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On the sharp regularity of solutions to hyperbolic boundary value problems

Corentin Audiard ^{*†}

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Abstract

We prove some sharp regularity results for solutions of classical first order hyperbolic initial boundary value problems. Our two main improvements on the existing litterature are weaker regularity assumptions for the boundary data and regularity in fractional Sobolev spaces. This last point is specially interesting when the regularity index belongs to $1/2 + \mathbb{N}$, as it involves nonlocal compatibility conditions.

1 Introduction

Everything in a toy model Consider the simplest hyperbolic initial boundary value problem (IBVP)

$$\begin{cases} \partial_t u + \partial_x u(x, t) = 0, & (x, t) \in (\mathbb{R}^+)^2 \\ u(x, 0) = u_0(x), \\ u(0, t) = g(t) \end{cases}$$

When $(u_0, g) \in L^2(\mathbb{R}^+)^2$, the solution is piecewise defined: $u(x, t) = u_0(x - t)$ for $x - t \geq 0$, $g(t - x)$ for $x - t < 0$, it belongs to $C_t L^2$.

It is well known that the smoothness of (u_0, g) is not enough to ensure the smoothness of u , compatibility conditions are required: for $k \in \mathbb{N}$, $u \in \cap_{j=0}^k C_t^j H^{k-j}$ if and only if

$$(u_0, g) \in (H^k)^2 \text{ and } \forall j \leq k - 1, u_0^{(j)}(0) = (-1)^j g^{(j)}(0).$$

These compatibility relations are trivial here due to the solution formula, but are more generally derived considering u (and its derivatives) at the corner $x = t = 0$, and writing $\partial^\alpha u|_{x=0}|_{t=0} = \partial^\alpha u|_{t=0}|_{x=0}$. A basic rule of thumb being that any compatibility condition that makes sense should be true.

For fractional regularity, not much changes except in the notoriously pathologic case $s \equiv 1/2[\mathbb{Z}]$. Indeed even if there is no trace in $H^{1/2}(\mathbb{R}^+)$, the gluing of two functions in $H^{1/2}(\mathbb{R}^+)$ is not $H^{1/2}(\mathbb{R})$. The simplest way to see this is to consider the map $f \in L^2(\mathbb{R}) \rightarrow f(\cdot) - f(-\cdot) \in L^2(\mathbb{R}^+)$. It is continuous $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}^+)$ and $H^1 \rightarrow H_0^1(\mathbb{R}^+)$ hence $H^{1/2}(\mathbb{R}) \rightarrow [L^2, H_0^1]_{1/2}$

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by interpolation. The interpolated space is the famous Lions-Magenes space $H_{00}^{1/2}(\mathbb{R}^+)$, and it is different (algebraically and topologically) from $H^{1/2}(\mathbb{R}^+)$: by interpolation of Hardy's inequality, any function $f \in H_{00}^{1/2}(\mathbb{R}^+)$ must satisfy

$$\int_{\mathbb{R}^+} \frac{f^2(x)}{x} dx < \infty,$$

this is obviously not the case for functions merely in $H^{1/2}(\mathbb{R}^+)$.

For the regularity of solutions of the BVP, this adds a "global" compatibility condition

$$u \in C_t H^{1/2} \Leftrightarrow (u_0, g) \in H^{1/2}(\mathbb{R}^+) \text{ and } \int_{\mathbb{R}^+} \frac{|g(x) - u_0(x)|^2}{x} dx < \infty.$$

Our aim here is to prove an analogous result for general hyperbolic boundary value problems.

Settings and results Let Ω be a smooth open set of \mathbb{R}^d , we consider first order boundary value problems of the form

$$\begin{cases} Lu := (\partial_t - \sum_{j=1}^d A_j \partial_j)u = 0, & (x, t) \in \Omega \times \mathbb{R}_t^+, \\ Bu|_{\partial\Omega} = g, & (x, t) \in \partial\Omega \times \mathbb{R}_t^+, \\ u|_{t=0} = u_0, & x \in \Omega. \end{cases} \quad (1.1) \quad \boxed{\text{IBVP}}$$

The index t in \mathbb{R}_t^+ has no meaning except to emphasize the time variable. The A_j 's are $q \times q$ matrices depending smoothly on (x, t) , B is a smooth $b \times q$ matrix, b is the number of boundary conditions.

For data $(u_0, g, f) \in L^2(\Omega) \times L^2(\partial\Omega \times \mathbb{R}_t^+) \times L^2(\mathbb{R}_t^+ \times \Omega)$, the well-posedness of such hyperbolic BVP has been obtained in a large variety of settings, that we will only shortly mention. After the pioneering results of Friedrichs ^{Friedrichs} [6] for symmetric dissipative systems, Kreiss ^{Kreiss} [10] proved the well-posedness of the BVP with zero initial data in the strictly hyperbolic case ($\sum A_j \xi_j$ has only real eigenvalues of algebraic multiplicity one) under the now standard Kreiss-Lopatinskiĭ condition on B . In Kreiss's framework, the case of L^2 initial data was then tackled by Rauch ^{Rauch} [17]. Well-posedness of constantly hyperbolic BVP was later obtained by Métivier ^{Métivier} [15] (zero initial data), the author then proved well-posedness with L^2 initial data [2]. A further generalization was obtained by Métivier ^{Métivier} [16] for a new class of hyperbolic operators, larger than the constantly hyperbolic ones. He also gave a new proof, both more general and simpler, of well-posedness with L^2 initial data.

For more references and results, in particular for characteristic BVP (that we do not consider here) the reader may refer to the book ^{Penzoni13} [4].

Let n be a normal on $\partial\Omega$, the problem (1.1) is said to be noncharacteristic when $\sum A_j n_j$ is invertible on $\partial\Omega$. For non characteristic boundary value problems, the main reference on the smoothness of solutions is the classical paper of Rauch and Massey ^{RauchMassey} [18], where, under no specific assumption (except of course well-posedness), the authors prove that the solution of (1.1) belongs to $\cap_{j=0}^k C_t^j(\mathbb{R}_t^+, H^{k-j}(\Omega))$ when $(u_0, g, f) \in H^k(\Omega) \times H^{k+1/2}(\partial\Omega \times \mathbb{R}_t^+) \times H^k(\Omega \times \mathbb{R}_t^+)$ and satisfy natural compatibility conditions that we describe now. For conciseness, when there is no ambiguity we will usually denote H^k instead of $H^k(X)$, $X = \Omega, \partial\Omega \times \mathbb{R}_t^+, \Omega \times \mathbb{R}_t^+$.

We denote $\mathcal{A} = \sum A_j \partial_j$ and define inductively v_j the formal value of $(\partial_t^j u)|_{t=0}$ by

$$v_0 = u_0, \quad v_{j+1} = (\partial_t^j \partial_t u)|_{t=0} = \partial_t^j (\mathcal{A}u + f)|_{t=0} = \sum_{l=0}^j \binom{j}{l} (\partial_t^l \mathcal{A}|_{t=0}) v_{j-l} + \partial_t^j f|_{t=0}. \quad (1.2) \quad \boxed{\text{taylor}}$$

The first order compatibility condition is $Bv_0|_{\partial\Omega} = g|_{t=0}$ and the generic compatibility condition of order j is

$$\text{Compatibility at order } j : \quad \partial_t^{j-1} g|_{t=0} = \sum_{l=0}^{j-1} \binom{j-1}{l} (\partial_t^l B) v_{j-1-l}|_{\partial\Omega}. \quad (1.3) \quad \boxed{\text{CCj}}$$

Note that $\boxed{\text{CCj}}$ makes sense as soon as $(u_0, g, f) \in (H^s)^3$, $s > j - 1/2$. If the smoothness of the data is $j - 1/2$, $j \in \mathbb{N}^*$, we define a special compatibility condition : when $\Omega = \mathbb{R}^{d-1} \times \mathbb{R}^+$, denote $x = (x', y)$; the condition is

Compatibility at order $j - 1/2$:

$$\partial_t^{j-1} g(x', t) - \left(\sum_{l=0}^{j-1} \binom{j-1}{l} (\partial_t^l B) v_{j-1-l}(x', t) \right) \in H_{00}^{1/2} \left(\mathbb{R}^{d-1} \times (\mathbb{R}^+) \right). \quad (1.4) \quad \boxed{\text{CCj-}}$$

For general smooth Ω , $\boxed{\text{CCj-}}$ is defined similarly through local maps and a partition of unity: near the boundary Ω is diffeomorphic to (a part of) $\mathbb{R}^{d-1} \times \mathbb{R}^+$ thanks to some map Φ , one simply requires $\boxed{\text{CCj}}$ to stand for $g(\Phi(x', 0), t)$, $(v_l \circ \Phi(x', t))_{0 \leq l \leq j-1}$.

Note that due to Hardy's inequality, the j -th condition implies the condition of order $j - 1/2$.

Definition 1.1. *If $s = k + \theta$, $-1/2 < \theta < 1/2$, $k \in \mathbb{N}^*$, $\theta \neq 1/2$, we say that data $(u_0, g, f) \in (H^s)^3$ satisfy the compatibility conditions at order s when $\boxed{\text{CCj}}$ is satisfied for $1 \leq j \leq k$.*

If $s = k - 1/2$, the compatibility conditions are satisfied at order s when $\boxed{\text{CCj-}}$ is true for $1 \leq j \leq k - 1$ and $\boxed{\text{CCj-}}$ is true for $j = k$.

