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Benefits of Rate-Sharing for Distributed Hypothesis Testing

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Abstract—We study distributed binary hypothesis testing with a single sensor and two remote decision centers that are also equipped with local sensors. The communication between the sensor and the two decision centers takes place over three links: a shared link to both centers and an individual link to each of the two centers. All communication links are subject to *expected* rate constraints. This paper characterizes the optimal exponents region of the type-II error for given type-I error thresholds at the two decision centers and further simplifies the expressions in the special case of having only the single shared link. The exponents region illustrates a gain under expected rate constraints compared to equivalent maximum rate constraints. Moreover, it exhibits a tradeoff between the exponents achieved at the two centers.

Index Terms—Broadcast channel, distributed hypothesis testing, error exponents, expected rate constraints, IoT, decision centers.

I. INTRODUCTION

We address a distributed hypothesis testing problem where different decision centers have to decide on the same hypothesis based on their local sensing and the messages they receive from remote sensors over rate-limited communication links. Motivated by systems that share bandwidth among several applications with variable instantaneous bandwidth for each application, we consider *expected-rate constraints* that limit only the expected bandwidth for each application.

In our work, we focus on distributed binary hypothesis testing against independence. The decision centers have to decide between a *i) null hypothesis* (normal situation) indicating that the centers' and the sensors' observations are correlated, and an *ii) alternative hypothesis* (alert situation) where the observations are independent, for example because one of the systems fails. Two types of errors can be distinguished: the *type-I error* indicates a wrong decision under the null hypothesis and the *type-II error* occurs if a wrong decision is made under the alternative hypothesis. Since the alternative hypothesis corresponds to a more critical situation, we aim at maximizing the exponential decay of the type-II error probability, called *error exponent*, subject to a type-I error that stays below a given threshold. Such a setup has been studied in many previous works focusing mostly on *maximum-rate constraints* [1]–[16]. *Expected-rate constraints* were introduced in [17], where the maximum error exponent for single-sensor single-decision center setup was characterized in the special case of testing-against independence. Extensions of this work were first proposed for a multi-sensor scenario in [18], for a multi-

hop scenario with multiple decision centers in [19], [20], for distributed sequential hypothesis testing with zero-rate [21], and most recently from a signal detection perspective in [22].

In this paper, we consider a single-sensor two-decision center scenario where the decision centers also have sensing capabilities. The communication takes place over three noise-free links: a common link to both decision centers and one private link to each decision center. For this one-to-many broadcast setup, we characterize the optimal exponents region under expected-rate constraints and we show that it improves over the exponents region under maximum-rate constraints, which we also establish in this paper. The optimal exponents region under expected-rate constraints illustrates two tradeoffs. The first tradeoff results from the shared link that has to serve both decision centers at the same time; this tradeoff is also present under maximum-rate constraints. The second tradeoff is particular to the setup with expected-rate constraints and stems from the rate-sharing between three different variants of the optimal coding scheme under maximum-rate constraints in [8], depending on the observations at the sensor. We show that two variants suffice when communication is only over a single shared link, leading to a significant reduction in the complexity of the optimal coding scheme.

Notation: We follow the notation in [23], [17]. In particular, we use sans serif font for bit-strings: e.g., m for a deterministic and M for a random bit-string, and we denote the length of m by $\text{len}(m)$. In addition, $\mathcal{T}_\mu^{(n)}(P)$ denotes the strongly μ -typical set with respect to P as defined in [24, Definition 2.8].

II. SYSTEM MODEL

Consider the distributed hypothesis testing problem in Figure 1 in the special case of testing against independence, i.e., depending on the binary hypothesis $\mathcal{H} \in \{0, 1\}$, the tuple (Y_0^n, Y_1^n, Y_2^n) is distributed as:

$$\text{under } \mathcal{H} = 0 : (Y_0^n, Y_1^n, Y_2^n) \text{ i.i.d. } \sim P_{Y_0} \cdot P_{Y_1 Y_2 | Y_0}; \quad (1a)$$

$$\text{under } \mathcal{H} = 1 : (Y_0^n, Y_1^n, Y_2^n) \text{ i.i.d. } \sim P_{Y_0} \cdot P_{Y_1 Y_2} \quad (1b)$$

for given probability mass functions (pmfs) P_{Y_0} and $P_{Y_1 Y_2 | Y_0}$ and where $P_{Y_1 Y_2}$ denotes the marginal of the joint pmf $P_{Y_0 Y_1 Y_2} := P_{Y_0} P_{Y_1 Y_2 | Y_0}$.

The system consists of a transmitter T_{Y_0} , and two receivers R_{Y_1}, R_{Y_2} . Transmitter T_{Y_0} observes the source sequence Y_0^n and computes three bit-string messages $(M_0, M_1, M_2) = \phi^{(n)}(Y_0^n)$,

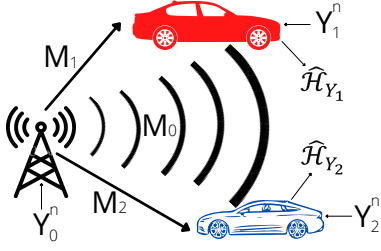


Fig. 1: Distributed hypothesis testing with a single sensor and two remote decision centers with integrated sensors.

