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### **► To cite this version:**

Marco Ripà, Emanuele Dalmasso. Patterns related to the Smarandache circular sequence primality problem. Notes on Number Theory and Discrete Mathematics, 2012, Notes on Number Theory and Discrete Mathematics, 18 (1), pp.29-48. <hal-03703339v2>

**HAL Id: hal-03703339**

**<https://hal.science/hal-03703339v2>**

Submitted on 16 Jul 2024

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# Patterns related to the Smarandache circular sequence primality problem

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**Abstract:** In this paper, we show the internal relations among the elements of the circular sequence (1, 12, 21, 123, 231, 312, 1234, 2341, 3412, ...). We illustrate one method to minimize the number of the “candidate prime numbers” up to a given term of the sequence. So, having chosen a particular prime divisor, it is possible to analyze only a fixed number of the smallest terms belonging to a given range, thus providing the distribution of that prime factor in a larger set of elements. Finally, we combine these results with another one, expanding the study to a few new integer sequences related to the circular one.

**Keywords:** Recurrence relations, Factorization, Patterns, Integer sequences, Permutations, Primes.

**2010 Mathematics Subject Classification:** 11B50 (Primary) 65Q30, 11Y05, 11Y11 (Secondary).

## 1 Introduction and main results

We introduce a few notations to indicate (synthetically) some special groups of terms of the sequence [A001292](#) of Sloane’s On-Line Encyclopedia [8]. First of all, we have to explain what the circular sequence is [9, 12].

**Definition 1.1** *Given the  $n$ -th term of the Smarandache consecutive sequence [A007908](#), constructed through the juxtaposition of the first  $n$  integers, we define as “circular permutations” the  $n$  elements that constitute the subset of the permutations of the form:*

$$\begin{aligned} a_{j-i+1} &:= 2\_3\_4\_ \dots \_n\_1 & \Rightarrow p_1 = 1, \\ a_{j-i+2} &:= 3\_4\_ \dots \_n\_1\_2 & \Rightarrow p_2 = 2, \\ &\vdots \\ a_j &:= (i+1)\_(i+2)\_ \dots \_(n-1)\_n\_1\_2\_ \dots \_(i-1)\_i & \Rightarrow p_i = i, \\ &\vdots \\ a_{j-i+n} &:= 1\_2\_ \dots \_(n-1)\_n & \Rightarrow p_n = n \end{aligned}$$

Now, we have to clarify the two parameters necessary to isolate a given class of elements from the rest [7] (see Fig. 1).



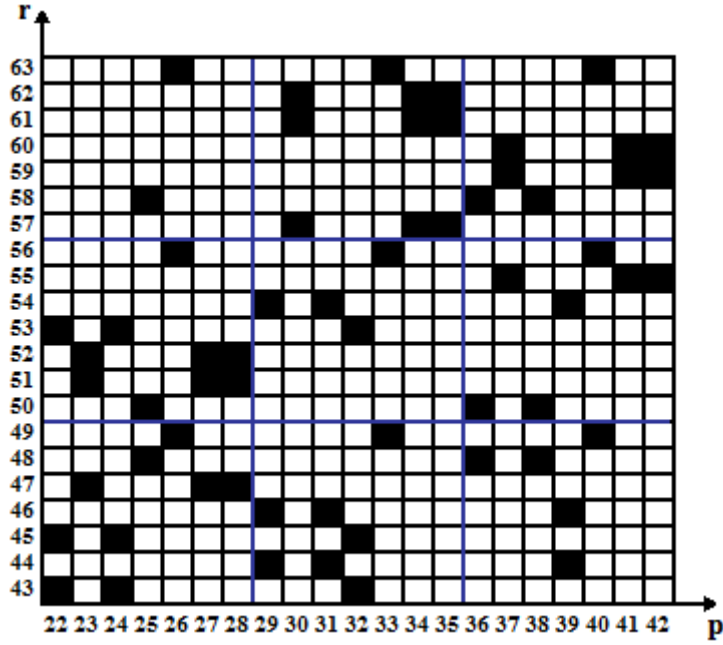


Figure 2: Pattern obtained using  $pr = 7$  and  $d(r) = d(p) = 2$ .

In this case,  $d(r) = d(p) \Rightarrow k = q = 7 \cdot m$ , particularly  $m = 3$ . The grid shows that each column and each row contains three black squares, but this is not a general law. It is possible to deduce that the total number of the terms of the sequence, divisible by a given “ $pr$ ” related to a pattern that recurs more than once in the chosen  $d(r) = \text{constant}$  interval, is at least  $\frac{1}{pr}$ . Under the previous restriction on  $pr$ , the non-strict inequality above includes all the possible combinations of  $d(r)$  and  $d(p)$ . So we could exploit this relation to synthesize a probabilistic formula that tries to estimate the ratio of “candidate prime terms” (the elements that are not divisible by the prime factors taken into account in the pattern analysis) in comparison with the numerosness of the elements belonging to the  $d(r) = \text{constant}$  subset of the circular sequence:

$$0.1\bar{3} - 0.1\bar{3} \cdot \frac{1}{7} - 0.1\bar{3} \cdot \frac{6}{7} \cdot \frac{1}{11} - 0.1\bar{3} \cdot \frac{6}{7} \cdot \frac{10}{11} \cdot \frac{1}{13} - \dots = 0.1\bar{3} \cdot \left[ \frac{6}{7} - \sum_{i=5}^h \frac{1}{pr_i} \cdot \prod_{j=4}^{i-1} \left( 1 - \frac{1}{pr_j} \right) \right]. \quad (1)$$

The index  $h$  refers to the  $h$ -th element of the prime number sequence **A000040**,  $pr_j := 2, 3, 5, 7, 11, 13, 17, \dots$  for  $j = 1, 2, 3, 4, 5, 6, 7, \dots$  so the first term of the summation above is  $pr_5 := 11$ , whereas the products start from  $pr_4 := 7$ .

Specifically referring to the patterns’ repetitiveness, the concrete constraint is that  $pr_h$  is forced to be less than  $9 \cdot 10^{d(r)-1}$ <sup>3</sup>.

Moreover, excluding from the previous formula the  $pr_j$  for which the corresponding pattern does not repeat at least once, for the given value of  $d(r)$ , we get a probabilistic overestimation of the candidate

<sup>3</sup>In Section 2, this result will be combined with another one to further reduce the size of the candidate prime numbers set.

primes ratio <sup>4</sup> (in such a case, we have to consider only the multiplication factors linked to the primes that could be used in the pattern analysis, while the remaining subset **A000040** must be deleted from the estimation formula (1)). To show an example of a pattern that does not respect the previous prediction, we can take  $pr = 11$  and  $d(r) = d(p) = 2$  (see Fig. 3).

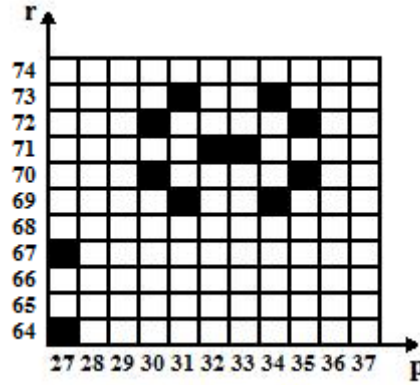


Figure 3: Pattern obtained using  $pr = 11$  and  $d(r) = d(p) = 2$ .

We could also think of the implicit result [7] related to  $pr = 37$  and  $r = 3$ . In that case  $m = 1$ , but there are  $37 \cdot 2$  terms divisible by 37 for each  $37 \times 37$  grid of elements. In fact, there are two different values of  $100 \leq r < 137$  such that, for each possible value of  $p \leq r$ , the factorization of  $a_j := (p+1) \dots (r-1) \cdot r \cdot 1 \cdot 2 \cdot 3 \dots (p-1) \cdot p$  includes the prime factor 37.

Section 4 contains a random collection of other patterns of the same kind, for  $pr \in \{7, 11, 13, 37\}$  and  $d(r) \leq 4$  (Figures 6 to 20).

