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Fractional modeling of Hansen's disease (Leprosy) transmission dynamics

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Abstract

In this work, we study a mathematical model for the Hansen's disease (leprosy) transmission dynamics with both integer and fractional derivatives in the Caputo sense. After the model formulation, we compute the leprosy reproduction number \mathcal{R}_0 and prove the existence of two steady states named the Leprosy-free equilibrium and the leprosy-endemic equilibrium which exists and is unique if and only if $\mathcal{R}_0 > 1$. Using the general theory of Lyapunov, we prove the global asymptotic stability of both steady states, for both models. The existence and uniqueness of the solutions of the fractional model are proved using fixed point theory. We finally perform numerical simulations to validate our analytical results, as well as to evaluate the impact of varying the fractional-order parameter on the disease dynamics.

Keywords

Hansen's disease (Leprosy), Fractional derivative, Caputo derivative, Asymptotic stability, Fixed point theory.

I INTRODUCTION

Hansen's disease, previously known colloquially as Leprosy, is a contagious disease that affects the skin, mucous membranes, and nerves, causing discoloration and lumps on the skin and, in severe cases, disfigurement and deformities. Hansen's disease is now mainly confined to tropical Africa and Asia [1]. In 2019, Hansen's disease has reported by 153 countries to the World Health Organization (WHO) among which 24 were from the Western Pacific region, 19 from the Eastern Mediterranean region, 31 from the region of the Americas, 11 from the South-East Asia region, 30 from the European region, and 38 from the African region [2].

Several model was developed to study the spread of Hansen's disease. To investigate the impact of relapse cases and non-compliance with WHO multi-drug therapy among leprosy patients, Mushayabasa *et al.* in [3] formulated a compartmental model. They computed the basic reproduction number and prove global stability of the steady states. Based on the understanding of the biology of the infection, Smith *et al.* [4] proposed a compartmental model for Hansen's disease dynamics. After computed the basic reproduction number, She fitted the transmission coefficients for the model and the rate of detection using real data of Hansen's disease in 5

regions of Brazil. In [5], Chiyaka et al. investigate non-compliant behavior by patients on treatment and possible inadequacy of the prescribed treatments as the reason for the persistence of the disease in the Sub-Saharan Africa region. To this aim, they construct compartmental mathematical models of the transmission dynamics of Hansen's disease, which are modified to encapsulate non-compliance and inadequate treatment. In order to predict the transmission of new patients and allow different groups of individuals to be analyzed in the face of exposure and infection, Dias et al. in [6] studied a compartmental model which illustrates the dispersion of leprosy in a population, seeking the impact on the understanding of transmission and planning of disease control strategies. To control the infection, Haroun et al. [7] studied the benefits of multi-drug therapy as a control measure. In this paper, we extend the recent work of Raza & Rafiq [8] by replacing: (1) mass action incidence by standard incidences; (2) integer derivative by fractional derivative in the Caputo's sense. After the formulation of the news models (with integer and fractional derivative), we compute the new basic reproduction number \mathcal{R}_0 as well as the steady states (the leprosy-free equilibrium and the unique endemic equilibrium). We prove the global asymptotic stability of these steady states depending on the value of \mathcal{R}_0 , using Lyapunov theory and LaSalle invariance principle [9]. The existence and uniqueness of solutions of the fractional model are also proved. Through the Adams-Bashforth-Moulton method. we construct a numerical scheme for the fractional model. Finally, several simulations are performed to validate our analytical results, as well as to see the impact of varying the fractional parameter on the disease spread.

The paper is organized as follows: In Section II, we present the definition of the Caputo operator, as well as two results required for model analysis. The new leprosy models are formulated in Section III, as well as the existence and stability analysis of steady states, the existence, and uniqueness of solutions of the fractional model. Numerical simulation results, are presented in Section IV. A conclusion round up the paper.

II PRELIMINARIES

Before formulating and studying the fractional model of Hansen's disease (Leprosy) transmission dynamics, we first present a definition of the Caputo operator, as well as two results required for model analysis (see [10]).

Definition 1:

Let $f \in C^l[0; a]$, $a > 0$, $\eta \in \mathbb{R}$, $l \in \mathbb{N}$ such that $l - 1 < \eta < l$. The fractional derivative in the Caputo sense of order l , of f is given by

$${}^C\mathbb{D}_t^\eta f(t) = \begin{cases} \frac{d^l f(t)}{dt^l}, & \eta = l \in \mathbb{N} \\ \frac{1}{\Gamma(l-\eta)} \int_0^t (t-\phi)^{l-\eta-1} f^{(l)}(\phi) d\phi, & l-1 < \eta < l, l \in \mathbb{N}. \end{cases} \quad (1)$$

Lemma 1:

If $0 < \eta < 1$ and $d \in \mathbb{N}$, then we can find nonnegative constants $W_{\eta,1}$ and $W_{\eta,2}$ only dependent on η , such that

$$(d+1)^\eta - d^\eta \leq W_{\eta,1}(d+1)^{\eta-1} \quad \text{and} \quad (d+2)^{\eta+1} - 2(d+1)^{\eta+1} + d^{\eta+1} \leq W_{\eta,2}(d+1)^{\eta-1}.$$

Lemma 2:

Assume that $x_{q,i} = (i-q)^{\eta-1} (q=1, 2, \dots, i-1)$ & $x_{q,i} = 0$ for $q \geq i$, $\eta, Q, h, Y > 0$, $kh \leq Y$

& $k \in \mathbb{N}$. Let $\sum_{q=k}^{q=i} x_{q,i}|e_q| = 0$ for $k > i \geq 1$. If $|e_i| \leq Qh^\eta \sum_{q=1}^{i-1} x_{q,i}|e_q| + |\eta_0|$, $i = 1, 2, \dots, k$, then $|e_k| \leq W|\eta_0|$, $k = 1, 2, \dots$, where $W \in \mathbb{R}_+$ is independent of k and h .

