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**Abstract**

The goal of this work is to study a ZIKA disease model using fractional derivatives in the Caputo sense. After formulating the model, we compute the basic reproduction number $\mathbf{R}_0$ and give rigorous proof of the existence of equilibrium points as well as stability analysis of these equilibrium points. Then, we study the existence and the uniqueness of the solutions of the fractional model using the Banach fixed point theory. Numerical simulations are performed to validate our analytical results, as well as to see the impact of varying the fractional parameter on the disease dynamics

**Keywords**

ZIKA disease; Fractional model; Caputo derivative; Asymptotic stability; Numerical simulations

## I INTRODUCTION

Primarily transmitted to humans through the bite of an infected Aedes species mosquito (Ae. aegypti and Ae. albopictus, which are are the same mosquitoes that spread dengue and chikungunya viruses), Zika disease is caused by a flavivirus called Zika virus (ZIKV). It is a public health problem in many Latina American countries which are experiencing the most serious known outbreaks of Zika virus (ZIKV), and this, since 2005 [**1**]. In October 2015, an association between Zika virus infection and microcephaly was reported in Brazil. From then on, epidemics and evidence of transmission quickly appeared in the Americas, Africa, and other parts of the world. To date, a total of 86 countries and territories have reported evidence of mosquito-borne Zika infection. The rise of microcephaly and Guillain-Barré syndrome is correlated with the rise of the spread of the Zika virus. Zika virus was first identified in Uganda in 1947 in monkeys, and later in humans in 1952. In 2007, the Island of Yap was the land of the first large outbreak of the disease [**1**, **2**]. Since then, outbreaks have appeared in many countries (United Republic of Tanzania, India, Indonesia, Malaysia, Pakistan, Pacific islands, Brazil...). On February 1, 2016, Zika disease became a Public Health Emergency of International Concern [**1**, **2**]. Since there is no available vaccine to prevent or treat Zika virus infection, the principal control means are vector controls and protection means against mosquito bites [**1**, **2**].

Mathematical modeling is a good tool for the projection and control of several infectious disease outbreaks. The current Covid-19 outbreak is an example [**3**, **4**]. Several works to model,

project, and control recent outbreaks of Zika virus disease in the world, were published [**5**, **6**, **7**, **8**]. Fractional derivative theory has been more and more used in the last decades to model natural phenomena like infectious diseases [**9**, **10**]. The fact that classical derivatives do not permit to describe fluctuations of different classes of population involved in infectious process, and thus heterogeneity between compartment classes, do that the use of Fractional derivatives is need nowadays [**11**]. Note that memory effect, which represents a great advantage of fractional derivatives compared to classical derivatives, permit to predict epidemic peaks for various diseases [**12**, **13**]. Several papers using fractional derivatives were developed for Zika transmission dynamics [**14**, **15**, **16**, **17**]. Most of them use Caputo's type derivatives. In this paper, we extend the model of Yuan et *al.* in [**8**] replacing integer derivatives with Caputo derivative. After the computation of the basic reproduction number $\mathbf{R}_0$, we prove the uniform asymptotically stability of the Zika-free equilibrium when $\mathbf{R}_0 \leq 1$, despite the fact that the latter coexists with an endemic equilibrium. For the case $\mathbf{R}_0 > 1$, we prove that the model admits two endemic equilibrium points, one of them is stable and the other is unstable. To simulate the fractional model, we construct two numerical schemes using the Adams-Bashforth Method (ABM) and the Laplace Transform Adomian Decomposition Method (LTADM), respectively.

The outline of the paper is as follows: In section II, we recall the definition of the Caputo derivative, and two precious results which is used top prove the existence and uniqueness of solutions. Section III is devoted to model formulation and mathematical results. Numerical simulation results are presented in Section IV. A conclusion round up the paper.

## II USEFUL DEFINITION AND RESULTS

In this section, we present useful definitions of Caputo derivative, as well ass useful lemmas which will be used in the model analysis (see [**18**, **19**]).

*Definition 1:*
Given a function $f : [a, b] \to \mathbb{R}$ of class $C^n$, the Caputo fractional derivative of $f$ of order $\alpha$ is defined by :

$$ {}^{C}_{t_0}D^{\alpha}_{t} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau \tag{1} $$

where $\alpha \notin \mathbb{N}, n = [\alpha] + 1$.

*Proposition 1:*
Let $f, g : [a, b] \to \mathbb{R}$ be such that ${}^{C}_{t_0}D^{\alpha}_{t} f(t)$ and ${}^{C}_{t_0}D^{\alpha}_{t} g(t)$ exist almost everywhere and let $c_1, c_2 \in \mathbb{R}$. Then ${}^{C}_{t_0}D^{\alpha}_{t}(c_1 f(t) + c_2 g(t))$ exists almost everywhere with ${}^{C}_{t_0}D^{\alpha}_{t}(c_1 f(t) + c_2 g(t)) = c_1 \times {}^{C}_{t_0}D^{\alpha}_{t} f(t) + c_2 \times {}^{C}_{t_0}D^{\alpha}_{t} g(t)$.

