

An attack of the Conjecture of Lehmer by the dynamical zeta function of the β -shift, and the modulo p problem

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Definition : Weil height : let $\alpha \in \overline{\mathbb{Q}}^*$, $P_\alpha(X) = a_0(X - \alpha_1)(X - \alpha_2)\dots(X - \alpha_n) = a_0X^n + a_1X^{n-1} + \dots + a_{n-1}X + a_n \in \mathbb{Z}[X]$, $a_0a_n \neq 0$, its minimal polynomial. The (abs. log.) Weil height of α is

$$h(\alpha) = \frac{1}{n} \text{Log} \left(|a_0| \prod_{i=1}^n \max\{1, |\alpha_i|\} \right)$$

Prop : $h(p/q) = \text{Log} \max(|p|, |q|)$, $(p, q) = 1$, $h(1) = 0$,
 $h(\alpha) \geq 0$ for all $\alpha \in \overline{\mathbb{Q}}^*$,
 $h(\alpha^r) = |r|h(\alpha)$, for $r \in \mathbb{Z}$, $\alpha \in \overline{\mathbb{Q}}^*$, $h(1/\alpha) = h(\alpha)$,
 $h(\sigma(\alpha)) = h(\alpha)$, for all $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Definition : Mahler measure : for

$$P(X) = a_0(X - \alpha_1)(X - \alpha_2)\dots(X - \alpha_n) = a_0X^n + a_1X^{n-1} + \dots + a_{n-1}X + a_n \in \mathbb{Z}[X], \quad a_0a_n \neq 0$$

then

$$M(P) := |a_0| \prod_{i, |\alpha_i| \geq 1} |\alpha_i|.$$

multiplicativity : $P = P_1 \times P_2 \times \dots \times P_m, \Rightarrow M(P) = M(P_1) \dots M(P_m).$

ex. : $P = \Phi_1 \times \dots \times \Phi_r \times R$ with R irr. pol., Φ_j cyclot. $\implies M(P) = M(R).$

α alg. number, $\deg \alpha = n$, P_α his minimal polynomial, $M(\alpha) := M(P_\alpha).$
Absolute logarithmic, Weil height of α :

$$h(\alpha) := \frac{\text{Log} M(\alpha)}{d}$$

facts : $M(\alpha) = M(\alpha^{-1})$,

$M(\alpha) = \alpha$ if $\alpha \in S$ (= set of Pisot numbers ; $|\alpha_j| < 1$),

$M(\alpha) = \alpha$ if $\alpha \in T$ (= set of Salem numbers ; $|\alpha_j| < 1$ with at least one $|\alpha_j| = 1$),

$M(\alpha) = 1$ if α is a root of unity.

Kronecker's Theorem(1857) : Let α be a nonzero algebraic integer. Then $M(\alpha) = 1$ iff α is a root of unity.

practice in Arithm. Geo. :

$M(\alpha)$ calculated \rightarrow useful to calculate $h(\alpha)$,

height h = sum of local contributions \rightarrow useful to prove Theorems.

Adler Marcus (1979) (topological entropy and equivalence of dynamical systems), Perron-Frobenius theory) :

$$\{M(\alpha) \mid \alpha \text{ alg. number}\} \subset \mathbb{P}_{Perron},$$

$$\{M(P) \mid P \in \mathbb{Z}[X]\} \subset \mathbb{P}_{Perron}.$$

Two strict inclusions (Dubickas 2004, Boyd 1981).

Definition : $\alpha \in \mathbb{P}_{Perron}$ if $\alpha = 1$ or if $\alpha > 1$ is a real algebraic integer, for which the conjugates $\alpha^{(i)}$ satisfy $|\alpha^{(i)}| < \alpha$ (i.e. dominant root > 1).

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Northcott's Theorem : for all $B \geq 0, d \geq 1,$

$$\#\{\alpha \in \overline{\mathbb{Q}} \mid h(\alpha) \leq B, [\mathbb{Q}(\alpha) : \mathbb{Q}] \leq d\} < +\infty.$$

in Dio. Geom. : bound on “degree” + bound on “ h ” gives finiteness property (Mordell eff., etc).

Conjecture of Lehmer : there exists $c > 0$ such that

$$M(\alpha) \geq 1 + c$$

for any algebraic number $\alpha \neq 0$ which is not a root of unity,

i.e. the interval $(1, 1 + c) \cap \mathbb{P}_{Perron}$ is deprived of any value of Mahler measure of any algebraic number.

– $>$ values : discontinuity at 1 (meaning, sense, of c ?).

Lehmer's problem (1933)

in the exhaustive search for large prime numbers : *if ε is a positive quantity, to find a polynomial of the form*

$$f(x) = x^r + a_1 x^{r-1} + \dots + a_r$$

where the a_i s are integers, such that the absolute value of the product of those roots of f which lie outside the unit circle, lies between 1 and $1 + \varepsilon$... Whether or not the problem has a solution for $\varepsilon < 0.176$ we do not know.

Lehmer's strategy : P_α with small M : useful to obtain large prime numbers p , in the Pierce numbers of α . Iwasawa theory : large powers of primes. Einsiedler, Everest and Ward : study of the density of such p s.

Lehmer's Problem is a limit problem + restrictions :

$$M(P) := |a_0| \prod_{i, |\alpha_i| \geq 1} |\alpha_i| \implies M(P) := |a_0| \geq |a_0|.$$

Let $\alpha \in \overline{\mathbb{Q}}$, $P = P_\alpha$:

- * if $\alpha \in \overline{\mathbb{Q}} \setminus \mathcal{O}_{\overline{\mathbb{Q}}}$, then $|a_0| \geq 2 \implies M(P) \geq 2$,
- * if α is an algebraic integer which is not reciprocal ($P_\alpha \neq P_\alpha^*$ with

$$P_\alpha^*(X) = X^{\deg P_\alpha} P_\alpha(1/X) \quad),$$

Smyth's Theorem '71 $\implies M(P_\alpha) \geq \Theta = 1.32\dots$ (= smallest Pisot number, $X^3 - X - 1$ mini. pol.).

- * restriction to real reciprocal algebraic integers is sufficient : if $\alpha \in \mathcal{O}_{\overline{\mathbb{Q}}}$, which is reciprocal ($P_\alpha = P_\alpha^*$), consider its house

$$\max\{|\alpha_i|\} =: |\overline{\alpha}| \in \mathcal{O}_{\overline{\mathbb{Q}}}$$

which is real, ≥ 1 .

Attack (context) : $\beta > 1$ real reciprocal algebraic integer,

$$1 < \beta \leq |\beta| \leq M(\beta).$$

β tends to 1^+ ,

$|\beta|$ tends to 1^+ .