A strong L^2 solution of $\boxed{\text{IBVP}}$ (I.1) is a function $u \in C_t L^2$ such that there exists a sequence u_n of smooth solutions of $\boxed{\text{IBVP}}$ (I.1) with data $(u_{0,n}, g_n, f_n)$ that converge to (u_0, g, f) in L^2 , and for any $T > 0$, $\|u - u_n\|_{C([0,T], L^2)} \rightarrow 0$.

Assumptions We need the smoothness of Ω and the well-posedness of $\boxed{\text{IBVP}}$ (I.1):

$\boxed{\text{assump1}}$

1. $\partial\Omega$ is a smooth hypersurface with normal ν , parametrized by local maps $(\phi_j(y'))_{1 \leq j \leq J}$, $y' \in \mathbb{R}^{d-1}$, and $\varphi_j(y', y_d) := \phi_j(y') + y_d \nu(\phi_j(y'))$ are local diffeomorphisms $V_j \rightarrow U_j$, with $\varphi_j((\mathbb{R}^{d-1} \times \mathbb{R}^{+*}) \cap V_j) \subset \Omega$, and $\cup_{j=1}^J U_j \supset \partial\Omega$. We do not assume that the U_j are bounded sets, but $D\varphi_j, D\varphi_j^{-1}$ must be uniformly bounded, and $d(\Omega \setminus \cup \text{Im}(\varphi_j), \partial\Omega) > 0$.

$\boxed{\text{charac}}$

2. The boundary is uniformly not characteristic, in the sense that $\sum A_j \nu_j$ is invertible on $\partial\Omega$, and the inverse is uniformly bounded.

3. For data $(u_0, g, f) \in (L^2)^3$, there exists a unique strong L^2 solution¹ to (I.1) ^{IBVP} that satisfies the semi-group estimate for γ large enough

$$\|e^{-\gamma \cdot} u\|_{C([0,t], L^2(\Omega))} + \sqrt{\gamma} \|e^{-\gamma \cdot} u|_{\partial\Omega}\|_{L^2(\partial\Omega \times [0,t])} \lesssim \|u_0\|_{L^2(\Omega)} + \|e^{-\gamma \cdot} g\|_{L^2(\partial\Omega \times [0,T])} + \frac{\|e^{-\gamma \cdot} f\|_{L^2([0,t] \times \Omega)}}{\sqrt{\gamma}}. \quad (1.5)$$

We use the convention that norms inside the domain are denoted $\|\cdot\|$ while norms on the boundary are denoted $|\cdot|$.

We point out that a consequence of the semi-group estimate is the *resolvent estimate*: for γ large enough (larger than for (I.5) ^{semigrroupe})

$$\gamma \|e^{-\gamma t} u\|_{L^2(\Omega \times \mathbb{R}_t^+)}^2 + \|e^{-\gamma t} u|_{\partial\Omega}\|_{L^2(\partial\Omega \times \mathbb{R}_t^+)}^2 \lesssim \left(\|u_0\|_{L^2(\Omega)}^2 + \|e^{-\gamma t} g\|_{L^2(\partial\Omega \times \mathbb{R}_t^+)}^2 + \frac{\|e^{-\gamma t} f\|_{L^2}^2}{\gamma} \right). \quad (1.6)$$

This is readily obtained by squaring (I.5) ^{semigrroupe} for some fixed γ_0 , multiplication by $e^{-2\gamma t}$, $\gamma > \gamma_0$ and integration in t . Higher regularity versions of the resolvent and the semi-group estimates are a bit more delicate to state. We define weighted Sobolev spaces H_γ^s in section 2, the weighted resolvent estimate is then

$$\gamma \|u\|_{H_\gamma^s}^2 + \|u|_{\partial\Omega}\|_{H_\gamma^s}^2 \lesssim \|u_0\|_{H^s(\Omega)}^2 + \|g\|_{H_\gamma^s}^2 + \frac{\|f\|_{H_\gamma^s}^2}{\gamma}. \quad (1.7)$$

The main point of this estimate is that it is sharp with respect to the parameter γ and allows to absorb commutators in a priori estimates. Moreover, it implies the following (simpler to read) estimate

$$\|e^{-\gamma t} u\|_{H^s(\Omega \times \mathbb{R}_t^+)}^2 + \|e^{-\gamma t} u|_{\partial\Omega}\|_{H^s(\partial\Omega \times \mathbb{R}_t^+)}^2 \lesssim \|u_0\|_{H^s(\Omega)}^2 + \|e^{-\gamma t} g\|_{H^s(\partial\Omega \times \mathbb{R}_t^+)}^2 + \|e^{-\gamma t} f\|_{H^s(\Omega \times \mathbb{R}_t^+)}^2. \quad (1.8)$$

We shall not need something as precise for the semi-group estimate: let $s = k + \theta$, $k \in \mathbb{N}$, $0 < \theta < 1$, then

$$\sum_{j=0}^k \|e^{-\gamma t} \partial_t^j u\|_{C(\mathbb{R}_t^+, H^{k-j+\theta}(\Omega))}^2 + \|e^{-\gamma t} u|_{\partial\Omega}\|_{H^s(\Omega \times \mathbb{R}_t^+)}^2 \lesssim \|u_0\|_{H^s(\Omega)}^2 + \|e^{-\gamma t} g\|_{H^s(\partial\Omega \times [0,T])}^2 + \|e^{-\gamma t} f\|_{H^s}^2. \quad (1.9)$$

Both estimates should be modified when $s = k + 1/2$, $k \in \mathbb{N}$: it is necessary to add in the right hand side the $H_{00}^{1/2}$ norm of $\partial_t^k g - \sum_0^k \binom{k}{i} (\partial_t^i B) v_{k-1-i}$, see page 15 for details. This is the (implicit) convention that we use in theorem 1.3, we refer to the proof for more details.

An interesting related feature is that the constant in \lesssim can not be uniform in θ , it blows up as $\theta \rightarrow 1/2$ and the estimates are actually not true for $\theta = 1/2$.

We can now state more precisely the regularity result of Rauch and Massey:

¹This assumption is somewhat too strong, as it is classical that in this framework, weak solutions are actually strong, see [11].

Theorem 1.2 ^{RauchMassey} ([18]). *If $(u_0, g, f) \in H^k(\Omega) \times H^{k+1/2}(\partial\Omega \times \mathbb{R}_t^+) \times H^k(\Omega \times \mathbb{R}_t^+)$ satisfy the compatibility condition up to order k , the solution of (1.1) ^{BVP} belongs to $\cap_{j=0}^k C_t^j H^{k-j}$.*

The only suboptimal part of the theorem is the regularity assumption on g . This is due to the fact that the theorem is deduced from the homogeneous case $g = 0$ with a lifting argument. It was already pointed out at the time by the authors that it could be improved (without proof), but quite unfortunately the result that remained in the litterature is the suboptimal one, see for example the reference book ^{Benzonis} [4], and in somewhat different settings the lecture notes ^{Metz} [14] or the interesting discussion in the introduction of ^{lanigu} [9]. Our result is that the same property holds with boundary data in H^k instead of $H^{k+1/2}$, moreover we allow k to be any nonnegative real number rather than an integer.

mainth

Theorem 1.3. *Let $s \in \mathbb{R}^+$. If $(u_0, g, f) \in H^s(\Omega) \times H^s(\partial\Omega \times \mathbb{R}_t^+) \times H^s(\Omega \times \mathbb{R}_t^+)$ satisfy the compatibility condition up to order s , the solution of (1.1) ^{BVP} belongs for any $T > 0$ to $H^s(\Omega \times [0, T])$, satisfies estimate ^{resolyreg} (1.7) for γ large enough, and if $s = k + \theta$, $k \in \mathbb{N}$, $0 \leq \theta < 1$, it satisfies ^{semireg} (1.9).*

The proof when s is an integer is quite similar to the original argument of Rauch and Massey, actually the fact that we handle directly nonzero boundary data leads to some slight simplifications due to the fact that it allows to avoid a reduction to the case where B is constant. The fractional case is essentially an interpolation argument, however it is not trivial due to the presence of the compatibility conditions. For example, in the model case described earlier instead of interpolating $[L^2 \times L^2, H^1 \times H^1]_\theta$ one must identify $[L^2 \times L^2, \{(u_0, g) \in H^1(\mathbb{R}^+) \times H^1(\mathbb{R}^+) : u_0(0) = g(0)\}]_\theta$.

The litterature on such problems is not very rich. Another related problem is the interpolation of Sobolev spaces with boundary conditions, that are in some sense between H^s and H_0^s . This issue appeared quite long ago for elliptic equations on non smooth domains or parabolic problems, see e.g. the last section of ^{Grisvard} [7], sections 14-17 of chapter 4 in ^{Lionsmagenes2} [13] (where most of the identification problems were left open), or the more recent (and much more involved) book ^{Amann} [1], in particular VIII.2.5. Due to the technicity of this last reference (anisotropic Besov spaces are studied), degenerate cases (in our settings $s \in \mathbb{N} + 1/2$) are not considered. The Schrödinger equation on a domain and related interpolation problems were also studied by the author in ^{Audiard} [3], where the natural spaces for the boundary data are Bourgain spaces.