*Lemma 1:*
If $0 < \chi < 1$ and $\mathbf{d} \in \mathbb{N}$, then there exist positive constants $W_{\chi,1}$ and $W_{\chi,2}$ only dependent on $\chi$, such that

$$ (\mathbf{d}+1)^{\chi} - \mathbf{d}^{\chi} \leq \mathcal{W}_{\chi,1}(\mathbf{d}+1)^{\chi-1} \quad \text{and} \quad (\mathbf{d}+2)^{\chi+1} - 2(\mathbf{d}+1)^{\chi+1} + \mathbf{d}^{\chi+1} \leq \mathcal{W}_{\chi,2}(\mathbf{d}+1)^{\chi-1}. $$

*Lemma 2:*
Assume that $\mathbf{y}_{q,m} = (m-q)^{\chi-1} (q = 1, 2, ..., m-1)$ & $\mathbf{y}_{q,m} = 0$ for $q \geq m, \chi, \mathbf{Q}, \mathbf{h}, \mathbf{Y} >$

$0, \mathbf{kh} \leq \mathbf{Y}$ & $\mathbf{k} \in \mathbb{N}$. Let $\sum_{q=\mathbf{k}}^{q=i} \mathbf{y}_{q,m}|\mathbf{e}_q| = 0$ for $\mathbf{k} > m \geq 1$. If

$$|\mathbf{e}_m| \leq \mathbf{Q}\mathbf{h}^{\chi} \sum_{q=1}^{m-1} \mathbf{y}_{q,m}|\mathbf{e}_q| + |\eta_0|, \quad m = 1, 2, ..., \mathbf{k},$$

then

$$|\mathbf{e}_k| \leq \mathcal{M}|\eta_0|, \quad \mathbf{k} = 1, 2, ...$$

where $\mathcal{M} \in \mathbb{R}_+$ independent of $\mathbf{k}$ and $\mathbf{h}$.

## III THE FRACTIONAL ZIKA MODEL AND ASYMPTOTIC STABILITY RESULTS

### 3.1 The fractional Zika model

The basic model comes from the works of Yuan et *al.* (2021) [**8**] which consists of a system of fives Ordinary Differential Equations. The human and mosquito populations are split into three and two compartment according to their clinical status. We denoted by: $S_h$ susceptible human, $I_h$ Infected humans, $R_h$ recovered human, $S_v$ susceptible mosquitoes, $I_v$ Infected mosquitoes. Since an infected mosquito is infected throughout its life, we do not consider removed class in this model.

$$\begin{cases} {}^{C}D^{\eta}S_h(t) &= \mu_h(H - S_h) - \left(\beta\dfrac{I_h}{H} + ab\dfrac{I_v}{H}\right)S_h, \\ {}^{C}D^{\eta}I_h(t) &= -(\gamma + \mu_h)I_h + \left(\beta\dfrac{I_h}{H} + ab\dfrac{I_v}{H}\right)S_h, \\ {}^{C}D^{\eta}R_h(t) &= -\mu_h R_h + \gamma I_h, \\ {}^{C}D^{\eta}S_v(t) &= \mu_v(M - S_v - \theta I_v) - ac\dfrac{I_h}{H}S_v, \\ {}^{C}D^{\eta}I_v(t) &= ac\dfrac{I_h}{H}S_v - \mu_v(1 - \theta)I_v, \end{cases} \tag{2}$$

with the initial conditions $S_h = S^{h0}$, $I_h = I^{h0}$, $R_h = R^{h0}$, $S_v = S^{v0}$, and $I_v = I^{v0}$.

In model (2): ${}^{C}D^{\eta}$ denotes the Caputo derivative with $\eta \in (0, 1]$; $H := S_h + I_h + R_h$ represents the total human population, $\mu_h$ is the human natural mortality rate, $\beta$ is the transmission coefficient from infected humans to susceptible humans, $a$ is the mosquitoes biting rate, $b$ is the transmission probability from an infectious mosquito to a susceptible human per bite, $\gamma$ is the recovery rate of humans, $M := S_v + I_v$ is the total mosquito population, $\mu_v$ is the mortality rate of mosquitoes, $\theta$ is the proportion of congenital infections in the offspring of infected female mosquitoes, and $c$ is the transmission probability from an infectious human to a susceptible mosquito per bite.

Note that, as in the case of mathematical modeling of infectious disease epidemics, we consider that human and mosquito populations are still constant during the Zika epidemic reported by the Guillain-Barré Syndrome (GBS) surveillance data. Using the fact that $S = H - I - R$ and $S_v = M - I_v$, the study of the model (2) is reduced to the study of the following system

$$\begin{cases} {}^{C}D^{\eta}I_h(t) &= -(\gamma + \mu_h)I_h + \left(\beta\dfrac{I_h}{H} + ab\dfrac{I_v}{H}\right)(H - I_h - R_h), \\ {}^{C}D^{\eta}R_h(t) &= -\mu_h R_h + \gamma I_h, \\ {}^{C}D^{\eta}I_v(t) &= ac\dfrac{I_h}{H}(M - I_v) - \mu_v(1 - \theta)I_v, \end{cases} \tag{3}$$

which define a dynamical system in the following bounded and absorbed set

$$\Sigma = \left\{(I_h, R_h, I_v) \in \mathbb{R}^3_+ : I_h + R_h \leq H,\ I_v \leq M\right\}.$$

### 3.2 The basic reproduction number and uniform asymptotic stability results

#### *3.2.1 The basic reproduction number and stability of the Zika-free equilibrium*

System (3) always has a Zika-free equilibrium $\mathcal{E}_0 = (0, 0, 0)$. Setting $\mathbf{R}_1 = \dfrac{\beta}{(\gamma + \mu_h)}$, $\mathbf{R}_2 = \dfrac{ab}{(1-\theta)\mu_v}$, and $\mathbf{R}_3 = \dfrac{acM}{H(\gamma + \mu_h)}$, we can express using the next generation method [**20**], the basic reproduction number of system (3) as follows:

$$\mathbf{R}_0 = \frac{\sqrt{4\mathbf{R}_2\mathbf{R}_3 + \mathbf{R}_1^2} + \mathbf{R}_1}{2}. \tag{4}$$