where the encoding function is of the form $\phi^{(n)} : \mathcal{Y}_0^n \rightarrow \{0, 1\}^* \times \{0, 1\}^* \times \{0, 1\}^*$. Message M_0 is sent to both receivers R_{Y_1}, R_{Y_2} , while message M_1 only to receiver R_{Y_1} and message M_2 only to receiver R_{Y_2} . The messages have to satisfy the *expected-rate* constraints

$$\mathbb{E}[\text{len}(M_i)] \leq nR_i, \quad i \in \{0, 1, 2\}. \quad (2)$$

Receiver $R_{Y_i}, i \in \{1, 2\}$, observes the source sequence Y_i^n and with messages M_0, M_i received from T_{Y_0} , it produces a guess $\hat{\mathcal{H}}_{Y_i}$ of the hypothesis \mathcal{H} using a decision function $g_i^{(n)} : \mathcal{Y}_i^n \times \{0, 1\}^* \times \{0, 1\}^* \rightarrow \{0, 1\}$:

$$\hat{\mathcal{H}}_{Y_i} = g_i^{(n)}(Y_i^n, M_0, M_i) \in \{0, 1\}, \quad i \in \{1, 2\}. \quad (3)$$

The goal is to design encoding and decision functions such that their type-I error probabilities

$$\alpha_{i,n} \triangleq \Pr[\hat{\mathcal{H}}_{Y_i} = 1 | \mathcal{H} = 0], \quad i \in \{1, 2\}, \quad (4)$$

stay below given thresholds $\epsilon_i > 0, i \in \{1, 2\}$, and the type-II error probabilities

$$\beta_{i,n} \triangleq \Pr[\hat{\mathcal{H}}_{Y_i} = 0 | \mathcal{H} = 1] \quad (5)$$

decay to 0 with largest possible exponential decay.

Definition 1: Fix maximum type-I error probabilities $\epsilon_1, \epsilon_2 \in [0, 1]$ and rates $R_1, R_2 \geq 0$. The exponent pair (θ_1, θ_2) is called (ϵ_1, ϵ_2) -*achievable* if there exists a sequence of encoding and decision functions $\{\phi^{(n)}, g_1^{(n)}, g_2^{(n)}\}_{n \geq 1}$ satisfying:

$$\mathbb{E}[\text{len}(M_i)] \leq nR_i, \quad i \in \{0, 1, 2\} \quad (6a)$$

$$\lim_{n \rightarrow \infty} \alpha_{i,n} \leq \epsilon_i, \quad i \in \{1, 2\} \quad (6b)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{1}{\beta_{i,n}} \geq \theta_i, \quad i \in \{1, 2\}. \quad (6c)$$

Definition 2: The closure of the set of all (ϵ_1, ϵ_2) -achievable exponent pairs (θ_1, θ_2) is called the (ϵ_1, ϵ_2) -*exponents region* and is denoted $\mathcal{E}^*(R_0, R_1, R_2, \epsilon_1, \epsilon_2)$.

III. MAIN RESULTS

Our main results are a complete characterization of the exponents region $\mathcal{E}^*(R_0, R_1, R_2, \epsilon_1, \epsilon_2)$ under the *expected-rate* constraints in (2) as well as a strong converse under analogous *maximum-rate* constraints. A simplified expression is provided for $\mathcal{E}^*(R_0, 0, 0, \epsilon_1, \epsilon_2)$.

A. Individual and Common Communication Links

Theorem 1: The (ϵ_1, ϵ_2) -exponents region $\mathcal{E}^*(R_0, R_1, R_2, \epsilon_1, \epsilon_2)$ is the set of all (θ_1, θ_2) pairs satisfying

$$\theta_i \leq \min \{I(U_0^0 U_i^0; Y_i), I(U_0^i U_i^i; Y_i)\}, \quad i \in \{1, 2\} \quad (7a)$$

for some non-negative numbers $\sigma_0, \sigma_1, \sigma_2$ with sum ≤ 1 and conditional pmfs $P_{U_0^0|Y_0}, P_{U_0^1|Y_0}, P_{U_0^2|Y_0}, P_{U_1^0|U_0^0 Y_0}, P_{U_1^1|U_0^1 Y_0}, P_{U_2^0|U_0^0 Y_0}, P_{U_2^2|U_0^2 Y_0}$ satisfying

$$R_0 \geq \sigma_0 I(U_0^0; Y_0) + \sigma_1 I(U_0^1; Y_0) + \sigma_2 I(U_0^2; Y_0), \quad (7b)$$

$$R_i \geq \sigma_0 I(U_i^0; Y_0 | U_0^0) + \sigma_i I(U_i^i; Y_0 | U_0^i), \quad i \in \{1, 2\}, \quad (7c)$$

$$\text{and } \sigma_0 + \sigma_i \geq 1 - \epsilon_i, \quad i \in \{1, 2\}, \quad (7d)$$

$$\sigma_0 \geq 1 - \epsilon_1 - \epsilon_2, \quad (7e)$$

and where the mutual information quantities are calculated according to the joint pmfs

$$P_{Y_0 Y_1 Y_2 U_0^0 U_1^0 U_2^0} \triangleq P_{Y_0 Y_1 Y_2} P_{U_0^0|Y_0} P_{U_1^0|U_0^0 Y_0} \quad (8)$$

$$P_{Y_0 Y_1 Y_2 U_0^i U_i^i} \triangleq P_{Y_0 Y_1 Y_2} P_{U_0^i|Y_0} P_{U_i^i|U_0^i Y_0}, \quad i \in \{1, 2\}. \quad (9)$$