An obvious consequence is that, once we have detected the value of  $q := q(pr, d(r), d(p))$  for a given  $(pr^*, d(r \leq r^*))$ , we automatically know what is the value of  $k := k(pr^*, d(r), d(p))$  for the whole set of terms identified by  $d(r) \leq d(r)^*$  (reminder:  $d(p) \leq d(r), \forall pr$ ). The final result is that, under the previous conditions we have stated,  $k$  is implicit in  $q$ . Another way to formalize the same result is to postulate that, for a given pair  $(pr, d(p))$ , the value of  $k$  is constant at the variation of the parameter  $r$ . This is graphically shown in Fig. 4.

<sup>4</sup>Due to the inequality stated a few lines before.

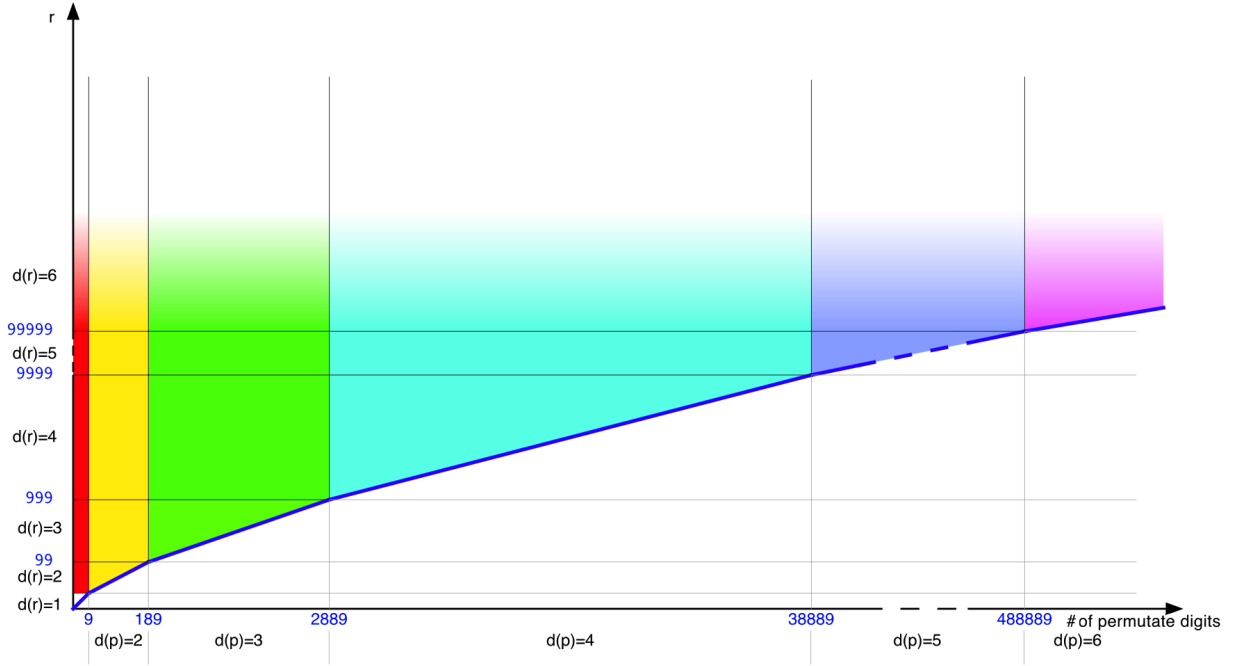


Figure 4: In this graph, sectors of the same color represent sets of elements characterized by an identical value of  $k$ .

There is another outcome (quite easy to demonstrate). Let  $pr = pr^*$  (i.e., any prime number). Taking two generic terms of the consecutive sequence, say  $a' := (p = n', r = n') \in M(n')$  and  $a'' := (p = n'', r = n'') \in M(n'')$  – where  $n' > n''$  –, which are both divisible by  $pr^*$ , the following element of the circular sequence is also divisible by  $pr^*$ :

$$a''' := (n'' + 1)_{-}(n'' + 2)_{-} \dots_{-}(n' - 1)_{-} n'_{-} 1_{-} 2_{-} 3_{-} \dots_{-}(n'' - 1)_{-} n'' \in M(n')$$

In fact, the number  $(n'' + 1)_{-} \dots_{-}(n' - 1)_{-}$  is divisible by  $pr^*$  as well.

Clearly we can be sure that  $pr^*$  divides the terms

$$(n'' + 1)_{-}(n'' + 2)_{-} \dots_{-}(n' - 1)_{-} n'_{-} \dots_{-}(n' + q - 1)_{-}(n' + q)_{-} 1_{-} 2_{-} 3_{-} \dots_{-}(n'' - 1)_{-} n'' \in M(n' + q),$$

$$(n'' - k + 1)_{-} \dots_{-}(n' - 1)_{-} n'_{-} 1_{-} 2_{-} 3_{-} \dots_{-}(n'' - k - 1)_{-}(n'' - k) \in M(n'), \text{ and}$$

$$(n'' - k + 1)_{-} \dots_{-} n'_{-}(n' + 1)_{-} \dots_{-}(n' + q)_{-} 1_{-} 2_{-} 3_{-} \dots_{-}(n'' - k - 1)_{-}(n'' - k) \in M(n' + q)$$

if and only if  $d(p)$  and  $d(r)$  will remain unchanged.

In this way, knowing the distribution of a small prime factor inside a portion of the consecutive sequence, we are able to exclude another subset of non-prime elements from the circular one. These data allow us to rule out some terms of the circular sequence characterized by  $d(p) = d(r)$  because the known period of the consecutive sequence is  $q$  and, in those sectors,  $q = k$  (Theorem 1.3).

To illustrate a practical application, we could adopt the same prime factors analyzed by Ripà [7] and study one particular square grid with  $37 \times 37$  terms characterized by  $d(r) = d(p) = 3$ . The pattern is shown in Fig. 5.

