{bmatrix} \leq 0\ .$$

where $\nu^{-1} = \epsilon$. By taking the Schur's complement again, we obtain

$$P\mathcal{A} + \mathcal{A}^\top P + \nu^{-1}I + \frac{1}{4}(H^\top + P\overline{G})\Gamma(H + \overline{G}^\top P) \leq 0$$

In other words, from 17, we obtain $\mathcal{L} \leq -\eta I$ with $\eta = \nu^{-1}$. Then, one can conclude that if relation (12) holds then system (1) is exponentially contractive in view of Theorem 1. To conclude the proof, by linearity of the Lie derivative, we have to show that ζ satisfies $L_\zeta P(x,\kappa) \preceq 0$ for all $\kappa \geq 1$ with

respect to the same K and P. As a consequence, compute, for any vector $b \in \mathbb{R}^n$,

$$\begin{aligned} b^\top L_\zeta P(x,\kappa) b = \\ &= (\kappa-1) b^\top \mathtt{He}\left\{ PBK + PBN \tfrac{\partial \varphi}{\partial y}(Hx) H \right\} b \\ &= (\kappa-1) b^\top \Big(PBK + (BK)^\top P \\ &\quad - \tfrac{1}{4}(PBN - H^\top) \mathtt{He}\{\tfrac{\partial \varphi}{\partial y}(Hx)\}((BN)^\top P - H) \\ &\quad + \tfrac{1}{4}(PBN + H^\top) \mathtt{He}\{\tfrac{\partial \varphi}{\partial y}(Hx)\}((BN)^\top P + H) \Big) b \\ &\leq (\kappa-1) b^\top (\mathtt{He}\{PBK\}) b \\ &\quad + \tfrac{1}{4} b^\top (PBN + H^\top) \Gamma ((BN)^\top P + H) b \\ &= (\kappa-1) b^\top \begin{bmatrix} PBK + (BK)^\top P & (PBN + H^\top) \\ (H + (BN)^\top P) & 4\Gamma^{-1} \end{bmatrix} b. \end{aligned}$$

By left and right-multiplying by $\mathtt{blkdiag}\{W, I\}$, we finally obtain

$$\begin{aligned} &b^\top L_\zeta P(x,\kappa) b \leq \\ &b^\top (\kappa-1) \mathtt{He} \left\{ \begin{bmatrix} BZ & 0 \\ HW + (BN)^\top & -2\Gamma^{-1} \end{bmatrix} \right\} b \leq 0 \end{aligned}$$

for all b and all $\kappa \geq 1$ by (13), concluding the proof. □

A dual result can be given in the case in which φ satisfies the incremental sector bound condition (8). This is highlighted in the following proposition.

Proposition 2. *Consider system* (6) *and assume that φ satisfies the incremental sector bound condition* (8) *for some matrices Ω and $S = S^\top \succ 0$. If there exist a matrix $W \in \mathbb{R}^{n\times n}$, $W = W^\top \prec 0$, two matrices $Z \in \mathbb{R}^{m\times n}$, $N \in \mathbb{R}^{m\times q}$ and a real number $\nu > 0$ such that the following LMIs hold*

$$\mathtt{He}\left\{ \begin{bmatrix} AW + BZ & 0 & 0 \\ G^\top + N^\top B^\top - S\Omega H W & -S & 0 \\ W & 0 & -\frac{\nu}{2} I \end{bmatrix} \right\} \preceq 0, \tag{18}$$

$$\mathtt{He}\left\{ \begin{bmatrix} BZ & 0 \\ N^\top B^\top - S\Omega H W & -S \end{bmatrix} \right\} \preceq 0, , \tag{19}$$

then Problem 1 is solved with $K = ZW^{-1}$ and such a N.

