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On the Realization of Impulse Invariant Low-rank Volterra Kernels

Phillip M. S. Burt and José Henrique de Morais Goulart

Abstract—Volterra models can accurately model numerous nonlinear systems of practical interest, but often at an unacceptable computational cost. If the Volterra kernels of a system have low-rank structure (like, e.g., kernels of bilinear systems), this major drawback can in principle be mitigated. Yet, when one seeks an exact discrete-time model of a mixed-signal chain involving that system, the existing formula that generalizes the impulse invariance principle to Volterra kernels yields discrete-time kernels that do not share the same low rank. At first sight this would seem to seriously complicate the otherwise simple discrete-time realization of low-rank kernels. We show here that this not the case. By defining a cascade operator, the structure of generalized impulse invariance can be unveiled, leading to a realization without an inordinate increase in computational complexity. Finally, we give a numerical example involving a physical system that shows the relevance of our proposal.

Index Terms—nonlinear systems, bilinear systems, Volterra model, impulse invariance

I. INTRODUCTION

VOLTERRA models are a popular choice for the modeling of non-linear systems of various kinds [1]–[7]. In particular, the most commonly used variant in digital signal processing applications, known as a Volterra filter (VF), is essentially a feedforward polynomial model whose output is linear in the model parameters—a desirable feature for system identification—and whose wide applicability has been established by Boyd and Chua in the eighties [8]. Unfortunately, though, the amount of parameters of a VF grows quite rapidly with the system memory length and the order (nonlinearity degree) of the model, whose practical realization thus often becomes too costly.

Motivated by this drawback, a whole line of research has been devoted to devising more practical alternatives which trade some generality of the VF—and often also the linearity in the parameters—by a lower parametric complexity [9]–[18]. One of the most effective and elegant proposals is based on the simple assumption that the p th-order Volterra kernel $h_p(n_1, \dots, n_p)$ (high-order analogues of the impulse response, see Section III for a definition) approximately decomposes as a sum of a few *separable* functions, that is, $h_p(n_1, \dots, n_p) \approx \sum_{r=1}^{R_p} h_r^{(1)}(n_1) \dots h_r^{(p)}(n_p)$ with a sufficiently small R_p . This amounts to a low-rank (specifically, rank- R_p) approximation of the kernels, viewed as tensors [13]. Besides the potential dramatic reduction in computational complexity, one major advantage of this approach is its straightforward realization by

a combination of linear blocks and instantaneous nonlinearities or multipliers.

However, in applications where an exact discrete-time modeling of a mixed-signal chain consisting of discrete-time and continuous-time parts is desirable (for instance, in the cancellation of signals originating from discrete-time, such as in acoustic echo cancellation or nonlinearity mitigation [10], [19]), the low-rank realization is not as simple anymore. The reason is that the well-known impulse invariance [20] between discrete-time and continuous-time linear time-invariant (LTI) systems does not generalize “cleanly” to nonlinear systems, but has to be somewhat modified so as to incorporate a factor which depends on the pattern of repeated kernel arguments, as pointed out in [21], [22]. We show that, as a consequence, the computational cost of a naive realization of generalized impulse invariance, while still much smaller than that of a VF realization (whose cost is not affected by generalized invariance), would increase by as much as 2^{p-1} times. We then deduce a much more efficient realization, preserving the great attractiveness of the low-rank approach.

To the best of our knowledge this problem has not been previously addressed. Apart from its practical relevance, as discussed above, the result we present is of interest in itself as an addition to the theory of signal processing.

