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# Beating the Sum-Rate Capacity of the Binary Adder Channel with Non-Signaling Correlations

Omar Fawzi\*    Paul Fermé†

## Abstract

We address the problem of coding for multiple-access channels (MACs) with the assistance of non-signaling correlations between parties. It is well-known that non-signaling assistance does not change the capacity of point-to-point channels. However, it was recently observed that one can construct MACs from two-player non-local games while relating the winning probability of the game to the capacity of the MAC. By considering games for which entanglement (a special kind of non-signaling correlation) increases the winning probability (e.g., the Magic Square game), this shows that for some specific kinds of channels, entanglement between the senders can increase the capacity.

Here, we show that the increase in capacity from non-signaling assistance goes beyond such special channels and applies even to a simple deterministic MAC: the binary adder channel. In particular, we show that, with non-signaling assistance, a sum-rate of  $\frac{\log_2(72)}{4} \simeq 1.5425$  can be reached with zero error, beating the maximum classical sum-rate capacity of  $\frac{3}{2}$ . Furthermore, we show that this capacity increase persists if a small amount of noise is added to the channel.

In order to achieve this, we show that efficient linear programs can be formulated to compute the success probability of the best non-signaling assisted code for a finite number of copies of a multiple-access channel. In particular, this can be used to give lower bounds on the zero-error non-signaling assisted capacity of multiple-access channels.

## 1 Introduction

Multiple-access channels (MACs for short) are one of the simplest model of network communication settings, where two senders aim to transmit individual messages to one receiver. The capacity of such channels has been entirely characterized by the seminal works by Liao [15] and Ahlswede [1] in terms of the following single-letter formula:

**Theorem 1.1.** The capacity region  $\mathcal{C}(W)$  of the MAC  $W$  is the closure of all rate pairs  $(R_1, R_2)$  such that:

$$R_1 \leq I(X_1 : Y|X_2) , \quad R_2 \leq I(X_2 : Y|X_1) , \quad R_1 + R_2 \leq I((X_1, X_2) : Y) ,$$

for  $(X_1, X_2) \in \mathcal{X}_1 \times \mathcal{X}_2$  following a product law  $P_{X_1} \times P_{X_2}$ , and  $Y \in \mathcal{Y}$  the outcome of  $W$  on inputs  $X_1, X_2$ .

From the point of view of quantum information, it is natural to ask whether additional resources, such as quantum entanglement between the parties, changes the capacity region. A well-known result [6] states that for classical point-to-point channels, entanglement and even more generally non-signaling correlations do not help; see also [18, 5]. However, in [14], it is shown that quantum entanglement shared between the two senders of a multiple access channel can strictly extend the capacity region. It occurs for some

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channels constructed from two-player non-local games, such as the Magic Square game [19, 21, 4, 7], translating known gaps between classical and quantum values of games into MAC capacity gaps. The MACs constructed this way have very special form though and one may wonder whether non-signaling correlations can help for more natural MACs. Here we focus on the simplest textbook MAC: the binary adder channel which maps  $(x_1, x_2) \in \{0, 1\}^2$  to  $x_1 + x_2 \in \{0, 1, 2\}$ . It is worth noting that, unlike the channels of [14], the BAC is deterministic. The capacity region of this channel is known to be given by  $\mathcal{C}(W_{\text{BAC}}) = \{(R_1, R_2) : R_1 \leq 1, R_2 \leq 1, R_1 + R_2 \leq \frac{3}{2}\}$ .

In this work, we focus on the quantity  $S^{\text{NS}}(W, k_1, k_2)$ , which denotes the success probability of the best non-signaling assisted  $(k_1, k_2)$ -code for the MAC  $W$ . Contrary to the unassisted value  $S(W, k_1, k_2)$ ,  $S^{\text{NS}}(W, k_1, k_2)$  can be formulated as a linear program; see Proposition 3.1. Furthermore, using symmetries, we have developed a linear program computing  $S^{\text{NS}}$  for a finite number of copies of a MAC  $W$  with a size growing polynomially in the number of copies; see Theorem 3.9 and Corollary 3.10. Applied to the binary adder channel, we have shown that the sum-rate  $\frac{\log_2(72)}{4} \simeq 1.5425$  can be reached with zero error, which beats the maximum classical sum-rate capacity of  $\frac{3}{2}$ ; see Theorem 4.1. Finally, for noisy channels, where the zero-error non-signaling assisted capacity region is trivial, we can use concatenated codes to obtain achievable points in the capacity region. Applied to a noisy version of the binary adder channel, we have shown that non-signaling assistance still improves the sum-rate capacity.

In Section 2, we define precisely the different notions of MAC capacities: the classical capacity (i.e. without any assistance) as well as the non-signaling assisted capacity. In Section 3, we address computational complexity questions concerning the probability of success of the best classical coding strategy and the best non-signaling strategy for a MAC. In Section 4, we apply these results to the binary adder channel. Finally, in Section 5, we handle the case of noisy channels through concatenated codes, and apply it to a noisy version of the binary adder channel.

## 2 Multiple Access Channels Capacities

### 2.1 Classical Capacities

Formally, a MAC  $W$  is a conditional probability distribution depending on two inputs in  $\mathcal{X}_1$  and  $\mathcal{X}_2$ , and an output in  $\mathcal{Y}$ , so  $W := (W(y|x_1x_2))_{x_1 \in \mathcal{X}_1, x_2 \in \mathcal{X}_2, y \in \mathcal{Y}}$  such that  $W(y|x_1x_2) \geq 0$  and  $\sum_{y \in \mathcal{Y}} W(y|x_1x_2) = 1$ . We will denote such a MAC by  $W : \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathcal{Y}$ . The tensor product of two MACs  $W : \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathcal{Y}$  and  $W' : \mathcal{X}'_1 \times \mathcal{X}'_2 \rightarrow \mathcal{Y}'$  is denoted by  $W \otimes W' : (\mathcal{X}_1 \times \mathcal{X}'_1) \times (\mathcal{X}_2 \times \mathcal{X}'_2) \rightarrow \mathcal{Y} \times \mathcal{Y}'$  and defined by  $(W \otimes W')(yy'|x_1x'_1x_2x'_2) := W(y|x_1x_2) \cdot W'(y'|x'_1x'_2)$ . We denote by  $W^{\otimes n}(y^n|x_1^n x_2^n) := \prod_{i=1}^n W(y_i|x_{1,i}x_{2,i})$ , for  $y^n := y_1 \dots y_n \in \mathcal{Y}^n$ ,  $x_1^n := x_{1,1} \dots x_{1,n} \in \mathcal{X}_1^n$  and  $x_2^n := x_{2,1} \dots x_{2,n} \in \mathcal{X}_2^n$ . We will use the notation  $[k] := \{1, \dots, k\}$ .

For a MAC  $W : \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathcal{Y}$ , we write  $S(W, k_1, k_2)$  for the maximal probability of successfully sending  $k_1$  messages for sender 1 and  $k_2$  messages for sender 2. This means that one can encode  $k_1$  messages in  $\mathcal{X}_1$  through  $e^1 : [k_1] \rightarrow \mathcal{X}_1$ ,  $k_2$  messages in  $\mathcal{X}_2$  through  $e^2 : [k_2] \rightarrow \mathcal{X}_2$ , and then decode these messages from the output in  $\mathcal{Y}$  with  $d : \mathcal{Y} \rightarrow [k_1] \times [k_2]$ , as depicted in Figure 2.1. This leads to the following optimization program for  $S(W, k_1, k_2)$ :

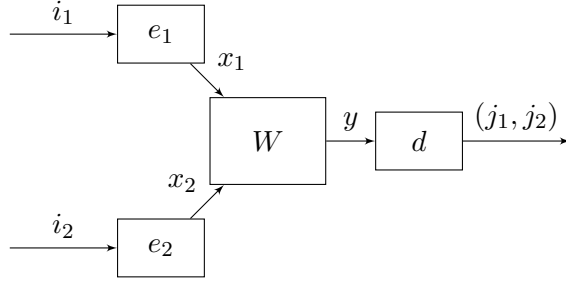


Figure 1: Coding for a MAC  $W$ .

$$\begin{aligned}
S(W, k_1, k_2) &:= \underset{e_1, e_2, d}{\text{maximize}} && \frac{1}{k_1 k_2} \sum_{x_1, x_2, y, i_1, i_2} W(y|x_1 x_2) e_1(x_1|i_1) e_2(x_2|i_2) d(i_1 i_2|y) \\
&\text{subject to} && \sum_{x_1 \in \mathcal{X}_1} e_1(x_1|i_1) = 1, \forall i_1 \in [k_1] \\
&&& \sum_{x_2 \in \mathcal{X}_2} e_2(x_2|i_2) = 1, \forall i_2 \in [k_2] \\
&&& \sum_{j_1 \in [k_1], j_2 \in [k_2]} d(j_1 j_2|y) = 1, \forall y \in \mathcal{Y} \\
&&& e_1(x_1|i_1), e_2(x_2|i_2), d(j_1 j_2|y) \geq 0
\end{aligned} \tag{1}$$

Since MACs are more general than point-to-point channels (by defining  $W(y|x_1 x_2) := \hat{W}(y|x_1)$  for  $\hat{W}$  a point-to-point channel and looking only at its first input), computing a single value  $S(W, k_1, k_2)$  is NP-hard, and it is even NP-hard to approximate  $S(W, k_1, k_2)$  within a better ratio than  $(1 - e^{-1})$ , as a consequence of the hardness result on  $S(W, k)$  shown in [5].

The (classical) capacity of a MAC, as defined for example in [9], can be reformulated in the following way:

**Definition 2.1** (Capacity Region  $\mathcal{C}(W)$  of a MAC  $W$ ). A rate pair  $(R_1, R_2)$  is achievable if:

$$\lim_{n \rightarrow +\infty} S(W^{\otimes n}, \lceil 2^{R_1 n} \rceil, \lceil 2^{R_2 n} \rceil) = 1 .$$

We define the (classical) capacity region  $\mathcal{C}(W)$  as the closure of the set of all achievable rate pairs.

For the zero-error (classical) capacity, this leads to the following definition:

**Definition 2.2** (Zero-Error Capacity Region  $\mathcal{C}_0(W)$  of a MAC  $W$ ). A rate pair  $(R_1, R_2)$  is achievable with zero-error if:

$$\exists n_0 \in \mathbb{N}^*, \forall n \geq n_0, S(W^{\otimes n}, \lceil 2^{R_1 n} \rceil, \lceil 2^{R_2 n} \rceil) = 1 .$$

We define the zero-error (classical) capacity region  $\mathcal{C}_0(W)$  as the closure of the set of all achievable rate pairs with zero-error.

## 2.2 Non-Signaling Assisted Capacities

When non-signaling assistance is given to a MAC, both encoders  $e_1, e_2$  and the decoder  $d$  are replaced by a non-signaling box  $P(x_1 x_2(j_1 j_2)|i_1 i_2 y)$ , i.e., each party's output is conditionally independent from

the inputs of the others, given this party's input. The way a non-signaling box can be used for coding is depicted in Figure 2. The maximal probability of successfully sending  $k_1$  messages for sender 1 and  $k_2$  messages for sender 2 with non-signaling assistance, which we call  $S^{\text{NS}}(W, k_1, k_2)$ , is given by the following linear program, where the constraints translate the fact that  $P$  is a non-signaling box:

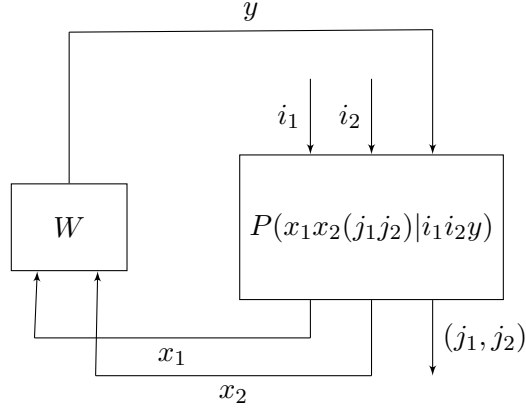


Figure 2: The use of a non-signaling box  $P$  as a coding strategy for the MAC  $W$ .

$$\begin{aligned}
S^{\text{NS}}(W, k_1, k_2) := & \underset{P}{\text{maximize}} \quad \frac{1}{k_1 k_2} \sum_{x_1, x_2, y, i_1, i_2} W(y|x_1 x_2) P(x_1 x_2(i_1 i_2)|i_1 i_2 y) \\
\text{subject to} & \sum_{x_1} P(x_1 x_2(j_1 j_2)|i_1 i_2 y) = \sum_{x_1} P(x_1 x_2(j_1 j_2)|i_1' i_2 y) \\
& \sum_{x_2} P(x_1 x_2(j_1 j_2)|i_1 i_2 y) = \sum_{x_2} P(x_1 x_2(j_1 j_2)|i_1 i_2 y') \\
& \sum_{j_1 j_2} P(x_1 x_2(j_1 j_2)|i_1 i_2 y) = \sum_{j_1 j_2} P(x_1 x_2(j_1 j_2)|i_1 i_2 y') \\
& \sum_{x_1, x_2, j_1, j_2} P(x_1 x_2(j_1 j_2)|i_1 i_2 y) = 1 \\
& P(x_1 x_2(j_1 j_2)|i_1 i_2 y) \geq 0
\end{aligned} \tag{2}$$

Since it is given as a linear program, the complexity of computing  $S^{\text{NS}}(W, k_1, k_2)$  is polynomial in the number of variables and constraints (see for instance Section 7.1 of [10]), which is a polynomial in  $|\mathcal{X}_1|, |\mathcal{X}_2|, |\mathcal{Y}|, k_1$  and  $k_2$ . Also, as it is easy to check that a classical strategy is a particular case of a non-signaling assisted strategy, we have that  $S^{\text{NS}}(W, k_1, k_2) \geq S(W, k_1, k_2)$ .