Proof. Similarly to the proof of Proposition 1, we can rewrite the closed loop as in (14) and define (15). By [12, Theorem 1] if (18) holds, then $\xi(x)$ defines a contraction with respect to the metric $P = W^{-1}$ for $K = ZW^{-1}$ and such N. Indeed, define $\mathcal{L}$ as in (16). In view of the sector bound property (8), it yields by pre-multiplying (resp. post-multiplying) inequality (8) by $H^\top$ (resp. H)

$$\begin{aligned} \mathcal{L} \leq \mathcal{L} &- H^\top \frac{\partial \varphi}{\partial \zeta}(\zeta)^\top S \left[\frac{\partial \varphi}{\partial \zeta}(\zeta) + \Omega \right] H \\ &- H^\top \left[\frac{\partial \varphi}{\partial \zeta}(\zeta) + \Omega \right]^\top S \frac{\partial \varphi}{\partial \zeta}(\zeta) H. \end{aligned} \tag{20}$$

By defining

$$\mathcal{Q} = \begin{bmatrix} \mathcal{A}^\top P + P\mathcal{A} + \eta I & P\overline{G} - H^\top \Omega^\top S \\ (P\overline{G} - H^\top \Omega^\top S)^\top & -2S \end{bmatrix}, \tag{21}$$

and using the definition of $\mathcal{L}$, inequality (20) gives

$$\mathcal{L} \leq -\eta I + \begin{bmatrix} I \\ \frac{\partial \varphi}{\partial \zeta}(\zeta) H \end{bmatrix}^\top \mathcal{Q} \begin{bmatrix} I \\ \frac{\partial \varphi}{\partial \zeta}(\zeta) H \end{bmatrix}. \tag{22}$$

Note that, by taking the Schur's complement of (18), by pre- and post-multiplying the resulting one by $\mathtt{blkdiag}\{P, I\}$ where $P = W^{-1}$ and $K = ZW^{-1}$ and by posing $\eta = \nu^{-1}$, we recover the left-hand-side of (22). Hence if (18) holds, one can conclude that $\mathcal{L} \leq -\eta I$, hence, $\xi(x)$ is a contraction. Then, by employing the sector bound condition (8), we have that, for $\kappa \geq 1$ and for any vector $b \in \mathbb{R}^n$,

$$\begin{aligned} &b^\top L_\zeta P(x,\kappa) b \leq \\ &\qquad b^\top L_\zeta P(x,\kappa) b + (\kappa-1) b^\top (\Psi(x) + \Psi^\top(x)) b \end{aligned}$$

with

$$\Psi(x) = H^\top \frac{\partial \varphi}{\partial y}(y) S \left[\frac{\partial \varphi}{\partial y}(y) + \Omega \right] H.$$

From the definition of ζ in (15), we get

$$b^\top L_\zeta P(x,\kappa) b \leq (\kappa-1) b^\top \begin{bmatrix} I \\ \frac{\partial \varphi}{\partial y}(y) \end{bmatrix}^\top \Lambda \begin{bmatrix} I \\ \frac{\partial \varphi}{\partial y}(y) \end{bmatrix} b$$

with

$$\Lambda := \mathtt{He}\left\{ \begin{bmatrix} PBK & 0 \\ N^\top B^\top - PS\Omega H & -S \end{bmatrix} \right\}.$$

By pre and post-multiplying (18) by $\mathtt{blkdiag}\{W, I\}$, we retrieve Λ. Hence $\Lambda \succeq 0$ and so $b^\top L_\zeta P(x,\kappa) b \leq 0$ for all b, x and $\kappa \geq 1$, concluding the proof. □

IV. Contraction and optimality

It is well known that there is a strong relationship between infinite gain margin and optimal control laws. See, for instance, [1, Section 3]. Consider therefore a cost function $\mathcal{J} : \mathbb{R}^n \times L^\infty_{loc}(\mathbb{R}_{>0})$ of the form

$$\mathcal{J}(x,u) := \int_0^\infty (\mathcal{Q}(X_u(x,t)) + u(t)^\top \mathcal{R}(X_u(x,t))^{-1} u(t)) dt \tag{23}$$

where X_u is the solution to the controlled system (6) for some chosen u, $\mathcal{Q} : \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$ is a positive definite function, $\mathcal{R} : \mathbb{R}^n \mapsto \mathbb{R}^{m\times m}$ is a matrix valued function such that $\mathcal{R}(x) > 0$ for all x in $\mathbb{R}^n$. Then a control law $u = \alpha(x)$ minimizing the cost function $\mathcal{J}$ is a stabilizing feedback which is of infinite gain margin.