III MODEL FORMULATION AND BASICS RESULTS

The proposed model, which is an extension of the model proposed by Raza & Rafiq [8], is presented as follows:

$$\begin{cases} D_t x(t) &= p - \frac{(\beta_m z + \beta_\rho w)x}{x + y + z + w} - \mu x, \\ D_t y(t) &= \frac{(\beta_m z + \beta_\rho w)x}{x + y + z + w} - \theta y - \mu y, \\ D_t z(t) &= f\theta y - \mu z, \\ D_t w(t) &= \theta(1 - f)y - \mu w. \end{cases} \quad (2)$$

In model (2), D_t denotes the integer derivative with respect to time t ; p is the annual recruitment rate; μ is the death rate; β_m is the Effective contact rate by multibacillary leprosy; β_ρ is the effective contact rate for paucibacillary leprosy; θ denotes the transmission rate of asymptomatic individuals to the symptomatic individuals; f is the fraction of individual developed by multibacillary leprosy while $1 - f$ is the fraction of individual developed by paucibacillary leprosy; x denotes susceptible individuals, y denotes asymptomatic infected individuals, w denotes infected individuals by paucibacillary leprosy; and z denotes infected individual by multibacillary leprosy.

The corresponding fractional model with Caputo derivatives is given by

$$\begin{cases} {}^C D_t^\eta x(t) &= p - \frac{(\beta_m z + \beta_\rho w)x}{x + y + z + w} - \mu x, \\ {}^C D_t^\eta y(t) &= \frac{(\beta_m z + \beta_\rho w)x}{x + y + z + w} - \theta y - \mu y, \\ {}^C D_t^\eta z(t) &= f\theta y - \mu z, \\ {}^C D_t^\eta w(t) &= \theta(1 - f)y - \mu w, \end{cases} \quad (3)$$

where ${}^C D_t^\eta$ denotes the Caputo-type fractional derivative of order η . To system (3), we add the following initial conditions: $x(0) = x_0 > 0$, $y(0) = y_0 \geq 0$, $z(0) = z_0 \geq 0$, and $w(0) = w_0 \geq 0$.

Without loss of generality, the above model (2) (resp. (3)) is defined in the following absorbed and bounded set $\Pi = \left\{ (x(t), y(t), z(t), w(t))' \in \mathbb{R}_+^4 : x(t) + y(t) + z(t) + w(t) \leq \frac{p}{\mu} \right\}$ [8].

3.1 Existence of steady states

Let us define the following threshold called the basic reproduction number

$$\mathcal{R}_0 = \frac{\theta(f\beta_m + (1 - f)\beta_\rho)}{(\theta + \mu)\mu}. \quad (4)$$

The following result holds.

Theorem 1:

Leprosy model (2) (resp. (3)) admits two steady states: the Leprosy-free equilibrium $\mathcal{E}_0 = \left(\frac{p}{\mu}, 0, 0, 0\right)'$, and an unique Leprosy-endemic equilibrium $\mathcal{E}_1 = \frac{p}{(\mu(\mathcal{R}_0 - 1) + \mu)} \left(1, \frac{\mu(\mathcal{R}_0 - 1)}{(\theta + \mu)}, \frac{f\theta\mu(\mathcal{R}_0 - 1)}{\mu(\theta + \mu)}, \frac{(1-f)\theta\mu(\mathcal{R}_0 - 1)}{\mu(\theta + \mu)}\right)$ which exists if and only if $\mathcal{R}_0 > 1$.

Proof. Equating the right-hand sides of (2) (resp. (3)) by zero, and noting that $x^* + y^* + z^* + w^* = \frac{p}{\mu}$, it follows that model (2) (resp. (3)) has two equilibrium points: the disease-free equilibrium $\mathcal{E}_0 = \left(\frac{p}{\mu}, 0, 0, 0\right)'$, and the endemic equilibrium $\mathcal{E}_1 = (x^*, y^*, z^*, w^*)'$ where $x^* = \frac{p}{\lambda^* + \mu}$, $y^* = \frac{\lambda x^*}{(\theta + \mu)}$, $z^* = \frac{f\theta y^*}{\mu}$, $w^* = \frac{\theta(1-f)y^*}{\mu}$, with $\lambda^* = \mu(\mathcal{R}_0 - 1)$. Note that \mathcal{E}_1 exists if and only if $\mathcal{R}_0 > 1$. This ends the proof. \square

3.2 Stability analysis of the Leprosy-free equilibrium

The Jacobian matrix of system (2) (resp. (3)) evaluated at the Leprosy-free equilibrium $\mathcal{E}_0 = \left(\frac{p}{\mu}, 0, 0, 0\right)'$, is given by $\mathcal{J}(\mathcal{E}_0) = \begin{pmatrix} -\mu & 0 & -\beta_m & -\beta_\rho \\ 0 & -(\theta + \mu) & \beta_m & \beta_\rho \\ 0 & f\theta & -\mu & 0 \\ 0 & \theta(1-f) & 0 & -\mu \end{pmatrix}$.

