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# Discrete and Continuous Curved Surface Antipodal Paths. Extensions of the Borsuk-Ulam Theorem and Application of the Feynman Path Integral and Wodehouse Contour Integral

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**Abstract Purpose:** This paper introduces discrete and continuous paths over simply-connected surfaces with non-zero curvature as means of comparing and measuring paths between antipodes with either a Feynman path integral or Wodehouse contour integral, resulting in a number of extensions of the Borsuk Ulam Theorem.

**Methods:** All paths originate on a Riemannian surface  $S$ , which is simply-connected and has non-zero curvature at every point in  $S$ . A surface is a **simply connected**, provided every cross-cut divides the surface into disjoint regions. A **cross cut** is an arc that runs through the interior of the surface  $S$  without self-intersections and joins one boundary to another.

**Results:** A fundamental result in this paper is that for any pair of antipodal surface points, a path that begins and ends at the antipodal points can be found. This result is extended to Feynman path integrals on trajectory-of-particle curves and to N.M.J. Wodehouse countour integrals for antipodal vectors on twistor curves. Another fundamental result in this paper is that the Feynman trajectory of a particle is realizable as a Lefschetz arc.

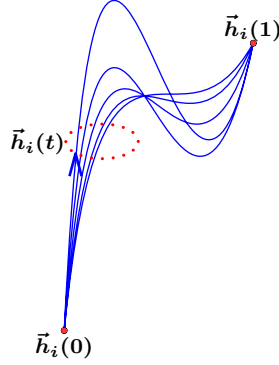
**Keywords** antipode, arc, cell, curvature, path, surface, vector

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## 1 Introduction

This paper introduces a path-Borsuk-Ulam Theorem, stemming from three main forms of paths over curved surfaces that have been identified, namely,



**Fig. 1** Discrete paths  $h_i : Id \rightarrow S$  with all  $t \in Id$ .

- 1° **Poincaré Contour paths** were introduced by Poincaré in 1892 in his analysis situs paper [1]. In a **contour path**, each subpath is an infinitely small contour on a manifold [1, p. 240]. Recently, N.M.J. Woodhouse [2] introduced contour integrals defined on twistor curves on a complex manifold.
- 2° **Whitehead Homotopic paths** were introduced during the late 1940s by J.H.C. Whitehead [3],[4] and S. Lefschetz [5]. For Whitehead, a **path** is a continuous map  $h : [0, 1] \rightarrow S$ , *i.e.*, a mapping from the unit interval to a space  $S$ . For Lefschetz, a **homotopic path**  $h$  in an arcwise connected space  $S$  is simply a mapping of a directed (= oriented) closed arc  $\widehat{v_0, v_1}$  into  $R$  [5, p. 158]. A space is **arcwise connected**, provided every vector in the space  $S$  is on a path containing an initial vector and a terminal vector such as the arcs in Fig. 1.
- 3° **Feynman paths** were introduced by R.P. Feynman in his thesis completed in 1942 [6, p. xiv]. A **Feynman path** is a trace of the trajectory of a particle between fixed endpoints [6, p. xiv], providing a framework for a path integral, also introduced by Feynman[6] and elaborated by R.P. Feynman and A.R. Hibbs in [7]. A **Penrose path** over a twistor curve (from R. Penrose's 1968 paper [8]) and its refinement by R.S. Ward in his 1977 thesis [9] supervised by Penrose, is a form of Feynman path in which the trajectory of a particle is over a twistor curve.

The original BUT [10] from K. Borsuk in 1933 is given in terms of antipodal vectors  $\vec{p}$ ,  $-\vec{p}$  on the surface of an  $n$ -dimensional Euclidean sphere  $S^n$ , defined

by

$$S^n = \{\vec{p}(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}, n \geq 2 | x_1^2 + \dots + x_{n+1}^2 = 1\}.$$

Points on the surface of a sphere are **antipodal**, provided the points are diametrically opposite each other. Examples of antipodal vectors are the poles on the surface of a planet.

In 1933, K. Borsuk introduced his Theorem 1.

#### BUT.

**Theorem 1 (Borsuk-Ulam Theorem)** [10, p. 178].

*Let vector  $\vec{p} \in S^n$ . For a continuous map  $f : S^n \rightarrow \mathbb{R}^n$ , there is an antipodal point  $-\vec{p}$  such that  $f(\vec{p}) = f(-\vec{p})$ .*

*Remark 1* Theorem 1 is a translation from German, which is given by J. Matousšek [11, p. 21]. ■

*Remark 2* The basis for Theorem 1 came from K. Borsuk's thesis completed in 1930 [12]. Ulam is credited by Borsuk (in a footnote [10, p. 178]) with the idea codified in Theorem 1, which Ulam stated as a conjecture. In effect, Borsuk proved Ulam's conjecture in 1933. In 1930, L. Lusternik and S. Shnirel'man introduced the nonvoid intersection of sets of closed surface curves that have antipodal vectors in common.