II. IMPULSE INVARIANCE OF LTI SYSTEMS

A continuous-time LTI system bandlimited to $1/2T$ Hz, with impulse response $h_c(t)$, can be implemented [20, p. 173] with a mixed-signal chain containing the *impulse invariant* discrete-time system with impulse response¹

$$h(n) = h_c(nT). \quad (1)$$

Impulse invariance also comes into question when modeling the mixed-signal chain depicted in Fig. 1, which is of greater concern here and is now described. From the input sequence $u(n)$, an impulsive D/A converter² with sampling period T provides

$$u_c(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)u(k). \quad (2)$$

After a reconstruction filter $h_r(t)$, an application-specific system $h_o(t)$, and an anti-aliasing filter $h_a(t)$, all LTI, it follows from (2) that

$$y_c(t) = \sum_{k=-\infty}^{\infty} h_c(t - kT)u(k), \quad (3)$$

¹Hereafter, the subscript c is used to distinguish a continuous-time signal from its discrete-time version. For convenience, we have dropped the factor T from the definition $h(n) = Th_c(nT)$ of impulse invariance of [20].

² $\delta(t)$ is the Dirac delta function. The assumption of ideal impulsive excitation is not restrictive, since the reconstruction filter $h_r(t)$ can absorb the rectangular impulse response of a real-world zero-order hold D/A.

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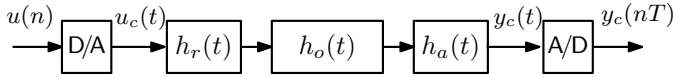


Fig. 1. Signal chain starting in discrete-time, passing through continuous-time and then returning to discrete-time.

where $h_c(t)$ is the overall impulse response given by the convolutions $h_r(t) * h_o(t) * h_a(t)$. Finally, an A/D sampler gives the output $y_c(nT)$. It follows that a system with impulse response $h(n) = h_c(nT)$, input $u(n)$ and output $y(n) = \sum_{k=-\infty}^{\infty} h(n-k)u(k)$ is an *exact* discrete-time model of the signal chain³ in the sense that

$$y(n) = y_c(nT), \quad (4)$$

as desired, for instance, in acoustic echo cancellation [23].

Remark. The sampling of $y_c(t)$ in (4) and of $h_c(t)$ in (1) must be consistent at discontinuities. For instance, let $h_c(t) = e^{-at}$ if $t > 0$ and $h_c(t) = 0$ if $t < 0$. From (3) then, $y_c(t)$ is discontinuous at $t = nT$ if $u(n) \neq 0$. Assuming the A/D sampler always provides the right-side limit $y_c(nT_+)$ (respectively, the left-side limit $y_c(nT_-)$ or $[y_c(nT_+) + y_c(nT_-)]/2$), it follows from (3) that, to achieve (4), $h(0)$ must be given by $h_c(0_+) = 1$ (respectively, $h_c(0_-) = 0$ or $[h_c(0_+) + h_c(0_-)]/2 = 1/2$).

It should be noted that when $h_c(t)$ represents an actual physical system, the impulse response $h(n) = h_c(nT)$ will, in general, have infinite duration. As long, though, as the system $h_c(t)$ is rational, the exact realization (with a finite number of operations) of a discrete-time system with impulse response $h(n)$ is straightforward [24].

III. GENERALIZATION OF IMPULSE INVARIANCE TO VOLTERRA KERNELS

Let the analog portion of the chain in Fig. 1 be now nonlinear. (This can arise, for instance, from a nonlinear loudspeaker in acoustic echo cancellation [19].) We assume then that its input/output relation is given by the (causal) Volterra series $y_c(t) = \sum_{p=1}^{\infty} y_{c,p}(t)$, with homogeneous outputs given by

$$y_{c,p}(t) = \int_0^{\infty} \cdots \int_0^{\infty} h_{c,p}(\tau_1, \dots, \tau_p) \prod_{i=1}^p u_c(t - \bar{\tau}_i) d\tau_1 \dots d\tau_p, \quad (5)$$

where $\bar{\tau}_i = \sum_{j=i}^p \tau_j$ and $h_{c,p}(\tau_1, \dots, \tau_p)$ is a *regular* Volterra kernel of order p [21, p. 15], continuous for $\tau_1, \dots, \tau_p > 0$. Although the existence of the realization problem addressed in this paper is independent of employing conventional⁴ or regular Volterra kernels, the latter are more convenient for the required algebraic manipulation.