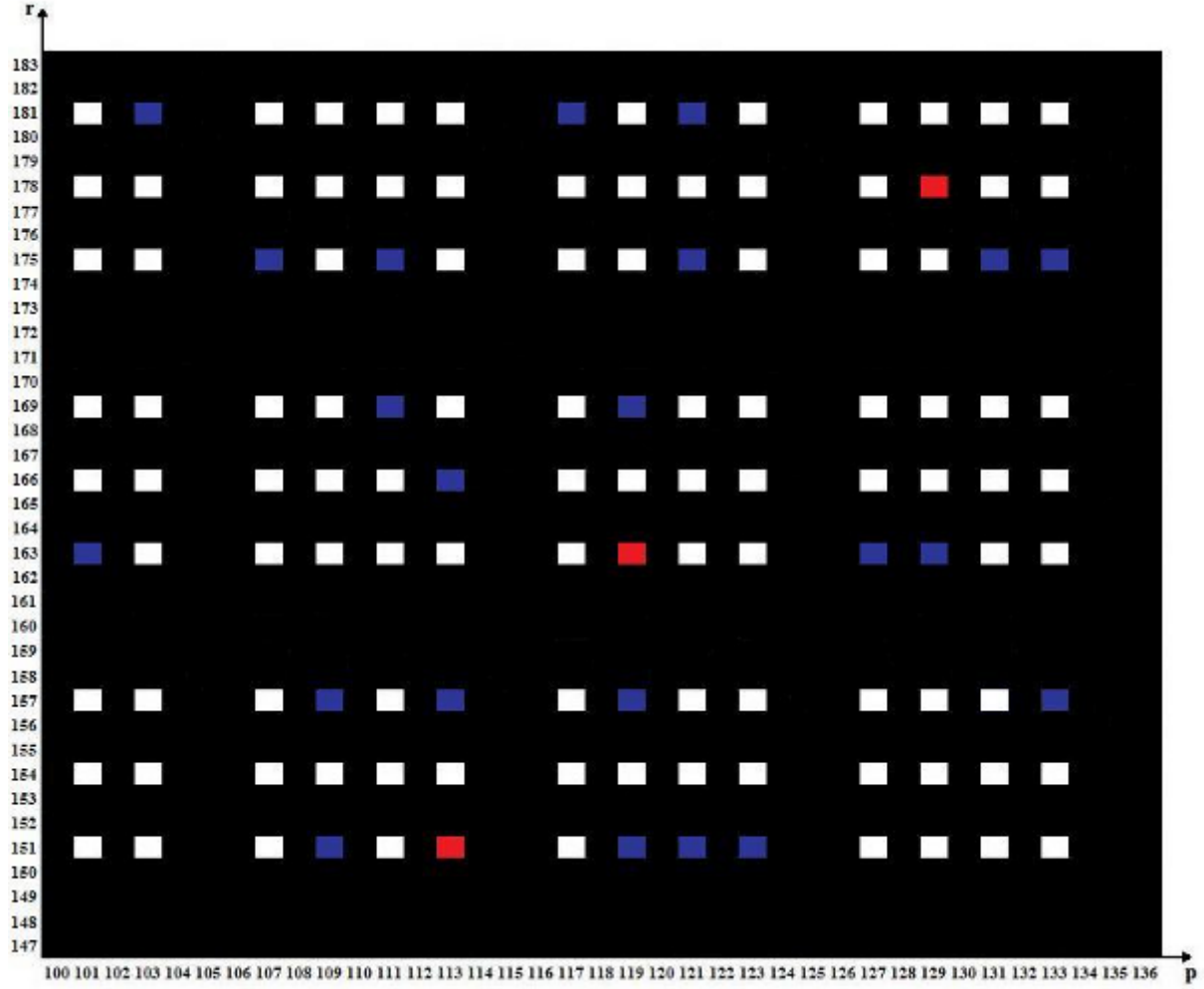


Figure 5: Pattern of a  $37 \times 37$  grid terms characterized by  $d(r) = d(p) = 3$ .

The white squares plus the red ones represent the candidate prime numbers (the red squares are effectively prime numbers), the black squares describe terms that are divisible by (at least) one prime factor belonging to the set  $\{2, 3, 5, 37\}$ , whereas the blue items represent the elements of the circular sequence divisible by 7, 11, or 13. In this case, the candidate primes are 101 (with 3 prime numbers **A181073**) in a total of 1369 terms. In a given  $(d(r), d(p))$  sector, overlapping the pattern of the terms divisible by  $pr_{b+1}$  (for  $b \leq h - 1$ ) on the pattern of the elements divisible by  $pr(j \leq b)$ , we are able to reduce the total of the candidate prime numbers to verify via a primality test [2], with the advantage of studying only a few of the “shortest” terms of the sequence in our  $(d(r), d(p))$  quadrant, for some  $7 \leq pr_j < 9 \cdot 10^{d(r)-1}$ .

## 2 The finite sequence of the circular digit permutations and others

In this section, we illustrate another important rule linked to the patterns of the circular sequence divisibility and then we present a few new sequences constructed starting from the result mentioned above.

**Definition 2.1** Let  $f_1-f_2-f_3-\dots-f_{(n-1)}-f_n$  be the  $n$ -tuple of consecutive digits resulting from the concatenation of the first  $n$  terms of the sequence **A007376**, where  $f_1 := 1$  and  $f_2 := f_1 + 1$ . We define as “circular permutations of the digits” the  $n$  elements that compound the subset of the permutations of the form

$$\begin{aligned} \bar{a}_{l+1} &:= f_2-f_3-f_4-\dots-f_{(n-1)}-f_n-f_1 && \equiv 2-3-4-\dots-f_{(n-1)}-f_n-1, \\ \bar{a}_{l+2} &:= f_3-f_4-\dots-f_{(n-1)}-f_n-f_1-f_2 && \equiv 3-4-\dots-f_{(n-1)}-f_n-1-2, \\ \vdots & && \\ \bar{a}_{l+i} &:= f_{(i+1)}-f_{(i+2)}-\dots-f_n-f_1-f_2-\dots-f_{i-1}-f_i && \equiv f_{(i+1)}-f_{(i+2)}-\dots-f_n-1-2-\dots-f_{i-1}-f_i, \\ \vdots & && \\ \bar{a}_{l+n} &:= f_1-f_2-f_3-\dots-f_{(n-1)}-f_n && \equiv 1-2-3-\dots-f_{(n-1)}-f_n. \end{aligned}$$

**Theorem 2.2** For some pairs  $(j, r), \exists M(r)|pr_j$ . For these  $(j, r)$ , all the circular permutations of the digits of  $M(r)$  are also divisible by  $pr_j$ .

Let  $d(r) = \text{constant}$ , by Theorem 1.3,  $M(r)|pr_j \Rightarrow M(r+q)|pr_j$ , but there exist others  $pr > 9 \cdot 10^{d(r)-1}$  which are in possession of the same property. Besides, from Theorem 2.2, it follows that all the possible circular permutations of the digits of  $M(r+q)$  are also divisible by  $pr_j$ .

On the next pages, we analyze the sequence by Definition 2.1, related to every circular permutation of the digits of the canonical consecutive sequence **A007908**. Hence, we limit the number of the terms of the sequence fixing  $d(r)$  at 3, so  $r \in [100, 999]$ . Unbundling a chunk of the extended circular digit permutations sequence, there are a lot more new elements than in the canonical circular sequence. Thus,

$$\begin{aligned} \bar{a}_1 &:= 12345\dots9899100 \in M(100), \\ \bar{a}_2 &:= 23456\dots98991001 \in M(100), \\ \vdots & \\ \bar{a}_{11} &:= 0111213\dots98991001234567891 \equiv 111213\dots98991001234567891 \in M(100), \\ \vdots & \\ \bar{a}_{1386450} &:= 91234\dots99799899 \in M(999). \end{aligned}$$

The total of the elements of such a sequence is  $192 + 195 + 198 + \dots + 2886 + 2889$ , that is

$$\sum_{m=0}^{899} (192 + m \cdot 3) = 192 \cdot 900 + 3 \cdot \frac{899 \cdot 900}{2} = 900 \cdot (192 + 1.5 \cdot 899) = \mathbf{1386450}. \quad (2)$$



This represents only the empirical evidence of the properties we have already seen in the first section: there are several  $pr_j$  (**A180346**) that divide the whole set of the digit permutations of  $M(\tilde{r})$ , for  $\tilde{r} \in [100, 999]$ . Let  $4 \leq j \leq 169$  (which implies  $7 \leq pr \leq 1009$ ), the  $\tilde{r}$  values are the following (and under our assumption, they belong to the sequence **A181373** of the OEIS [8]):

$$\begin{aligned}
M(\tilde{r})|7 &\Rightarrow \tilde{r} = 100 + 14 \cdot v & (v = 0, 1, 2, \dots, 64), \\
M(\tilde{r})|11 &\Rightarrow \tilde{r} = 106 + 22 \cdot v & (v = 0, 1, 2, \dots, 40), \\
M(\tilde{r})|13 &\Rightarrow \tilde{r} = 120 + 26 \cdot v & (v = 0, 1, 2, \dots, 33), \\
M(\tilde{r})|17 &\Rightarrow \tilde{r} = 196 + 272 \cdot v & (v = 0, 1, 2), \\
M(\tilde{r})|19 &\Rightarrow \tilde{r} = 102 + 114 \cdot v & (v = 0, 1, 2, 3, 4, 5, 6, 7), \\
M(\tilde{r})|23 &\Rightarrow \tilde{r} = 542, \\
M(\tilde{r})|29 &\Rightarrow \tilde{r} = 400, \\
M(\tilde{r})|31 &\Rightarrow \tilde{r} = 181 + 155 \cdot v & (v = 0, 1, 2, 3, 4, 5), \\
M(\tilde{r})|37 &\Rightarrow \tilde{r} = 123 + \sum_s d_s, & (\text{where } d_s = 0, 12, 25, 12, 25, \dots \text{ for } s = 0, 1, 2, \dots, 47), \\
M(\tilde{r})|41 &\Rightarrow \tilde{r} = 216 + 205 \cdot v & (v = 0, 1, 2, 3), \\
M(\tilde{r})|43 &\Rightarrow \tilde{r} = 372 + 301 \cdot v & (v = 0, 1, 2), \\
M(\tilde{r})|53 &\Rightarrow \tilde{r} = 127 + 689 \cdot v & (v = 0, 1), \\
M(\tilde{r})|61 &\Rightarrow \tilde{r} = 616, \\
M(\tilde{r})|67 &\Rightarrow \tilde{r} = 399, \\
M(\tilde{r})|73 &\Rightarrow \tilde{r} = 196 + 584 \cdot v & (v = 0, 1), \\
M(\tilde{r})|83 &\Rightarrow \tilde{r} = 118, \\
M(\tilde{r})|97 &\Rightarrow \tilde{r} = 516, \\
M(\tilde{r})|101 &\Rightarrow \tilde{r} = 416 + 404 \cdot v & (v = 0, 1), \\
M(\tilde{r})|107 &\Rightarrow \tilde{r} = 884, \\
M(\tilde{r})|127 &\Rightarrow \tilde{r} = 106, \\
M(\tilde{r})|163 &\Rightarrow \tilde{r} = 576, \\
M(\tilde{r})|211 &\Rightarrow \tilde{r} = 306, \\
M(\tilde{r})|271 &\Rightarrow \tilde{r} = 936, \\
M(\tilde{r})|277 &\Rightarrow \tilde{r} = 174, \\
M(\tilde{r})|1009 &\Rightarrow \tilde{r} = 960.
\end{aligned}$$