We have then the same definitions of capacity and zero-error capacity:

**Definition 2.3** (Non-Signaling Assisted Capacity Region  $\mathcal{C}^{\text{NS}}(W)$  of a MAC  $W$ ). A rate pair  $(R_1, R_2)$  is achievable with non-signaling assistance if:

$$\lim_{n \rightarrow +\infty} S^{\text{NS}}(W^{\otimes n}, \lceil 2^{R_1 n} \rceil, \lceil 2^{R_2 n} \rceil) = 1 .$$

We define the non-signaling assisted capacity region  $\mathcal{C}^{\text{NS}}(W)$  as the closure of the set of all achievable rate pairs with non-signaling assistance.

**Definition 2.4** (Zero-Error Non-Signaling Assisted Capacity Region  $\mathcal{C}_0^{\text{NS}}(W)$  of a MAC  $W$ ). A rate pair  $(R_1, R_2)$  is achievable with zero-error and non-signaling assistance if:

$$\exists n_0 \in \mathbb{N}^*, \forall n \geq n_0, S^{\text{NS}}(W^{\otimes n}, \lceil 2^{R_1 n} \rceil, \lceil 2^{R_2 n} \rceil) = 1 .$$

We define the zero-error non-signaling assisted capacity region  $\mathcal{C}_0^{\text{NS}}(W)$  as the closure of the set of all achievable rate pairs with zero-error and non-signaling assistance.

### 3 Basic Properties and Computational Aspects of Non-Signaling Assisted Codes for MACs

#### 3.1 A Smaller Linear Program Computing $S^{\text{NS}}(W, k_1, k_2)$

One can prove an equivalent formulation of the linear program computing  $S^{\text{NS}}(W, k_1, k_2)$  with a number of variables and constraints polynomial in only  $|\mathcal{X}_1|, |\mathcal{X}_2|$  and  $|\mathcal{Y}|$  and independent of  $k_1$  and  $k_2$ :

**Proposition 3.1.** For a MAC  $W : \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathcal{Y}$  and  $k_1, k_2 \in \mathbb{N}^*$ , we have:

$$\begin{aligned} S^{\text{NS}}(W, k_1, k_2) = & \underset{r, r^1, r^2, p}{\text{maximize}} && \frac{1}{k_1 k_2} \sum_{x_1, x_2, y} W(y|x_1 x_2) r_{x_1, x_2, y} \\ \text{subject to} & && \sum_{x_1, x_2} r_{x_1, x_2, y} = 1 \\ & && \sum_{x_1} r_{x_1, x_2, y}^1 = k_1 \sum_{x_1} r_{x_1, x_2, y} \\ & && \sum_{x_2} r_{x_1, x_2, y}^2 = k_2 \sum_{x_2} r_{x_1, x_2, y} \\ & && \sum_{x_1} p_{x_1, x_2} = k_1 \sum_{x_1} r_{x_1, x_2, y}^2 \\ & && \sum_{x_2} p_{x_1, x_2} = k_2 \sum_{x_2} r_{x_1, x_2, y}^1 \\ & && 0 \leq r_{x_1, x_2, y} \leq r_{x_1, x_2, y}^1, r_{x_1, x_2, y}^2 \leq p_{x_1, x_2} \\ & && p_{x_1, x_2} - r_{x_1, x_2, y}^1 - r_{x_1, x_2, y}^2 + r_{x_1, x_2, y} \geq 0 \end{aligned} \tag{3}$$

*Proof.* One can check that given a solution of the original program, the following choice of variables is a valid solution of the second program achieving the same objective value:

$$\begin{aligned} r_{x_1, x_2, y} &:= \sum_{i_1, i_2} P(x_1 x_2(i_1 i_2) | i_1 i_2 y) , & r_{x_1, x_2, y}^1 &:= \sum_{j_1, i_1, i_2} P(x_1 x_2(j_1 i_2) | i_1 i_2 y) , \\ r_{x_1, x_2, y}^2 &:= \sum_{j_2, i_1, i_2} P(x_1 x_2(i_1 j_2) | i_1 i_2 y) , & p_{x_1, x_2} &:= \sum_{j_1, j_2, i_1, i_2} P(x_1 x_2(j_1 j_2) | i_1 i_2 y) . \end{aligned} \tag{4}$$

Note that  $p_{x_1, x_2}$  is well-defined since  $\sum_{j_1, j_2, i_1, i_2} P(x_1 x_2(j_1 j_2) | i_1 i_2 y)$  is independent from  $y$  by NS conditions.

For the other direction, given those variables, a non-signaling probability distribution  $P(x_1 x_2(j_1 j_2) | i_1 i_2 y)$

achieving the same objective value is given by, for  $j_1 \neq i_1$  and  $j_2 \neq i_2$ :

$$\begin{aligned}
P(x_1 x_2(i_1 i_2)|i_1 i_2 y) &:= \frac{r_{x_1, x_2, y}}{k_1 k_2}, \\
P(x_1 x_2(j_1 i_2)|i_1 i_2 y) &:= \frac{r_{x_1, x_2, y}^1 - r_{x_1, x_2, y}}{k_1 k_2 (k_1 - 1)}, \\
P(x_1 x_2(i_1 j_2)|i_1 i_2 y) &:= \frac{r_{x_1, x_2, y}^2 - r_{x_1, x_2, y}}{k_1 k_2 (k_2 - 1)}, \\
P(x_1 x_2(j_1 j_2)|i_1 i_2 y) &:= \frac{p_{x_1, x_2} - r_{x_1, x_2, y}^1 - r_{x_1, x_2, y}^2 + r_{x_1, x_2, y}}{k_1 k_2 (k_1 - 1)(k_2 - 1)}.
\end{aligned} \tag{5}$$

□

### 3.2 Properties of $S^{\text{NS}}(W, k_1, k_2)$ , $\mathcal{C}^{\text{NS}}(W)$ and $\mathcal{C}_0^{\text{NS}}(W)$

**Definition 3.2.** We say that a conditional probability distribution  $Q(a^n|x^n)$  defined on  $(\mathcal{A}_1 \times \dots \times \mathcal{A}_n) \times (\mathcal{X}_1 \times \dots \times \mathcal{X}_n)$  and  $Q'(a^m|x^m)$  is *non-signaling* if for all  $a^n, x^n, \hat{x}^n$ , we have

$$\forall i \in [n], \sum_{a_i} Q(a_1 \dots a_i \dots a_n | x_1 \dots x_i \dots x_n) = \sum_{a_i} Q(a_1 \dots a_i \dots a_n | x_1 \dots \hat{x}_i \dots x_n).$$

**Definition 3.3.** Let  $Q(a^n|x^n)$  be a conditional probability distribution defined on  $(\mathcal{A}_1 \times \dots \times \mathcal{A}_n) \times (\mathcal{X}_1 \times \dots \times \mathcal{X}_n)$  and  $Q'(a^m|x^m)$  a conditional probability distribution defined on  $(\mathcal{A}'_1 \times \dots \times \mathcal{A}'_n) \times (\mathcal{X}'_1 \times \dots \times \mathcal{X}'_n)$ . We define  $P := Q \otimes Q'$  the tensor product conditional probability distribution defined on  $((\mathcal{A}_1 \times \mathcal{A}'_1) \times \dots \times (\mathcal{A}_n \times \mathcal{A}'_n)) \times ((\mathcal{X}_1 \times \mathcal{X}'_1) \times \dots \times (\mathcal{X}_n \times \mathcal{X}'_n))$  by  $P(a_1 a'_1 \dots a_n a'_n | x_1 x'_1 \dots x_n x'_n) := Q(a^n|x^n) \cdot Q'(a^m|x^m)$ .

**Lemma 3.4.** If both  $Q$  and  $Q'$  are non-signaling, then  $P = Q \otimes Q'$  is non-signaling.

*Proof.* Let  $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n \in \mathcal{A}_1 \times \dots \times \mathcal{A}_{i-1} \times \mathcal{A}_{i+1} \times \dots \times \mathcal{A}_n$ ,  $a'_1, \dots, a'_{i-1}, a'_{i+1}, \dots, a'_n \in \mathcal{A}'_1 \times \dots \times \mathcal{A}'_{i-1} \times \mathcal{A}'_{i+1} \times \dots \times \mathcal{A}'_n$ ,  $x_1, \dots, x_n \in \mathcal{X}_1 \times \dots \times \mathcal{X}_n$  and  $x'_1, \dots, x'_n \in \mathcal{X}'_1 \times \dots \times \mathcal{X}'_n$ . We have:

$$\begin{aligned}
&\sum_{a_i a'_i} P(a_1 a'_1 \dots a_i a'_i \dots a_n a'_n | x_1 x'_1 \dots x_i x'_i \dots x_n x'_n) \\
&= \sum_{a_i a'_i} Q(a_1 \dots a_i \dots a_n | x_1 \dots x_i \dots x_n) \cdot Q'(a'_1 \dots a'_i \dots a'_n | x'_1 \dots x'_i \dots x'_n) \\
&= \left( \sum_{a_i} Q(a_1 \dots a_i \dots a_n | x_1 \dots x_i \dots x_n) \right) \cdot \left( \sum_{a'_i} Q'(a'_1 \dots a'_i \dots a'_n | x'_1 \dots x'_i \dots x'_n) \right) \\
&= \left( \sum_{a_i} Q(a_1 \dots a_i \dots a_n | x_1 \dots \hat{x}_i \dots x_n) \right) \cdot \left( \sum_{a'_i} Q'(a'_1 \dots a'_i \dots a'_n | x'_1 \dots \hat{x}'_i \dots x'_n) \right) \\
&\quad \text{since } Q \text{ and } Q' \text{ are non-signaling} \\
&= \sum_{a_i a'_i} P(a_1 a'_1 \dots a_i a'_i \dots a_n a'_n | x_1 x'_1 \dots \hat{x}_i \hat{x}'_i \dots x_n x'_n),
\end{aligned} \tag{6}$$

so  $P$  is non-signaling. □

**Proposition 3.5.** For a MAC  $W : \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathcal{Y}$  and  $k_1, k_2 \in \mathbb{N}^*$ , we have:

$$1. \frac{1}{k_1 k_2} \leq S^{\text{NS}}(W, k_1, k_2) \leq 1.$$

$$2. \text{S}^{\text{NS}}(W, k_1, k_2) \leq \min \left( \frac{|\mathcal{X}_1|}{k_1}, \frac{|\mathcal{X}_2|}{k_2}, \frac{|\mathcal{Y}|}{k_1 k_2} \right).$$

$$3. \text{If } k'_1 \leq k_1 \text{ and } k'_2 \leq k_2, \text{ then } \text{S}^{\text{NS}}(W, k'_1, k'_2) \geq \text{S}^{\text{NS}}(W, k_1, k_2).$$

$$4. \text{For any MAC } W' : \mathcal{X}'_1 \times \mathcal{X}'_2 \rightarrow \mathcal{Y} \text{ and } k_1, k_2 \in \mathbb{N}^*, \text{ we have } \text{S}^{\text{NS}}(W \otimes W', k_1 k'_1, k_2 k'_2) \geq \text{S}^{\text{NS}}(W, k_1, k_2) \cdot \text{S}^{\text{NS}}(W', k'_1, k'_2). \text{ In particular, for any positive integer } n, \text{S}^{\text{NS}}(W^{\otimes n}, k_1^n, k_2^n) \geq [\text{S}^{\text{NS}}(W, k_1, k_2)]^n \text{ and } \text{S}^{\text{NS}}(W \otimes W', k_1, k_2) \geq \text{S}^{\text{NS}}(W, k_1, k_2).$$

*Proof.* 1. Let us first show that  $\text{S}^{\text{NS}}(W, k_1, k_2) \geq \frac{1}{k_1 k_2}$ . Take  $p_{x_1, x_2} := \frac{k_1 k_2}{|\mathcal{X}_1| |\mathcal{X}_2|}$ ,  $r_{x_1, x_2, y}^1 := \frac{p_{x_1, x_2}}{k_2}$ ,  $r_{x_1, x_2, y}^2 := \frac{p_{x_1, x_2}}{k_1}$  and  $r_{x_1, x_2, y} := \frac{p_{x_1, x_2}}{k_1 k_2} = \frac{1}{|\mathcal{X}_1| |\mathcal{X}_2|}$ . One can easily check that it is indeed a valid solution of the linear program computing  $\text{S}^{\text{NS}}(W, k_1, k_2)$ . Thus we have:

$$\begin{aligned} \text{S}^{\text{NS}}(W, k_1, k_2) &\geq \frac{1}{k_1 k_2} \sum_{x_1, x_2, y} W(y|x_1 x_2) r_{x_1, x_2, y} = \frac{1}{k_1 k_2} \sum_{x_1, x_2} \frac{1}{|\mathcal{X}_1| |\mathcal{X}_2|} \sum_y W(y|x_1 x_2) \\ &= \frac{1}{k_1 k_2} \sum_{x_1, x_2} \frac{1}{|\mathcal{X}_1| |\mathcal{X}_2|} = \frac{1}{k_1 k_2}. \end{aligned} \quad (7)$$

Furthermore, in order to show that it is at most 1, let us consider an optimal solution of  $\text{S}^{\text{NS}}(W, k_1, k_2)$ . We have:

$$\begin{aligned} \text{S}^{\text{NS}}(W, k_1, k_2) &= \frac{1}{k_1 k_2} \sum_{x_1, x_2, y} W(y|x_1 x_2) r_{x_1, x_2, y} \leq \frac{1}{k_1 k_2} \sum_{x_1, x_2, y} W(y|x_1 x_2) p_{x_1, x_2} \\ &= \frac{1}{k_1 k_2} \sum_{x_1, x_2} p_{x_1, x_2} \sum_y W(y|x_1 x_2) = \frac{1}{k_1 k_2} \sum_{x_1, x_2} p_{x_1, x_2} = 1, \end{aligned} \quad (8)$$

$$\text{since } \sum_{x_1, x_2} p_{x_1, x_2} = k_1 \sum_{x_1, x_2} r_{x_1, x_2, y}^2 = k_1 k_2 \sum_{x_1, x_2} r_{x_1, x_2, y} = k_1 k_2.$$

