A RESOLUTION TO HILBERT'S FIRST PROBLEM

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A Resolution to Hilbert's First Problem By Applying Cantor's Diagonal Argument with Conditioned Subsets of $\mathbb{R}$, with that of $(\mathbb{N}, \mathbb{R})$.

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Abstract

*Herein, we introduce a novel approach to the Continuum Hypothesis (**CH**). Our methodology involves the utilization of string conditioning, a technique aimed at constraining the scope of a string across specific segments of its sub-domain. This, in turn, facilitates the creation of subsets denoted as K within the set of real numbers, $\mathbb{R}$.*

Our objective is to establish the well-defined nature of these subsets, confirming their status as proper subsets of $\mathbb{R}$. To achieve this, we harness Cantor's Diagonal Argument in its original formulation. This enables us to determine the cardinality of the subset K," positioning it within the cardinality spectrum between that of $\mathbb{N}$ (the set of natural numbers) and $\mathbb{R}$.

Keywords

Diagonal Argument, Continuum Hypothesis CH, Resolution to CH

1 Preface

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The Continuum Hypothesis (CH) stands as one of, if not the most, pivotal unsolved conundrums within the realm of set theory. Its significance spans both mathematical and philosophical domains.

From a philosophical perspective, and potentially in practical terms as well, mathematicians find themselves divided regarding the potential resolution of CH. The enigmatic persistence of this quandary has given rise to multiple prevailing viewpoints concerning its potential solution. Noteworthy discussions revolve around the plausibility of finding a resolution. One notable instance is the discourse emanating from the Institute for Advanced Study at Princeton, which provides a concise overview of the various thoughts surrounding the problem, along with a comprehensive summary of the progress made thus far, and exploration of what could potentially constitute a solution (see [13]).
Numerous comparable discussions articulate the prevailing sequence of thoughts on this subject, highlighting the division among mathematicians regarding their perspectives on a resolution, the nature of that resolution, and the implications of a solution to CH. Several prevailing viewpoints can be succinctly summarized as follows:

Finitist Mathematicians: This perspective asserts that we are exclusively concerned with the finite, making it challenging to make definitive claims about the infinite. Put simply, the infinite realm remains largely inscrutable from this standpoint.
Pluralists Mathematicians: Pluralists hold that various outcomes of CH are equally plausible. Despite the groundbreaking achievements of Cohen and Gödel, who demonstrated the consistency of ZFC + /CH and ZFC + CH respectively, Cohen adhered to a robust pluralist stance. His establishment that CH cannot be conclusively determined from ZFC alone was, in his view, a significant resolution of the matter. (For Cohen's independence results, refer to [11].) In contrast, Gödel contended that a well-justified extension of ZFC was the requisite approach to settle CH. Gödel's program, a promising pathway, aimed to develop ZFC extensions capable of resolving CH. Gödel himself proposed the large cardinal axioms as a potential candidate.
Gödel's Program: *To decide mathematically intriguing questions independently of ZFC within well-justified ZFC extensions.*
However, this extension proved inadequate for the task of settling CH, as evidenced by the work of Levy and Solovay. Cohen's pioneering forcing techniques, meanwhile, paved the way for establishing a range of consistency results, including the inadequacy of the large cardinal extension of ZFC in addressing CH—also demonstrated by Levy and Solovay (see [12]).
A pivotal turning point emerged through the contributions of W.H. Woodin, whose work demonstrated the effective failure of CH (based on large cardinals) using a canonical model in which CH does not hold [9]. It is widely acknowledged that *forcing* cannot be wielded to resolve CH, prompting us to explore innovative concepts beyond the confines of *forcing*.

2 PREVIOUS ATTEMPTS AT A RESOLUTION TO CH

The origins of the Continuum Hypothesis (CH) can be traced back to the late 19th and early 20th centuries when mathematicians were exploring the concept of infinity and the cardinality of infinite sets.

The Continuum Hypothesis was formally stated by Cantor in 1878. It asserts that there is no cardinality between the cardinality of the natural numbers and the cardinality of the real numbers. In other words, there is no set with cardinality strictly greater than $\aleph_0$ and strictly smaller than $\aleph_1$. He developed the theory of cardinality, which assigns a cardinal number to each set, indicating the size or "bigness" of the set. Cantor discovered that not all infinite sets have the same cardinality, and he introduced the concept of different infinite sizes or "degrees of infinity." Cantor also showed that the cardinality of the natural numbers (denoted by $\aleph_0$ or aleph-null) is the smallest infinite cardinality, and he conjectured that the next larger cardinality is the cardinality of the real numbers (denoted by $\aleph_1$ or aleph-one). This conjecture is essentially equivalent to the Continuum Hypothesis. This initial insight was demonstrated by a *diagonal* argument bearing his name. The basic idea of the argument is as follows:
Assume, for the sake of contradiction, that $\mathbb{R}$ is countable, meaning its elements can be

listed as a sequence: $r_1, r_2, r_3, r_4, ...$ Form a new real number x by constructing its decimal representation in a specific way. In the ith decimal place, choose a digit different from the ith digit of the number ri. For example, if the ith digit of r_i is 3, select 7 for the ith digit of x. The resulting number x is guaranteed to be different from every number in the assumed countable list $r_1, r_2, r_3, r_4, ...$ because it differs from each of them in at least one decimal place. Thus, we have constructed a real number x that does not appear in the assumed countable list, contradicting the assumption that $\mathbb{R}$ is countable. The key insight of Cantor's diagonal argument is that by constructing a new element that differs from each element of a given countable list, we can demonstrate the existence of elements outside the list. This shows that there are "more" real numbers than can be enumerated in a countable manner. For elementary texts on this topic, see for instance [1], [22].