Plan of the article Section 2 is devoted to notations and a brief reminder on interpolation. The proof of theorem ^{mainth} 1.3 is then organized in three sections : in section 3 we recall a standard smoothness result for the pure boundary value problem posed for $t \in \mathbb{R}$, due to Tartakov. For completeness, we include a sketch of proof that follows an argument of the (unfortunately depleted) book ^{Chap1} [5]. Theorem ^{mainth} 1.3 in the case s integer is proved in section 4. An important point is a basic lifting lemma which proves also useful for the general case. In section 5, smoothness is first proved for $0 \leq s \leq 1$ with an interpolation argument, then for any s with a non trivial differentiation argument.

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2 Notations and basic results

notations

Basic notations Proofs are often reduced to the case $\Omega = \mathbb{R}^{d-1} \times \mathbb{R}^+$. In such settings, we denote the variable $x = (x', y)$ $x' \in \mathbb{R}^{d-1}$. The variables x', t are called tangential, while y is the normal variable.

Partial differential operators acting on functions of (x, t) are written as ∂^α , $\alpha \in \mathbb{N}^{d+1}$, by convention α_{d+1} is the order of differentiation in time. A multi-index, or a differential operator, is said to be tangential when $\alpha_d = 0$.

We denote $[L_1, L_2] = L_1L_2 - L_2L_1$ the commutator between two linear operators.

Sobolev spaces Ω is assumed to be a smooth open set as in definition page [3](#).^{assump1} The Sobolev spaces $H^s(\Omega)$, are defined when s is an integer as

$$\{u \in L^2 : \|u\|_{H^s}^2 = \sum_{|\alpha| \leq s} \int_{\Omega} |\partial^\alpha u|^2 dx < \infty\}.$$

When s is not an integer, they are defined by (complex) interpolation, $H^s = [L^2, H^k]_{s/k}$ for any integer k larger than s . This definition does not depend on k .

The Sobolev spaces for functions defined on $\partial\Omega, \Omega \times \mathbb{R}_t^+$ etc are defined in the same standard way.

$H_0^s(\Omega)$ is the closure of $C_c^\infty(\Omega)$. We do have $[L^2, H_0^1]_s = H_0^s$ for $0 < s < 1$, except for $s = 1/2$, where $H_0^{1/2} = H^{1/2}$ and $[L^2, H_0^1]_{1/2} = H_{00}^{1/2}$ is different algebraically and topologically from $H^{1/2}$. It is a Banach space endowed with the norm

$$\|u\|_{H_{00}^{1/2}}^2 = \|u\|_{H^{1/2}}^2 + \int_{\Omega} \frac{|u(x)|^2}{d(x)} dx,$$

where d is the distance to $\partial\Omega$ (see [\[I2\]](#)).^{liensmagenes} The only important fact, regularly used in the article, is that if X_0, X_1 are Banach spaces, an operator $T : X_0 \rightarrow L^2, X_1 \rightarrow H_0^1$ maps $[X_0, X_1]_{1/2}$ to $H_{00}^{1/2}$. For example, $u \in H^s(\mathbb{R}^d) \rightarrow u(x', y) - u(x', -y)$ maps $H^{1/2}(\mathbb{R}^d)$ to $H_{00}^{1/2}(\mathbb{R}^{d-1} \times \mathbb{R}^+)$. The weighted Sobolev spaces H_γ^s are defined as follows :

Definition 2.1. *When s is a nonnegative integer we define $H_\gamma^s(\Omega \times \mathbb{R}_t^+)$ as the the set of functions in L^2 such that the following norm is finite*

$$\|u\|_{H_\gamma^s} = \sum_{|\alpha| \leq s} \|e^{-\gamma t} \partial^\alpha u\|_{L^2}.$$

When s is not an integer, H_γ^s is defined by complex interpolation : if k is an integer larger than s , $H_\gamma^s = [L_\gamma^2, H_\gamma^k]_{s/k}$.

$H_\gamma^s(\partial\Omega \times \mathbb{R}_t^+)$ is defined similarly.

When s is an integer, it is a straightforward consequence of Leibniz formula $\partial_t^j(e^{-\gamma t} u) = \sum \binom{j}{i} (-\gamma)^i e^{-\gamma t} \partial_t^{j-i} u$ that the H_γ^s norm is equivalent to $\|e^{-\gamma t} u\|_{H^s}$, though with constants that depend on γ , hence the H_γ^s spaces coincide algebraically and topologically with the set of functions such that $e^{-\gamma t} u \in H^s$.

Traces Sobolev spaces on $\partial\Omega$ are defined with local maps. The trace operator is an isomorphism:

$$\begin{cases} H^s(\Omega) \rightarrow \prod_{k < s-1/2} H^{s-1/2-k}(\partial\Omega), \\ u \rightarrow (\partial_n^k u|_{\partial\Omega})_{k < s-1/2}, \end{cases}$$

where ∂_n is the normal derivative on $\partial\Omega$.

For functions defined in $H^s(\Omega \times \mathbb{R}^{+*})$, the trace operator on $\partial\Omega \times \mathbb{R}^{+*}$ and $\Omega \times \{0\}$ is more subtle, the map

$$\begin{cases} H^s(\Omega \times \mathbb{R}_t^{+*}) \rightarrow \left(\prod_{k < s-1/2} H^{s-1/2-k}(\partial\Omega \times \mathbb{R}_t^{+*}) \right) \times \left(\prod_{k < s-1/2} H^{s-1/2-k}(\Omega \times \{0\}) \right), \\ u \rightarrow (\partial_n^k u|_{\partial\Omega \times \mathbb{R}^{+*}}, \partial_t^k u|_{\Omega \times \{0\}})_{k < s-1/2}, \end{cases} \quad (2.1) \quad \text{trace2}$$

is continuous but not surjective: if $s \notin \mathbb{N}$, *local compatibility conditions* between $(g_k, v_k) \in \left(\prod H^{s-1/2-k}(\partial\Omega \times \mathbb{R}_t^{+*}) \right) \times \left(\prod H^{s-1/2-k}(\Omega \times \{0\}) \right)$ are required as follows

$$\forall k + j < s - 1, \partial_t^j g_k|_{t=0} = \partial_n^k v_j|_{\partial\Omega}, \quad (2.2) \quad \text{compaclassiq}$$

(see [Lionsmagenes2](#) [\[13\]](#)).

In the case $s = 1$, and $\Omega = \mathbb{R}^{d-1} \times \mathbb{R}^{+*}$, surjectivity requires the *global compatibility condition*

$$v_0(x', t) - g_0(x', t) \in H_{00}^{1/2}(\partial\Omega). \quad (2.3) \quad \text{compaclassiq}$$

This condition extends to smooth Ω , see the short comment after [\(1.4\)](#) [\[CCj-\]](#).

Provided such compatibility conditions are added, the trace map is a surjection and has a right inverse, this very well known fact will be proved later in the article in some basic cases where it is needed with more precise estimates.

3 Regularity for the pure boundary value problem

Consider the boundary value problem

$$\begin{cases} Lu = f, (x, t) \in \Omega \times \mathbb{R}_t^+ \\ Bu|_{\partial\Omega} = g, \\ u|_{t=0} = 0. \end{cases} \quad (3.1) \quad \text{bvp}$$

When g, f can be smoothly extended by 0 for $t < 0$, the smoothness of u is well known [\[19\], \[5\]](#) [\[Tart&Lap\]](#). The classical proof is done by first studying the pure boundary value problem posed on $t \in \mathbb{R}$, the case $t \in \mathbb{R}^+$ is then deduced by an extension by 0 for $t < 0$. We give here a minor variation of this argument that directly tackles [\(3.1\)](#) [\[bvp\]](#).

regBVP

Proposition 3.1. *Let $k \in \mathbb{N}$. If the extension of f and g by 0 for $t < 0$ belongs to H^k , then for γ large enough the solution of [\(3.1\)](#) [\[bvp\]](#) satisfies $e^{-\gamma t} u \in H^k(\Omega \times \mathbb{R}_t^+)$. In particular, it belongs to $H^k(\Omega \times [0, T])$ for any $T > 0$.*

Proof. The classical plan is to straighten the boundary through local maps, then use a tangential regularization. It is done by induction on k , it suffices to prove the final step where we assume $u \in H^{k-1}(\mathbb{R}^d \times \mathbb{R}_t)$ and prove $u \in H^k$.

