General remarks on the propagation of chaos in wave turbulence and application to the incompressible Euler dynamics

Anne-Sophie de Suzzoni

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Abstract

In this paper, we prove propagation of chaos in the context of wave turbulence for a generic quasisolution. We then apply the result to full solutions to the incompressible Euler equation.

1 Introduction

We address the question of propagation of chaos in the context of *wave* turbulence.

The issue at stake is the following : we consider the solution to a Hamiltonian equation with a random initial datum whose Fourier coefficients are initially independent and we want to know if this independence remains satisfied at later times. These Fourier modes must satisfy what is called in the Physics literature *Random Phase Approximation*, which is something satisfied by Gaussian variables. Here, we address also the following question : assuming that the initial Fourier modes are Gaussian, do the Fourier modes at later times conserve some sort of Gaussianity.

In the context of weak turbulence and for Schrödinger equations, these questions have been successfully adressed by Deng and Hani in [10]. The Gaussianity in these papers consists in proving that at later times the moments of the Fourier modes still behave like Gaussian moments.

Of course, the independance and Gaussianity are asymptotic in some sense. In the work by Deng and Hani, the cubic Schrödinger equation is considered on a torus of size $L \gg 1$ and with an initial datum of size $\varepsilon(L) \ll 1$ but at very big times in terms of ε, passed the deterministic nonlinear time, at the so-called kinetic time, where nonlinear effects start appearing in the dynamics of the statistics. They prove that the correlations between different Fourier modes tend to 0 as $L \to \infty$ and that if the initial datum is a Gaussian field, then the Fourier conserve Gaussian moments. They deduce this result from their successful derivation of the so-called *kinetic* equation, see [8].

Here, we do not adress the issue of the derivation of the kinetic equation. However, we mention the pioneer work by Peierls, [19], the following works by Brout and Prigogine or Prigogine alone, [3, 20], and the works on fluid mechanics by Hasselman [15, 16], Zakharov and Filonenko or Zakharov alone, [23, 24, 22]. For a review, we mention the book by Nazarenko, [18]. Mathematical works on the derivation of kinetic equations for the Schrödinger equations include [7, 9, 6, 1, 4, 12, 11, 13]. For Korteweg de Vries type equations, we mention [21]. Finally, we mention a result on discrete Schrödinger equations [17].

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In this paper, what we call asymptotic Gaussianity is the fact that the formula of cumulants remain asymptotically valid at later times.

In the first part of this paper, we adress these issues on a generic Hamiltonian equation. We work, similarly to [12, 11, 13], and to [18], in the context of *quasi-solutions* and of wave turbulence. We do not assume that the initial datum is somewhat small but we let the size of the torus go to ∞ and this is our asymptotic regime. The proof is mainly combinatorial.

In the second part of this paper, we pass from quasisolutions to full solutions to the incompressible Euler equation. For this part, we need a functional framework that fits both the initial datum and the Euler equation. We adapt the analytic functional framework of [2] keeping in mind that for our problem the initial datum is not localised. We also need to render explicit the abstract Cauchy-Kowaleskaia theorem, and for this, we use [5]. Finally, we need to estimate probabilities on the initial datum, we use a strategy very close to proving Fernique's theorem, see [14].

1.1 Framework and results

We consider a generic equation :

$$\partial_t u_L = K u_L + J_L(u_L, \dots, u_L) \tag{1}$$ genEq

on the torus $L\mathbb{T}^d$ of size L and in dimension d. Here, K is a skew-symmetric operator, and J_L a N-linear map, the map u has values in $\mathbb{C}^D$.

We assume that K and J_L take the following form in Fourier mode : for any test functions, $u_L, u_{L,1}, \dots, u_{L,N}$, we set

$$\widehat{Ku_L}(\xi) = i\omega(\xi)\hat{u}_L(\xi)$$

$$\widehat{J_L(u_{L,1}, \dots, u_{L,N})}(\xi) = \frac{1}{(2\pi L)^{d(N-1)/2}} \sum_{\xi_1+\dots+\xi_N=\xi} \Psi(\xi_1, \dots, \xi_N)(\hat{u}_{L,1}(\xi_1), \dots, \hat{u}_N(\xi_{L,N})) \tag{2}$$ defOmegaPsi

where $\Psi(\xi_1, \dots, \xi_N)$ is a N-linear map from $(\mathbb{C}^D)^N$ to $\mathbb{C}^D$, where for all $\xi \in \frac{1}{L}\mathbb{Z}^d$,

$$\hat{u}_L(\xi) := \frac{1}{(2\pi L)^{d/2}} \int_{L\mathbb{T}^d} u(x) e^{-i\xi x} dx.$$

We also assume that for all $u_{L,1}, \dots u_{L,N}$,

$$\widehat{J_L(u_{L,1}, \dots, u_{L,N})}(0) = 0$$

such that the quantity

$$\int_{L\mathbb{T}^d} dx u(x)$$

is conserved under the action of the flow of (1) and thus can be chosen null.