Cantor made significant efforts to prove or disprove the Continuum Hypothesis but was unable to settle the question. He corresponded with fellow mathematicians and worked on the problem for many years, but a definitive answer eluded him.
The Continuum Hypothesis remained a major open question in set theory until the early 20th century. In 1900, David Hilbert included the problem of the Continuum Hypothesis as one of his famous 23 unsolved problems presented at the International Congress of Mathematicians in Paris. This helped elevate the status of the problem and spurred further investigations. For a full detailed account of this see for instance [15]. The search for a resolution to the Continuum Hypothesis continued throughout the 20th century, with numerous mathematicians attempting to prove or disprove it. Notable mathematicians such as Kurt Gödel, Paul Cohen, and Saharon Shelah made significant contributions to the study of the Continuum Hypothesis and its independence from the standard axioms of set theory.

Gödel, famous for the independence results, specifically the Gödel's Incompleteness Theorems which demonstrate the existence of undecidable propositions in Peano Arithmetic (PA), is a famous result in mathematical logic. Before we outline Gödels ideas on CH, we will outline his brilliant proof of the *Incompleteness Theorems*. At a high level, Gödel's proof begins by representing the syntax and semantics of PA within the system itself. This encoding allows the system to reason about its own statements and proofs. Gödel constructs a method to encode formulas and proofs of PA as numbers. This encoding enables the system to manipulate and reason about its own syntactic objects. Gödel assigns unique numbers (Gödel numbers) to formulas and proofs in PA. This encoding is recursive and captures the structure of formulas and proofs. Gödel uses a diagonalization argument (Diagonal argument as is more conventional), to construct a formula that asserts its own unprovability within PA. This formula is referred to as the Gödel sentence or the diagonal lemma. By establishing the unprovability of the Gödel sentence within PA, Gödel demonstrates that there exists a true statement that is not provable in the system. This shows the incompleteness of PA. Gödel's proof shows that any consistent formal system that can encode arithmetic, such as PA, will have undecidable propositions. These undecidable propositions cannot be proven or disproven within the system itself. See for instance [22], and [23].

To provide an outline of this fascinating result, we take from *Smulliyan*: Let X be some

encoded-expression, then the following is possible:

Let P stand for printable,
N norm of,
and ! not.
$P(X) \rightarrow True$ if X is 'printable'.
$P(N(X)) \rightarrow True$ if $N(X)$ is 'printable'.
$!P(X) \rightarrow True$ if X is NOT 'printable'.
$!P(N(X)) \rightarrow True$ if $N(X)$ is NOT 'printable'.
Given that 'the machine' never prints false sentences:
The sentence $PN(!PN(X))$ is true if the norm of $(!PN(X))$ is printable, as $PN(..)$ means 'Printable, Norm of that which lies within (..)'. But this means that if we place $!PN$ within, the statement then translates to 'Printable, Norm of that which lies within (Norm of this not Printable(X))'. This either means that: the sentence is true and not printable, or it is printable and not true. The latter violates our hypothesis that the machine is only capable of printing true statements.
The significance of this is that all systems 'morphic' to the above in a manner of setting up statements, then Gödels argument is made. The infinitely more significant result is that Arithmetic is one such formal system.

Gödel's proof of the independence of CH builds upon this earlier work on incompleteness theorems. His ideas were highly inspired by the work of Cantor, as was the case for Turing as well. This was a truly revolutionary period of mathematical enlightenment. Gödel established that within any consistent formal system that is sufficiently 'powerful' to express arithmetic, there are true statements that cannot be proven within that system. Gödel used a technique called the constructible universe, denoted by L, which is a particular model of set theory. In this model, sets are constructed in a step-by-step fashion using a hierarchy of stages. Gödel then introduced a hierarchy of sets called the constructible hierarchy. Each stage of this hierarchy represents a level of the cumulative hierarchy of sets, and it is constructed based on the previous stages. A notion of constructible sets is then defined within his constructible hierarchy. These sets are built using formulas of set theory, and each constructible set is associated with a particular formula. The reflection principle ensures that if a statement is true at one stage of the constructible hierarchy, then it continues to be true at later stages. Gödel then showed that within the constructible universe L, the continuum hypothesis holds. In other words, within L, it is true that there is no set whose cardinality is strictly between that of the natural numbers and the real numbers. Finally, Gödel constructs a different model of set theory (referred to as the "Gödel model") in which CH is false. This model is obtained by considering a larger universe of sets that extends beyond L and introducing certain additional sets that violate CH.

In 1963, Paul Cohen presented his ground breaking proof that the Continuum Hypothesis is independent of the standard axioms of set theory. This meant that the hypothesis cannot be proved or disproved within the existing framework of set theory. The results of Cohen were inspired by those of Gödel and Cantor.
The independence of the Continuum Hypothesis had a profound impact on the field of set theory and the understanding of mathematical infinity. It highlighted the inherent com-

plexity and richness of infinite set theory and paved the way for further investigations into different cardinalities and the structure of the continuum. For a detailed account of independence results see for instance, [3], [4]. We here give a small account of the work done by Cohen on the Continuum Hypothesis.