? minimum of $\beta \rightarrow M(\beta)$, of $|\beta| \rightarrow M(|\beta|)$.

assumption : existence of a real reciprocal algebraic integer $\beta > 1$ having :
 $M(\beta) < 1.176280\dots$ Lehmer's number.

Previously : Dobrowolski's inequality ('79) : for any reciprocal algebraic integer α of degree d ,

$$M(\alpha) > 1 + (1 - \varepsilon) \left(\frac{\text{LogLog } d}{\text{Log } d} \right)^3, \quad d > d_1(\varepsilon).$$

(Dobrowolski, 1/1200, Schinzel, $1 - \varepsilon$ for $d > d_1$)

here,
 the lower bound in the rhs tends to 1 when d tends to infinity. Remarkable inequality. Not satisfying. Need to improve Dobrowolski's inequality by another method.

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Why use dynamical zeta functions $\zeta_\beta(z)$ of the β -shift ?

Because the Conjecture of Lehmer can be proved by this means for β running over

$$\{\theta_n^{-1}; n \geq 3\}, \quad (\text{seq. of Perron numbers } > 1 \text{ tending to } 1)$$

where

θ_n is the unique root of $-1 + x + x^n$ in $(0, 1)$.

- * the information lies in the poles inside $D(0, 1)$ the open unit disk,
- * the poles of modulus < 1 form a **lenticulus**.

goal : extend the method to all β s > 1 tending to 1.

Let $1 < \beta < (1 + \sqrt{5})/2$ be a **real number**.

- Consider the dynamical system :

$$(X = [0, 1], B, T_\beta, \mu)$$

where

$$T_\beta : x \mapsto \beta x \pmod{1} = \{\beta x\}.$$

is the β -transformation.

Invariant measure (Parry, '60) abs cont Lebesgue, pure jump, density :

$$h_\beta(x) = C \sum_{n, x < T_\beta^n(1)} \frac{1}{\beta^{n+1}}$$

unique (Rényi, '57), ergodic (Parry, '60), maximal (Hofbauer, '78)

- Later on, we will specialize $\beta > 1$ to run over reciprocal algebraic integers.

The “second” analytic function uniquely associated with β :

$$(X, B, T, \mu) = ([0, 1], T_\beta, h_\beta(x)dx)$$

Theorem

Let $\beta \in (1, \theta_2^{-1})$. Then, the Artin-Mazur dynamical zeta function

$$\zeta_\beta(z) := \exp\left(\sum_{n=1}^{\infty} \frac{\#\{x \in [0, 1] \mid T_\beta^n(x) = x\}}{n} z^n\right), \quad (1)$$

counting the number of periodic points of period dividing n , is nonzero and meromorphic in $\{z \in \mathbb{C} : |z| < 1\}$, and such that $1/\zeta_\beta(z)$ is holomorphic in $\{z \in \mathbb{C} : |z| < 1\}$,

Theorem (Takahashi, Ito-Takahashi)

Let $\beta \in (1, \theta_2^{-1})$ be a real number. Then

$$\zeta_\beta(z) = \frac{1 - z^N}{(1 - \beta z) \left(\sum_{n=0}^{\infty} T_\beta^n(1) z^n \right)} \quad (2)$$

where N is the minimal positive integer such that $T_\beta^N(1) = 0$; in the case where $T_\beta^j(1) \neq 0$ for all $j \geq 1$, “ z^N ” has to be replaced by “0”.

Up to the sign, the denominator of (2) is the Parry Upper function $f_\beta(z)$ at β . It satisfies

$$(i) \quad f_\beta(z) = -\frac{1 - z^N}{\zeta_\beta(z)} \quad \text{in the first case,} \quad (3)$$

$$(ii) \quad f_\beta(z) = -\frac{1}{\zeta_\beta(z)} \quad \text{in the second case,} \quad (4)$$

and, denoting by $t_1, t_2, \dots \in \{0, 1\}$ the coefficients in

$$-1 + t_1 z + t_2 z^2 + t_3 z^3 + \dots = f_\beta(z) = -(1 - \beta z) \left(\sum_{n=0}^{\infty} T_\beta^n(1) z^n \right), \quad (5)$$

$f_\beta(z)$ is such that $0.t_1 t_2 t_3 \dots$ is the Rényi β -expansion of unity $d_\beta(1)$.

The Parry Upper function $f_\beta(z)$ has no zero in $\{z \in \mathbb{C} : |z| \leq 1/\beta\}$ except $z = 1/\beta$ which is a simple zero.

The total ordering $<$ on $(1, +\infty)$ is uniquely in correspondence with the lexicographical ordering $<_{lex}$ on the vector coefficients, i.e. on the Rényi expansions of 1 by (Parry, '60) :

In Rényi β -expansions, **the correspondence $\beta \leftrightarrow (t_i = t_i(\beta))$ is a bijection.**

Proposition[Parry] : Let $\alpha > 1$ and $\beta > 1$. If the Rényi α -expansion of 1 is

$$d_\alpha(1) = 0.t'_1 t'_2 t'_3 \dots, \quad \text{i.e.} \quad 1 = \frac{t'_1}{\alpha} + \frac{t'_2}{\alpha^2} + \frac{t'_3}{\alpha^3} + \dots$$

and the Rényi β -expansion of 1 is

$$d_\beta(1) = 0.t_1 t_2 t_3 \dots, \quad \text{i.e.} \quad 1 = \frac{t_1}{\beta} + \frac{t_2}{\beta^2} + \frac{t_3}{\beta^3} + \dots,$$

then $\alpha < \beta$ if and only if $(t'_1, t'_2, t'_3, \dots) <_{lex} (t_1, t_2, t_3, \dots)$.

Theorem

Let $n \geq 2$. A real number $\beta \in (1, \frac{1+\sqrt{5}}{2}]$ belongs to $[\theta_{n+1}^{-1}, \theta_n^{-1})$ if and only if the Rényi β -expansion of unity $d_\beta(1)$ is of the form

$$d_\beta(1) = 0.10^{n-1} 10^{n_1} 10^{n_2} 10^{n_3} \dots, \quad (6)$$

with $n_k \geq n - 1$ for all $k \geq 1$.

Pf. : Since $d_{\theta_{n+1}^{-1}}(1) = 0.10^{n-1}1$ and $d_{\theta_n^{-1}}(1) = 0.10^{n-2}1$, the condition $\beta \in [\theta_{n+1}^{-1}, \theta_n^{-1})$ implies that the condition is sufficient.