2. First let us show that  $\text{S}^{\text{NS}}(W, k_1, k_2) \leq \frac{|\mathcal{X}_1|}{k_1}$  (the case  $\text{S}^{\text{NS}}(W, k_1, k_2) \leq \frac{|\mathcal{X}_2|}{k_2}$  is symmetric):

$$\begin{aligned} \text{S}^{\text{NS}}(W, k_1, k_2) &= \frac{1}{k_1 k_2} \sum_{x_1, x_2, y} W(y|x_1 x_2) r_{x_1, x_2, y} \leq \frac{1}{k_1 k_2} \sum_{x_1, x_2, y} W(y|x_1 x_2) r_{x_1, x_2, y}^2 \\ &\leq \frac{1}{k_1 k_2} \sum_{x_2, y} \left( \sum_{x'_1} W(y|x'_1 x_2) \right) \cdot \left( \sum_{x_1} r_{x_1, x_2, y}^2 \right) \quad \text{since nonnegative terms.} \\ &= \frac{1}{k_1 k_2} \sum_{x_2, y} \left( \sum_{x'_1} W(y|x'_1 x_2) \right) \cdot \left( \frac{1}{k_1} \sum_{x_1} p_{x_1, x_2} \right) \\ &= \frac{1}{k_1^2 k_2} \sum_{x_1, x_2} p_{x_1, x_2} \sum_{x'_1} \left( \sum_y W(y|x'_1 x_2) \right) = \frac{|\mathcal{X}_1|}{k_1^2 k_2} \sum_{x_1, x_2} p_{x_1, x_2} = \frac{|\mathcal{X}_1|}{k_1}. \end{aligned} \quad (9)$$

Let us show now that  $\text{S}^{\text{NS}}(W, k_1, k_2) \leq \frac{|\mathcal{Y}|}{k_1 k_2}$ :

$$\begin{aligned} \text{S}^{\text{NS}}(W, k_1, k_2) &= \frac{1}{k_1 k_2} \sum_{x_1, x_2, y} W(y|x_1 x_2) r_{x_1, x_2, y} \leq \frac{1}{k_1 k_2} \sum_y \left( \max_{x_1, x_2} W(y|x_1 x_2) \right) \sum_{x_1, x_2} r_{x_1, x_2, y} \\ &\leq \frac{1}{k_1 k_2} \sum_y \sum_{x_1, x_2} r_{x_1, x_2, y} = \frac{|\mathcal{Y}|}{k_1 k_2}. \end{aligned} \quad (10)$$



3. Let us assume that  $k'_1 \leq k_1$  and that  $k'_2 = k_2$ , since this latter case will follow by symmetry. Consider an optimal solution of  $S^{\text{NS}}(W, k_1, k_2) = \frac{1}{k_1} \sum_{i_1 \in [k_1]} f(i_1)$  with:

$$f(i_1) := \frac{1}{k_2} \sum_{x_1, x_2, y, i_2} W(y|x_1 x_2) P(x_1 x_2(i_1 i_2)|i_1 i_2 y) ,$$

and  $P$  non-signaling. Let us consider  $S \in \operatorname{argmax}_{S' \subseteq [k_1]: |S'|=k'_1} \sum_{i_1 \in S'} f(i_1)$ . Then, by construction, we have that  $\frac{1}{k'_1} \sum_{i_1 \in S} f(i_1) \geq \frac{1}{k_1} \sum_{i_1 \in [k_1]} f(i_1) = S^{\text{NS}}(W, k_1, k_2)$ , since we have taken the average of the  $k'_1$  largest values of the sum.

Let us define the strategy  $P'$  on the smallest set  $\mathcal{X}_1 \times \mathcal{X}_2 \times (S \times [k_2]) \times S \times [k_2] \times \mathcal{Y}$ :

$$\begin{aligned} P'(x_1 x_2(j_1 j_2)|i_1 i_2 y) &:= P(x_1 x_2(j_1 j_2)|i_1 i_2 y) + C(x_1 x_2 j_2|i_1 i_2 y) , \\ \text{with } C(x_1 x_2 j_2|i_1 i_2 y) &:= \frac{1}{k'_1} \sum_{j'_1 \in [k_1] - S} P(x_1 x_2(j'_1 j_2)|i_1 i_2 y) . \end{aligned} \quad (11)$$

$P'$  is a correct conditional probability distribution. Indeed, it is nonnegative by construction, and we have that:

$$\begin{aligned} \sum_{x_1, x_2, j_1 \in S, j_2} P'(x_1 x_2(j_1 j_2)|i_1 i_2 y) &= \sum_{x_1, x_2, j_1 \in S, j_2} P(x_1 x_2(j_1 j_2)|i_1 i_2 y) + \sum_{x_1, x_2, j_1 \in S, j_2} C(x_1 x_2 j_2|i_1 i_2 y) \\ &= \sum_{x_1, x_2, j_2} \sum_{j_1 \in S} P(x_1 x_2(j_1 j_2)|i_1 i_2 y) + \sum_{x_1, x_2, j_2} \sum_{j_1 \in S} \frac{1}{k'_1} \sum_{j'_1 \in [k_1] - S} P(x_1 x_2(j'_1 j_2)|i_1 i_2 y) \\ &= \sum_{x_1, x_2, j_2} \sum_{j_1 \in S} P(x_1 x_2(j_1 j_2)|i_1 i_2 y) + \sum_{x_1, x_2, j_2} \sum_{j'_1 \in [k_1] - S} P(x_1 x_2(j'_1 j_2)|i_1 i_2 y) \\ &= \sum_{x_1, x_2, j_1, j_2} P(x_1 x_2(j_1 j_2)|i_1 i_2 y) = 1 . \end{aligned} \quad (12)$$

Let us show that  $P'$  is non-signaling:

(a) First with  $x_1$ :

$$\begin{aligned} \sum_{x_1} P'(x_1 x_2(j_1 j_2)|i_1 i_2 y) &= \sum_{x_1} P(x_1 x_2(j_1 j_2)|i_1 i_2 y) + \sum_{x_1} C(x_1 x_2 j_2|i_1 i_2 y) \\ &= \sum_{x_1} P(x_1 x_2(j_1 j_2)|i_1 i_2 y) + \frac{1}{k'_1} \sum_{j'_1 \in [k_1] - S} \sum_{x_1} P(x_1 x_2(j'_1 j_2)|i_1 i_2 y) \\ &= \sum_{x_1} P(x_1 x_2(j_1 j_2)|i'_1 i_2 y) + \frac{1}{k'_1} \sum_{j'_1 \in [k_1] - S} \sum_{x_1} P(x_1 x_2(j'_1 j_2)|i'_1 i_2 y) \end{aligned} \quad (13)$$

since  $P$  is non-signaling.

$$= \sum_{x_1} P'(x_1 x_2(j_1 j_2)|i'_1 i_2 y) .$$

(b) Then with  $x_2$ :

$$\begin{aligned}
\sum_{x_2} P'(x_1 x_2(j_1 j_2)|i_1 i_2 y) &= \sum_{x_2} P(x_1 x_2(j_1 j_2)|i_1 i_2 y) + \sum_{x_2} C(x_1 x_2 j_2|i_1 i_2 y) \\
&= \sum_{x_2} P(x_1 x_2(j_1 j_2)|i_1 i_2 y) + \frac{1}{k'_1} \sum_{j'_1 \in [k_1] - S} \sum_{x_2} P(x_1 x_2(j'_1 j_2)|i_1 i_2 y) \\
&= \sum_{x_2} P(x_1 x_2(j_1 j_2)|i_1 i'_2 y) + \frac{1}{k'_1} \sum_{j'_1 \in [k_1] - S} \sum_{x_2} P(x_1 x_2(j'_1 j_2)|i_1 i'_2 y) \quad (14)
\end{aligned}$$

since  $P$  is non-signaling.

$$= \sum_{x_2} P'(x_1 x_2(j_1 j_2)|i_1 i'_2 y) .$$

(c) Finally with  $(j_1 j_2)$ :

$$\begin{aligned}
\sum_{j_1 \in S, j_2} P'(x_1 x_2(j_1 j_2)|i_1 i_2 y) &= \sum_{j_2} \sum_{j_1 \in S} P(x_1 x_2(j_1 j_2)|i_1 i_2 y) + \sum_{j_2} \sum_{j_1 \in S} C(x_1 x_2 j_2|i_1 i_2 y) \\
&= \sum_{j_2} \sum_{j_1 \in S} P(x_1 x_2(j_1 j_2)|i_1 i_2 y) + \sum_{j_2} \sum_{j_1 \in S} \frac{1}{k'_1} \sum_{j'_1 \in [k_1] - S} P(x_1 x_2(j'_1 j_2)|i_1 i_2 y) \\
&= \sum_{j_2} \sum_{j_1 \in S} P(x_1 x_2(j_1 j_2)|i_1 i_2 y) + \sum_{j_2} \sum_{j'_1 \in [k_1] - S} P(x_1 x_2(j'_1 j_2)|i_1 i_2 y) \\
&= \sum_{j_1 j_2} P(x_1 x_2(j_1 j_2)|i_1 i_2 y) \\
&= \sum_{j_1 j_2} P(x_1 x_2(j_1 j_2)|i_1 i_2 y') \text{ since } P \text{ is non-signaling.} \\
&= \sum_{j_1 \in S, j_2} P'(x_1 x_2(j_1 j_2)|i_1 i_2 y') . \quad (15)
\end{aligned}$$

Thus  $P'$  is a correct solution of the program computing  $S^{\text{NS}}(W, k'_1, k_2)$ , and it leads to the value:

$$\begin{aligned}
S^{\text{NS}}(W, k'_1, k_2) &\geq \frac{1}{k'_1 k_2} \sum_{x_1, x_2, y, i_1 \in S, i_2} W(y|x_1 x_2) P'(x_1 x_2(i_1 i_2)|i_1 i_2 y) \\
&\geq \frac{1}{k'_1 k_2} \sum_{x_1, x_2, y, i_1 \in S, i_2} W(y|x_1 x_2) P(x_1 x_2(i_1 i_2)|i_1 i_2 y) \quad (16) \\
&= \frac{1}{k'_1} \sum_{i_1 \in S} f(i_1) \geq \frac{1}{k_1} \sum_{i_1 \in [k_1]} f(i_1) = S^{\text{NS}}(W, k_1, k_2) .
\end{aligned}$$

4. Consider optimal non-signaling probability distributions  $P$  and  $P'$  reaching respectively the values  $S^{\text{NS}}(W, k_1, k_2)$  and  $S^{\text{NS}}(W', k'_1, k'_2)$ . Then by Lemma 3.4,  $P \otimes P'$  is a non-signaling probability distribution on  $(\mathcal{X}_1 \times \mathcal{X}'_1) \times (\mathcal{X}_2 \times \mathcal{X}'_2) \times (([k_1] \times [k'_1]) \times ([k_2] \times [k'_2])) \times ([k_1] \times [k'_1]) \times ([k_2] \times [k'_2]) \times (\mathcal{Y} \times \mathcal{Y}')$ , which is trivially in bijection with  $(\mathcal{X}_1 \times \mathcal{X}'_1) \times (\mathcal{X}_2 \times \mathcal{X}'_2) \times ([k_1 k'_1] \times [k_2 k'_2]) \times [k_1 k'_1] \times [k_2 k'_2] \times (\mathcal{Y} \times \mathcal{Y}')$ . This gives a valid solution of the program computing  $S^{\text{NS}}(W \otimes W', k_1 k'_1, k_2 k'_2)$ .

Thus, we get that  $S^{\text{NS}}(W \otimes W', k_1 k'_1, k_2 k'_2)$  is larger than or equal to:

$$\begin{aligned}
& \sum_{x_1 x'_1, x_2 x'_2, y y', i_1 i'_1, i_2 i'_2} (W \otimes W') (y y' | x_1 x'_1 x_2 x'_2) (P \otimes P') (x_1 x'_1 x_2 x'_2 (i_1 i'_1 i_2 i'_2) | i_1 i'_1, i_2 i'_2 y y') \\
&= \sum_{x_1 x'_1, x_2 x'_2, y y', i_1 i'_1, i_2 i'_2} (W(y | x_1 x_2) \cdot W'(y' | x'_1 x'_2)) (P(x_1 x_2 (i_1 i_2) | i_1 i_2 y) \cdot P'(x'_1 x'_2 (i'_1 i'_2) | i'_1 i'_2 y')) \\
&= \left( \sum_{x_1, x_2, y, i_1, i_2} W(y | x_1 x_2) P(x_1 x_2 (i_1 i_2) | i_1 i_2 y) \right) \cdot \left( \sum_{x'_1, x'_2, y', i'_1, i'_2} W'(y' | x'_1 x'_2) P'(x'_1 x'_2 (i'_1 i'_2) | i'_1 i'_2 y') \right) \\
&= S^{\text{NS}}(W, k_1, k_2) \cdot S^{\text{NS}}(W', k'_1, k'_2) .
\end{aligned} \tag{17}$$

In particular, applying this  $n$  times on the same MAC  $W$  gives the first corollary, and the second one comes from the fact that  $S^{\text{NS}}(W \otimes W', k_1, k_2) \geq S^{\text{NS}}(W, k_1, k_2) \cdot S^{\text{NS}}(W', 1, 1) = S^{\text{NS}}(W, k_1, k_2)$ , since  $S^{\text{NS}}(W', 1, 1) = 1$  by the first property of Proposition 3.5.  $\square$

**Corollary 3.6.** 1.  $\mathcal{C}^{\text{NS}}(W)$  is convex.

2. If  $(R_1, R_2)$  is achievable with non-signaling assistance, then  $R_1 \leq \log_2 |\mathcal{X}_1|$ ,  $R_2 \leq \log_2 |\mathcal{X}_2|$  and  $R_1 + R_2 \leq \log_2 |\mathcal{Y}|$ .
3. If  $(R_1, R_2)$  is achievable with non-signaling assistance, then for all  $R'_i \leq R_i$ ,  $(R'_1, R'_2)$  is achievable with non-signaling assistance.