First, let us define some notation. For any countable ordinal α, let 2^α denote the set of all functions from α to 2, and let $2^{<\alpha}$ denote the set of all finite functions from α to 2. We can think of $2^{<\alpha}$ as the set of "partial" functions from α to 2, i.e., functions that are only defined on a finite initial segment of α. We order $2^{<\alpha}$ by extension, so $p \leq q$ means that p extends q, i.e., p is a stronger condition than q. We say that p and q are compatible (written $p \vdash q$) if there exists r such that $r \leq p$ and $r \leq q$.
Now, let V be a model of ZFC, and let G be a generic filter over V for the forcing notion $(2^{<\alpha}, \leq)$. We say that G is a Cohen generic filter if it has the following two properties:
G is downward-closed: if $p \in G$ and $p \leq q$, then $q \in G$. G intersects every maximal antichain in $2^{<\alpha}$, i.e., every collection A of pairwise incompatible elements of $2^{<\alpha}$ has a common extension in G. Note that property (2) implies that G is maximal with respect to the ordering $\leq$. In other words, if $p \notin G$, then there exists a q such that $q \leq p$ and q is incompatible with every element of G.
We can now define the Cohen generic extension $V[G]$ of V. The universe $V[G]$ consists of all sets that can be constructed using elements of V and elements of G. Specifically, for each name τ in V, we define its interpretation τ^G in $V[G]$ as follows:
If τ is a ground set, then $\tau^G = \tau$. If τ is a name for an element of $2^{<\omega}$, then τ^G is the function in 2^ω defined by $\tau^G(n) = 1$ if and only if $m < n : \tau(m) = 1 \in G$. The key fact about Cohen forcing is that it adds a new subset of ω to V. Specifically, the set $n \in \omega : \tau^G(n) = 1$ is a new subset of ω that is not in V. This new subset has the property that it is not constructible from any set in V. In particular, it is not constructible from any countable sequence of sets in V.
To see why this is the case, suppose for contradiction that there exists a sequence $(S_n : n < \omega)$ of sets in V such that $n \in \omega : \tau^G(n) = 1 = \bigcup_{n<\omega} S_n$. Then each S_n is constructible from a countable sequence of sets in V, say $(T_{n,m} : m < \omega)$. Since V is a model of ZFC, there exists a formula $\phi(x)$ such that for each n, the set $m < \omega : T_{n,m} \in x$ is the n-th element of the sequence S_n if $\phi(x)$ is true, and the empty set otherwise. Since the sequence $(S_n : n < \omega)$ is not in V, there exists a Cohen condition p such that p forces $\neg\phi(G)$. But this contradicts the fact that G intersects every maximal antichain in $2^{<\omega}$.
Finally, it is a well-known result that the addition of a Cohen subset of ω to V is independent of ZFC (See [6], [7], [8]).

One way to visualize the Cohen forcing notion is to imagine a binary tree whose nodes correspond to partial functions from ω to 2. The root of the tree corresponds to the empty function $\{\}$, and the children of a node corresponding to a partial function f are obtained by extending f with a new pair (n,b), where n is a natural number not already in $dom(f)$ and b is either 0 or 1. The nodes are formed in a manner that have a chain for each ordinal in the base model.
At each level of the tree, we have a finite number of choices to make, corresponding to the possible values of the next unused natural number and the next bit in the binary representation of the function. At the limit levels of the tree, we have a branch for each possible

function from ω to 2. The partial order on P is defined by saying that a node correspond-

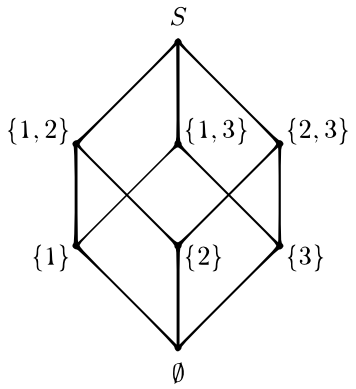

Figure 1: Example of chains and anti chains via the use of Hasse Diagrams.

ing to a partial function f is less than or equal to a node corresponding to a partial function g if and only if g extends f, that is, $dom(f)$ is a subset of $dom(g)$ and $g(x) = f(x)$ for all x in $dom(f)$.
A generic filter for the Cohen forcing notion can be thought of as a path through the tree that includes all the branches that correspond to a condition in the filter. Intuitively, a generic filter "chooses" one branch from each level of the tree in a way that is consistent with the ordering relation.
The beauty of the *forcing* technique is that, should the models constructed, behave transitively, then the model is one of ZFC. However since we can find two models that model ZFC, one supporting CH and one supporting ¬CH, one has that CH is undecidable from ZFC alone.

Forcing, as a method in set theory, allows us to construct mathematical models (forcing extensions) in which certain statements are either true or false. However, forcing cannot definitively resolve the Continuum Hypothesis (CH) because it does not provide a conclusive answer as to whether CH is true or false in the standard set-theoretic universe. The main reason forcing cannot settle CH is that it does not add any new information about the truth value of CH in the original set-theoretic universe. Instead, forcing allows us to construct additional models of set theory, called forcing extensions, in which we have more freedom to manipulate certain properties and values.
When applying forcing to the Continuum Hypothesis, we can construct forcing extensions in which CH is true and others in which CH is false. This shows that CH is independent of the standard axioms of set theory because both possibilities can be consistently realized.
In other words, forcing demonstrates that there are models of set theory in which CH is true and models in which CH is false. This independence result implies that CH cannot be settled within the confines of the standard axioms of set theory alone. It indicates that additional axioms or principles beyond the standard ones are needed to establish the truth or falsity of CH (Again, see [3], [4]).