It is also necessary : $d_\beta(1)$ begins as $0.10^{n-1}1$ for all β such that $\theta_{n+1}^{-1} \leq \beta < \theta_n^{-1}$. For such β s we write $d_\beta(1) = 0.10^{n-1}1u$ with digits in the alphabet $\mathcal{A}_\beta = \{0, 1\}$ common to all β s, that is

$$u = 1^{h_0}0^{n_1}1^{h_1}0^{n_2}1^{h_2} \dots$$

and $h_0, n_1, h_1, n_2, h_2, \dots$ integers ≥ 0 . The Conditions of Parry (Parry '60, ref : Frougny, Lothaire) applied to the sequence $(1, 0^{n-1}, 1^{1+h_0}, 0^{n_1}, 1^{h_1}, 0^{n_2}, 1^{h_2}, \dots)$, which characterizes uniquely the base of numeration β , readily implies $h_0 = 0$ and $h_k = 1$ and $n_k \geq n-1$ for all $k \geq 1$.

Def. : The polynomials of the class \mathcal{B}

are all the polynomial sections of the power series $f_\beta(z)$ for β in the interval $(1, (1 + \sqrt{5})/2)$. Indeed, from the above, for

$$\beta \in [\theta_{n+1}^{-1}, \theta_n^{-1}),$$

the power series $f_\beta(x)$ takes the form :

$$-1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s} + \dots$$

with the **distanciation conditions** :

$$m_1 - n \geq n - 1, m_{q+1} - m_q \geq n - 1 \quad \text{for } 1 \leq q$$

let θ_n be the unique root of the trinomial $G_n(z) := -1 + z + z^n$ in $(0, 1)$.

 α $f_\alpha(z)$ θ_n^{-1} $-1 + z + z^n$ $\theta_n^{-1} < \beta < \theta_{n-1}^{-1}$ $(-1 + x + x^n) + (x^{m_1} + x^{m_2} + \dots + x^{m_s} + \dots)$ where $m_1 - n \geq n-1$, $m_{q+1} - m_q \geq n-1$ θ_{n-1}^{-1} $-1 + z + z^{n-1}$

Def. : n is called the **dynamical degree of β** , denoted by $\text{dyg}(\beta)$.

The **Parry Upper function** at β

$$\theta_n^{-1} < \beta < \theta_{n-1}^{-1} \quad -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s} + \dots$$

satisfies (Conditions of admissibility of Parry) :

$$m_1 - n \geq n - 1, m_{q+1} - m_q \geq n - 1 \text{ for } 1 \leq q$$

[lexicographical ordering implies moderate gappiness]

its zeroes in $D(0, 1) = \text{eigenvalues}^{-1}$ of the transfer operators (generalized Fredholm determinants - BaladiKeller)
= poles of $\zeta_\beta(z)$.

$$\beta \rightarrow 1^+ \quad \iff \quad n = \text{dyg}(\beta) \rightarrow \infty.$$

Poles of $\zeta_\beta(z)$ in $D(0, 1)$: are limits of zeroes of polynomials of the class \mathcal{B} (Hurwitz). In $D(0, 1)$, the poles of $\zeta_\beta(z)$ are separated into two subcollections :

- the **lenticular poles**,
- the non-lenticular poles.

Description of the lenticular zeroes of $f \in \mathcal{B}$:

- leave the comfort of Taylor series (including the formulation as hypergeometric functions as functions of the coefficients, as in Mellin '15),
- enter “Poincaré asymptotic expansions as a function of the dynamical degree $n = \text{dyg}(\beta)$, i.e. the integer controlling the lacunarity a minima.
- replace the usual degree $\text{deg}(\beta)$ by the dynamical degree $\text{dyg}(\beta)$.

Classification not canonical : due to the method of Rouché.

From the structure of the asymptotic expansions of the roots of G_n it is natural to restrict the angular sector to

$$-\pi/18 < \arg \omega < +\pi/18.$$

More precisely,

Theorem (VG, '17)

Let $n \geq 260$. There exists two positive constants c_n and $c_{A,n}$, $c_{A,n} < c_n$, such that the roots of $f \in \mathcal{B}_n$,

$$f(x) - 1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s},$$

where $s \geq 1$, $m_1 - n \geq n - 1$, $m_{j+1} - m_j \geq n - 1$ for $1 \leq j < s$, lying in $-\pi/18 < \arg z < +\pi/18$ either belong to

$$\left\{ z \mid \left| |z| - 1 \right| < \frac{c_{A,n}}{n} \right\}, \quad \text{or to} \quad \left\{ z \mid \left| |z| - 1 \right| \geq \frac{c_n}{n} \right\}.$$

true even though n is smaller than 260 on many examples.

Definition

Let $n \geq 260$. Let $\beta > 1$ be a real number such that $\text{dyg}(\beta) = n$. The poles of $\zeta_\beta(z)$ which belong to the angular sector

$$\left\{ z \in \mathbb{C} : |z| < 1 - \frac{C_{\text{lent}}}{n}, |\arg z| \leq +\frac{\pi}{18} \right\} \quad (7)$$

are called the *lenticular poles* of $\zeta_\beta(z)$.

The lenticuli of lenticular poles, relative to θ_n^{-1} and β respectively, are :

$$\mathcal{L}_{\theta_n^{-1}} := \{ \overline{z_{\lfloor n/6 \rfloor, n}}, \dots, \overline{z_{2, n}}, \overline{z_{1, n}}, \theta_n, z_{1, n}, z_{2, n}, \dots, z_{\lfloor n/6 \rfloor, n} \},$$

$$\mathcal{L}_\beta = \{ \overline{\omega_{J_n, n}}, \dots, \overline{\omega_{1, n}}, \beta^{-1}, \omega_{1, n}, \dots, \omega_{J_n, n} \}.$$

Solomyaks's fractal

The lenticular roots lie on (universal) continuous curves stemming from $z = 1$, including the boundary of Solomyak's fractal.

Let

$$\mathscr{W} := \left\{ h(z) = 1 + \sum_{j=1}^{\infty} a_j z^j \mid a_j \in [0, 1] \right\}$$

be the class of power series defined on $|z| < 1$ equipped with the topology of uniform convergence on compact sets of $|z| < 1$. Let $\mathscr{W}_{0,1} \subset \mathscr{W}$ denote functions whose coefficients are in $\{0, 1\}$. The space \mathscr{W} is compact and convex. Let

$$\mathscr{G} := \{ \lambda \mid |\lambda| < 1, \exists h(z) \in \mathscr{W} \text{ such that } h(\lambda) = 0 \} \subset \{ z \mid |z| < 1 \}$$

The domain $D(0, 1) \setminus \mathscr{G}$ is star-convex due to the fact that :

$$h(z) \in \mathscr{W} \implies h(z/r) \in \mathscr{W}, \quad \text{for any } r > 1.$$

The zeroes of any $f \in \mathcal{B}$ lie in \mathcal{G} since :

$t_i := \lfloor \beta T_\beta^{i-1}(1) \rfloor = \beta T_\beta^{i-1}(1) - T_\beta^i(1)$, for $i \geq 1$, implies the factorization

$$-1 + t_1x + t_2x^2 + t_3x^3 + \dots = -(1 - \beta x)(1 + \sum_{j \geq 1} T_\beta^j(1)x^j)$$

For every $\phi \in (0, 2\pi)$, there exists $\lambda = re^{i\phi} \in \mathcal{G}$; the point of minimal modulus with argument ϕ is denoted $\lambda_\phi = \rho_\phi e^{i\phi} \in \mathcal{G}$, $\rho_\phi < 1$.