The characteristic polynomial of $\mathcal{J}(\mathcal{E}_0)$ is given by $\mathcal{P}(X) := -(X + \mu)(a_3 X^3 + a_2 X^2 + a_1 X + a_0)$, where $a_3 = 1$, $a_2 = 2\mu + (\theta + \mu)$, $a_1 = \mu[\mu + (\theta + \mu)(2 - \mathcal{R}_0)]$, and $a_0 = \mu^2(\theta + \mu)(1 - \mathcal{R}_0)$. The roots of $\mathcal{P}(X)$ are $X_1 = -\mu$ and those of the polynomial $\mathcal{P}_1(X) := (a_3 X^3 + a_2 X^2 + a_1 X + a_0)$. Note that $a_0 > 0 \iff \mathcal{R}_0 < 1$. Also, $\mathcal{R}_0 < 1$ ensures that $a_3, a_2, a_1 > 0$.

For the model (2), the characteristic equation [11] is given by $\det[s(I_4 - (1 - \eta)\mathcal{J}(\mathcal{E}_0)) - \eta\mathcal{J}(\mathcal{E}_0)] = 0$, which is equivalent to $[s(1 + \mu\eta_1) + \mu\eta]^2 (A_2 s^2 + A_1 s + A_0) = 0$, where $A_2 = [k_1(1 - \mathcal{R}_0)(\eta - 1)^2 \mu + (1 - \eta)\mu + k_1(1 - \eta) + 1]$, $A_0 = k_1 \eta^2 \mu(1 - \mathcal{R}_0)$, and $A_1 = [2k_1 \eta(1 - \mathcal{R}_0)(1 - \eta)\mu + \eta(\mu + k_1)]$, with $\eta_1 = 1 - \eta$, $k_1 = \theta + \mu$.

Solutions of the above equation are $s = -\frac{\mu\eta}{1 + \mu\eta_1} < 0$ and those of $(A_2 s^2 + A_1 s + A_0)$. The conditions $\mathcal{R}_0 < 1$ ensures that coefficients A_2, A_1 , and A_0 are positive, and thus the solutions of $(A_2 s^2 + A_1 s + A_0) = 0$ are always negative. We thus resume the above analysis as follows:

Theorem 2:

For the model (2) (resp. (3)), the Leprosy-free equilibrium $\mathcal{E}_0 = \left(\frac{p}{\mu}, 0, 0, 0\right)'$ is locally asymptotically stable in Π whenever $\mathcal{R}_0 < 1$.

We also claim the following result.

Theorem 3:

For the model (2) (resp. (3)), the Leprosy-free equilibrium $\mathcal{E}_0 = \left(\frac{p}{\mu}, 0, 0, 0\right)'$ is globally (resp. uniformly) asymptotically stable in Π whenever $\mathcal{R}_0 < 1$.

Proof. Let us consider the following Lyapunov function:

$$\mathcal{M}(y(t), z(t), w(t)) := y(t) + \frac{\beta_m}{\mu} z(t) + \frac{\beta_\rho}{\mu} w(t), \quad (5)$$

The derivative of \mathcal{M} is given by

$$\begin{aligned} \dot{\mathcal{M}}(y(t), z(t), w(t)) &= a_1 \dot{y}(t) + a_2 \dot{z}(t) + a_3 \dot{w}(t) \\ &= a_1 \left(\frac{(\beta_m z + \beta_\rho w) x}{x + y + z + w} - (\theta + \mu) y \right) + a_2 (f\theta y - \mu z) + a_3 (\theta(1 - f)y - \mu w) \\ &\leq (a_2 f\theta - a_1(\theta + \mu) + a_3 \theta(1 - f)) y + (a_1 \beta_m - a_2 \mu) z + (a_1 \beta_\rho - a_3 \mu) w \\ &= a_1(\theta + \mu) \left(\frac{\beta_m}{\mu(\theta + \mu)} f\theta + \frac{\beta_\rho}{\mu(\theta + \mu)} \theta(1 - f) - 1 \right) y \\ &= a_1(\theta + \mu) (\mathcal{R}_0 - 1) y. \end{aligned} \quad (6)$$

It then follows that $\mathcal{R}_0 < 1 \iff \dot{\mathcal{M}} < 0$, and $\dot{\mathcal{M}} = 0$ if and only if $\mathcal{R}_0 = 1$ or $y = 0$. Replacing $y = 0$ in (2) (resp. (3)) gives $x = x^* = \frac{p}{\mu}$ and $z = w = 0$. So that,

$\lim_{t \rightarrow \infty} (x(t), y(t), z(t), w(t)) \rightarrow \left(\frac{p}{\mu}, 0, 0, 0 \right)' := \mathcal{E}_0$. Thus (by [12]), we conclude that the Leprosy-free equilibrium point \mathcal{E}_0 of model (2) (resp. (3)) is globally (resp. uniformly) asymptotically stable in Π whenever $\mathcal{R}_0 < 1$. \square