The matrix of the system (3) evaluated at the disease-free equilibrium point $\mathcal{E}_0$ is given by

$$\mathcal{J}(\mathcal{E}_0) = \begin{bmatrix} -(\gamma + \mu_h) + \beta & 0 & ab \\ \gamma & -\mu_h & 0 \\ ac\dfrac{M}{H} & 0 & -\mu_v(1-\theta) \end{bmatrix}. \tag{5}$$

The characteristic equation of $J(\mathcal{E}_0)$ is therefore $\det\left(s[I - (1-\eta)J(\mathcal{E}_0)] - \eta J(\mathcal{E}_0)\right) = 0$ [**21**].

*Theorem 1:*
The Zika-free equilibrium point $\mathcal{E}_0$ of the model (3) is locally asymptotically stable if $\mathcal{R}_0 < 1$ and unstable otherwise.

*Proof.* Setting $M/H = \alpha$, $1 - \theta = \theta_1$, $1 - \eta = \rho$, $\mathcal{M} = s[I - \rho J(\mathcal{E}_0)] - \eta J(\mathcal{E}_0)$. The characteristic equation $\det(\mathcal{M}) = 0$ is equivalent to $\left[(\mu_h\rho + 1)\, s + \mu_h\eta\right]\Phi(s) = 0$, where $\Phi(s) := e_2 s^2 + e_1 s + e_0$ with $e_2 = ((1 - \mathbf{R}_0)(1 - \mathbf{R}_1 + \mathbf{R}_0)k_1\theta_1\rho^2 + \theta_1\rho)\,\mu_v + (1 - \mathbf{R}_1)\,k_1\rho + 1$, $e_1 = (2(1 - \mathbf{R}_0)(1 - \mathbf{R}_1 + \mathbf{R}_0)k_1\theta_1\eta\rho + \theta_1\eta)\,\mu_v + (1 - \mathbf{R}_1)\,k_1\eta$, and $e_0 := (1 - \mathbf{R}_0)(1 - \mathbf{R}_1 + \mathbf{R}_0)k_1\theta_1\eta^2\mu_v$.

Note that $s_1 = \dfrac{-\eta\mu_h}{1 + \rho\mu_h} < 0$ is a solution of $\det(\mathcal{M}) = 0$. Since $\mathbf{R}_0 < 1 \Longrightarrow \mathbf{R}_1 < 1$, it follows that all coefficients $e_0$, $e_1$, and $e_2$ of $\Phi(s)$ is always positive. Thus, we conclude that all solutions of the equation $\det(\mathcal{M}) = 0$ have negative real part, which means that the Zika-free equilibrium is locally asymptotically stable. This ends the proof. □

*Theorem 2:*
The Zika-free equilibrium point $\mathcal{E}_0$ of the model (3) is uniformly asymptotically stable if $\mathcal{R}_0 < 1$ and unstable otherwise.

*Proof.* Let us consider the following Lyapunov function $\mathbf{L} = \dfrac{\mu_v(1-\theta)}{ab} I_h + I_v$. The derivative of $\mathbf{L}$ along the trajectories of (3), using Proposition 1, gives:

$$\begin{aligned}
D^\eta \mathbf{L} &= \frac{\mu_v(1-\theta)}{ab} D^\eta I_h + D^\eta I_v \\
&= -\frac{\mu_v(1-\theta)}{ab}\left(\beta\frac{I_h^2}{H} + \beta\frac{I_h R_h}{H} + ab\frac{I_h I_v}{H} + ab\frac{R_h I_v}{H}\right) + ac\frac{M}{H} I_h - \xi_2\mu_v(1-\theta) I_v \\
&- ac\frac{I_h I_v}{H} + \frac{\mu_v(1-\theta)}{ab}\left(\beta - (\gamma+\mu_h)\right) I_h + \frac{\mu_v(1-\theta)}{ab} ab I_v \\
&\leq \left[\xi_1\left(\beta - (\gamma+\mu_h)\right) + \xi_2 ac\frac{M}{H}\right] I_h + \left[\xi_1 ab - \xi_2\mu_v(1-\theta)\right] I_v \\
&= \xi_2\frac{(\gamma+\mu_h)\mu_(1-\theta)}{ab}(\mathbf{R}_0 - 1)(\mathbf{R}_0 + 1 - \mathbf{R}_1) I_h
\end{aligned}$$

Finally, $D^\eta \mathbf{L} \leq 0$ if and only if $\mathbf{R}_0 \leq 1$. Thus, from [**22**], it follows that the Zika-free equilibrium $\mathcal{E}_0$ of the fractional model (2) is uniformly asymptotically stable whenever $\mathbf{R}_0 \leq 1$. □

### *3.2.2 The endemic equilibrium and its uniform stability*

Equating the right-hand side of system (3) by zero, and solving the obtained system in term of $I_v$, we have

$$I_h = \frac{\mu_v\theta_1 H I_v}{ac\left(H\alpha - I_v\right)}, R_h = \frac{\gamma}{\mu_h} I_h, \tag{6}$$

where $I_v$ is the solutions of the following equation $I_v\left(\sigma_2 I_v^2 + \sigma_1 I_v + \sigma_0\right) = 0$, with $\sigma_2 = a^3b^2c\left(\theta_1\mu_v k_1 + ac\mu_h\right)$, $\sigma_0 = R_0\left(R_0 - 1\right)\left(R_1 - R_0 - 1\right)\left(R_1 - R_0\right) k_1^2\theta_1^2 H^2\mu_v^2\mu_h$,
$\sigma_1 = k_1\theta_1 Hab\mu_v\left\{\theta_1\mu_v k_1\left(\left(R_0 - 1\right)R_1 - R_0^2\right) + ac\mu_h\left[2R_0R_1 - 2R_0^2 - R_1 + 1\right]\right\}$.