In this section our aim is to extend these notions to the contraction framework in the case where φ is a monotonic function.

Theorem 2. *Consider system* (6) *with φ with a symmetric Jacobian satisfying the following monotonic condition*

$$0 \preceq \frac{\partial \varphi}{\partial y}(y) \qquad \forall y \in \mathbb{R}^p. \tag{24}$$

Assume that there exist two positive definite matrices $P \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$, two matrices $M \in \mathbb{R}^{n\times q}$, $N \in \mathbb{R}^{m\times q}$, and real numbers (μ_1, μ_2) and $\nu > 0$ such that

$$\begin{aligned} \mathtt{He}\left\{P\left(A + M\frac{\partial\varphi}{\partial\zeta}(\zeta)H\right)\right\} - PBRB^\top P \leq -\nu I\,, \\ P(G-M) = \mu_1 H^\top, \quad PBN = \mu_2 H^\top. \end{aligned} \tag{25}$$

Then there exists a positive real number γ such that the control law

$$\alpha(x) = -\gamma\left(RB^\top Px + \mu_2 N\varphi(Hx)\right), \tag{26}$$

is a solution to Problem 1, namely makes the closed-loop system (6) *a contraction with infinite gain margin. Moreover if $\varphi(0) = 0$ then for all $\kappa \geq 2$ there exist $\mathcal{Q} : \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$ and $\mathcal{R} : \mathbb{R}^n \mapsto \mathbb{R}^{m\times m}$ both depending on κ such that $u = \kappa\alpha(x)$ minimizes the cost function* (23)*.*

Proof. Noting that the control law $\alpha(x)$ defined in (26) depends on the parameter γ, we define

$$\begin{aligned} &f(x) := Ax + G\varphi(Hx), \;\; \mathfrak{L}(\gamma, x) := \mathtt{He}\left\{P\frac{\partial f_\gamma}{\partial x}\right\}, \\ &f_\gamma(x) := Ax + G\varphi(Hx) + B\alpha(x). \end{aligned} \tag{27}$$

About infinite gain margin. Let γ be defined as

$$\gamma := \max\left\{\frac{\mu_1}{\mu_2^2},\ \frac{1}{2}\right\}. \tag{28}$$

Using the definition of $\mathfrak{L}$ in (27), we obtain

$$\begin{aligned} \mathfrak{L}(\gamma, x) := \mathtt{He}\left\{P\left[A + G\frac{\partial\varphi}{\partial y}(y)H\right]\right\} \\ - 2\gamma PBRB^\top P - \gamma\mu_2 \mathtt{He}\left\{PBN\frac{\partial\varphi}{\partial y}(y)H\right\}\,. \end{aligned}$$

Adding and subtracting the term $M\frac{\partial\varphi}{\partial y}(y)H$ to the right-hand side of the above definition of $\mathfrak{L}$ yields

$$\begin{aligned} \mathfrak{L}(\gamma, x) = \mathtt{He}\left\{P\left[A + M\frac{\partial\varphi}{\partial y}(y)H\right]\right\} \\ + \mathtt{He}\left\{P(G-M)\frac{\partial\varphi}{\partial y}(y)H\right\} - 2\gamma PBRB^\top P \\ - \gamma\mu_2 \mathtt{He}\left\{PBN\frac{\partial\varphi}{\partial y}(y)H\right\}\,. \end{aligned}$$