3.3 Stability analysis of the endemic equilibrium

The jacobian matrix of system (2) (resp. (3)) evaluated at the endemic equilibrium point $\mathcal{E}_1 = (x^*, y^*, z^*, w^*)'$ is given by

$$\mathcal{J}(\mathcal{E}_1) = \begin{pmatrix} -\mu - \frac{(y^* + z^* + w^*)\lambda^*}{N^*} & -\frac{\lambda^* x^*}{N^*} & \frac{-\beta_m x^*(x^* + y^* + w^*) + \beta_\rho x^* w^*}{(N^*)^2} & \frac{-\beta_\rho x^*(x^* + y^* + z^*) + \beta_m x^* z^*}{(N^*)^2} \\ \frac{\lambda^*(y^* + z^* + w^*)}{N^*} & -k_1 - \frac{\lambda^* x^*}{N^*} & \frac{\beta_m x^*(x^* + y^* + w^*) - \beta_\rho x^* w^*}{(N^*)^2} & \frac{\beta_\rho x^*(x^* + y^* + z^*) - \beta_m x^* z^*}{(N^*)^2} \\ 0 & f\theta & -\mu & 0 \\ 0 & (1 - f)\theta & 0 & -\mu \end{pmatrix}$$

where $N^* = x^* + y^* + z^* + w^* = \frac{p}{\mu}$. The characteristic equation of $\mathcal{J}(\mathcal{E}_1)$ is given by $-(X + \mu)^2 (c_2 X^2 + c_1 X + c_0) = 0$, where $c_2 = R_0^2 k_1 (b_\rho(1 - f) + f b_m) > 0$,

$$\begin{aligned} c_1 &= (1 - f)((R_0 - 1)^2 R_0 \mu^2 + (2R_0 - 1)R_0 k_1 \mu + R_0^2 k_1^2) \beta_\rho \\ &\quad + (R_0 - 1)^2 R_0 (f \beta_m + R_0 k_1) \mu^2 + R_0 (2R_0 - 1) k_1 f \beta_m \mu + R_0^2 k_1^2 f \beta_m \\ &> 0, \end{aligned}$$

$$\begin{aligned} c_0 &= (R_0 - 1) \{ \mu(2(R_0 - 1)^2 \mu^2 + (R_0 - 1)R_0 k_1 \mu + R_0 k_1^2)(1 - f) \beta_\rho \\ &\quad + 2(R_0 - 1)^2 (f \beta_m + R_0 k_1) \mu^3 + R_0 k_1 (R_0 - 1) (f \beta_m + R_0 k_1) \mu^2 + R_0 k_1^2 f \beta_m \mu \}, \end{aligned}$$

Note that c_2 and c_1 are always positive, and $c_0 > 0 \iff \mathcal{R}_0 > 1$. Thus, we claim the following result:

Lemma 3:

The unique endemic equilibrium point \mathcal{E}_1 of model (2) is locally asymptotically stable in Π whenever $\mathcal{R}_0 > 1$.

We also claim what follows.

Theorem 4:

If $\mathcal{R}_0 > 1$, then the unique endemic equilibrium point \mathcal{E}_1 of model (2) (resp. (3)) is globally asymptotically stable in Π provided that

$$\left(1 - \frac{\lambda}{\lambda^*}\right) \left(1 - \frac{x y^*}{x^* y}\right) \geq 0. \quad (7)$$

Proof. To prove the global (uniformly) asymptotic stability of the unique endemic equilibrium point $\mathcal{E}_1 = (x^*, y^*, z^*, w^*)'$ of model (2) (resp. (3)) when $\mathcal{R}_0 > 1$, we define the following Lyapunov function ([13])

$$\begin{aligned} \mathcal{N} = & \frac{(\mu z^* + \mu w^*)}{\lambda^* x^*} \left(x - x^* - x^* \ln \frac{x}{x^*}\right) + \frac{(\mu z^* + \mu w^*)}{\lambda^* x^*} \left(y - y^* - y^* \ln \frac{y}{y^*}\right) \\ & + \left(z - z^* - z^* \ln \frac{z}{z^*}\right) + \left(w - w^* - w^* \ln \frac{w}{w^*}\right) \end{aligned} \quad (8)$$

The time derivative of \mathcal{N} is given by

$$\begin{aligned} \dot{\mathcal{N}} = & \frac{(\mu z^* + \mu w^*)}{\lambda^* x^*} \left(1 - \frac{x^*}{x}\right) x^* + \frac{(\mu z^* + \mu w^*)}{\lambda^* x^*} \left(1 - \frac{y^*}{y}\right) y^* + \left(1 - \frac{z^*}{z}\right) z^* + \left(1 - \frac{w^*}{w}\right) w^* \\ = & b_1 \left(1 - \frac{x^*}{x}\right) (p - \lambda x - \mu x) + b_2 \left(1 - \frac{y^*}{y}\right) (\lambda x - \theta y - \mu y) \\ & + \left(1 - \frac{z^*}{z}\right) (f\theta y - \mu z) + \left(1 - \frac{w^*}{w}\right) (\theta(1 - f)y - \mu w) \end{aligned} \quad (9)$$