Using these relations, we can reduce the percentage of candidate prime terms of the chart at the end of Section 1, since  $M(181)|31$  and the numerousness of candidate primes among our 1369 elements crashes at **90** (with a ratio of 0.06574).

**Lemma 2.3** *In the interval we have set, the “fixed factors” from Theorem 2.2 recur with regularity when  $\tilde{r}$  grows: they are periodical and the period is a multiple of  $pr_j$  itself (as stated in Corollary 1.5).*

The inclusion of  $pr_{169} := 1009$  in the list, which is above the upper limit of  $\tilde{r}$ , represents the proof of the existence of primes that divide all the possible circular digit permutations of a given  $M(r)$ , for which the method presented in the first section cannot be applied. It is possible to combine this property with the method previously shown. The synergy of these two techniques would lead us to a very small percentage of candidate prime terms to test in the direct way<sup>5</sup>.

We should initially know for which pairs  $(pr, r)$  there is a full match. For this purpose, it is sufficient to apply Pascal's method [3, 4] to derive the divisibility criteria of the prime factors we want to consider.

Under the restriction on  $r$  made at the beginning, we can find a lot of values of  $pr > 9 \cdot 10^{d(r)-1} = 900$  characterized by the applicability of Lemma 2.3. For example, we can take  $pr_{2157} := 18973$  and  $r = 903$ .

Specifically referring to the fixed  $pr$  analysis (derived from Lemma 2.3), and considering the finite interval we are studying, we can synthesize the previous conditions (for all the given  $pr_{j \leq 169}$  – implying the constraint  $a_i(\text{mod } 10) \equiv 1, 3, 7, 9 -$ ) as follows:

$$\left\{ \begin{array}{l} \tilde{r} = 100 + 3 \cdot v \\ \tilde{r} \neq 100 + 14 \cdot v \\ \tilde{r} \neq 106 + 22 \cdot v \\ \tilde{r} \neq 120 + 26 \cdot v \\ \tilde{r} \neq 123 + 37 \cdot v \\ \tilde{r} \neq 135 + 37 \cdot v \\ \tilde{r} \neq 118, 127, 181, 196, 400, 421, 673, 820. \end{array} \right.$$

So, there are only 233  $M(\tilde{r})$  in this finite sequence, that do not have (at least) a  $pr < 1009$  that divides all the elements of the set  $r = \text{costant}$ , and these are

103, 109, 112, 115, 121, 124, 130, 133, 139, 145, 151, 154, 157, 163, 166, 169, 175, 178, 187, 190, 193, 199, 202, 205, 208, 211, 214, 217, 220, 223, 229, 232, 235, 241, 244, 247, 256, 259, 262, 265, 274, 277, 280, 283, 286, 289, 292, 295, 298, 301, 307, 313, 316, 319, 322, 325, 331, 334, 337, 340, 343, 346, 349, 355, 358, 361, 364, 367, 373, 376, 379, 385, 388, 391, 397, 403, 409, 412, 415, 418, 424, 427, 430, 433, 439, 442, 445, 448, 451, 454, 457, 460, 463, 466, 469, 472, 475, 481, 487, 490, 496, 499, 508, 511, 514, 517, 523, 526, 529, 532, 535, 538, 541, 544, 547, 550, 553, 556, 559, 565, 571, 574, 577, 580, 583, 586, 589, 592, 595, 598, 601, 607, 610, 613, 619, 622, 625, 628, 631, 637, 643, 649, 652, 655, 658, 661, 664, 667, 670, 676, 679, 682, 685, 694, 697, 703, 706, 709, 712, 721, 724, 733, 736, 739, 742, 745, 748, 751, 754, 757, 760, 763, 769, 775, 778, 781, 784, 787, 790, 793, 799, 802, 805, 808, 811, 817, 823, 829, 835, 841, 844, 847, 850, 853, 859, 862, 865, 868, 871, 877, 880, 883, 886, 889, 892, 895, 901, 904, 907, 910, 913, 916, 919, 922, 925, 928, 931, 934, 943, 946, 955, 958, 961, 967, 970, 973, 976, 979, 985, 988, 991, 994, 997.

Considering the canonical circular sequence bounded by  $d(r) = 3$ , without using the pattern analysis on the smallest  $pr_j$ , we are able to exclude most of the elements of the sequence, limiting the candidate prime terms subset to under 11.1% of the total sum of the elements. In fact, we have:

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<sup>5</sup>We are clearly referring to primality tests.

$\frac{56039-1176}{139950-2936} \approx 0.400418$  and  $\frac{137014}{494550} \approx 0.2770478$ . This implies a percentage of candidate prime terms equal to  $0.4004189 \cdot 0.2770478 \cdot 100 = \mathbf{11.0935\%}$ .

*In the case of the extended circular sequence to 1386450 terms, the percentage is even lower than in the previous one (well below 11%).<sup>6</sup>*

At this point, we could modify our finite sequence a little, without altering its representative properties. We have to arrange, in ascending order, the 1386450 elements of (2). Cutting out the details, it is clear that, for a given  $r$ , the first terms of the corresponding sequence subset start with two zeros. In fact, they have 2 significant digits fewer than most of the other “room-mates” (in short, for a given value of the parameter  $r$ ). At the end of this operation, we know that the result does not vary, because the order of the subsets  $r_i$  remains unchanged, due to the maximum spread (between one pair of the elements of this set) of 2 digits, since each unitary increase of  $r$  adds 3 new digits to the “standard term” linked to the preceding value of  $r$ . It is also superfluous to specify that the numerosness of the candidate primes does not change appreciably, as well as the considerations we have already made about the “fixed prime factors”.

Remember that the criterion at the bottom of the previous relations is general: for example, taking  $r \in [1000, 9999]$  it will only vary the linear rules that describe the periodicity of the fixed factors, but the concept will survive the new  $d(r)$ . The only constraint to adhere to is that  $d(r)$  must be constant.

