



HAL
open science

On compound mixed concepts

Michael Freund

► **To cite this version:**

Michael Freund. On compound mixed concepts. *Journal of Mathematical Psychology*, 2022, 109, pp.102690. 10.1016/j.jmp.2022.102690 . hal-03702783

HAL Id: hal-03702783

<https://hal.science/hal-03702783>

Submitted on 23 Jun 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On compound mixed concepts

Michael Freund

IHPST

University of Paris Sorbonne

13 rue du Four 75006 Paris, France

micfreund@gmail.com

Journal of Mathematical Psychology 109 (2022)

Abstract

We extend to a certain class of compound concepts the binary model classically used for the representation of simple concepts. This class consists of concepts that are determined or modified by a single feature. The treatment of such a mixed composition shows the need to differentiate between exceptional and non exceptional modifiers. In the first case, typicality is easily retrieved from the components of the composition, while, in the second case, it is necessary to isolate the characteristic features of the initial concept that contradict the modifier. The distinction between exceptional and non exceptional modifiers plays a key role in the evaluation of resemblance with mixed compositions.

Keywords categorization, prototype theory, categorial membership, typicality, resemblance theory, concept, feature, relative clause.

1 Introduction

The study of concept combination may be seen as the touchstone of concept theories. Indeed, while elementary concepts can be suitably represented through different models (see [2], [5], [17], [29], [22], [34], [37] or [27] for a general overview), the extension to compound concepts of such representations

requires a robustness of the model that may be lacking in the original theory. Except for some particular cases, the basic notions linked with concept theories are not compositional. The evidence of some paradoxical results showed the difficulty to treat concept composition with the tools of set theory, and led to alternative approaches ([38], [1], [10], [41]).

In [13] and [14], we studied a certain family of concepts, which we called *featured concepts*. This family consists of concepts that can be described by a simple version of the binary model proposed by [37] and [34]. We showed that, for this class of concepts, the notions of categorial membership and typicality could be modeled by simple order relations. Such an order adequately translates the fact that an object may be *more or less* a member of a category, or that, inside this category, it appears as *more or less* typical. After having been a ‘matter of degree’ (see [33]), categorial membership and typicality could be therefore treated as intuitive ordinal notions. The membership and the typicality orders that a concept induces among the objects of the universe were shown to provide a satisfactory solution, in the framework of featured concepts, to the problems classically linked with categorial membership, typicality, resemblance and concept induction.

One might think it odd to build a mathematical theory on items - e.g. concepts - for which there exists no mathematical definition. This apparent contradiction however disappears if one keeps in mind that we work on a *representation* of concepts and not on concepts themselves. As in the case of almost all theories of concept, we propose a mathematical model and use mathematical tools to investigate problems like categorial membership, typicality, concept composition and concept based induction. The solutions brought by this model can be then evaluated in the perspective of cognitive psychology.

The model built to account for simple featured concepts easily extends to the combination of two such concepts, although the resulting concept is generally not a featured concept. An important side-result is the possibility of describing and explaining in our model the guppy effect as well as the conjunction fallacy (see [29] and [40]), showing that our approach successfully models concept conjunction.

In our representation of compound concepts, a problem was however left aside, which is that of *compound mixed* concepts. These concepts result from the conjunction of a single attribute with a featured concept, e.g. a *black-dog*, an *aquatic-bird*, an *English-physician*. The class of compound mixed concepts deserves a study of its own because these concepts are omnipresent in every-

day language as adjective-noun combinations. Furthermore, feature-concept combination seems simpler to investigate than concept-concept combination.

We shall devote this paper to the analysis of this kind of composed concepts.

It is clear that compound mixed concepts cannot receive a uniform treatment: it will be indeed necessary to distinguish between the simple *determination* of a concept, where the head feature can be integrated as a supplementary attribute of the concept (e.g. *to-be-a-black-dog*), and a *modification* of the concept, where the head feature is considered as exceptional because it stands in contradiction with one or several attributes of the original concept (e.g. *to-be-a-silent-dog*, *to-be-a-walking-bird*). For each type of feature, we shall build a set of characteristic attributes of the resulting mixed concept. We shall see that, in the principal cases, this concept can be given the structure of a featured concept, to which we can apply our preceding results. We shall devote a particular attention to questions linked with *resemblance*. In particular, the presence of the head feature in resemblant objects will be debated, as it has no obvious solution: an animal that is said to resemble a *black-dog* is expected to be black itself, but a plant that resembles a *tropical-flower* need not have grown in the tropics.

Plan of the paper

In order to make this paper self-contained, we recall in section 2 the basic notions and results that are necessary to understand the questions evoked in the remainder of the paper. Section 3 introduces the notion of mixed concepts and explains the difference between exceptional and non-exceptional modifiers. In Section 4, after having recalled some results on simple concept resemblance, we examine the problem of resemblance relative to compound mixed concepts. We conclude in Section 5.

The proofs of the propositions and theorems are given in the Appendix.

2 Tools for categorization

2.1 Concepts, features and objects

The universe of discourse consists of a set of objects \mathcal{O} that a human agent has at his disposal at a given time, a set of concepts \mathcal{C} , and a set of features

\mathcal{F} . The set of concepts reflects the agent’s knowledge of his environment and is used by the agent to build his reasoning process. For simplification, we shall adopt the original presentation of Frege [11] who assimilated concepts with one-place predicates. In this perspective, concepts will be generally introduced through the auxiliary *to-be*, followed by a noun: *to-be-a-bird*, *to-be-a-vector-space*, *to-be-a-democracy*. Objects are entities that are identified as such by the agent. The process of identification gives rise to a binary relation between concepts and objects. This relation is satisfied between an object x and a concept α when the agent considers that x falls under α , or that α applies to x . For example, one may say that ‘Beethoven’s opus 111 falls under the concept *to-be-a-piano-sonata*’, or that ‘The dog Toby does not fall under the concept *to-be-a-cat*’.

An object that falls under a concept is said to be an *exemplar* or an *instance* of this concept. The set $Ext\ \alpha$ of exemplars of a concept α forms its associated *category* or *extension*.

Features are usually introduced in the form of a verb (e.g. *to-fly*), of the auxiliary *to-have* followed by a noun (e.g. *to-have-a-beak*), or of the auxiliary *to-be* followed by an adjective (e.g. *to-be-tall*).

Features, like concepts, apply to the objects at hand but, contrary to concepts, they are context-sensitive: they borrow part of their significance from the concept to which they are attached. Properties like *to-be-tall*, *to-be-rich* or *to-be-red* take their full meaning only in a given context, that is when qualifying a well-defined entity. Even simple verbal forms like *to-fly*, *to-run*, *to-live-in-water*, *to-be-made-of-metal* need a principal referent concept to fully seize the strength with which they apply to different items. The concept to which a feature applies may be seen itself as a contextual determination of this feature. To summarize, the meaning of a feature depends on the context where the feature is used, contrary to the meaning of a concept, which, for a given agent, exists by itself.

To account for the fact that we are dealing with a *phenomenal* representation of cognitive structures, we suggested in [13] and [14] to evaluate the applicability of a feature f to objects by a degree function δ_f from \mathcal{O} to a finite subset of $[0, 1]$ (see [16] for the difference between scientific and phenomenal representations).

The mathematical model we presented in [14] and which we recall in this section describes the structure of concepts as well as the main relations that exist between concepts, like hierarchy, resemblance or induction. We did not

- and we shall not - examine the question of concept formation. Research in this domain may be found in [4], [6], or [7]. The question of how agents *acquire* concepts will be therefore set aside. *Our purpose is to investigate in a mathematical framework what it is, for an agent, to have a concept.*

In the classical view of categorization, concepts are apprehended by their features: vertebrates that have beaks and feathers will be labelled *birds*, a *coat* is an outer garment with sleeves that opens down the front, and, (following Plato), a *man* is a featherless biped. Exemplar and Prototype theory evaluate categorial membership through the common features an item may have with a concept. Features also play a fundamental role in Formal Concept Analysis ([15]), where, in a given context, a concept is defined by a pair (A, B) , where A is a set of objects and B a set of features. Features may be also retrieved in the *quality dimensions* of the geometrical model of Gärdenfors ([17] and, more recently, [30]), where a concept is characterized by a region in the space of its quality dimensions.

In this paper, we shall start from the *binary model* of [34] and [37]. There, a concept α is described by means of two auxiliary sets of features, the *defining feature set* and the *characteristic set*, which we shall detail now.

First of all, the *defining feature set* $\Delta(\alpha)$ of a concept α gathers features that, in the agent's mind, are *individually necessary and collectively sufficient* to decide, in a given context, whether or not an item is an exemplar of α . Categorization relative to a concept is then evaluated through categorization relative to its defining features. For example, if, for some agent, the defining feature set associated with the concept *to-be-a-bird* consists of the four features *to-be-warm-blooded*, *to-have-a-beak*, *to-have-feathers* and *to-have-wings*, then, for this agent, the degree with which each of these features apply to a given item will determine its *birdhood*. Similarly, to quote an example of Putnam [32], the *meaning* of the term *tiger* is to be searched by reference to the terms *yellow feline*, *fierce*, *black stripes*, and *jungle*: the defining features associated with the concept *to-be-a-tiger* consist of the properties *to-be-a-yellow-feline*, *to-have-black-stripes*, *to-live-in-the-jungle* and *to-be-fierce*. The defining set of a concept may also include as features the potential actions that are expected from instances of the concept and that our perception is aware of ([28]). Thus, *to-drive-in-nails* may be listed among the defining features of the concept *to-be-a-hammer*.

We have to underline that, given a concept, the choice of its associated defining feature set is purely subjective. For a same concept α , the set $\Delta(\alpha)$ may vary from an agent to another, as it may vary over time for a same agent.

It may also be the case that, for some agent, a concept has a corresponding set of defining features, while being deprived of such a set for another agent.

Concepts defined by a conjunction of features form an exceptional subclass. Fodor for instance [9] argued that there exists practically no examples of successful definition around. Without being so radical, it is clear that a great deal of concepts are deprived of any set of defining features: what list of attributes could be attached to concepts like *to-be-a-game*, *to-be-a-lie*, or *to-be-a-heap* ? However, it appears that the hypothesis of a defining set of features is justified for certain well-defined families of concepts: such is for instance the case for most *nominal* concepts, i.e. concepts that are conventionally defined, like *to-be-a-mammal*, *to-be-a-theft* or *to-be-a-refugee*. In particular, this remains true for scientific concepts, like *to-be-a-vortex* or *to-be-a-square*. Finally, it may happen that a concept, first grasped through its exemplars, is thereafter sharpened with the help of a set of defining features, like the concept *to-be-a-bird*, a natural kind concept that was revisited by naturalists and turned into the pseudo-nominal *to-have-feathers + to-have-a-beak + to-have-wings*.

All the features that the agent has chosen to form the set $\Delta(\alpha)$ do not necessarily have the same importance. For example, as far as birdhood is concerned, *to-have-wings* may be judged as a more important feature than *to-have-a-beak*. To account for this fact, it appears necessary to equip the set $\Delta(\alpha)$ with a *salience ordering* $>_{\Delta(\alpha)}$ that reflects the relative importance of its elements. This is done in the classical models through a *salience degree* that is attached to every defining feature. Then, the degree with which a concept applies to an object is evaluated through the degree with which every defining feature applies to this object, together with the relative salience degree of this feature (for an example of such a calculation, see [19] or [21]). In fact, as we argued in [14], comparing the salience of the defining features is easier than assigning to each of them a numerical degree. For this reason, we shall only suppose that, in the agent's mind, the set $\Delta(\alpha)$ is equipped with a strict partial order $>_{\Delta(\alpha)}$ which, in this paper, we shall suppose non empty. This *salience order* is, again, subjective, depending on the state of mind, at a given time, of a given agent. Moreover, it is independant of the relations that may exist between two features: an agent may consider that a feature f is more salient than a feature g even in the case where falling under f implies falling under g . Our model is supposed to reflect the agent's mind as it is, with its possible apparent contradictions.

The defining feature set associated with a concept is not enough to fully understand the link between this concept and the objects of the agent's universe. Indeed, all instances of a concept do not have the same status: some of them can be considered as *typically* representing this concept, while other ones may be viewed as exceptions among the concepts exemplars. Thus, a robin may be judged as typical in the category of *birds*, contrary to an ostrich or a penguin.

To account for this difference among the elements of $Ext\alpha$, the binary model adds to the defining feature set a second set of features, called the *characteristic set* of α , which will be denoted by $\chi(\alpha)$. This set consists of the features that an exemplar of a concept should have to be qualified as *typical* of this concept. Thus, an example of a characteristic set relative to the concept *to-be-a-bird* could be $\{to-fly, to-sing, to-eat-seeds, to-live-in-the-trees\}$: any bird to which apply all these feature will be qualified as typical. Note that the sets $\Delta(\alpha)$ and $\chi(\alpha)$ are disjoint : contrary to a defining feature, a characteristic feature is not supposed to apply to each instance of a concept.

As was the case for the defining feature set, salience has to be taken into account among the set of characteristic features. This will be done by means of a non-empty strict partial order $>_{\chi(\alpha)}$.

The above considerations lead to the following notion of *featured concepts* (see [14], definition 1 and 5):

Definition 1 *A featured concept α is a concept for which there exists a finite non-empty strictly ordered set of defining features $(\Delta(\alpha), >_{\Delta(\alpha)})$ and a finite non-empty strictly ordered set of characteristic features $(\chi(\alpha), >_{\chi(\alpha)})$ that satisfy the following three properties:*

1. $\Delta(\alpha) \cap \chi(\alpha) = \emptyset$
2. *For every defining or characteristic feature f , the corresponding applicability function δ_f takes a finite number of values in $[0, 1]$*
3. *There exists at least one object z such that $\delta_f(z) = 1$ for all defining and characteristic features f .*

The extension $Ext\alpha$ of α can be retrieved as the set of all objects z such that $\delta_f(z) = 1$ for all f in $\Delta(\alpha)$.

The set $Typ\alpha$ of *typical instances* of α gathers all objects z such that $\delta_f(z) = 1$ for all f in $\Delta(\alpha) \cup \chi(\alpha)$. Condition 3 above guarantees that $Typ\alpha \neq \emptyset$ for every featured concept α .