#### LST.

**Theorem 2 (Lusternik-Shnirel'man Theorem)** [13].

*For any cover  $F_1, \dots, F_{n+1}$  of the sphere  $S^n$  by  $n+1$  closed sets, there is at least one set containing a pair of antipodal points common to  $F_i, -F_i$  (i.e.,  $F_i \cap -F_i \neq \emptyset$ ).*

*Remark 3* Theorem 2 is a translation from Russian, which is given by J. Matousšek [11, p. 21]. ■

Theorem 2 contrasts with Theorem 1. In the Lusternik-Shnirel'man Theorem 2, there is a closed set  $F_i$  that is a cover of a sphere  $S^n$  and that has an opposite set  $-F_i$ , in which the sets  $F_i, -F_i$  contain antipodal points such that  $F_i \cap -F_i \neq \emptyset$ . This sharply contrasts with the Borsuk-Ulam Theorem, which asserts there is a continuous map  $f$  from  $S^n$  into  $\mathbb{R}^n$  over a surface containing antipodal surface vectors  $\vec{p}, -\vec{p}$  such that  $f(\vec{p}) = f(-\vec{p})$ . Also, Theorem 2 concludes with the observation that the intersection of  $F_i, -F_i$  is nonvoid but the values of the shared antipodal points are not given. In the LS theorem formulation, it is possible that the antipodal points in  $F_i \cap -F_i$  have different values. By contrast, in the Theorem 1 formulation, it is asserted that the antipodal points map to the same value. ■

Let  $Id$  be a discrete form of a unit interval  $[0, 1]$  defined by

$$\begin{aligned} i &\in \mathbb{Z}^+ \text{ natural numbers } 1, 2, 3, \dots \\ &\text{i.e., there is a gap between } \mathbf{x}_i \text{ and } \mathbf{x}_{i+1} \\ X &= \overbrace{\{X_i \in S^n \mid |x_{i+1} - x_i| > 0\}} \\ Id &= \{0, 1\} \cup \{X : X_i \in S^n\}. \end{aligned}$$

*Example 1* Let  $X_{0.00001}$  be a set of surface points on an  $n$ -dimensional Riemannian sphere  $S^n$ , defined by

$$\begin{aligned} &\text{i.e., there is a } 0.00001 \text{ gap between } \mathbf{x}_i \text{ and } \mathbf{x}_{i+1}. \\ X_{0.00001} &= \overbrace{\{X_i \in S^n \mid |x_{i+1} - x_i| = 0.00001\}} \\ Id &= \{0, 1\} \cup \{X_{0.1} : X_i \in S^n\}. \quad \blacksquare \end{aligned}$$

## 2 Preliminaries

More recent versions of BUT [14, §68, p.405] require the map  $f : S^n \rightarrow R^n$  to be continuous, *i.e.*, the map  $f$  is continuous, provided, for each subset  $E \subset S^n$ , if a point  $\vec{p}$  is arbitrarily close to  $E$ , then  $f(\vec{p})$  is arbitrarily close to  $E$ . However, in keeping with an interest in Riemannian surface surface points with gaps between them, we consider discrete maps.

### Definition 1 Discrete map.

**Discretely close surface points**  $\vec{p}, \vec{q}$  such as water molecules always have a minute gap between them. A map  $f : S^n \rightarrow R^n$  is discrete, provided, for each subset  $E \subset S^n$ , if a point  $\vec{p}$  is discretely close to  $E$ , then  $f(\vec{p}) = \vec{q}$  is close to  $E$ .

*Example 2* **Discretely close surface points**  $\vec{p}, \vec{q}$  such as close water molecules always have a minute gap between them. ■

*Example 3* The discrete unit interval  $Id$  is a collection **Discretely close surface points**  $\vec{p}, \vec{q} \in Id$  such that  $|\vec{p} - \vec{q}| > 0$ . ■

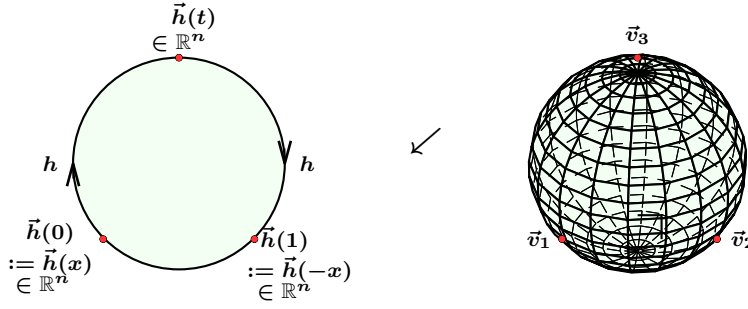
### Definition 2 Discrete Path.