For these reasons, we will be turning to some new practical techniques that arose from a study into *partial bit encryption*. This study is strongly coupled with Information The-

ory which for the interested reader is a branch of mathematics and computer science that deals with the quantification, storage, and communication of information. As this is not the topic of this article, we leave some references for the interested reader. See for instance [17], [18], [19] and [20].

3 INTRODUCTION TO THE MAIN IDEA

Let us pause at this juncture to clarify the underlying purpose of the upcoming paragraphs. Firstly, our goal is to devise a formal language capable of effectively constructing well-defined subsets of $F := \bigcup_{\forall i} f_i : \omega_< \to 0,1$, where ($\omega_<$ signifies a finite set of ordinals or ordered numbers). This entails restricting the range of functions f_i to 0 over arbitrary domain values within $dom(f_i) := \omega_< := 0,1,2,..,n$. These functions, in essence, compose the elements within the sets of our interest. An essential observation, albeit straightforward, is that the collection of all sequences of length n exhibiting 'conditioned segments' serves as an exemplar of a subset within F.

Secondly, our objective is to devise a methodology for determining the cardinality of these *conditioned sequence sets*, which now encompass sequences of non-finite length, in relation to the entirety of $SEQ := \bigcup_{\forall i} f_i : \mathbb{N} \to 0,1$. To attain this outcome, we will employ inductive reasoning.

The conceptual ambiguity inherent in the first point becomes distinctly evident through an illustrative example. Suppose our aim is to construct a finite set containing elements of the form $120005, 340004, 710004, ...$, implying that each element constitutes a 6-digit sequence mapping from $0,..,5$ to $0,..,9$, with s_2 to s_4 consistently being 0. Such sequences represent a subset of the broader collection of all sequences with a length of 6.

It becomes evident that a succinct and mathematically precise language capable of effectively generating these conditioned sequence sets holds significant value, particularly in the context of establishing cardinality.

Returning to point two, the fundamental diagonal argument operates effectively when one can demonstrate that for each element within a set S, there exist infinitely more elements within, let's say, $\mathbb{R}_2$, as implied by the $Diag(S, \mathbb{R}_2)$ argument (where $\mathbb{N}_2, \mathbb{R}_2$ represent naturals and reals in base two, respectively). A subtle yet significant nuance to recognize is that the mapping $\mathbb{N}_2 \to S$ need not be surjective for the *Diag* argument to hold. While seemingly trivial, this detail holds importance, signalling that when searching for a set S with a cardinal existence between $\mathbb{N}_2$ and $\mathbb{R}_2$, it suffices that S is a subset of $\mathbb{R}_2$ without necessitating a surjection onto $\mathbb{N}_2$ for the *Diag* argument to retain its validity.
Considering the conventional application of the diagonal argument, one can discern that $\mathbb{R}$ is traditionally treated as encompassing all infinite sequences in the form of $\mathbb{N} \to 0,1$ (Refer, for example, to Chapter 5 in [2]). In essence, every element within $\mathbb{R}$ can be represented as an infinite binary sequence within $\mathbb{R}_2$. This naturally prompts the question: Can subsets of SEQ be constructed?
With a touch of creativity, several ideas emerge, aligning with the concepts outlined at the outset of this section. One potential avenue involves constraining the *Range* values of f_i to 0 over specific segments of its domain. Indeed, all elements of the form

$S := \bigcup M_1 0 M_2 00 M_3 000 M_4 0000...$ where $M_i := f_i : 0,..,e \to 0,1 | e \in \mathbb{N}$ form a subset of SEQ.
This premise invites intriguing inquiries: What is the cardinality of S concerning $\mathbb{N}$ and $\mathbb{R}$ respectively? How can the diagonal argument be adapted or reused to facilitate this comparison? These are some of the questions that we endeavour to address within this letter, intended for the consideration of experts.
An essential point to contemplate pertains to the strict exclusion of S from the realms of $\mathbb{N}$ and $\mathbb{Q}$—an aspect we will delve into subsequently.
With our temporary designation of the conditioned sets as S, our objective is twofold: firstly, to establish $Diag(\mathbb{N}_2, S)$ and $Diag(S, \mathbb{R}_2)$—here, employing "Diag" as a shorthand reference to Cantor's diagonal argument—between the sets denoted within the parentheses (analogous to the well-established diagonal argument between $(\mathbb{N}, \mathbb{R})$). Secondly, we aim to present a compelling rationale for the interchangeable use of the diagonal argument in the subsequent sentences. The elucidation of this rationale will become apparent in subsequent sections, representing the primary focus of this article.

A) *Given* $0,1,2,....,12,...,1000,..,$ *(i.e., all naturals) there are infinitely many more Reals*

B) *Given some arbitrary collection* $e_1,....,e_n,.... \subset \mathbb{R}$ *there are infinitely many more Reals*

C) *Given some arbitrary collection* $1,2,....,12,...,1000,..,$ *(i.e., all naturals), there are infinitely many more* $e_1,....,e_n,.... \in S \subset \mathbb{R}$

All of which should be establish-able via $Diag(\mathbb{N}, S)$ and $Diag(S, \mathbb{R})$.

The task at hand would involve selecting a collection that possesses the requisite attributes to facilitate the execution of $Diag(\mathbb{N}, S)$ and $Diag(S, \mathbb{R})$. Fortunately, this collection S can indeed be any collection meeting these criteria.