To provide a discrete-time model of the signal chain, let $\bar{n}_i = \sum_{j=i}^p n_j$, where n_j always represents discrete time, and, for some causal discrete-time kernel $v_p: \mathbb{Z}^p \rightarrow \mathbb{R}$ let

$$y_p(n) = \sum_{n_p=0}^{\infty} \cdots \sum_{n_1=0}^{\infty} v_p(n_1, \dots, n_p) \prod_{i=1}^p u(n - \bar{n}_i), \quad (6)$$

³In this case, is not required that system $h_c(t)$ be bandlimited.

⁴With a conventional kernel $h_{c,p}^{(\text{conv})}(\tau_1, \tau_2, \dots, \tau_p) = h_{c,p}^{(\text{reg})}(\tau_1 - \tau_2, \dots, \tau_{p-1} - \tau_p, \tau_p)$, (5) assumes the more familiar form with τ_i instead of $\bar{\tau}_i$.

with $u(n)$ and $u_c(t)$ still being related by (2). It follows from [22] that achieving $y_p(n) = y_{c,p}(nT)$, $p > 1$, assuming right-side sampling at discontinuities, requires generalizing (1) as

$$v_p(n_1, \dots, n_p) = \frac{h_{c,p}(n_1T, \dots, n_pT)}{m_1! \dots m_q!}, \quad (7)$$

where q is the number of groups of consecutive null indices among n_1, \dots, n_{p-1} and $m_1 - 1, \dots, m_q - 1$ are the numbers of indices in each group⁵. For instance, if $p = 5$, $n_1 = n_2 = 0 \neq n_3$ and $n_4 = 0$, then $q = 2$, $m_1 = 3$ and $m_2 = 2$.

This generalized impulse invariance results from the impulsive terms of the integrand in (5) and the possible discontinuity of the kernel on the border of the domain $\tau_1, \dots, \tau_{p-1} \geq 0$. In the interior of this domain we retrieve a direct extension of the invariance condition (1), that is, we have $v_p(n_1, \dots, n_p) = h_{c,p}(n_1T, \dots, n_pT)$ for $n_1, \dots, n_{p-1} > 0$. For more details on the steps leading to (7), the reader is referred to [22].

Similarly to the discretized $h(n)$ of (1), (7) gives in general an infinite duration $v_p(n_1, \dots, n_p)$. This raises the issue of its realization with a finite number of operations, not addressed in [22]. We do this now, for the class of low-rank kernels.

IV. REALIZATION OF LOW-RANK KERNELS

We consider systems with *low-rank* Volterra kernels

$$h_{c,p}(\tau_1, \dots, \tau_p) = \sum_{r=1}^{R_p} \mathbf{H}_{c,r}^{(p)}(\tau_p) \cdots \mathbf{H}_{c,r}^{(1)}(\tau_1), \quad (8)$$

for any set of (vector- and) matrix-valued functions $\mathbf{H}_{c,r}^{(i)}(\tau_i)$ of compatible dimensions⁶, and a given $R_p \in \mathbb{N}^*$, termed the *rank* of $h_{c,p}$. Of particular practical interest are bilinear systems [21], [25], further discussed in Section V, for which $R_p = 1$. We consider $R_p = 1$ from here onward, and thus look into the realization of one of the parallel branches of (8).