Furthermore, using the second line in (25), the latter gives

$$\begin{aligned} \mathfrak{L}(\gamma, x) = \mathtt{He}\left\{P\left(A + M\frac{\partial\varphi}{\partial y}(y)H\right)\right\} \\ - 2\gamma PBRB^\top P + (\mu_1 - \gamma\mu_2^2)H^\top \mathtt{He}\left\{\frac{\partial\varphi}{\partial y}(y)\right\}H\,. \end{aligned} \tag{29}$$

Using the first line of (25), it implies for all (γ, x)

$$\begin{aligned} \mathfrak{L}(\gamma, x) \preceq -\nu I - (2\gamma - 1)PBRB^\top P \\ + (\mu_1 - \gamma\mu_2^2)H^\top \mathtt{He}\left\{\frac{\partial\varphi}{\partial y}(y)\right\}H\,. \end{aligned} \tag{30}$$

Hence, with γ defined in in (28), and using inequality (24), it follows that the right-hand term of (30) is negative definite for all x. Consequently, α makes the flow of the system a contraction. Note moreover that for all $\kappa \geq 1$, $u = \kappa\alpha(x)$ makes also the flow of the system a contraction. Hence, α is with infinite gain margin.

About optimality. The idea of the proof is first to show that the Lyapunov function $V(x) = x^\top Px$ is the value function associated to an optimal control problem and satisfies a Hamilton-Jacobi-Bellman (HJB) equation. Second, one proves that the control law $u = \kappa\alpha(x)$ is the optimal one. To this end, note that since $\varphi(0) = 0$, we have

$$\varphi(Hx) = \int_0^1 \frac{\partial\varphi}{\partial y}(sy)ds Hx\ ,\forall\ x\ . \tag{31}$$

Hence, the function α given in (26) reads

$$\alpha(x) = -\gamma\left(RB^\top Px + \mu_2 N\int_0^1 \frac{\partial\varphi}{\partial y}(sy)ds Hx\right)\ . \tag{32}$$

Consequently, employing $PBN = \mu_2 H^\top$ from (25), the following equation may be obtained.

$$\begin{aligned} \kappa\alpha(x) &= -\kappa\gamma RB^\top Px - \kappa\gamma\mu_2 N\int_0^1 \frac{\partial\varphi}{\partial y}(sy)ds Hx \\ &= -\kappa\gamma\left[R + \mu_2 N\int_0^1 \frac{\partial\varphi}{\partial y}(sy)ds N^\top\right]B^\top Px \\ &= -\frac{1}{2}\mathcal{R}(x)(2B^\top Px)\ , \end{aligned}$$

where $\mathcal{R} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ depends on κ and is defined as

$$\mathcal{R}(x) := \kappa\gamma\left(R + \mu_2 N\int_0^1 \frac{\partial\varphi}{\partial y}(sy)ds N^\top\right)\ .$$

Note that φ being monotonic and with a symmetric Jacobian (24), we have that $\mathcal{R}$ takes symmetric positive definite values. Then, recalling the definition of f given in (27), let the function $\mathcal{Q} : \mathbb{R}^n \mapsto \mathbb{R}$ be defined as

$$\mathcal{Q}(x) := -L_f V(x) + \frac{1}{4}(2x^\top PB)\mathcal{R}(x)(2B^\top Px)\ , \tag{33}$$

The former equation (33) is a HJB equation with respect to the cost function (23) if $\mathcal{Q}$ takes positive semi-definite values. Recalling the definitions of V and f, the term $L_f V$ is computed as

$$L_f V(x) = x^\top(PA + A^\top P)x + \mathtt{He}\Big\{x^\top PG\varphi(Hx)\Big\},$$

which gives by using (31)

$$L_f V(x) = x^\top\left[\int_0^1 \mathtt{He}\left\{P\left[A + G\frac{\partial\varphi}{\partial y}(sy)H\right]\right\}ds\right]x.$$