2.2 The impact of a featured concept in the binary model

A featured concept structures an agent's universe of discourse. In this section, we will show how such a structure can be represented by order relations among objects.

2.2.1 Induced orders

We first describe how to build an preorder relation in \mathcal{O} from a finite ordered subset of features. This will enable us to build the membership and typicality orders of a concept from its defining and characteristic sets.

Given a finite set of features X equipped with a non-empty strict partial order $>_X$, we define on a subset T of \mathcal{O} the relation \preceq_T^X by:

$$(1) \quad x \preceq_T^X y \text{ iff } \forall f \in X : \delta_f(y) < \delta_f(x), \exists g \in X, g >_X f : \delta_g(x) < \delta_g(y).$$

That is, we have $x \preceq_T^X y$ if every feature of X that applies less to y than to x is dominated in X by a feature that applies more to y than to x .

It may be helpful to interpret this relation in the framework of Social Choice Theory: supposing that a certain number of voters have to decide between two candidates x and y , and supposing that the voters voices do not matter the same, the candidate y will be preferred over the candidate x if, for each voice in favor of x , there exists a more important voice in favor of y .

Lemma 1 *The relation \preceq_T^X is a preorder. For two elements x and y of T , one has simultaneously $x \preceq_T^X y$ and $y \preceq_T^X x$ iff, $\forall f \in X, \delta_f(x) = \delta_f(y)$. The strict partial order \prec_T^X associated with \preceq_T^X satisfies:*

$$(2) \quad x \prec_T^X y \text{ iff } x \preceq_T^X y \text{ and } \exists f \in X : \delta_f(x) < \delta_f(y).$$

We shall refer to the relation \preceq_T^X as to *the order induced on T by X* .

Note that the \preceq_T^X -maximal elements z of T are those for which one has $\delta_f(z) = 1 \forall f \in X$. These elements satisfy $x \prec_T^X z$ for all elements x that are not \preceq_T^X -maximal.

The above construction leads to the two notions that govern concept representation.

The first one is the *membership order* \preceq_α^μ . It is induced on \mathcal{O} by the defining set $\Delta(\alpha)$, that is $\preceq_\alpha^\mu = \preceq_{\mathcal{O}}^{\Delta(\alpha)}$. For x and y in \mathcal{O} , we have therefore

$$(3) \quad x \preceq_\alpha^\mu y \text{ iff } \forall f \in \Delta(\alpha) : \delta_f(y) < \delta_f(x), \exists g \in \Delta(\alpha), g >_{\Delta(\alpha)} f : \delta_g(x) < \delta_g(y).$$

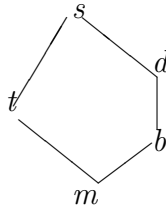
The following example was proposed in [14]

Example 1 *Let α be the concept to-be-a-bird, and suppose that, from the point of view of a particular agent, its defining feature set in the context of living beings is the set {to-have-two-legs, to-lay-eggs, to-have-a-beak, to-have-wings}, equipped the salience order: to-have-a-beak $>_s$ to-lay-eggs $>_s$ to-have-two-legs, and to-have wings $>_s$ to-lay-eggs $>_s$ to-have-two-legs. Suppose for the sake of simplicity that, in the agent's mind, membership to any of these features is a two-valued function. Let s , m , t , b and d respectively stand for a sparrow, a mouse, a tortoise, a bat and a dragonfly. Then the induced membership order is determined by the following array :*

	<i>two – legs</i>	<i>lay – eggs</i>	<i>beak</i>	<i>wings</i>
<i>sparrow</i>	★	★	★	★
<i>mouse</i>				
<i>tortoise</i>		★	★	
<i>bat</i>	★			★
<i>dragonfly</i>		★		★

One readily checks that $d \prec_\alpha s$, $m \prec_\alpha t$, and $m \prec_\alpha b$. Note that one has $b \preceq_\alpha d$, since the concept to-have-two-legs under which the bat falls, contrary to the dragonfly, is dominated by the concept to-lay-eggs that applies to the dragonfly and not to the bat. On the other hand, one does not have $d \preceq_\alpha b$, as nothing compensates the fact that the dragonfly lays eggs and the bat does not. This yields $b \prec_\alpha d$. Note also that the tortoise and the bat are incomparable: one has neither $b \preceq_\alpha t$, nor $t \preceq_\alpha b$.

The strict α -membership order therefore reads:



The maximal elements of \preceq_α^μ form the *extension* $Ext\alpha$ of α . One has $x \prec_\alpha^\mu z$ for every element $z \in Ext\alpha$ and $x \notin Ext\alpha$.

The second order we shall deal with is the *typicality order* \preceq_α^τ . It is induced on $Ext\alpha$ by the characteristic set $\chi(\alpha)$, that is $\preceq_\alpha^\tau = \preceq_{Ext\alpha}^{\chi(\alpha)}$. For x and y in $Ext\alpha$, we have then:

$$(4) \quad x \preceq_\alpha^\tau y \text{ iff } \forall f \in \chi(\alpha) : \delta_f(y) < \delta_f(x), \exists g \in \chi(\alpha), g >_{\chi(\alpha)} f : \\ \delta_g(x) < \delta_g(y).$$

The \preceq_α^τ -maximal elements of $Ext\alpha$ form the set $Typ\alpha$ of *typical* instances of α . One has $x \prec_\alpha^\tau z$ for every element $z \in Typ\alpha$ and $x \notin Typ\alpha$.

The sets $Ext\alpha$ and $Typ\alpha$ can be used to investigate the notion of *subconcept*. A subconcept of α is a concept β such that $Ext\beta \subseteq Ext\alpha$. In [14] we showed the necessity of refining this notion and isolating two particularly interesting kinds of subconcepts. The first one is that of *smooth subconcept* ([14] definition 11): a subconcept β of α is *smooth* in α if it satisfies $Typ\beta \subseteq Typ\alpha$. For instance, *to-be-a-white-bird* is smooth in *to-be-a-bird*. The second one is that of a *typical subconcept*: β is a typical subconcept of α if $Ext\beta \subseteq Typ\alpha$. Thus, *to-be-a-robin* is a typical subconcept of *to-be-a-bird*. These notions play an important role in concept-based induction. The interested reader may report to the chapter 7 of [14].

Remark 1 *It is interesting to compare the notions of salience and the resulting typicality order with the Typicality Criterion defined by Osta-Vélez and Gärdenfors in [31]. There, contrary to our approach, the authors define salience from typicality: they propose a salience quasi order \geq defined on $\chi(\alpha)$ by $f \geq g$ iff $\forall x : \delta_f(x) = 0, \exists y : \delta_g(y) = 0 : x \preceq_\alpha^\tau y$. Note that $f >_{\chi(\alpha)} g$ implies $f \geq g$, provided there exists an object y that does not fall under g but falls under all other characteristic features of α .*

Given a feature f , let us denote by $Ext f$ the set of elements $x \in \mathcal{O}$ such that $\delta_f(x) = 1$. We have then clearly $Ext\alpha = \bigcap_{f \in \Delta(\alpha)} Ext f$. One has similarly $Typ\alpha = Ext\alpha \cap \bigcap_{f \in \chi(\alpha)} Ext f$: an object x is a *typical* exemplar of a concept α if and only if $\delta_f(x) = 1 \forall f \in \Delta(\alpha) \cup \chi(\alpha)$

The knowledge of the membership and the typicality orders induced by a concept renders possible the process of categorization. Not only does it choose among the set \mathcal{O} the objects that fall under the concept, and, among

them, those that can be considered as typical, but it also establishes a relation among the enviring objects, comparing their distance to the referred category.¹ Note however that two agents may agree on the membership and the typicality orders of a concept, while having in mind a different defining set or a different characteristic one. This observation leads to the following notion:

Definition 2 *Two concepts α and β are equivalent, written $\alpha \equiv \beta$, if they induce the same membership and typicality orders.*

One has clearly $Ext \alpha = Ext \beta$ and $Typ \alpha = Typ \beta$ whenever $\alpha \equiv \beta$, but the converse need not be true. For instance, we may have $\Delta(\alpha) = \Delta(\beta)$, while $>_{\Delta(\alpha)} \neq >_{\Delta(\beta)}$. In this case, the extensions will agree, but the associated membership orders may not be the same.

2.2.2 Concept determination

Concept composition is a blending operation that yields a new concept from two components. The problem of how to form such a composition and of determining the properties inherited by the resulting concept has been at the origin of numerous articles (see in particular [29], [20], [23], [25] and [41]).

We proposed on [12] to introduce a *determination connective* that accounts for a specific kind of concept combination. This idea was developed in [14].

The determination of a concept α by a concept β is possible whenever $Ext \beta \cap Ext \alpha \neq \emptyset$. It yields a compound concept, denoted by $\beta \star \alpha$, that corresponds to a noun-noun combination - like in *to-be-a-pet-fish*, or *to-be-a-barnyard-bird* - and, more generally, to a relative clause that will be globally encapsulated by the concept β (e.g. *to-be-an-American-who-lives-in-Paris*). Typically, the determiner β becomes a simple feature of the compound concept $\beta \star \alpha$. Its role can be considered as secondary, compared with that played by the principal concept α : *to-be-a-woman* becomes a feature of *to-be-physician-that-is-a-woman*. Unlike conjunction, concept determination is not commutative : to take a well-known example, the concept of *games-that-are-sports* differs from the concept of *sports-that-are-games*. It is important

¹The distance of an object x to a category $Ext \alpha$ may be defined as the maximal length of a chain $x \prec_{\alpha}^{\mu} x_1 \prec_{\alpha}^{\mu} x_2 \prec_{\alpha}^{\mu} \dots \prec_{\alpha}^{\mu} x_n$ with $x_n \in Ext \alpha$.

to keep in mind that only the *intersective* conceptual combinations are accounted for: we consider the determination of α by β in the only case where $Ext\ \alpha \cap Ext\ \beta \neq \emptyset$ (see [25] for the distinction between intersective and non-intersective determiner). This shows that the determination connective \star is only a *partial* operator: given arbitrary α and β , it may be meaningless to build the concept $\beta \star \alpha$. For instance, there is no sense in talking of a *violin-bus* or a *wolf-frog*. Such pseudo-concepts correspond to nothing, and no object, real or fictitious, can be thought of falling under them, contrary to imaginary concepts like a *plane-car*, or a *Martian-teacher*: these latter definitely have a non-empty extension, because we can imagine a car that flies and a teacher who comes from Mars.

Given two featured concepts α and β such that $Ext\ \alpha \cap Ext\ \beta \neq \emptyset$, the *determination* $\beta \star \alpha$ of α by β can be described by a membership order $\preceq_{\beta \star \alpha}^{\mu}$ and a typicality order $\preceq_{\beta \star \alpha}^{\tau}$ defined as follows:

$$(5) \quad \text{for } x, y \in \mathcal{O}, x \preceq_{\beta \star \alpha}^{\mu} y \text{ iff } x \preceq_{\alpha}^{\mu} y \text{ and } (x \prec_{\alpha}^{\mu} y \text{ or } x \preceq_{\beta}^{\mu} y).$$

$$(6) \quad \text{for } x, y \in Ext\ \alpha \cap Ext\ \beta, x \preceq_{\beta \star \alpha}^{\tau} y \text{ iff } x \preceq_{\alpha}^{\tau} y \text{ and } (x \prec_{\alpha}^{\tau} y \text{ or } x \preceq_{\beta}^{\tau} y).$$

Note that both orders give priority to α over β .

The set of $\preceq_{\beta \star \alpha}^{\mu}$ -maximal elements of \mathcal{O} forms the extension $Ext(\beta \star \alpha)$ of the concept $(\beta \star \alpha)$. One has $Ext(\beta \star \alpha) = Ext\ \alpha \cap Ext\ \beta$.

The set of $\preceq_{\beta \star \alpha}^{\tau}$ -maximal elements of $Ext(\beta \star \alpha)$ forms the set $Typ(\beta \star \alpha)$ of typical instances of $(\beta \star \alpha)$.

Typicality is *not* compositional: typical pet-fish are not typical fish that are typical pets. However, in the particular case where $Typ\ \alpha \cap Typ\ \beta \neq \emptyset$, it can be shown that $Typ(\beta \star \alpha) = Typ\ \alpha \cap Typ\ \beta$ ([14], Theorem 2). In this case, $\beta \star \alpha$ has the structure of a featured concept with membership order equal to $\preceq_{\beta \star \alpha}^{\mu}$ and typicality order equal to $\preceq_{\beta \star \alpha}^{\tau}$.

As shown in [13], the above construction provides a faithful interpretation of conjunction fallacy and is free of some shortcomings encountered in classical theories. Our task is now to extend this construction to the case where a *feature* is used as a concept modifier.

3 Features as concept determiners

3.1 The impact of a mixed compound concept

The representation of compound concepts proposed in the preceding section can be extended to the determination of a featured concept α by a single feature h . As we shall see, it is indeed possible to build an ordinal model that accounts for simple constructions of adjective-noun type.

Definition 3 *Let α be a featured concept and h a feature. Suppose that $Ext\ h \cap Ext\ \alpha \neq \emptyset$. The membership order $\preceq_{h\star\alpha}^\mu$ on \mathcal{O} is defined by :*

$$(7) \quad x \preceq_{h\star\alpha}^\mu y \text{ iff } x \preceq_\alpha^\mu y \text{ and } x \prec_\alpha^\mu y \text{ or } \delta_h(x) \leq \delta_h(y).$$

We denote by $Ext(h \star \alpha)$ the set of $\preceq_{h\star\alpha}^\mu$ -maximal elements of \mathcal{O} . It consists of the elements x of \mathcal{O} such that $\delta_f(x) = 1 \forall f \in \Delta(\alpha) \cup \{h\}$. One has immediately $Ext(h \star \alpha) = Ext\ h \cap Ext\ \alpha$.