*Proof.* 1. Let  $(R_1, R_2)$  and  $(\tilde{R}_1, \tilde{R}_2)$ , two pairs of rational rates achievable with non-signaling assistance for  $W$ , ie:

$$S^{\text{NS}}(W^{\otimes n}, \lceil 2^{R_1 n} \rceil, \lceil 2^{R_2 n} \rceil) \xrightarrow{n \rightarrow +\infty} 1 \text{ and } S^{\text{NS}}(W^{\otimes n}, \lceil 2^{\tilde{R}_1 n} \rceil, \lceil 2^{\tilde{R}_2 n} \rceil) \xrightarrow{n \rightarrow +\infty} 1 .$$

Let  $\lambda \in (0, 1)$  rational and define  $R_{\lambda, i} := \lambda \cdot R_i + (1 - \lambda) \cdot \tilde{R}_i$ , let us show that  $(R_{\lambda, 1}, R_{\lambda, 2})$  is achievable with non-signaling assistance. Let us call respectively  $k_i := 2^{R_i}$ ,  $\tilde{k}_i := 2^{\tilde{R}_i}$ ,  $k_{\lambda, i} := 2^{R_{\lambda, i}} = k_i^\lambda \cdot \tilde{k}_i^{(1-\lambda)}$ . We have  $R_{\lambda, i} n = \lambda \cdot R_i n + (1 - \lambda) \cdot \tilde{R}_i n = (\lambda n) \cdot R_i + (1 - \lambda) n \cdot \tilde{R}_i$ . This is the idea of *time-sharing*: for  $\lambda n$  copies of the MAC, we use the strategy with rate  $(R_1, R_2)$  and for the  $(1 - \lambda)n$  other copies of the MAC, we use the strategy with rate  $(\tilde{R}_1, \tilde{R}_2)$ . There exists some  $n$  such that  $\lambda n, (1 - \lambda)n, \lambda n R_i, (1 - \lambda)n \tilde{R}_i$  are integers, since everything is rational. This implies that  $k_i^{\lambda n}, \tilde{k}_i^{(1-\lambda)n}, k_{\lambda, i}^n$  are integers. Thus, thanks to the fourth property of Proposition 3.5, we have:

$$\begin{aligned}
S^{\text{NS}}(W^{\otimes n}, k_{\lambda, 1}^n, k_{\lambda, 2}^n) &\geq S^{\text{NS}}(W^{\otimes (\lambda n)}, k_1^{\lambda n}, k_2^{\lambda n}) \cdot S^{\text{NS}}(W^{\otimes ((1-\lambda)n)}, \tilde{k}_1^{(1-\lambda)n}, \tilde{k}_2^{(1-\lambda)n}) \\
&\xrightarrow{n \rightarrow +\infty} 1 \cdot 1 = 1 .
\end{aligned} \tag{18}$$

Thus in particular, since we have  $S^{\text{NS}}(W^{\otimes n}, k_{\lambda, 1}^n, k_{\lambda, 2}^n) \leq 1$ , we get that  $S^{\text{NS}}(W^{\otimes n}, k_{\lambda, 1}^n, k_{\lambda, 2}^n) \xrightarrow{n \rightarrow +\infty} 1$ , so  $(R_{\lambda, 1}, R_{\lambda, 2})$  is achievable with non-signaling assistance. Finally, since  $\mathcal{C}^{\text{NS}}(W)$  is defined as the closure of achievable rates with non-signaling assistance, we get that  $\mathcal{C}^{\text{NS}}(W)$  is convex.

2. By the second property of Proposition 3.5, we have that  $S^{\text{NS}}(W^{\otimes n}, k_1^n, k_2^n) \leq \frac{|\mathcal{X}_1|^n}{k_1^n}$ . In particular, if one takes  $R_1 > \log_2 |\mathcal{X}_1|$ , then  $k_1 > |\mathcal{X}_1|$  and we get that  $S^{\text{NS}}(W^{\otimes n}, k_1^n, k_2^n) \leq \left( \frac{|\mathcal{X}_1|}{k_1} \right)^n \xrightarrow{n \rightarrow +\infty} 0$ , so

$R_1 > \log_2 |\mathcal{X}_1|$  is not achievable with non-signaling assistance. Symmetrically,  $R_2 > \log_2 |\mathcal{X}_2|$  is not achievable with non-signaling assistance.

Furthermore, if one takes  $R_1 + R_2 > \log_2 |\mathcal{Y}|$ , then in particular  $k_1 k_2 > |\mathcal{Y}|$ , so by the second property of Proposition 3.5,  $S^{\text{NS}}(W^{\otimes n}, k_1^n, k_2^n) \leq \frac{|\mathcal{Y}^n|}{k_1^n k_2^n} = \left(\frac{|\mathcal{Y}|}{k_1 k_2}\right)^n \xrightarrow{n \rightarrow +\infty} 0$ . Thus,  $R_1 + R_2 > \log_2 |\mathcal{Y}|$  is not achievable with non-signaling assistance.

3. Since  $(R_1, R_2)$  is achievable with non-signaling assistance, we have  $S^{\text{NS}}(W^{\otimes n}, \lceil 2^{nR_1} \rceil, \lceil 2^{nR_2} \rceil) \xrightarrow{n \rightarrow +\infty} 1$ .  
 1. But, for all positive integer  $n$ , we have that  $\lceil 2^{nR'_1} \rceil \leq \lceil 2^{nR_1} \rceil$  and  $\lceil 2^{nR'_2} \rceil \leq \lceil 2^{nR_2} \rceil$ , so by the third property of Proposition 3.5, we have that  $S^{\text{NS}}(W^{\otimes n}, \lceil 2^{nR'_1} \rceil, \lceil 2^{nR'_2} \rceil) \geq S^{\text{NS}}(W^{\otimes n}, \lceil 2^{nR_1} \rceil, \lceil 2^{nR_2} \rceil)$ . Thus  $S^{\text{NS}}(W^{\otimes n}, \lceil 2^{nR'_1} \rceil, \lceil 2^{nR'_2} \rceil) \xrightarrow{n \rightarrow +\infty} 1$  since it is upper bounded by 1, and so  $(R'_1, R'_2)$  is achievable with non-signaling assistance. □

**Proposition 3.7.**  $\mathcal{C}_0^{\text{NS}}(W)$  is the closure of the set of rate pairs  $(R_1, R_2)$  such that:

$$\exists n \in \mathbb{N}^*, S^{\text{NS}}(W^{\otimes n}, \lceil 2^{R_1 n} \rceil, \lceil 2^{R_2 n} \rceil) = 1.$$

*Proof.* It is clear that if  $(R_1, R_2)$  is such that  $\exists n_0 \in \mathbb{N}^*, \forall n \geq n_0, S^{\text{NS}}(W^{\otimes n}, \lceil 2^{R_1 n} \rceil, \lceil 2^{R_2 n} \rceil) = 1$ , then in particular  $\exists n \in \mathbb{N}^*, S^{\text{NS}}(W^{\otimes n}, \lceil 2^{R_1 n} \rceil, \lceil 2^{R_2 n} \rceil) = 1$ . So,  $\mathcal{C}_0^{\text{NS}}(W)$ , which is the closure of the former rate pairs, is in particular included in the closure of the latter rate pairs.

For the other inclusion, consider a rate pair  $(R_1, R_2)$  and let us assume that there exists some positive integer  $n$  such that  $S^{\text{NS}}(W^{\otimes n}, \lceil 2^{R_1 n} \rceil, \lceil 2^{R_2 n} \rceil) = 1$ . Let us show that for any  $(R'_1, R'_2)$  such that  $R'_1 < R_1$  and  $R'_2 < R_2$ :

$$\exists n_0 \in \mathbb{N}^*, \forall n \geq n_0, S^{\text{NS}}(W^{\otimes n}, \lceil 2^{R'_1 n} \rceil, \lceil 2^{R'_2 n} \rceil) = 1,$$

which is enough to conclude, since we consider only closure of such sets.

First, for all positive integer  $m$ , we have that  $S^{\text{NS}}(W^{\otimes nm}, \lceil 2^{R_1 nm} \rceil, \lceil 2^{R_2 nm} \rceil) = 1$ . Indeed, by the fourth property of Proposition 3.5, we have that  $S^{\text{NS}}((W^{\otimes n})^{\otimes m}, \lceil 2^{R_1 n} \rceil^m, \lceil 2^{R_2 n} \rceil^m) \geq [S^{\text{NS}}(W^{\otimes n}, \lceil 2^{R_1 n} \rceil, \lceil 2^{R_2 n} \rceil)]^m = 1$ , so  $S^{\text{NS}}((W^{\otimes n})^{\otimes m}, \lceil 2^{R_1 n} \rceil^m, \lceil 2^{R_2 n} \rceil^m) = 1$  since  $S^{\text{NS}}(W, k_1, k_2) \leq 1$  by the first property of Proposition 3.5. But  $(W^{\otimes n})^{\otimes m} = W^{\otimes nm}$ , and  $\lceil 2^{R_1 n} \rceil^m \geq \lceil 2^{R_1 nm} \rceil, \lceil 2^{R_2 n} \rceil^m \geq \lceil 2^{R_2 nm} \rceil$ , so by the third property of Proposition 3.5, we have  $S^{\text{NS}}(W^{\otimes nm}, \lceil 2^{R_1 nm} \rceil, \lceil 2^{R_2 nm} \rceil) \geq 1$ , so  $S^{\text{NS}}(W^{\otimes nm}, \lceil 2^{R_1 nm} \rceil, \lceil 2^{R_2 nm} \rceil) = 1$ .

Then, consider some  $r \in \{0, \dots, n-1\}$ . By the fourth property of Proposition 3.5, we have that:

$$\begin{aligned} S^{\text{NS}}(W^{\otimes(nm+r)}, \lceil 2^{R_1 nm} \rceil, \lceil 2^{R_2 nm} \rceil) &= S^{\text{NS}}(W^{\otimes nm} \otimes W^{\otimes r}, \lceil 2^{R_1 nm} \rceil, \lceil 2^{R_2 nm} \rceil) \\ &\geq S^{\text{NS}}(W^{\otimes nm}, \lceil 2^{R_1 nm} \rceil, \lceil 2^{R_2 nm} \rceil) = 1, \end{aligned} \tag{19}$$

so  $S^{\text{NS}}(W^{\otimes(nm+r)}, \lceil 2^{R_1 nm} \rceil, \lceil 2^{R_2 nm} \rceil) = 1$ . But  $\lceil 2^{R_1 nm} \rceil = \lceil 2^{\frac{R_1 nm}{nm+r}(nm+r)} \rceil = \lceil 2^{\frac{R_1}{1+\delta}(nm+r)} \rceil$  with  $\delta = \frac{r}{nm} \leq \frac{1}{m}$ , and symmetrically  $\lceil 2^{R_2 nm} \rceil = \lceil 2^{\frac{R_2}{1+\delta}(nm+r)} \rceil$ . Thus in particular, for all  $R'_1 \leq \frac{R_1}{1+\frac{1}{m}}$  and  $R'_2 \leq \frac{R_2}{1+\frac{1}{m}}$ , we have that for all  $n' \geq nm$ ,  $S^{\text{NS}}(W^{\otimes n'}, \lceil 2^{R'_1 n'} \rceil, \lceil 2^{R'_2 n'} \rceil) = 1$ . So for any  $(R'_1, R'_2)$  such that  $R'_1 < R_1$  and  $R'_2 < R_2$ , there is large enough  $m$  such that  $R'_1 \leq \frac{R_1}{1+\frac{1}{m}}$  and  $R'_2 \leq \frac{R_2}{1+\frac{1}{m}}$ , and thus we get the expected property on  $(R'_1, R'_2)$  for  $n_0 := nm$ . □

### 3.3 Symmetrization

Although  $S^{\text{NS}}(W, k_1, k_2)$  can be computed in polynomial time in  $W, k_1$  and  $k_2$ , a channel of the form  $W^{\otimes n}$  has exponential size in  $n$ . Thus, the linear program for  $S^{\text{NS}}(W^{\otimes n}, k_1, k_2)$  grows exponentially with  $n$ . However, using the invariance of  $W^{\otimes n}$  under permutations, one can find a much smaller linear program computing  $S^{\text{NS}}(W^{\otimes n}, k_1, k_2)$ .

**Definition 3.8.** Let  $G$  a group acting on  $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}$ . We say that a MAC  $W : \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathcal{Y}$  is *invariant under the action of  $G$*  if:

$$\forall g \in G, W(g \cdot y | g \cdot x_1 g \cdot x_2) = W(y | x_1 x_2) .$$

In particular, for  $W^{\otimes n} : \mathcal{X}_1^n \times \mathcal{X}_2^n \rightarrow \mathcal{Y}^n$ , the symmetric group  $G := S_n$  acts in a natural way in any set  $\mathcal{A}$  raised to power  $n$ . So for  $\sigma \in S_n$ , we have that:

$$W^{\otimes n}(\sigma \cdot y^n | \sigma \cdot x_1^n \sigma \cdot x_2^n) = \prod_{i=1}^n W(y_{\sigma(i)} | x_{1,\sigma(i)} x_{2,\sigma(i)}) = \prod_{i=1}^n W(y_i | x_{1,i} x_{2,i}) = W^{\otimes n}(y^n | x_1^n x_2^n) ,$$

and so  $W^{\otimes n}$  is invariant under the action of  $S_n$ .