4 FORMULATION

The objective of this section is to present definitions that are employed to establish precisely defined subsets of $\bigcup_{\forall i} f_i : \omega_< \to 0,1$, adhering to the previously outlined properties.
Our ultimate goal is to construct an inductive argument that initially pertains to finite sequences, but inevitably extends its scope to encompass infinite sequences.

Definition (Sequence-Function)

We establish the concept of a **finite** sequence-function, characterized as a function denoted by $f : \omega_< \to 0,1$, where $\omega_<$ signifies the ordered set of numbers commencing from 1 and extending up to a certain arbitrary $n \in \mathbb{N}$, represented as $1,..,n$ within its domain. Additionally, the range of f is confined to $range(f) := 1,0$.
Throughout this article, these particular functions, as well as sets composed of them, are symbolized as f, s, S, and S'—unless expressly specified otherwise.

Definition (Sequence-Function Sets)

We define the term "sequence-function set" to describe any set of the form $S := \bigcup_{\forall i} f_i : \omega_< \to 0,1$, designated by the symbols S and S'.

Definition (Sum)

For any given sequence-function f within the set S, we define the binary operation $+_A$ as follows: For $f +_A f$, the operation involves applying the inverse binary numeral function $Bnum^{-1}$ to the sum of the binary number representations obtained from $Bnum(f)$ and $Bnum(f)$. Here, $Bnum(f)$ is a function that maps elements in $Bnum(f)$ to binary values in $\mathbb{B}$, where $\mathbb{B}$ represents the set of binary numbers. The notation $+$ signifies the standard arithmetic addition operation.

To elaborate further, let b denote the binary number equivalent of the image produced by the function $Bnum(f)$. Similarly, let $Bnum^{-1}(b \in \mathbb{B})$ be a function that maps binary numbers b to the corresponding ordered image f.
In concise terms, the expression $f +_A f = Bnum^{-1}(Bnum(f) + Bnum(f))$ captures the operation's essence, utilizing these functions and concepts to combine two instances of the sequence-function f through arithmetic addition.

Remarks

$Bnum(f) + Bnum(f)$ is written A_2, and in general A_j for many such sums.
As an example: $A_2(1001) = 10010$. It is to be clearly mentioned that the domain of the function(singular) remains unaffected after *Summing* which strictly affects the image of f alone.

Definition (Conditioned sub-sequence)

Given an arbitrary sequence-function f, we define a segment I_K of the $Image(f)$ where $Range(f) := 0$ holds for two or more domain values. This segment is referred to as a *conditioned sub-sequence*.

Remarks

These conditioned sub-sequences, denoted by the functions f_K, are strictly partial functions associated with f. Both f_K and f share identical domain values over the interval I_K. The domain sets linked to these image intervals are denoted as $D(I_i)$ and $D(/I_i)$, where $D(/I_i)$ encompasses the domain values not associated with f_K.

Definition (Length)

Given an arbitrary sequence-function, let $s_k := s[n_1, n_2] \in s \in S$ represent a random image segment of s, where $Dom(s_K) = [n_1, n_2]$ signifies the domain interval of the segment s_K. The notation $\mathscr{M}(s_K[n_1, n_2])$, denoted as $\mathscr{M}(s_k)$ when unambiguous, is defined to represent

the length of the segment, expressed as $|n_2 - n_1|$ with $n_i \in \mathbb{N}$.

Remarks

When referring to the length of the entire function, we succinctly write $\mathscr{M}(f)$.

In order to facilitate the selection of such elements for $S \subset \mathbb{R}$, our focus is directed towards sequences that cannot be reduced to a natural number through finite summation. This pursuit, as we will soon discover, presents a challenging task.

It is widely recognized that specific rational numbers, like 1/3 = 0.3333..., are linked to fractional components displaying sequence-like attributes, specifically, non-terminating decimals. Therefore, when our focus is exclusively on *irrational sequences*, it becomes essential to establish a method for eliminating these cases from the power set $2^{\aleph_0}$. Intuitively one way of attempting this is to have some idea of what may constitute an *'irrational sequence'*. It turns out that binary sequences following a certain schema (SH) belong (Or are comparable to) to a subset SS, of the irrational numbers . Numbers such as $(Y.11111111...), (Y.333333...)$ having a sequence-like fractional portion,i.e., an infinite sequence of numbers, can be transformed into to a natural number via finite Sum, thus In searching for such a schema, it should be noted that such periodically repetitive sequences are to be discounted.
The formulation of schema (SH), which represents a generalization encompassing various arrangements of possible 0's and 1's, thus constructing binary sequences, establishes a robust connection with the impact of addition on binary numbers. To illustrate this relationship, let's consider the straightforward case of $010010 + 010010 = 100100$. Here, the alignment of (1) symbols in such summations results in the shifting of the (1) by one position to the left. The presence of (0)'s in the alignment does not influence the outcome, except for contributing to the final sum of 0.
Consequently, the formulation of schema *SH* necessitates a specific arrangement of binary sequences tied to *irrational sequences*. This arrangement is designed to ensure that it remains impossible to generate a result like $A_j(s) = X.111...$ through finite self-additions of some sequence s, where $j, X \in \mathbb{N}$.
On another note, schema *SH* prompts the consideration of an alternative approach compared to those that exhibit periodic recursion. For instance, a sequence like $(110001100011000..)$ displays periodic recursive characteristics, leading to a finite summation result of $111111...$
As an example: $(110001100011000..) +_A ... +_A ... +_A (111001110011100..) = (111111111111111..)$
It is the above consideration that sparked the idea that targeting the number of $0's$ between pairs of $1's$ forming a binary sequence is what holds the key to forming (SH) i.e., a schema not having a recursive property). Noteworthy is the observation that a means of forming a non-recursive binary sequence is by increasing the number $0's$ between pairs of $1's$, and as a natural extension to this is having arbitrary finite-length sequences M_i spaced suchlike forming the sequence.
It is almost arbitrary why such sequences would form part of the *irrational sequences*, as, conditioned sub-sequences larger in length so to say, require more Summing in the way of resulting in 1111.. spanning its length. If there is **always** in existence one such conditioned sub-sequence greater in length than all preceding conditioned sub-sequences, then

no amount of 'Sum' on such a sequence is sufficient in the way of resulting in 111... .