Moreover,

$$\begin{aligned} &x^\top PB\mathcal{R}(x)B^\top Px = \\ &\kappa\gamma x^\top\left[\int_0^1 \left(PBRB^\top P + \frac{\mu_2}{2}\mathtt{He}\left\{PBN\frac{\partial\varphi}{\partial y}(sy)H\right\}\right)ds\right]x. \end{aligned}$$

Hence, using the definition of $\mathfrak{L}$ given in (27) and equation (29), we obtain

$$\mathcal{Q}(x) = -x^\top \left(\int_0^1 \mathfrak{L}(\kappa\gamma, sx) ds \right) x - x^\top PB\mathcal{R}(x)B^\top Px \ .$$

With (30), it implies

$$\begin{aligned} -\mathfrak{L}(\kappa\gamma, x) \succeq & \nu I + (2\kappa\gamma - 1)PBRB^\top P \\ & - (\mu_1 - \kappa\gamma\mu_2^2) H^\top \mathtt{He}\left\{ \frac{\partial \varphi}{\partial y}(y) \right\} H \ . \end{aligned} \quad (34)$$

Hence,

$$\begin{aligned} \mathcal{Q}(x) \succeq & \nu |x|^2 + (\kappa\gamma - 1) x^\top PBRB^\top Px \\ & - (\mu_1 - \kappa\gamma \frac{\mu_2^2}{2}) H^\top \mathtt{He}\left\{ \frac{\partial \varphi}{\partial y}(y) \right\} H \ . \end{aligned} \quad (35)$$

With the definition of γ in (28), previous inequality implies

$$\begin{aligned} \mathcal{Q}(x) \succeq & \nu |x|^2 + \left(\frac{\kappa}{2} - 1 \right) x^\top PBRB^\top Px \\ & + \mu_1 \left(\frac{\kappa}{2} - 1 \right) H^\top \mathtt{He}\left\{ \frac{\partial \varphi}{\partial y}(y) \right\} H \ . \end{aligned} \quad (36)$$

Consequently $\mathcal{Q}$ is a positive definite function if $\kappa \geq 2$. Hence, $u = \kappa\alpha(x)$ minimizes the cost function (23). □

Remark 1. *Using the monotonic condition* (24)*, the conditions in* (25) *can be also rewritten as*

$$\begin{gathered} WA^\top + AW - BRB^\top \prec 0 \, , \\ G = WH^\top \, , \qquad BN = \mu_2 WH^\top \, , \end{gathered}$$

with unknown $W = W^\top \succ 0$, μ_2 and N so that inequality (25) *is satisfied with $P = W^{-1}$, $M = G$, $\mu_1 = 0$. Furthermore, if one employs the incremental monotonic condition* (7)*, the condition* (25) *can be written as*

$$\begin{gathered} \begin{bmatrix} A^\top P + PA - PBRB^\top P + \nu I & H^\top + PM \\ H + M^\top P & -4\Gamma^{-1} \end{bmatrix} \preceq 0 \, , \\ P(G - M) = \mu_1 H^\top , \qquad PBN = \mu_2 H^\top . \end{gathered}$$

Again, with the choice $P = W^{-1}$, $M = G$, $\mu_1 = 0$, by matrix manipulation one can recover (12), (13)*. Finally, if φ is a monotonic function satisfying* (24) *and the sector bound condition* (8) *instead of* (7)*, similar LMIs conditions can be derived.*

V. Example

To show that the proposed design can possibly be applied to practical case-studies, in this section we consider a single-link robot arm system (see [26, Section 4.10]). Its model is given in the form (1) with

$$\begin{gathered} A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k+mgd\ell}{J_2} & -\frac{F_2}{J_2} & \frac{k}{J_2 b} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J_1 b} & 0 & -\frac{k}{J_2 b} & -\frac{F_1}{J_1} \end{bmatrix} , \ B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} , \\ G^\top = \begin{bmatrix} 0 & -\frac{mgd}{J_2} & 0 & 0 \end{bmatrix} , \ H = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \\ \varphi(s) = \cos(s) + \ell s \end{gathered}$$