We now turn to the definition of the typical order on the set $Ext(h \star \alpha)$. Note that if x and y are two elements of this set, one has $\delta_h(x) = \delta_h(y) = 1$. This justifies the following definition:

Definition 4 *The typicality order $\preceq_{h\star\alpha}^\tau$ of the concept $h \star \alpha$ is the restriction to the set $Ext\ h \cap Ext\ \alpha$ of the order \preceq_α^τ .*

Using the notation paragraph 2.2.1, we have $\preceq_{h\star\alpha}^\tau = \preceq_{Ext\ h \cap Ext\ \alpha}^{\chi(\alpha)}$.

Definition 5 *The $\preceq_{h\star\alpha}^\tau$ -maximal elements of $Ext\ h \cap Ext\ \alpha$ form the set $Typ(h \star \alpha)$ of typical instances of the concept $h \star \alpha$.*

In the general case, there exists no link between $Typ\ \alpha$ and $Typ(h \star \alpha)$. As we shall see now, the nature of h , considered as a modifier of α , plays an important role in evaluating the typicality of $Typ(h \star \alpha)$.

3.2 Exceptional and non-exceptional modifiers

There exists an important distinction concerning the structural changes that the determination of a concept by a feature may bring to this concept. This question was evoked in [5] and [24], where was studied the impact of a *blank predicate* on a predicate-noun combination, and in [35] (thereafter developed

in [36]), where was operated a distinction between *compatible*, *not compatible* and *neutral* adjectives.

To give an idea of what is involved, consider for instance the concept *to-be-a-bird*. If we compose this concept with the feature *to-be-black*, we get the concept *to-be-a-black-bird*, whose typical instances are the typical birds that are black. But if we compose it with the feature *to-live-on-water*, we get as a result the concept *to-be-an-aquatic-bird*, whose typical instances have little to do with typical birds. The difference between these two examples is that, in the first example, the feature *to-be-black* does not stand in contradiction with the features that are specifically attributed to typical birds. In the second example, on the contrary, the feature *to-live-on-water* (or *to-be-aquatic*) stands in contradiction with the feature *to-live-in-the-trees* that an agent may consider as applying to all typical birds: for this agent, no typical bird lives on water.

This observation lead to the following definition, proposed in [12]:

Definition 6 *A feature h is exceptional for α if $Typ\alpha \cap Ext h = \emptyset$, that is if h applies to no typical instance of α . On the contrary, h is said to be non-exceptional for α if $Typ\alpha \cap Ext h \neq \emptyset$.*

Recalling that $Typ\alpha = Ext\alpha \cap (\bigcap_{f \in \chi(\alpha)} Ext f)$, we see that a feature h is exceptional for α if and only if $Ext h \cap Ext\alpha \cap (\bigcap_{f \in \chi(\alpha)} Ext f) = \emptyset$.

An important result concerning the determination of a featured concept α by a non-exceptional feature h is given by the following:

Theorem 1 *Let h be a feature that is not exceptional for α . Then one has $Typ(h \star \alpha) = Ext h \cap Typ\alpha$. The concept $h \star \alpha$ is equivalent to a featured concept whose defining feature set $\Delta(h \star \alpha)$ is equal to $\{h\} \cup \Delta(\alpha)$, and whose characteristic set $\chi(h \star \alpha)$ is equal to $\chi(\alpha)$ if $h \notin \chi(\alpha)$, and to $\chi(\alpha) \setminus \{h\}$ if $h \in \chi(\alpha)$.*

By the first part of the theorem, we see that $h \star \alpha$ is a smooth subconcept of α . More precisely, the typical instances of $(h \star \alpha)$ are the typical instances of α to which applies the attribute h : thus, a typical *black-bird* is simply a typical bird that is black. In this particular case, typicality is compositional.

Example 2 *Suppose that the defining features of the concept to-be-a-bird consist of the set {to-be-warm-blooded, to-have-a-beak, to-have-feathers, to-have-wings}, and that its characteristic set is equal to {to-fly, to-sing, to-live-in-the-trees, to-eat-seeds}. Then the concept to-be-a-flying-bird is equivalent*

to a featured concept with defining feature set equal to {to-fly, to-be-warm-blooded, to-have-a-beak, to-have-feathers, to-have-wings}, and the characteristic set is {to-sing, to-live-in-the-trees, to-eat-seeds}: the feature to-fly has simply moved from the characteristic set of to-be-a-bird to the defining set of to-be-a-flying-bird.

Theorem 1 has attractive consequences for a particular class of features:

Definition 7 *Given a featured concept α and a feature g , we say that g is a typical attribute of α if g applies to all typical elements of α , that is if $Typ\alpha \subseteq Ext\,g$.*

By Theorem 1, we see that if h is not exceptional for α , a typical attribute for α remains a typical attribute for $h\star\alpha$. Thus, to take a well-known example, knowing that *ducks have webbed feet*, the model predicts that *quacking ducks have webbed feet*.

It is interesting to translate this result in the framework of inference relations: given a concept α and a feature g , let us say that α generally implies g , or that α typically implies g , written $\alpha|\sim g$, if g is a typical attribute of α . Then the meaning of a conditional like $\alpha|\sim\neg h$ is that h applies to no typical instance of α , that is that h is exceptional for α . It is therefore natural to write $\alpha\not|\sim\neg h$ to express the fact that h is not exceptional for α . Using this formalism, we see that Theorem 1 can be translated into the well-known rule of *Rational Monotony* (see [18], or [26] for a general presentation of preferential inference relations).

$$\text{Rational Monotony} \quad \frac{\alpha|\sim g, \alpha\not|\sim\neg h}{h\star\alpha|\sim g}$$

For example, knowing that *birds generally live in the trees* and that *there exists typical birds that are black*, our model predicts that *black birds generally live in the trees*.

It is clear that a typical attribute of α is non exceptional for α . Applying Theorem 1 in the case where h is a typical attribute of α , we get the interesting

Corollary 1 *For any typical attribute h of α , one has $Typ(h\star\alpha) = Typ\alpha$.*

It follows from this that a feature g is a typical attribute of α if and only if g is a typical attribute of $h\star\alpha$. Translating this result in the framework of non-monotonic inference relations enables us to retrieve two fundamental rules of preferential logic :

Cautious Monotony $\frac{\alpha \sim h, \alpha \sim f}{h \star \alpha \sim f}$

Cut $\frac{\alpha \sim h, h \star \alpha \sim f}{\alpha \sim f}$

These two rules may be reinterpreted in the framework of Concept Based Induction. The first one provides a solution to the problem of Within-Category Induction: the subconcept $h \star \alpha$ inherits the typical attributes of α . Thus, knowing that *birds generally eat seeds* and *birds generally fly*, we may deduce that *flying-birds generally eat seeds*. The second one is an answer to the Over-Category induction problem: the concept α inherits the typical attributes of its subconcept $h \star \alpha$. For instance, knowing that *flying birds are generally small* and that *birds generally fly*, we can conclude that *birds are generally small*.

As we shall see, the result of Theorem 1 can be extended to the exceptional case, provided this exceptionality is due to a single characteristic feature of α . Before examining this situation, we need to establish a technical result:

Proposition 1 *Let h be a feature and k a characteristic feature of α such that $Ext\ h \cap Ext\ \alpha \cap Ext\ k = \emptyset$ but $Ext\ h \cap Ext\ \alpha \cap (\bigcap_{f \in \chi(\alpha) \setminus \{k\}} Ext\ f) \neq \emptyset$. Suppose that δ_k is a two-valued function, so that we have $\delta_k : \mathcal{O} \rightarrow \{0, 1\}$. Then one has $Typ(h \star \alpha) = Ext\ h \cap Ext\ \alpha \cap (\bigcap_{f \in \chi(\alpha) \setminus \{k\}} Ext\ f)$.*

It follows from this proposition that the typical elements of $h \star \alpha$ are easily determined, even in this exceptional case: they correspond to the instances of α to which h applies, and to which applies every characteristic feature of α different from k . For instance, if the characteristic set of *to-be-a-dog* consists of the features *to-bark*, *to-run-after-sticks* and *to-gnaw-bones*, we deduce from this proposition that typical *silent* dogs run after sticks and gnaw bones, although they do not bark.

Definition 8 *A feature h for which the hypotheses of proposition 1 are satisfied is said to be k -exceptional for α ; the resulting concept $h \star \alpha$ will be called a k -exceptionally modified concept.*

We shall speak of an *elementary feature* when referring to a feature whose applicability is given by a two-valued function.

We now prove that, as in the non-exceptional case, a k -exceptionally modified concept is equivalent to a featured concept:

Theorem 2 *If k is an elementary feature, the k -exceptionally modified concept $h \star \alpha$ is equivalent to a featured concept whose defining feature set $\Delta(h \star \alpha)$ is equal to $\Delta(\alpha) \cup \{h\}$ and characteristic set $\chi(h \star \alpha)$ equal to $\chi(\alpha) \setminus \{k\}$.*

Theorem 1 and Theorem 2 show that the family of featured concepts is stable under feature determination when the determiner contradicts at most one of the characteristic features of the original concept. This is an interesting result, as it renders possible the study of k -exceptionally modified concepts with the tools used for elementary featured concepts.

4 Resemblance to modified concepts

We now turn to an important problem of the categorization process, which concerns the link that exists between categorial membership, typicality and resemblance. The notion of resemblance (or similarity) plays a key role in most theories of concepts (see for instance [33], [40], [42], and more recently [30] or [3]).

We shall briefly recall the basis of our approach before examining how resemblance theory can be applied to the case of mixed compound concepts.

4.1 The stereotypical set as a tool for resemblance evaluation

As noted in [14], resemblance may be considered as a binary relation between objects (*Henry resembles his brother*), between concepts (*the wolf resembles a dog*), or between an object and a concept (*Henry's dog resembles a wolf*). We shall only treat the two latter cases, in which the 'referent' of the comparison is a concept.

We observed in [14] that resemblance between an object and a concept has to be evaluated through the *typical instances* of this concept. Thus, looking at a bat, one may say it resembles a bird just because it has wings, flies, and has the size or the shape of a bird. Conversely, an animal may be declared *not* to resemble a bird if it does not resemble a *typical* bird, even though this animal is known to be a bird. For instance, looking at a penguin, an assertion like 'this animal does not resemble a bird' is perfectly justified. Resemblance first deals with the typical attributes of a concept.

Similarly to what was done for the notions of categorial membership and typicality, resemblance theory can be modeled through a qualitative approach, so that it is possible to *compare* the resemblance of two objects to a given concept. Such an ordinal model would account for judgments like *Henry's dog resembles more a wolf than Jack's wolfhound*. For this purpose, given a featured concept α , we could equip the set $\Delta(\alpha) \cup \chi(\alpha)$ with the salience order that extends the salience orders of $\Delta(\alpha)$ and $\chi(\alpha)$ and makes every element of $\chi(\alpha)$ more salient than the elements of $\Delta(\alpha)$. Then the corresponding resemblance order could be defined as the order $\preceq_{\mathcal{O}}^{\Delta(\alpha) \cup \chi(\alpha)}$ induced on \mathcal{O} by $\Delta(\alpha) \cup \chi(\alpha)$. However, the preeminence this order would give to hierarchy over number, together with the fact that $\Delta(\alpha) \cup \chi(\alpha)$ contains more elements than $\Delta(\alpha)$ and $\chi(\alpha)$, may pose a problem. For instance, if α is the concept *to-be-a-bird*, and if *to-fly* has maximal salience in $\chi(\alpha)$, bird-resemblance will principally rest of the ability of an item to fly; consequently, bats will be more bird-resemblant than kiwis, although *having feathers, singing, having a beak* and *building nests* taken together should, at least compensate the fact that kiwis do not fly.

Another reason to consider changing our paradigm and letting aside the qualitative model is that it makes more sense to speak of a *degree of resemblance* than to speak of a *membership* or a *typicality* degree. For instance, to evaluate the α -membership degree of an item x , one has to take into account not only x and α , but also all the objects that stand between x and the category $Ext \alpha$ (see footnote 1). In contrast, to evaluate resemblance of an object with a concept, one has only to deal with the properties the concept shares with the object. It is a fact that, since the first work of Tversky ([39]), it is exclusively with quantitative approaches that resemblance has been studied.

For these reasons, we chose in [14] to work in a *unitary model*, more adapted than the binary model that was used in the representation of categorial membership and typicality. Thus, we proposed to attach with every featured concept its *stereotypical set* $\Pi(\alpha)$ (see [5], [8], or [24]). This set consists of the union of the defining feature set $\Delta(\alpha)$ and the characteristic set $\chi(\alpha)$, on which we put a salience order that gives priority to the characteristic features over the defining ones.

More precisely, we equip the set $\Pi(\alpha) = \Delta(\alpha) \cup \chi(\alpha)$ with the order $>_{\Pi(\alpha)}$ that extends the orders $>_{\chi(\alpha)}$ on $\chi(\alpha)$ and $>_{\Delta(\alpha)}$ on $\Delta(\alpha)$ and satisfies furthermore $f >_{\Pi(\alpha)} g$ for all $f \in \chi(\alpha)$ and $g \in \Delta(\alpha)$. This latter condition is justified by the fact that, as previously noted, resemblance with a concept

is first resemblance with the typical instances of this concept.

This salience order on $\Pi(\alpha)$ yields, for each $f \in \Pi(\alpha)$, a *salience degree* $s_\alpha(f)$ that reflects the number of stereotypical features that f dominates. This degree is defined by $s_\alpha(f) = 1 + |\{h \in \Pi(\alpha); f >_{\Pi(\alpha)} h\}|$.

We define the *weight* $w(\alpha)$ of the concept α by $w(\alpha) = \sum_{f \in \Pi(\alpha)} s_\alpha(f)$, and we denote by $\epsilon(\alpha)$ the number $\epsilon(\alpha) = \frac{|\Delta(\alpha)|}{w(\alpha)}$.

It is now possible to introduce the α -resemblance degree of an object x :

Definition 9 *The degree to which an object x resembles a featured concept α is the number $\rho_\alpha(x) = \frac{\sum_{f \in \Pi(\alpha)} s_\alpha(f) \delta_f(x)}{w(\alpha)}$.*

It is immediate that $\rho_\alpha(x) = 1$ if and only if $x \in Typ\alpha$. Note that one has $\rho_\alpha(x) \geq \epsilon(\alpha)$ for all objects $x \in Ext\alpha$: an object that falls under a concept cannot be totally dissimilar to this concept.