Proof: The converse is proved in Section IV. To prove achievability, define three sets $\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2 \subseteq \mathcal{Y}_0^n$ with probabilities (under $P_{Y_0}^n$) equal to $\sigma_0, \sigma_1, \sigma_2$, respectively. For each set \mathcal{D}_i ($i \in \{0, 1, 2\}$), we apply the optimal coding scheme under maximum rate constraints in [8], but for each set \mathcal{D}_i we construct different codebooks and use different auxiliaries U_0^i, U_1^i, U_2^i . In particular, we choose U_1^2 and U_2^1 constants, indicating that when $Y_0^n \in \mathcal{D}_i$ then only messages (M_0, M_i) are sent, for $i \in \{1, 2\}$. When $Y_0^n \in \mathcal{D}_0$, then all three messages M_0, M_1, M_2 are sent. For a detailed analysis, see [25]. ■

Theorem 1 shows a tradeoff between the two achievable exponents θ_1 and θ_2 . (Figure 2 ahead illustrates this tradeoff at hand of a numerical example in the special case $R_1 = R_2 = 0$.) The tradeoff stems from the common random variable U_0^0 that is included in the exponent constraint (7a) for both $i \in \{1, 2\}$, and from the rate-sharing of the coding scheme in [8] for three different choices of $(\sigma_i, U_0^i, U_1^i, U_2^i)$, for $i = 0, 1, 2$.

To see the effect of the expected-rate constraint in (2), we compare above exponents region $\mathcal{E}^*(R_0, R_1, R_2, \epsilon_1, \epsilon_2)$ with the exponents region $\mathcal{E}_{\text{fix}}^*(R_0, R_1, R_2, \epsilon_1, \epsilon_2)$ under more stringent maximum-length constraints

$$\text{len}(M_i) \leq nR_i, \quad i \in \{0, 1, 2\}. \quad (10)$$

In the limit $\epsilon_1, \epsilon_2 \downarrow 0$, the exponents region $\mathcal{E}_{\text{fix}}^*(R_0, R_1, R_2, \epsilon_1, \epsilon_2)$ was determined in [8]. Here, we strengthen this result by providing a strong converse, whose proof follows similar steps (but with the expected rate replaced by the maximum rate) as the converse to Theorem 1.

Theorem 2: Under the maximum rate constraints (10), the exponents region $\mathcal{E}_{\text{fix}}^*(R_0, R_1, R_2, \epsilon_1, \epsilon_2)$ is independent of (ϵ_1, ϵ_2) and equals the set of (θ_1, θ_2) pairs satisfying:

$$\theta_i \leq I(U_0 U_i; Y_i), \quad i \in \{1, 2\}, \quad (11a)$$

for some conditional pmfs $P_{U_0|Y_0}, P_{U_i|Y_0}$ satisfying

$$R_0 \geq I(U_0; Y_0), \quad (11b)$$

$$R_i \geq I(U_i; Y_0 | U_0), \quad i \in \{1, 2\}. \quad (11c)$$

Proof: Achievability is proved in [8]. The converse is proved in [25]. ■

Notice that (11) is obtained from (7) by setting $\sigma_0 = 1$ and $U_0^0, U_0^1, U_0^2, U_0^2$ constants. Moreover, $\mathcal{E}_{\text{fix}}^*(R_0, R_1, R_2, \epsilon_1, \epsilon_2) = \mathcal{E}^*(R_0, R_1, R_2, 0, 0)$. Since $\mathcal{E}^*(R_0, R_1, R_2, \epsilon_1, \epsilon_2)$ is generally increasing in (ϵ_1, ϵ_2) , expected-rate constraints allow to boost the exponents region compared to maximum-rate constraints.

B. Only a Common Communication Link

For $R_1 = R_2 = 0$, i.e., without individual communication links, we can simplify the expression for $\mathcal{E}^*(R_0, R_1, R_2, \epsilon_1, \epsilon_2)$.

Definition 3: Define the two functions

$$\eta_i(R_0^i) := \max_{\substack{P_{U_0^i|Y_0}: \\ R_0^i \geq I(U_0^i; Y_0)}} I(U_0^i; Y_i), \quad i \in \{1, 2\}, \quad (12)$$

where the mutual information quantities are calculated with respect to the joint pmf $P_{U_0^i Y_0 Y_1 Y_2} \triangleq P_{U_0^i|Y_0} P_{Y_0 Y_1 Y_2}$.

Corollary 1: Let $\pi: \{1, 2\} \rightarrow \{1, 2\}$ be a permutation ordering the ϵ -values in decreasing order:

$$\epsilon_{\pi(1)} \geq \epsilon_{\pi(2)}. \quad (13)$$

Then $\mathcal{E}^*(R_0, 0, 0, \epsilon_1, \epsilon_2)$ is the set of all (θ_1, θ_2) pairs satisfying

$$\theta_{\pi(1)} \leq I(U_0; Y_{\pi(1)}), \quad (14a)$$

$$\theta_{\pi(2)} \leq \min \{I(U_0; Y_{\pi(2)}), \eta_{\pi(2)}(R_0^{\pi(2)})\}, \quad (14b)$$

for some conditional pmf $P_{U_0|Y_0}$ and rate $R_0^{\pi(2)}$ satisfying

$$R_0 \geq (1 - \epsilon_{\pi(1)})I(U_0; Y_0) + (\epsilon_{\pi(1)} - \epsilon_{\pi(2)})R_0^{\pi(2)}. \quad (14c)$$

Proof: See Appendix A. ■

The following example illustrates the benefits of expected-rate constraints versus maximum-rate constraints, and the trade-off between the two exponents when $R_1 = R_2 = 0$.