Note that $\sigma_2$ is always positive. Since $\mathbf{R}_0 - \mathbf{R}_1 = \dfrac{\mathbf{R}_2\mathbf{R}_3}{\mathbf{R}_0} > 0 \Longrightarrow \mathbf{R}_1 - \mathbf{R}_0 - 1 = \dfrac{\mathbf{R}_2\mathbf{R}_3}{\mathbf{R}_0} - 1 = \dfrac{\mathbf{R}_2\mathbf{R}_3 - \mathbf{R}_0}{\mathbf{R}_0} < 0$, it follows that the sign of $\sigma_0$ depends of the value of $\mathbf{R}_0$. Indeed, $\mathbf{R}_0 > 1 \Longleftrightarrow \sigma_0 > 0$ and $\mathbf{R}_0 < 1 \Longleftrightarrow \sigma_0 < 0$. Also, we have $\mathbf{R}_0 > 1 \Longrightarrow \sigma_1 < 0$.

Since $\mathbf{R}_0 > \mathbf{R}_1$, it follows that $\Delta > 0$. We resume the above analysis as follows:

*Theorem 3:*
The number of equilibrium points os system (3) depend of the basic reproduction number as follows:

1. If $\mathbf{R}_0 \leq 1$, then system (3) has two equilibrium points: the Zika-free equilibrium $\mathcal{E}_0 = (0,0,0)$ and an endemic equilibrium point $\mathcal{E}^\dagger = \left(I_h^\dagger, R_h^\dagger, I_v^\dagger\right) = \left(\dfrac{\mu_v\theta_1 H I_v^\dagger}{ac\left(H\alpha - I_v^\dagger\right)}, \dfrac{\gamma}{\mu_h}\dfrac{\mu_v\theta_1 H I_v^\dagger}{ac\left(H\alpha - I_v^\dagger\right)}, I_v^\dagger\right)$, where $I_v^\dagger = \dfrac{-\sigma_1 + \sqrt{\sigma_1^2 - 4\sigma_2\sigma_0}}{2}$;
2. If $\mathbf{R}_0 > 1$, then system (3) has three equilibrium points: the Zika-free equilibrium $\mathcal{E}_0 = (0,0,0)$ and two endemic equilibrium points $\mathcal{E}^\pm = \left(I_h^\pm, R_h^\pm, I_v^\pm\right)$, with $I_v^\pm = \dfrac{-\sigma_1 \pm \sqrt{\sigma_1^2 - 4\sigma_2\sigma_0}}{2}$.

The above first case ($\mathbf{R}_0 \leq 1$) means that there is a possibility that the Zika-free equilibrium co-exists with an endemic equilibrium $\mathcal{E}^{\dagger} = \left(I_h^{\dagger}, R_h^{\dagger}, I_v^{\dagger}\right)$ whenever the $\mathbf{R}_0 \leq 1$. Nevertheless, by Theorem 2, we conclude that this endemic equilibrium is always unstable whenever $\mathbf{R}_0 \leq 1$. Now, suppose that $\mathbf{R}_0 > 1$, which means that the Zika-free equilibrium $\mathcal{E}_0 = (0, 0, 0)$ becomes unstable, system (3) admits two endemic equilibrium points $\mathcal{E}^{\pm} = \left(I_h^{\pm}, R_h^{\pm}, I_v^{\pm}\right)$ and $\mathcal{E}^{\ddagger} = \left(I_h^{\ddagger}, R_h^{\ddagger}, I_v^{\ddagger}\right)$. In what follows, we will show that one of these endemic equilibrium points is stable which implies that the other is unstable. To this aim, we follow Yuan et *al.* [**8**] by defining the following Lyapunov function for system (3)

$$\mathcal{L} = \frac{S_h^*}{H}\left(\frac{S_h}{S_h^*} - 1 - \ln\frac{S_h}{S_h^*}\right) + \frac{I_h^*}{H}\left(\frac{I_h}{I_h^*} - 1 - \ln\frac{I_h}{I_h^*}\right) + \frac{abS_h^*I_v^*}{\mu_v\theta_1}\left(\frac{I_v}{I_v^*} - 1 - \ln\frac{I_v}{I_v^*}\right),$$

where $(S_h^*, I_h^*, I_v^*)$ represents one of the two endemic equilibrium points $\mathcal{E}^{\pm}$.