Let  $S$  be a Riemannian surface. A **discrete path**  $h$  is a discrete mapping, defined by

$$h : Id \rightarrow S, \text{ such that}$$

$h(0) \in S$  is the initial point in path  $h$ ,  $h(1) \in S$  the endpoint of  $h$  and  $h(t) \in S$  for all  $t \in \{x : x \in X \subset Id\}$ .

*Example 4* A sample discrete path  $h : Id \rightarrow S$  on the surface of a Riemannian sphere is shown in Fig. 2. This path begins at vector  $\vec{h}(0) \in \mathbb{R}^n$  at  $\vec{v}_1$  on the surface of  $S$  and ends at vector  $\vec{h}(1) \in \mathbb{R}^n$ , which is the value of antipode of  $\vec{v}_1$ . The assumption made here is that  $\vec{h}(0)$  and  $\vec{h}(1)$  have the same value such as identical temperature. ■



**Fig. 2** The left-slanting arrow  $\swarrow$  reads **collapses to**, e.g.,  $\blacktriangleleft \swarrow \blacktriangleright$ , i.e., collapse a left-pointing solid triangle to its boundary. For example, collapse a sphere  $S$  to a circle containing a discrete path  $h : Id \rightarrow S$  with  $\vec{h}(0) = \vec{h}(x) \in \mathbb{R}^2$ , antipodal to  $\vec{h}(1) = \vec{h}(-x) \in \mathbb{R}^2$ , with  $\vec{h}(t) \in \mathbb{R}^3$  for  $t \in Id \setminus \{0, 1\}$ .

That is, a **discrete path**  $h : Id \rightarrow S$  is a mapping from the discrete unit interval  $Id \subset I$  (for  $I = [0, 1]$ ) to a bounded, simply connected surface  $S$  with non-zero curvature. Path  $h$  is discrete, since there are gaps between all points  $\vec{h}(t) \in S$  between 0 and 1 in  $Id \subset [0, 1]$ . The surface  $S$  is **simply connected**, provided every path  $h$  has end points  $h(0), h(1) \in S$  and  $h$  has no self-loops.

Paths either lie entirely on a surface in the planar case or lie on a surface and, possibly, puncture a surface in the non-planar case. Paths that puncture a surface are called cross-cuts. A **cross cut path**  $P$  (also called an **ideal arc** [15, §3, p.11]) has both ends in  $P$  and path interior in the interior of  $S$ .

*Remark 4* Homotopic paths were introduced by J.H.C. Whitehead[3]. For Whitehead, a path  $h : [0, 1] \rightarrow X$  is a continuous mapping from the unit interval to a cell complex  $X$ . In the pursuit of discrete paths in a curved space, the focus is on 0-cells (single points) and 1-cells (arcs) in an  $n$ -dimensional Riemannian space  $S$ . A single surface vector is a 0-cell. A **Lefschetz arc** is a curvilinear line segment attached to a pair of 0-cells [16]. ■

**Definition 3 (Path-Connected)** A pair of vectors  $v_0, v_1$  are **path-Connected**, provided there is a sequence of arcs between the vectors.

*Example 5* All vectors on the circle in Fig. 2 are path-connected, since there is a sequence of arcs between each pair of vectors. ■

### 3 Antipodal and Non-Antipodal Path Borsuk-Ulam Theorem

This section introduces results for paths on a Riemann surface.

**Theorem 3** *The endpoints of a discrete Lefschetz arc can be the same.*

*Proof* Let  $\widehat{v_0, v_1}$  be a discrete Lefschetz arc in which the endpoints  $v_0 = v_1$  are the same. There is an arc between  $v_0, v_1$ , namely, itself. Hence, from Def. 3, the endpoints are path-connected.

**Lemma 1** *Every discrete path constructs a discrete arc.*

*Proof* Every  $h(t)$  is a point on an arc  $\widehat{h(0), h(1)}$  between  $h(0)$  and  $h(1)$  is a discrete path  $h : Id \rightarrow S$ .  $Id$  is discrete. Hence,  $\widehat{h(0), h(1)}$  is also discrete.