Example 1: Consider the following joint pmf $P_{Y_0 Y_1 Y_2}$:

	(Y_1, Y_2)			
Y_0	(0, 0)	(0, 1)	(1, 0)	(1, 1)
0	0.05	0.05	0.15	0.083325
1	0.05	0.15	0.05	0.08335
2	0.15	0.05	0.05	0.083325

For this pmf, Figure 2 shows the optimal exponents regions under maximum- and expected-rate constraints when $R_0 = 0.1$ and $\epsilon_1 = 0.15 > \epsilon_2 = 0.05$. The figure illustrates the boost in the exponents region due to the *expected*-rate constraints. It also emphasizes the benefits of sharing the rate in (14c) between two summands, which relate to the fact that depending on the observation Y_0^n we use two variants of the coding scheme in [8], one with auxiliary U_0 and the other with an auxiliary $U_0^{\pi(2)}$ that satisfies $I(U_0^{\pi(2)}; Y_0) \leq R_0^{\pi(2)}$ and $I(U_0^{\pi(2)}; Y_1) = \eta_{\pi(2)}(R_0^{\pi(2)})$. Restricting to a single auxiliary U_0 in (14) (i.e., setting $R_0^{\pi(2)} = I(U_0; Y_0)$) results in an exponents region, denoted $\mathcal{E}_{\text{no-RS}}(R_0, 0, 0, \epsilon_1, \epsilon_2)$ which coincides with $\mathcal{E}^*(R_0, 0, 0, \epsilon_2, \epsilon_2)$ and $\mathcal{E}_{\text{fix}}^*((1 - \epsilon_2)^{-1}R_0, 0, 0, \epsilon_1, \epsilon_2)$.

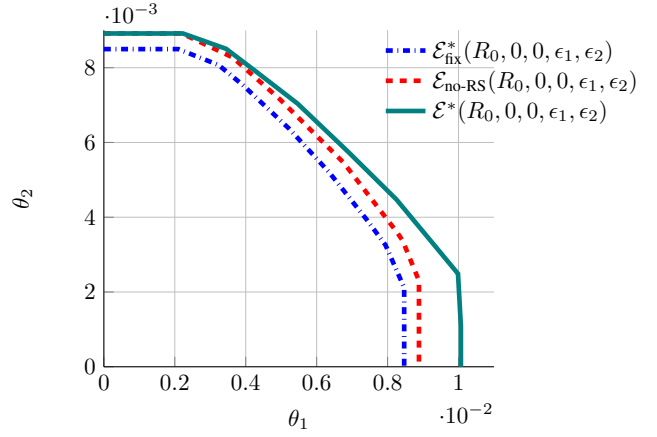


Fig. 2: Optimal error exponents regions under expected- and maximum-rate constraints for $R_0 = 0.1, \epsilon_1 = 0.15, \epsilon_2 = 0.05$.

IV. CONVERSE PROOF TO THEOREM 1

Fix an exponent pair in $\mathcal{E}^*(R_0, R_1, R_2, \epsilon_1, \epsilon_2)$ and a sequence (in n) of encoding and decision functions $\{(\phi^{(n)}, g_1^{(n)}, g_2^{(n)})\}$ satisfying the constraints on the rate and the error probabilities in (6). Our proof relies on the following lemma:

Lemma 1: Fix a blocklength n and a set $\mathcal{D} \subseteq \mathcal{Y}_0^n$ of positive probability, and let the tuple $(\tilde{M}_0, \tilde{M}_1, \tilde{M}_2, \tilde{Y}_0^n, \tilde{Y}_1^n, \tilde{Y}_2^n)$ follow the pmf

$$P_{\tilde{M}_0 \tilde{M}_1 \tilde{M}_2 \tilde{Y}_0^n \tilde{Y}_1^n \tilde{Y}_2^n}(\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2, y_0^n, y_1^n, y_2^n) \triangleq P_{Y_0^n Y_1^n Y_2^n}(y_0^n, y_1^n, y_2^n) \cdot \frac{\mathbb{1}\{y_0^n \in \mathcal{D}\}}{P_{Y_0^n}(\mathcal{D})} \cdot \mathbb{1}\{\phi^{(n)}(y_0^n) = (\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2)\}. \quad (15)$$

Further, define $U_0 \triangleq (\tilde{M}_0, \tilde{Y}_0^{T-1}, T)$, $U_1 \triangleq \tilde{M}_1$, $U_2 \triangleq \tilde{M}_2$, $\tilde{Y}_i \triangleq \tilde{Y}_{i,T}$ (for $i \in \{0, 1, 2\}$), where T is uniform over $\{1, \dots, n\}$ and independent of all other random variables. Notice the Markov chain $(U_0, U_1, U_2) \rightarrow \tilde{Y}_0 \rightarrow (\tilde{Y}_1, \tilde{Y}_2)$. Then the following inequalities hold:

$$H(\tilde{M}_0) \geq nI(U_0; \tilde{Y}_0) + \log P_{Y_0^n}(\mathcal{D}), \quad (16)$$

$$H(\tilde{M}_i) \geq nI(U_i; \tilde{Y}_0 | U_0), \quad i \in \{1, 2\}. \quad (17)$$

Let $\eta > 0$ be arbitrary. For $i \in \{1, 2\}$, if

$$\Pr[\hat{\mathcal{H}}_{Y_i} = 0 | \mathcal{H} = 0, Y_0^n = y_0^n] \geq \eta, \quad \forall y_0^n \in \mathcal{D}, \quad (18)$$

then

$$-\frac{1}{n} \log \beta_{i,n} \leq I(U_0 U_i; \tilde{Y}_i) + \phi_i(n), \quad (19)$$

where $\phi_i(n)$ is a function that tends to 0 as $n \rightarrow \infty$.