The equilibrium equations of system (2) lead to $\mu_h(H - S_h^*) = \left(\beta\frac{I_h^*}{H} + ab\frac{I_v^*}{H}\right)S_h^*$, $(\gamma+\mu_h)I_h^* = \left(\beta\frac{I_h^*}{H} + ab\frac{I_v^*}{H}\right)S_h^*$, $ac\frac{I_h^*}{H}S_v^* = \mu_v(1-\theta)I_v^*$. By using these equalities in the derivative of $\mathcal{L}$ along of trajectories of (2) leads to

$$\begin{aligned}\mathcal{L}' \leq & -\frac{\mu_h}{S_h}\left(S_h - S_h^*\right)^2 - \frac{a^2bcS_h^*I_h}{\mu_v\theta_1H^2I_v}\left(I_v - I_v^*\right)^2 + \frac{abS_h^*I_v^*}{H}\left(3 - \frac{S_h^*}{S_h} - \frac{S_hI_h^*I_v}{S_h^*I_hI_v^*} - \frac{I_hI_v^*}{I_h^*I_v}\right)\\ & + \beta\frac{S_h^*I_h^*}{H^2}\left(2 - \frac{S_h}{S_h^*} - \frac{S_h^*}{S_h}\right),\end{aligned}$$

which, thanks to the inequality of arithmetic and geometric means, is negative. Also, $\mathcal{L}' = 0 \Longrightarrow (S_h, I_h, I_v) = (S_h^*, I_h^*, I_v^*)$. Using [**22**], we conclude that one of the both endemic equilibrium points is globally stable. We therefore affirm the following result:

*Theorem 4:*
The Zika model (2) admits a unique endemic equilibrium in $\Sigma$ which is globally asymptotically stable whenever $\mathbf{R}_0 > 1$.

### 3.3 Existence and uniqueness of solutions

The objective of this part is to show by using the theory of the fixed points the existence and the uniqueness of the solutions of the Zika fractional model (in the Caputo sense) with the initial conditions. In this part, $\mathbf{B}$ is the Banach space for continuous functions with real values over an interval $\mathbf{I}$ with the associated norms

$$\| (S_h, I_h, R_h, S_v, I_v) \| = \| S_h \| + \| I_h \| + \| R_h \| + \| S_v \| + \| I_v \|,$$

where $\| S_h \| = \sup\{| S_h(t) | : t \in \mathbf{I}\}$, $\| I_h \| = \sup\{| I_h(t) | : t \in \mathbf{I}\}$, $\| R_h \| = \sup\{| R_h(t) | : t \in \mathbf{I}\}$, $\| S_v \| = \sup\{| S_v(t) | : t \in \mathbf{I}\}$, $\| I_v \| = \sup\{| I_v(t) | : t \in \mathbf{I}\}$ and $\mathbf{B} = \varepsilon(\mathbf{I}) \times \varepsilon(\mathbf{I}) \times \varepsilon(\mathbf{I}) \times \varepsilon(\mathbf{I}) \times \varepsilon(\mathbf{I})$, with $\varepsilon(\mathbf{I})$ the Banach space for continuous functions with real values on $\mathbf{I}$ and the above associated norms. The fractional system (2) can be written in the form:

$$\begin{aligned}&{}^{C}D_t^{\eta}S_h(t) = \mathbf{G}_1(t, S_h), {}^{C}D_t^{\eta}I_h(t) = \mathbf{G}_2(t, I_h), {}^{C}D_t^{\eta}R_h(t) = \mathbf{G}_3(t, R_h),\\ &{}^{C}D_t^{\eta}S_v(t) = \mathbf{G}_1(t, S_v), {}^{C}D_t^{\eta}I_v(t) = \mathbf{G}_1(t, I_v),\end{aligned} \tag{7}$$

where $G_i(t,x)$, $i \in [0;5] \cap \mathbb{N}$, $x \in \{S_h, I_h, R_h, S_v, I_v\}$, represent the right-hand sides of model (2).

By applying the fractional integral operator in equation (7), we obtain:

$$\begin{aligned} S_h(t) - S_h(0) &= \frac{1}{\Gamma(\eta)} \int_0^t (t-\chi)^{\eta-1} G_1(\chi, S_h) d\chi, I_h(t) - I_h(0) = \frac{1}{\Gamma(\eta)} \int_0^t (t-\chi)^{\eta-1} G_2(\chi, I_h) d\chi, \\ R_h(t) - R_h(0) &= \frac{1}{\Gamma(\eta)} \int_0^t (t-\chi)^{\eta-1} G_3(\chi, R_h) d\chi, S_v(t) - S_v(0) = \frac{1}{\Gamma(\eta)} \int_0^t (t-\chi)^{\eta-1} G_4(\chi, S_v) d\chi, \\ I_v(t) - I_v(0) &= \frac{1}{\Gamma(\eta)} \int_0^t (t-\chi)^{\eta-1} G_5(\chi, I_v) d\chi, \end{aligned} \tag{8}$$

The kernels $\mathbf{G}_i$, $i = 1, 2, 3, 4, 5$, satisfy the Lipschitz and contraction conditions if the following inequality holds:

$$0 \leq V_i < 1, \qquad i = 1, 2, 3, 4, 5, \tag{9}$$

where $V_1 = \left(\mu_h + \beta + ab\frac{\nu}{H}\right)$, $V_2 = (\mu_h + \gamma) + \beta$, $V_3 = \mu_h$, $V_4 = \mu_v + ac$, $V_5 = \mu_v(1-\theta)$. We thus claim the following result

*Theorem 5:*
The kernel $\mathbf{G}_1$ satisfies the Lipschitz condition if the following inequality is respected

$$0 \leq V_1 := \left(\mu_h + \beta + ab\frac{\nu}{H}\right) \leq 1 \qquad \text{with} \qquad \| I_v \| = \nu. \tag{10}$$

*Proof.* Consider the kernel $\mathbf{G}_1$, and $S_h$, $S_{1h}$ any two functions. We have

$$\| \mathbf{G}_1(t, S_h) - \mathbf{G}_1(t, S_{1h}) \| = \| -\mu_h (S_h - S_{1h}) - \left(\beta\frac{I_h}{H} + ab\frac{I_v}{H}\right)(S_h - S_{1h}) \| . \tag{11}$$