We fix local maps φ_j as in assumption [I](#). Let $(\psi_j)_{0 \leq j \leq J}$ be a partition of unity associated to $\Omega \cup (\cup_j \text{Im}(\varphi_j))$. We denote the new variable $y = (y', y_d)$, $u_j = (\psi_j e^{-\gamma t} u) \circ \varphi_j$, and $u_0 = \psi_0 u$, $L_j = \partial_t + \gamma + \sum_i (\sum_k A_k (D_y \varphi_j)^{-1}_{ik}(y)) \partial_{y_i}$. For $1 \leq j \leq J$, u_j satisfies

$$\begin{cases} L_j u_j + ([\psi_j, L] e^{-\gamma t} u) \circ \varphi_j = e^{-\gamma t} (\psi_j f) \circ \varphi_j := f_j, (y', y_d, t) \in \mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R}_t^+, \\ B(\varphi_j(y', 0)) u_j(y', 0, t) = e^{-\gamma t} (\psi_j g) \circ \varphi_j(y', 0, t) := g_j. \end{cases} \quad (3.2)$$

eqredresse

For simplicity we still denote B for $B \circ \varphi_j(\cdot, 0)$. The regularization procedure was introduced by Hörmander [\[8\]](#): for $v \in L^2(\mathbb{R}^p)$, $p \geq 1$, define

$$\|v\|_{H^{s,\delta}(\mathbb{R}^p)}^2 = \int_{\mathbb{R}^d} |\widehat{v}(\xi)|^2 \frac{(1 + |\xi|^2)^{s+1}}{1 + |\delta \xi|^2} d\xi \xrightarrow{\delta \rightarrow 0} \|v\|_{H^{s+1}}^2.$$

Let $\rho(x) \in C_c^\infty(\mathbb{R}^p)$, such that $|\widehat{\rho}(\xi)| \lesssim |\xi|^m$, $m > k$ and $\widehat{\rho}$ does not cancel on a neighborhood outside 0 (such functions are easily constructed, for example using $\Delta^{m/2}(\rho_0(t)\rho'(y'))$, m even). Define $\rho_\varepsilon = \rho(\cdot/\varepsilon)/\varepsilon^d$. It is an exercise in calculus that for $0 \leq s \leq k-1$, an equivalent norm to $\|\cdot\|_{H^{s,\delta}}$ -uniformly in δ - is

$$\|v\|_{L^2} + \left(\int_0^1 \|v * \rho_\varepsilon\|_{L^2}^2 \frac{1}{\varepsilon^{2(s+1)}(1 + \delta^2/\varepsilon^2)} \frac{d\varepsilon}{\varepsilon} \right)^{1/2} \sim \|v\|_{H^{s,\delta}}. \quad (3.3)$$

equivsobo

Friedrich's lemma can be generalized in such settings: for P a first order differential operator with smooth coefficients

$$\int_0^1 \|[P, \rho_\varepsilon *]v\|_{L^2}^2 \frac{1}{\varepsilon^{2(s+1)}(1 + \delta^2/\varepsilon^2)} \frac{d\varepsilon}{\varepsilon} \lesssim \|v\|_{H^{s,\delta}}^2. \quad (3.4)$$

friedrichs

For details, we refer to [\[5\]](#) chapter 2 section 6.

We shall use tangential mollifiers $\rho_\varepsilon(x', t)$ for the functions u_j , $1 \leq j \leq J$, and full mollifiers $\rho_\varepsilon(x, t)$ for u_0 . Everything in (3.2) is extended by 0 for $t < 0$. Note that due to the assumptions on f, g , the extensions of (f_j, g_j) are still in H^k . We apply $\rho_\varepsilon *$ to (3.2) for $1 \leq j \leq J$:

$$\begin{cases} L_j \rho_\varepsilon * u_j = \rho_\varepsilon * f_j - \rho_\varepsilon * [\psi_j, L_j] e^{-\gamma t} u \circ \varphi_j - [\rho_\varepsilon *, L_j] e^{-\gamma t} u_j, \\ B(\rho_\varepsilon * u_j)|_{y_d=0} = \rho_\varepsilon * g_j - [\rho_\varepsilon *, B] u_j|_{y_d=0}. \end{cases} \quad (3.5)$$

eqreg

Since $\rho_\varepsilon * u_j$ belongs to $L^2(\mathbb{R}^+, H^\infty(\mathbb{R}^{d-1} \times \mathbb{R}_t))$ it is in H^∞ due to non-characteristicity, we can use the resolvent estimate [\(1.6\)](#).

$$\begin{aligned} \gamma \|\rho_\varepsilon * u_j\|_{L^2}^2 + |\rho_\varepsilon * u_j|_{L^2}^2 &\lesssim \frac{\|\rho_\varepsilon * f_j\|_{L^2}^2 + \|\rho_\varepsilon * [\psi_j, L_j] e^{-\gamma t} u \circ \varphi_j\|_{L^2}^2 + \|[\rho_\varepsilon *, L_j] u_j\|_{L^2}^2}{\gamma} \\ &\quad + |\rho_\varepsilon * g_j - [\rho_\varepsilon *, B] u_j|_{L^2}^2. \end{aligned}$$

Multiplying by $\varepsilon^{-2k-1} (1 + (\delta/\varepsilon)^2)^{-1}$, integrating in ε and using Friedrich's lemma we have

$$\gamma \|u_j\|_{L^2 H^{k-1,\delta}}^2 + |u_j|_{H^{k-1,\delta}}^2 \lesssim \frac{\|f_j\|_{H^k}^2 + \|[\psi_j, L_j] e^{-\gamma t} u \circ \varphi_j\|_{L^2 H^{k-1,\delta}}^2}{\gamma} + \|g_j\|_{H^k}^2. \quad (3.6)$$

estimtan

The commutator $[\psi_j, L_j]$ is the multiplication by a smooth matrix θ_j . Due to the special structure of the local maps, $\varphi_i^{-1} \circ \varphi_j$ has the form $(\varphi_{i,j}(y'), y_d)$ hence

$$\theta_j e^{-\gamma t} u \circ \varphi_j = \sum_1^J \psi_i \theta_j u_i(\varphi_{i,j}(y'), y_d) + \theta_j u_0 \circ \varphi_j.$$

Thanks to composition rules (in $H^{s,\delta}$, again see [\[5\]](#)),

$$\|[\psi_j, L_j] e^{-\gamma t} u \circ \varphi_j\|_{L^2 H^{k-1,\delta}} \lesssim \sum_{i=1}^J \|u_i\|_{L^2 H^{k-1,\delta}} + \|u_0\|_{H^{k-1,\delta}}$$

For γ large enough, this can be absorbed in (the sum over j of) the left-hand side of [\(3.6\)](#):

$$\sum_{j=1}^J \gamma \|u_j\|_{L^2 H^{k-1,\delta}}^2 + \|u_j\|_{H^{k-1,\delta}}^2 \lesssim \frac{\sum_1^J \|f_j\|_{H^k}^2 + \|u_0\|_{H^{k-1,\delta}}^2}{\gamma} + \sum_1^J |g_j|_{H^k}^2. \quad (3.7) \quad \text{estimtan2}$$

It seems ‘‘moral’’ that noncharacteristicity should imply the same bound for $\|u_j\|_{H^{k-1,\delta}}$, however the $H^{k-1,\delta}$ norm is a non local norm for functions defined on $\mathbb{R}^d \times \mathbb{R}_t$, hence such an assertion is not clear. Instead we first obtain interior estimates with similar, simpler computations

$$\gamma \|u_0\|_{H^{k-1,\delta}}^2 \lesssim \frac{\|f_0\|_{H^k}^2 + \|e^{-\gamma t} \widetilde{\psi}_0 u\|_{H^{k-1,\delta}}^2}{\gamma}, \quad \text{supp}(\widetilde{\psi}_0) \subset \Omega, \quad \widetilde{\psi}_0 \equiv 1 \text{ on } \text{supp}(\psi_0). \quad (3.8) \quad \text{interieur}$$

Decomposing again $\widetilde{\psi}_0 u = \sum_{j=0}^J \widetilde{\psi}_0 \psi_j u$, and following the same lines that led to [\(3.7\)](#),

$$\sum_{j=1}^J \gamma \|\widetilde{\psi}_0 \psi_j u \circ \varphi_j\|_{L^2 H^{k-1,\delta}}^2 \lesssim \frac{\sum_1^J \|f_j\|_{H^k}^2 + \|u_0\|_{H^{k-1,\delta}}^2}{\gamma} + \sum_1^J |g_j|_{H^k}^2. \quad (3.9)$$

A simple consequence of the definition of the $H^{s,\delta}$ spaces is that for any tangential differential operator D of order 1 and $s \geq 1$

$$\|Dv\|_{H^{s-2,\delta}} \lesssim \frac{1}{C} \|v\|_{H^{s-1,\delta}} + C \|v\|_{L^2 H^{s-1,\delta}}. \quad (3.10) \quad \text{transfert}$$

Now for $j \geq 1$, each function $\widetilde{\psi}_0 \psi_j u \circ \varphi_j$ is compactly supported in $\mathbb{R}^{d-1} \times \mathbb{R}^{+*} \times \mathbb{R}_t$, and on its support L_j is (uniformly) non characteristic, so we may extend it by zero for $y_d < 0$ and use [\(3.10\)](#) to deduce

$$\sum_{j=1}^J \gamma \|\widetilde{\psi}_0 \psi_j u \circ \varphi_j\|_{H^{k-1,\delta}}^2 \lesssim \frac{\|f_j\|_{H^k}^2 + \|u_0\|_{H^{k-1,\delta}}^2}{\gamma} + \sum_1^J |g_j|_{H^k}^2 + \gamma \|u\|_{H_\gamma^{k-1}}^2. \quad (3.11) \quad \text{interieur2}$$

Note that the term $\gamma \|e^{-\gamma t} u\|_{H_\gamma^{k-1}}^2$ is present due to the factor γ in the definition of L_j . Thanks to the induction assumption, this lower order term is bounded by $\|g\|_{H_\gamma^{k-1}}^2 + \|f\|_{H_\gamma^{k-1}}^2$. Putting together [\(3.7\)](#), [\(3.8\)](#), [\(3.11\)](#) we have

$$\left(\sum_1^J \|u_j\|_{L^2 H^{k-1,\delta}}^2 + \|u_0\|_{H^{k-1,\delta}}^2 \right) + \sum_1^J |u_j|_{H^{k-1,\delta}}^2 \lesssim \|e^{-\gamma t} f\|_{H^k}^2 + \|e^{-\gamma t} g\|_{H^k}^2.$$

Letting $\delta \rightarrow 0$ we have $u_j \in L^2 H^k$, $1 \leq j \leq J$ and $u_0 \in H^k$. We conclude that $u \in H^k$ again thanks to the uniform non characteristicity. \square

4 Smoothness of the IBVP: the integer case

We assume in this section that $(u_0, g, f) \in (H^k)^3$ satisfy the compatibility conditions (I.3) up to order k , and we prove theorem I.3 in these settings.