Now we could once again extend the sequence of the circular digits permutations mentioned before, defining another sequence with the same features as the previous one, but in which, after each set of permutations, only one new digit is added. Starting from the infinite extended consecutive sequence 1, 12, 123, ..., 123456789, 1234567891, 12345678910, 123456789101, ... we consider the set of all the circular permutations of each one of these terms, as we have already done starting from the canonical circular sequence. We could also repeat the same process for a few other Smarandache sequences (e.g., the reverse one **A000422** [1, 10]). Referring to the expanded circular digit permutations sequence, we might transpose the same questions introduced by Ripà [7]: the results are pretty interesting. Without taking away the fun of the discovery, choosing as an example the extended consecutive sequence that we have just described, we can only say that it contains some prime terms [11] (it does not even require deep analysis). In fact, the shortest primes are **A176942**

$$\begin{aligned}
Sm'_{10} &:= 1234567891, \\
Sm'_{14} &:= 12345678910111, \\
Sm'_{24} &:= 123456789101112131415161, \\
Sm'_{235} &:= 12345 \dots 1101111121131141, \\
Sm'_{2804} &:= 12345 \dots 96696796896997097, \\
Sm'_{4347} &:= 12345 \dots 1359136013611362136313, \\
\text{and so forth} &\quad (\text{we observe that } Sm'_n \text{ is exactly } n \text{ digits long}).
\end{aligned}$$

Referring to the circular sequence extended to include all the circular permutations of its digits, it is clear that the outcome stated in the first section remains valid considering that,  $\forall n' > n''$ , if  $1.2.3 \dots f_{(n'-1)}-f_{n'}$

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<sup>6</sup>We have taken into account the constraint for the candidate primes that implies the congruence modulo 10 to  $\{1, 3, 7, 9\}$ .

is divisible by  $pr_j$  and that  $1\_2\_3\_ \dots \_f_{(n''-1)}\_f_{n''}$  is also divisible by the same prime factor, then  $f_{(n''+1)}\_ \dots \_f_{(n'-1)}\_f_{n'}\_1\_2\_3\_ \dots \_f_{(n''-1)}\_f_{n''}$  results in being divisible by  $pr_j$  (and obviously  $pr_j$  is a prime factor of  $f_{(n''+1)}\_ \dots \_f_{(n'-1)}\_f_{n'}$  as well). This is a general property of each number, not only for those belonging to the extended consecutive sequence set: taking  $T'$  (a random sequence composed of  $t'$  figures divisible by  $pr_j$ ) and  $T''$  (another  $t''$  digit long sequence that is divisible by the same  $pr_j$ ),  $T' \_T''$  and  $T'' \_T'$  are both divisible by  $pr_j$ .

If we take a look at the sequence constructed from the circular permutations of the digits of the extended consecutive sequence (1, 12, 123,  $\dots$ , 123456789, 1234567891, 12345678910, 1234456789101,  $\dots$ ), we can easily find a lot of sets of circular arrangements that are entirely divisible by the same  $pr_j$ . So, 1\\_2\\_3\\_  $\dots$  1005\\_1006\\_1007\\_1 plus the remaining 2921 circular permutations of its digits are divisible by  $pr_4 = 7$  and  $pr_5 = 11$ , whereas every circular permutation of the digits of 1\\_2\\_3\\_  $\dots$  36\\_37\\_3 is divisible by 7, 11 and 13. Section 3 contains further comments on the topic.

### 3 Some observations and remarks

We could study what happens if we try to expand the set, giving the usual interval  $d(r) = 3$ , of the circular digit permutations of the 900 terms belonging to the canonical consecutive sequence to the circular digit permutations of the 2698 elements that characterize the extended consecutive one (constructed – as already seen – adding only one figure of the canonical sequence at the end of the previous element). Thus,  $\bar{a}_1 := 1234 \dots 9899100$ ,  $\bar{a}_2 := 234 \dots 98991001$ ,  $\bar{a}_{11} := 0111213 \dots 98991001234567891$ ,  $\bar{a}_{4153381} := 1234 \dots 997998999$  and  $\bar{a}_{4156269} := 91234 \dots 99799899$ : only approximately one-third of the terms belonging to this sequence are part of the sequence formed by the circular permutations of the digits of the canonical consecutive one, even if Lemma 2.3 is satisfied as well. In this case, we will find new fixed factors not included in the “short version” of the sequence, e.g.,  $pr_{17} := 59$  (resulting from the circular digit permutations of 1234  $\dots$  1121131), mixed with other prime factors that we have already met, as  $pr_5 = 11$ , which is a fixed factor for 1234  $\dots$  10710810.

Nevertheless, one problem arises: we cannot arrange the terms in ascending order or we will lose the validity of Lemma 2.3. Moreover some  $\bar{a}_j$  are coincident and their position inside the sequence could not be univocally identified, e.g.,  $\bar{a}_{101470} := 1234 \dots 1981992$  is equal to  $\bar{a}_{102450} := 01234 \dots 1981992$  and  $\bar{a}_{102941} := 001234 \dots 1981992$ .

The prime factors  $\geq 7$  that divide all the terms of the general (unlimited) sequence, constructed from the circular permutations of the digits of the extended consecutive one, exist starting from 16 digit long numbers (all the circular digits permutations of 1234567891011121 are divisible by 17) without an upper limit. This is testified by the example we have already introduced: each one of the 2922 circular permutations of the string 1234  $\dots$  1005100610071 is divisible by  $pr = 11$  and we can say the same for the circular permutations of the elements formed by  $2922 + 44 \cdot v$  digits, for  $v = 0, 1, 2, 3, \dots, 817$ . In this case, the period of  $pr = 11$  is identical to the one linked to the sequence derived by an integer value of  $r \in [1000, 9999]$ , for  $d(p) = d(r) = 4$  and  $pr = 11$ .

The consideration above leads us to the following observation involving all the subsequences constructed, as previously described, from the extended consecutive sequence. Starting from the sequence that includes every circular permutation of the extended consecutive one (1, 12, 123,  $\dots$ , 123456789,

1234567891, 12345678910, 1234456789101, ...), given the value of  $d(r)$  plus the initial  $\bar{a}_j$ , we obtain exactly “ $d(r)$ ” interrupted subsequences (one of which – the canonical circular sequence – is complete). For these subsequences, most of the properties already explained in Section 1, plus Theorem 2.2, still remain valid.

In conclusion, we might even give free rein to the imagination and transpose some of the questions [6, 9] (totally or partially) answered by Ripà [7] to the new sequences designed “ex novo” starting from the considerations we have made in these pages: thus we can define a new sequence obtained taking into account only the odd figures of the circular one (considering all the variations we have seen so far) plus a lot of others on the same line.

## 4 Patterns belonging to the Smarandache circular sequence

This is a small collection of the patterns of some  $pr_j$  that divides the corresponding terms of the canonical circular sequence (i.e., 1, 12, 21, 123, 231, 312, 1234, 2341, ...).

In the following charts, each dark square represents an element of the sequence that is divisible by the given  $pr$  (remember that the generic term can be written as  $(p + 1) \cdot (p + 2) \cdot \dots \cdot (r - 1) \cdot r \cdot 1 \cdot 2 \cdot 3 \cdot \dots \cdot (p - 1) \cdot p$ ).

Often these patterns show some intrinsic regularity, that depends, for the most part, on the chosen  $pr_j$  (a clear example is represented by  $pr_6 := 13$ ).

$$pr_4 := 7$$

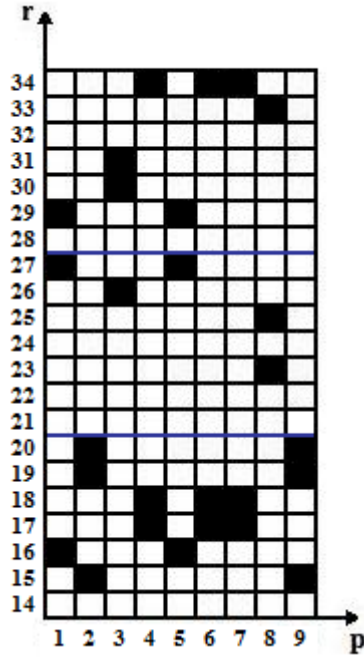


Figure 6:  $d(r) = 2, d(p) = 1$ .

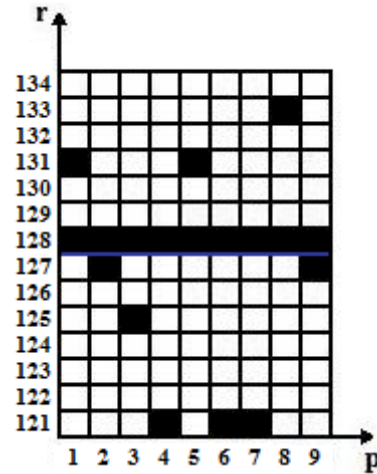


Figure 7:  $d(r) = 3, d(p) = 1$ .