A function $h \in \mathcal{W}$ is called ϕ -optimal if $h(\lambda_\phi) = 0$.

Denote by \mathcal{K} the subset of $(0, \pi)$ for which there exists a ϕ -optimal function belonging to $\mathcal{W}_{0,1}$.

Denote by $\partial\mathcal{G}_S$ the “spike” : $[-1, \frac{1}{2}(1 - \sqrt{5})]$ on the negative real axis.

Theorem (Solomyak, '94)

- (i) *The union $\mathcal{G} \cup \mathbb{T} \cup \partial\mathcal{G}_S$ is closed, symmetrical with respect to the real axis, has a cusp at $z = 1$ with logarithmic tangency.*
- (ii) *the boundary $\partial\mathcal{G}$ is a **continuous curve**, given by $\phi \rightarrow |\lambda_\phi|$ on $[0, \pi)$, taking its values in $[\frac{\sqrt{5}-1}{2}, 1)$, with $|\lambda_\phi| = 1$ if and only if $\phi = 0$. It admits a left-limit at π^- , $1 > \lim_{\phi \rightarrow \pi^-} |\lambda_\phi| > |\lambda_\pi| = \frac{1}{2}(-1 + \sqrt{5})$, the left-discontinuity at π corresponding to the extremity of $\partial\mathcal{G}_S$.*
- (iii) *at all points $\rho_\phi e^{i\phi} \in \mathcal{G}$ such that ϕ/π is rational in an **open dense subset** of $(0, 2)$, $\partial\mathcal{G}$ is non-smooth,*
- (iv) *there exists a nonempty subset of transcendental numbers L_{tr} , of Hausdorff dimension zero, such that $\phi \in (0, \pi)$ and $\phi \notin \mathcal{X} \cup \pi\mathbb{Q} \cup \pi L_{tr}$ implies that the boundary curve $\partial\mathcal{G}$ has a tangent at $\rho_\phi e^{i\phi}$ (smooth point).*

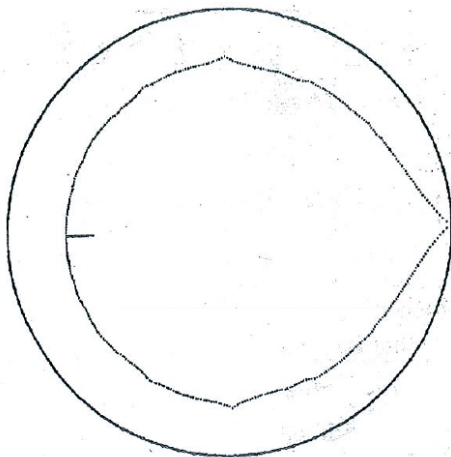


FIGURE: Solomyak's fractal.

Importance of the angular sector $|\arg(z)| < \pi/18$ containing the point 1 was already guessed by :

M. LANGEVIN, *Méthode de Fekete-Szegő et Problème de Lehmer*, C.R. Acad. Sci. Paris Série I Math. **301** (1) (1985), 463–466.

M. LANGEVIN, *Minorations de la Maison et de la Mesure de Mahler de Certains Entiers Algébriques*, C.R. Acad. Sci. Paris Série I Math. **303** (12) (1986), 523–526.

M. LANGEVIN, *Calculs Explicites de Constantes de Lehmer*, in *Groupe de travail en Théorie Analytique et Élémentaire des nombres*, 1986–1987, Publ. Math. Orsay, Univ. Paris XI, Orsay **88** (1988), 52–68.

A. DUBICKAS and C. SMYTH, *The Lehmer Constant of an Annulus*, J. Théorie Nombres Bordeaux **13** (2001), 413–420.

G. RHIN and C.J. SMYTH, *On the Absolute Mahler Measure of Polynomials Having all Zeros in a Sector*, Math. Comp. **64** (1995), 295–304.

G. RHIN and Q. WU, *On the Absolute Mahler Measure of Polynomials Having all Zeros in a Sector II*, Math. Comp. **74** (2005), 383–388.

lenticulus $\mathcal{L}_{\theta_n^{-1}}$ of simple zeroes in $\arg(z) \in (-\pi/3, +\pi/3)$, $n = 71$ and $= 12$.

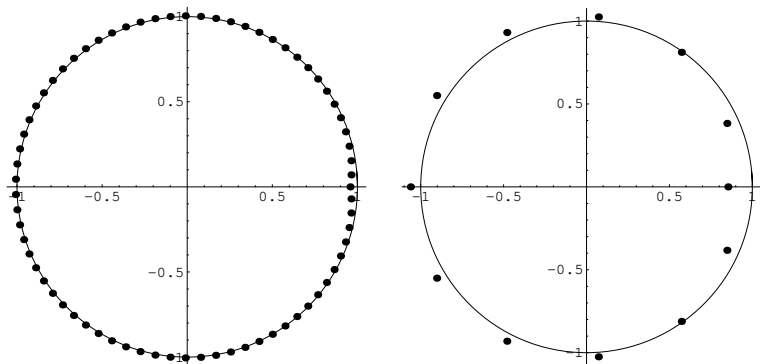


FIGURE: Roots of $G_{71}(z)$, $G_{12}(z)$.

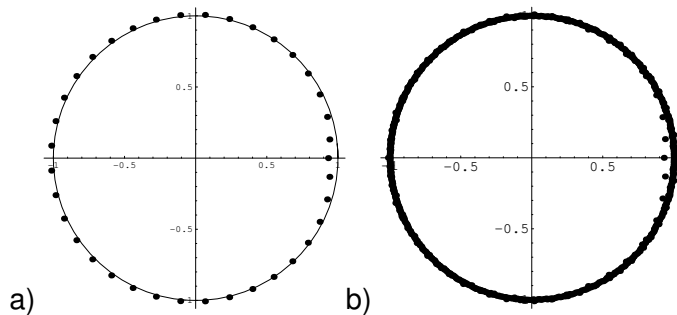


FIGURE: a) The 37 zeroes of $G_{37}(x) = -1 + x + x^{37}$, b) The 649 zeroes of $f(x) = G_{37}(x) + x^{81} + x^{140} + x^{184} + x^{232} + x^{285} + x^{350} + x^{389} + x^{450} + x^{514} + x^{550} + x^{590} + x^{649} = G_{37}(x) + x^{81} + \dots + x^{649}$. The lenticulus of roots of f (having 3 simple zeroes) is obtained by a very slight deformation of the restriction of the lenticulus of roots of G_{37} to the angular sector $|\arg z| < \pi/18$, off the unit circle. The other roots (nonlenticular) of f can be found in a narrow annular neighbourhood of $|z| = 1$.