Let  $\mathcal{Z} := \{\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, \mathcal{X}_1 \times \mathcal{Y}, \mathcal{X}_2 \times \mathcal{Y}, \mathcal{X}_1 \times \mathcal{X}_2, \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y}\}$ . Let us call  $\mathcal{O}_G(\mathcal{A})$  the set of orbits of  $\mathcal{A}$  under the action of  $G$ . Then, one can find an equivalent smaller linear program for  $S^{\text{NS}}(W, k_1, k_2)$ :

**Theorem 3.9.** Let  $W : \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathcal{Y}$  a MAC invariant under the action of  $G$ . Let us name systematically  $w \in \mathcal{O}_G(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y})$ ,  $u \in \mathcal{O}_G(\mathcal{X}_1 \times \mathcal{X}_2)$ ,  $u^1 \in \mathcal{O}_G(\mathcal{X}_1)$ ,  $u^2 \in \mathcal{O}_G(\mathcal{X}_2)$ ,  $v^1 \in \mathcal{O}_G(\mathcal{X}_1 \times \mathcal{Y})$ ,  $v^2 \in \mathcal{O}_G(\mathcal{X}_2 \times \mathcal{Y})$ ,  $v \in \mathcal{O}_G(\mathcal{Y})$ . We will also call  $z_{\mathcal{A}}$  the projection of  $z \in \mathcal{O}_G(\mathcal{B})$  on  $\mathcal{A}$ , for  $\mathcal{A}, \mathcal{B} \in \mathcal{Z}$  and  $\mathcal{A}$  projection of  $\mathcal{B}$ ; note that  $z_{\mathcal{A}} \in \mathcal{O}_G(\mathcal{A})$ , since by definition of the action, the projection of an orbit is an orbit. Let us finally call  $W(w) := W(y | x_1 x_2)$  for any  $(x_1, x_2, y) \in w$ , which is well-defined since  $W$  is invariant under  $G$ . We have that  $S^{\text{NS}}(W, k_1, k_2)$  is the solution of the following linear program:

$$\begin{aligned} S^{\text{NS}}(W, k_1, k_2) = & \underset{r, r^1, r^2, p}{\text{maximize}} \quad \frac{1}{k_1 k_2} \sum_{w \in \mathcal{O}_G(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y})} W(w) r_w \\ & \text{subject to} \quad \sum_{w: w_{\mathcal{Y}}=v} r_w = |v|, \forall v \in \mathcal{O}_G(\mathcal{Y}) \\ & \quad \sum_{w: w_{\mathcal{X}_2 \mathcal{Y}}=v^2} r_w^1 = k_1 \sum_{w: w_{\mathcal{X}_2 \mathcal{Y}}=v^2} r_w, \forall v^2 \in \mathcal{O}_G(\mathcal{X}_2 \times \mathcal{Y}) \\ & \quad \sum_{w: w_{\mathcal{X}_1 \mathcal{Y}}=v^1} r_w^2 = k_2 \sum_{w: w_{\mathcal{X}_1 \mathcal{Y}}=v^1} r_w, \forall v^1 \in \mathcal{O}_G(\mathcal{X}_1 \times \mathcal{Y}) \\ & \quad \sum_{u: u_{\mathcal{X}_2}=v_{\mathcal{X}_2}^2} p_u = \frac{|v_{\mathcal{X}_2}^2|}{|v^2|} k_1 \sum_{w: w_{\mathcal{X}_2 \mathcal{Y}}=v^2} r_w^2, \forall v^2 \in \mathcal{O}_G(\mathcal{X}_2 \times \mathcal{Y}) \\ & \quad \sum_{u: u_{\mathcal{X}_1}=v_{\mathcal{X}_1}^1} p_u = \frac{|v_{\mathcal{X}_1}^1|}{|v^1|} k_2 \sum_{w: w_{\mathcal{X}_1 \mathcal{Y}}=v^1} r_w^1, \forall v^1 \in \mathcal{O}_G(\mathcal{X}_1 \times \mathcal{Y}) \\ & \quad 0 \leq r_w \leq r_w^1, r_w^2 \leq \frac{|w|}{|w_{\mathcal{X}_1 \mathcal{X}_2}|} p_{w_{\mathcal{X}_1 \mathcal{X}_2}}, \forall w \in \mathcal{O}_G(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y}) \\ & \quad \frac{|w|}{|w_{\mathcal{X}_1 \mathcal{X}_2}|} p_{w_{\mathcal{X}_1 \mathcal{X}_2}} - r_w^1 - r_w^2 + r_w \geq 0, \forall w \in \mathcal{O}_G(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y}) . \end{aligned} \tag{20}$$

**Corollary 3.10.** For a channel  $W : \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathcal{Y}$ ,  $S^{\text{NS}}(W^{\otimes n}, k_1, k_2)$  is the solution of a linear program of size bounded by  $O(n^{|\mathcal{X}_1|+|\mathcal{X}_2|+|\mathcal{Y}|-1})$ , thus it can be computed in polynomial time in  $n$ .

*Proof.* We use the linear program obtained in Theorem 3.9 with  $G := S_n$  acting on  $W^{\otimes n}$  as described before. The number of variables and constraints is linear in the number of orbits of the action of  $S^n$  on the different sets  $\mathcal{A} \in \mathcal{Z}$ , where here  $\mathcal{Z} = \{\mathcal{X}_1^n, \mathcal{X}_2^n, \mathcal{Y}^n, \mathcal{X}_1^n \times \mathcal{Y}^n, \mathcal{X}_2^n \times \mathcal{Y}^n, \mathcal{X}_1^n \times \mathcal{X}_2^n, \mathcal{X}_1^n \times \mathcal{X}_2^n \times \mathcal{Y}^n\}$ . For example, for  $\mathcal{A} \in \mathcal{X}_1^n \times \mathcal{X}_2^n \times \mathcal{Y}^n$ , we have that:

$$|\mathcal{O}_{S_n}(\mathcal{X}_1^n \times \mathcal{X}_2^n \times \mathcal{Y}^n)| = \binom{n + |\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}| - 1}{|\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}| - 1} \leq (n + |\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}| - 1)^{|\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}| - 1}.$$

So the number of variables and constraints is  $O(n^{|\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}| - 1})$ . Note also that all the numbers occurring in this linear program are integers or fractions of integers, with those integers ranging in  $[(|\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}|)^n]$ , thus of size  $O(n \log(|\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}|))$ . So the size of this linear program is bounded by  $O(n^{|\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}| - 1})$ , and thus  $S^{\text{NS}}(W^{\otimes n}, k_1, k_2)$  can be computed in polynomial time in  $n$ ; see for instance Section 7.1 of [10].  $\square$

In order to prove Theorem 3.9, we will need several lemmas. For all of them,  $\mathcal{A}$  and  $\mathcal{B}$  will denote finite sets on which a group  $G$  is acting, and  $x^G$  will denote the orbit of  $x$  under  $G$ :

**Lemma 3.11.** Let  $\tau \in \mathcal{O}_G(\mathcal{A} \times \mathcal{B})$ , and call  $\nu := \tau_{\mathcal{A}}$  and  $\mu := \tau_{\mathcal{B}}$ . For  $x \in \nu$ , let us call  $B_{\tau}^x := \{y : (x, y) \in \tau\}$ . Then,  $|B_{\tau}^x| = |B_{\tau}^{x'}| =: c_{\tau}^x$  for any  $x, x' \in \nu$ , and furthermore, we have that  $c_{\tau}^x = \frac{|\tau|}{|\nu|}$ . Symmetrically, the same occurs for  $A_{\tau}^y := \{x : (x, y) \in \tau\}$  with  $y \in \mu$ , where one gets that  $|A_{\tau}^y| = |A_{\tau}^{y'}| =: c_{\tau}^y = \frac{|\tau|}{|\mu|}$  for  $y, y' \in \mu$ .

*Proof.* Let  $x, x' \in \nu$ . Thus there exists  $g \in G$  such that  $x' = g \cdot x$ . Let:

$$\begin{aligned} f &: B_{\tau}^x &\rightarrow & B_{\tau}^{x'} \\ &y &\mapsto & g \cdot y. \end{aligned}$$

First,  $f$  is well defined. Indeed, if  $y \in B_{\tau}^x = \{y : (x, y) \in \tau\}$ , then  $g \cdot y \in \{y : (g \cdot x, y) \in \tau\} = B_{\tau}^{x'}$ , since  $\tau \in \mathcal{O}_G(\mathcal{A} \times \mathcal{B})$ . Let us show that  $f$  is injective. If  $g \cdot y = g \cdot y'$ , then  $g^{-1} \cdot (g \cdot y) = (g^{-1}g) \cdot y = y$ ,  $g^{-1} \cdot (g \cdot y') = y'$ , so  $y = y'$ . Thus we get that  $|B_{\tau}^x| \leq |B_{\tau}^{x'}|$ . By a symmetric argument with  $x'$  replacing  $x$  and  $g^{-1}$  replacing  $g$ , we get that  $|B_{\tau}^{x'}| \leq |B_{\tau}^x|$ , and so  $|B_{\tau}^x| = |B_{\tau}^{x'}| =: c_{\tau}^x$ .

Furthermore,  $\{B_{\tau}^x\}_{x \in \nu}$  is a partition of  $\tau$ , so  $\sum_{x \in \nu} |B_{\tau}^x| = |\nu| c_{\tau}^x = |\tau|$ , and thus  $c_{\tau}^x = \frac{|\tau|}{|\nu|}$ .  $\square$

**Lemma 3.12.** For any  $(x, y) \in \mathcal{A} \times \mathcal{B}$  and  $v_{(x, y)^G}$  variable indexed by orbits of  $\mathcal{A} \times \mathcal{B}$ , let us define the variable  $v_{x, y} := \frac{v_{(x, y)^G}}{|(x, y)^G|}$ . We have:

$$\sum_{x \in \mathcal{A}} v_{x, y} = \frac{1}{|y^G|} \sum_{\tau \in \mathcal{O}_G(\mathcal{A} \times \mathcal{B}) : \tau_{\mathcal{B}} = y^G} v_{\tau}, \forall y \in \mathcal{B}.$$

*Proof.*

$$\begin{aligned} \sum_{x \in \mathcal{A}} v_{x, y} &= \sum_{\tau \in \mathcal{O}_G(\mathcal{A} \times \mathcal{B}) : \tau_{\mathcal{B}} = y^G} \sum_{x \in \mathcal{A} : (x, y) \in \tau} v_{x, y} \\ &= \sum_{\tau \in \mathcal{O}_G(\mathcal{A} \times \mathcal{B}) : \tau_{\mathcal{B}} = y^G} \sum_{x \in \mathcal{A} : (x, y) \in \tau} \frac{v_{\tau}}{|\tau|} \quad \text{since } (x, y)^G = \tau \\ &= \sum_{\tau \in \mathcal{O}_G(\mathcal{A} \times \mathcal{B}) : \tau_{\mathcal{B}} = y^G} c_{\tau}^{y^G} \frac{v_{\tau}}{|\tau|} \quad \text{by Lemma 3.11, since } y \in \tau_{\mathcal{B}} \\ &= \sum_{\tau \in \mathcal{O}_G(\mathcal{A} \times \mathcal{B}) : \tau_{\mathcal{B}} = y^G} \frac{|\tau|}{|y^G|} \frac{v_{\tau}}{|\tau|} = \frac{1}{|y^G|} \sum_{\tau \in \mathcal{O}_G(\mathcal{A} \times \mathcal{B}) : \tau_{\mathcal{B}} = y^G} v_{\tau}. \end{aligned} \tag{21}$$

$\square$

**Lemma 3.13.** For any  $\tau \in \mathcal{O}_G(\mathcal{A} \times \mathcal{B})$ ,  $\mu \in \mathcal{O}_G(\mathcal{B})$  and  $v_{x, y}$  variable indexed by elements of  $\mathcal{A} \times \mathcal{B}$ , let us define  $v_{\tau} := \sum_{(x, y) \in \tau} v_{x, y}$ . We have:

$$\sum_{\tau \in \mathcal{O}_G(\mathcal{A} \times \mathcal{B}) : \tau_{\mathcal{B}} = \mu} v_{\tau} = \sum_{y \in \mu} \sum_{x \in \mathcal{A}} v_{x, y}.$$

*Proof.*

$$\sum_{\tau \in \mathcal{O}_G(\mathcal{A} \times \mathcal{B}) : \tau_{\mathcal{B}} = \mu} v_{\tau} = \sum_{\tau \in \mathcal{O}_G(\mathcal{A} \times \mathcal{B}) : \tau_{\mathcal{B}} = \mu} \sum_{(x,y) \in \tau} v_{x,y} = \sum_{y \in \mu} \sum_{x \in \mathcal{A}} v_{x,y} .$$

□

*Proof of Theorem 3.9.* Let  $r_{x_1, x_2, y}, r_{x_1, x_2, y}^1, r_{x_1, x_2, y}^2, p_{x_1, x_2}$  a feasible solution of the program defined in Proposition 3.1, and call  $S := \frac{1}{k_1 k_2} \sum_{x_1, x_2, y} W(y|x_1 x_2) r_{x_1, x_2, y}$  its value. Define:

$$\begin{aligned} r_w &:= \sum_{(x_1, x_2, y) \in w} r_{x_1, x_2, y} , & r_w^1 &:= \sum_{(x_1, x_2, y) \in w} r_{x_1, x_2, y}^1 , \\ r_w^2 &:= \sum_{(x_1, x_2, y) \in w} r_{x_1, x_2, y}^2 , & p_u &:= \sum_{(x_1, x_2) \in u} p_{x_1, x_2} . \end{aligned} \quad (22)$$

Let us show that  $r_w, r_w^1, r_w^2, p_u$  is a feasible solution of the program defined in Theorem 3.9, and that its value  $S^* := \frac{1}{k_1 k_2} \sum_w W(w) r_w = S$ .