**Theorem 4** *Every discrete path constructs a vector field.*

*Proof* Let  $h : Id \rightarrow S$  be a discrete path. From Lemma 1,  $h$  constructs a discrete arc  $\widehat{h(0), h(1)}$  on a surface  $S$ . Consequently, each  $h(t) \in \widehat{h(0), h(1)}$  has a location  $(x_1, \dots) \in S$  with its own magnitude and direction  $S$ , i.e., every  $h(t)$  is a vector in  $S$ . Hence,  $h$  constructs a vector field.

**Lemma 2** *Let  $\vec{v}_1, \vec{v}_2$  be antipodal vectors on the surface of a sphere  $S$ . There exists a discrete path  $h$  with endpoints are antipodal on a surface  $S$ .*

*Proof* Let  $\vec{v}_1, \vec{v}_2$  be antipodal vectors on the surface of a sphere  $S$ . Then define a discrete path  $h : Id \rightarrow S$  with  $\vec{h}(0) = \vec{v}_1$  and  $\vec{h}(1) = \vec{v}_2$ . Hence, a discrete path can be defined for every pair of antipodal points on  $S$ .

From what we have observed about discrete paths on the surface of a sphere, we obtain

#### path-BUT.

##### **Theorem 5 (Path-Borsuk-Ulam Theorem).**

Let the discrete unit interval  $Id$  be an index set for vectors  $v_0, \dots, v_t, \dots, v_1, t \in Id$  in  $S^n$  in a path  $h : S^n \rightarrow \mathbb{R}^n$ , i.e., a path discretely maps  $v_0, \dots, v_t, \dots, v_1$  to a set of values  $\{h(v_0), \dots, h(v_t), \dots, h(v_1)\}$  in  $\mathbb{R}^n$ . For a discrete path  $h : S^n \rightarrow \mathbb{R}^n$  and a vector  $\vec{p} \in S^n$ , there is an antipodal vector  $-\vec{p} \in S^n$  such that  $h(\vec{p}) = h(-\vec{p})$ .

*Proof* From Lemma 1, a path  $h : S^n \rightarrow \mathbb{R}^n$  constructs of values from vectors on an arc with endpoints  $\vec{v}, \vec{v}'$  on  $S^n$ . Let the endpoints of this arc be antipodal. Hence, from Lemma 2, the desired result follows.

*Remark 5* An immediate consequence of Theorem 5 is that, for any pair of antipodal surface points, we can always introduce a discrete path  $h$  that begins and ends at the antipodal points such as places that have same latitude and longitude. For example, the antipode of Winnipeg, Manitoba, Canada with coordinates  $49^\circ.53'N, 97^\circ.8'W$  is Port-aux-Français, Kerguelen, French Southern Territories. ■

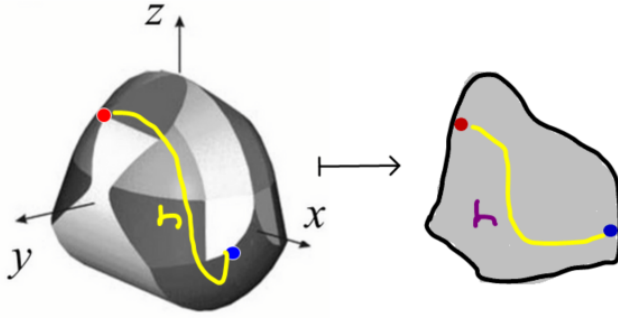
*Example 6* An example of a discrete path that begins and ends at antipodal surface points is shown in Fig. 2. ■

Observe that a path can be constructed between any pair of surface vectors. This observation leads to more general form of Theorem 5.

**Theorem 6** (Non-antipodal path-BUT).

Let the discrete unit interval  $Id$  be an index set for vectors  $v_0, \dots, v_t, \dots, v_1, t \in Id$  in  $S^n$  in a path  $h : S^n \rightarrow \mathbb{R}^n$ . There is a discrete path  $h : S^n \rightarrow \mathbb{R}^n$  with endpoints  $h(v_0), h(v_1)$  that are values in  $\mathbb{R}^n$  such that  $h(v_0) = h(v_1)$ .

*Proof* In the proof of Theorem 5, replace antipodal points on a Leftschetz arc with endpoints that are any pair of surface points. In addition, notice that the initial point and the ending point of any discrete path can be the same. Hence, the desired result follows.