Definition (Unconditioned sub-sequence)

Given an arbitrary sequence-function f of arbitrary length, an *unconditioned sub-sequence* P_i of f, is defined to mean a segment of $Image(f)$ where $Range(f) := \{0,1\}$.

Remarks

A formidable task in set theory is precision in defining sets. *Whereof we do not know, thereof we must remain silent.* Present theory struggles in the ability to form precisely a collection of $\bigcup f_i : \mathbb{N} \to \{0,1\}$, primarily because it requires the existence of a choice function and the acceptance of ZFC-axioms of arithmetic. As its existence is highly debated we need to be precise and relay caution. What we wish to do is condition the functions both inductively and in a precisely defined manner, allowing for set formation. In order to achieve this, we will need to make use of these definitions.

Definition (Imposed Sequence Sets)

Let I be the union of non-overlapping conditioned sub-sequence domain intervals $D(I_i)$ for an arbitrary sequence-function f of sufficient length. An *imposed* sequence-function, denoted as $f \leftarrow (I_i)$, is defined as any arbitrary sequence-function $f : \omega_< \to 0,1$ with the following conditioning:

$$s = \begin{cases} Range(f(d)) := \{0,1\} & \text{if } d \in \bigcup_{\forall i} \mathrm{D}(\backslash I_i) \\ Range(f(d)) := \{0\} & \text{if } d \in \bigcup_{\forall i} D(I_i)_{Zero-Segments} \end{cases}$$

We use the notation $S \leftarrow (I_i)$ to indicate that the entire set S follows the same conditioning as the intervals I_i. In this context, S is said to be imposed by I_i, avoiding confusion in meaning.
It's worth noting that we only require conditioned sub-sequences to define an imposition. However, in certain cases, we provide additional information to ensure clarity by including the domain values outside the interval segments in the subsequent arguments.

Definition (Unconditioned sub-sequence Combination $C_i(g)$)

The set $Ci(g)$ is defined to be the set of functions $\bigcup_{\forall i} f_i : \{1,..,g\} \to \{0,1\}$, $g \in \mathbb{N}$.

Definition (Span Sets, we denote as $span(S \leftarrow (I_i, P_i))$, (Simply $span(I_i, P_i)$))

Consider an arbitrary sequence-function f of varying length, where each sub-sequence is classified as either Conditioned or Unconditioned. Let $f \leftarrow (I_i, P_i)$ denote the conditioning applied to each sub-sequence, with $\mathscr{M}(P_i)$ and $\mathscr{M}(I_i)$ being non-zero for all relevant indices i.
In this context, the term $span(f)$ is introduced. This refers to the collection of functions

formed by taking the union of all functions $f_j \leftarrow (I_i, P_i)$, where each function f_j corresponds to distinct sub-sequences of f.
In summary, the concept of $span(f)$ encompasses the entire array of functions resulting from the union of conditioned and unconditioned sub-sequences in f, each subject to their respective conditionings.

Definition (Sequence-Function Set Product)

Consider three sets of sequences and functions, denoted as S, S', and U. Let Π be the union of these sets, i.e., $\Pi = S \cup S' \cup U$. For any indices i and j, where $f_i \in S$ and $f'_j \in S'$, the multiplication operation (S, S') is introduced as a binary operation $\otimes : \Pi \times \Pi \rightarrow \Pi$. Specifically, for all possible combinations of indices i and j, the result of the multiplication operation is the set $f_i +_A f'_j$, where $+_A$ represents a certain defined operation, and this resultant set belongs to the set U.
Such sets are comparable to a direct product set.
Written $S \otimes S'$, denoted $S \otimes S' = U := \{f_1 +_A f'_1, f_1 +_A f'_2, .., f_1 +_A f'_j, f_2 +_A f'_1, f_2 +_A f'_2, .., f_2 +_A f'_j$ ... ... $f_i +_A f'_1, f_i +_A f'_2, .., f_i +_A f'_j\}$

Remarks

A direct consequence of span multiplication is a shift in the composition of conditioned and unconditioned sub-sequences within the resultant set.

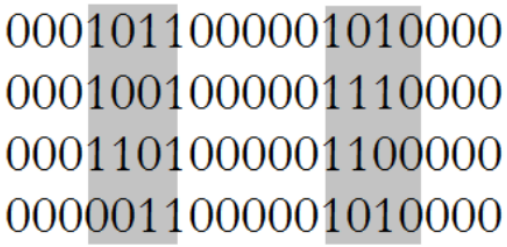

Figure 2: The diagram above illustrates a sequence of arbitrary elements belonging to $span(S)$, where $S \leftarrow (I_i, P_i)$. The units within the shaded region indicate the unconditioned sub-sequences associated with each depicted sequence.