Finally, we assume that Ψ has at most linear growth : there exists $r \in [0, 1]$ such that for all $(\xi_1, \dots, \xi_N) \in (\mathbb{R}^d)^N$, in operator norm

$$|\Psi(\xi_1, \dots, \xi_N)| \leq \max_{j=1}^{N} \langle \xi_j \rangle^r.$$

We set the following initial datum for (1) :

$$u(t = 0)(x) = a_L(x) := \sum_{k \in \mathbb{Z}^d_*} \frac{e^{ikx/L}}{(2\pi L)^{d/2}} g_k a_{L,k} \tag{3}$$ InitialDatum

where $\mathbb{Z}^d_* = \mathbb{Z}^d \setminus \{(0,\dots,0)\}$. We write u_L the solution to (1) with initial datum a_L.

In (3), $(g_k)_{k\in\mathbb{Z}^d_*}$ is a sequence of centred and normalized complex Gaussian variables such that for all $k \in \mathbb{Z}^d_*$,

$$g_k = \bar{g}_{-k},$$

and such that if $k \neq l, -l$, then g_k and g_l are independent.

Finally, $(a_{L,k})_k$ is a sequence with values in $(\mathbb{R}^D)$ with finite support such that $a_{L,-k} = a_{L,k}$ for all $k \in \mathbb{Z}^d_*$.

We define by induction for $n \in \mathbb{N}$, $t \in \mathbb{R}$,

$$u_{L,0}(t) = e^{tK}a_L, \quad u_{L,n+1}(t) = \sum_{n_1+\dots n_N=n} \int_0^t e^{(t-\tau)K}[J(u_{L,n_1}(\tau),\dots,u_{L,n_N}(\tau))]d\tau. \tag{4}$$

Picard expansi

For $M \in \mathbb{N}$,

$$\sum_{n=0}^{M} u_{L,n}$$

is called a quasi-solution.

For a given $\xi \in \frac{1}{L}\mathbb{Z}^d_*$ and a given t, $\hat{u}_{L,n}(t)(\xi)$ is a vector in $\mathbb{C}^D$, we write $\hat{u}^{(i)}_{L,n}(t)(\xi)$ its i-th component.

rem:spaceInv **Remark 1.1.** *We note that the law of the initial datum is invariant under the action of space translations. For any space translation τ, we also have*

$$\tau K = K\tau, \quad \tau J(\cdot,\dots,\cdot) = J(\tau\cdot,\dots,\tau\cdot).$$

Therefore, by induction on n the law of $(u_{L,n})_n$ is invariant under space translations and therefore, for all n, m, i, j, t,

$$\mathbb{E}(u^{(i)}_{L,n}(t)(\xi)u^{(j)}_{L,m}(t)(\eta))$$

is equal to 0 unless $\eta = -\xi$.

In this framework, we prove Theorem 1.1.

th:genEq **Theorem 1.1.** *There exists $C = C(\Psi, N)$ such that for all $R \in \mathbb{N}^*$, $(n_1,\dots,n_R) \in \mathbb{N}^R$, $(i_1,\dots,i_R) \in [|1,d|]]^R$, $(\xi_1,\dots,\xi_R) \in (\frac{1}{L}\mathbb{Z}^d_*)^R$, all $t \in \mathbb{R}$, we have*

$$\Big|\mathbb{E}\Big(\prod_{l=1}^{R} \hat{u}^{(i_l)}_{L,n_l}(t)(\xi_l)\Big) - \sum_{O\in\mathcal{P}_R} \prod_{\{l,l'\}\in O} \mathbb{E}(\hat{u}^{(i_l)}_{L,n_l}(t)(\xi_l)\hat{u}^{(i_{l'})}_{L,n_{l'}}(t)(\xi_{l'}))\Big| \leq \frac{S!}{(S/2)!}\|(a_{L,k})_k\|_{\ell^2\cap\ell^\infty}\frac{(CA^r_L)^{\sum n_l}}{(2\pi L)^{d/2}} \tag{5}$$

EstTh

if $S = \sum_l n_l(N-1) + R$ is even, otherwise

$$\mathbb{E}\Big(\prod_{l=1}^{R} \hat{u}^{(i_l)}_{L,n_l}(t)(\xi_l)\Big) = \sum_{O\in\mathcal{P}_R} \prod_{\{l,l'\}\in O} \mathbb{E}(\hat{u}^{(i_l)}_{L,n_l}(t)(\xi_l)\hat{u}^{(i_{l'})}_{L,n_{l'}}(t)(\xi_{l'})).$$

Above we used the notations

$$A_L = \sup\{\langle\frac{k}{L}\rangle \mid k \in \mathbb{Z}^d_*, a_{L,k} \neq 0\},$$

the set $\mathcal{P}_R$ is the set of partitions of $[|1,R|]$ that contains only pairs (hence it is empty if R is odd).

Remark 1.2. *Taking $a_{L,k} = a(\frac{k}{L})$ where a is a bounded, compactly supported function, we have*

$$\|(a_{L,k})_k\|_{\ell^2\cap\ell^\infty} \lesssim \|a\|_{L^2\cap L^\infty(\mathbb{R}^d)}$$

and

$$A_L \leq A_\infty = \sup\{\langle\xi\rangle \mid a(\xi) \neq 0\}.$$

Hence in this context the difference in (5) is a $O(L^{-d/2})$.