Example 3 *Suppose that the defining feature set associated with to-be-a-bird is the set $\{l = \text{to-have-two-legs}, e = \text{to-lay-eggs}, b = \text{to-have-a-beak}, w = \text{to-have-wings}\}$ with salience order: $b >_{\Delta(\alpha)} e >_{\Delta(\alpha)} l$, and $w >_{\Delta(\alpha)} e >_{\Delta(\alpha)} l$. Suppose also that the corresponding characteristic set consists of the three features: $n = \text{to-build-nests}$, $s = \text{to-sing}$, and $f = \text{to-fly}$, with the order: $f >_{\chi(\alpha)} n$, and $f >_{\chi(\alpha)} s$. We have $s_\alpha(l) = 1, s_\alpha(e) = 2, s_\alpha(b) = s_\alpha(w) = 3, s_\alpha(n) = s_\alpha(s) = 5$ and $s_\alpha(f) = 7$. This yields $w(\alpha) = 26$ and $\epsilon(\alpha) = 4/26$. If the applicability of all features is given by elementary functions, the bird-resemblance degrees of a bat ba , a penguin pe , a kiwi k and a plane pl is given by $\rho_\alpha(ba) = 11/26, \rho_\alpha(pe) = 9/26, \rho_\alpha(k) = 19/26$ and $\rho_\alpha(pl) = 10/26$. Note that the bat, which is not a bird, is more bird-resemblant than the penguin - which is a bird.*

Remark 2 *The notion of concept resemblance that results from Definition 8 is not based on the impact of a concept, that is on its induced membership and typicality orders. Rather, it rests on its Gestalt, giving priority to its defining and characteristic set.*

Note that resemblance degrees allow comparing the resemblance that an object may have with different concepts: it makes possible to express judgments like ‘Henry’s dog resembles more a fox than a wolf’.

The notion of resemblance between objects and concepts can be easily enlarged to that of resemblance between concepts. Indeed, the resemblance

between two concepts is best evaluated as a resemblance of their typical instances: to say that a concept β resembles a concept α amounts to saying that every typical instance of β resembles α . This justifies the following definition:

Definition 10 *The α -resemblance degree $\rho_\alpha(\beta)$ of a featured concept β is the number $\text{Min}_{x \in \text{Typ } \beta} \rho_\alpha(x)$.*

One has immediately $\rho_\alpha(\beta) = 1$ if and only if $\text{Typ } \beta \subseteq \text{Typ } \alpha$.

The resemblance degree provides a *similarity measure* that can be used in Concept-Based Induction (see [30] for an overview of different ways of modeling CBI and a geometrical approach of similarity). Note that the value of this resemblance degree is context-dependent: it rests on the features that, at a given time, an agent decides to consider as characteristic of a concept. For instance, the attributes of the concept *to-be-a-piano* are not the same if considered by an agent in the context of *music* or in the context of *removal*.

The following result shows that there exists a *resemblance threshold*: any object whose α -resemblance is large enough must fall, at least partially, under the characteristic features of α :

Theorem 3 *Let x be an object such that $\rho_\alpha(x) > 1 - \epsilon(\alpha)$. Then one has $\delta_g(x) > 0$ for all elements g of $\Pi(\alpha)$ such that $|\Delta(\alpha)| \leq s_\alpha(g)$.*

In particular, we see by this theorem that an object that is sufficiently α -resemblant will inherit every elementary characteristic feature of α . Thus, in Example 3, the resemblance threshold is equal to $22/26$ and any item whose bird-resemblance degree is greater than $22/26$ has, to some degree, the properties of flying, singing and building a nest...²

From the construction of the order $>_{\Pi(\alpha)}$, it follows from Theorem 3 that the elementary characteristic features of α apply to any typical instance of a concept that sufficiently resembles α . This result can be translated in the framework of concept-based induction: using the formalism described in section 3.2, we can display the following similarity rule, valid when g is an elementary feature:

$$\text{Similarity Rule } \frac{\rho_\alpha(\beta) > 1 - \epsilon(\alpha), g \in \chi(\alpha)}{\beta \sim g}$$

²If the characteristic features of Example 3 are elementary features, noting that only birds fly, sing and build nests, an item with resemblance degree greater than $22/26$ is necessarily a (typical) bird.

4.2 The case of compound mixed concepts

The results of the preceding section can be applied to evaluate resemblance to compound mixed concepts, either in the non-exceptional case, or in the case of a k -exceptionally modified concept. To do this, we can directly make use of the results of Theorem 1 and Theorem 2, assimilating $h \star \alpha$ with a featured concept and applying the results of the preceding section. This leads us to define the $h \star \alpha$ -resemblance degree of an object x by

$$\rho_{h\star\alpha}(x) = \frac{\sum_{f \in \Pi(h\star\alpha)} s_{h\star\alpha}(f) \delta_f(x)}{w(h\star\alpha)}.$$

We have then, as an analogue of theorem 3, the

Corollary 2 *Let x be an object such that $\delta_{h\star\alpha}^p(x) > 1 - \epsilon(h \star \alpha)$, and g an elementary feature of $\Pi(h\star\alpha)$ such that $|\Delta(h\star\alpha)| \leq s_{h\star\alpha}(g)$. Then g applies to x .*

A particular attention has to be given to the state of the modifier h in the set $\Pi(h \star \alpha)$. Observe indeed that, whether h is a non-exceptional or a k -exceptional feature, it results from the proofs of Theorem 1 and Theorem 2 that the salience order $>_{\Pi(h\star\alpha)}$ gives to h a salience $s_{(h\star\alpha)}(h)$ that is always less or equal than $|\Delta(\alpha)|$. By this we see that h does not generally satisfy the conditions of corollary 2, and nothing guarantees that this feature should apply to objects that are sufficiently $h \star \alpha$ -resemblant. This may lead to paradoxical results. For instance, our model built on the salience order $>_{\Pi(h\star\alpha)}$ predicts that an animal that strongly resembles a *bird* should fly, but that this will not be the case for an animal that strongly resembles a *flying bird* ! Indeed, the most salient characteristic feature of α , $f = to\text{-}fly$, became the least salient defining feature of $f \star \alpha$, and we have $s_{(h\star\alpha)}(f) = 1$.

This remark provides an important clue concerning the choice of the right salience order in $\Pi(h \star \alpha)$. If the feature h is not expected to apply to objects that are $h \star \alpha$ -resemblant, the original order $>_{\Pi(h\star\alpha)}$ need not be changed. But if h is expected to apply to objects similar to $h \star \alpha$, it is necessary to increase the salience degree of h to a number greater than $|\Delta(h \star \alpha)|$.

Another problem may occur in the case of a modification of α by a feature that is part of $\chi(\alpha)$: to take the above example, reducing to 1 the salience of f will make a bat more *bird*-resemblant ($\rho_\alpha(b) = 11/26$) than *flying-bird* resemblant ($\rho_{h\star\alpha}(b) = 7/26$). In any way, we expect that determining a concept by one of its characteristic features emphasizes its importance.

To remedy these shortcomings, in the case where h is expected to apply to $h \star \alpha$ -resemblant objects, we propose to consider the salience degree function $s_{h \star \alpha}^*$ that agrees with $s_{h \star \alpha}$ on $\Pi(h \star \alpha) \setminus \{h\}$ and gives to h a salience $s_{h \star \alpha}^*(h)$ equal to $|\Delta(h \star \alpha)|$. With these changes indeed, we will have in all cases $s_{h \star \alpha}^*(h) \geq |\Delta(h \star \alpha)|$, so that Corollary 2 will apply to h , provided h is an elementary feature.

To give an idea of what is involved, consider the following examples:

1. α stands for *to-be-a-dog* and h for *to-be-black*
2. α stands for *to-be-a-flower* and h for *to-be-striped*
3. α stands for *to-be-a-dog* and h for *to-be-young*
4. α stands for *to-be-a-flower* and h for *to-be-tropical*.

In the first two cases, we expect the modifier to apply to any item similar to the compound concept: one would not say that an animal resembles a black dog unless this animal is black; similarly, one would not say that an object resembles a striped flower unless this object has stripes. On the contrary, in the two last examples, the modifier is *not* expected to apply to objects that are $h \star \alpha$ resemblant: an animal that is said to resemble a young dog need not be young itself, and an item may resemble a tropical flower without having grown in the tropics.

It is difficult to decide which type of modified concept requires a model of the first or of the second kind. We conjecture that when the feature used to determine a concept has a simple ‘physical’ significance, directly related with one of our senses, strong resemblance to the modified concept implies the presence of this feature in any resemblant item. The situation is different when the modifier is more sophisticated. This may happen in particular when it shows, in the context of the principal concept, a certain number of proper characteristic attributes; it is these attributes, more than the feature itself, that are expected to be retrieved through resemblance. An animal that resembles a *young dog* has the behavior of a young dog, likely to be *clumsy*, *cute* and *playful*, a plant that resembles a *tropical flower* is expected to have *large leaves* and *flourishing colors*. When used in a comparison, such a feature may be considered as the conjunction of some undisplayed attributes. In the *young dog* example, these attributes would be *to-be-clumsy*, *to-be-cute* and *to-be-playful*. Then, (*to-be-young-dog*) may be assimilated to the simple concept (*to-be-a-puppy*).

More generally, if the modifier h shows, in the context of α , a certain number of characteristic attributes, this modifier may be assimilated with a featured concept. Then the composition $h \star \alpha$ can be operated as described in section 2.2.2. This shows that features with attributes deserve a special treatment. On one hand, they cannot be considered as simple features, because their applicability cannot be measured by a degree function. On the other hand, they cannot be considered as concepts because they have no significance outside of a precise context. This tends to show that our model finds its limits in the representation of such a kind of hybrid item.

Another point that deserves to be underlined is that, apart from its simple significance, the link between a feature and the concept it modifies has to be taken into account. An exceptional modifier has more weight in the modified concept than a non exceptional one: *to-be-unfriendly* is expected from an animal that resembles an unfriendly dog, although *to-be-unfriendly*, in the context of dogs, is an aggregate of several attributes.

From now on, we shall denote by $\Pi^*(h \star \alpha)$ the set $\Pi(h \star \alpha)$ endowed with the salience order induced by $s_{h \star \alpha}^*$, and by $\rho_{h \star \alpha}^*$ the corresponding degree function.

4.3 Comparing α and $h \star \alpha$ -resemblance

In this paragraph, we shall compare the α -resemblance and the $h \star \alpha$ -resemblance degree of an object. Making use of Definition 10, we shall also determine to what extent a concept α resembles its modified version $h \star \alpha$, and consequently to what extent the modified version differs from the head concept.

4.3.1 The non-exceptional case

Given a non exceptional modifier h , our aim is to compare the resemblance degrees ρ_α and $\rho_{h \star \alpha}$.

We need first to establish some technical results:

Lemma 2 *The salience degrees $s_{h \star \alpha}$ and $s_{h \star \alpha}^*$ are given by:*

- *In the case where $h \notin \Delta(\alpha) \cup \chi(\alpha)$:*

$$s_{h \star \alpha}(h) = 1,$$

$$s_{h \star \alpha}(f) = 1 + s_\alpha(f) \quad \forall f \in \Pi(\alpha) \setminus \{h\},$$

$$s_{h \star \alpha}^*(h) = 1 + |\Delta(\alpha)|,$$

$$s_{h\star\alpha}^*(f) = s_\alpha(f) \forall f \in \Pi(\alpha) \setminus \{h\}$$

- In the case where $h \in \Delta(\alpha)$:

$$s_{h\star\alpha}(f) = s_\alpha(f) \forall f \in \Pi(\alpha),$$

$$s_{h\star\alpha}^*(h) = |\Delta(\alpha)|,$$

$$s_{h\star\alpha}^*(f) = s_\alpha(f) \forall f \in \Pi(\alpha) \setminus \{h\}$$

- In the case where $h \in \chi(\alpha)$:

$$s_{h\star\alpha}(h) = 1,$$

$$s_{h\star\alpha}(f) = 1 + s_\alpha(f) \forall f \in \Delta(\alpha),$$

$$s_{h\star\alpha}(f) = s_\alpha(f) \text{ if } f \in \chi(\alpha), f \succ_{\chi(\alpha)} h,$$

$$s_{h\star\alpha}(f) = 1 + s_\alpha(f) \text{ if } f \in \chi(\alpha), f \not\succ_{\chi(\alpha)} h,$$

$$s_{h\star\alpha}^*(f) = s_\alpha(f) \forall f \in \Pi(\alpha).$$

This result enables us determining the link between the weights $w(\alpha)$ and $w(h \star \alpha)$.