To prove that $u \in \cap_{j=0}^k C_t^j H^{k-j}$, the strategy is to use the regularity for the pure boundary value problem by subtracting an approximate solution (actually a Taylor expansion at $t = 0$) to u . For technical reasons, it is necessary to use much more regular data that satisfy compatibility conditions to higher order. The construction of such data requires the following lifting lemma that is also used in the next section.

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Lemma 4.1. *For $m \in \mathbb{N}$, there exists a lifting map $R_m : H^s(\partial\Omega) \rightarrow H^{m+s+1/2}(\partial\Omega \times \mathbb{R}_t)$, continuous for any $s > 0$ such that*

$$\partial_t^m R_m g|_{t=0} = g, \quad \partial_t^j R_m g|_{t=0} = 0, \quad j < m + s, \quad j \neq m - 1, \quad (4.1)$$

trace

and for $r < m + 1/2$, $\|R_m\|_{L^2 \rightarrow H^r} \ll 1$ is arbitrarily small.

Proof. Up to the use of local maps, the problem is reduced to $\partial\Omega = \mathbb{R}^{d-1}$, and to construct a lifting valued in $H^{m+s+1/2}(\mathbb{R}^{d-1} \times \mathbb{R}_t)$. The variables are denoted (x', t) .

We choose $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi^{(k)}(0) = 0$, $k \neq m$, $\chi^{(m)}(0) = 1$. We use the Fourier transform on $\mathbb{R}^{d-1} \times \mathbb{R}_t$ and denote ξ the dual variable of x' , τ the dual variable of t , and λ is a large parameter to fix later:

$$\widehat{R_m g} = \frac{\widehat{\chi}(\tau/(\lambda\langle\xi\rangle))}{(\lambda\langle\xi\rangle)^{m+1}} \widehat{g}(\xi), \quad \text{equivalently } \mathcal{F}_{x'}(R_m(g))(\xi, y) = \frac{\chi(\lambda t\langle\xi\rangle)}{\lambda^m \langle\xi\rangle^m} \widehat{g}(\xi), \quad \langle\xi\rangle = \sqrt{1 + |\xi|^2}.$$

The trace relations (4.1) are obvious from the second formula. The $H^{m+s+1/2}$ norm is easily bounded

$$\begin{aligned} \|R_m g\|_{H^{m+s+1/2}(\mathbb{R}^d)}^2 &= \int \frac{|\widehat{\chi}(\tau/(\lambda\langle\xi\rangle))|^2 |\widehat{g}|^2}{(\lambda\langle\xi\rangle)^{2(m+1)}} (\langle\xi\rangle^2 + \tau^2)^{m+s+1/2} d\xi d\tau \\ &= \int \frac{|\widehat{\chi}(\tau)|^2 |\widehat{g}|^2}{(\lambda\langle\xi\rangle)^{2(m+1)}} (\langle\xi\rangle^2 (1 + \lambda^2 \tau^2))^{m+s+1/2} d\xi \lambda \langle\xi\rangle d\tau \\ &\leq \int |\widehat{g}|^2 \langle\xi\rangle^{2s} \int |\widehat{\chi}(\tau)|^2 \frac{(1 + \lambda^2 \tau^2)^{m+s+1/2}}{\lambda^{2m+1}} d\tau d\xi \\ &\lesssim \lambda^{2s} \|g\|_{H^s}^2. \end{aligned}$$

With the same computation

$$\|\widehat{R_m g}\|_{H^r}^2 \leq \int \frac{|\widehat{g}|^2}{(\lambda\langle\xi\rangle)^{2(m-r)+1}} \int |\widehat{\chi}(\tau)|^2 (1 + \lambda^2 \tau^2)^r d\tau d\xi \lesssim \frac{\|g\|_{L^2}^2}{\lambda^{2(m-r)+1}}.$$

It is therefore sufficient to choose λ large enough to ensure the smallness of $\|R_m\|_{L^2 \rightarrow H^r}$. \square

compahaute

Lemma 4.2 (Construction of smooth compatible data). *Let $k \geq 0$, $(u_0, g, f) \in (H^k)^3$ satisfying the compatibility conditions up to order k . For any $m > k$, there exists $(u_{0,n}, g_n, f_n) \in (H^\infty)^3$ satisfying the compatibility conditions up to order m , and such that*

$$\|(u_0, g, f) - (u_{0,n}, g_n, f_n)\|_{(H^k)^3} \rightarrow 0.$$

Proof. By density of smooth functions, there exists a sequence $(u_{0,n}, g_n, f_n) \in (H^\infty)^3$ converging to (u_0, g, f) in $(H^k)^3$. We denote $v_{j,n}$ the corresponding functions in (1.2). For $j \geq 1$ the “compatibility error” is defined as

$$\varepsilon_{j,n} := \partial_t^{j-1} g_n|_{t=0} - \sum_{l=0}^{j-1} \binom{j}{l} (\partial_t^l B) v_{j-1-l,n}|_{\partial\Omega}.$$

Due to the compatibility conditions and continuity of traces we have

$$\forall 1 \leq j \leq k, \|\varepsilon_{j,n}\|_{H^{k-j+1/2}} \rightarrow_n 0.$$

As a consequence, given a lifting operator R_{j-1} as in lemma 4.1, $\|R_{j-1}\varepsilon_{j,n}\|_{H^k} \xrightarrow{\text{releve}} 0$. For $k < j \leq m$, $\varepsilon_{j,n}$ is not small in any Sobolev space, nevertheless from lemma 4.1 there exists a lifting $R_{j-1,n}$ such that $\|R_{j-1,n}\varepsilon_{j,n}\|_{H^k} \leq 1/n$. We then define

$$\tilde{g}_n := g_n - \sum_{j=1}^m R_{j-1,n}(\varepsilon_{j,n}).$$

This choice ensures that compatibility conditions are satisfied by $(u_{0,n}, \tilde{g}_n, f_n)$ up to order m and $\|\tilde{g}_n - g\|_{H^k} \rightarrow 0$. \square

Proof of theorem 1.3 (integer case) ^{mainth} We follow the notations of lemma 4.2; ^{compahaute} $v_{j,n}$ are smooth functions defined by (1.2) ^{taylor} for smooth data $(u_{0,n}, g_n, f_n)$. We define the approximate solution

$$u_{app,n}(x, t) = \sum_{j=0}^{m-1} \frac{t^j}{j!} v_{j,n}(x) \chi(t), \quad \chi \in C_c^\infty(\mathbb{R}^+), \quad \chi \equiv 1 \text{ near } 0.$$

We solve then

$$\begin{cases} Lw_n = f_n - Lu_{app,n}, \\ w_n|_{t=0} = 0, \\ Bw_n = g_n - Bu_{app,n}, \end{cases}$$

By construction, the data $(0, g_n - Bu_{app,n}, f_n - Lu_{app,n})$ are smooth and it is easily seen that $\partial_t^j (g_n - Bu_{app,n})|_{t=0} = 0$, $\partial_t^j (f_n - Lu_{app,n}) = 0$, $j \leq k+1$ provided $m \geq k+4$. Hence according to proposition 3.1, ^{regBVP} the solution w_n belongs to H^{k+2} , this implies by Sobolev embedding $w_n \in \cap_{j=0}^{k+1} C_t^j H^{k+1-j}$. Therefore $u_n := w_n + u_{app,n}$ is also in $\cap_{j=0}^{k+1} C_t^j H^{k+1-j}$, and it is a solution of (1.1) ^{IBVP} with data $(u_{0,n}, g_n, f_n)$.

Using a differentiation argument similar to the proof of proposition 3.1, ^{regBVP} but much simpler since no regularization is needed, we see that u_n satisfies ^{semireg} 1.9:

$$\sum_{j=0}^k \|\partial_t^j (e^{-\gamma t} u_n)\|_{C(\mathbb{R}^+, H^{k-j}(\Omega))} + |e^{-\gamma t} u_n|_{\partial\Omega}|_{H^k} \lesssim \left(\|u_{0,n}\|_{H^k(\Omega)} + |e^{-\gamma t} g_n|_{H^k(\partial\Omega \times [0, T])} + \|e^{-\gamma t} f_n\|_{H^k} \right),$$

as well as (1.7). ^{resolvreg} The same estimates, applied to $u_p - u_q$, $(p, q) \in \mathbb{N}^2$, shows that (u_n) is a Cauchy sequence in $\cap_{j=0}^k C_t^j H^{k-j}$, but since (u_n) converges (in L^2) to the solution u of (1.1) ^{IBVP} with

data (u_0, g, f) , this ensures that $u \in \cap_{j=0}^k C_t^j H^{k-j}$. The estimate $\stackrel{\text{resolvreg}}{(\text{I.7})}$ is then an elementary differentiation argument : tangential regularity is obtained directly by differentiation (which is now legal) and use of the L^2 estimate, while normal regularity uses the non characteristicity.