Remark 1.3. *This theorem contains the asymptotic formula of cumulants for the quasisolutions, but considering Remark 1.1, it also implies asymptotic independence.*

In the context of the Euler incompressible equation :

$$\begin{cases} \partial_t u_L + u_L \cdot \nabla u_L = P_L \\ \nabla \cdot u_L = 0 \\ u_L(t=0) = a_L \end{cases} \tag{6}$$

where P_L is the pressure and $\nabla\cdot$ is the divergence, we assume that the sequences $(a_{L,k})_k$ take the form :

$$a_{L,k} = \varepsilon(L) a(\frac{k}{L})$$

where $\varepsilon(L) = O(\frac{1}{\sqrt{\ln L}})$, such that ε^{-1} has at most polynomial growth in L and where a is a bounded, compactly supported function. In order to have initially $\nabla \cdot u_L(t=0) = 0$, we impose that for all $\xi \in \mathbb{R}^d$, $\xi \cdot a(\xi) = 0$. We prove (local) well-posedness of (6) in the analytical framework presented in Subsection 3.1. In this analytical framework, the size of the initial datum can be up to $\varepsilon(L)\sqrt{\ln L}$, we refer to Appendix A. But if one looks at the initial datum locally, it is as small as $\varepsilon(L)$. Indeed, we have that for a given $x \in L\mathbb{T}^d$, the random variable

$$\sum_{k\in\mathbb{Z}^d_*} \frac{e^{ikx/L}}{(2\pi L)^{d/2}} g_k a(k/L)$$

converges in law towards the Wiener integral

$$\frac{1}{(2\pi)^{d/2}} \int e^{i\xi x} a(\xi) dW(\xi)$$

where W is a multidimensional Brownian motion. However, the regime we impose on $\varepsilon(L)$ is quite different that the ones in [10, 8], which are imposed by the dispersion of the Schrödinger equation. What is more, we do not claim that we reach derivation of the kinetic equation, or that we reach kinetic times. The result is valid for any time if $\varepsilon(L) = o((\ln L)^{-1/2})$, or for small times if $\varepsilon(L) = O((\ln L)^{-1/2})$ but we do not rescale the time. In this context, we prove Theorem 1.2.

Theorem 1.2. *There exist Banach spaces $(\mathcal{X}, \|\cdot\|_\mathcal{X})$ and $\mathcal{Y}_\theta$ such that*

$$\mathcal{X} \subseteq C(\mathbb{R}^d, \mathbb{C}^d), \quad \mathcal{Y}_\theta \subseteq C([-\theta,\theta]\times\mathbb{R}^d, \mathbb{C}^d)$$

such that for all $\theta \in \mathbb{R}_+$, there exists $A(\theta) > 0$ such that the Cauchy problem

$$\begin{cases} \partial_t u + u \cdot \nabla u = P \\ \nabla \cdot u = 0 \\ u(t=0) = u_0 \end{cases}$$

is well-posed in $\mathcal{Y}_\theta$ for all u_0 in the ball of $\mathcal{X}$ of center 0 and radius $A(\theta)$. The map $\theta \mapsto A(\theta)$ can be chosen nonincreasing. The flow hence defined conserves periodicity.

What is more, seeing a_L as a periodic function of $\mathbb{R}^d$ we get that there exists $c > 0$ such that if $A(\theta) \geq \frac{\sqrt{\ln L}\varepsilon(L)}{c}$, we have that a_L belongs to χ and

$$\mathbb{P}(\|a_L\|_\chi > A(\theta)) \leq e^{-cA(\theta)^2\varepsilon^{-2}(L)}.$$

Writing

$$\mathcal{E}_{L,\theta} = \{\|a_L\| \leq A(\theta)\},$$

we get that for all $R \in \mathbb{N}^*$, there exists $C = C(R, a, \theta, \mathcal{X}, \varepsilon)$ (the constant depends on the functional framework and the function ε but not on L) and $c_1 = c_1(a, \theta, R)$, $c_2 = c_2(a, \theta, \mathcal{X})$ such that for all $(i_1, \dots, i_R) \in [|1, d|]^R$, $(\xi_1, \dots, \xi_R) \in (\frac{1}{L}\mathbb{Z}^d_*)^R$, all $t \in [-\theta, \theta]$, and for all L, assuming

$$\varepsilon(L)\sqrt{\ln L} \leq c_1(a, \theta, R), \quad A(\theta) \geq \frac{\sqrt{\ln L}\varepsilon(L)}{c_2},$$

we have

$$\left|\mathbb{E}\Big(\mathbf{1}_{\mathcal{E}_{L,\theta}} \prod_{l=1}^{R} \hat{u}_L^{(i_l)}(t)(\xi_l)\Big) - \sum_{O \in \mathcal{P}_R} \prod_{\{l,l'\} \in O} \mathbb{E}(\mathbf{1}_{\mathcal{E}_{L,\theta}} \hat{u}^{(i_l)}_{L,n_l}(t)(\xi_l)\hat{u}^{(i_{l'})}_{L,n_{l'}}(t)(\xi_{l'}))\right| \leq C\varepsilon(L)^R L^{-d/2}. \tag{7}$$ EstEuler

What is more, if $\xi_l \neq -\xi_{l'}$, we have

$$\mathbb{E}(\mathbf{1}_{\mathcal{E}_{L,\theta}} \hat{u}^{(i_l)}_{L,n_l}(t)(\xi_l)\hat{u}^{(i_{l'})}_{L,n_{l'}}(t)(\xi_{l'})) \lesssim_{a,\theta,\mathcal{X},\varepsilon} \varepsilon(L)^R L^{-d/2}.$$

Remark 1.4. *If $\varepsilon(L) = o((\ln L)^{-1/2})$, then the result is global, because the inequalities*

$$A(\theta) \geq \frac{\sqrt{\ln L}\varepsilon(L)}{c_2}, \quad \varepsilon(L)\sqrt{\ln L} \leq c_1$$

are satisfied for L big enough. Otherwise, we need,

$$A(\theta) > \frac{\limsup(\varepsilon(L)\sqrt{\ln L})}{c_2},$$

which requires that θ has to be small enough. In other words, if $\limsup \varepsilon(L)\sqrt{\ln L} = c$, we need both θ to be smaller than a constant depending on the functional framework, the function a and c (non-increasing with c). But we also need that c is smaller than a constant depending on R, a and the functional framework.