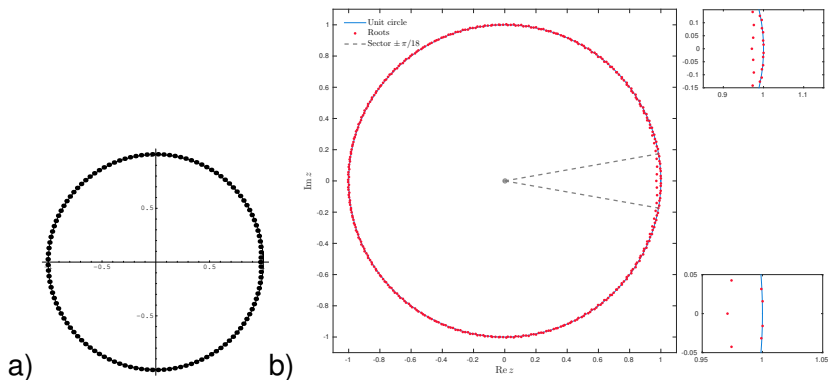


FIGURE: a) Zeroes of G_{121} , b) Zeroes of $f(x) = -1 + x + x^{121} + x^{250} + x^{385}$. On the right the distribution of the roots of f is zoomed twice in the angular sector $-\pi/18 < \arg(z) < \pi/18$. The lenticulus of roots of f has 7 zeroes.

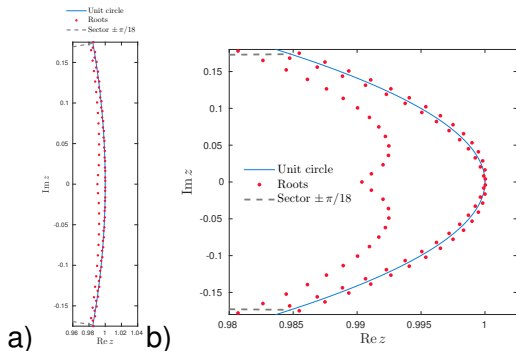


FIGURE: The representation of the 27 zeroes of the lenticulus of $f(x) = -1 + x + x^{481} + x^{985} + x^{1502}$ in the angular sector $-\pi/18 < \arg z < \pi/18$ in two different scalings in x and y (in a) and b)). In this angular sector the other zeroes of f can be found in a thin annular neighbourhood of the unit circle. The real root $1/\beta > 0$ of f is such that β satisfies : $1.00970357\dots = \theta_{481}^{-1} < \beta = 1.0097168\dots < \theta_{480}^{-1} = 1.0097202\dots$

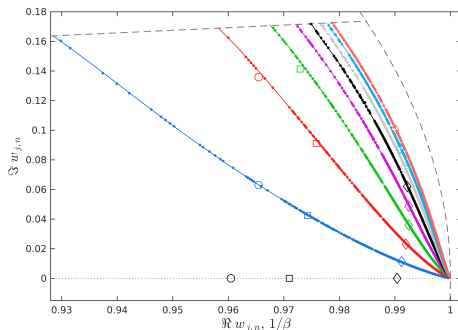


FIGURE: Universal curves stemming from 1 which constitute the lenticular zero locus of all the polynomials of the class \mathcal{B} . These curves are continuous, semi-fractal. The first one above the real axis, corresponding to the zero locus of the first lenticular roots, lies in the boundary of Solomyak's fractal [?]. The lenticular roots of the previous polynomials f are represented by the respective symbols \circ , \square , \diamond . The dashed lines represent the unit circle and the top boundary of the angular sector $|\arg z| < \pi/18$.

A Dobrowolski-type inequality for the lenticular measure

Def. : If $\theta_n^{-1} < \beta < \theta_{n-1}^{-1}$ for some n large enough, and

$$\mathcal{L}_\beta = \{\overline{\omega_{J_n, n}}, \dots, \overline{\omega_{1, n}}, \beta^{-1}, \omega_{1, n}, \dots, \omega_{J_n, n}\}$$

denotes the set of the lenticular poles of $\zeta_\beta(z)$, all depending upon β , then we attribute to \mathcal{L}_β a measure, that we call *lenticular measure* of β , by the expression

$$\mathfrak{S}_{lent}(\beta) := \prod_{\omega \in \mathcal{L}_\beta} |\omega|^{-1} = \beta \prod_{j=1}^{J_n} |\omega_{j, n}|^{-2}. \quad (8)$$

By construction, $\mathfrak{S}_{lent}(\beta) \geq 1$.

If $\beta = \theta_n^{-1}$, then the identification with the Mahler measure of θ_n^{-1} holds :

$$\mathfrak{S}_{lent}(\theta_n^{-1}) = M(\theta_n^{-1}).$$

Denote by $a_{\max} = 5.87433\dots$ the abscissa of the maximum of the function

$$a \mapsto (1 - \exp(-\frac{\pi}{a})) / (2 \exp(\frac{\pi}{a}) - 1)$$

on $(0, \infty)$. Let $\kappa := 0.171573\dots$ be the value of its maximum, at $a = a_{\max}$. From a numerical viewpoint we have : $2 \arcsin(\kappa/2) = 0.171784\dots$

Denote

$$C := \exp\left(\frac{-1}{\pi} \int_0^{2 \arcsin(\frac{\kappa}{2})} \text{Log} \left[\frac{1 + 2 \sin(\frac{x}{2}) - \sqrt{1 - 12 \sin(\frac{x}{2}) + 4(\sin(\frac{x}{2}))^2}}{4} \right] dx\right)$$

$$= \mathbf{1.15411\dots}$$

Theorem (VG '17)

There exists an integer $\eta \geq 260$ such that the following inequality holds :

$$\mathfrak{S}_{lent}(\beta) \geq C - C \frac{\arcsin(\kappa/2)}{\pi} \frac{1}{\text{Log}(n)}, \quad \text{for all } n \geq \eta \quad (9)$$

and any $\beta \in (\theta_n^{-1}, \theta_{n-1}^{-1})$.