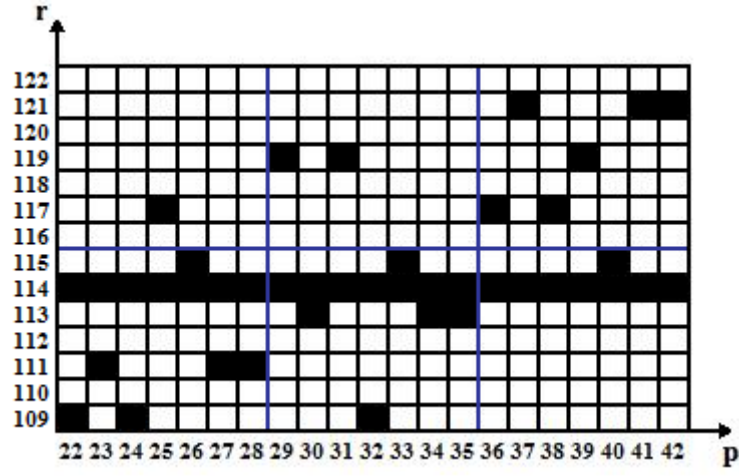


Figure 8:  $d(r) = 3, d(p) = 2$ .

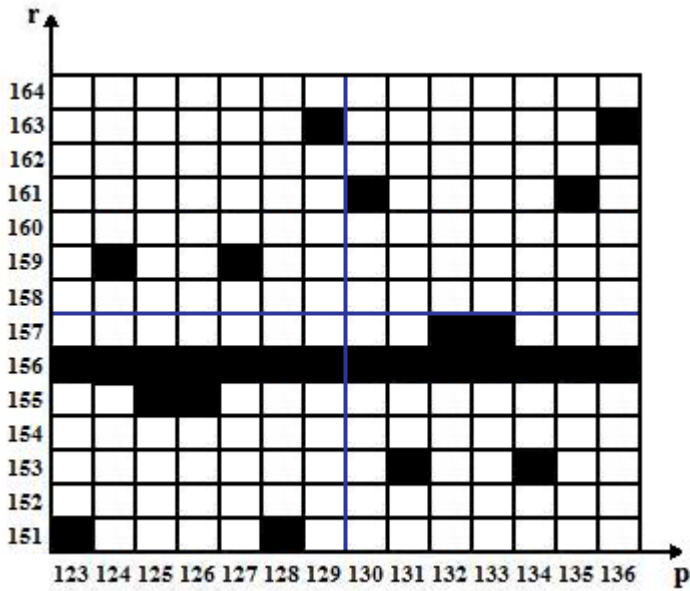


Figure 9:  $d(r) = 3, d(p) = 3$ .

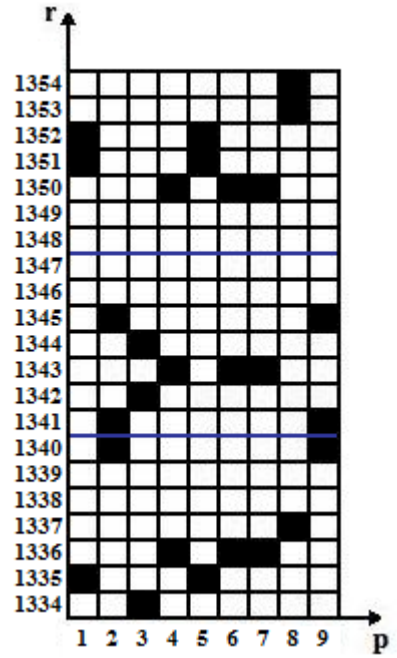


Figure 10:  $d(r) = 4, d(p) = 1$ .

$$pr_5 := 11$$

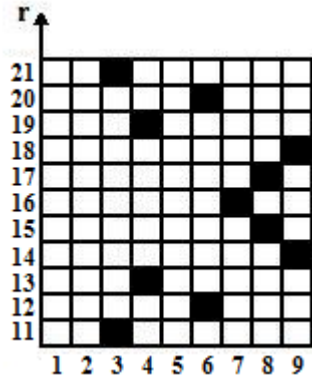


Figure 11:  $d(r) = 2, d(p) = 1$ .

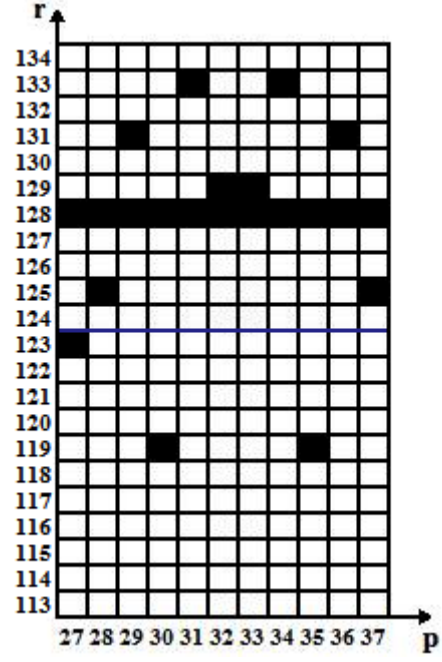


Figure 12:  $d(r) = 3, d(p) = 2$ .

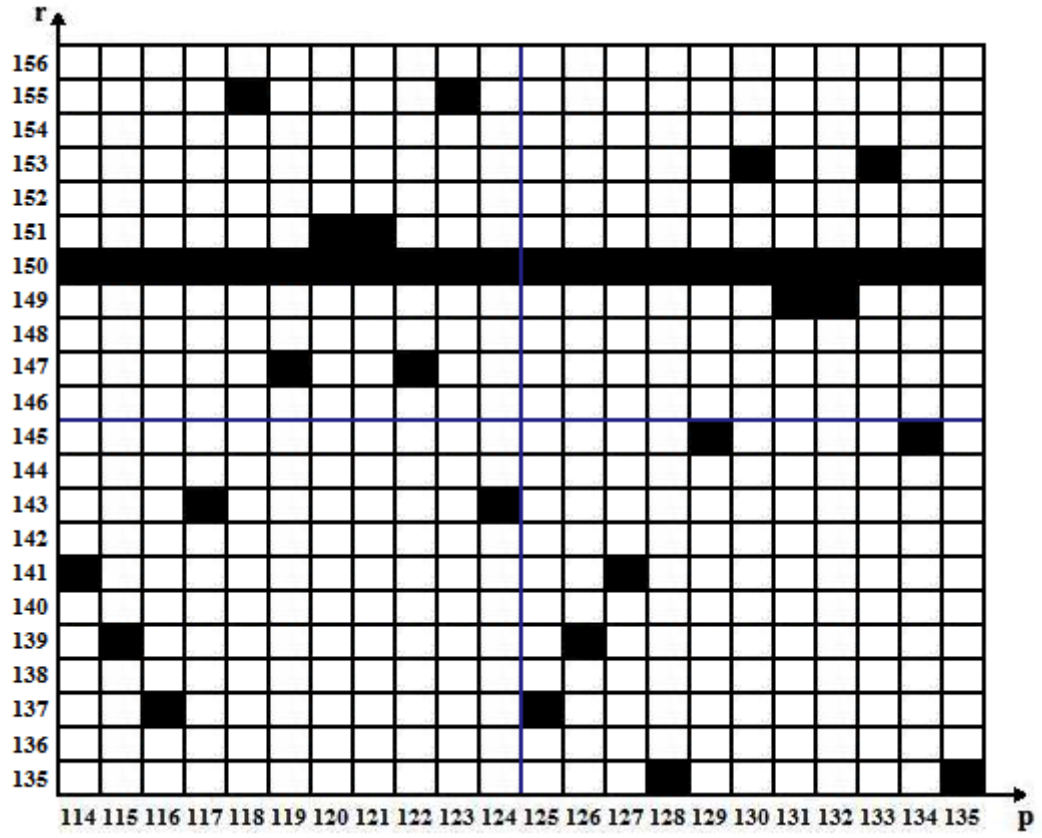


Figure 13:  $d(r) = 3, d(p) = 3$ .