1.2 Notations

By $\langle\cdot\rangle$, we denote the Japanese bracket, that is for $x \in \mathbb{R}^d$,

$$\langle x\rangle = \sqrt{1 + \sum_{i=1}^{d} x_i^2}.$$

By $[|a, b|]$ with $a \leq b \in \mathbb{R}$, we denote $[a, b] \cap \mathbb{N}$.

By the lexicographical order on $\mathbb{N}^2$, we mean the order defined for (l_1, j_1) and $(l_2, j_2) \in \mathbb{N}^2$ as

$$(l_1, j_1) < (l_2, j_2) \quad \Leftrightarrow \quad l_1 < l_2 \text{ or } (l_1 = l_2 \text{ and } j_1 < j_2).$$

For the norms on the sequence $(a_{L,k})_k$, we denote

$$\|(a_{L,k})_k\|_{\ell^\infty} = \sup_{k\in\mathbb{Z}^d_*} |a_{L,k}|, \quad \|(a_{L,k})_k\|_{\ell^2} = \frac{1}{(2\pi L)^{d/2}} \sqrt{\sum_{k\in\mathbb{Z}^d_*} |a_{L,k}|^2}, \quad A_L = \sup\{\langle \frac{k}{L} \rangle \mid a_{L,k} \neq 0\},$$

such that if $a_{L,k} = a(k/L)$ with $a \in L^\infty$ with compact support, setting

$$A_\infty = \sup\{\langle \xi \rangle \mid a(\xi) \neq 0\},$$

we have, for all L,

$$A_L \leq A_\infty, \quad \|(a_{L,k})_k\|_{\ell^\infty} \leq \|a\|_{L^\infty}, \quad \|(a_{L,k})_k\|_{\ell^2} \leq A_\infty^{d/2} \pi^{-d/2} \|a\|_{L^\infty}$$

when $L \to \infty$. We also denote

$$\|(a_{L,k})_k\|_{\ell^\infty \cap \ell^2} = \|(a_{L,k})_k\|_{\ell^\infty} + \|(a_{L,k})_k\|_{\ell^2}.$$

The spaces $L^p(\mathbb{R}^d)$ are the standard Lebesgue spaces.

Finally, in all the paper but Subsection 3.1, we consider Fourier transforms for L-periodic functions, or for functions of the torus $L\mathbb{T}^d$. We use the previously mentioned convention

$$\hat{u}_L(\xi) = \frac{1}{(2\pi L)^{d/2}} \int_{L\mathbb{T}^d} u_L(x) e^{-i\xi x} dx$$

for u_L defined on $L\mathbb{T}^d$, $\xi \in \frac{1}{L}\mathbb{Z}^d_*$. With this convention, we have

$$\hat{a}_L(\frac{k}{L}) = a_{L,k} g_k.$$

When u_L also depends on time, we set for all $t \in \mathbb{T}$, $\hat{u}_L(t)(\xi) = \widehat{u_L(t)}(\xi)$.

In Subsection 3.1, we consider functions of the full $\mathbb{R}^d$, without conditions of periodicity, we use the convention that the Fourier transform of a Schwartz class function f at $\xi \in \mathbb{R}^d$ is defined as

$$\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-ix\xi} dx.$$

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2 Asymptotic independence of the quasi solutions

2.1 N-trees

We introduce the notion of N-trees.

def:Ntrees **Definition 2.1.** Let $\mathcal{A}_0 = \{()\}$ and define by induction for all $n \in \mathbb{N}$,

$$\mathcal{A}_{n+1} = \{(A_1, \dots, A_N) \mid \forall j, A_j \in \mathcal{A}_{n_j}, \sum_{n_j} = n\}$$

We call the elements in $\mathcal{A}_n$ the N-trees with n nodes. We call () the trivial tree.