Proposition 2 *One has*

$$w(h \star \alpha) = 1 + w(\alpha) + |\Pi(\alpha)| \text{ if } h \notin \Delta(\alpha) \cup \chi(\alpha),$$

$$w(h \star \alpha) = w(\alpha) \text{ if } h \in \Delta(\alpha) \text{ if } h \in \chi(\alpha),$$

$$w(h \star \alpha) = w(\alpha) + 1 + |\Delta(\alpha)| - s_\alpha(h) + |\{f \in \chi(\alpha) \setminus \{h\}, f \not\succ_{\chi(\alpha)} h\}|.$$

The weights $w(h \star \alpha)$ being given as above, we can now compare the $h \star \alpha$ -resemblance of an object and its α -resemblance:

Proposition 3 *The $h \star \alpha$ resemblance degree $\rho_{h\star\alpha}(x)$ of an element $x \in \mathcal{O}$ satisfies:*

$$w(h \star \alpha)\rho_{h\star\alpha}(x) = \sum_{f \in \chi(\alpha)} \delta_f(x) + w(\alpha)\rho_\alpha(x) + (1 + |\Delta(\alpha)|)\delta_h(x) \text{ if } h \notin \Pi(\alpha)$$

$$w(h \star \alpha)\rho_{h\star\alpha}(x) = \sum_{f \in \Pi(\alpha)} \delta_f(x) + w(\alpha)\rho_\alpha(x) + \delta_h(x) \text{ if } h \notin \Pi(\alpha),$$

$$\rho_{h\star\alpha}(x) = \rho_\alpha(x) \text{ if } h \in \Delta(\alpha), \text{ and, if } h \in \chi(\alpha),$$

$$w(h \star \alpha)\rho_{h\star\alpha}(x) = w(\alpha)\rho_\alpha(x) - s_\alpha(h)\delta_h(x) + \delta_h(x) + \sum_{f \in \Pi(\alpha) \setminus \{h, f \not\succ_{\chi(\alpha)} h\}} \delta_f(x).$$

Example 4 *In the context of birds, consider the feature $h = \text{to-be-european}$, and let us compute how much a bat ba resembles an european-bird. Since we do not expect an animal that resembles an european bird to be itself european, we use the salience function $s_{h\star\alpha}(f)$. We have then $w(h \star \alpha) = 34$; ba falls under the three characteristic features to-have two-legs, to-have-wings, to-fly. We saw that $w(\alpha) = 26$, and $\delta_\alpha(\text{ba}) = 11/26$. This shows that $\delta_{h\star\alpha}(\text{ba}) = 14/34$.*

Applying Proposition 3 to the elements x of $Typ\ \alpha$ enables us to determine the $h \star \alpha$ resemblance of the concept α :

Theorem 4 *In the non exceptional case, given the order $>_{\Pi(h\star\alpha)}$ on $\Pi(h\star\alpha)$, the $h \star \alpha$ resemblance of α is given by*

$$\begin{aligned}\rho_{h\star\alpha}(\alpha) &= 1 - \frac{1}{w(h\star\alpha)} \text{ if } h \notin \Pi(\alpha), \\ \rho_{h\star\alpha}(\alpha) &= 1 \text{ if } h \in \Pi(\alpha).\end{aligned}$$

Concerning the α -resemblance degree of $h \star \alpha$, things are much simpler: indeed, given an element x of $Typ(h \star \alpha)$, one has $x \in Typ\ \alpha$ by Theorem 1, and therefore $\rho_\alpha(x) = 1$. It follows that $\rho_\alpha(h \star \alpha) = 1$: a *black dog* maximally resembles a *dog*, although a *dog* does not maximally resemble a *black dog*.

The model with salience functions $s_{h\star\alpha}(f)$ is inadequate when the modifier h is expected to apply to items that are sufficiently $h \star \alpha$ -resemblant, and in particular when h is a characteristic feature of α . Suppose for example that, in the context of birds, h is the feature *to-fly*. Then a computation of the *flying-bird*-resemblance of a penguin through the functions $s_{h\star\alpha}(f)$ shows that its degree is equal to $13/26$, and therefore greater than its *bird*-resemblance degree, which we found equal to $9/26$. This paradox is a consequence of the fact that resemblance with a concept is, before all, resemblance with the typical instances of the concept. The feature *to-fly*, which became a defining feature of *to-be-a-flying-bird*, is no more a salient characteristic feature of this concept. As we shall see now, the use of the salience degree $s_{h\star\alpha}^*$ can remedy this situation.

Suppose that the salience on $\Pi(h \star \alpha)$ is given by the function $s_{h\star\alpha}^*$. Then the analogue of Proposition 2 for the weight $w^*(h \star \alpha)$ reads:

Proposition 4 *One has*

$$\begin{aligned}w^*(h \star \alpha) &= 1 + w(\alpha) + |\Delta(\alpha)| \text{ for } h \notin \Delta(\alpha) \cup \chi(\alpha), \\ w^*(h \star \alpha) &= w(\alpha) - s_\alpha(h) + |\Delta(\alpha)| \text{ if } h \in \Delta(\alpha), \\ w^*(h \star \alpha) &= w(\alpha) \text{ if } h \in \chi(\alpha).\end{aligned}$$

The link between $\rho_{h\star\alpha}^*(x)$ and $\rho_\alpha(x)$ is then given by the

Proposition 5 *The $h \star \alpha$ -resemblance degree of an element $x \in \mathcal{O}$ satisfies:*

$$\begin{aligned}w^*(h \star \alpha)\rho_{h\star\alpha}^*(x) &= \sum_{f \in \chi(\alpha)} \delta_f(x) + w(\alpha)\rho_\alpha(x) + (1 + |\Delta(\alpha)|)\delta_h(x) \text{ if } h \notin \Pi(\alpha), \\ w^*(h \star \alpha)\rho_{h\star\alpha}^*(x) &= w(\alpha)\rho_\alpha(x) - s_\alpha(h)\delta_h(x) - \sum_{f \in \Delta(\alpha), f >_{\Delta(\alpha)} h} \delta_f(x) + |\Delta(\alpha)|\delta_h(x) \\ &\text{when } h \in \Delta(\alpha), \text{ and} \\ \rho_{h\star\alpha}^*(x) &= \rho_\alpha(x) \text{ if } h \in \chi(\alpha).\end{aligned}$$

The paradox evoked above disappears when salience is evaluated by the function $s_{h\star\alpha}^*$: we find that the *flying-bird*-resemblance of a penguin becomes equal to its *bird*-resemblance.

Example 5 *We take again the bird example and compute the resemblance of a bat with the concept to-be-a-white-bird. An animal resembling a white-bird is expected to be white, and we shall therefore use the salience $s_{h\star\alpha}^*$. We have $w^*(h\star\alpha) = 31$, and it follows from the first line of the above proposition that $\rho_{h\star\alpha}^*(ba) = 12/31$. A bat resembles less a white-bird than an european-bird.*

From Proposition 5 we deduce the $h\star\alpha$ -resemblance of a featured concept α :

Theorem 5 *In the non exceptional case, we have:*

$$\begin{aligned}\rho_{h\star\alpha}^*(\alpha) &= \frac{|\chi(\alpha)|+w(\alpha)}{w^*(h\star\alpha)} \text{ if } h \notin \Pi(\alpha), \\ \rho_{h\star\alpha}^*(\alpha) &= 1 \text{ if } h \in \Pi(\alpha).\end{aligned}$$

Example 6 *If h is the feature to-be-white and α stands for to-be-a-bird we have $w^*(h\star\alpha) = 31$, $|\chi(\alpha)| = 3$ and $w(\alpha) = 26$. Birds resemble white birds to the degree $29/31$.*

4.3.2 The exceptional case

In the case where h is k -exceptional, we recall that it follows from the proof of Theorem 2, that $\chi(h\star\alpha) = \chi(\alpha) \setminus \{k\}$ and $\Delta(h\star\alpha) = \{h\} \cup \Delta(\alpha)$. The order in this latter set makes h less salient than any element of $\Delta(\alpha)$. The salience order $>_{\Pi(h\star\alpha)}$ is defined as in the case of simple featured concepts (see the beginning of section 4.1). Similarly to what was done in the preceding paragraph, we will also consider the salience order $s_{\Pi(h\star\alpha)}^*$, in which h is given a salience equal to $1 + |\Delta(\alpha)|$.

For k -exceptionally modified concepts, the analogue of Lemma 2 reads:

Lemma 3 *If h is k -exceptional, the salience degrees in $\Pi(h\star\alpha)$ are given by*

$$\begin{aligned}s_{h\star\alpha}(h) &= 1, \\ s_{h\star\alpha}(f) &= 1 + s_\alpha(f) \forall f \in \Pi(h\star\alpha) \setminus \{h\}, f \not\triangleright_{\Pi(h\star\alpha)} k \\ s_{h\star\alpha}(f) &= s_\alpha(f) \text{ for all other elements of } \Pi(h\star\alpha) \setminus \{h\}. \\ s_{h\star\alpha}^*(h) &= |\Delta(\alpha)| + 1, \\ s_{h\star\alpha}^*(f) &= s_\alpha(f) \forall f \in \Pi(\alpha) \setminus \{k\}.\end{aligned}$$

From this we draw the relation between the weight of $h \star \alpha$ and the weight of α .

Proposition 6 *Suppose that h is k -exceptional for α . Then*
 $w(h \star \alpha) = w(\alpha) + |\Delta(\alpha)| + 1 - s_\alpha(k) + |\{f \in \chi(\alpha) \setminus \{k\}, f \not\triangleright_{\chi(\alpha)} k\}|$.
 $w^*(h \star \alpha) = w(\alpha) - s_\alpha(k) + 1 + |\Delta(\alpha)|$.

The two above results allow the calculation of the $h \star \alpha$ resemblance degrees when h is k -exceptional for α :

Proposition 7 *In the k -exceptional case, the $h \star \alpha$ -resemblance degrees of an object x are given by:*

$$w(h \star \alpha)\rho_{h\star\alpha}(x) = w(\alpha)\rho_\alpha(x) + \delta_h(x) - s_\alpha(k)\delta_k(x) + \sum_{f \in \Pi(\alpha) \setminus \{k\}, f \not\triangleright_k} \delta_f(x)$$

$$w^*(h \star \alpha)\delta_{h\star\alpha}^{\rho^*}(x) = w(\alpha)\rho_\alpha(x) - s_\alpha(k)\delta_k(x) + (\Delta(\alpha) + 1)\delta_h(x).$$

Example 7 *Let us compute how much a penguin pe resembles a walking-bird. By Proposition 6, we have $w(h \star \alpha) = 26 + 4 + 1 - 7 + 2 = 26$, and by Proposition 7 and Example 3, $26 \cdot \rho_{h\star\alpha}(\text{pe}) = 26 \cdot (9/26) + 1 - 0 + 4 = 14$, showing that $\rho_{h\star\alpha}(\text{pe}) = 14/26$. Then $w^*(h \star \alpha) = 26 - 7 + 14 = 24$, so $24 \cdot \delta_{h\star\alpha}^{\rho^*}(\text{pe}) = 26 \cdot 9/26 - 0 + 4 + 1 = 14$, which yields $\delta_{h\star\alpha}^{\rho^*}(\text{pe}) = 14/24$.*

Let us finally evaluate the resemblance between a concept and its modification by an exceptional feature:

Proposition 8 *The resemblance of a concept with its modification by a k -exceptional feature is given by:*

$$\rho_{h\star\alpha}(\alpha) = 1 - \frac{1}{w(h\star\alpha)}$$

$$\delta_{h\star\alpha}^{\rho^*}(\alpha) = 1 - \epsilon(h \star \alpha)$$

Contrary to what we saw in the non-exceptional case, the resemblance of a k -exceptionally modified concept with its original head concept is no longer equal to 1:

Proposition 9 *One has $\rho_\alpha(h \star \alpha) = 1 - \frac{s_\alpha(k)}{w(\alpha)}$*

The α resemblance of $h \star \alpha$ is maximal when the relative salience of k in $\chi(\alpha)$ is $>_{\chi(\alpha)}$ -minimal. This translates the fact that, due to the k -exceptionality of h , k is no longer an element of the characteristic set of $h \star \alpha$. For instance, if we think that *to-live-in-the-trees* is not a salient attribute of birds, we are ready to consider that typical aquatic birds are not really dissimilar to birds.

5 Conclusion

The model that we studied in [14] to represent the basic notions linked with categorization can be carried over to mixed concept composition. It proposes a rigorous definition of notions that were shown to play an important role in concept study, like those of *smooth subconcept*, *concept equivalence* and *concept resemblance*, as well as the necessary distinction between *exceptional* and *non-exceptional modifiers*. It provides interesting results in the domain of concept composition, even in the exceptional case where the modifier stands in contradiction with some characteristic feature of the original concept. Problems linked with categorial membership, typicality, and resemblance can be then treated as in the elementary case, leading to interesting and non-trivial results. At the same time, such an ordinal model has the advantage of being intuitive and user-friendly, as it does not require the use of sophisticated mathematical tools.

However this study needs to be improved in several points. It appeared for instance that the notion of feature is not fully determined. In particular, we noticed an ambiguity concerning the treatment features should receive within the classical binary model. Apart from the limit case where the modifier has an hybrid status that stands half-way between a feature and a concept, two possible issues were considered, each leading to a different solution. In the first one, the modifier of a concept will apply to objects that sufficiently resemble the modified concept. In the second case, the modifier is not preserved by resemblance. In the end, it appears that the criteria on which we should base feature representation remain unclear at this stage, showing the necessity of refining our primitive model and investigate more closely the different ways a feature relates with a concept.

We are also aware that this model lacks experimental confirmations. Thus, we considered several times as granted some common sense statements. We claimed for instance that ordering the elements of a set provides a simpler and more accurate way of comparing them than attributing to each of them a degree number. This should be supported by experiments. Above all, the main results proposed by the model concerning membership ordering and resemblance evaluation definitely need to be confirmed by experiments. At this stage, this work can only be taken as an hypothetical construction.

A Proofs

Lemma 1 *The relation \preceq_T^X is a preorder. For two elements x and y of T , one has simultaneously $x \preceq_T^X y$ and $y \preceq_T^X x$ iff, $\forall f \in X$, $\delta_f(x) = \delta_f(y)$. The strict partial order \prec_T^X associated with \preceq_T^X satisfies:*

$$(8) \quad x \prec_T^X y \text{ iff } x \preceq_T^X y \text{ and } \exists f \in X : \delta_f(x) < \delta_f(y).$$

Proof: For the first part of the lemma, the reflexivity is immediate, and we have only to prove transitivity.