Definition (Set of Irrational Sequences)

A collection denoted as SR, encompassing all distinct non-repetitive sequence-functions $s \leftarrow (I_i, P_i)$, where the condition $\mathscr{M}(I_i) > \mathscr{M}(I_{i-1})$ holds for every $i \in \mathbb{N}$, is termed a set of irrational sequences.

Definition ($1_{q_1,t_1}$)

For any arbitrary sequence-function $f \in S$, the term $1_{q_1,t_1}$ refers to an unconditioned segment of f, denoted as $P(f)[q_1, t_1]$, where the segment is characterized by a sequence of

consecutive $1's$ spanning its length.

Lemma 0

For every s belonging to the set SR, there does not exist any natural number j for which $A_j(s)$ equals a sequence of consecutive $1's$.

proof

Consider the arbitrary consecutive conditioned sub-sequences $I_{q_1}[q_1,t_1]$ and $I_{q_2}[q_2,t_2]$ within the sequence $s \in SR$. If the outcome of $A_j(S)$ results in $P_{q_1}[q_1,t_1] = 1_{q_1,t_1}$, then it becomes necessary for $A_g(S)$ with $g > j$ to materialize, ensuring that $P_{q_2}[q_2,t_2]$ yields $1_{q_2,t_2}$.
Due to the arbitrariness of the chosen conditioned sub-sequences, when this method is applied to an infinite set of sequence-functions, there exists no finite collection of sums such that $A_g(S) = 11111....$
It is assured that any sequence following an *irrational* progression cannot be reduced to the form $0.1111...$. This particular trait is precisely what safeguards these sequences from exhibiting properties akin to $\mathbb{Q}$ when extended to an unrestricted length, so to speak.

Definition (Conditioned sub-sequence Removed Sequence Set)

Consider an arbitrary sequence-function s belonging to a sequence-function set S. This set, denoted as S, comprises both conditioned sub-sequences and unconditioned sub-sequences denoted by (I_i,P_i) for all i respectively. Now, let the sequence-function s' be formed by arranging the images of P_i in the order of their occurrence.
This sequence-function s' is termed a "conditioned sub-sequence removed sequence-function." The collection of all such sequence-functions within the set S is defined as the "conditioned sub-sequence removed sequence set" associated with S, denoted as $S_{/I}$.

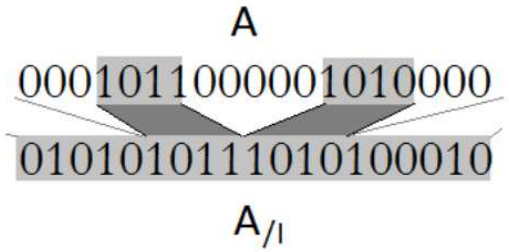

Figure 3: Conditioned sub-sequence Removed sequences.

Definition (Half Paired Elements)

Consider an arbitrary sequence function $s \leftarrow (I_i,P_i)$. We define a new function by con-

ditioning the first half of each unconditioned sub-sequence P_i (for all i) of s with a conditioned sub-sequence. This conditioning process involves taking a sub-sequence of length $\mathscr{M}(P_i),(\mathrm{Mod}_L),2$ from the conditioned sub-sequence. The resulting function is denoted as a "half paired function element" $/s$ of s.

Definition (Half Conditioned sub-sequence Extended Sequence Set)

We define a "half conditioned sub-sequence extended sequence set" as the collection denoted by $/S$. This set encompasses all distinct non-repetitive half-paired function elements that are associated with a given sequence set S.

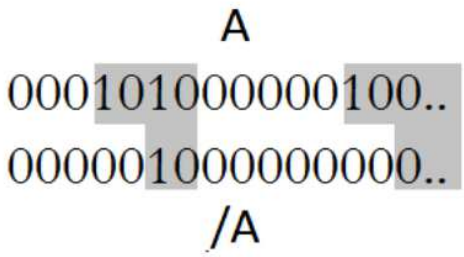

Figure 4: Conditioned sub-sequence Extended sequences.

5 THE MAIN ARGUMENT

The latter portion of this article will be dedicated to establishing the cardinality of $\mathbb{SR}$ concerning both $\mathbb{N}$ and $\mathbb{R}$. To accomplish this, we will formulate a ratio in the form: $|(\mathbb{N})|:|(\mathbb{R})|$ which will be represented as $|(\mathbb{N})|:|(H)|$, where $H \subset \mathbb{SR} \subset \mathbb{R}$. For the subset $H \subset \mathbb{SR}$, we will establish a ratio $|(H)|:|(\mathbb{R})|$ equivalent to $|(\mathbb{N})|:|(\mathbb{R})|$. (Here, $\mathbb{SR}$ denotes the collection of all Irrational-Sequences of indefinite length).
The complexity of this endeavour arises from H being a subset of $\mathbb{S}_{\mathbb{R}}$, rendering any attempt to pair elements from H with those from $\mathbb{R}$ quite challenging for evident reasons. Nonetheless, we will devise a technique to surmount this challenge by employing a method of pairing and *shrinking* through a symbolic representation that maintains a one-to-one correspondence with the sequence set H.
Subsequently, we will put forth the argument that $\mathbb{R}$ possesses a spanning-set of larger cardinality compared to H, achieved by demonstrating that the unpaired residual subset $K \subset \mathbb{S}_{\mathbb{R}}$ exhibits a pairing difficulty akin to the task of pairing $(\mathbb{N},\mathbb{R})$ when attempting to match its elements to $\mathbb{R}$, respectively.

Before proceeding further, we prove the following important lemma.