Proof: Similar to the proof of [20, Lemma 1]. For details, see Appendix B in [25]. ■

We now proceed to prove the converse to Theorem 1. Fix a positive $\eta > 0$. Denote for each blocklength n , the set of strongly typical sequences in \mathcal{Y}_0^n by $\mathcal{T}_{\mu_n}^{(n)}(P_{Y_0})$. Set $\mu_n = n^{-1/3}$ and define for $i \in \{1, 2\}$, the sets

$$\mathcal{B}_i(\eta) \triangleq \{y_0^n \in \mathcal{T}_{\mu_n}^{(n)}(P_{Y_0}) :$$

$$\Pr[\hat{\mathcal{H}}_{Y_i} = 0 | Y_0^n = y_0^n, \mathcal{H} = 0] \geq \eta\}, \quad i \in \{1, 2\}, \quad (20)$$

$$\mathcal{D}_0(\eta) \triangleq \mathcal{B}_1(\eta) \cap \mathcal{B}_2(\eta), \quad (21)$$

$$\mathcal{D}_i(\eta) \triangleq \mathcal{B}_i(\eta) \setminus \mathcal{D}_0(\eta). \quad (22)$$

Further define for each n the probabilities

$$\Delta_j \triangleq P_{Y_0^n}(\mathcal{D}_j(\eta)), \quad j \in \{0, 1, 2\}, \quad (23)$$

and notice that by the laws of probability

$$\Delta_0 + \Delta_i = P_{Y_0^n}(\mathcal{B}_i(\eta)), \quad i \in \{1, 2\}, \quad (24)$$

$$\Delta_0 \geq P_{Y_0^n}(\mathcal{B}_1(\eta)) + P_{Y_0^n}(\mathcal{B}_2(\eta)) - 1. \quad (25)$$

By (6b), it can be shown that

$$1 - \epsilon_i \leq \eta(1 - P_{Y_0^n}(\mathcal{B}_i(\eta))) + P_{Y_0^n}(\mathcal{B}_i(\eta)) + P_{Y_0^n}(\bar{\mathcal{T}}_{\mu_n}^{(n)}). \quad (26)$$

Thus, by (26) and [24, Lemma 2.12]:

$$P_{Y_0^n}(\mathcal{B}_i(\eta)) \geq \frac{1 - \epsilon_i - \eta}{1 - \eta} - \frac{|\mathcal{Y}_0|}{(1 - \eta)4\mu_n^2 n}, \quad i \in \{1, 2\}, \quad (27)$$

and we conclude that in the limit $n \rightarrow \infty$ and $\eta \downarrow 0$:

$$\lim_{\eta \downarrow 0} \lim_{n \rightarrow \infty} (\Delta_0 + \Delta_i) \geq 1 - \epsilon_i, \quad i \in \{1, 2\} \quad (28a)$$

$$\lim_{\eta \downarrow 0} \lim_{n \rightarrow \infty} \Delta_0 \geq 1 - \epsilon_1 - \epsilon_2 \quad (28b)$$

$$\lim_{\eta \downarrow 0} \lim_{n \rightarrow \infty} \sum_{j=0}^2 \Delta_j \leq 1. \quad (28c)$$

We proceed by applying Lemma 1 to the set \mathcal{D}_j for any $j \in \{0, 1, 2\}$ with $\Delta_j > 0$, and conclude that for any $j \in \{0, 1, 2\}$ with $\Delta_j > 0$ there is a tuple (U_0^j, U_1^j, U_2^j) satisfying

$$H(\tilde{M}_0^j) \geq nI(U_0^j; \tilde{Y}_0^j) + \log P_{Y_0^n}(\mathcal{D}_j), \quad j \in \{0, 1, 2\}, \quad (29)$$

$$H(\tilde{M}_i^j) \geq nI(U_i^j; \tilde{Y}_0^j | U_0^j), \quad i \in \{1, 2\}, j \in \{0, i\}, \quad (30)$$

and for $i \in \{1, 2\}, j \in \{0, i\}$:

$$-\frac{1}{n} \log \beta_{i,n} \leq I(U_0^j U_i^j; \tilde{Y}_i^j) + \phi_i^j(n), \quad (31)$$

where for each pair (i, j) , the function $\phi_i^j(n) \rightarrow 0$ as $n \rightarrow \infty$ and the random variables $\tilde{Y}_0^j, \tilde{Y}_i^j, \tilde{M}_0^j, \tilde{M}_i^j$ are defined as in the lemma applied to the subset \mathcal{D}_j .

To summarize:

$$-\frac{1}{n} \log \beta_{i,n} \leq \min\{I(U_0^0 U_i^0; \tilde{Y}_i^0); I(U_0^i U_i^i; \tilde{Y}_i^i)\} + \phi_i(n), \quad (32)$$

where $\phi_i(n)$ is a function tending to 0 as $n \rightarrow \infty$.