This theorem extends (VG'16) :

$$M(-1 + x + x^n) = M(\theta_n^{-1}) > \Lambda - \frac{\Lambda}{6} \left(\frac{1}{\text{Log } n} \right), \quad n \geq 2,$$

where Λ is the following constant

$$\begin{aligned} \Lambda &:= \exp\left(\frac{3\sqrt{3}}{4\pi} L(2, \chi_3)\right) = \exp\left(\frac{-1}{\pi} \int_0^{\pi/3} \text{Log}\left(2 \sin\left(\frac{x}{2}\right)\right) dx\right) \\ &= \mathbf{1.38135\dots}, \end{aligned}$$

higher than $C = 1.1541\dots$, and $L(s, \chi_3) := \sum_{m \geq 1} \frac{\chi_3(m)}{m^s}$ the Dirichlet L-series for the character χ_3 , with χ_3 the uniquely specified odd character of conductor 3 ($\chi_3(m) = 0, 1$ or -1 according to whether $m \equiv 0, 1$ or $2 \pmod{3}$), equivalently $\chi_3(m) = \left(\frac{m}{3}\right)$ the Jacobi symbol).

In 1979, Dobrowolski, using an auxiliary function, obtained the general asymptotic minoration, with $n = \deg(\alpha)$ the degree of the nonzero algebraic integer α , which is not a root of unity,

$$M(\alpha) > \mathbf{1} + (1 - \varepsilon) \left(\frac{\text{Log Log } n}{\text{Log } n} \right)^3, \quad n > n_0(\varepsilon), \quad (10)$$

with $1 - \varepsilon$ replaced by $1/1200$ for $n \geq 2$, for an effective version of the minoration.

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 - Identification, rewriting trails, Galois conjugation and convergence
- 5 Universal minorant

Hyp. : existence of a reciprocal algebraic integer β in the interval $(\theta_n^{-1}, \theta_{n-1}^{-1})$ for some integer n , large enough, such that

$$M(\beta) < 1.176280\dots,$$

if

$$P_\beta(X) = \widetilde{P}_\beta(X^r) \quad \text{implies } r = 1,$$

for some \mathbb{Z} -minimal integer polynomial $\widetilde{P}_\beta(X)$.

Proposition : for any lenticular pole $\omega_{j,n}$ of $\zeta_\beta(z)$,

$$f_\beta(\omega_{j,n}) = 0 \quad \implies \quad P_\beta(\omega_{j,n}) = 0.$$

This identification gives the following minoration to the Mahler measure of β :

$$M(\beta) = M(\beta^{-1}) = \prod_{\omega \notin \mathcal{L}_\beta, |\omega| < 1} |\omega|^{-1} \times \prod_{\omega \in \mathcal{L}_\beta} |\omega|^{-1} \geq \prod_{\omega \in \mathcal{L}_\beta} |\omega|^{-1} = \mathfrak{G}_{lent}(\beta).$$

Pf. : rewriting trails.

Two Theorems : Factorization of the polynomials of the class \mathcal{B} and Kala-Vávra's Theorem

Theorem (Dutykh - VG '18)

For any $f \in \mathcal{B}_n$, $n \geq 3$, denote by

$$f(x) = A(x)B(x)C(x) = -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s},$$

where $s \geq 1$, $m_1 - n \geq n - 1$, $m_{j+1} - m_j \geq n - 1$ for $1 \leq j < s$, the factorization of f where A is the cyclotomic part, B the reciprocal noncyclotomic part, C the nonreciprocal part. Then

- (i) the nonreciprocal part C is nontrivial, irreducible and **never vanishes on the unit circle**,
- (ii) if $\gamma > 1$ denotes the real algebraic number uniquely determined by the sequence $(n, m_1, m_2, \dots, m_s)$ such that $1/\gamma$ is the unique real root of f in (θ_{n-1}, θ_n) , $-C^*(X)$ is the minimal polynomial $P_\gamma(X)$ of γ , and γ is a nonreciprocal algebraic integer.

For a general complex number $\beta \in \mathbb{C}$, $|\beta| > 1$, and a finite alphabet $\mathcal{A} \subset \mathbb{C}$, we define the (β, \mathcal{A}) -representations as expressions of the form

$$\sum_{k \geq -L} a_k \beta^{-k}, \quad a_k \in \mathcal{A},$$

for some integer $L \in \mathbb{Z}$. They are Laurent series of $1/\beta$. We define

$$\text{Per}_{\mathcal{A}}(\beta) := \{x \in \mathbb{C} \mid x \text{ has an eventually periodic } (\beta, \mathcal{A})\text{-representation}\}.$$

Those β s which are the real roots > 1 of the polynomials of the class \mathcal{B} will be of special interest in the next section.

Theorem (Kala -Vávra '19)

Let $\beta \in \mathbb{C}$ be an algebraic number of degree d , $|\beta| > 1$, and let $a_d x^d - a_{d-1} x^{d-1} - \dots - a_1 x - a_0 \in \mathbb{Z}[x]$ be its minimal polynomial.

Suppose that $|\beta'| \neq 1$ for any conjugate β' of β . Then there exists a finite alphabet $\mathcal{A} \subset \mathbb{Z}$ such that

$$\mathbb{Q}(\beta) = \text{Per}_{\mathcal{A}}(\beta). \tag{11}$$

Rewriting polynomials - identification of lenticular roots as conjugates

[Res. Number Theory 7 :64 (2021)]

Take $\beta \in (\theta_n^{-1}, \theta_{n-1}^{-1})$ a **reciprocal** algebraic integer for some integer n large enough. Then

$$f_\beta(z) = -1 + z + z^n + z^{m_1} + z^{m_2} + \dots + z^{m_j} + z^{m_{j+1}} + \dots,$$

where $m_1 - n \geq n - 1$, $m_{j+1} - m_j \geq n - 1$ for $j \geq 1$, written $= -1 + \sum_{i \geq 1} t_i z^i$, is a power series which is never a polynomial (Descartes's rule).

For every $s \geq 1$, let $S_{\gamma_s}(X) := X^s - \sum_{i=0}^{s-1} t_{s-i} X^i$ such that

$$S_{\gamma_s}^*(X) = X^s S_{\gamma_s}(1/X) = 1 - t_1 X - t_2 X^2 - \dots - t_{s-1} X^{s-1} - t_s X^s$$

and $-S_{\gamma_s}^*(z)$ is the s th polynomial section of $f_\beta(z)$.

Def. : $\gamma_s > 1$ unique zero of $S_{\gamma_s}(z)$. : $S_{\gamma_s}^*(\gamma_s^{-1}) = 0$, $\lim_{s \rightarrow +\infty} \gamma_s^{-1} = \beta^{-1}$.