First, we have  $S^* = S$ . Indeed:

$$\begin{aligned} S^* &= \frac{1}{k_1 k_2} \sum_w W(w) r_w = \frac{1}{k_1 k_2} \sum_w W(w) \sum_{(x_1, x_2, y) \in w} r_{x_1, x_2, y} \\ &= \frac{1}{k_1 k_2} \sum_w \sum_{(x_1, x_2, y) \in w} W(y|x_1 x_2) r_{x_1, x_2, y} \quad \text{since } W(w) = W(y|x_1 x_2) \text{ for all } (x_1, x_2, y) \in w \\ &= \frac{1}{k_1 k_2} \sum_{x_1, x_2, y} W(y|x_1 x_2) r_{x_1, x_2, y} = S . \end{aligned} \quad (23)$$

Then, all the constraints are satisfied. Indeed, thanks to Lemma 3.13, we have for the first constraint:

$$\sum_{w: w_{\mathcal{Y}} = v} r_w = \sum_{y \in v} \sum_{x_1, x_2} r_{x_1, x_2, y} = \sum_{y \in v} 1 = |v| . \quad (24)$$

For the second constraint (and symmetrically for the third constraint), we have:

$$\sum_{w: w_{\mathcal{X}_2 \mathcal{Y}} = v^2} r_w^1 = \sum_{(x_2, y) \in v^2} \sum_{x_1} r_{x_1, x_2, y}^1 = \sum_{(x_2, y) \in v^2} k_1 \sum_{x_1} r_{x_1, x_2, y} = k_1 \sum_{w: w_{\mathcal{X}_2 \mathcal{Y}} = v^2} r_w . \quad (25)$$

For the fourth (and symmetrically for the fifth), we have:

$$\begin{aligned} \sum_{w: w_{\mathcal{X}_2 \mathcal{Y}} = v^2} r_w^2 &= \sum_{(x_2, y) \in v^2} \sum_{x_1} r_{x_1, x_2, y}^2 = \sum_{(x_2, y) \in v^2} \frac{1}{k_1} \sum_{x_1} p_{x_1, x_2} = \frac{1}{k_1} \sum_{x_2 \in v_{\mathcal{X}_2}^2} \sum_{y: (x_2, y) \in v^2} \sum_{x_1} p_{x_1, x_2} \\ &= \frac{1}{k_1} \sum_{x_2 \in v_{\mathcal{X}_2}^2} \frac{|v^2|}{|v_{\mathcal{X}_2}^2|} \sum_{x_1} p_{x_1, x_2} \quad \text{thanks to Lemma 3.11} \\ &= \frac{1}{k_1} \frac{|v^2|}{|v_{\mathcal{X}_2}^2|} \sum_{u: u_{\mathcal{X}_2} = v_{\mathcal{X}_2}^2} p_u . \end{aligned} \quad (26)$$

Finally for the last constraints, we only need to compute:

$$\sum_{(x_1, x_2, y) \in w} p_{x_1, x_2} = \sum_{(x_1, x_2) \in w \mathcal{X}_1 \mathcal{X}_2} \sum_{y: (x_1, x_2, y) \in w} p_{x_1, x_2} = \sum_{(x_1, x_2) \in w \mathcal{X}_1 \mathcal{X}_2} \frac{|w|}{|w \mathcal{X}_1 \mathcal{X}_2|} p_{x_1, x_2} = \frac{|w|}{|w \mathcal{X}_1 \mathcal{X}_2|} p_{w \mathcal{X}_1 \mathcal{X}_2}, \quad (27)$$

which implies that the linear inequalities on  $p_{x_1, x_2}, r_{x_1, x_2, y}, r_{x_1, x_2, y}^1, r_{x_1, x_2, y}^2$  get transposed respectively to the values  $\frac{|w|}{|w \mathcal{X}_1 \mathcal{X}_2|} p_{w \mathcal{X}_1 \mathcal{X}_2}, r_w, r_w^1, r_w^2$ . Indeed, for instance, one has for any  $x_1, x_2, y$  that  $p_{x_1, x_2} - r_{x_1, x_2, y}^1 - r_{x_1, x_2, y}^2 + r_{x_1, x_2, y} \geq 0$ . Thus for some orbit  $w$ :

$$\sum_{(x_1, x_2, y) \in w} (p_{x_1, x_2} - r_{x_1, x_2, y}^1 - r_{x_1, x_2, y}^2 + r_{x_1, x_2, y}) \geq 0,$$

and then  $\frac{|w|}{|w \mathcal{X}_1 \mathcal{X}_2|} p_{w \mathcal{X}_1 \mathcal{X}_2} - r_w^1 - r_w^2 + r_w \geq 0$ , which was what we wanted to show.

Now let us consider a feasible solution  $r_w, r_w^1, r_w^2, p_w$  of the program defined in Theorem 3.9, with a value  $S^* := \frac{1}{k_1 k_2} \sum_w W(w) r_w$ . Define:

$$\begin{aligned} r_{x_1, x_2, y} &:= \frac{r_{(x_1, x_2, y)^G}}{|(x_1, x_2, y)^G|}, & r_{x_1, x_2, y}^1 &:= \frac{r_{(x_1, x_2, y)^G}^1}{|(x_1, x_2, y)^G|}, \\ r_{x_1, x_2, y}^2 &:= \frac{r_{(x_1, x_2, y)^G}^2}{|(x_1, x_2, y)^G|}, & p_{x_1, x_2} &:= \frac{p_{(x_1, x_2)^G}}{|(x_1, x_2)^G|}. \end{aligned} \quad (28)$$

Let us show that  $r_{x_1, x_2, y}, r_{x_1, x_2, y}^1, r_{x_1, x_2, y}^2, p_{x_1, x_2}$  is a feasible solution of the program defined in Proposition 3.1, and that its value  $S := \frac{1}{k_1 k_2} \sum_{x_1, x_2, y} W(y|x_1 x_2) r_{x_1, x_2, y} = S^*$ .

First we have  $S = S^*$ . Indeed:

$$\begin{aligned} S &= \frac{1}{k_1 k_2} \sum_{x_1, x_2, y} W(y|x_1 x_2) r_{x_1, x_2, y} = \frac{1}{k_1 k_2} \sum_{x_1, x_2, y} W(y|x_1 x_2) \frac{r_{(x_1, x_2, y)^G}}{|r_{(x_1, x_2, y)^G}|} \\ &= \frac{1}{k_1 k_2} \sum_w \sum_{(x_1, x_2, y) \in w} W(y|x_1 x_2) \frac{r_w}{|w|} = \frac{1}{k_1 k_2} \sum_w \sum_{(x_1, x_2, y) \in w} W(w) \frac{r_w}{|w|} \\ &= \frac{1}{k_1 k_2} \sum_w |w| W(w) \frac{r_w}{|w|} = \frac{1}{k_1 k_2} \sum_w W(w) r_w = S^*. \end{aligned} \quad (29)$$

Then, all the constraints are satisfied. Indeed, thanks to Lemma 3.12, we have for the first constraint:

$$\sum_{x_1, x_2} r_{x_1, x_2, y} = \frac{1}{|y^G|} \sum_{w: w \mathcal{Y} = y^G} r_w = \frac{|y^G|}{|y^G|} = 1. \quad (30)$$

For the second constraint (and symmetrically for the third constraint), we have:

$$\sum_{x_1} r_{x_1, x_2, y}^1 = \frac{1}{|(x_2, y)^G|} \sum_{w: w \mathcal{X}_2 \mathcal{Y} = (x_2, y)^G} r_w^1 = \frac{k_1}{|(x_2, y)^G|} \sum_{w: w \mathcal{X}_2 \mathcal{Y} = (x_2, y)^G} r_w = k_1 \sum_{x_1} r_{x_1, x_2, y}. \quad (31)$$



For the fourth (and symmetrically for the fifth), we have:

$$\begin{aligned}
\sum_{x_1} r_{x_1, x_2, y}^2 &= \frac{1}{|(x_2, y)^G|} \sum_{w: w\mathcal{X}_2 y = (x_2, y)^G} r_w^2 = \frac{1}{|(x_2, y)^G|} \frac{1}{k_1} \frac{|(x_2, y)^G|}{|(x_2, y)_{\mathcal{X}_2}^G|} \sum_{u: u\mathcal{X}_2 = (x_2, y)_{\mathcal{X}_2}^G} p_u \\
&= \frac{1}{k_1} \frac{1}{|(x_2, y)_{\mathcal{X}_2}^G|} \sum_{u: u\mathcal{X}_2 = (x_2, y)_{\mathcal{X}_2}^G} p_u = \frac{1}{k_1} \frac{1}{|x_2^G|} \sum_{u: u\mathcal{X}_2 = x_2^G} p_u \text{ since } (x_2, y)_{\mathcal{X}_2}^G = x_2^G \quad (32) \\
&= \frac{1}{k_1} \sum_{x_1} p_{x_1, x_2} .
\end{aligned}$$

Finally, to conclude with the last constraints, one has only to see that for any  $x_1, x_2, y$ :

$$\frac{|(x_1, x_2, y)^G|}{|(x_1, x_2, y)_{\mathcal{X}_1 \mathcal{X}_2}^G|} p_{(x_1, x_2, y)_{\mathcal{X}_1 \mathcal{X}_2}^G} = \frac{|(x_1, x_2, y)^G|}{|(x_1, x_2)^G|} p_{(x_1, x_2)^G} = |(x_1, x_2, y)^G| p_{x_1, x_2} , \quad (33)$$

which implies that the linear inequalities on  $\frac{|w|}{|w\mathcal{X}_1 \mathcal{X}_2|} p_{w\mathcal{X}_1 \mathcal{X}_2}, r_w, r_w^1, r_w^2$  get transposed respectively to the values  $p_{x_1, x_2}, r_{x_1, x_2, y}, r_{x_1, x_2, y}^1, r_{x_1, x_2, y}^2$ . Indeed, for instance, one has for any  $w$  that  $\frac{|w|}{|w\mathcal{X}_1 \mathcal{X}_2|} p_{w\mathcal{X}_1 \mathcal{X}_2} - r_w^1 - r_w^2 + r_w \geq 0$ . But for any  $(x_1, x_2, y) \in w$ , one has that  $r_{x_1, x_2, y} = \frac{r_w}{|w|}, r_{x_1, x_2, y}^1 = \frac{r_w^1}{|w|}, r_{x_1, x_2, y}^2 = \frac{r_w^2}{|w|}$ . Thanks to the previous inequality, we have that  $p_{x_1, x_2} = \frac{p_{w\mathcal{X}_1 \mathcal{X}_2}}{|w\mathcal{X}_1 \mathcal{X}_2|}$ , and thus:

$$p_{x_1, x_2} - r_{x_1, x_2, y}^1 - r_{x_1, x_2, y}^2 + r_{x_1, x_2, y} = \frac{p_{w\mathcal{X}_1 \mathcal{X}_2}}{|w\mathcal{X}_1 \mathcal{X}_2|} - \frac{r_w^1}{|w|} - \frac{r_w^2}{|w|} + \frac{r_w}{|w|} \geq 0 ,$$

which was what we wanted to show.  $\square$

## 4 Application to the Binary Adder Channel

The binary adder channel  $W_{\text{BAC}}$  is the following MAC:

$$\forall x_1, x_2 \in \{0, 1\}, \forall y \in \{0, 1, 2\}, W_{\text{BAC}}(y|x_1 x_2) := \delta_{y, x_1 + x_2} .$$

Its classical capacity region  $\mathcal{C}(W_{\text{BAC}})$  is well known and consists of all  $(R_1, R_2)$  such that  $R_1 \leq 1, R_2 \leq 1, R_1 + R_2 \leq \frac{3}{2}$ , as a consequence of Theorem 1.1. Its zero-error classical capacity  $\mathcal{C}_0(W_{\text{BAC}})$  is not yet characterized. A lot of work has been done in finding upper and lower bounds on this region [16, 24, 12, 11, 13, 23, 8, 22, 2, 17, 20]. To date, the best lower bound on the sum-rate capacity is  $\log_2(240/6) \simeq 1.3178$  [17].

Thanks to Corollary 3.10, we were able to compute all the values of  $\text{S}^{\text{NS}}(W_{\text{BAC}}^n, k_1, k_2)$  up to  $n = 6$ . In particular, taking the largest  $k_1, k_2$  where  $\text{S}^{\text{NS}}(W_{\text{BAC}}^n, k_1, k_2) = 1$  gave us lower bounds on the zero-error non-signaling assisted capacity region  $\mathcal{C}_0^{\text{NS}}(W_{\text{BAC}})$ . Finally, for  $n = 7$ , we were able to compute directly the border of the zero-error non-signaling assisted capacity region, which led to Figure 3. The code can be found on GitHub. It uses Mosek linear programming solver [3].

Note that the linear program from Theorem 3.9 has still a large number of variables and constraints although polynomial in  $n$ . Specifically, for  $n = 2$ , it has 244 variables and 480 constraints; for  $n = 3$ , it has 1112 variables and 2054 constraints; for  $n = 7$ , it has 95592 variables and 162324 constraints; finally, for  $n = 8$ , it has 226911 variables and 383103 constraints.