At the equilibrium we have the following relations $p = \lambda^* x^* + \mu x^*$, $\lambda^* x^* = (\mu + \theta)y^*$, $f\theta y^* = \mu z^*$, and $\theta(1 - f)y^* = \mu w^*$. Using the these relations in Eq. (9), and after few algebraic computations, we obtain

$$\begin{aligned} \dot{\mathcal{N}} = & \frac{1}{\lambda^* x^*} (\mu z^* + \mu w^*) \mu x^* \left(2 - \frac{x}{x^*} - \frac{x^*}{x}\right) \\ & + \mu z^* \left[4 - \frac{x y^*}{x^* y} - \frac{z}{z^*} - \frac{z^* y}{z y^*} - \frac{x^*}{x}\right] + \mu w^* \left[4 - \frac{x y^*}{x^* y} - \frac{w}{w^*} - \frac{w^* y}{w y^*} - \frac{x^*}{x}\right] \\ & - \mu z^* \left(1 - \frac{\lambda}{\lambda^*}\right) \left(1 - \frac{x y^*}{x^* y}\right) - \mu w^* \left(1 - \frac{\lambda}{\lambda^*}\right) \left(1 - \frac{x y^*}{x^* y}\right). \end{aligned} \quad (10)$$

The first three terms of (10) are negative since the arithmetic mean exceeds the geometric mean. Together with condition (7), it follows that the remaining last two terms of (10) are also negative. Thus, $\dot{\mathcal{N}} < 0$. So $\lim_{t \rightarrow \infty} (x(t), y(t), z(t), w(t)) \rightarrow (x^*, y^*, z^*, w^*)$. By the LaSalle invariance principle [9], it follows that the unique endemic equilibrium point \mathcal{E}_1 of Leprosy model (2) is globally asymptotically stable (GAS) in Π whenever $\mathcal{R}_0 > 1$.

From [14], we note that the Lyapunov function \mathcal{N} for (2) is also a Lyapunov function for the Leprosy fractional model (3). Thus The unique endemic equilibrium of the fractional model (3) is globally asymptotically stable whenever $\mathcal{R}_0 > 1$. \square

3.4 Existence and uniqueness of solutions of the fractional model

The objective of this part is to show, by using the theory of the Banach fixed point, the existence and the uniqueness of the solutions of the fractional model (3) with the initial conditions. To this aim, we denote by \mathbf{Q} the Banach space for continuous functions with real values over an interval \mathbf{K} with the associated norms $\| (x, y, z, w) \| = \| x \| + \| y \| + \| z \| + \| w \|$, where $\| x \| = \sup \{ |x(t)| : t \in \mathbf{K} \}$, $\| y \| = \sup \{ |y(t)| : t \in \mathbf{K} \}$, $\| z \| = \sup \{ |z(t)| : t \in \mathbf{K} \}$, $\| w \| = \sup \{ |w(t)| : t \in \mathbf{K} \}$, and $\mathbf{Q} = \varepsilon(\mathbf{K}) \times \varepsilon(\mathbf{K}) \times \varepsilon(\mathbf{K}) \times \varepsilon(\mathbf{K})$, with $\varepsilon(\mathbf{K})$ the Banach space for continuous functions with real values on \mathbf{K} and the above associated norms. The fractional system (3) can be written in the form :

$$D_t^\eta x(t) = \mathbf{G}_1(t, x), D_t^\eta y(t) = \mathbf{G}_2(t, y), D_t^\eta z(t) = \mathbf{G}_3(t, z), D_t^\eta w(t) = \mathbf{G}_4(t, w), \quad (11)$$

with

$$\begin{aligned} \mathbf{G}_1(t, x) &= p - \frac{(\beta_m z + \beta_\rho w) x}{x + y + z + w} - \mu x, \quad \mathbf{G}_2(t, y) = \frac{(\beta_m z + \beta_\rho w) x}{x + y + z + w} - \theta y - \mu y, \\ \mathbf{G}_3(t, z) &= f\theta y - \mu z, \quad \mathbf{G}_4(t, w) = \theta(1 - f)y - \mu w. \end{aligned} \quad (12)$$

By applying the fractional integral operator in equation (11), we obtain:

$$\begin{aligned} x(t) - x(0) &= \frac{1}{\Gamma(\eta)} \int_0^t (t - \chi)^{\eta-1} \mathbf{G}_1(\chi, x) d\chi, \quad y(t) - y(0) = \frac{1}{\Gamma(\eta)} \int_0^t (t - \chi)^{\eta-1} \mathbf{G}_2(\chi, y) d\chi, \\ z(t) - z(0) &= \frac{1}{\Gamma(\eta)} \int_0^t (t - \chi)^{\eta-1} \mathbf{G}_3(\chi, z) d\chi, \quad w(t) - w(0) = \frac{1}{\Gamma(\eta)} \int_0^t (t - \chi)^{\eta-1} \mathbf{G}_4(\chi, w) d\chi \end{aligned} \quad (13)$$

The kernels \mathbf{G}_i , $i = 1, 2, 3, 4$, satisfy the Lipschitz and contraction conditions if the following inequality holds:

$$0 \leq V_i < 1, \quad i = 1, 2, 3, 4, \quad (14)$$

where

$$\begin{aligned} V_1 &= (\beta_m \kappa_3 + \beta_\rho \kappa_4 + \mu), \quad V_2 = \kappa_1 (\beta_m \kappa_3 + \beta_\rho \kappa_4 + \mu), \quad V_3 = V_4 = \mu, \\ \| x \| &= \kappa_1, \quad \| y \| = \kappa_2, \quad \| z \| = \kappa_3, \quad \| w \| = \kappa_4. \end{aligned} \quad (15)$$

We thus claim the following result

Theorem 5:

The kernel \mathbf{G}_1 satisfies the Lipschitz condition if the following inequality is respected

$$0 \leq V_1 = (\beta_m \kappa_3 + \beta_\rho \kappa_4 + \mu) \leq 1. \quad (16)$$

Proof. Consider the kernel \mathbf{G}_1 , and x, x_1 any two functions. We have

$$\begin{aligned} &\| \mathbf{G}_1(t, x) - \mathbf{G}_1(t, x_1) \| \\ &= \left\| -\frac{(\beta_m z + \beta_\rho w) x}{x + y + z + w} - \mu x + \frac{(\beta_m z + \beta_\rho w) x_1}{x_1 + y + z + w} + \mu x_1 \right\| \\ &\leq \| (\beta_m z + \beta_\rho w) \left(\frac{(y + z + w)}{(x + y + z + w)(x_1 + y + z + w)} \right) + \mu \| \| (x - x_1) \| \\ &\leq (\beta_m \| z \| + \beta_\rho \| w \| + \mu) \| (x - x_1) \| \\ &= \overbrace{(\beta_m \kappa_3 + \beta_\rho \kappa_4 + \mu)}^{V_1} \| (x - x_1) \| . \end{aligned} \quad (17)$$

Therefore, the Lipschitz condition is fulfilled for \mathbf{G}_1 , and if in addition, $0 \leq V_1 < 1$, then it is also a contraction. We get similar results for the other kernels:

$$\begin{aligned} \|\mathbf{G}_2(t, y) - \mathbf{G}_2(t, y_1)\| &\leq V_2 \|y(t) - y_1(t)\|, \|\mathbf{G}_3(t, z) - \mathbf{G}_3(t, z_1)\| \leq V_3 \|z(t) - z_1(t)\|, \\ \|\mathbf{G}_4(t, w) - \mathbf{G}_4(t, w_1)\| &\leq V_4 \|w(t) - w_1(t)\|. \end{aligned} \quad (18)$$

□

The expressions of the equation (13) can be written recursively as:

$$\begin{aligned} x_n(t) - x(0) &= \frac{1}{\Gamma(\eta)} \int_0^t (t - \chi)^{\eta-1} \mathbf{G}_1(\chi, x_{n-1}) d\chi, \\ y_n(t) - y(0) &= \frac{1}{\Gamma(\eta)} \int_0^t (t - \chi)^{\eta-1} \mathbf{G}_2(\chi, y_{n-1}) d\chi, \\ z_n(t) - z(0) &= \frac{1}{\Gamma(\eta)} \int_0^t (t - \chi)^{\eta-1} \mathbf{G}_3(\chi, z_{n-1}) d\chi, \\ w_n(t) - w(0) &= \frac{1}{\Gamma(\eta)} \int_0^t (t - \chi)^{\eta-1} \mathbf{G}_4(\chi, w_{n-1}) d\chi \end{aligned} \quad (19)$$

The difference between successive terms for recursive formulas is written:

$$\begin{aligned} \phi_{1n}(t) = x_n(t) - x_{n-1}(t) &= \frac{1}{\Gamma(\eta)} \int_0^t (t - \chi)^{\eta-1} (\mathbf{G}_1(\chi, x_{n-1}) - \mathbf{G}_1(\chi, x_{n-2})) d\chi, \\ \phi_{2n}(t) = y_n(t) - y_{n-1}(t) &= \frac{1}{\Gamma(\eta)} \int_0^t (t - \chi)^{\eta-1} (\mathbf{G}_2(\chi, y_{n-1}) - \mathbf{G}_2(\chi, y_{n-2})) d\chi, \\ \phi_{3n}(t) = z_n(t) - z_{n-1}(t) &= \frac{1}{\Gamma(\eta)} \int_0^t (t - \chi)^{\eta-1} (\mathbf{G}_3(\chi, y_{n-1}) - \mathbf{G}_3(\chi, y_{n-2})) d\chi, \\ \phi_{4n}(t) = w_n(t) - w_{n-1}(t) &= \frac{1}{\Gamma(\eta)} \int_0^t (t - \chi)^{\eta-1} (\mathbf{G}_4(\chi, w_{n-1}) - \mathbf{G}_4(\chi, w_{n-2})) d\chi, \end{aligned} \quad (20)$$

with initial conditions: $x_0(t) = x(0)$, $y_0(t) = y(0)$, $z_0(t) = z(0)$, $w_0(t) = w(0)$. Let us take the norm of the first expression of the equation (20)

$$\begin{aligned} \|\phi_{1n}(t)\| &= \|x_n(t) - x_{n-1}(t)\| \\ &= \left\| \frac{1}{\Gamma(\eta)} \int_0^t (t - \chi)^{\eta-1} (\mathbf{G}_1(\chi, x_{n-1}) - \mathbf{G}_1(\chi, x_{n-2})) d\chi \right\|, \\ &\leq \frac{1}{\Gamma(\eta)} \left\| \int_0^t (t - \chi)^{\eta-1} (G_1(\chi, x_{n-1}) - G_1(\chi, x_{n-2})) d\chi \right\|, \end{aligned} \quad (21)$$

Using the Lipschitz condition we have:

$$\|x_n(t) - x_{n-1}(t)\| \leq \frac{1}{\Gamma(\eta)} V_1 \int_0^t (t - \chi)^{\eta-1} \|x_{n-1} - x_{n-2}\| d\chi. \quad (22)$$