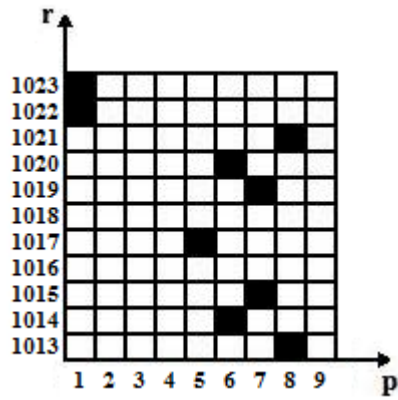


Figure 14:  $d(r) = 4, d(p) = 1$ .

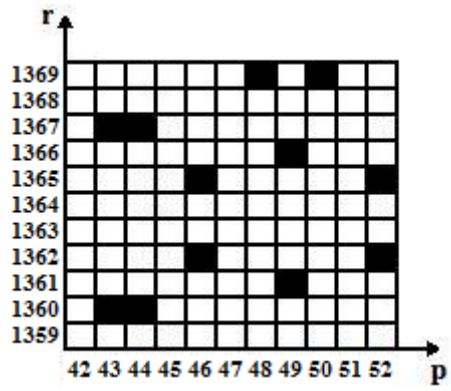


Figure 15:  $d(r) = 4, d(p) = 2$ .



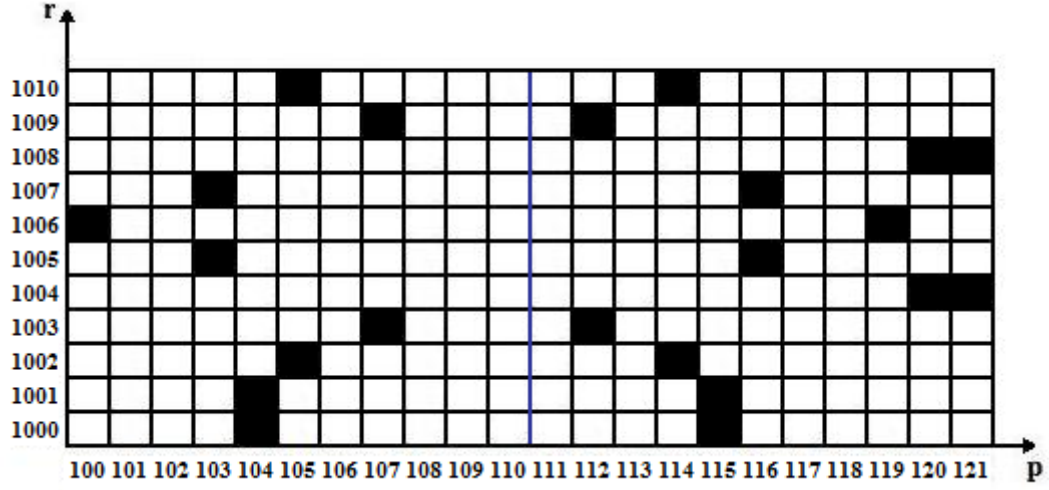


Figure 16:  $d(r) = 4$ ,  $d(p) = 3$ .

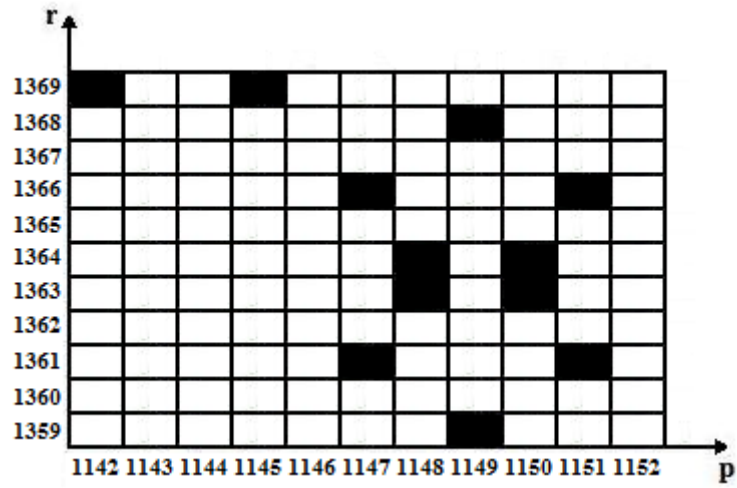


Figure 17:  $d(r) = 4$ ,  $d(p) = 4$ .

$$pr_6 := 13$$

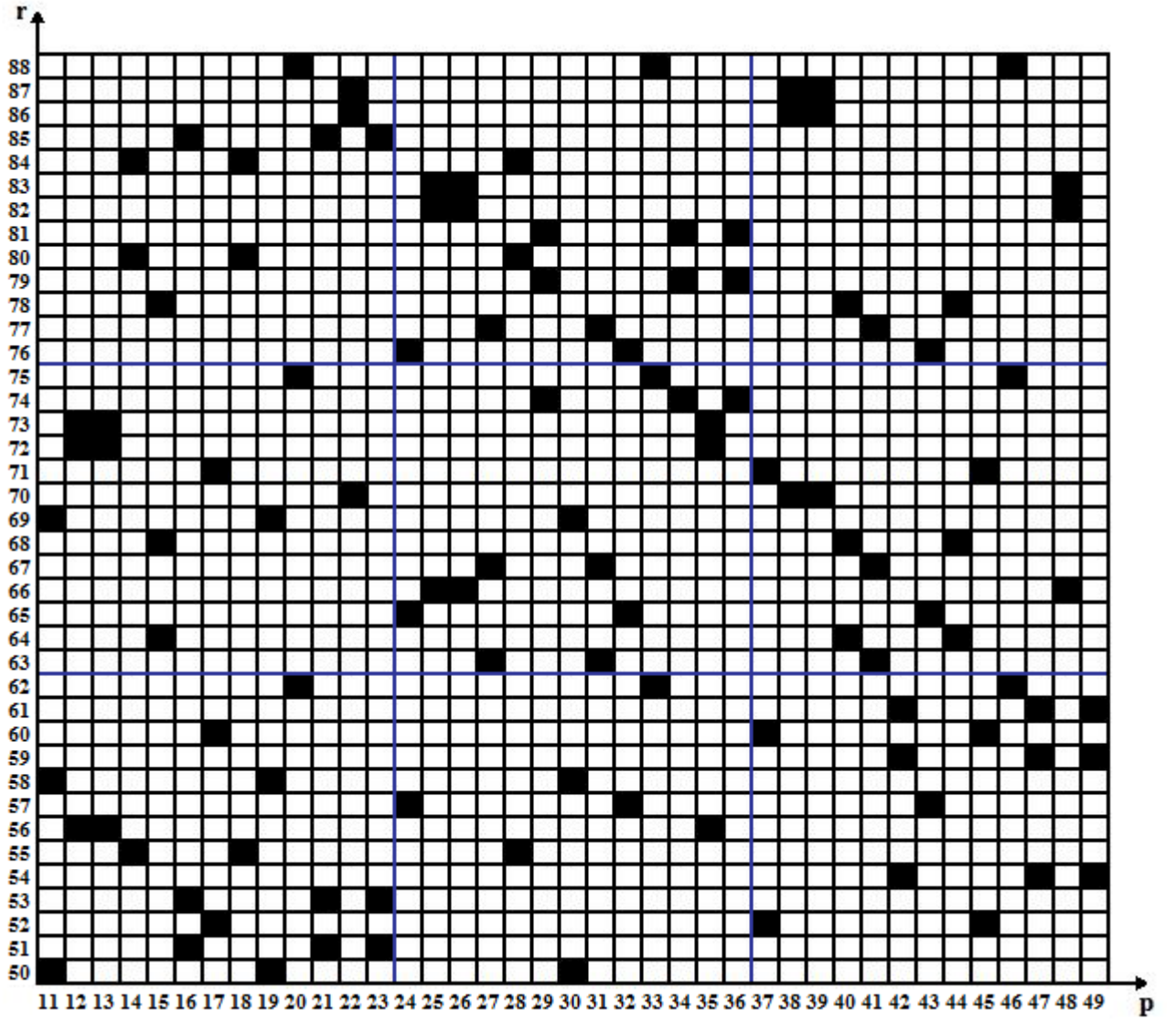


Figure 18:  $d(r) = 2, d(p) = 2$ .