5 Regularity for positive s

For ease of presentation, we only detail the case $\Omega = \mathbb{R}^{d-1} \times \mathbb{R}^+$. The general case can be obtained by using a partition of unity as in the previous section.

In this section, we follow the (non standard) convention that H_0^s is $H_{00}^{1/2}$ if $s = 1/2$.

Under such settings, we can assume that A_d is invertible and A_d^{-1} is uniformly bounded. Furthermore since $B : \mathbb{R}^p \rightarrow \mathbb{R}^b$ has maximal rang b , there exists a smooth basis of $\text{Ker } B$ (as a smooth vector bundle over the contractible space $\mathbb{R}^{d-1} \times \mathbb{R}_t^+$) that we denote (k_1, \dots, k_{p-b}) . A basis $(v_j)_{1 \leq j \leq b}$ of $(\text{Ker } B)^\perp$ is then obtained easily:

$$\tilde{B} = \begin{pmatrix} B \\ k_1^t \\ \vdots \\ k_{p-b}^t \end{pmatrix} \text{ is an isomorphism } \mathbb{R}^p \rightarrow \mathbb{R}^p, \text{ we can choose } v_j = \tilde{B}^{-1}(e_j), \ 1 \leq j \leq b.$$

We remind that compatibility conditions of order $s = k + \theta$, $k \in \mathbb{N}^*$, $0 < \theta < 1$ are defined as follows:

1. If $\theta < 1/2$, then compatibility conditions $\stackrel{\text{CCj}}{(\text{I.3})}$ up to order k are satisfied.
2. If $\theta > 1/2$, then compatibility conditions $\stackrel{\text{CCj}}{(\text{I.3})}$ up to order $k + 1$ are satisfied.
3. If $\theta = 1/2$, compatibility conditions up to order k are satisfied and

$$\int_{\mathbb{R}^{d-1}} \left| \partial_t^{k-1} g(x', y) - \sum_{j=0}^{k-1} \binom{k-1}{j} (\partial_t^j B) (A_{k-1-j} u_0 + B_{k-1-j} f|_{t=0}) |(x', y) \right|^2 \frac{dy}{y} < \infty.$$

The case $0 < s < 1$ From the previous section, the map $(u_0, g, f) \rightarrow u$ solution of $\stackrel{\text{IBVP}}{(\text{I.1})}$ is continuous

$$\begin{aligned} X_0 \times L^2 &:= (L^2)^3 \rightarrow C_t L^2 \text{ and} \\ X_1 \times H^1 &:= \{(u_0, g) \in (H^1)^2 : B u_0|_{\partial\Omega} = g|_{t=0}\} \times H^1 \rightarrow C_t H^1 \cap C_t^1 L^2. \end{aligned}$$

Let us define for $0 \leq \theta \leq 1$

$$X_\theta = \left\{ (u_0, g) \in (H^\theta)^2 : \text{the compatibility condition of order } \theta \text{ is satisfied} \right\},$$

(note that compatibility conditions of order less than $3/2$ do not involve f).

Both the semi-group estimate $\stackrel{\text{semigreg}}{(\text{I.9})}$ and the resolvent estimate $\stackrel{\text{resolvreg}}{(\text{I.7})}$ follow from an interpolation argument if we can prove that

$$X_\theta = [X_0, X_1]_\theta. \tag{5.1} \quad \boxed{\text{interpX}}$$

More precisely, since the resolvent estimate implies for $s = 0, 1$

$$\begin{aligned} \gamma \|u\|_{L^2_\gamma}^2 + \|u|_{\partial\Omega}\|_{L^2_\gamma}^2 &\lesssim \|(u_0, e^{-\gamma t}g)\|_{X_0}^2 + \frac{\|f\|_{L^2_\gamma}^2}{\gamma} \\ \gamma \|u\|_{H^1_\gamma}^2 + \|u|_{\partial\Omega}\|_{H^1_\gamma}^2 &\lesssim C(\gamma)\|(u_0, e^{-\gamma t}g)\|_{X_1}^2 + \frac{\|f\|_{H^1_\gamma}^2}{\gamma}, \end{aligned}$$

the interpolation identity (5.1) implies

$$\gamma \|u\|_{H^\theta_\gamma}^2 + \|u|_{\partial\Omega}\|_{H^\theta_\gamma}^2 \lesssim C'(\gamma)\|(u_0, e^{-\gamma t}g)\|_{X_\theta}^2 + \frac{\|f\|_{H^\theta_\gamma}^2}{\gamma}. \quad (5.2)$$

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(a better estimate would require to use weighted X^θ spaces, a course that we chose not to follow).

Proof of (5.1) We extend \tilde{B} on $\Omega \times \mathbb{R}^+$ as $\tilde{B}(x', y, t) = \tilde{B}(x', t)$, and consider the map $u_0 \rightarrow \tilde{B}u_0 := \tilde{u}_0$. It is an isomorphism $(H^s(\Omega))^p \rightarrow (H^s(\Omega))^p$, and the compatibility condition can be rewritten

$$Bu_0|_{\partial\Omega} = g|_{t=0} \Leftrightarrow B\tilde{B}^{-1}\tilde{B}u_0|_{\partial\Omega} = g|_{t=0} \Leftrightarrow (I_b \ 0) \tilde{u}_0|_{\partial\Omega} = g|_{t=0}, \text{ with } \tilde{u}_0 = \tilde{B}u_0.$$

This transformation “diagonalizes” (5.1), and we are reduced to determine

$$[L^2 \times L^2, H^1 \times H^1]_\theta, \text{ and } [L^2 \times L^2, \{(u_0, g) \in H^1 \times H^1 : u_0|_{y=0} = g|_{t=0}\}]_\theta = [Y_0, Y_1]_\theta,$$

where u_0 and g are now *scalar* functions.

Of course, it is well-known that $[L^2, H^1]_\theta = H^\theta$, so the first case is immediate. In the second case, surprisingly, we were not able to find results in the litterature except in the simplest case $\theta < 1/2$, which is in [13] section 14.

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Lemma 5.1. For $\theta < 1/2$, $[Y_0, Y_1]_\theta = Y_\theta$.

Proof. The following inclusions are clear : $H_0^1 \times H_0^1 \subset Y_1 \subset H^1(\Omega) \times H^1(\partial\Omega \times \mathbb{R}^+)$. On the other hand, for $\theta < 1/2$ we have $[L^2, H_0^1]_\theta = H^\theta$ ([12], chapter 1 section 11), and we can conclude

$$H^\theta \times H^\theta = [L^2 \times L^2, H_0^1 \times H_0^1]_\theta \subset [Y_0, Y_1]_\theta \subset [L^2 \times L^2, H^1 \times H^1]_\theta = H^\theta \times H^\theta. \quad \square$$

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Lemma 5.2. For $0 < \theta \leq 1$, there exists an universal (independent of θ) operator R

$$R : Y_\theta \rightarrow H^{\theta+1/2}(\Omega \times \mathbb{R}^+), \quad \forall 0 < \theta \leq 1.$$

Proof. This is a result due to Grisvard [7], for completeness we include a simple proof. Given $(u_0, g) \in (H^\theta)^2$, from lemma 4.1 there exists an opertor $R_b : g \rightarrow R_b(g) \in H^{\theta+1/2}$ which is independent of θ . By construction, $R_b g|_{t=0} - u_0 \in H_0^\theta$. If $\theta = 1/2$, we also notice

$$R_b g(x', y, 0) - u_0(x', y) = \underbrace{R_b g(x', y, 0) - g(x', y)}_{H_0^{1/2} \text{ by interpolation}} + \underbrace{g(x', y) - u_0(x', y)}_{H_0^{1/2} \text{ by assumption}}.$$

If there exists an universal lifting $R_0 : H_0^\theta(\Omega) \rightarrow \{u \in H^{\theta+1/2}(\Omega \times \mathbb{R}^+) \mid u|_{\partial\Omega} = 0\}$, R can be defined as $R(u_0, g) = R_b g + R_0(u_0 - R_b g|_{t=0})$ so we focus on the construction of R_0 .

For $u_0 \in H_0^\theta$ ($H_{00}^{1/2}$ for $\theta = 1/2$), we extend it as an odd function of y , $I(u_0)$ defined on \mathbb{R}^d . The map $I : H_0^\theta(\mathbb{R}^{d-1} \times \mathbb{R}^+) \rightarrow H^\theta(\mathbb{R}^d)$ is continuous as it is clearly the case for $\theta = 0, 1$. Define now

$$R_I(\widehat{I(u_0)})(\xi, \delta) = \chi(\langle \xi \rangle t) \widehat{I(u_0)}(\xi),$$

where χ is as in lemma [4.1](#). According to the proof of lemma [4.1](#), $R_I \circ I : H^\theta \rightarrow H^{\theta+1/2}(\mathbb{R}^d \times \mathbb{R}^+)$ is continuous, moreover by construction $R_I \circ I(u_0)$ is an odd function of y , therefore necessarily $R_I \circ I(u_0)|_{y=0} = 0$. Thus by taking the restriction on $\mathbb{R}^{d-1} \times \mathbb{R}_y^+ \times \mathbb{R}_t^+$, $R_0 := R_I \circ I$ solves the problem. \square

Proposition 5.3. For $0 < \theta < 1$, $[Y_0, Y_1]_\theta = Y_\theta$.