Thus, we obtain

$$\|\phi_{1n}(t)\| \leq \frac{1}{\Gamma(\eta)} V_1 \int_0^t (t - \chi)^{\eta-1} \|\phi_{1(n-1)}(\chi)\| d\chi. \quad (23)$$

Similar reasoning makes it possible to have:

$$\begin{aligned}
\| \phi_{2n}(t) \| &\leq \frac{1}{\Gamma(\eta)} V_2 \int_0^t (t - \chi)^{\eta-1} \| \phi_{2(n-1)}(\chi) \| d\chi, \\
\| \phi_{3n}(t) \| &\leq \frac{1}{\Gamma(\eta)} V_3 \int_0^t (t - \chi)^{\eta-1} \| \phi_{3(n-1)}(\chi) \| d\chi, \\
\| \phi_{4n}(t) \| &\leq \frac{1}{\Gamma(\eta)} V_4 \int_0^t (t - \chi)^{\eta-1} \| \phi_{4(n-1)}(\chi) \| d\chi.
\end{aligned} \tag{24}$$

Note that

$$x_n(t) = \sum_{i=1}^n \phi_{1i}(t), \quad y_n(t) = \sum_{i=1}^n \phi_{2i}(t), \quad z_n(t) = \sum_{i=1}^n \phi_{3i}(t), \quad w_n(t) = \sum_{i=1}^n \phi_{4i}(t). \tag{25}$$

Now let's show the uniqueness of the model solution. We claim the following

Theorem 6:

The fractional model (3) admits a unique solution on $t \in [0, T]$ if the following inequality is satisfied

$$\frac{1}{\Gamma(\eta)} V_i b^\eta \leq 1, \quad i = 1, 2, \dots, 4. \tag{26}$$

Proof. The proof follows the proof of Theorem 4 in [15]. □

IV NUMERICAL SCHEME AND RESULTS

4.1 Numerical scheme

In what follows, we will use the Adams-Bashforth-Moulton method (see [15, 16]) to construct a numerical scheme for the fractional model (3). To this aim, let us consider the uniform discretization of $[0, a]$ given by $t_m = m\aleph$, $m \in [0; N] \cap \mathbb{N}$ where $\aleph = a/m$ denotes the step size. For a given approximation $\mathcal{Z}_\aleph(t_i) \approx \mathcal{Z}(t_i)$, the next approximation $\mathcal{Z}_\aleph(t_{i+1})$ is obtain (using the predictor-corrector method) as follows:

$$\text{Predictor: } \mathcal{Z}_\aleph^p(t_{m+1}) = \sum_{l=0}^{[\varphi]-1} \frac{t_{n+1}^l}{l!} \mathcal{Z}_0^l + \frac{1}{\Gamma(\eta)} \sum_{l=0}^m b_{l,m+1} \mathbf{G}(t_l, \mathcal{Z}_\aleph(t_l));$$

$$\text{Corrector: } \mathcal{Z}_\aleph(t_{m+1}) = \sum_{l=0}^{[\varphi]-1} \frac{t_{n+1}^l}{l!} \mathcal{Z}_0^l + \frac{h^\eta}{\Gamma(2+\eta)} \mathbf{G}(t_{l+1}, \mathcal{Z}_\aleph(t_{l+1})) + \frac{\aleph^\eta}{\Gamma(2+\eta)} \sum_{l=0}^m c_{l,m+1} \mathbf{G}(t_l, \mathcal{Z}_\aleph(t_l));$$

with

$$c_{l,m+1} = \begin{cases} m^{1+\varphi} - (m-\varphi)(m+1)^\varphi, & \text{if } l = 0, \\ (m-l+2)^{1+\varphi} + (m-l)^{1+\varphi} - 2(m-l+1)^{1+\varphi}, & \text{if } 1 \leq l \leq m, \\ 1 & \text{if } l = m+1, \end{cases}$$

$$\text{and } b_{l,m+1} = \frac{\aleph^\varphi}{\varphi} [(m-l+1)^\varphi - (m-l)^\varphi].$$

4.2 Numerical results

For numerical simulations, we consider the following set of parameter values:

1. $p = 100$; $m = 1/55.5$; $\beta_m = \beta_p = 0.019$; $\theta = 0.19$ and $f = 0.7$ which give $\mathcal{R}_0 = 0.9632 < 1$;
2. $p = 100$; $m = 1/55.5$; $\beta_m = \beta_p = 0.03$; $\theta = 0.19$ and $f = 0.7$ which give $\mathcal{R}_0 = 1.5208 > 1$;