Define the following random variables for $i \in \{1, 2\}$ and $j \in \{0, 1, 2\}$

$$\tilde{L}_{i,j} \triangleq \text{len}(\tilde{M}_i^j). \quad (33)$$

By the rate constraints (2), and the definition of the random variables \tilde{M}_i^j , we obtain by the total law of expectations

$$nR_0 \geq \mathbb{E}[L_0] \geq \sum_{j \in \{0, 1, 2\}} \mathbb{E}[\tilde{L}_{0,j}] \Delta_j. \quad (34)$$

Moreover,

$$H(\tilde{M}_0^j) = H(\tilde{M}_0^j, \tilde{L}_{0,j}) \quad (35)$$

$$= \sum_{l_j} \Pr[\tilde{L}_{0,j} = l_j] H(\tilde{M}_0^j | \tilde{L}_{0,j} = l_j) + H(\tilde{L}_{0,j}) \quad (36)$$

$$\leq \sum_{l_j} \Pr[\tilde{L}_{0,j} = l_j] l_j + H(\tilde{L}_{0,j}) \quad (37)$$

$$= \mathbb{E}[\tilde{L}_{0,j}] + H(\tilde{L}_{0,j}), \quad (38)$$

which combined with (34) establishes

$$\sum_{j \in \{0, 1, 2\}} \Delta_j H(\tilde{M}_0^j) \leq \sum_{j \in \{0, 1, 2\}} \Delta_j \mathbb{E}[\tilde{L}_{0,j}] + \Delta_j H(\tilde{L}_{0,j}) \quad (39)$$

$$\leq nR_0 \left(1 + \sum_{j \in \{0, 1, 2\}} h_b \left(\frac{\Delta_j}{nR_0} \right) \right), \quad (40)$$

where (40) holds by (34) and because the entropy of a discrete and positive random variable $\tilde{L}_{0,j}$ of mean $\mathbb{E}[\tilde{L}_{0,j}] \leq \frac{nR_0}{\Delta_j}$ is bounded by $\frac{nR_0}{\Delta_j} \cdot h_b \left(\frac{\Delta_j}{nR_0} \right)$, see [26, Theorem 12.1.1].

In a similar way we obtain for $i \in \{1, 2\}$

$$\sum_{j \in \{0, i\}} \Delta_j H(\tilde{M}_i^j) \leq nR_i \left(1 + \sum_{j \in \{0, i\}} h_b \left(\frac{\Delta_j}{nR_i} \right) \right). \quad (41)$$

Notice that when $\Delta_j = 0$, the trivial choice $U_i^j = \tilde{Y}_i^j$ satisfies the inequalities (32), (40), and (41). Therefore, above conclusions hold for (U_0^j, U_1^j, U_2^j) for any $j \in \{0, 1, 2\}$.

Combining (40) and (41) with (29) and (30), noting (24) and (27), and considering also (32), we have proved so far that for all $n \geq 1$ there exist joint pmfs $P_{U_0^j U_1^j U_2^j \tilde{Y}_0^j \tilde{Y}_1^j \tilde{Y}_2^j} = P_{\tilde{Y}_0^j} P_{\tilde{Y}_1^j \tilde{Y}_2^j | \tilde{Y}_0^j} P_{U_0^j U_1^j U_2^j | \tilde{Y}_0^j}$ (abbreviated as $P_j^{(n)}$) for $j \in \{0, 1, 2\}$ so that the following conditions hold for $i \in \{1, 2\}$ (where I_P indicates that the mutual information should be calculated according to a pmf P):

$$R_0 \geq \sum_{j \in \{0, 1, 2\}} (I_{P_j^{(n)}}(U_0^j; \tilde{Y}_0^j) + g_{1,j}(n)) \cdot g_{2,j}(n, \eta), \quad (42a)$$

$$R_i \geq \sum_{j \in \{0, i\}} (I_{P_j^{(n)}}(U_i^j; \tilde{Y}_0^j | U_0^j)) \cdot g_{2,j}(n, \eta), \quad (42b)$$

$$\theta_i \leq \min\{I_{P_0^{(n)}}(U_0^0 U_i^0; \tilde{Y}_i^0), I_{P_i^{(n)}}(U_0^i U_i^i; \tilde{Y}_i^i)\} + g_{3,i}(n), \quad (42c)$$

for some nonnegative functions $g_{1,j}(n), g_{2,j}(n, \eta), g_{3,i}(n)$ with the following asymptotic behaviors:

$$\lim_{n \rightarrow \infty} g_{1,j}(n) = 0, \quad \forall j \in \{0, 1, 2\}, \quad (43)$$

$$\lim_{n \rightarrow \infty} g_{3,i}(n) = 0, \quad \forall i \in \{1, 2\}, \quad (44)$$

$$\lim_{n \rightarrow \infty} (g_{2,0}(n, \eta) + g_{2,i}(n, \eta)) \geq \frac{1 - \epsilon_i - \eta}{1 - \eta}, \quad \forall i \in \{1, 2\}. \quad (45)$$

By Carathéodory's theorem [23, Appendix C], there exist for each n , random variables $U_0^0, U_1^0, U_2^0, U_1^1, U_1^2, U_2^1, U_2^2$ satisfying (42) over alphabets of sizes

$$|\mathcal{U}_0^0| \leq |\mathcal{Y}_0| + 3, \quad (46)$$

$$|\mathcal{U}_0^j| \leq |\mathcal{Y}_0| + 2, \quad j \in \{1, 2\}, \quad (47)$$

$$|\mathcal{U}_i^j| \leq |\mathcal{U}_0^j| \cdot |\mathcal{Y}_0| + 1, \quad i \in \{1, 2\}, j \in \{0, i\}. \quad (48)$$