The first noticeable result coming from these curves is that the zero-error non-signaling assisted sum-rate capacity beats with only 7 copies the classical sum-rate capacity of  $\frac{3}{2}$ , even without a zero-error

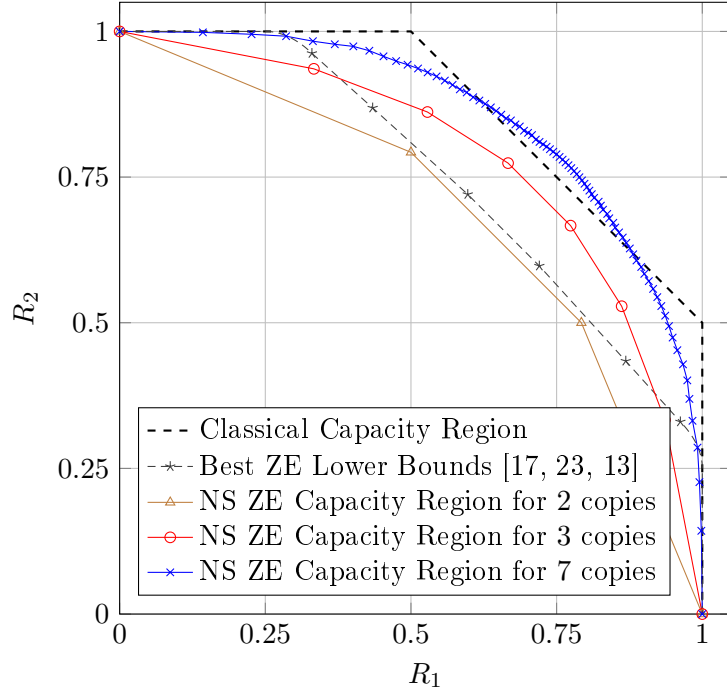


Figure 3: Capacity regions of the binary adder channel  $W_{\text{BAC}}$ . The black dashed curve depicts the classical capacity region  $\mathcal{C}(W_{\text{BAC}})$ , whereas the grey dashed curve shows the best known lower bound border on the zero-error classical capacity region  $\mathcal{C}_0(W_{\text{BAC}})$ , made from results by [17, 23, 13]; see [17] for a description of this border. On the other hand, the continuous curves depict the best zero-error non-signaling assisted achievable rate pairs for respectively 2, 3 and 7 copies of the binary adder channel.

constraint, with a value of  $\frac{2\log_2(42)}{7} \simeq 1.5406$ , coming from the fact that  $S^{\text{NS}}(W_{\text{BAC}}^{\otimes 7}, 42, 42) = 1$  and Proposition 3.7. This implies that  $\mathcal{C}_0^{\text{NS}}(W_{\text{BAC}})$  has larger sum-rate pairs than  $\mathcal{C}(W_{\text{BAC}})$ , and that  $\mathcal{C}^{\text{NS}}(W_{\text{BAC}})$  is strictly larger than  $\mathcal{C}(W_{\text{BAC}})$ . This sum-rate can even be increased up to  $\frac{\log_2(72)}{4} \simeq 1.5425$ , since we have computed  $S^{\text{NS}}(W_{\text{BAC}}^{\otimes 8}, 72, 72) = 1$ , which is the largest number of copies we have been able to manage with our efficient version of the linear program from Theorem 3.9. This should be compared with the only upper bound known on the non-signaling assisted sum-rate coming from Corollary 3.6 and given by  $\log_2(|\mathcal{Y}|) = \log_2(3) \simeq 1.5850$ .

Another astonishing result is the speed at which one obtains efficient zero-error non-signaling assisted codes compared to classical zero-error codes. Indeed, with only three copies of the binary adder channel, one gets that  $S^{\text{NS}}(W_{\text{BAC}}^{\otimes 3}, 4, 5) = 1$ , which corresponds to a sum-rate of  $\frac{2+\log_2(5)}{3} \simeq 1.4406$ , which already largely beats the best known zero-error achieved sum-rate of  $\log_2(240/6) \simeq 1.3178$  [17]. These results are summarized in the following theorem:

**Theorem 4.1.** We have that  $\left(\frac{\log_2(72)}{8}, \frac{\log_2(72)}{8}\right) \in \mathcal{C}_0^{\text{NS}}(W_{\text{BAC}})$  but  $\left(\frac{\log_2(72)}{8}, \frac{\log_2(72)}{8}\right) \notin \mathcal{C}(W_{\text{BAC}})$ , and as a consequence, we have that  $\mathcal{C}(W_{\text{BAC}}) \subsetneq \mathcal{C}^{\text{NS}}(W_{\text{BAC}})$ .

*Proof.* Since  $2^{8\frac{\log_2(72)}{8}} = 72$  and numerically  $S^{\text{NS}}(W_{\text{BAC}}^{\otimes 8}, 72, 72) = 1$  thanks to Corollary 3.10, we get that  $\left(\frac{\log_2(72)}{8}, \frac{\log_2(72)}{8}\right) \in \mathcal{C}_0^{\text{NS}}(W_{\text{BAC}})$  by Proposition 3.7. However,  $\frac{\log_2(72)}{8} + \frac{\log_2(72)}{8} > \frac{3}{2}$  so  $\left(\frac{\log_2(72)}{8}, \frac{\log_2(72)}{8}\right) \notin \mathcal{C}(W_{\text{BAC}})$  by Theorem 1.1 applied to  $W_{\text{BAC}}$ . Since  $\mathcal{C}(W_{\text{BAC}}) \subseteq \mathcal{C}^{\text{NS}}(W_{\text{BAC}})$  and  $\mathcal{C}_0^{\text{NS}}(W_{\text{BAC}}) \subseteq \mathcal{C}^{\text{NS}}(W_{\text{BAC}})$ , we thus get that  $\mathcal{C}(W_{\text{BAC}}) \subsetneq \mathcal{C}^{\text{NS}}(W_{\text{BAC}})$ .  $\square$

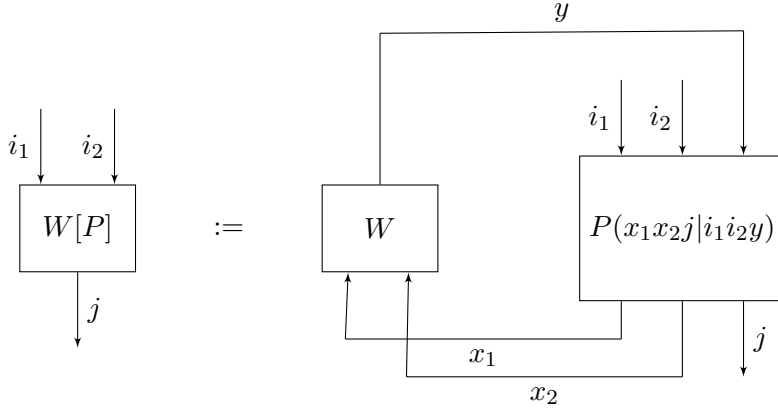
## 5 Handling Errors with Concatenated Codes

In Section 4, we have analyzed the non-signaling assisted capacity region through zero-error strategies in the case of the BAC, which is in particular deterministic. However, if some noise is added to that channel, its zero-error non-signaling assisted capacity region becomes trivial (see Proposition 5.3). Thus, the previous method fails to find significant lower bounds on the non-signaling assisted capacity region of noisy MACs.

In this section, we will present a new technique to overcome this difficulty using concatenated codes, and apply it to a noisy version of the BAC.

### 5.1 Concatenated Codes

Given a MAC  $W$  and a non-signaling code  $P$ , define  $W[P] : [k_1] \times [k_2] \rightarrow [\ell]$  with  $W[P](j|i_1 i_2) := \sum_{x_1, x_2, y} W(y|x_1 x_2) P(x_1 x_2 j|i_1 i_2 y)$ :



Note that  $W[P]$  is a MAC since  $W[P](j|i_1 i_2) \geq 0$  and:

$$\begin{aligned} \sum_j W[P](j|i_1 i_2) &= \sum_{x_1, x_2, y} W(y|x_1 x_2) \sum_j P(x_1 x_2 j|i_1 i_2 y) \\ &= \sum_{x_1, x_2} \left( \sum_y W(y|x_1 x_2) \right) P(x_1 x_2|i_1 i_2) \text{ since } P \text{ is non-signaling} \\ &= \sum_{x_1, x_2} P(x_1 x_2|i_1 i_2) = 1 . \end{aligned}$$

The following proposition states that combining a classical code to a non-signaling strategy leads to lower bounds on the non-signaling assisted capacity region of a MAC:

**Proposition 5.1.** If  $P$  is a non-signaling code for the MAC  $W$ , we have that  $\mathcal{C}(W[P]) \subseteq \mathcal{C}^{\text{NS}}(W)$ .

*Proof.* Let  $(R_1, R_2) \in \mathcal{C}(W[P])$ . Then, by definition, we have that:

$$\lim_{n \rightarrow +\infty} S(W[P]^{\otimes n}, \lceil 2^{R_1 n} \rceil, \lceil 2^{R_2 n} \rceil) = 1 .$$

Let us fix  $\varepsilon > 0$ . There exists some rank  $N$  such that  $S(W[P]^{\otimes N}, \lceil 2^{R_1 N} \rceil, \lceil 2^{R_2 N} \rceil) \geq 1 - \varepsilon$ . Let us call  $\ell_1 := \lceil 2^{R_1 N} \rceil$  and  $\ell_2 := \lceil 2^{R_2 N} \rceil$ . Thus, there exists encoders  $e_1 : [\ell_1] \rightarrow [k_1]$ ,  $e_2 : [\ell_2] \rightarrow [k_2]$  and a decoder  $d : [\ell] \rightarrow [\ell_1] \times [\ell_2]$  such that:

$$\frac{1}{\ell_1 \ell_2} \sum_{i_1, i_2, j} W[P](j|i_1 i_2) \sum_{a_1 \in [\ell_1], a_2 \in [\ell_2]} e_1(i_1|a_1) e_2(i_2|a_2) d(a_1 a_2|j) \geq 1 - \varepsilon .$$

In particular, we have:

$$\frac{1}{\ell_1 \ell_2} \sum_{x_1, x_2, y} W(y|x_1 x_2) \left[ \sum_{i_1, i_2, j, a_1, a_2} P(x_1 x_2 j | i_1 i_2 y) e_1(i_1 | a_1) e_2(i_2 | a_2) d(a_1 a_2 | j) \right] \geq 1 - \varepsilon .$$

Let us define  $\hat{P}(x_1 x_2 (b_1 b_2) | a_1 a_2 y) := \sum_{i_1, i_2, j} P(x_1 x_2 j | i_1 i_2 y) e_1(i_1 | a_1) e_2(i_2 | a_2) d(b_1 b_2 | j)$ . Then, one can easily check that  $\hat{P}$  is non-signaling, and thus:

$$S^{\text{NS}}(W^{\otimes n}, \ell_1, \ell_2) \geq \frac{1}{\ell_1 \ell_2} \sum_{x_1, x_2, y} W(y|x_1 x_2) \sum_{a_1, a_2} \hat{P}(x_1 x_2 (a_1, a_2) | a_1 a_2 y) \geq 1 - \varepsilon .$$

This implies that  $\lim_{n \rightarrow +\infty} S^{\text{NS}}(W^{\otimes n}, [2^{R_1 n}], [2^{R_2 n}]) = 1$ , ie.  $(R_1, R_2) \in \mathcal{C}^{\text{NS}}(W)$ .  $\square$

Thanks to Proposition 5.1, we have for any non-signaling code  $P$ ,  $\mathcal{C}(W^{\otimes n}[P]) \subseteq \mathcal{C}^{\text{NS}}(W^{\otimes n})$ . But if  $(R_1, R_2) \in \mathcal{C}^{\text{NS}}(W^{\otimes n})$ , we have that  $(\frac{R_1}{n}, \frac{R_2}{n}) \in \mathcal{C}^{\text{NS}}(W)$ . Thus, applying Theorem 1.1 on  $W^{\otimes n}[P]$  leads to lower bounds on  $\mathcal{C}^{\text{NS}}(W)$ :

**Proposition 5.2** (Numerical Method to find lower bounds on  $\mathcal{C}^{\text{NS}}(W)$ ). For any number of copies  $n$ , number of inputs  $k_1 \in [|\mathcal{X}_1|^n]$  and  $k_2 \in [|\mathcal{X}_2|^n]$ , non-signaling codes  $P$  on inputs in  $[k_1], [k_2]$  for  $W^{\otimes n}$ , and distributions  $q_1, q_2$  on  $[k_1], [k_2]$ , we have that the following  $(R_1, R_2)$  are in  $\mathcal{C}^{\text{NS}}(W)$ :

$$R_1 \leq \frac{I(I_1 : J | I_2)}{n} , R_2 \leq \frac{I(I_2 : J | I_1)}{n} , R_1 + R_2 \leq \frac{I((I_1, I_2) : J)}{n} ,$$

for  $(I_1, I_2) \in [k_1] \times [k_2]$  following the product law  $q_1 \times q_2$ , and  $J \in [\ell]$  the outcome of  $W^{\otimes n}[P]$  on inputs  $I_1, I_2$ . In particular, the corner points of this capacity region are given by:

$$\left( \frac{I(I_1 : J | I_2)}{n}, \frac{I(I_2 : J)}{n} \right) \text{ and } \left( \frac{I(I_1 : J)}{n}, \frac{I(I_2 : J | I_1)}{n} \right) .$$

## 5.2 Application to the Noisy Binary Adder Channel

We will now apply this strategy to a noisy version of the BAC. We will consider flip errors  $\varepsilon_1, \varepsilon_2$  of inputs  $x_1, x_2$  on  $W_{\text{BAC}}$ , which leads to the following definition of  $W_{\text{BAC}, \varepsilon_1, \varepsilon_2}$ :

$$\begin{aligned} \forall y, x_1, x_2, W_{\text{BAC}, \varepsilon_1, \varepsilon_2}(y|x_1 x_2) &:= (1 - \varepsilon_1)(1 - \varepsilon_2) W_{\text{BAC}}(y|x_1 x_2) \\ &+ \varepsilon_1(1 - \varepsilon_2) W_{\text{BAC}}(y|\bar{x}_1 x_2) \\ &+ (1 - \varepsilon_1)\varepsilon_2 W_{\text{BAC}}(y|x_1 \bar{x}_2) \\ &+ \varepsilon_1 \varepsilon_2 W_{\text{BAC}}(y|\bar{x}_1 \bar{x}_2) . \end{aligned}$$

First, let us note that the zero-error non-signaling assisted capacity region of  $W_{\text{BAC}, \varepsilon_1, \varepsilon_2}$  is trivial, which won't help us to understand  $\mathcal{C}^{\text{NS}}(W_{\text{BAC}, \varepsilon_1, \varepsilon_2})$ :

**Proposition 5.3.** If  $\varepsilon_1, \varepsilon_2 \in (0, 1)$ , then  $\mathcal{C}_0^{\text{NS}}(W_{\text{BAC}, \varepsilon_1, \varepsilon_2}) = \{(0, 0)\}$ .