Let therefore x , y and z be three elements of T such that $x \preceq_X^T y$ and $y \preceq_X^T z$; suppose that there exists $f \in X$ such that $\delta_f(z) < \delta_f(x)$. We have to show that there exists a feature $g \in X$, g more salient than f , such that $\delta_g(x) < \delta_g(z)$. We consider two cases:

- Suppose first that $\delta_f(x) \leq \delta_f(y)$. Then we have $\delta_f(z) < \delta_f(y)$, and there exists a feature g of X , $g >_X f$, such that $\delta_g(y) < \delta_g(z)$. We can suppose that g is maximally salient in X for this property (recall that X is finite). If $\delta_g(x) \leq \delta_g(y)$, we get $\delta_g(x) < \delta_g(z)$ and we are done. Otherwise, we have $\delta_g(y) < \delta_g(x)$. Since we supposed $x \preceq_X^T y$, this implies that there exists a feature $h \in X$, h more salient than g , such that $\delta_h(x) < \delta_h(y)$. We cannot have $\delta_h(z) < \delta_h(y)$ otherwise there would exist $k \in X$, k more salient than h , and therefore more salient than g , such that $\delta_k(y) < \delta_k(z)$ contradicting the choice of g . We have therefore $\delta_h(y) \leq \delta_h(z)$, hence $\delta_h(x) < \delta_h(z)$ as desired.
- Suppose now that $\delta_f(y) < \delta_f(x)$. Then there exists $k \in X$, k more salient than f such that $\delta_k(x) < \delta_k(y)$, and we can again suppose k maximally salient for this property. In the case where $\delta_k(y) \leq \delta_k(z)$, we get $\delta_k(x) < \delta_k(z)$ and we are through. Otherwise, we have $\delta_k(z) < \delta_k(y)$ and there exists g more salient than k such that $\delta_g(y) < \delta_g(z)$. Let us show that $\delta_g(x) \leq \delta_g(y)$: if this were not the case, we would have $\delta_g(y) < \delta_g(x)$, so that there would exist h more salient than g such that $\delta_h(x) < \delta_h(y)$. But then h would be more salient than k , which is impossible. We have therefore $\delta_g(x) \leq \delta_g(y)$, hence $\delta_g(x) < \delta_g(z)$, which completes the proof of the transitivity.

Concerning the second part of the lemma, we observe that one has $x \preceq_T^X y$ and $y \preceq_T^X x$ whenever $\delta_f(x) = \delta_f(y) \forall f \in X$. Conversely, suppose that

one has $x \preceq_T^X y$ and $y \preceq_T^X x$, and let us show that $\delta_f(x) = \delta_f(y)$ for all $f \in X$. If this were not the case, there would exist a feature f of X such that $\delta_f(x) \neq \delta_f(y)$, f being $>_X$ -maximal for this property. We have either $\delta_f(x) < \delta_f(y)$, or $\delta_f(y) < \delta_f(x)$. In the first case, the hypothesis $y \preceq_T^X x$ implies the existence of a feature g of X , $g >_X f$, such that $\delta_g(y) < \delta_g(x)$, which contradicts the choice of f . In the second case, the hypothesis $x \preceq_T^X y$ leads again to a contradiction.

For the last part of the lemma, by definition of the strict partial order associated with a preorder, we have $x \prec_T^X y$ iff $x \preceq_T^X y$ and not $y \preceq_T^X x$. This latter condition is equivalent to the existence of a feature f of X that satisfies $\delta_f(x) < \delta_f(y)$ and such that $\delta_g(x) \leq \delta_g(y)$ for all features $g \in X$, $g >_X f$. The RHS of equation (1) then clearly follows from the LHS. If now the RHS is satisfied, that is if we have $x \preceq_T^X y$ and $\exists f \in X$ such that $\delta_f(x) < \delta_f(y)$, then, by what precedes, we cannot have $y \preceq_T^X x$. This completes the proof of the lemma. \blacksquare

Theorem 1 *Let h be a feature that is not exceptional for α . Then one has $Typ(h \star \alpha) = Ext h \cap Typ \alpha$. The concept $h \star \alpha$ is equivalent to a featured concept whose defining feature set $\Delta(h \star \alpha)$ is equal to $\{h\} \cup \Delta(\alpha)$, and whose characteristic set $\chi(h \star \alpha)$ is equal to $\chi(\alpha)$ if $h \notin \chi(\alpha)$, and to $\chi(\alpha) \setminus \{h\}$ if $h \in \chi(\alpha)$.*

Proof: Let z be an element of $Ext h \cap Typ \alpha$. Choose an element x in $Ext h \cap Ext \alpha$ that is not in $Typ \alpha$. We have then $x \prec_\alpha^\tau z$. Since x and z are elements of $Ext h \cap Ext \alpha$, it follows from Definition 4 that $x \prec_{h \star \alpha}^\tau z$. This shows that the $\preceq_{h \star \alpha}^\tau$ -maximal elements of $Ext h \cap Ext \alpha$ are the elements of $Ext h \cap Typ \alpha$.

We now prove the second part of the theorem.

A) We suppose first that $h \notin \Delta(\alpha) \cup \chi(\alpha)$.

To build the membership order $\preceq_{\mathcal{O}}^{\{h\} \cup \Delta(\alpha)}$, we define on the set $\{h\} \cup \Delta(\alpha)$ the salience order $>_{\{h\} \cup \Delta(\alpha)}$ that extends the salience order on $\Delta(\alpha)$ and makes every element of $\Delta(\alpha)$ more salient than h . We claim that one has $x \preceq_{h \star \alpha}^\mu y$ if and only if $x \preceq_{\mathcal{O}}^{\{h\} \cup \Delta(\alpha)} y$.

Given two objects x and y such that $x \preceq_{\mathcal{O}}^{\{h\} \cup \Delta(\alpha)} y$, let us indeed show that $x \preceq_{h \star \alpha}^\mu y$. If $f \in \Delta(\alpha)$ is a feature such that $\delta_f(y) < \delta_f(x)$, the hypothesis implies that there exists $g \in \{h\} \cup \Delta(\alpha)$, g more salient than f , such that $\delta_g(x) < \delta_g(y)$. By the choice of the salience order in $\{h\} \cup \Delta(\alpha)$,

we have necessarily $g \neq h$, so $g \in \Delta(\alpha)$, and we have proven that $x \preceq_{\alpha}^{\mu} y$. If $\delta_h(x) \leq \delta_h(y)$, this implies $x \preceq_{h\star\alpha}^{\mu} y$. If, on the contrary, $\delta_h(y) < \delta_h(x)$, by the hypothesis, this implies the existence of a feature $k \in \{h\} \cup \Delta(\alpha)$, k more salient than h , such that $\delta_k(x) < \delta_k(y)$. We have then $k \in \Delta(\alpha)$. By Lemma 1, this leads to $x \prec_{\alpha}^{\mu} y$, from which follows $x \prec_{h\star\alpha}^{\mu} y$. In any case, we have shown that $x \preceq_{h\star\alpha}^{\mu} y$ whenever $x \preceq_{\mathcal{O}}^{\{h\} \cup \Delta(\alpha)} y$.

Conversely, suppose that one has $x \preceq_{h\star\alpha}^{\mu} y$. To show that $x \preceq_{\mathcal{O}}^{\{h\} \cup \Delta(\alpha)} y$, we suppose that $\delta_f(y) < \delta_f(x)$ for some feature f of $\{h\} \cup \Delta(\alpha)$. We have to prove that there exists $g \in \{h\} \cup \Delta(\alpha)$, g more salient than f , such that $\delta_g(x) < \delta_g(y)$.

-If $f \in \Delta(\alpha)$, the hypothesis $x \preceq_{h\star\alpha}^{\mu} y$, which implies $x \preceq_{\alpha}^{\mu} y$, shows the existence of a feature $g \in \Delta(\alpha)$, more salient than f , such that $\delta_g(x) < \delta_g(y)$.

-If $f = h$, it follows from the hypothesis $x \preceq_{h\star\alpha}^{\mu} y$ that we have $x \prec_{\alpha}^{\mu} y$. Then, there exists $g \in \Delta(\alpha)$ such that $\delta_g(x) < \delta_g(y)$. By the choice of the salience order on $\Delta(h\star\alpha)$, g is more salient than h . In any case, we have shown that $x \preceq_{\mathcal{O}}^{\{h\} \cup \Delta(\alpha)} y$.

The equality $\preceq_{h\star\alpha}^{\mu} = \preceq_{\mathcal{O}}^{\{h\} \cup \Delta(\alpha)}$ is therefore established.

To check that $\preceq_{h\star\alpha}^{\tau} = \preceq_{Ext\,h \cap Ext\,\alpha}^{\chi(h\star\alpha)}$, recall that, by definition 4, the orders $\preceq_{h\star\alpha}^{\tau}$ and \preceq_{α}^{τ} agree on the set $Ext\,h \cap Ext\,\alpha$. Since $h \notin \chi(\alpha)$, we have by hypothesis $\chi(h\star\alpha) = \chi(\alpha)$. The order $\preceq_{h\star\alpha}^{\tau}$ is therefore the restriction to $Ext\,h \cap Ext\,\alpha$ of the order \preceq_{α}^{τ} , and thus equal to $\preceq_{Ext\,h \cap Ext\,\alpha}^{\chi(h\star\alpha)}$.

B) In the case where $h \in \Delta(\alpha)$.

The salience order on $\Delta(h\star\alpha)$ is that of $\Delta(\alpha)$, and we have $\preceq_{\alpha}^{\mu} = \preceq_{\mathcal{O}}^{\{h\} \cup \Delta(\alpha)}$. Let us show that $\preceq_{\alpha}^{\mu} = \preceq_{h\star\alpha}^{\mu}$: We know that $x \preceq_{h\star\alpha}^{\mu} y$ implies $x \preceq_{\alpha}^{\mu} y$. Conversely, suppose that we have $x \preceq_{\alpha}^{\mu} y$ and let us show that $x \preceq_{h\star\alpha}^{\mu} y$. We have to prove that $x \prec_{\alpha}^{\mu} y$ or $\delta_h(x) \leq \delta_h(y)$. But if we have $\delta_h(y) < \delta_h(x)$ there exists $g >_{\Delta(\alpha)} h$ such that $\delta_g(x) < \delta_g(y)$. This together with the fact that $x \preceq_{\alpha}^{\mu} y$ implies $x \prec_{\alpha}^{\mu} y$ as desired.

As for the typical order, the proof of the equality $\preceq_{h\star\alpha}^{\tau} = \preceq_{Ext\,h \cap Ext\,\alpha}^{\chi(h\star\alpha)}$ mirrors that made in case A).

C) In the case where $h \in \chi(\alpha)$.

We have $h \notin \Delta(\alpha)$ and the proof that $\preceq_{h\star\alpha}^{\mu} = \preceq_{\mathcal{O}}^{\{h\} \cup \Delta(\alpha)}$ mirrors that made in A).

Concerning the typicality order, note first that that $\chi(\alpha) \neq \{h\}$, since we supposed the existence of a non-empty salience order on this set. We put on $\chi(\alpha) \setminus \{h\}$ the restriction of the order $>_{\chi(\alpha)}$. Given two element x and y of $Ext h \cap Ext \alpha$, we have to prove that $x \preceq_{\alpha}^{\tau} y$ iff $x \preceq_{Ext h \cap Ext \alpha}^{\chi(\alpha) \setminus \{h\}} y$.

Suppose first that $x \preceq_{\alpha}^{\tau} y$ and let f be a feature of $\chi(\alpha) \setminus \{h\}$ such that $\delta_f(y) < \delta_f(x)$. Then $f \in \chi(\alpha)$, and there exists $g \in \chi(\alpha)$, g more salient than f , such that $\delta_g(x) < \delta_g(y)$. We cannot have $g = h$ because $\delta_h(x) = \delta_h(y) = 1$. We have then $g \in \chi(\alpha) \setminus \{h\}$, showing that $x \preceq_{Ext h \cap Ext \alpha}^{\chi(\alpha) \setminus \{h\}} y$.

Conversely, suppose that $x \preceq_{Ext h \cap Ext \alpha}^{\chi(\alpha) \setminus \{h\}} y$. If $\delta_f(y) < \delta_f(x)$ for some feature $f \in \chi(\alpha)$, we have $f \neq h$ by the choice of x and y , so $f \in \chi(\alpha) \setminus \{h\}$. It follows that there exists $g \in \chi(\alpha) \setminus \{h\}$, g more salient than f , such that $\delta_g(x) < \delta_g(y)$. We have therefore $x \preceq_{\alpha}^{\tau} y$ as desired, and this completes the proof of the theorem. \blacksquare

Corollary 1 *For any typical attribute g of α , one has $Typ(g \star \alpha) = Typ \alpha$.*

Proof: Immediate. \blacksquare

Proposition 1 *Let h be a feature and k a characteristic feature of α such that $Ext h \cap Ext \alpha \cap Ext k = \emptyset$ but $Ext h \cap Ext \alpha \cap (\bigcap_{f \in \chi(\alpha) \setminus \{k\}} Ext f) \neq \emptyset$. Suppose that δ_k is a two-valued function, so that we have $\delta_k : \mathcal{O} \rightarrow \{0, 1\}$. Then one has $Typ(h \star \alpha) = Ext h \cap Ext \alpha \cap (\bigcap_{f \in \chi(\alpha) \setminus \{k\}} Ext f)$.*

Proof: Let x be an element of $Typ(h \star \alpha)$, and suppose that there exists a characteristic feature $g \neq k$ such that $\delta_g(x) < 1$. Take an element y of $Ext h \cap Ext \alpha \cap (\bigcap_{f \in \chi(\alpha) \setminus \{k\}} Ext f)$. We would have then $x \prec_{\alpha}^{\tau} y$ and therefore $x \prec_{h \star \alpha}^{\tau} y$, which is impossible because, by the choice of x , x is $\preceq_{h \star \alpha}^{\tau}$ -maximal in $Ext(h \star \alpha)$. We have therefore proven that all characteristic features $g \neq k$ apply to the typical instances of $h \star \alpha$. This shows that $Typ(h \star \alpha) \subseteq Ext h \cap Ext \alpha \cap (\bigcap_{f \in \chi(\alpha) \setminus \{k\}} Ext f)$.