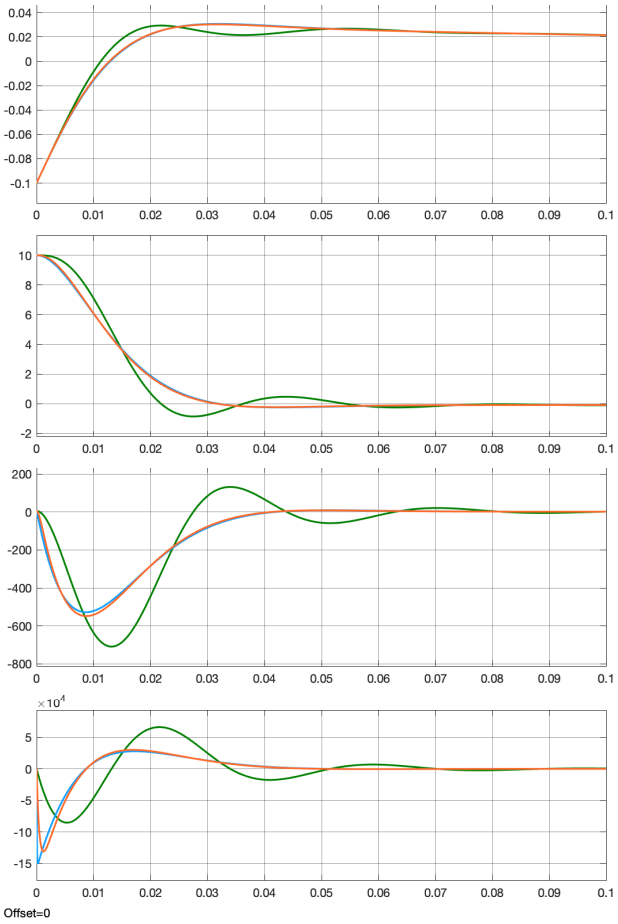

Fig. 3: x-dynamics for $\kappa = 1$ (green), $\kappa = 10$ (red) and $\kappa = 100$ (blue)

where $k = 0.4, m = 0.8, g = 9.81, d = 0.6, J_2 = 0.2, F_2 = 0.15, b = 2, J_1 = 0.15, F_1 = 0.1$ are plant parameters and ℓ is a degree of freedom. Note that if $\ell > 1$, then the monotonic condition (7) holds with $\Gamma = \frac{\ell+1}{2}$. With $\ell = 1.01$, it's possible to check that the LMIs (12), (13) are satisfied with $N = 6.15 \cdot 10^3$ and

$$K = \begin{bmatrix} -1.03 \cdot 10^6 & -0.35 \cdot 10^6 & -0.5 \cdot 10^8 & -2.22 \cdot 10^3 \end{bmatrix} .$$

We obtain that the closed-loop system with the control law $u = \kappa(Kx + \varphi(Hx))$ for any $\kappa \geq 1$ is a contraction. Figure 3 shows the dynamics of the closed-loop system for different values of κ. As the closed-loop system is a contraction, the trajectories converge to a globally exponentially stable equilibrium point, that is the origin (see [9]).

VI. Conclusions

In this work we considered the classes of systems defined by a linear term and a nonlinearity satisfying some (incremental) monotonic or sector bound condition. We provided a set of sufficient conditions based on LMIs for the design of a state-feedback control law possessing the infinite gain margin property for the closed-loop to define a contraction. Then we showed how such control design can be associated to the solution of an optimization problem with respect to

a cost function. To conclude, we applied our results with a practical case of study, namely a single-link robot arm system.

Future works include the design of output feedback controllers which may be obtained following an observer-based design, e.g., [25], or extending LMI optimization methods, e.g., [24]. Furthermore, we aim at generalizing the proposed result to more general classes of nonlinear systems, investigate the deep connection between infinite gain margin feedbacks, optimality, and dissipativity [27] and study the connections with respect to positive real systems [28, Section 7].

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