The minimal polynomial P_β of β is monic and reciprocal. Denote it

$$P_\beta(x) = 1 + a_1x + a_2x^2 + \dots + a_{d-1}x^{d-1} + a_dx^d \quad (a_{d-i} = a_i).$$

Let $H := \max\{|a_i| : 1 \leq i \leq d-1\} \geq 1$ be the naïve height of P_β .

We have 2 “characterizers” of the same object β :

$$P_\beta \text{ and } f_\beta.$$

How to pass from one to the other ? rewriting trail and control of the alphabet in Kala-Vávra’s Theorem.

We proceed in two steps :

- (i) first we express $P_\beta(\gamma_s)$ as a (γ_s, \mathcal{A}) - eventually periodic representation with the symmetrical alphabet $\mathcal{A} = \{-m, \dots, 0, \dots, +m\} \subset \mathbb{Z}$ which does not depend upon s , with $m = \lceil 2((2^d - 1)H + 2^d)/3 \rceil$. This expression of $P_\beta(\gamma_s)$, as a Laurent series of $1/\gamma_s$, is obtained using the two above Theorems ,
- (ii) we allow s to tend to infinity to obtain the convergence of $P_\beta(\sigma(\gamma_s))$ to 0 which will imply $P_\beta(\sigma(\beta)) = 0$.

Starting point : $1 = 1$,
 to which we add $0 = -S_\gamma^*(\gamma^{-1})$ in the right hand side. Then we define a rewriting trail from

$$1 = 1 - S_\gamma^*(\gamma^{-1}) = t_1\gamma^{-1} + t_2\gamma^{-2} + \dots + t_{s-1}\gamma^{-(s-1)} + t_s\gamma^{-s}$$

to

$$-a_1\gamma^{-1} - a_2\gamma^{-2} + \dots - a_{d-1}\gamma^{-(d-1)} - a_d\gamma^{-d} = 1 - P(\gamma^{-1}).$$

A rewriting trail will be a sequence of integer polynomials, whose role will consist in **“restoring” the coefficients** of $1 - P(\gamma^{-1})$ one after the other, from the left, by adding “0” conveniently at each step to both sides.

At the first step we add $0 = (-a_1 - t_1)\gamma^{-1} S_\gamma^*(\gamma^{-1})$; and we obtain

$$1 = -\mathbf{a}_1\gamma^{-1} \\ + (-(-a_1 - t_1)t_1 + t_2)\gamma^{-2} + (-(-a_1 - t_1)t_2 + t_3)\gamma^{-3} + \dots$$

so that the height of the polynomial

$$(-(-a_1 - t_1)t_1 + t_2)X^2 + (-(-a_1 - t_1)t_2 + t_3)X^3 + \dots$$

is $\leq H + 2$.

At the second step we add $0 = (-a_2 - (-(a_1 - t_1)t_1 + t_2))\gamma^{-2} S_\gamma^*(\gamma^{-1})$.
Then we obtain

$$1 = -\mathbf{a}_1\gamma^{-1} - \mathbf{a}_2\gamma^{-2}$$

$$+ [(-a_2 - (-(a_1 - t_1)t_1 + t_2))t_1 + (-(a_1 - t_1)t_2 + t_3)]\gamma^{-3} + \dots$$

where the height of the polynomial

$$[(-a_2 - (-(a_1 - t_1)t_1 + t_2))t_1 + (-(a_1 - t_1)t_2 + t_3)]X^3 + \dots$$

is $\leq H + (H + 2) + (H + 2) = 3H + 4$. Iterating this process d times we obtain

$$1 = -\mathbf{a}_1\gamma^{-1} - \mathbf{a}_2\gamma^{-2} - \dots - \mathbf{a}_d\gamma^{-d}$$

$$+ \textit{polynomial remainder in } \gamma^{-1}.$$

To summarize, we obtain a sequence $(A'_q(X))_{q \geq 1}$ of rewriting polynomials involved in this rewriting trail; for $q \geq 1$, $A'_q \in \mathbb{Z}[X]$, $\deg(A'_q) \leq q$ and $A'_q(0) = -1$. The first polynomial $A'_1(X)$ is $-1 + (-a_1 - t_1)X$. The second polynomial $A'_2(X)$ is $-1 + (-a_1 - t_1)X + (-a_2 - (-(a_1 - t_1)t_1 + t_2))X^2$, etc.

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- 5 Universal minorant**
- 6 Search for very small Mahler measures from \mathcal{B} ; Modulo p Problem ; Hypergeometric functions over \mathbb{Q}

Theorem

There exists an integer $\eta \geq 260$ such that, for any nonzero real reciprocal algebraic integer β , which is $\neq \pm 1$,

$$M(\beta) \geq \theta_{\eta}^{-1}.$$

$$\theta_{260}^{-1} \approx 1.01 \dots$$

Pf. : Let $\beta \neq 0$ be a reciprocal algebraic integer which is not a root of unity, such that $\text{dyg}(\beta) \geq \eta$ with $\eta \geq 260$. Since $M(\beta) = M(\beta^{-1})$ there are three cases to be considered :

- (i) the house of β satisfies $|\overline{\beta}| \geq \theta_5^{-1}$,
- (ii) the dynamical degree of β satisfies : $6 \leq \text{dyg}(\beta) < \eta$, with $M(\beta) < 1.176280\dots$,
- (iii) the dynamical degree of β satisfies : $\text{dyg}(\beta) \geq \eta$, with $M(\beta) < 1.176280\dots$

In the first case, $M(\beta) \geq \theta_5^{-1} > \theta_{260}^{-1} \geq \theta_\eta^{-1} > 1$. In the second case, $M(\beta) \geq \theta_\eta^{-1}$. In the third case, the **Dobrowolski-type inequality** gives the following lower bound of the Mahler measure

$$\begin{aligned} M(\beta) &\geq C - C \frac{\arcsin(\kappa/2)}{\pi \text{Log}(\text{dyg}(\beta))} \\ &\geq C - C \frac{\arcsin(\kappa/2)}{\pi \text{Log}(\eta)} \geq C - C \frac{\arcsin(\kappa/2)}{\pi \text{Log}(260)}, \approx 1.14\dots \end{aligned}$$

This lower bound is numerically greater than $\theta_{260}^{-1} = 1.01\dots$, itself greater than θ_η^{-1} . In any case, the universal lower bound θ_η^{-1} of $M(\beta)$ holds true.

improving the method : Rouché to “spirals”

better minorant, better constants in Dobro.-type ineq.

Possible goal : Lehmer's number is the smallest (non-trivial) possible Mahler measure of reciprocal algebraic integers.