A. Cascade structure and operator

From (8) and (5) it follows directly that low-rank kernels can be realized quite simply by a cascade of linear blocks and multipliers. This is depicted in Fig. 2 ($p = 3$, $R_3 = 1$), where

$$\begin{aligned} \mathbf{z}_1(t) &= [\int_0^{\infty} \mathbf{H}_c^{(1)}(\tau) u_c(t - \tau) d\tau] u_c(t) \\ &= [\mathbf{H}_c^{(1)} * u_c(t)] u_c(t) = \mathbf{H}_c^{(1)} \circ u_c(t), \\ \mathbf{z}_2(t) &= [\mathbf{H}_c^{(2)} * \mathbf{z}_1(t)] u_c(t) = \mathbf{H}_c^{(2)} \circ \mathbf{z}_1(t), \\ y_{c,3}(t) &= \mathbf{H}_c^{(3)} * \mathbf{z}_2(t). \end{aligned}$$

Here, $*$ stands for convolution and \circ stands for the *cascade* operator defined, given $u_c(t)$, by $h \circ x(t) \triangleq [h * x(t)] u_c(t)$, which is linear in h and in x . Also, with it we can write

$$y_{c,3}(t) = \mathbf{H}_c^{(3)} * \{ \mathbf{H}_c^{(2)} \circ [\mathbf{H}_c^{(1)} \circ u_c(t)] \}.$$

Hence, the realization can be expressed as a sequential calculation with $p-1$ applications of the cascade operator, followed by a convolution at the final stage.

⁵A related result is stated without proof in [21, p. 254]. For simplicity, the dependence of m_i on n_1, \dots, n_{p-1} is omitted. If $y_p(n) = y_{c,p}(nT_-)$ or $y_p(n) = [y_{c,p}(nT_-) + y_{c,p}(nT_+)]/2$ at discontinuities, the result is similar.

⁶For uniformity of notation, $\mathbf{H}_{c,r}^{(p)}$ (resp., $\mathbf{H}_{c,r}^{(1)}$) is denoted as a matrix, though being a row (resp., column) vector. If $R_p = 1$, we drop subscripts r .

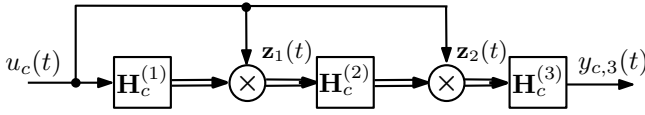


Fig. 2. Cascade realization of a separable kernel, $p = 3$, $R_3 = 1$. Double arrows denote vector-valued signals.

B. Parallel-cascade realization of impulse invariance

Consider the sampled kernel factors $\mathbf{H}^{(i)}(n_i) = \mathbf{H}_c^{(i)}(n_i T)$. We can readily verify that, assuming the input has the form (2), the cascade structure has this very particular property:

Property. Replacing $\mathbf{H}_c^{(i)}(\tau_i)$ with $\mathbf{H}^{(i)}(n_i)$ and $u_c(t)$ with $u(n)$ in the cascade structure that realizes $h_{c,p}(\tau_1, \dots, \tau_p) = \mathbf{H}_c^{(p)}(\tau_p) \dots \mathbf{H}_c^{(1)}(\tau_1)$, gives a realization of⁷

$$\tilde{v}_p(n_1, \dots, n_p) = \mathbf{H}^{(p)}(n_p) \dots \mathbf{H}^{(1)}(n_1). \quad (9)$$

This discrete-time kernel, however, is not impulse invariant in relation to $h_{c,p}(\tau_1, \dots, \tau_p)$, since the term $1/m_1! \dots m_q!$ in (7) is missing. To include the missing term, let us initially rewrite the invariance condition (7), assuming $R_p = 1$, as

$$v_p(n_1, \dots, n_p) = c_p(n_1, \dots, n_{p-1}) \mathbf{H}^{(p)}(n_p) \dots \mathbf{H}^{(1)}(n_1), \quad (10)$$

where $c_p(n_1, \dots, n_{p-1}) \triangleq 1/m_1! \dots m_q!$. We note then that c_p can be decomposed as

$$c_p(n_1, \dots, n_{p-1}) = \sum_{r=1}^{2^{p-1}} a_r c_r^{(1)}(n_1) \dots c_r^{(p-1)}(n_{p-1}), \quad (11)$$

where $c_r^{(i)}(n_i)$ is either the unit impulse⁸ $\delta(n)$ or its complement $\bar{\delta}(n) = [1 - \delta(n)]$.