Using the triangular inequality for the norms of (11) above, we get:

$$\begin{aligned} \| \mathbf{G}_1(t, s_h) - \mathbf{G}_1(t, s_{1h}) \| &\leq \| \mu_h (S_h - S_{1h}) \| + \| \left(\beta\frac{I_h}{H} + ab\frac{I_v}{H}\right)(S_h - S_{1h}) \| \\ &\leq \| \left(\mu_h + \beta\frac{I_h}{H} + ab\frac{I_v}{H}\right) \| \| (S_h(t) - S_{1h}(t) \|, \\ &\leq \left(\mu_h + \beta + ab\frac{\| I_v \|}{H}\right) \| (S_h(t) - S_{1h}(t) \|, \\ &= \overbrace{\left(\mu_h + \beta + ab\frac{\nu}{H}\right)}^{V_1} \| S_h(t) - S_{1h}(t) \| . \end{aligned} \tag{12}$$

Therefore, the Lipschitz condition is fulfilled for $\mathbf{G}_1$, and if in addition, $0 \leq V_1 < 1$, then it is also a contraction. We get similar results for the other kernels:

$$\begin{aligned} &\| \mathbf{G}_2(t, I_h) - \mathbf{G}_2(t, I_{1h}) \| \leq V_2 \| I_h(t) - I_{1h}(t) \| \\ &\| \mathbf{G}_3(t, R_h) - \mathbf{G}_3(t, R_{1h}) \| \leq V_3 \| R_h(t) - R_{1h}(t) \| \\ &\| \mathbf{G}_4(t, S_v) - \mathbf{G}_4(t, S_{1v}) \| \leq V_4 \| S_v(t) - S_{1v}(t) \| \\ &\| \mathbf{G}_5(t, I_v) - \mathbf{G}_5(t, I_{1v}) \| \leq V_5 \| I_v(t) - I_{1v}(t) \| . \end{aligned} \tag{13}$$

□

The expressions of the equation (8) can be written recursively as:

$$\begin{aligned}
S_{hn}(t) - S_h(0) &= \frac{1}{\Gamma(\eta)} \int_0^t (t-\chi)^{\eta-1} \mathbf{G}_1(\chi, S_{h(n-1)}) d\chi, \\
E_{hn}(t) - E_h(0) &= \frac{1}{\Gamma(\eta)} \int_0^t (t-\chi)^{\eta-1} \mathbf{G}_2(\chi, E_{h(n-1)}) d\chi, \\
I_{hn}(t) - I_h(0) &= \frac{1}{\Gamma(\eta)} \int_0^t (t-\chi)^{\eta-1} \mathbf{G}_3(\chi, I_{h(n-1)}) d\chi, \\
S_{vn}(t) - S_v(0) &= I \frac{1}{\Gamma(\eta)} \int_0^t (t-\chi)^{\eta-1} \mathbf{G}_4(\chi, S_{v(n-1)}) d\chi, \\
I_{vn}(t) - I_v(0) &= \frac{1}{\Gamma(\eta)} \int_0^t (t-\chi)^{\eta-1} \mathbf{G}_5(\chi, I_{v(n-1)}) d\chi,
\end{aligned} \tag{14}$$

The difference between successive terms for recursive formulas is written:

$$\begin{aligned}
\phi_{1n}(t) = S_{hn}(t) - S_{h(n-1)}(t) &= \frac{1}{\Gamma(\eta)} \int_0^t (t-\chi)^{\eta-1} \left( \mathbf{G}_1(\chi, S_{h(n-1)}) - \mathbf{G}_1(\chi, S_{h(n-2)}) \right) d\chi, \\
\phi_{2n}(t) = I_{hn}(t) - I_{h(n-1)}(t) &= \frac{1}{\Gamma(\eta)} \int_0^t (t-\chi)^{\eta-1} \left( \mathbf{G}_2(\chi, I_{h(n-1)}) - \mathbf{G}_2(\chi, I_{h(n-2)}) \right) d\chi, \\
\phi_{3n}(t) = R_{hn}(t) - R_{h(n-1)}(t) &= \frac{1}{\Gamma(\eta)} \int_0^t (t-\chi)^{\eta-1} \left( \mathbf{G}_3(\chi, R_{h(n-1)}) - \mathbf{G}_3(\chi, R_{h(n-2)}) \right) d\chi, \\
\phi_{4n}(t) = S_{vn}(t) - S_{v(n-1)}(t) &= \frac{1}{\Gamma(\eta)} \int_0^t (t-\chi)^{\eta-1} \left( \mathbf{G}_4(\chi, S_{v(n-1)}) - \mathbf{G}_4(\chi, S_{v(n-2)}) \right) d\chi, \\
\phi_{5n}(t) = I_{vn}(t) - I_{v(n-1)}(t) &= \frac{1}{\Gamma(\eta)} \int_0^t (t-\chi)^{\eta-1} \left( \mathbf{G}_5(\chi, I_{v(n-1)}) - \mathbf{G}_5(\chi, I_{v(n-2)}) \right) d\chi,
\end{aligned} \tag{15}$$

with initial conditions:

$$S^{h0}(t) = S_h(0), \quad I^{h0}(t) = I_h(0), \quad R^{h0}(t) = R_h(0), \quad S^{v0}(t) = S_v(0), \quad I^{v0}(t) = I_v(0). \tag{16}$$