Proof. On one hand, the map $(u_0, g) \rightarrow u_0(x', y) - g(x', y)$ is continuous $Y_i \rightarrow H_0^i$ for $i = 0, 1$, therefore by interpolation it is continuous $[Y_0, Y_1]_\theta \rightarrow H_0^\theta$. This gives the first inclusion

$$[Y_0, Y_1]_\theta \subset Y_\theta. \quad (5.3) \quad \boxed{\text{inclusion}}$$

On the other hand, from Lions-Peetre reiteration theorem, for any $0 < s, \theta < 1$

$$[[Y_0, Y_1]_s, Y_1]_\theta = [Y_0, Y_1]_{\theta+s(1-\theta)}.$$

If we have for some $s < 1/2$, $[Y_s, Y_1]_\theta \supset Y_{\theta+s(1-\theta)}$ for any $0 < \theta < 1$, then by reiteration [this](#) implies $[Y_0, Y_1]_\theta = Y_\theta$ for $\theta > s$. On the other hand, the case $\theta \leq s$ is contained in lemma [5.1](#).

For any $0 < r < 1$ we define the map

$$u \in H^{r+1/2}(\Omega \times \mathbb{R}_t^+) \rightarrow \text{Tr}(u) = (u|_{t=0}, u|_{y=0}).$$

It is easily seen that Tr is continuous $H^{3/2} \rightarrow Y_1$ and $H^{1/2+s} \rightarrow Y_s$ for $0 < s < 1/2$. As it is well known that $[H^{s+1/2}, H^{3/2}]_\theta = H^{1/2+\theta+(1-\theta)s}$, we deduce by interpolation

$$\text{Tr} : H^{1/2+(1-\theta)s+\theta} = [H^{s+1/2}, H^{3/2}]_\theta \rightarrow [Y_s, Y_1]_\theta \text{ is continuous.}$$

We observe now that the lifting R from lemma [5.2](#) is a right inverse for Tr : for fixed $0 < s < 1/2$ and any $0 < \theta < 1$, we have $\text{Tr} \circ R = I_d : Y_{\theta+s(1-\theta)} \rightarrow Y_{\theta+s(1-\theta)}$. Since R maps $Y_{\theta+s(1-\theta)}$ to $H^{s(1-\theta)+\theta+1/2}$, this implies

$$Y_{\theta+s(1-\theta)} \subset [Y_s, Y_1]_\theta,$$

which was the required converse inclusion. \square

The case $s > 1$ We denote $s = k + \theta$, $0 \leq \theta < 1$. According to the integer case, we already have $u \in \cap C^{k-j} H^j$. For any tangential multi-index α of order k (that is, $\alpha_d = 0, |\alpha| = k$), $\partial^\alpha u$ satisfies

$$\begin{cases} L(\partial^\alpha u) = \partial^\alpha f + [L, \partial^\alpha]u, \\ B\partial^\alpha u|_{\partial\Omega} = \partial^\alpha g + [B, \partial^\alpha]u|_{\partial\Omega}, \\ \partial^\alpha u|_{t=0} = L_\alpha(u_0) + L'_\alpha(f)|_{t=0}. \end{cases} \quad (5.4) \quad \boxed{\text{derBVP}}$$

where L_α, L'_α are differential operators of respective order $\alpha, \alpha - 1$. Regularity will again be obtained by regularization of the data, we distinguish three cases:

The case $0 < \theta < 1/2$ With the same argument as in the integer case (note that the condition $\theta < 1/2$ allows to use lemma 4.1), there exists regularized data $(u_{0,n}, g_n, f_n) \in (H^{k+1})^3$, converging to (u_0, g, f) that satisfy the compatibility conditions up to order $k+1$. The corresponding solution u_n belongs to $\cap_0^{k+1} C_t^j H^{k+1-j}$ so that we may apply the resolvent estimate (1.7) to $\partial^\alpha u_n$ with $s = \theta$, combined with basic trace estimates and the commutator estimate $\|[\partial^\alpha, L]u_n\|_{H_\gamma^\theta} \lesssim \|u_n\|_{H_\gamma^s}$:

$$\gamma \|\partial^\alpha u_n\|_{H_\gamma^\theta}^2 \lesssim \|u_{0,n}\|_{H^s}^2 + \|g_n\|_{H_\gamma^s}^2 + \frac{\|f_n\|_{H_\gamma^s}^2 + \|u_n\|_{H_\gamma^s}^2}{\gamma}.$$

Due to the boundary being non characteristic, we deduce as for the integer case (note that the fractional regularity gained here includes conormal regularity) for γ large enough only depending on s

$$\gamma \|u_n\|_{H_\gamma^s}^2 \lesssim \|u_{0,n}\|_{H^s}^2 + \|g_n\|_{H_\gamma^s}^2 + \frac{\|f_n\|_{H_\gamma^s}^2}{\gamma}.$$

With the resolvent estimate available, the semi group estimate is now an immediate consequence of the case $0 < s < 1$ applied to (5.4):

$$\begin{aligned} \|e^{-\gamma t} \partial^\alpha u_n\|_{C_t H^\theta}^2 &\lesssim \|u_{0,n}\|_{H^s(\Omega)}^2 + \|f_n\|_{H_\gamma^s([0,T] \times \Omega)}^2 + \|[\partial^\alpha, L]u_n\|_{H_\gamma^\theta}^2 + \|g_n\|_{H_\gamma^s}^2 \\ &\lesssim \|u_{0,n}\|_{H^s(\Omega)}^2 + \|f_n\|_{H_\gamma^s([0,T] \times \Omega)}^2 + \|g_n\|_{H_\gamma^s}^2. \end{aligned}$$

Once more, normal regularity is then obtained thanks to the boundary being non characteristic. Letting $n \rightarrow \infty$, we deduce that $e^{-\gamma t} u$ is in $H^s(\mathbb{R}^+ \times \Omega) \cap (\cap_{j=0}^k C^j(\mathbb{R}^+, H^{s-j}(\Omega)))$ and satisfies the semi group estimate and the resolvent estimate.

The case $1/2 < \theta < 1$ This can be done with exactly the same argument. Actually, the construction of regularized data $(u_{0,n}, g_n, f_n) \in (H^{k+1})^3$ that satisfy compatibility conditions up to order $k+1$ and converging to (u_0, g, f) in $(H^s)^3$ is even simpler. Indeed (u_0, g, f) satisfy compatibility conditions up to order $k+1$, hence any regularization of (u_0, g, f) satisfies

$$\forall 1 \leq j \leq k+1, \left\| \underbrace{\partial_t^{j-1} g_n|_{t=0} - \sum_{l=0}^{k-1} \binom{j}{l} (\partial_t^l B) v_{j-1-l,n}|_{\partial\Omega}}_{:=\varepsilon_{j,n}} \right\|_{H^{s-j+1/2}} \rightarrow_n 0,$$

and it suffices to modify g_n as $g_n - \delta_n$ where δ_n is a function in $H^{k+1}(\partial\Omega \times \mathbb{R}_t^+)$ that satisfies for $1 \leq j \leq k+1$, $\partial_t^{j-1} \delta_n|_{t=0} = \varepsilon_{j,n}$

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The case $\theta = 1/2$ When $s = k+1/2$, the compatibility conditions are satisfied in particular up to order k . From the previous study, we have $e^{-\gamma t} u \in (\cap_{j=0}^k C_t^j H^{k+\theta-j}) \cap H^{k+\theta}$ for any $\theta < 1/2$, with the estimate

$$\|e^{-\gamma t} u\|_{(\cap_{j=0}^k C_t^j H^{j+\theta-j})} + \|e^{-\gamma t} u\|_{H^{k+\theta}} \leq C(\theta) \|(u_0, g, f)\|_{(H^s)^3}.$$

Of course this is not enough to conclude, but the estimate can be sharpened: apply estimate ^{derBVP}(5.2) to ^{derBVP}(5.4) for $\theta < 1$ and any tangential multi-index $\alpha \in \mathbb{N}^d$, $|\alpha| = k$, this reads

$$\gamma \|\partial^\alpha u\|_{H_\gamma^\theta}^2 \lesssim \|(L_\alpha u_0 + L'_\alpha f|_{t=0}, e^{-\gamma t}(\partial^\alpha g + [B, \partial^\alpha]u|_{\partial\Omega}))\|_{X_\theta}^2 + \frac{\|f\|_{H_\gamma^{k+\theta}}^2 + \|u\|_{H_\gamma^{k+\theta}}^2}{\gamma}.$$

Recall that the compatibility conditions at order j are are

$$\forall 1 \leq j \leq k, \partial_t^{j-1} g|_{t=0} - \sum_{l=0}^{j-1} \binom{j}{l} (\partial_t^l B) v_{j-1-l}|_{\partial\Omega} = 0,$$

and at order $k + 1/2$

$$\partial_t^k g(x', t) - \left(\sum_{l=0}^k \binom{k}{l} (\partial_t^l B) v_{k-1-l}(x', t) \right) \in H_{00}^{1/2}(\mathbb{R}^{d-1} \times (\mathbb{R}^+)).$$