Example. Since $c_3(n_1, n_2) = 1$ when $n_1, n_2 > 0$, $c_3(n_1, n_2) = 1/2$ if either $n_1 = 0$ or $n_2 = 0$ (but not both), and $c_3(n_1, n_2) = 1/3!$ if $n_1 = n_2 = 0$, it decomposes as

$$c_3(n_1, n_2) = \bar{\delta}(n_1)\bar{\delta}(n_2) + \frac{1}{2}\delta(n_1)\bar{\delta}(n_2) + \frac{1}{2}\bar{\delta}(n_1)\delta(n_2) + \frac{1}{3!}\delta(n_1)\delta(n_2). \quad (12)$$

Now, from (10) and (11) we get

$$v_p(n_1, \dots, n_p) = \sum_{r=1}^{2^{p-1}} a_r \mathbf{G}_r^{(p)}(n_p) \dots \mathbf{G}_r^{(1)}(n_1), \quad (13)$$

where $\mathbf{G}_r^{(i)}(n_i)$ is either given by $\bar{\delta}(n_i)\mathbf{H}^{(i)}(n_i)$ or by $\delta(n_i)\mathbf{H}^{(i)}(n_i)$, $1 \leq i < p$, and $\mathbf{G}_r^{(p)}(n_p) = \mathbf{H}^{(p)}(n_p)$. It follows that v_p can be realized by summing the output of 2^{p-1} parallel cascade structures. Nevertheless, because all blocks $\mathbf{G}_r^{(i)}(n_i)$ in (13) come from the decomposition $(\delta(n_i) + \bar{\delta}(n_i))\mathbf{H}^{(i)}(n_i)$, we can largely mitigate this increase in computational complexity by “sharing” computations among the branches of the parallel structure. We describe next a systematic way of implementing this strategy.

⁷This is not true in general for interconnections of linear blocks. For instance, two discrete-time linear systems $f(n) = f_c(nT)$ and $g(n) = g_c(nT)$ in series have impulse response $\sum_k f(k)g(n-k) \neq \int f_c(\tau)g_c(nT-\tau)d\tau$.

⁸ $\delta(n) = 1$ if $n = 0$ and $\delta(n) = 0$ if $n \neq 0$.

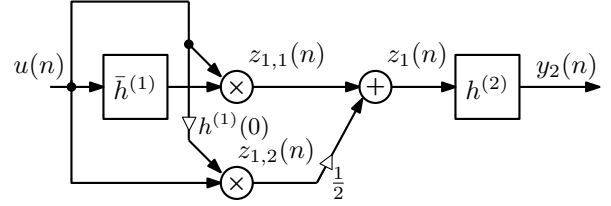


Fig. 3. Realization of impulse invariant kernel of order $p = 2$.

C. Efficient realization of impulse invariance

For simplicity, we consider henceforth low-rank kernels $h_p(\tau_1, \dots, \tau_p) = h_c^{(1)}(\tau_1) \dots h_c^{(p)}(\tau_p)$ with scalar (instead of matrix) factors, the extension to matrices being straightforward. With $h^{(i)}(n_i) = h_c^{(i)}(n_i T)$, we define:

$$\bar{h}^{(i)}(n_i) \triangleq \bar{\delta}(n_i)h^{(i)}(n_i), \quad h_0^{(i)}(n_i) \triangleq \delta(n_i)h^{(i)}(0). \quad (14)$$

With $p = 2$, initially, from (10) and (14) we have $c_2(n_1) = \bar{\delta}(n_1) + 1/2\delta(n_1)$, and, therefore,

$$\begin{aligned} v_2(n_1, n_2) &= [\bar{\delta}(n_1) + \frac{1}{2}\delta(n_1)] h^{(1)}(n_1)h^{(2)}(n_2) \\ &= [\bar{h}^{(1)}(n_1) + \frac{1}{2}h_0^{(1)}(n_1)] h^{(2)}(n_2). \end{aligned}$$