**Fig. 3** 2D and 3D views of discrete paths on a Gomboc Riemannian surface

*Example 7* An example of a discrete path that begins and ends at antipodal surface vectors on a bumpy Riemannian sphere (aka Gomboc sphere) is shown in Fig. 3. ■

*Example 8* An example of a discrete path  $h : S^2 \rightarrow R^3$  on a 3D Gomboc Riemannian surface is shown in Fig. 3. The same path is also depicted on a 2D slice of the 3D surface. In keeping with Theorem 6, each vector  $\vec{h}(v_t)$  is a signal value from the path  $h$ . For example, if we let the discrete path be an optical field flow containing a stream of photons reflected from a Riemannian surface, then there are number of possible signal values for  $\vec{h}(v_t)$ , e.g.,  
 1° wavelength of  $\vec{h}(v_t)$ .  
 2° frequency of  $\vec{h}(v_t)$ .

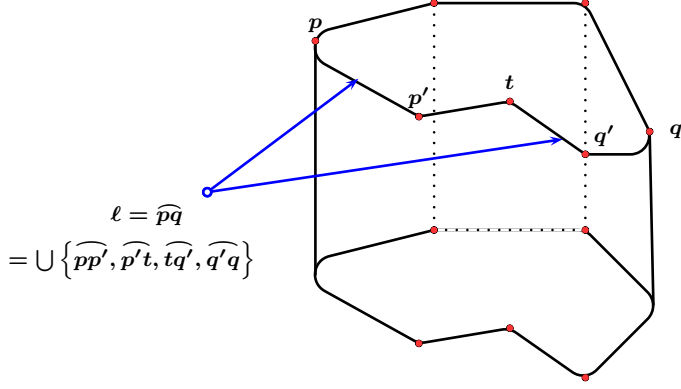


3<sup>o</sup> electron voltage of  $\vec{h}(v_t)$ .

4<sup>o</sup> lumens (luminosity) of  $\vec{h}(v_t)$ .

5<sup>o</sup> gradient of  $\vec{h}(v_t), t \in Id$ , which would be perpendicular to the surface at  $(x, y, z)$ , defined by

$$\mathbf{grad}(\vec{h}(v_t)) = \frac{\partial \vec{h}}{\partial x} i + \frac{\partial \vec{h}}{\partial y} j + \frac{\partial \vec{h}}{\partial z} k.$$



**Fig. 4** Trajectory of a particle over twistor curve realizable as the union of a sequence of sub-arcs on Lefschetz arc  $\ell = \widehat{pq}$  on a R.S. Ward hypersurface  $\mathbb{CS}$  [9, p.62].

#### 4 Feynman Trajectories of a Particle

This section introduces particle trajectories as continuous paths over the curvature of space-time, which leads to the counterpart of the discrete path results already given. The transition from discrete paths results from the geometry of space-time generated by quantum processes [17], which is in keeping with the observation by R. Penrose [8] that the link between space-time curvature and quantum processes such as those found in Feynman trajectory of a particle is supplied by the use of twistors. A **twistor space** is a complex manifold  $\mathbb{CM}$ . For example, a Lefschetz arc in curved space-time is a R.S. Ward hypersurface  $\mathcal{S}$  twistor [9, p.56], which is a complex curve  $\ell$  in  $\mathbb{CS}$ .

*Example 9* A sample twistor curve  $\ell \in \mathbb{CS}$  is shown in Fig. 4, which is a geometric realization of a Feynman trajectory of a particle (see Def. 4), which leads to a space-time view of a Lefschetz arc (see Def. 4 and Lemma 3). ■

**Definition 4** The **trajectory of a particle** in a 2-plane in curved space-time is a mapping

$$h : \{\mathbb{R}^2 \times S^2\} \rightarrow \{\mathbb{R}^2 \times S^2\}$$

defined by

$$h(\{t^{t_{x_i}}\}) = \widehat{t_{x_0}, t_{x_i}} \cup \bigcup \widehat{t_{x_i}, t'_{x_{i'}}} = \ell, t, t' \in \mathbb{R}, x_i \in S^2, i \in I,$$

in which each  $\widehat{t_{x_i}, t'_{x_{i'}}}$  is a space-time line segment in a curve  $\ell$  starting with subarc  $\widehat{t_{x_0}, t_{x_i}}$  in a Lefschetz arc at times  $t_t$  (instant  $t$  in region time  $t$ ) with index  $i$  in the unit interval  $I = [0, 1]$  is mapped to an arcwise-connected set, *i.e.*, the line segments in the trajectory are attached to each other and starting with  $\widehat{t_{x_0}, t'_{x_i}}$ , there is a path from any subarc a sequence of subarcs can be traversed to reach an ending subarc  $\widehat{t_{x_n}, t'_{x_{n'}}}$  in a N.M.J. Woodhouse [2] twistor space  $\mathbb{R}^2 \times S^2$  with metric signature  $++--$ . ■