Remark 2.1. *A N-tree is a sequence of parenthesis and commas. Another way of defining N-trees is to use Polish notation and write*

$$\mathcal{A}_0 = \{0\}, \quad \mathcal{A}_{n+1} = \{1A_1 \dots A_N \mid \forall j, A_j \in \mathcal{A}_{n_j}, \sum_{n_j} = n\}$$

and see the N-trees as sequences of 0 *and* 1*. In this case, the decomposition $1A_1 \dots A_N$ is unique (see Appendix B).*

app:Polish

prop:PicardExp **Proposition 2.2.** *Define by induction on the N-trees, for all $t \in \mathbb{R}$, $A \in \cup_n \mathcal{A}_n$,*

$$F_{L,A}(t) = \begin{cases} u_{L,0}(t) & \text{if } A = () \\ \int_0^t e^{(t-\tau)K}[J_L(F_{L,A_1}(\tau), \dots, F_{L,A_N}(\tau))]d\tau & \text{if } A = (A_1, \dots, A_N) \end{cases}.$$

We have for $n \in \mathbb{N}$,

$$u_{L,n}(t) = \sum_{A \in \mathcal{A}_n} F_{L,A}(t).$$

Proof. The proof follows by induction on n. For $n = 0$, this is by definition. Otherwise, we have

$$u_{L,n+1}(t) = \sum_{n_1+\dots+n_N=n} \int_0^t e^{(t-\tau)K}[J_L(u_{L,n_1}(\tau), \dots, u_{L,n_N}(\tau))]d\tau.$$

Using the induction hypothesis and the fact that all the sums are finite, we get

$$u_{L,n+1}(t) = \sum_{n_1+\dots+n_N=n} \sum_{A_j \in \mathcal{A}_{n_j}} \int_0^t e^{(t-\tau)K}[J_L(F_{L,A_1}(\tau), \dots, F_{L,A_N}(\tau))]d\tau.$$

We recognize

$$u_{L,n+1}(t) = \sum_{n_1+\dots+n_N=n} \sum_{A_j \in \mathcal{A}_{n_j}} F_{L,(A_1,\dots,A_N)}(t)$$

and we use the definition of N-trees to conclude. □

def:labelling **Definition 2.3.** Let $n \in \mathbb{N}$ and $\vec{k} = (k_1, \dots, k_{(N-1)n+1}) \in (Z_*^d)^{(N-1)n+1}$. Let $A \in \mathcal{A}_n$. Define $F_{L,A,\vec{k}}$ by induction on n in the following way. If $n = 0$ then $A = ()$ and $\vec{k} = (k_1)$, we set

$$F_{L,A,\vec{k}}(t) = e^{it\omega(\frac{k_1}{L})} g_{k_1} a_{L,k_1}.$$

If $n = m+1$ with $m \in \mathbb{N}$, there exists $n_1, \dots n_N$ such that $\sum n_j = m$ and $A_j \in \mathcal{A}_{n_j}$ such that $A = (A_1, \dots, A_N)$. We set $\tilde{n}_j = \sum_{l<j}((N-1)n_l + 1)$ and

$$\vec{k}_j = (k_{(N-1)\tilde{n}_j+1}, \dots, k_{\tilde{n}_{j+1}}) \in (\mathbb{Z}_*^d)^{n_j(N-1)+1}.$$

(Note that $\tilde{n}_{N+1} = \sum_{j=1}^{N}((N-1)n_j + 1) = (N-1)m + N = (N-1)n + 1$.) We set also $R(\vec{k}) = \frac{1}{L}\sum_{j=1}^{(N-1)n+1} k_j$.

We now set

$$F_{L,A,\vec{k}}(t) = \frac{1}{(2\pi L)^{d(N-1)/2}} \int_0^t e^{i(t-\tau)\omega(R(\vec{k}))} \Psi(R(\vec{k}_1), \dots, R(\vec{k}_N))(F_{L,A_1,\vec{k}_1}(\tau), \dots, F_{L,A_N,\vec{k}_N}(\tau))d\tau.$$

:PicardExpFourier

Proposition 2.4. *We have for all $n \in \mathbb{N}$ and $A \in \mathcal{A}_n$,*

$$\widehat{F_{L,A}(t)}(\xi) = \sum_{R(\vec{k})=\xi} F_{L,A,\vec{k}}(t).$$

Remark 2.2. *The sum is finite because $(a_{L,k})$ has finite support.*

Proof. By induction on n. For $n = 0$, we have

$$F_{L,A,\vec{k}}(t) = e^{it\omega(R(\vec{k}))} g_{k_1} a_{L,k_1} = \hat{u}_0(t)(R(\vec{k})).$$

For $n = m+1$ with $m \in \mathbb{N}$ with the above construction. We have

$$F_{L,A}(t) = \int_0^t e^{(t-\tau)K}[J_L(F_{L,A_1}(\tau), \dots, F_{L,A_N}(\tau))]d\tau.$$

In Fourier mode, this transforms as

$$\widehat{F_{L,A}(t)}(\xi) = \frac{1}{(2\pi L)^{d(N-1)/2}} \int_0^t e^{i(t-\tau)\omega(\xi)} \sum_{\xi_1+\dots\xi_N=\xi} \Psi(\xi_1, \dots, \xi_N)(\widehat{F_{L,A_1}(\tau)}(\xi_1), \dots, \widehat{F_{L,A_N}(\tau)}(\xi_N))d\tau.$$

We use the induction hypothesis to get that

$$\widehat{F_{L,A_j}(\tau)}(\xi_j) = \sum_{R(\vec{k}_j)=\xi_j} F_{L,A,\vec{k}_j}(\tau).$$

We see now that

$$\{\vec{k} \mid R(\vec{k}_j) = \xi_j \wedge \sum \xi_j = \xi\} = \{\vec{k} \mid R(\vec{k}) = \xi\}.$$