Using this in (6) and a discrete-time version of the cascade operator $h \circ x(n) \triangleq [h * x(n)]u(n)$, we can write the output as

$$y_2(n) = h^{(2)} * \left[(\bar{h}^{(1)} + \frac{1}{2}h_0^{(1)}) \circ u(n) \right]. \quad (15)$$

Defining now $z_1(n) \triangleq (\bar{h}^{(1)} + 1/2h_0^{(1)}) \circ u(n)$, it follows from the linearity in h of the cascade operator that

$$z_1(n) = z_{1,1}(n) + \frac{1}{2}z_{1,2}(n), \quad (16)$$

where

$$z_{1,1}(n) = \bar{h}^{(1)} \circ u(n), \quad (17)$$

$$z_{1,2}(n) = h_0^{(1)} \circ u(n) = h^{(1)}(0)u^2(n), \quad (18)$$

and the output is $y_2(n) = h^{(2)} * z_1(n)$. This realization is depicted in Fig. 3.

Let, now, $p = 3$. From (10) and (12) we can write, initially,

$$\begin{aligned} v_3(n_1, n_2, n_3) &= \left\{ [\bar{\delta}(n_1) + \frac{1}{2}\delta(n_1)] \bar{\delta}(n_2) + \frac{1}{2}\bar{\delta}(n_1)\delta(n_2) \right. \\ &\quad \left. + \frac{1}{3!}\delta(n_1)\delta(n_2) \right\} h^{(1)}(n_1)h^{(2)}(n_2)h^{(3)}(n_3). \end{aligned}$$

Moving $h^{(1)}(n_1)h^{(2)}(n_2)$ into the brackets, we get then

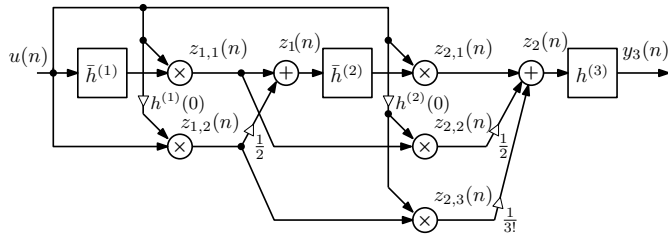
$$\begin{aligned} v_3(n_1, n_2, n_3) &= \left\{ [\bar{h}^{(1)}(n_1) + \frac{1}{2}h_0^{(1)}(n_1)] \bar{h}^{(2)}(n_2) \right. \\ &\quad \left. + \frac{1}{2}\bar{h}^{(1)}(n_1)h_0^{(2)}(n_2) + \frac{1}{3!}h_0^{(1)}(n_1)h_0^{(2)}(n_2) \right\} h^{(3)}(n_3), \end{aligned}$$

so that, using this in (6), introducing another cascade operator for the second stage, and with (16)–(18), we can write

$$\begin{aligned} y_3(n) &= h^{(3)} * \left[\bar{h}^{(2)} \circ z_1(n) + \frac{1}{2} \left(h_0^{(2)} \circ z_{1,1}(n) \right) \right. \\ &\quad \left. + \frac{1}{3!} \left(h_0^{(2)} \circ z_{1,2}(n) \right) \right]. \end{aligned}$$

Organizing as for $p = 2$ then, $y_3(n) = h^{(3)} * z_2(n)$, where

$$z_2(n) = z_{2,1}(n) + \frac{1}{2}z_{2,2}(n) + \frac{1}{3!}z_{2,3}(n),$$

Fig. 4. Realization of impulse invariant kernel of order $p = 3$.

with

$$\begin{aligned} z_{2,1}(n) &= \bar{h}^{(2)} \circ z_1(n), \\ z_{2,2}(n) &= h_0^{(2)} \circ z_{1,1}(n) = h^{(2)}(0)z_{1,1}(n)u(n) \\ z_{2,3}(n) &= h_0^{(2)} \circ z_{1,2}(n) = h^{(2)}(0)z_{1,2}(n)u(n), \end{aligned}$$

which is depicted in Fig. 4. Generalizing, we should compute:

- For $i = 1, \dots, p-1$ and with $z_0(n) = z_{0,1}(n) = u(n)$,
$$z_{i,1}(n) = [\bar{h}^{(i)} * z_{i-1}(n)] u(n),$$

$$z_{i,j}(n) = h^{(i)}(0)z_{i-1,j-1}(n)u(n), \quad j = 2, \dots, i+1,$$

$$z_i(n) = \sum_{j=1}^{i+1} \frac{1}{j!} z_{i,j}(n).$$
- $y_p(n) = h^{(p)} * z_{p-1}(n).$

V. EXAMPLE: BILINEAR SYSTEMS

Bilinear systems have state-space equations of the form

$$\begin{aligned} \mathbf{x}'_c(t) &= \mathbf{F}\mathbf{x}_c(t) + \mathbf{G}\mathbf{x}_c(t)u_c(t) + \mathbf{b}u_c(t) \\ y_c(t) &= \mathbf{c}^\top \mathbf{x}_c(t), \end{aligned}$$

and can approximate, up to any kernel order p , the large class known as linear-analytic systems [21], [26], [27]. Their kernels read $h_{c,p}(\tau_1, \dots, \tau_p) = \mathbf{c}^\top e^{\mathbf{F}\tau_p} \mathbf{G} e^{\mathbf{F}\tau_{p-1}} \mathbf{G} \dots \mathbf{G} e^{\mathbf{F}\tau_1} \mathbf{b}$, $\tau_i \geq 0$, thus having the low-rank form of (8) with $R_p = 1$ (that is, rank one), and $\mathbf{H}_c^{(1)}(\tau_1) = e^{\mathbf{F}\tau_1} \mathbf{b}$, $\mathbf{H}_c^{(i)}(\tau_i) = e^{\mathbf{F}\tau_i} \mathbf{G}$, $1 < i < p$, and $\mathbf{H}_c^{(p)}(\tau_p) = \mathbf{c}^\top e^{\mathbf{F}\tau_p} \mathbf{G}$.

As an example, consider the bilinear model of a bass loudspeaker [22], sampled at a rate of 1.5 kHz. An infinite-memory discrete-time realization of its fourth-order impulse invariant Volterra kernel was derived as in Section IV-C, and the corresponding output $y_4(n)$ obtained for an unit-power AWGN input $u(n)$. For validation, the output $\hat{y}_4(n)$ of the time-truncated kernel was directly calculated using (7) in (6) (in other words, a VF realization) with $\bar{n}_i \leq 120$, aiming at a small discrepancy $\epsilon(n) = y_4(n) - \hat{y}_4(n)$. Indeed, as seen in Fig. 5, $\epsilon(n)$ is of the order of 10^{-16} (mainly due to computing with 64 bit precision), validating the proposed procedure. Also displayed is the output $\tilde{y}_4(n)$ of the kernel given by (9). Its large discrepancy in relation to $y_4(n)$ shows that the invariance principle of (7) can be very relevant in practice.

To compare computational costs now, a VF filter realization of the truncated kernel, with $\bar{n}_i < N$ in (6), requires at least $\binom{N+p-1}{p}$ multiplications [28, p. 36]⁹. In the previous

⁹Excluding the computation of $\prod_{i=1}^p u(n - \bar{n}_i)$, for simplicity. All multiplication figures refer to the computation of one output sample. In [28] a triangular kernel equivalent to v_p is considered.