Then we invoke the Bolzano-Weierstrass theorem and consider for each $j \in \{0, 1, 2\}$ a sub-sequence $P_{U_0^j U_1^j U_2^j Y_0^j Y_1^j Y_2^j}^{(n_k)}$ that converges to a limiting pmf $P_{U_0^j U_1^j U_2^j Y_0^j Y_1^j Y_2^j}^*$. For these limiting pmfs, which we abbreviate by P_j^* , we conclude by (42a)–(42c) and (28) that for all $i \in \{1, 2\}$:

$$R_0 \geq \sigma_0 \cdot I_{P_0^*}(U_0^0; Y_0^0) + \sigma_1 \cdot I_{P_1^*}(U_0^1; Y_0^1) + \sigma_2 \cdot I_{P_2^*}(U_0^2; Y_0^2), \quad (49)$$

$$R_i \geq \sigma_0 \cdot I_{P_0^*}(U_0^i; Y_0^i | U_0^0) + \sigma_i \cdot I_{P_i^*}(U_0^i; Y_0^i | U_0^i), \quad (50)$$

$$\theta_i \leq \min\{I_{P_0^*}(U_0^0 U_i^0; Y_i^0), I_{P_i^*}(U_0^0 U_i^i; Y_i^i)\}, \quad (51)$$

where numbers $\sigma_0, \sigma_1, \sigma_2 > 0$ satisfy $\sigma_0 + \sigma_1 + \sigma_2 \leq 1$ and

$$\sigma_0 + \sigma_i \geq 1 - \epsilon_i, \quad i \in \{1, 2\}, \quad (52a)$$

$$\sigma_0 \geq 1 - \epsilon_1 - \epsilon_2. \quad (52b)$$

Notice further that since for any $j \in \{0, 1, 2\}$ and any k , the sequence \tilde{Y}_0^{j, n_k} lies in the typical set $\mathcal{T}_{\mu_{n_k}}^{(n_k)}(P_{Y_0})$, we have for all $j \in \{0, 1, 2\}$, $|P_{\tilde{Y}_0^j} - P_{Y_0}| \leq \mu_{n_k}$ and thus the limiting pmf satisfies $P_{Y_0^j}^* = P_{Y_0}$. Moreover, since for each n_k the pair of random variables $(\tilde{Y}_1^j, \tilde{Y}_2^j)$ is drawn according to $P_{Y_1 Y_2 | Y_0}$ given \tilde{Y}_0^j , the limiting pmf also satisfies $P_{Y_1^j Y_2^j | Y_0^j}^* = P_{Y_1 Y_2 | Y_0}$. We also notice for all $j \in \{0, 1, 2\}$ that under P_j^* the Markov chain $(U_0^j, U_1^j, U_2^j) \rightarrow Y_0 \rightarrow (Y_1, Y_2)$ holds. This concludes the converse proof.

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APPENDIX PROOF OF COROLLARY 1

By Theorem 1, $\mathcal{E}^*(R_0, 0, 0, \epsilon_1, \epsilon_2)$ is the set of all (θ_1, θ_2) pairs satisfying

$$\theta_i \leq \min\{I(U_0; Y_i), \eta_i(R_0^i)\}, \quad i \in \{1, 2\}. \quad (53a)$$

for some non-negative numbers $\sigma_0, \sigma_1, \sigma_2$ with sum ≤ 1 and satisfying (7d) and (7e), a conditional pmf $P_{U_0 | Y_0}$, and nonnegative rates R_0^1, R_0^2 such that

$$R_0 \geq \sigma_0 I(U_0; Y_0) + \sigma_1 R_0^1 + \sigma_2 R_0^2. \quad (53b)$$

Notice that without loss in optimality, in the evaluation of above region, we can restrict to tuples $(P_{U_0 | Y_0}, R_0^1, R_0^2)$ satisfying

$$I(U_0; Y_i) \geq \eta_i(R_0^i), \quad (54)$$

which by the maximum in the definition of function η_i implies

$$I(U_0; Y_0) \geq R_0^i, \quad i \in \{1, 2\}. \quad (55)$$

In fact, if (54) is violated, rates R_0^1 and/or R_0^2 can be reduced without changing (53a) and so that (54) holds.

We next show that any exponent pair (θ_1, θ_2) and tuple $(P_{U_0 | Y_0}, R_0^1, R_0^2)$ satisfying (53), (54), and

$$I(U_0; Y_0) \leq R_0^{\pi(1)} + R_0^{\pi(2)} \quad (56)$$

also satisfies (14). The exponents' constraints (14a) and (14b) are easily verified. To verify (14c), notice that when $\sigma_0 > 1 - \epsilon_{\pi(1)}$:

$$R_0 \geq \sigma_0 I(U_0; Y_0) + \sigma_{\pi(1)} R_0^{\pi(1)} + \sigma_{\pi(2)} R_0^{\pi(2)} \quad (57)$$

$$= (1 - \epsilon_{\pi(1)}) I(U_0; Y_0) + \sigma_{\pi(1)} R_0^{\pi(1)} + (\sigma_0 - 1 + \epsilon_{\pi(1)}) I(U_0; Y_0) + \sigma_{\pi(2)} R_0^{\pi(2)} \quad (58)$$

$$\geq (1 - \epsilon_{\pi(1)}) I(U_0; Y_0) + (\epsilon_{\pi(1)} - \epsilon_{\pi(2)}) R_0^{\pi(2)} \quad (59)$$

where (59) holds because $\sigma_{\pi(1)} R_0^{\pi(1)} \geq 0$, because $I(U_0; Y_0) \geq R_0^{\pi(2)}$ by (55), and $\sigma_0 + \sigma_{\pi(2)} \geq 1 - \epsilon_{\pi(2)}$ by (7d).