*Remark 6* From Def. 4, the vectors in  $h(\{t_{x_i}\})$  are J.H.C. Wodehouse zero cells in an arcwise-connected space  $\mathbb{R} \times S^2 \times \mathbb{R}$ . ■

**Definition 5 (Lefschetz arc).**

A **Lefschetz arc**  $E$  is a curve  $\ell$  attached between a pair of 0-cells  $p, p'$ . The curve  $\ell$  is dense and the points in  $\ell$  are path-connected, *i.e.*, between every pair of points  $q, q'$  in  $\ell$ , there is a sequence of sub-arcs traversable between  $q$  and  $q'$ . ■

**Lemma 3 (Lefschetz trajectory arc).**

*A trajectory of a particle is realizable as a Lefschetz arc.*

*Proof* From Def. 4, a trajectory  $h$  is a curve  $\ell$  that starts and ends with a 0-cell and is the union of subarcs in an arcwise-connected space. Hence, from Def. 5, the trajectory  $h$  is realizable as a Lefschetz arc.

*Example 10* A sample trajectory of a particle as a Lefschetz arc over a twistor curve realized as a Lefschetz arc  $\ell = \widehat{p\vec{c}}$  with endpoints (0-cells)  $\vec{p}, \vec{q}$  and which is the union of sub-arcs is shown in Fig. 4. ■

**Definition 6 (Unit interval).**

The unit  $I = [0, 1] \in \mathbb{R}$  is the set of all real values in the closed interval with initial value 0 and ending 1 and an unbounded number of consecutive everywhere dense subintervals of real values between 0 and 1. That is, every real number  $x$  in a subinterval of  $A \subset I$  has another real number  $x' \in A$  that is arbitrarily close to  $x$ . ■

**Lemma 4** *The trajectory of a particle is continuous.*

*Proof* From Def. 6,  $I$  is dense and is the index set for the points in the trajectory of a particle. From L. Susskind and A. Friedman [18, p. 238], a particle moving along the Lefschetz curve can be found at any real value in the unit interval  $I = [0, 1]$ . Let  $h$  be the trajectory of a particle. Then if  $i, i' \in I$  are close, then  $t_{x_i}, t_{x_{i'}}$  are close. Hence, the trajectory  $h$  is continuous. ■

**Example 11 (Trajectory of a Particle).**

Let  $g : I \rightarrow \mathbb{R}$  be defined by  $g(t_i) = t_i, i \in I, g(t_i) \in \mathbb{R}$  for each time  $t_i$  in the trajectory of a particle. That is,  $I$  is a index set for the instants of time of occurrence of the points in the trajectory of a particle over a vector field. The map  $f$  is continuous, since, for every pair  $t_i, t_j, i \neq j$ , if  $i, j$  are arbitrarily close, then  $g(t_i), g(t_j)$  are arbitrarily close. Hence, the  $g$  is continuous. ■

## 5 Feynman Path Integral

In this section, it is observed that a Feynman path is continuous (Lemma 5), which leads to the results in Theorem 7 and Theorem 8 for Feynman paths, which are consequences of the Borsuk-Ulam Theorem.

**Definition 7** A Feynman path is a function  $h(t_{x_i}) = x_i$  for a particle at point  $x_i$  at time  $t$  [7, p.31]. ■

**Lemma 5 (Feynman Path).**

*Every Feynman path is continuous.*

*Proof* Let  $h : R^2 \times \mathbb{R}^2 \rightarrow S^2$  be a Feynman path, defined by  $h(t_{x_a}) = x_a$  which is the trajectory of a particle at point  $x_a$  at time  $t_t$ . Let  $\ell$  represent that a particle travels over during its trajectory and let  $h(t_{x_a}) = x_a$  be a point in  $\ell$ . For simplicity, the curve  $\ell$  is referred to as the trajectory of a particle and  $p$  is point in  $\ell$ . During the passage of a particle over  $\ell$ ,  $\ell$  has no gaps in it. That is, for each point  $h(t_{x_a}) = x_a \in \ell$  at time  $t_t$ , if  $x_a$  is arbitrarily close to point  $h(t_{x_{k+1}}) = x_{k+1} \in \ell$  at time  $t_t$ , then  $h(t_{x_a})$  is arbitrarily close to  $h(t_{x_{k+1}})$ . Hence, a Feynman path  $h$  is continuous.