Conversely, given an element $x \in Ext h \cap Ext \alpha \cap (\bigcap_{f \in \chi(\alpha) \setminus \{k\}} Ext f)$, let us prove that x is $\preceq_{h \star \alpha}^{\tau}$ -maximal in $Ext(h \star \alpha)$. If this were not the case, there would exist $y \in Ext(h \star \alpha)$ such that $x \prec_{h \star \alpha}^{\tau} y$, that is $x \prec_{\alpha}^{\tau} y$. By the choice of x , this could be possible only if $\delta_k(x) < \delta_k(y)$, that is $\delta_k(y) = 1$. But then, this would imply $y \in Ext h \cap Ext \alpha \cap Ext k = \emptyset$, a contradiction. We have therefore shown that $Ext h \cap Ext \alpha \cap (\bigcap_{f \in \chi(\alpha) \setminus \{k\}} Ext f)$ is a subset

of $Typ(h \star \alpha)$, and this completes the proof of the proposition. \blacksquare

Theorem 2 *If k is an elementary feature, the k -exceptionally modified concept $h \star \alpha$ is equivalent to a featured concept whose defining feature set $\Delta(h \star \alpha)$ is equal to $\Delta(\alpha) \cup \{h\}$ and characteristic set $\chi(h \star \alpha)$ equal to $\chi(\alpha) \setminus \{k\}$.*

Proof: Since h is exceptional for α , we cannot have $h \in \Delta(\alpha) \cup \chi(\alpha)$. We attach to $(h \star \alpha)$ the defining set $\Delta(h \star \alpha) = \Delta(\alpha) \cup \{h\}$, equipped with the order $>_{\Delta(\alpha) \cup \{h\}}$ that extends the salience order of $\Delta(\alpha)$ and makes h less salient than every element of $\Delta(\alpha)$. Then the proof that $\preceq_{\mathcal{O}}^{\Delta(h \star \alpha)} = \preceq_{h \star \alpha}^{\mu}$ is similar to that made in Theorem 1.

We equip the set $\chi(h \star \alpha)$ with the restriction to $\chi(\alpha) \setminus \{k\}$ of the order $>_{\chi(\alpha)}$. Recalling that, on $Ext(h \star \alpha)$, the order $\preceq_{h \star \alpha}^{\tau}$ is equal to \preceq_{α}^{τ} , we have to prove that, for x and y in $Ext h \cap Ext \alpha$, one has $x \preceq_{\alpha}^{\tau} y$ iff $x \preceq_{Ext h \cap Ext \alpha}^{\chi(\alpha) \setminus \{k\}} y$.

Suppose first that one has $x \preceq_{\alpha}^{\tau} y$, and let f be an element of $\chi(\alpha) \setminus \{k\}$ such that $\delta_f(y) < \delta_f(x)$. There exists $g \in \chi(\alpha)$, g more salient than f , such that $\delta_g(x) < \delta_g(y)$. If we had $g = k$, we would have $\delta_k(x) < \delta_k(y)$, that is, since k is an elementary feature, $\delta_k(x) = 0$ and $\delta_k(y) = 1$. But then y would be an element of $Ext h \cap Ext \alpha \cap Ext k$, contradicting the k -exceptionality of h . This shows that $g \in \chi(\alpha) \setminus \{k\}$. Since g is more salient than f in this set, we have proven that $x \preceq_{Ext h \cap Ext \alpha}^{\chi(\alpha) \setminus \{k\}} y$.

Conversely, suppose that we have $x \preceq_{Ext h \cap Ext \alpha}^{\chi(\alpha) \setminus \{k\}} y$ for two elements of $Ext h \cap Ext \alpha$. We have to show that $x \preceq_{\alpha}^{\tau} y$. If $f \in \chi(\alpha)$ is such that $\delta_f(y) < \delta_f(x)$, the same reasoning that before shows that $f \neq k$, that is $f \in \chi(\alpha) \setminus \{k\}$. The conclusion follows. \blacksquare

Theorem 3 *Let x be an object such that $\rho_{\alpha}(x) > 1 - \epsilon(\alpha)$. Then one has $\delta_g(x) > 0$ for all elements g of $\Pi(\alpha)$ such that $|\Delta(\alpha)| \leq s_{\alpha}(g)$.*

Proof: Suppose that g is a stereotypical feature such that $|\Delta(\alpha)| \leq s_{\alpha}(g)$. From $0 \leq s_{\alpha}(g) - |\Delta(\alpha)|$, we get $\sum_{f \in \Pi(\alpha) \setminus \{g\}} s_{\alpha}(f) \leq \sum_{f \in \Pi(\alpha)} s_{\alpha}(f) - |\Delta(\alpha)|$. If $\delta_g(x) = 0$, we would have $\sum_{f \in \Pi(\alpha)} s_{\alpha}(f) \delta_f(x) \leq \sum_{f \in \Pi(\alpha)} s_{\alpha}(f) - |\Delta(\alpha)|$, and therefore $\rho_{\alpha}(x) \leq 1 - \frac{|\Delta(\alpha)|}{w(\alpha)} = 1 - \epsilon(\alpha)$, a contradiction. \blacksquare

Corollary 2 *Let x be an object such that $\rho_{h \star \alpha}(x) > 1 - \epsilon(h \star \alpha)$, and g an elementary feature of $\Pi(h \star \alpha)$ such that $|\Delta(h \star \alpha)| \leq s_{h \star \alpha}(g)$. Then g applies*

to x .

Proof: Immediate. ■

Lemma 2 *The salience degrees $s_{h\star\alpha}$ and $s_{h\star\alpha}^*$ are given by:*

- In the case where $h \notin \Delta(\alpha) \cup \chi(\alpha)$: $s_{h\star\alpha}(h) = 1$,
 $s_{h\star\alpha}(f) = 1 + s_\alpha(f) \forall f \in \Pi(\alpha) \setminus \{h\}$,
 $s_{h\star\alpha}^*(h) = 1 + |\Delta(\alpha)|$,
 $s_{h\star\alpha}^*(f) = s_\alpha(f) \forall f \in \Pi(\alpha) \setminus \{h\}$
- In the case where $h \in \Delta(\alpha)$:
 $s_{h\star\alpha}(f) = s_\alpha(f) \forall f \in \Pi(\alpha)$,
 $s_{h\star\alpha}^*(h) = |\Delta(\alpha)|$,
 $s_{h\star\alpha}^*(f) = s_\alpha(f) \forall f \in \Pi(\alpha) \setminus \{h\}$
- In the case where $h \in \chi(\alpha)$:
 $s_{h\star\alpha}(h) = 1$,
 $s_{h\star\alpha}(f) = 1 + s_\alpha(f) \forall f \in \Delta(\alpha)$,
 $s_{h\star\alpha}(f) = s_\alpha(f)$ if $f \in \chi(\alpha)$, $f >_{\chi(\alpha)} h$,
 $s_{h\star\alpha}(f) = 1 + s_\alpha(f)$ if $f \in \chi(\alpha)$, $f \not>_{\chi(\alpha)} h$,
 $s_{h\star\alpha}^*(f) = s_\alpha(f) \forall f \in \Pi(\alpha)$.

Proof: Straightforward from the definitions of $s_{h\star\alpha}$ and $s_{h\star\alpha}^*$. ■

Proposition 2 *One has*

$$w(h \star \alpha) = 1 + w(\alpha) + |\Pi(\alpha)| \text{ if } h \notin \Delta(\alpha) \cup \chi(\alpha),$$

$$w(h \star \alpha) = w(\alpha) \text{ if } h \in \Delta(\alpha),$$

$$w(h \star \alpha) = w(\alpha) + 1 + |\Delta(\alpha)| - s_\alpha(h) + |\{f \in \chi(\alpha) \setminus \{h\}, f \not>_{\chi(\alpha)} h\}| \text{ if } h \in \chi(\alpha).$$

Proof: The proof directly follows from the preceding lemma. As an example we shall only examine the last case where $h \in \chi(\alpha)$. We have

$$w(h \star \alpha) = \sum_{f \in \Pi(h\star\alpha)} s_{h\star\alpha}(f) = \sum_{f \in \chi(\alpha) \setminus \{h\}} s_{h\star\alpha}(f) + \sum_{f \in \Delta(\alpha)} s_{h\star\alpha}(f) + s_{h\star\alpha}(h)$$

$$= |\{f \in \chi(\alpha) \setminus \{h\}, f \not>_{\chi(\alpha)} h\}| + \sum_{f \in \chi(\alpha) \setminus \{h\}} s_\alpha(f) + |\Delta(\alpha)| + \sum_{f \in \Delta(\alpha)} s_\alpha(f).$$

The result follows, noting that $w(\alpha) = s_\alpha(h) + \sum_{f \in \chi(\alpha) \setminus \{h\}} s_\alpha(f)$. ■

Proposition 3 *The $h \star \alpha$ resemblance degree $\rho_{h\star\alpha}(x)$ of an element $x \in \mathcal{O}$ satisfies:*

$$\begin{aligned} w(h \star \alpha)\rho_{h\star\alpha}(x) &= \sum_{f \in \Pi(\alpha)} \delta_f(x) + w(\alpha)\rho_\alpha(x) + \delta_h(x) \text{ if } h \notin \Pi(\alpha), \\ \rho_{h\star\alpha}(x) &= \rho_\alpha(x) \text{ if } h \in \Delta(\alpha), \text{ and, if } h \in \chi(\alpha), \\ w(h \star \alpha)\rho_{h\star\alpha}(x) &= w(\alpha)\rho_\alpha(x) - s_\alpha(h)\delta_h(x) + \delta_h(x) + \sum_{f \in \Pi(\alpha) \setminus \{h, f \not\prec_{\chi(\alpha)} h\}} \delta_f(x). \end{aligned}$$

Proof: By Theorem 1, $w(h\star\alpha)\rho_{h\star\alpha}(x) = \sum_{f \in \Pi(\alpha)} s_{h\star\alpha}(f)\delta_f(x) + s_{h\star\alpha}(h)\delta_h(x)$ whenever $h \notin \chi(\alpha)$. If $h \notin \Pi(\alpha)$ this yields

$$\begin{aligned} w(h \star \alpha)\rho_{h\star\alpha}(x) &= \sum_{f \in \Pi(\alpha)} (1 + s_\alpha(f))\delta_f(x) + \delta_h(x), \text{ that is} \\ w(h \star \alpha)\rho_{h\star\alpha}(x) &= \sum_{f \in \Pi(\alpha)} \delta_f(x) + w(\alpha)\rho_\alpha(x) + \delta_h(x) \text{ as desired.} \end{aligned}$$

If $h \in \Delta(\alpha)$, we have $s_{h\star\alpha}(f) = s_\alpha(f) \forall f \in \Pi(\alpha)$, $w(h \star \alpha) = w(\alpha)$, and therefore $\rho_{h\star\alpha}(x) = \rho_\alpha(x)$.

If finally $h \in \chi(\alpha)$, we get

$$w(h\star\alpha)\delta_{h\star\alpha}(x) = \sum_{f \in \chi(\alpha) \setminus \{h\}} s_{h\star\alpha}(f)\delta_f(x) + s_{h\star\alpha}(h)\delta_h(x) + \sum_{f \in \Delta(\alpha)} s_{h\star\alpha}(f)\delta_f(x).$$

By Lemma 2, the first summand is equal to

$$\sum_{f \in \chi(\alpha), f >_{\chi(\alpha)} h} s_\alpha(f)\delta_f(x) + \sum_{f \in \chi(\alpha) \setminus \{h\}, f \not\prec_{\chi(\alpha)} h} (1 + s_\alpha(f))\delta_f(x).$$

We have $s_{h\star\alpha}(h) = 1$.

The last summand is equal to $\sum_{f \in \Delta(\alpha)} (1 + s_\alpha(f))\delta_f(x)$. We observe that $\sum_{f \in \chi(\alpha), f >_{\chi(\alpha)} h} s_\alpha(f)\delta_f(x) + \sum_{f \in \chi(\alpha) \setminus \{h\}, f \not\prec_{\chi(\alpha)} h} (s_\alpha(f))\delta_f(x) = \sum_{f \in \chi(\alpha) \setminus \{h\}} s_\alpha(f)\delta_f(x)$, and that $\sum_{f \in \chi(\alpha) \setminus \{h\}, f \not\prec_{\chi(\alpha)} h} \delta_f(x) + \sum_{f \in \Delta(\alpha)} \delta_f(x) = \sum_{f \in \Pi(\alpha) \setminus \{h, f \not\prec_{\chi(\alpha)} h\}} \delta_f(x)$. The result follows. \blacksquare

Theorem 4 *If h is not exceptional for α , the $h \star \alpha$ resemblance of α is given by*

$$\begin{aligned} \rho_{h\star\alpha}(\alpha) &= 1 - \frac{1}{w(h\star\alpha)} \text{ if } h \notin \Pi(\alpha), \\ \rho_{h\star\alpha}(\alpha) &= 1 \text{ if } h \in \Pi(\alpha). \end{aligned}$$

Proof: By definition 10, we have $\rho_{h\star\alpha}(\alpha) = \text{Min}_{x \in \text{Typ } \alpha} \rho_{h\star\alpha}(x)$. If x is an element of $\text{Typ } \alpha$, we have $\rho_\alpha(x) = 1$ and $\delta_f(x) = 1 \forall f \in \Pi(\alpha)$. The result then follows from Proposition 3, noticing that $\text{Min}_{x \in \text{Typ } \alpha} \rho_{h\star\alpha}(x)$ is reached when $\delta_h(x) = 0$.

If $h \in \Delta(\alpha)$, one has $\rho_{h\star\alpha}(\alpha) = 1$ from Proposition 2.