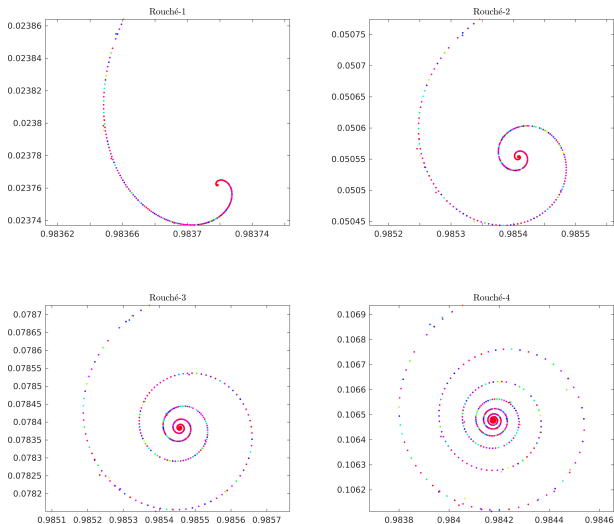


FIGURE: Curves of the lenticular j th-zeroes, containing the lenticular Galois conjugates $j = 1, 2, 3, 4$, $n = 220$)

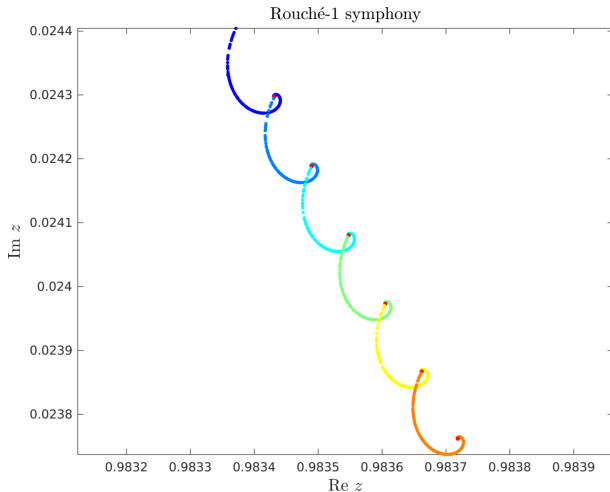


FIGURE: Curve of the lenticular first-zeroes ($j = 1$, $n = 215$ to 220)

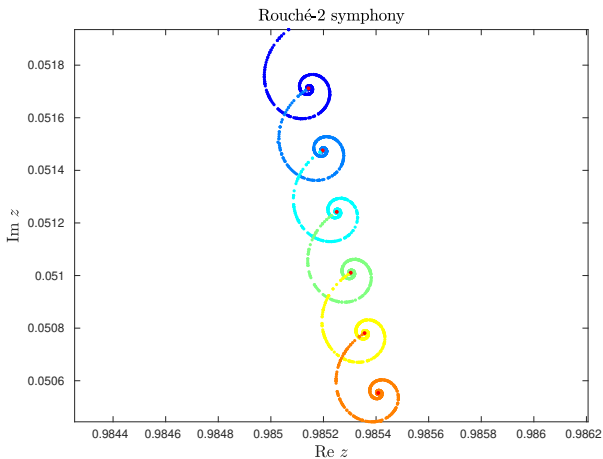


FIGURE: Curve of the lenticular 2nd-zeroes ($j = 2$, $n = 215$ to 220)

Does there exist reciprocal non-cyclotomic factors in the decomposition of $f \in \mathcal{B}$?

Recall

$$f(x) = A(x) \times B(x) \times C(x).$$

Two Conjectures have appeared :

- (i) the “Asymptotic Reducibility Conjecture” says that 75% of the polynomials $f(x) \in \mathcal{B}$ are irreducible (then non-reciprocal),
- (ii) the Conjecture B, which says that the factorization of any $f \in \mathcal{B}$ never contains factors which are reciprocal non-cyclotomic.

Conjecture B has been established (Dutykh VG, '22) for some infinite families of polynomials f of \mathcal{B} , using a criterium of Sawin, Shusterman and Stoll complementing some results of Schinzel.

Pb. : Factorization problem, number of factors ?

Theorem

Let $f \in \mathbb{Z}[X]^c$. If $f = \prod_{i \in J} f_i^{v_i}$ is the decomposition of f into irreducible factors, with all $v_i \geq 1$, then

$$\lim_{x \rightarrow \infty} \frac{1}{\pi(x)} \sum_{p \leq x} N_p(f) = \sum_{i \in J} v_i$$

which is the number of irreducible factors in the decomposition of f .

c stands for “content=1”

p prime number

$N_p(f)$ = number of zeroes of f in \mathbb{F}_p .

Stated in 1880 by Kronecker without proof. Proof by Rosen in 2012.

Apply to class \mathcal{B} :

A subset S of the set of primes \mathbb{P} has density c if

$$\lim_{x \rightarrow \infty} \frac{\text{number of } p \in S \text{ with } p \leq x}{\pi(x)} = c. \quad \text{The limit is denoted by } \delta(S).$$

Theorem

Let $n \geq 3$. Let $f(x) := -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s} \in \mathcal{B}$, where $s \geq 0$, $m_1 - n \geq n - 1$, $m_{q+1} - m_q \geq n - 1$ for $1 \leq q < s$. The polynomial f is assumed irreducible. Denote by G the Galois group of $f(x)$ and $g := \#G$. Then

- (i) [Serre, '04 - Chebotarev density Thm] the set $\mathcal{P}_0 := \{p \in \mathbb{P} \mid N_p(f) = 0\}$ is infinite, has a density and its density satisfies

$$\delta(\mathcal{P}_0) \geq \frac{1}{m_s},$$

with strict inequality if m_s is not a power of a prime,

- (ii) [Rosen, '12 - Frobenius density Thm] the set $\mathcal{P}_{\max} := \{p \in \mathbb{P} \mid N_p(f) = \deg(f)\}$ is infinite and has density

$$\delta(\mathcal{P}_{\max}) = \frac{1}{g}.$$

From Beukers, Cohen, Mellit (“Finite Hypergeometric Functions”) :

Cor. 1.6 : Let $f(t) = x^3 + 3x^2 - 4t$ with $t \in \mathbb{F}_q$ and $t \neq 0, 1$. Let $N_f(t)$ be the number of zeroes of $f(x)$ in \mathbb{F}_q . Suppose that q is not divisible by 2 or 3. Then

$$N_f(t) = 1 + H_q(1/3, 2/3; 1, 1/2 \mid t).$$

Here (context Lehmer - the curve of the lenticular poles, having infinitely many branches emitted by $z = 1$ is known) $\beta > 1$ is not used as a real variable, but by its coding :

$$\beta \leftrightarrow (t_i = t_i(\beta)).$$

Pb. : find analogue of BCM theory : $N_f(t) \equiv N_p(f) \equiv N_p(\beta)$.

$$N_p(\beta) = 1 + ??_p(?? \mid \beta).$$

Thank you