*Proof.* If  $S^{\text{NS}}(W^{\otimes n}, k_1, k_2) = 1$ , then  $\forall y^n, x_1^n, x_2^n : W^{\otimes n}(y^n | x_1^n x_2^n) > 0 \implies r_{x_1^n, x_2^n, y^n} = p_{x_1^n, x_2^n}$ . Indeed, we have for an optimal  $p, r$  that:

$$S^{\text{NS}}(W^{\otimes n}, k_1, k_2) = \frac{1}{k_1 k_2} \sum_{x_1^n, x_2^n, y^n} W^{\otimes n}(y^n | x_1^n x_2^n) r_{x_1^n, x_2^n, y^n} \leq \frac{1}{k_1 k_2} \sum_{x_1^n, x_2^n, y^n} W^{\otimes n}(y^n | x_1^n x_2^n) p_{x_1^n, x_2^n} = 1 ,$$

which implies the previous statement. But, for  $W_{\text{BAC},\varepsilon_1,\varepsilon_2}^{\otimes n}$ , one can easily check that for all  $y^n, x_1^n, x_2^n$ ,  $W^{\otimes n}(y^n|x_1^n x_2^n) > 0$  since  $\varepsilon_1, \varepsilon_2 \in (0, 1)$ . Indeed, you just have to flip the inputs to a valid preimage of the output. Thus if  $S^{\text{NS}}(W_{\text{BAC},\varepsilon_1,\varepsilon_2}^{\otimes n}, k_1, k_2) = 1$ , we have that  $\forall y^n, x_1^n, x_2^n, r_{x_1^n, x_2^n, y^n} = p_{x_1^n, x_2^n}$ . In particular, this implies that  $\sum_{x_1^n, x_2^n} r_{x_1^n, x_2^n, y^n} = \sum_{x_1^n, x_2^n} p_{x_1^n, x_2^n}$ , therefore  $1 = k_1 k_2$ , so  $k_1 = 1$  and  $k_2 = 1$ . Thus  $S^{\text{NS}}(W^{\otimes n}, 2^{nR_1}, 2^{nR_2}) = 1$  implies that  $(R_1, R_2) = (0, 0)$ .  $\square$

We have then applied the numerical method described in Proposition 5.2 to  $W_{\text{BAC},\varepsilon_1,\varepsilon_2}$  for the symmetric case  $\varepsilon_1 = \varepsilon_2 = \varepsilon := 10^{-3}$ . Since it is hard to go through all non-signaling codes  $P$  and product distributions  $q_1, q_2$ , we have applied the heuristic of using non-signaling codes obtained while optimizing  $S^{\text{NS}}(W^{\otimes n}, k_1, k_2)$  in the symmetrized linear program. We have combined them with uniform  $q_1, q_2$ , as the form of those non-signaling codes coming from our optimization program is symmetric. We have evaluated the achievable corner points for all  $k_1, k_2 \leq 2^n$  for  $n \leq 5$  copies which led to Figure 4:

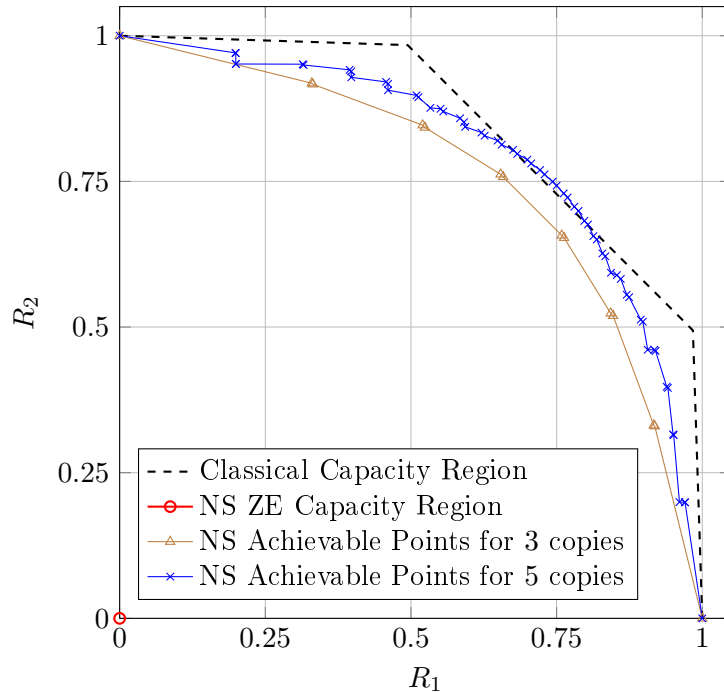


Figure 4: Capacity regions of the noisy binary adder channel  $W_{\text{BAC},\varepsilon,\varepsilon}$  for  $\varepsilon = 10^{-3}$ . The black dashed curve depicts the classical capacity region  $\mathcal{C}(W_{\text{BAC},\varepsilon,\varepsilon})$  which was found numerically using Theorem 1.1. The red point depicts the zero-error non-signaling assisted capacity region (Proposition 5.3). The blue curve depicts achievable non-signaling rates pairs obtained from  $\mathcal{C}(W_{\text{BAC},\varepsilon,\varepsilon}^{\otimes 5}[P])$  through the numerical method described in Proposition 5.2.

Compared to the noiseless BAC, we can first notice that the classical capacity region is slightly smaller, with a classical sum-rate capacity of 1.478 at most. On the other hand, although the zero-error non-signaling assisted capacity of  $W_{\text{BAC},\varepsilon,\varepsilon}$  is completely trivial, we have with our concatenated codes strategy found significant rate pairs achievable with non-signaling assistance. In particular, we have reached a non-signaling assisted sum-rate capacity of 1.493 which beats the best classical sum-rate capacity. Thus, it shows that non-signaling assistance can improve the capacity of the noisy binary adder channel as well.

## 6 Conclusion

In this work, we have studied the impact of non-signaling assistance on the capacity of multiple-access channels. We have developed an efficient linear program computing the success probability of the best non-signaling assisted code for a finite number of copies of a multiple-access channel. In particular, this gives lower bounds on the zero-error non-signaling assisted capacity of multiple-access channels. Applied to the binary adder channel, these results were used to prove that a sum-rate of  $\frac{\log_2(72)}{4} \simeq 1.5425$  can be reached with zero error, which beats the maximum classical sum-rate capacity of  $\frac{3}{2}$ . For noisy channels, we have developed a technique giving lower bounds through the use of concatenated codes. Applied to the noisy binary adder channel, this technique was used to show that non-signaling assistance still improves the sum-rate capacity.

Our results suggest that quantum entanglement may also increase the capacity of such channels. However, even for the binary adder channel, this question remains open. One could also ask if such efficient methods to compute the best non-signaling assisted codes can be extended to Gaussian multiple-access channels. Finally, the existence of a single-letter formula for the non-signaling assisted capacity of multiple-access channels remains open, even for the binary adder channel. An intriguing question is whether the non-signaling assisted capacity of multiple access channels is given by the same expression as in Theorem 1.1 and dropping the requirement that  $X_1$  and  $X_2$  are independent.

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We would like to thank Mario Berta and Andreas Winter for discussions about the use of non-signaling correlations for multiple access channels. This work is funded by the European Research Council (ERC Grant Agreement No. 851716).

## References

- [1] Rudolf Ahlswede. Multi-way communication channels. In *2nd International Symposium on Information Theory, Tsahkadsor, Armenia, USSR, September 2-8, 1971*, pages 23–52. Publishing House of the Hungarian Academy of Science, 1973.
- [2] Rudolf Ahlswede and Vladimir B. Balakirsky. Construction of uniquely decodable codes for the two-user binary adder channel. *IEEE Trans. Inf. Theory*, 45(1):326–330, 1999. doi:10.1109/18.746834.
- [3] Erling D Andersen and Knud D Andersen. The Mosek interior point optimizer for linear programming: an implementation of the homogeneous algorithm. In *High performance optimization*, pages 197–232. Springer, 2000. doi:10.1007/978-1-4757-3216-0\_8.
- [4] PK Aravind. A simple demonstration of Bell’s theorem involving two observers and no probabilities or inequalities. *CoRR*, abs/quant-ph/0206070, 2002. arXiv:quant-ph/0206070.

- [5] Siddharth Barman and Omar Fawzi. Algorithmic aspects of optimal channel coding. *IEEE Trans. Inf. Theory*, 64(2):1038–1045, 2018. doi:10.1109/TIT.2017.2696963.
- [6] Charles H Bennett, Peter W Shor, John A Smolin, and Ashish V Thapliyal. Entanglement-assisted classical capacity of noisy quantum channels. *Physical Review Letters*, 83(15):3081, 1999. doi:10.1103/PhysRevLett.83.3081.
- [7] Gilles Brassard, Anne Broadbent, and Alain Tapp. Quantum pseudo-telepathy. *Foundations of Physics*, 35(11):1877–1907, 2005. doi:10.1007/s10701-005-7353-4.
- [8] Shraga I. Bross and Ian F. Blake. Upper bound for uniquely decodable codes in a binary input N-user adder channel. *IEEE Trans. Inf. Theory*, 44(1):334–340, 1998. doi:10.1109/18.651062.
- [9] Thomas M. Cover and Joy A. Thomas. *Elements of Information Theory*. Wiley, 2001. doi:10.1002/0471200611.
- [10] Bernd Gärtner and Jirí Matousek. *Understanding and using linear programming*. Universitext. Springer, 2007. doi:10.1007/978-3-540-30717-4.
- [11] E. J. Weldon Jr. Coding for a multiple-access channel. *Inf. Control.*, 36(3):256–274, 1978. doi:10.1016/S0019-9958(78)90312-1.
- [12] Tadao Kasami and Shu Lin. Bounds on the achievable rates of block coding for a memoryless multiple-access channel. *IEEE Trans. Inf. Theory*, 24(2):187–197, 1978. doi:10.1109/TIT.1978.1055860.
- [13] Tadao Kasami, Shu Lin, Victor K.-W. Wei, and Saburo Yamamura. Graph theoretic approaches to the code construction for the two-user multiple-access binary adder channel. *IEEE Trans. Inf. Theory*, 29(1):114–130, 1983. doi:10.1109/TIT.1983.1056614.
- [14] Felix Leditzky, Mohammad A Alhejji, Joshua Levin, and Graeme Smith. Playing games with multiple access channels. *Nature communications*, 11(1):1–5, 2020. doi:10.1038/s41467-020-15240-w.
- [15] Henry Herg-Jiunn Liao. Multiple access channels (Ph.D. Thesis abstr.). *IEEE Trans. Inf. Theory*, 19(2):253, 1973. doi:10.1109/TIT.1973.1054960.
- [16] Bernt Lindström. Determination of two vectors from the sum. *Journal of Combinatorial Theory*, 6(4):402–407, 1969. doi:10.1016/S0021-9800(69)80038-4.
- [17] M. Mattas and Patric R. J. Östergård. A new bound for the zero-error capacity region of the two-user binary adder channel. *IEEE Trans. Inf. Theory*, 51(9):3289–3291, 2005. doi:10.1109/TIT.2005.853309.
- [18] William Matthews. A linear program for the finite block length converse of Polyanskiy-Poor-Verdú via nonsignaling codes. *IEEE Trans. Inf. Theory*, 58(12):7036–7044, 2012. doi:10.1109/TIT.2012.2210695.
- [19] N David Mermin. Simple unified form for the major no-hidden-variables theorems. *Physical review letters*, 65(27):3373, 1990. doi:10.1103/PhysRevLett.65.3373.
- [20] Or Ordentlich and Ofer Shayevitz. A VC-dimension-based outer bound on the zero-error capacity of the binary adder channel. In *IEEE International Symposium on Information Theory, ISIT 2015, Hong Kong, China, June 14-19, 2015*, pages 2366–2370. IEEE, 2015. doi:10.1109/ISIT.2015.7282879.

- [21] Asher Peres. Incompatible results of quantum measurements. *Physics Letters A*, 151(3-4):107–108, 1990. doi:10.1016/0375-9601(90)90172-K.
- [22] Rüdiger Urbanke and Quinn Li. The zero-error capacity region of the 2-user synchronous BAC is strictly smaller than its Shannon capacity region. In *1998 Information Theory Workshop (Cat. No. 98EX131)*, page 61. IEEE, 1998. doi:10.1109/ITW.1998.706434.
- [23] P. A. B. M. Coebergh van den Braak and Henk C. A. van Tilborg. A family of good uniquely decodable code pairs for the two-access binary adder channel. *IEEE Trans. Inf. Theory*, 31(1):3–9, 1985. doi:10.1109/TIT.1985.1057004.
- [24] Henk C. A. van Tilborg. An upper bound for codes in a two-access binary erasure channel (Corresp.). *IEEE Trans. Inf. Theory*, 24(1):112–116, 1978. doi:10.1109/TIT.1978.1055814.