Let us take the norm of the first expression of the equation (15)

$$\begin{aligned}
\| \phi_{1n}(t) \| &= \| S_{hn}(t) - S_{h(n-1)}(t) \| \\
&= \| \frac{1}{\Gamma(\eta)} \int_0^t (t-\chi)^{\eta-1} (\mathbf{G}_1(\chi, S_{h(n-1)}) - \mathbf{G}_1(\chi, S_{h(n-2)}) d\chi \|, \\
&\leq \frac{1}{\Gamma(\eta)} \| \int_0^t (t-\chi)^{\eta-1} (\mathbf{G}_1(\chi, S_{h(n-1)}) - \mathbf{G}_1(\chi, S_{h(n-2)}) d\chi \| .
\end{aligned} \tag{17}$$

Using the Lipschitz condition we have:

$$\| S_{hn}(t) - S_{h(n-1)}(t) \| \leq \frac{1}{\Gamma(\eta)} V_1 \int_0^t (t-\chi)^{\eta-1} \| S_{h(n-1)} - S_{h(n-2)} \| d\chi. \tag{18}$$

Thus, we obtain

$$\| \phi_{1n}(t) \| \leq \frac{1}{\Gamma(\eta)} V_1 \int_0^t (t-\chi)^{\eta-1} \| \phi_{1(n-1)}(\chi) \| d\chi. \tag{19}$$

Similar reasoning makes it possible to have:

$$\begin{aligned}
\| \phi_{2n}(t) \| &\leq \frac{1}{\Gamma(\eta)} V_2 \int_0^t (t-\chi)^{\eta-1} \| \phi_{2(n-1)}(\chi) \| d\chi, \\
\| \phi_{3n}(t) \| &\leq \frac{1}{\Gamma(\eta)} V_3 \int_0^t (t-\chi)^{\eta-1} \| \phi_{3(n-1)}(\chi) \| d\chi, \\
\| \phi_{4n}(t) \| &\leq \frac{1}{\Gamma(\eta)} V_4 \int_0^t (t-\chi)^{\eta-1} \| \phi_{4(n-1)}(\chi) \| d\chi, \\
\| \phi_{5n}(t) \| &\leq \frac{1}{\Gamma(\eta)} V_5 \int_0^t (t-\chi)^{\eta-1} \| \phi_{5(n-1)}(\chi) \| d\chi.
\end{aligned} \tag{20}$$

Note that $S_{hn}(t) = \sum_{i=1}^{n} \phi_{1i}(t)$, $E_{hn}(t) = \sum_{i=1}^{n} \phi_{2i}(t)$, $I_{hn}(t) = \sum_{i=1}^{n} \phi_{3i}(t)$, $S_{vn}(t) = \sum_{i=1}^{n} \phi_{4i}(t)$, $I_{vn}(t) = \sum_{i=1}^{n} \phi_{5i}(t)$.

Now let's show the uniqueness of the model solution. We claim the following

*Theorem 6:*
The fractional model (2) admits a unique solution on $t \mid [0, T]$ if the following inequality is satisfied

$$\frac{1}{\Gamma(\eta)} V_i b^{\eta} \leq 1, \quad i = 1, 2, ...5. \tag{21}$$

*Proof.* The proof follows the proof of [**18**]. □

## IV NUMERICAL SCHEME AND RESULTS

### 4.1 Numerical scheme

In what follows, we will use the Adams-Bashforth-Moulton method (see [**15**, **23**]) to construct a numerical scheme for the fractional model (3). To this aim, let us consider the uniform discretization of $[0, a]$ given by $t_m = m\aleph$, $m \in [0; N] \cap \mathbb{N}$ where $\aleph = a/m$ denotes the step size. For a given approximation $\mathcal{U}_{\aleph}(t_i) \approx \mathcal{U}(t_i)$, the next approximation $\mathcal{U}_{\aleph}(t_{i+1})$ is obtain (using the predictor-corrector method) as follows:

$$\text{Predictor: } \mathcal{U}^{p}_{\aleph}(t_{m+1}) = \sum_{l=0}^{\lceil \varphi \rceil - 1} \frac{t_{n+1}^{l}}{l!} \mathcal{U}_0^l + \frac{1}{\Gamma(\eta)} \sum_{l=0}^{m} b_{l,m+1} \mathbf{G}\left(t_l, \mathcal{U}_{\aleph}(t_l)\right);$$

$$\text{Corrector: } \mathcal{U}_{\aleph}(t_{m+1}) = \sum_{l=0}^{\lceil \varphi \rceil - 1} \frac{t_{n+1}^{l}}{l!} \mathcal{U}_0^l + \frac{\aleph^{\eta}}{\Gamma(2+\eta)} \mathbf{G}\left(t_{l+1}, \mathcal{U}^{\aleph}_{\aleph}(t_{l+1})\right) + \frac{\aleph^{\eta}}{\Gamma(2+\eta)} \sum_{l=0}^{m} c_{l,m+1} \mathbf{G}\left(t_l, \mathcal{U}_{\aleph}(t_l)\right);$$

with

$$c_{l,m+1} = \begin{cases} m^{1+\varphi} - (m-\varphi)(m+1)^{\varphi}, & \text{if} \quad l = 0, \\ (m-l+2)^{1+\varphi} + (m-l)^{1+\varphi} - 2(m-l+1)^{1+\varphi}, & \text{if} \quad 1 \leq l \leq m, \\ 1 & \text{if} \quad l = m+1, \end{cases}$$

and $b_{l,m+1} = \dfrac{\aleph^{\varphi}}{\varphi} \left[(m-l+1)^{\varphi} - (m-l)^{\varphi}\right]$.