Lemma 1.0

If a sequence-function s is constructed such that its image aligns with the diagonal values d of a set of sequence-functions $s_1, \ldots, s_n \in D \leftarrow (I_i, P_i)$, then s is a member of the set D, following the structure of $s \leftarrow (I_i, P_i)$.

Proof
A conditioned sub-sequence with a length of k involves utilizing k elements, thereby contributing k diagonal entries to generate a sequence of length k. When such a collection of conditioned sub-sequences is appropriately aligned, the outcome is evidently a conditioned sub-sequence with the same length. This equivalence applies correspondingly to any random arrangement of aligned unconditioned sub-sequences.

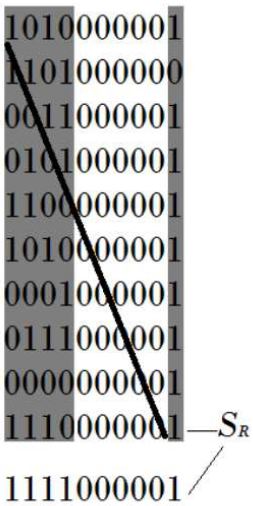

Figure 5: Diagonal argument as applied to induced sets.

6 FINAL PROPOSAL

Consider the set of all s in SR defined as $s \leftarrow (I_i, P_i)$, where for all i, $\mathscr{M}(I_{i+1}) = \mathscr{M}(I_i) + 1$ and $\mathscr{M}(P_i) = g, |, g \in 2n, |, n \in \mathbb{N}$.
Let I_i' and P_i^2 represent the conditioned and unconditioned sub-sequences, respectively, of all sequences $s' \in S'$ that belong to $s' \in hs(SR) = \text{span}(I_i', P_i^2)$.
Furthermore, let $\mathscr{M}(P_i^1) + \mathscr{M}(P_i^2) = g$, where $\mathscr{M}(P_i^2) = \frac{g}{2}$. With these conditions, we can express the relationship as follows:

$$SR = \text{span}(I_i, P_i^1) \otimes \text{span}(I_i', P_i^2) \tag{1}$$

Redefining the components of $(\Gamma_1 := \text{span}(I_i, P_i^1), \Gamma_2 := \text{span}(I_i', P_i^2))$ as $\alpha_i \in \Gamma_1$ and $\alpha_i' \in \Gamma_2$ respectively, we can deduce from equation (2) that $SR := \Gamma_2 \otimes \text{span}(I_i, P_i^1)$.

Now, if we endeavour to pair the sets $\bigcup_{\forall i} \alpha_i$ and $\bigcup_{\forall i} \alpha_i \otimes \text{span}(I'i, P_i^2) := \bigcup \forall i \alpha_i \otimes \bigcup_{\forall i} \alpha_i'$, we observe that the elements within both corresponding sequence-sets have already been subjected to a process of *shrinking*. (Here, "shrinking via denotation" refers to the re-representation of a sub-sequence using a variable.)
When attempting to pair $\bigcup_{\forall i} \alpha_i$ with $\text{span}(I'i, P_i^2)$ and $\bigcup \forall i \alpha_i$, it parallels the challenge of pairing elements from $(\mathbb{N}, \mathbb{R})$, respectively. This analogy becomes apparent when substituting $\text{span}(I_i', P_i^2)$ into the diagonal argument in place of $\mathbb{R}$, while considering the insights from Lemma 1.0 in the following manner:
For any arbitrary set of paired elements $(\alpha_1 : \alpha_1), (\alpha_2 : \alpha_2), \ldots, (\alpha_n : \alpha_n)$, attempting to map elements from $s_1, s_2, \ldots \in \text{span}(I'i, P_i^2)$ alongside such a pairing – with the awareness that $\bigcup \forall i \alpha_i \to \bigcup_{\forall i} \alpha_i$ is evidently surjective – swiftly reveals that an unpaired element $s_j \in \text{span}(I_i', P_i^2)$ can be effortlessly found by employing a diagonal argument based on $s_1, s_2, \ldots \in \text{span}(I_i', P_i^2)$.

$$\begin{gathered}
\alpha_1 : (\alpha_1, 100100101111...), \\
\alpha_2 : (\alpha_2, 001010100001...), \\
..., \\
\alpha_n : (\alpha_n, 010101011110...)
\end{gathered}$$

(Example) An endeavour to establish a pairing between Γ_1 and SR.

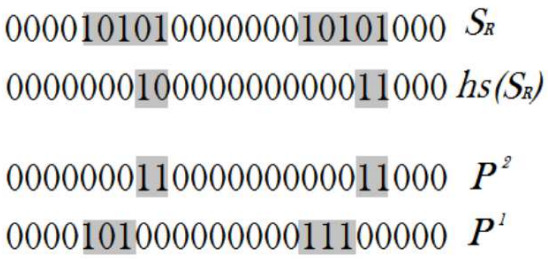

Figure 6: Elements and Half Paired Elements.

7 CONCLUSION

Since the choice of $\omega_<$ is arbitrary, the aforementioned reasoning can be recursively extended to encompass all ω_j. Consequently, the results naturally align with the overarching principles introduced in the earlier section.
The application of a diagonal argument involving $(\mathbb{N}_2, SR)$ becomes relevant. This is particularly valid given that the inclusion $SR \subset \mathbb{R}_2$ has already been substantiated. In this manner, we sufficiently establish the task of determining the cardinality of SR in relation to $(\mathbb{N}_2, \mathbb{R}_2)$.

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