We deduce the result. □

PicardExpFourier2

Proposition 2.5. *We have that for all $A \in \mathcal{A}_n$ and all $\vec{k} \in (\mathbb{Z}_*^d)^{(N-1)n+1}$,*

$$F_{L,A,\vec{k}}(t) = \frac{1}{(2\pi L)^{d(N-1)n/2}} G_{L,A,\vec{k}}(t) \prod_{j=1}^{(N-1)n+1} g_{k_j}$$

where $G_{L,(),(k_1)}(t) = e^{i\omega(k_1/L)t} a_{L,k_1}$ and with the notations of Proposition 2.4 prop:PicardExpFourier

$$G_{L,A,\vec{k}}(t) = \int_0^t e^{i(t-\tau)\omega(R(\vec{k}))} \Psi(R(\vec{k}_1), \dots, R(\vec{k}_N))(G_{L,A_1,\vec{k}_j}(\tau), \dots, G_{L,A_N,\vec{k}_N}(\tau))d\tau.$$

Proof. By induction on n. □

Summing up, we have the following formula :

$$\hat{u}_n(\xi) = \frac{1}{(2\pi L)^{d(N-1)n/2}} \sum_{A\in\mathcal{A}_n} \sum_{R(\vec{k})=\xi} G_{A,\vec{k}}(t) \prod_{j=1}^{(N-1)n+1} g_{k_j}. \tag{8}$$

sumupPicardExp

2.2 Expectations

For the rest of this section, we set $R \in \mathbb{N}^*$, $(n_1, \dots, n_R) \in \mathbb{N}^R$, $i_1, \dots, i_R \in [|1, D|]^R$ and $(\xi_1, \dots, \xi_R) \in (\frac{1}{L}\mathbb{Z}^d_*)^R$. We also set

$$S = \{(l, j) \mid l \in [1, R] \cap \mathbb{N}, j \in [|1, n_l(N-1)+1|]\}$$

and

$$\mathfrak{S}$$

the set of involutions of S without fixed points.

Using Equation (8), we get

$$\mathbb{E}\Big(\prod_{l=1}^{R} \hat{u}^{(i_l)}_{n_l}(t)(\xi_l)\Big) = \frac{1}{(2\pi L)^{d(N-1)(\sum n_l)/2}} \sum_{A_l \in \mathcal{A}_{n_l}} \sum_{R(\vec{k}_l)=\xi_l} \prod_{l=1}^{R} G^{(i_l)}_{A_l,\vec{k}_l}(t) \mathbb{E}\Big(\prod_{m \in S} g_{k_m}\Big).$$

By the formula of cumulants, we have

$$\mathbb{E}\Big(\prod_{m \in S} g_{k_m}\Big) = \sum_{\sigma \in \mathfrak{S}} \prod_{m \in S_\sigma} \mathbb{E}(g_{k_m} g_{k_{\sigma(m)}})$$

where $S_\sigma = \{m \in S \mid m < \sigma(m)\}$ (using the lexicographical order). We get the following proposition.

Proposition 2.6. *We have that*

$$\mathbb{E}\Big(\prod_{l=1}^{R} \hat{u}^{(i_l)}_{n_l}(t)(\xi_l)\Big) = \frac{1}{(2\pi L)^{d(N-1)(\sum n_l)/2}} \sum_{A_l \in \mathcal{A}_{n_l}} \sum_{\sigma \in \mathfrak{S}} \sum_{\Sigma_\sigma} \prod_{l=1}^{R} G^{(i_l)}_{L,A_l,\vec{k}_l}(t)$$

where

$$\Sigma_\sigma = \{\vec{k} \in (\mathbb{Z}^d)^S \mid \forall l \in [|1, R|],\ R(\vec{k}_l) = \xi_l,\ \forall m \in S, k_m = -k_{\sigma(m)},$$

and where

$$G^{(i_l)}_{L,A_l,\vec{k}_l}(t) := 0$$

whenever there exists $j \in [|1, n_l(N-1)+1|]$ such that $k_{l,j} = 0$, and where we used the notation

$$\vec{k}_l = (k_{l,1}, \dots, k_{l,(N-1)n_l+1}).$$

Proof. We have

$$\mathbb{E}(g_{k_m} g_{k_{\sigma(m)}}) = \begin{cases} 1 & \text{if } k_m = -k_{\sigma(m)} \\ 0 & \text{otherwise.} \end{cases}$$

□

Remark 2.3. *If the cardinal of S, that is, $\sum_l n_l(N-1) + R$ is odd, then the expectation is* 0.

We also set for $l \in [|1, R|]$, and $j \in [|1, n_l(N-1)+1|]$,

$$\sigma(l, j) = (\tilde{\sigma}(l, j), j')$$

for some $j' \in [|1, n_{l'}(N-1)+1|]$.

We now compute the dimension of Σ_σ. For this, we introduce the notion of orbits of σ.

def:orbits **Definition 2.7.** Let $A \subset [|1, R|]$. We set

$$\sigma(A) = \{l \in [|1, R|] \cap \mathbb{N} \mid \exists l' \in A, \exists j' \in [|1, n_{l'}(N-1)+1|], l = \tilde{\sigma}(l', j')\}.$$

This defines a map of the parts of $[|1, R|]$ to itself.

We call the orbit of l in σ and we write $o_\sigma(l)$ the set

$$o_\sigma(l) = \bigcup_{n \in \mathbb{N}} \sigma^n(\{l\}).$$

We write $\mathcal{O}_\sigma$ the set whose elements are the orbits of σ.

prop:partition **Proposition 2.8.** *The orbits of σ form a partition of* $[|1, R|]$.