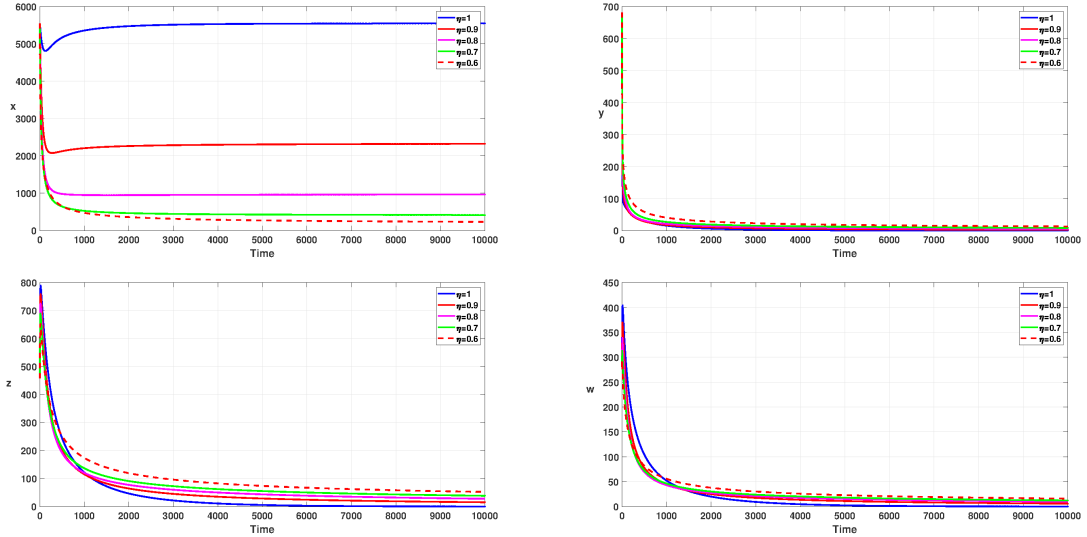


Figure 1: Time-series of state variables for $\mathcal{R}_0 = 0.9632 < 1$ and different values of the fractional-order $\eta \in \{1; 0.9; 0.7; 0.6\}$.

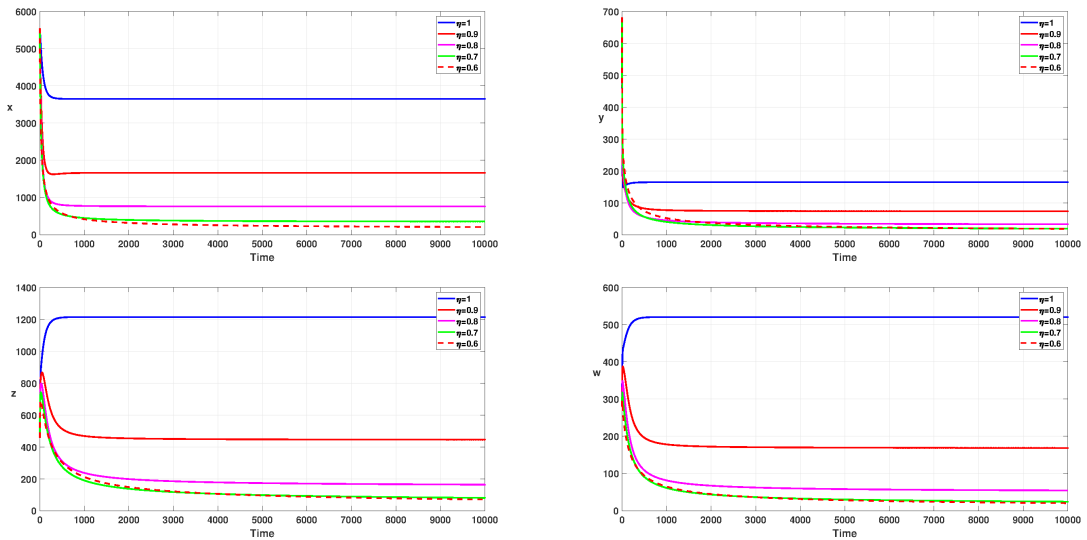


Figure 2: Time-series of state variables for $\mathcal{R}_0 = 1.5208 > 1$ and different values of the fractional-order $\eta \in \{1; 0.9; 0.7; 0.6\}$.

The theoretical results of the fractional-order model are verified by varying the fractional-order parameter η . Indeed, we demonstrate our obtained analytical results numerically in Section III. In Figure 1, we see that when $\mathcal{R}_0 < 1$, the solutions converge to the Leprosy-free equilibrium, while in Figure 2, the solutions converge to the endemic equilibrium point whenever $\mathcal{R}_0 > 1$.

In Figure 2, we also see that the population of each class or compartment decreases when the fractional-order decreases.

V CONCLUSION

In this work, we extended the works of Raza & Rafiq [8], which consists of a system of Ordinary Differential Equations, by replacing integer derivatives with fractional derivative in the Caputo sense, and mass action incidence by standard incidence. We computed the new basic reproduction number \mathcal{R}_0 and proved the existence of two steady states named the Leprosy-free equilibrium which always exists, and the unique endemic equilibrium point which exists if and only if $\mathcal{R}_0 > 1$. We also proved the global stability of both equilibrium points for both models. Indeed, by constructing two Lyapunov functions, we proved that the leprosy disease-free equilibrium (resp. the unique endemic equilibrium point) is globally asymptotically stable whenever the basic reproduction number is less than unity ($\mathcal{R}_0 < 1$) (resp. greater than unity ($\mathcal{R}_0 > 1$)). Then, we studied the existence and the uniqueness of the solutions of the fractional model using fixed point theory. Using the Adams-Bashforth-Moulton method to construct the numerical scheme, we performed numerical simulations to validate our analytical results, as well as to see the impact of varying the fractional parameter on the disease spread. Indeed, when $\mathcal{R}_0 < 1$, the solutions converge to the Leprosy-free equilibrium while the solutions converge to the endemic equilibrium point whenever $\mathcal{R}_0 > 1$. Also, the population of each class or compartment decreases with the decrease of the fractional-order.

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