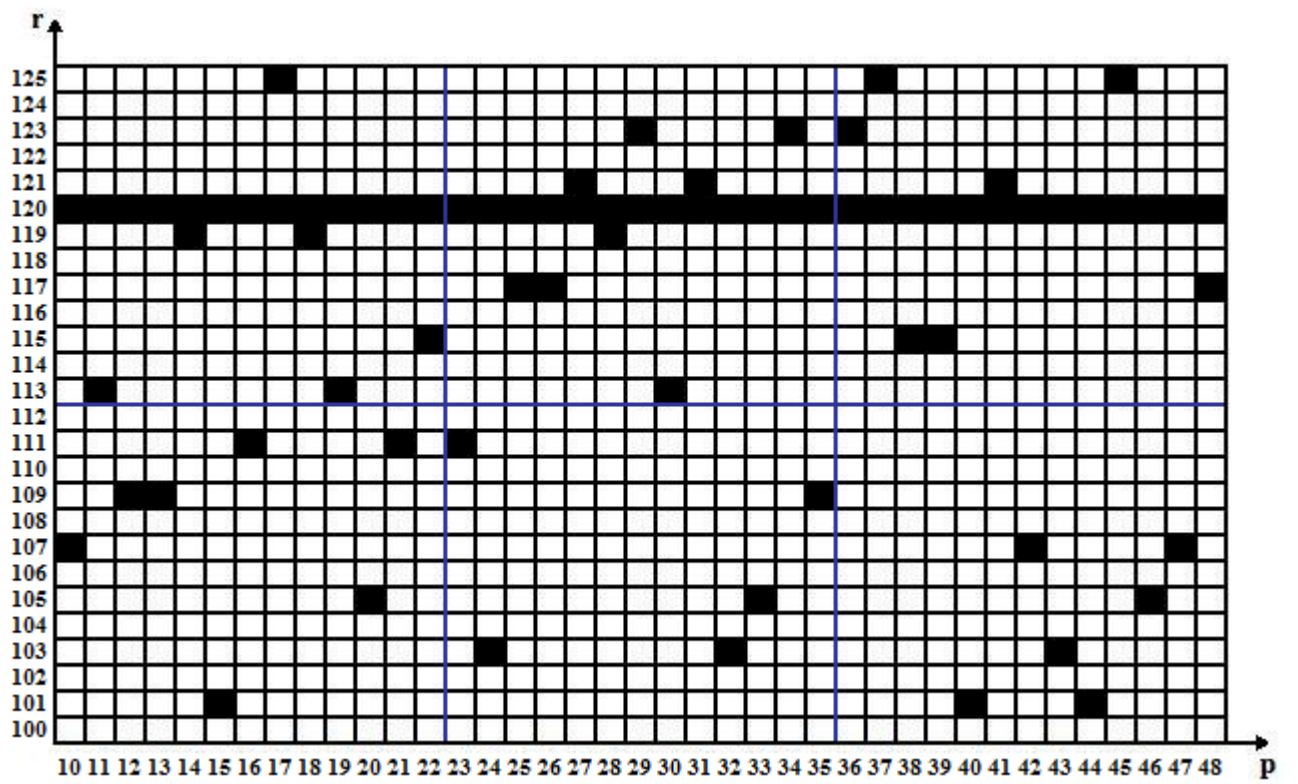


Figure 19:  $d(r) = 3, d(p) = 2$ .

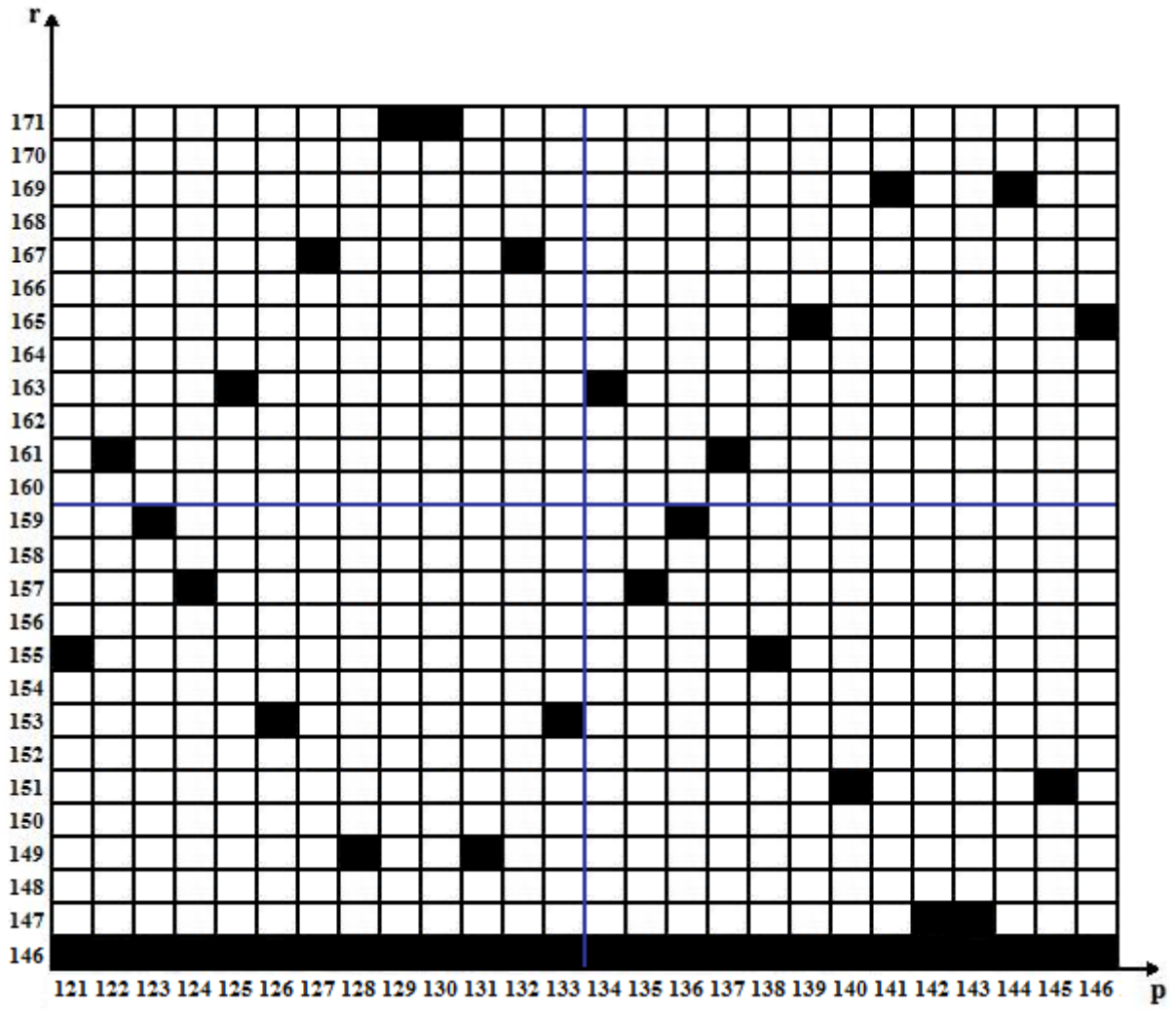


Figure 20:  $d(r) = 3, d(p) = 3$ .

## Acknowledgements

The authors sincerely thank Dr. Paolo Magrassi, Jacqueline Slade, Graham Powell, and Stanislav Riha for their help and invaluable support.

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2010 Mathematics Subject Classification: Primary 11B50; Secondary 65Q30, 11Y05, 11Y11.  
Concerned with sequences A000040, A000422, A001292, A007376, A007908, A176942, A180346, A181073, A181129, and A181373.