For $\sigma_0 \leq 1 - \epsilon_{\pi(1)}$, (14c) can be verified as follows:

$$R_0 \geq \sigma_0 I(U_0; Y_0) + \sigma_{\pi(1)} R_0^{\pi(1)} + \sigma_{\pi(2)} R_0^{\pi(2)} \quad (60)$$

$$\geq \sigma_0 I(U_0; Y_0) + (1 - \epsilon_{\pi(1)} - \sigma_0) R_0^{\pi(1)} + \sigma_{\pi(2)} R_0^{\pi(2)} \quad (61)$$

$$\geq \sigma_0 I(U_0; Y_0) + (1 - \epsilon_{\pi(1)} - \sigma_0) (R_0^{\pi(1)} + R_0^{\pi(2)}) + (\epsilon_{\pi(1)} - \epsilon_{\pi(2)}) R_0^{\pi(2)} \quad (62)$$

$$\geq (1 - \epsilon_{\pi(1)}) I(U_0; Y_0) + (\epsilon_{\pi(1)} - \epsilon_{\pi(2)}) R_0^{\pi(2)} \quad (63)$$

where (61) holds by (7d), (62) holds because $\sigma_{\pi(2)} \geq 1 - \epsilon_{\pi(2)} - \sigma_0$ by (7d), and (63) holds by (56) and $\sigma_0 \leq 1 - \epsilon_{\pi(1)}$. This establishes that (53) holds under condition (56).

The proof is concluded by showing that for any tuple $(\theta_1, \theta_2, P_{U_0 | Y_0}, R_0^1, R_0^2)$ satisfying (53), (54), and

$$I(U_0; Y_0) > R_0^{\pi(1)} + R_0^{\pi(2)}, \quad (64)$$

we can find a pmf $P_{\tilde{U}_0 | Y_0}$, satisfying (14) when U_0 is replaced by \tilde{U}_0 . Choose a bivariate $\tilde{U}_0 = (\tilde{U}_0^1, \tilde{U}_0^2)$ such that $\tilde{U}_0^1 \rightarrow Y_0 \rightarrow \tilde{U}_0^2$ forms a Markov chain and for each $i \in \{1, 2\}$ the new random-variable \tilde{U}_0^i achieves $\eta_i(R_0^i)$, i.e.,

$$R_0^i \geq I(Y_0; \tilde{U}_0^i) \quad \text{and} \quad \eta_i(R_0^i) = I(\tilde{U}_0^i; Y_i). \quad (65)$$

Since for any $i \in \{1, 2\}$ we have $I(\tilde{U}_0; Y_i) \geq I(\tilde{U}_0^i; Y_i) = \eta_i(R_0^i)$, the exponents satisfy

$$\theta_{\pi(1)} \leq \min\{I(U_0; Y_{\pi(1)}), \eta_{\pi(1)}(R_0^{\pi(1)})\} = \eta_{\pi(1)}(R_0^{\pi(1)}) \quad (66)$$

$$\leq I(\tilde{U}_0; Y_{\pi(1)}), \quad (67)$$

$$\theta_{\pi(2)} \leq \min\{I(U_0; Y_{\pi(2)}), \eta_{\pi(2)}(R_0^{\pi(2)})\} = \eta_{\pi(2)}(R_0^{\pi(2)}) \quad (68)$$

$$= \min\{I(\tilde{U}_0; Y_{\pi(2)}), \eta_{\pi(2)}(R_0^{\pi(2)})\}, \quad (69)$$

where the inequalities in (66) and (68) hold by (54). Similarly,

$$R_0 \geq \sigma_0 I(U_0; Y_0) + \sigma_{\pi(1)} R_0^{\pi(1)} + \sigma_{\pi(2)} R_0^{\pi(2)} \quad (70)$$

$$> (1 - \epsilon_{\pi(1)}) R_0^{\pi(1)} + (1 - \epsilon_{\pi(2)}) R_0^{\pi(2)} \quad (71)$$

$$= (1 - \epsilon_{\pi(1)}) I(\tilde{U}_0^{\pi(1)}; Y_0) + (1 - \epsilon_{\pi(2)}) I(\tilde{U}_0^{\pi(2)}; Y_0) \quad (72)$$

$$\geq (1 - \epsilon_{\pi(1)}) I(\tilde{U}_0^{\pi(1)}; Y_0) + (1 - \epsilon_{\pi(1)}) I(\tilde{U}_0^{\pi(2)}; Y_0 | \tilde{U}_0^{\pi(1)}) \quad (73)$$

$$+ (\epsilon_{\pi(1)} - \epsilon_{\pi(2)}) I(\tilde{U}_0^{\pi(2)}; Y_0) \quad (73)$$

$$= (1 - \epsilon_{\pi(1)}) I(\tilde{U}_0; Y_0) + (\epsilon_{\pi(1)} - \epsilon_{\pi(2)}) I(\tilde{U}_0^{\pi(2)}; Y_0) \quad (74)$$

where inequality (71) holds by the assumption that $I(U_0; Y_0) > R_0^1 + R_0^2$ and by condition (7d); equality (72) holds by (65); inequality (73) holds by the Markov chain $\tilde{U}_0^1 \rightarrow Y_0 \rightarrow \tilde{U}_0^2$; and (74) by the chain rule and the definition of \tilde{U}_0 .

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