### 4.2 Numerical results

For numerical simulation, we consider the same parameter values which came from parameter estimation of the model (3) using the Guillain-Barré Syndrome (GBS) surveillance data (see [**8**]). In [**8**], Yuan et *al.* obtained after model calibration $a = 0.45$, $b = 0.3$, $c = 0.4$, $\theta = 0.0034$, $\gamma = 0.14$, $\mu_h = 4 \times 10^{(-5)}$, $\mu_v = 0.125$, $\beta = 0.05$, $\rho = 0.0024\%$, $H = 10^5$, and $M = 1.5H = 15 \times 10^4$. With these parameter values,$\mathbf{R}_0 = 1.6349$, which means that we are in endemic state. Figure 1&2 illustrate the results of Theorems 1 and 4, respectively. It is

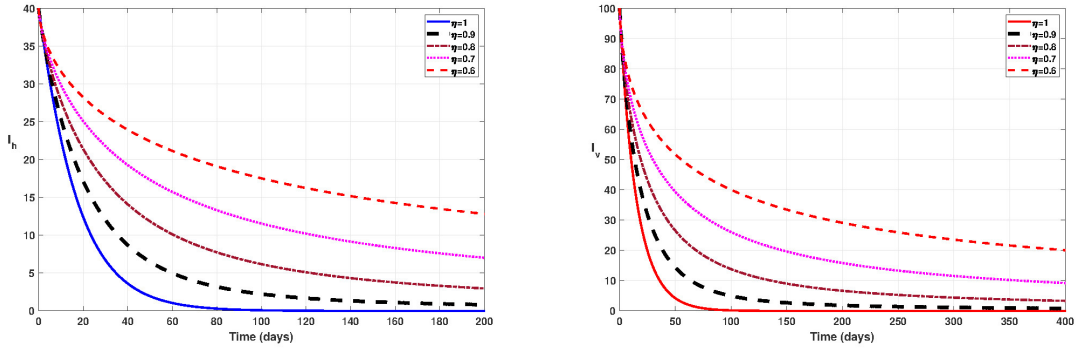


Figure 1: Time-series of $I_h$ and $I_v$ with $\mathcal{R}_0 = 0.92$ (we have set $a = 0.225$) for different values of the fractional order $\eta \in \{1; 0.9; 0.8; 0.7; 0.6\}$. Initial conditions are $(I_h(0), R_h(0), I_v(0)) = (40, 60000, 100)$.

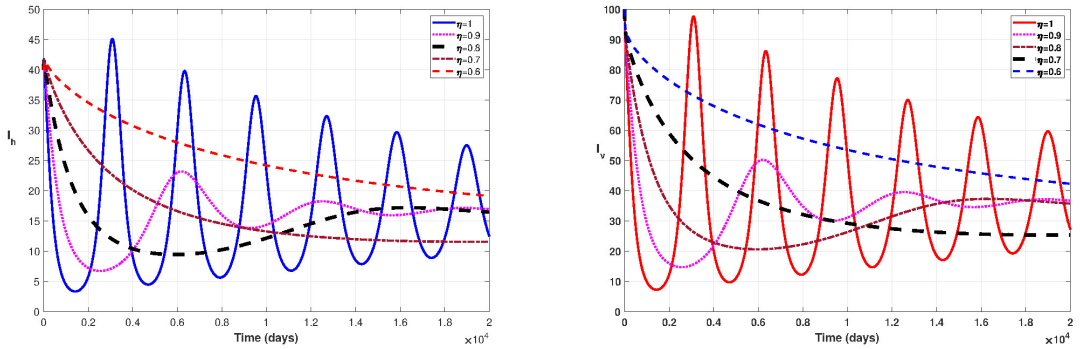


Figure 2: Time-series of $I_h$ and $I_v$ with $\mathcal{R}_0 = 1.63$ for different values of the fractional order $\eta \in \{1; 0.9; 0.8; 0.7; 0.6\}$. Initial conditions are $(I_h(0), R_h(0), I_v(0)) = (40, 60000, 100)$.

clear that, when $\mathbf{R}_0 < 1$, the solution approaches the Zika-free equilibrium for $\eta = 1$, and the needed time is forward delayed as the fractional order $\eta$ decreases (Fig. 1), and when $\mathbf{R}_0 > 1$, the solution approaches the endemic equilibrium (Fig. 2).

Figures 1&2 also illustrates the impact of varying fractional order $\eta$ on the Zika dynamics. Indeed, we see that oscillations decrease when the fractional order decreases.

## V CONCLUSION

In this work, we studied a transmission model of Zika disease with fractional derivatives in the Caputo sense. After the formulation of the model which consists to replace integer derivative

with fractional derivative in the Caputo sense, we computed the basic reproduction number $\mathbf{R}_0$ and gave rigorous proof of the existence of equilibrium points of both models as well as stability analysis of these equilibrium points. Then, we studied the existence and the uniqueness of the solutions of the fractional model using the Banach fixed point theory. We finally performed numerical simulations to validate our analytical results, as well as to see the impact of varying the fractional parameter $\eta$ on the disease dynamics. Indeed, when the fractional order decreases, the oscillation amplitude also decreases.

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## A ACKNOWLEDGEMENTS

The first author thanks the Zukunftskolleg of the University of Konstanz for their hospitality during his research stay which permits him to improve this work. The authors express their gratitude to the anonymous reviewers for their comments and suggestions which helped to improve the manuscript.