As a consequence, for any $j \leq k + 1$ and any $\beta \in \mathbb{N}^{d-1}$, $|\beta| = k + 1 - j$,

$$\partial_{x'}^\beta \partial_t^{j-1} g(x', t) - \partial_{x'}^\beta \sum_{l=0}^{j-1} \binom{j-1}{l} (\partial_t^l B) v_{j-1-l}(x', t) \in H_{00}^{1/2}(\mathbb{R}^{d-1} \times \mathbb{R}^+). \quad (5.5) \quad \boxed{\text{compfrac}}$$

Furthermore, $e^{-\gamma t} u \in H^k(\Omega \times \mathbb{R}^+)$, hence for any multi-index of order $k - 1$

$$\|e^{-\gamma t} \partial^\alpha u|_{y=0} - e^{-\gamma t} \partial^\alpha u|_{t=0}\|_{H_{00}^{1/2}(\mathbb{R}^{d-1} \times \mathbb{R}^+)} \lesssim \|e^{-\gamma t} u\|_{H^k(\mathbb{R}^{d-1} \times (\mathbb{R}^+)^2)}. \quad (5.6) \quad \boxed{\text{compagratuite}}$$

Now to make ^{derBVP}(5.4) more explicit, let us write $\partial^\alpha = \partial_t^j \partial_{x'}^\beta$, $\beta \in \mathbb{N}^{d-1}$, $|\beta| = k - j$. Then $\partial^\alpha u|_{t=0} = \partial_{x'}^\beta v_j \in H^{1/2}(\mathbb{R}^{d-1} \times \mathbb{R}^+)$, the compatibility condition of order $1/2$ for ^{derBVP}(5.4) is thus

$$e^{-\gamma t} (\partial^\alpha g + [B, \partial^\alpha]u|_{\partial\Omega}) - B \partial_{x'}^\beta v_j \in H_{00}^{1/2}(\mathbb{R}^{d-1} \times \mathbb{R}^+).$$

With basic computations, we now check that it is implied by ^{compfrac}(5.5), ^{compagratuite}(5.6):

$$\begin{aligned} & e^{-\gamma t} (\partial^\alpha g + [B, \partial^\alpha]u|_{\partial\Omega}) - B \partial_{x'}^\beta v_j \\ &= e^{-\gamma t} \left(\partial^\alpha g - \partial_{x'}^\beta \sum_{l=0}^j \binom{j}{l} (\partial_t^l B) \partial_t^{j-l} u|_{\partial\Omega} \right) + e^{-\gamma t} B \partial^\alpha u|_{\partial\Omega} - B \partial_{x'}^\beta v_j \\ &= e^{-\gamma t} \partial^\alpha g - \partial_{x'}^\beta \sum_{l=1}^j \binom{j}{l} (\partial_t^l B) v_{j-l} - B \partial_{x'}^\beta v_j \\ &\quad - \partial_{x'}^\beta \sum_{l=1}^j \binom{j}{l} (\partial_t^l B) (e^{-\gamma t} \partial_t^{j-l} u|_{\partial\Omega} - v_{j-l}) \\ &\quad - \partial_{x'}^\beta (B e^{-\gamma t} \partial_t^j u|_{\partial\Omega}) + e^{-\gamma t} B \partial^\alpha u|_{\partial\Omega} \\ &= e^{-\gamma t} \left(\partial^\alpha g - \partial_{x'}^\beta \sum_{l=0}^j \binom{j}{l} (\partial_t^l B) v_{j-l} \right) \\ &\quad - \partial_{x'}^\beta \sum_{l=1}^j \binom{j}{l} (\partial_t^l B) (e^{-\gamma t} \partial_t^{j-l} u|_{\partial\Omega} - v_{j-l}) \\ &\quad + [B, \partial_{x'}^\beta] e^{-\gamma t} \partial_t^j u|_{\partial\Omega} - [B, \partial_{x'}^\beta] v_j. \end{aligned}$$

For $j \leq k$, due to the compatibility condition (5.5), the first line in the last equality is in $H_{00}^{1/2}$. The $H_{00}^{1/2}$ norm of the second line is easily controlled by writing

$$e^{-\gamma t} \partial_t^{j-l} u|_{\partial\Omega} - v_{j-l} = e^{-\gamma t} (\partial_t^{j-l} u|_{\partial\Omega} - v_{j-l}) + (1 - e^{-\gamma t}) v_{j-l},$$

the first term can be bounded thanks to (5.6) while for the second we simply use $(1 - e^{-\gamma t})/t \lesssim 1$. The same argument is used for the third line. We deduce that for $\theta < 1/2$, α tangential, $|\alpha| \leq k$

$$\begin{aligned} \gamma \|\partial^\alpha u\|_{H_\gamma^\theta(\Omega \times \mathbb{R}_t^+)}^2 &\lesssim C(\gamma) \left(\|(u_0, g, f)\|_{(H^{k+1/2})^3} + \left\| g - \sum_0^k \binom{k}{l} (\partial_t^l B) v_{k-1-l} \right\|_{H_{00}^{1/2}(\mathbb{R}^{d-1} \times \mathbb{R}^+)} \right) \\ &\quad + \frac{\|u\|_{H_\gamma^{k+\theta}}^2}{\gamma}. \end{aligned}$$

Using non characteristicity, we recover

$$\gamma \|u\|_{H_\gamma^{k+\theta}(\Omega \times \mathbb{R}_t^+)}^2 \lesssim \|(u_0, g, f)\|_{(H^{k+1/2})^3} + \left\| g - \sum_0^k \binom{k}{l} (\partial_t^l B) v_{k-1-l} \right\|_{H_{00}^{1/2}(\mathbb{R}^{d-1} \times \mathbb{R}^+)}.$$

This estimate is uniform in $\theta < 1/2$, we deduce that the same estimate holds for $\theta = 1/2$. Finally we deduce that the semi group estimate is true with the same argument as for the end of the case $0 < \theta < 1/2$. This ends the proof of theorem [1.3](#).

References

- [1] Herbert Amann. *Anisotropic function spaces and maximal regularity for parabolic problems. Part 1*. Jindřich Nečas Center for Mathematical Modeling Lecture Notes, 6. Matfyzpress, Prague, 2009. Function spaces.
- [2] Corentin Audiard. On mixed initial-boundary value problems for systems that are not strictly hyperbolic. *Appl. Math. Lett.*, 24(5):757–761, 2011.
- [3] Corentin Audiard. Global Strichartz estimates for the Schrödinger equation with non zero boundary conditions and applications. *Ann. Inst. Fourier (Grenoble)*, 69(1):31–80, 2019.
- [4] Sylvie Benzoni-Gavage and Denis Serre. *Multidimensional hyperbolic partial differential equations*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, Oxford, 2007. First-order systems and applications.
- [5] Jacques Chazarain and Alain Piriou. *Introduction à la théorie des équations aux dérivées partielles linéaires*. Gauthier-Villars, Paris, 1981.
- [6] K. O. Friedrichs. Symmetric positive linear differential equations. *Comm. Pure Appl. Math.*, 11:333–418, 1958.
- [7] P. Grisvard. Caractérisation de quelques espaces d'interpolation. *Arch. Rational Mech. Anal.*, 25:40–63, 1967.

- [8] Lars Hörmander. Pseudo-differential operators and non-elliptic boundary problems. *Ann. of Math. (2)*, 83:129–209, 1966.
- [9] Tatsuo Iguchi and David Lannes. Hyperbolic free boundary problems and applications to wave-structure interactions. *Indiana Univ. Math. J.*, 70(1):353–464, 2021.
- [10] Heinz-Otto Kreiss. Initial boundary value problems for hyperbolic systems. *Comm. Pure Appl. Math.*, 23:277–298, 1970.
- [11] P. D. Lax and R. S. Phillips. Local boundary conditions for dissipative symmetric linear differential operators. *Comm. Pure Appl. Math.*, 13:427–455, 1960.
- [12] J.-L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications. Vol. 1.* Travaux et Recherches Mathématiques, No. 17. Dunod, Paris, 1968.
- [13] J.-L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications. Vol. 2.* Travaux et Recherches Mathématiques, No. 18. Dunod, Paris, 1968.
- [14] G. Métivier. Stability of multidimensional shocks. In *Advances in the theory of shock waves*, volume 47 of *Progr. Nonlinear Differential Equations Appl.*, pages 25–103. Birkhäuser Boston, Boston, MA, 2001.
- [15] Guy Métivier. The block structure condition for symmetric hyperbolic systems. *Bull. London Math. Soc.*, 32(6):689–702, 2000.
- [16] Guy Métivier. On the L^2 well posedness of hyperbolic initial boundary value problems. *Ann. Inst. Fourier (Grenoble)*, 67(5):1809–1863, 2017.
- [17] Jeffrey Rauch. \mathcal{L}_2 is a continuable initial condition for Kreiss’ mixed problems. *Comm. Pure Appl. Math.*, 25:265–285, 1972.
- [18] Jeffrey B. Rauch and Frank J. Massey, III. Differentiability of solutions to hyperbolic initial-boundary value problems. *Trans. Amer. Math. Soc.*, 189:303–318, 1974.
- [19] David S. Tartakoff. Regularity of solutions to boundary value problems for first order systems. *Indiana Univ. Math. J.*, 21:1113–1129, 1971/72.