**Remark 7** In Lemma 5, the continuity of a Feynman path  $h$  is explained in terms of the closeness (nearness) paradigm from [19, §1.5, p. 8], instead of the abstract (less intuitive)  $\epsilon - \delta$  view of continuity. This approach befits the character of the trajectory of a particle over a curve  $\ell$ , where both the trajectory of a particle and the curve  $\ell$  (without gaps) traced by particle trajectory. Just as pairs of points in the curve  $\ell$  can be arbitrarily close, so too, from Lemma 4, the vectors  $h(t_{x_a}), h(t_{x_{k+1}})$  in the trajectory of a particle can be arbitrarily close. ■

The value of a path between points  $a$  and  $b$  on a curve  $\ell$  (the positions of a particle trajectory at times  $t_a, t_b$ , respectively), is  $K(b, a)$ , defined in a complex space  $\mathbb{CS}$  with respect to Planck's constant  $\hbar$  by Feynman and Hibbs [7, p. 45] by

$$\begin{aligned} V(x, t) &= \text{Potential energy of particle with mass } m. \\ L &= \frac{m}{2} \dot{x}^2 - V(x, y) (\text{Lagrangian for the system}). \\ S &= \int_{t_a}^b L(\dot{x}, x, t) dt. \\ K(b, a) &= \int_a^b e^{(\frac{i}{\hbar})S[b, a]} \mathcal{D}x(t). \end{aligned}$$

**Theorem 7 (Feynman Path Theorem).**

Let  $h : S^3 \times \mathbb{R}^3 \rightarrow S^3$  be a Feynman path over a curved surface  $S^3$  and let  $K(b_k, x_a), K(-x_a, b_k)$  be the value of path  $h$  containing points  $b_k, x_a$  in a segment  $\overline{b_k, x_a}$  in a curve  $\ell$  starting at point  $x_a$  and with endpoint  $b_k$  attached to a segment  $\overline{-x_a, b_k}$  starting at point  $b_k$  and with endpoint  $-x_a$ . For each initial point  $x_a$ , there exists an antipodal point  $-x_a$  in path  $h$  such that  $K(b_k, x_a) = K(-x_a, b_k)$ .

*Proof* From Lemma 5, a Feynman path  $h$  is continuous. Hence, from Theorem 1, we obtain the desired result for antipodal points  $x_a, x_b$  in path  $h$ .

**Trajectory Results.**

**Theorem 8 (Feynman Trajectory-of-Particle Theorem).**

The Feynman trajectory of a particle satisfies Borsuk-Ulam Theorem 1. Let  $h : S^n \rightarrow \mathbb{R}^n$  be the trajectory of a particle on the surface of sphere. There is at least one pair vectors  $\vec{p}, \vec{p}' \in S^n$  such that  $h(\vec{p}) = h(\vec{p}')$ .

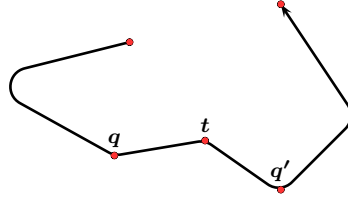
*Proof* From Lemma 5, a Feynman trajectory is continuous. Hence, from Theorem 1, we obtain the desired result for antipodal points  $\vec{p}, \vec{p}' \in S^n$  in Feynman trajectory  $h$ .

**Theorem 9 (Feynman Path Integral Theorem).**

There exists a Feynman path with an initial path integral  $K(b_k, x_a)$  for an initial vector  $\vec{x}_a$  that equals the path integral  $K(b_k, -x_a)$  for a later vector  $-\vec{x}_a$ , which may or may not be the antipode of vector  $\vec{x}_a$ .

*Proof*  $K(b_k, x_a)$  are Feynman path integrals that resonate (have values) for a particle that has gradients on either two different surface curvatures along a surface curve  $\ell$  or on the same surface curvature on a path  $\ell'$  for a boomerang trajectory that follows a path that is a cycle. In either case, choose an intermediate point  $b_k$  in the path between  $\vec{x}_a$  and  $b_k$  so that the two segments on  $\ell$  have the same length. In that case,  $K(b_k, x_a) = K(b_k, -x_a)$

*Remark 8* The significance of Theorem 9 is that the endpoints on a particle trajectory curve  $\ell$  need not be antipodal points. That is, Theorem 9 is more general than Theorem 7. ■