Proof. We prove that the relation $l \in o_\sigma(l')$ is an equivalence relation.

This relation is reflexive since $l \in \sigma^0(\{l\})$ for all l.

This relation is symmetric. Indeed, let $l, l' \in [|1, R|]$. We prove that $l \in o(l')$ implies $l' \in o(l)$. Since $l \in o(l')$, there exists n such that $l \in \sigma^n(\{l'\})$. Therefore, there exists $j_1, \dots, j_n, k_0, \dots, k_{n-1}$ and $l' = l_0, l_1, \dots, l_{n-1}, l_n = l$ such that for all $m = 0, \dots, n-1$,

$$(l_{m+1}, j_{m+1}) = \sigma(l_m, k_m).$$

Because σ is an involution, this also reads as

$$(l_m, k_m) = \sigma(l_{m+1}, j_{m+1})$$

and thus $l' \in o(l)$.

This relation is transitive. Indeed, if $l \in o_\sigma(l')$ and if $l' \in o_\sigma(l'')$ then there exist n_1 and $n_2 \in \mathbb{N}$, such that

$$l \in \sigma^{n_1}(\{l'\}) \quad l' \in \sigma^{n_2}(\{l''\}).$$

Therefore, we have

$$l \in \sigma^{n_1+n_2}(\{l''\}) \subseteq o_\sigma(l'').$$

□

prop:dimSigma **Proposition 2.9.** *If for all $o \in \mathcal{O}_\sigma$, we have*

$$\sum_{l \in o} \xi_l = 0$$

then

$$\Sigma_\sigma \sim (Z^d)^{s_\sigma}$$

with $s_\sigma = \frac{1}{2}\#S + \#\mathcal{O}_\sigma - R$.

Otherwise, $\Sigma_\sigma = \emptyset$.

Remark 2.4. *By $\Sigma_\sigma \sim (Z^d)^{s_\sigma}$, we mean that within the $\#S$ parameters of the elements of Σ_σ, s_σ of them are free and $\#S - s_\sigma$ are fixed by the values of the s_σ free parameters. More precisely, we mean that up to a reordering of the parameters in Σ_σ,*

$$\Sigma_\sigma = \{(\xi_1, \dots, \xi_{s_\sigma}, L_{s_\sigma+1}(\xi_1, \dots, \xi_{s_\sigma}), \dots, L_{\#S}(\xi_1, \dots, \xi_{s_\sigma})) \mid (\xi_1, \dots, \xi_{s_\sigma}) \in \mathbb{Z}^d)^{s_\sigma}\}$$

where $L_{s_\sigma+1}, \dots, L_{\#S}$ are linear maps.

Proof. For all $l \in [|1, R|]$, set

$$S_{\sigma,l,+} = \{j \in [1, n_l(N-1)+1] \mid l < \tilde{\sigma}(l,j)\}$$

and

$$S_{\sigma,l,-} = \{j \in [1, n_l(N-1)+1] \mid l > \tilde{\sigma}(l,j)\}.$$

Note that $S_{\sigma,l,+} \subseteq S_\sigma$ and that $S_{\sigma,l,-}$ is included in the complementary of S_σ in S and that $\sigma(S_{\sigma,l,-}) \subseteq S_\sigma$.

By definition, we have

$$\Sigma_\sigma = \{\vec{k} \in (\mathbb{Z}_*^d)^S | \forall l \in [1,R] \cap \mathbb{N},\ R(\vec{k_l}) = \xi_l,\ \forall m \in S, k_m = -k_{\sigma(m)}\}.$$

By taking only half the ks (the ones in S_σ, the others being entirely determined by the ones in S_σ), we get

$$\Sigma_\sigma \sim \Sigma'_\sigma = \{\vec{k} \in (\mathbb{Z}_*^d)^{S_\sigma} \mid \forall l \in [1,R] \cap \mathbb{N},\ \sum_{j \in S_{\sigma,l,+}} k_{(l,j)} - \sum_{j \in S_{\sigma,l,-}} k_{\sigma(l,j)} = L\xi_l\}.$$

Because the orbits of σ form a partition of $[1,R] \cap \mathbb{N}$ and because equations

$$\sum_{j \in S_{\sigma,l,+}} k_{(l,j)} - \sum_{j \in S_{\sigma,l,-}} k_{\sigma(l,j)} = L\xi_l$$

involve only ls from the same orbit. Indeed, we have that $\tilde{\sigma}(l,j) \in o_\sigma(l)$. We have the decomposition

$$\Sigma'_\sigma \sim \prod_{o \in O_\sigma} \Sigma_{\sigma,o}$$

with

$$\Sigma_{\sigma,o} = \{\vec{k} \in (\mathbb{Z}_*^d)^{S_{\sigma,o}} \mid \forall l \in o,\ \sum_{j \in S_{\sigma,l,+}} k_{(l,j)} - \sum_{j \in S_{\sigma,l,-}} k_{\sigma(l,j)} = L\xi_l\}$$

where

$$S_{\sigma,o} = \{(l,j) \in S \mid l \in o \wedge (l,j) < \sigma(l,j)\}$$

where we used the lexicographical order. We have

$$\sum_{l \in o}\Big(\sum_{j \in S_{\sigma,l,+}} k_{(l,j)} - \sum_{j \in S_{\sigma,l,-}} k_{\sigma(l,j)}\Big) = \sum_{l \in o, j \in S_{\sigma,l,+}} k_{l,j} - \sum_{l \in o, j \in S_{\sigma,l,-}} k_{\sigma(l,j)}.$$