In the case where $h \in \chi(\alpha)$, the result follows from Proposition 2 and Proposition 3. One may also directly apply Corollary 1 of Theorem 1. \blacksquare

Proposition 4 *One has*

$$\begin{aligned}
w^*(h \star \alpha) &= 1 + w(\alpha) + |\Delta(\alpha)| \text{ for } h \notin \Delta(\alpha) \cup \chi(\alpha), \\
w^*(h \star \alpha) &= w(\alpha) - s_\alpha(h) + |\Delta(\alpha)| \text{ if } h \in \Delta(\alpha), \\
w^*(h \star \alpha) &= w(\alpha) \text{ if } h \in \chi(\alpha).
\end{aligned}$$

Proof: If $h \notin \Delta(\alpha)$, the result is immediate from Lemma 2. Suppose $h \in \Delta(\alpha)$. We have $\Sigma_{\chi(\alpha)} s_{h \star \alpha}^*(f) = \Sigma_{\chi(\alpha)} s_\alpha(f)$. Writing then the sum $\Sigma_{f \in \Delta(\alpha)} s_{h \star \alpha}^*(f)$ as $\Sigma_{f \in \Delta(\alpha) \setminus \{h\}, f > \Pi(\alpha)h} s_{h \star \alpha}^*(f) + \Sigma_{f \in \Delta(\alpha) \setminus \{h\}, f \not> \Pi(\alpha)h} s_{h \star \alpha}^*(f) + s_{h \star \alpha}^*(h)$, we apply the second part of Lemma 2. ■

Proposition 5 In the non-exceptional case, the $h \star \alpha$ resemblance degree of an element $x \in \mathcal{O}$ satisfies:

$$\begin{aligned}
w^*(h \star \alpha) \rho_{h \star \alpha}^*(x) &= \Sigma_{f \in \chi(\alpha)} \delta_f(x) + w(\alpha) \rho_\alpha(x) + (1 + |\Delta(\alpha)|) \delta_h(x) \text{ if } h \notin \Pi(\alpha), \\
w^*(h \star \alpha) \rho_{h \star \alpha}^*(x) &= w(\alpha) \rho_\alpha(x) - s_\alpha(h) \delta_h(x) + |\Delta(\alpha)| \delta_h(x) \text{ if } h \in \Delta(\alpha), \\
\rho_{h \star \alpha}^*(x) &= \rho_\alpha(x) \text{ if } h \in \chi(\alpha).
\end{aligned}$$

Proof: If $h \notin \Pi(\alpha)$, we write

$$w^*(h \star \alpha) \delta_{h \star \alpha}^*(x) = \Sigma_{\chi(\alpha)} s_{h \star \alpha}^*(f) \delta_f(x) + \Sigma_{\Delta(\alpha)} s_{h \star \alpha}^*(f) \delta_f(x) + s_{h \star \alpha}^*(h) \delta_h(x),$$

and apply the first part of Lemma 2.

If $h \in \Delta(\alpha)$, the second part of Lemma 2 yields

$$w^*(h \star \alpha) \rho_{h \star \alpha}^*(x) = \Sigma_{\Pi(\alpha) \setminus \{h\}} s_{h \star \alpha}(f) \delta_f(x) + |\Delta(\alpha)| \delta_h(x), \text{ that is } w^*(h \star \alpha) \rho_{h \star \alpha}^*(x) = w(\alpha) \rho_{h \star \alpha}(x) - s_\alpha(h) \delta_h(x) + |\Delta(\alpha)| \delta_h(x).$$

If $h \in \chi(\alpha)$, we write $w^*(h \star \alpha) \rho_{h \star \alpha}^*(x) = \Sigma_{\Pi(\alpha)} s_{h \star \alpha}(f) \delta_f(x)$ and conclude by Lemma 2, noting that $w^*(h \star \alpha) = w(\alpha)$. ■

Theorem 5 In the non exceptional case, the salience $s_{h \star \alpha}^*$ yields:

$$\begin{aligned}
\rho_{h \star \alpha}^*(\alpha) &= \frac{|\chi(\alpha)| + w(\alpha)}{w^*(h \star \alpha)} \text{ if } h \notin \Pi(\alpha), \\
\rho_{h \star \alpha}^*(\alpha) &= 1 \text{ if } h \in \Pi(\alpha).
\end{aligned}$$

Proof: Apply Proposition 5 for $x \in \text{Typ } \alpha$ and take the minimum possible value: If $h \notin \Pi(\alpha)$, we have $\delta_h(x) = 0$. If $h \in \Pi(\alpha)$, we have $\delta_h(x) = 1$. This yields the result.

One may also remark that one has $\text{Typ}(h \star \alpha) = \text{Ext } h \cap \text{Typ } \alpha$ (Theorem 5): any typical instance of α that falls under h is a typical instance of $h \star \alpha$. ■

Lemma 3 *If h is k -exceptional, the salience degrees in $\Pi(h \star \alpha)$ are given by*

$$\begin{aligned} s_{h \star \alpha}(h) &= 1, \\ s_{h \star \alpha}(f) &= 1 + s_\alpha(f) \quad \forall f \in \Pi(h \star \alpha) \setminus \{h\}, f \not\triangleright_{\Pi(\alpha)} k, \\ s_{h \star \alpha}(f) &= s_\alpha(f) \quad \text{for all other elements of } \Pi(h \star \alpha) \setminus \{h\}, \\ s_{h \star \alpha}^*(h) &= |\Delta(\alpha)| + 1, \\ s_{h \star \alpha}^*(f) &= s_\alpha(f) \quad \forall f \in \Pi(\alpha) \setminus \{k\}. \end{aligned}$$

Proof: Straightforward. ■

Proposition 6 *Suppose that h is k -exceptional for α . Then*

$$\begin{aligned} w(h \star \alpha) &= w(\alpha) + |\Delta(\alpha)| + 1 - s_\alpha(k) + |\{f \in \chi(\alpha) \setminus \{k\}, f \not\triangleright_{\chi(\alpha)} k\}|. \\ w^*(h \star \alpha) &= w(\alpha) - s_\alpha(k) + 1 + |\Delta(\alpha)|. \end{aligned}$$

Proof: The first equality is a direct consequence of the preceding Lemma. For the second equality, we use once more the fact that

$$\sum_{f \in \Pi(\alpha) \setminus \{k\}} s_\alpha(f) = w(\alpha) \rho_\alpha - s_\alpha(k). \quad \blacksquare$$

Proposition 7 *In the k -exceptional case, the $h \star \alpha$ -resemblance degree of an object x is given by:*

$$\begin{aligned} w(h \star \alpha) \rho_{h \star \alpha}(x) &= w(\alpha) \rho_\alpha(x) + \delta_h(x) - s_\alpha(k) \delta_k(x) + \sum_{f \in \Pi(\alpha) \setminus \{k\}, f \not\triangleright_k} \delta_f(x), \\ w^*(h \star \alpha) \delta_{h \star \alpha}^*(x) &= w(\alpha) \rho_\alpha(x) - s_\alpha(k) \delta_k(x) + \Delta(\alpha) + 1. \end{aligned}$$

Proof: For the first equality; we have

$$\begin{aligned} w(h \star \alpha) \rho_{h \star \alpha}(x) &= \delta_h(x) + \sum_{f \in \Pi(h \star \alpha) \setminus \{h\}} s_{h \star \alpha}(f) \delta_f(x). \text{ We then write} \\ \Pi(h \star \alpha) \setminus \{h\} &= \{f \in \Pi(h \star \alpha) \setminus \{h\}, f \not\triangleright_{\Pi(\alpha)}\} \cup \{f \in \Pi(h \star \alpha) \setminus \{h\}, f >_{\Pi(\alpha)}\} \end{aligned}$$

and apply Lemma 2. The second equality directly follows from Lemma 3.

Proposition 8 *The resemblance of a concept with its modification by a k -exceptional feature is given by:*

$$\begin{aligned} \rho_{h \star \alpha}(\alpha) &= 1 - \frac{1}{w(h \star \alpha)} \\ \delta_{h \star \alpha}^*(\alpha) &= 1 - \epsilon(h \star \alpha) \end{aligned}$$

Proof: If z is an element of $Typ \alpha$, it follows from Proposition 7 that

$$\begin{aligned} w(h \star \alpha) \rho_{h \star \alpha}(z) &= w(\alpha) - s_\alpha(k) + |\{f \in \Pi(\alpha) \setminus \{k\}, f \not\triangleright k\}|. \text{ Writing} \\ \{f \in \Pi(\alpha) \setminus \{k\}, f \not\triangleright k\} &= \{f \in \chi(\alpha) \setminus \{k\}, f \not\triangleright k\} \cup \{f \in \Delta(\alpha)\}, \text{ we get,} \end{aligned}$$

by Proposition 6

$w(h \star \alpha)\rho_{h\star\alpha}(z) = w(h \star \alpha) - 1$ whence the result.

Similarly, we have by Proposition 7 $w^\rho(h \star \alpha)\rho_{h\star\alpha}(z) = w(\alpha) - s_\alpha(k) = w^\rho(h \star \alpha) - (|\Delta(\alpha) + 1|)$. ■

Proposition 9 One has $\rho_\alpha(h \star \alpha) = 1 - \frac{s_\alpha(k)}{w(\alpha)}$.

Proof: For any $z \in \mathcal{O}$,

$w(\alpha)\rho_\alpha(h \star \alpha) = \sum_{\Pi(\alpha)\setminus\{k\}} s_\alpha(f)\delta_f(z) + s_\alpha(k)\delta_k(z)$. If $z \in Typ(h \star \alpha)$, we have $\delta_f(z) = 1 \forall f \in \Sigma_{\Pi(\alpha)\setminus\{k\}}$ and $\delta_k(z) = 0$ since h is k -exceptional. ■

References

- [1] D.Aerts. Quantum structure in cognition. *Journal of Mathematical Psychology*, (53-5):314–348, 2009.
- [2] M.E. Barton and L.K. Komatsu. Defining features of natural kinds and artifacts. *Journal of Psycholinguistic Research*, (18-5):433–447, 1989.
- [3] S.Clark. Vector space models of lexical meaning. *Handbook of Contemporary Semantics*, 08 2015.
- [4] R.Confalonieri, E.Plaza, and M.Schorlemmer. Computational aspects of concept invention. In *Concept Invention*, 2018.
- [5] A.C. Connolly, J.A. Fodor, L.R. Gleitman, and H.Gleitman. Why stereotypes don’t even make good defaults. *Cognition*, (103-1):1–22, 2007.
- [6] P.Egré and C.O.Madagain. Concept utility. *The Journal of Philosophy*, 116:525–554, 01 2019.
- [7] G.Fauconnier and Mark Turner. The way we think: Conceptual blending and the mind’s hidden complexities. basic books. 2002.
- [8] J.Fodor. Concepts: a potboiler. *Cognition*, (50):95–113, 1994.
- [9] J.A. Fodor. *Concepts: where cognitive science went wrong*. Oxford University Press, 1998.
- [10] R.Franco. The conjunction fallacy and interference effects. *Journal of Mathematical Psychology*, (53-5):415–442, 2009.

- [11] G.Frege. *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*. Halle, 1879.
- [12] M.Freund. On the notion of concept I. *Artificial Intelligence*, (172):570–590, 2008.
- [13] M.Freund. On categorial membership. *Erkenntnis*, (70), 2014.
- [14] M.Freund. Ordered models for concept representation. *Journal of Logic and Computation*, 30(6):1143–1181, 2020.
- [15] Bernhard Ganter and Rudolf Wille. *Formal Concept Analysis: Mathematical Foundations*. Springer, 1999.
- [16] P.Gärdenfors. Conceptual spaces- the geometry of thought. 2000.
- [17] P.Gärdenfors. Conceptual spaces as a framework for knowledge representation. *Mind and Matter*, (Vol.2(2)):9–27, 2004.
- [18] Peter Gärdenfors and David Makinson. Nonmonotonic inference based on expectations. *Artificial Intelligence*, 65(2):197–245, 1994.
- [19] J.Hampton. Prototype models of concept representation. *in I. Van Mechelen, J. Hampton, R. S. Michalski, and P. Theuns (Eds.), Categories and concepts: Theoretical views and inductive data analysis Academic Press*, pages 67–95, 1993.
- [20] J.A. Hampton. Overextension of conjunctive effects: Evidence for a unitary model of concept typicality and class inclusion. *Journal for Experimental Psychology : Learning, Memory, and Cognition*, (14):12–32, 1988.
- [21] J.A. Hampton. Similarity-based categorisation: the development of prototype theory. *Psychological Belgica*, (35):103–125, 1995.
- [22] J.A. Hampton. Conceptual combination: conjunction and negation of natural concepts. *Memory and Cognition*, (25):55–71, 1997.
- [23] J.A. Hampton. Similarity-based categorisation and fuzziness of natural categories. *Cognition*, (65):137–165, 1998.

- [24] M.L. Jönsson and J.A. Hampton. On prototypes as defaults (comment on Connolly, Fodor, Gleitman and Gleitman, 2007). *Cognition*, 2007.
- [25] H.Kamp and B.Partee. Prototype theory and compositionality. *Cognition*, (57):129–191, 1995.
- [26] S.Kraus, D.Lehmann, and M.Magidor. Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence*, (44):167–207, 1990.
- [27] G.L. Murphy. *The Big Book of Concepts*. MIT Press, Cambridge, 2002.
- [28] A.Noë. *Action in Perception*. Cambridge MIT Press, 2004.
- [29] D.Osherson and E.E. Smith. On the adequacy of prototype theory as a theory of concepts. *Cognition*, (11):237–262, 1981.
- [30] M.Osta-Velez and P.Gärdenfors. Category-based induction in conceptual spaces. *Journal of Mathematical Psychology*, (96):102357, 2020.
- [31] M.Osta-Velez and P.Gärdenfors. Nonmonotonic reasoning, expectations orderings and conceptual spaces. *Journal of Logic, Language and Information*, pages 1–21, 2021.
- [32] HPutnam. The meaning of meaning. In *Mind, language and reality*, pages 115–120. Cambridge University Press, 1975/1985.
- [33] E.Rosch. Cognitive representations of semantic categories. *Journal of Experimental Psychology*, (104):192–233, 1975.
- [34] E.E. Smith and D.L.Medin. *Categories and concepts*. Harvard University Press, Cambridge, MA, 1981.
- [35] E.E. Smith and D.Osherson. Conceptual combination with prototype concepts. *Cognitive Science*, (8):357–361, 1984.
- [36] E.E. Smith, D.Osherson, L.Rips, and M.Keane. Combining prototypes: a selective modification model. *Cognitive Science*, (12):357–361, 1988.
- [37] E.E. Smith, E.J. Shoben, and L.J. Rips. Structure and process in semantic memory: a featural model for semantic decisions. *Psychological Review*, (81):214–241, 1974.

- [38] Sandro Sozzo. A quantum probability explanation in fock space for borderline contradictions. *Journal of Mathematical Psychology*, 58, 11 2013.
- [39] A.Tversky. Features of similarity. *Psychological review*, (84):327–352, 1977.
- [40] A.Tversky and D.Kahneman. Extension versus intuitive reasoning: The conjunction fallacy in probability judgement. *Psychological review*, (90):141–168, 1983.
- [41] E.J. Wisniewski. When concepts combine. *Psychonomic bulletin & review*, 4:167–183, 1997.
- [42] L.Wittgenstein. *Philosophical Investigations*. New York, Macmillan, 1953.