**Fig. 5** Wodehouse contour integrals on sub-twistor curve antipodes  $q, q'$  with  $\oint_{qt()} = \oint_{tq'()}$ .

## 6 Woodhouse Borsuk-Ulam Theorem

This section briefly introduces N.M.J. Woodhouse contour integrals [2, p. 198], defined with respect to real  $\alpha$ -planes in a twistor space  $\mathbb{R}^2 \times S^1$  in which  $S^1$  represents the  $\alpha$ -planes that lie in the null cone at  $\infty$ , over which we have

$$\begin{aligned} \xi &= x_1 + ix_2 \text{ and } \tau = t_1 + it_2, \text{ representing } \alpha\text{-planes as surfaces, with} \\ w &= \xi + \bar{z}r, \bar{w} = \bar{\xi} + zr, \text{ constant for } z = e^{i\theta}. \\ \phi(\xi, \tau, \bar{\xi}, \bar{\tau}) &= \frac{1}{2\pi} \oint_{|z|=1} f(\xi + \bar{z}\tau, \bar{\xi} + z\bar{\tau}, z) \frac{dz}{z}. \end{aligned}$$

**Definition 8 (Contour Integral as a Smooth function).**

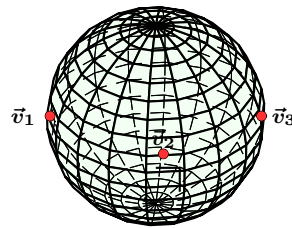
The contour integral  $\phi$  is a smooth function on a twistor space [2]. That is,  $\phi$  is continuous. ■

### Antipodal Contour Integrals Result.

**Theorem 10** *The contour integral  $\phi$  satisfies the Borsuk-Ulam Theorem.*

*Proof* From Def. 8, the contour integral  $\phi$  is a continuous function. Let  $p, -p$  be antipodes on a twistor curve in  $\mathbb{R}^2 \times S^1$ . Replace the Feynman path integral with the Wodehouse contour integral in the proof of Theorem 9, and the desired result follows.

*Example 12* Sample contour integrals on sub-twistor vectors that are antipodal are shown in Fig. 5. ■



**Fig. 6** Wodehouse contour integrals on sub-twistor curves  $\widehat{\vec{p}_1 \vec{p}_2}, \widehat{\vec{p}_2 \vec{p}_3}$  with  $\widehat{\vec{p}_1 \vec{p}_2}(\cdot) = \widehat{\vec{p}_2 \vec{p}_3}(\cdot)$ .

### Non-Antipodal Contour Integrals Result.

**Theorem 11** *Let  $\phi_p, \phi_{p'}$  be the Wodehouse contour integral over a twistor curve  $\ell$  and let  $p, p'$  be any two distinct points on  $\ell$ . Then there are  $\phi, \phi'$  such that  $\phi_p = \phi_{p'}$ .*

*Proof* Again, replace the Feynman path integral with the Wodehouse contour integral in the proof of Theorem 9, and the desired result follows. That is, we can always find a point  $q$  between  $p, p'$  on the twistor  $\ell$  such that  $\phi_p = \phi_{p'}$ .

*Remark 9* Theorem 11 covers a broader spectrum of twistor length measurements than Theorem 10. That is, for any pair of distinct vectors on a twistor curve, we can always find an intermediate vector so that the contour integrals over the resulting twistor sub-arcs have equal value. ■

*Example 13* Sample contour integrals on sub-twistor curves  $\widehat{\vec{p}, \vec{q}}, \widehat{\vec{p}', \vec{q}}$  with endpoints  $\vec{p}, \vec{p}'$  that may or may not be non-antipodal are shown in Fig. 6. ■

## 7 Concluding Remarks

The Borsuk-Ulam Theorem is topological with an implicit surface geometry. That is, the focus in the Borsuk-Ulam Theorem is on a continuous map from the surface of a sphere  $S^n$  to real values of antipodes in  $\mathbb{R}^n$ . The geometry underlying the Borsuk-Ulam Theorem looms up, when we consider where to look for antipodes such as the endpoints on an arc stretching over the curved surface of the Earth between a place such as the University of Manitoba and another place at varying space-times with the same latitude and longitude.

The beauty of the Borsuk-Ulam Theorem is that it serves as a roadmap, telling us how to look for surface antipodes (with continuous maps), where to look (*e.g.*, endpoints of twistor curves that are antipodal) and how to measure antipodal distances. In this paper, the Borsuk-Ulam Theorem is an emperor with new clothes to wear, namely,

- 1° **How to look:** consider either a discrete or continuous map.
- 2° **Where to look:** endpoints of twistor curves that are either antipodal or non-antipodal.
- 3° **Length-of-arc measure:** *e.g.*, measure with either a Feynman path integral or Wodehouse contour integral over arcs having antipodal endpoints.

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## 8.3 Author Contributions

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