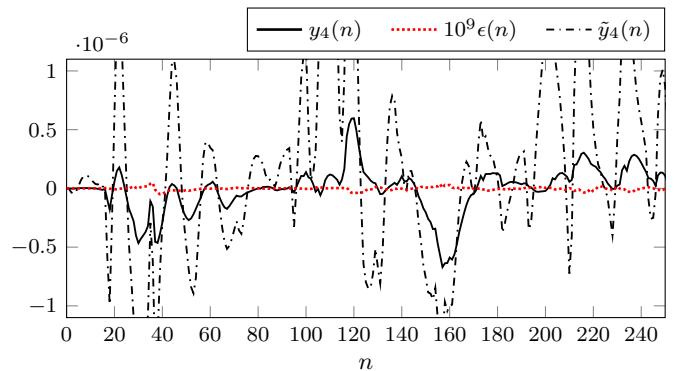


Fig. 5. Output $y_4(n)$ of the realization of an impulse invariant fourth-order kernel and its discrepancy $\epsilon(n)$ to a time-truncated VF realization. For comparison, we also depict the output $\tilde{y}_4(n)$ of the non-impulse invariant kernel of (9).

loudspeaker example, even allowing for less precision such that $\{E[\epsilon^2(n)]/E[y_4^2(n)]\}^{1/2} = 10^{-3}$, still requires $N = 48$ and, thus, 249900 multiplications.

Consider next the last stage in Fig. 4, now with matrix factors. For a bilinear kernel, $\mathbf{H}^{(p)}(n) = \mathbf{c}^\top e^{\mathbf{F}Tn} \mathbf{G}$, $n \geq 0$, so $y_p(n) = \mathbf{H}^{(p)} * \mathbf{z}_{p-1}(n) = \sum_{k=0}^{\infty} \mathbf{c}^\top e^{\mathbf{F}Tk} \mathbf{G} \mathbf{z}_{p-1}(n-k)$, which we readily see is realized by

$$\mathbf{x}_p(n+1) = \mathbf{A}\mathbf{x}_p(n) + \mathbf{B}\mathbf{z}_{p-1}(n) \quad (19)$$

$$y_p(n) = \mathbf{c}^\top \mathbf{x}_p(n) + \mathbf{d}^\top \mathbf{z}_{p-1}(n), \quad (20)$$

where $\dim[\mathbf{x}_p(n)] = M \triangleq \dim[\mathbf{x}_c(t)]$, $\mathbf{A} = e^{\mathbf{F}T}$, $\mathbf{B} = e^{\mathbf{F}T} \mathbf{G}$ and $\mathbf{d} = \mathbf{G}\mathbf{c}$. Assuming \mathbf{A} , \mathbf{B} and \mathbf{d} are pre-calculated and have no structure to be exploited for reducing computational cost, (19) and (20) require $2(M^2 + M)$ multiplications. Proceeding similarly for the outputs of the remaining linear blocks gives then a sub-total cost of $C_O = (2p-1)M^2 + 3M$. Finally, the computation of the inputs $\mathbf{z}_i(n)$ of the linear blocks requires $[(p-3)(p/2+1)+1]M^2 + [(p-1)(p/2+1)+3]M$ multiplications [29]¹⁰. In the loudspeaker example, $p=4$ requires $M = 34$ [22], giving a total of 13226 multiplications, much less than the at least 249900 required by the VF (and with no loss in precision) and the slightly over $2^{p-1}[C_O + (p-1)M] = 66368$ required by the parallel-cascade of Section IV-B.

VI. CONCLUSION

By defining a cascade operator, we have shown how to construct a realization of discrete-time kernels obtained from continuous-time low-rank regular kernels by the generalized impulse invariance principle. This construction is required because such discrete-time kernels are not themselves of the same low-rank and thus cannot be realized by the same cascade structures that realize their continuous-time counterparts. The proposed structure requires additional multipliers, not incurring however in an inordinate increase of computational complexity. The low-rank property is found in kernels with practical relevance, and holds in particular for kernels of bilinear systems.

¹⁰This results from eq. (27) in [29], with all $M_i = M$, taking the upper bound $M_i M_{i-1}$ for all μ_i , and adding the $(p-1)M$ multiplications by $u(n)$ required for $\mathbf{z}_{i,1}$, $1 \leq i < p$.

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