Let $l \in o$ and $j \in S_{\sigma,l,+}$. By definition, $\sigma(l,j) = (l',j')$ with $l' > l$. By definition of the orbits, we also have $l' \in o$. Because σ is an involution, we have

$$(l,j) = \sigma(l',j').$$

Finally, by definition of $S_{\sigma,l',-}$, we have $j' \in S_{\sigma,l',-}$. In other words, there exists (a unique) couple (l',j') such that $l' \in o$ and $j' \in S_{\sigma,l',-}$ such that

$$(l,j) = \sigma(l',j').$$

Conversely, if (l',j') is such that $l' \in o$, $j' \in S_{\sigma,l,-}$ then $(l,j) := \sigma(l',j')$ is such that $l \in o$ and $j \in S_{\sigma,l,+}$. Therefore,

$$\bigsqcup_{l \in o} S_{\sigma,l,+} = \sigma\Big(\bigsqcup_{l \in o} S_{\sigma,l,-}\Big).$$

We deduce

$$\sum_{l\in o}\Big(\sum_{j\in S_{\sigma,l,+}} k_{(l,j)} - \sum_{j\in S_{\sigma,l,-}} k_{\sigma(l,j)}\Big) = 0$$

and thus $\Sigma_{\sigma,o} \neq \emptyset$ implies

$$\sum_{l\in o} \xi_l = 0.$$

Assume now that $\sum_{l\in o}\xi_l = 0$. We write (E_l) the equation

$$\sum_{j\in S_{\sigma,l,+}} k_{(l,j)} - \sum_{j\in S_{\sigma,l,-}} k_{\sigma(l,j)} = L\xi_l.$$

We know that these equations are not independent since

$$\sum_{l\in o}(E_l) = 0.$$

We prove now that at least $\#o - 1$ of them are independent. We argue by contradiction. By contradiction, we assume that there exists $(\alpha_l)_{l\in o}$ such that the sequence is not constant and

$$\sum_{l\in o}\alpha_l(E_l) = 0.$$

This would imply that $\sum_l \alpha_l\xi_l = 0$ and

$$\sum_{l\in o}\sum_{j\in S_{\sigma,j,+}} \alpha_l k_{l,j} - \sum_{l\in o}\sum_{j\in S_{\sigma,l,-}} \alpha_l k_{\sigma(l,j)} = 0$$

for all $\vec{k} \in (\mathbb{Z}_*^d)^{S_{\sigma,o}}$. This may be rewritten as

$$\sum_{l\in o}\sum_{j\in S_{\sigma,j,+}} k_{l,j}(\alpha_l - \alpha_{\tilde{\sigma}(l,j)}) = 0$$

for all $\vec{k} \in (\mathbb{Z}^d)^{S_{\sigma,o}}$.

We deduce that if there exists j such that $\tilde{\sigma}(l, j) = l'$, then

$$\sum_{l}\alpha_l(E_l) = 0$$

implies $\alpha_l = \alpha_{l'}$. In other words, for all $l' \in \sigma(\{l\})$, $\alpha_{l'} = \alpha_l$. By induction, we get α is constant on the whole orbit which yields a contradiction. We get indeed that at least $\#o - 1$ equations are independent. We deduce

$$\Sigma_{\sigma,o} \sim (\mathbb{Z}^d)^{\#S_{\sigma,o}-\#o+1}$$

and thus

$$\Sigma_\sigma \sim (\mathbb{Z}^d)^{s_\sigma}$$

with

$$s_\sigma = \sum_o(\#S_{\sigma,o} - \#o + 1) = \#S_\sigma - R + \#O_\sigma$$

hence the result. □

2.3 Estimates

We estimate the cardinal of $\mathcal{A}_n$ and $G_{L,A,\vec{k}}$ for a given $\vec{k}$.

prop:cardinalA **Proposition 2.10.** *Let $n \in \mathbb{N}^*$, we have that*

$$\#\mathcal{A}_n \leq \begin{cases} 4^{n-1} & \text{if } N = 1 \\ (3eN)^{n-1} & \text{otherwise.} \end{cases}$$

Proof. This is a classical computation that we detail here for the seek of completeness.

Using Polish notation, the trees in $\mathcal{A}_n$ are sequences of $n(N-1)+1$ zeros and n ones, knowing that the first character is a one and the last a zero. Therefore, it remains to place $n-1$ ones into $n(N-1)+1+n-2 = nN-1$ slots. There are of course extra rules than the ones we mention but this leaves at most

$$\binom{nN-1}{n-1}$$

possibilities and thus

$$\#\mathcal{A}_n \leq \frac{(nN-1)!}{(n-1)!(n(N-1))!}.$$

We start with $N = 2$. In this case, we have

$$\#\mathcal{A}_n \leq \frac{(2n-1)!}{n!(n-1)!} = \prod_{k=1}^{n-1} \frac{2k}{k} \prod_{k=2}^{n} \frac{2k-1}{k}$$

which yields the result using that $2k-1 \leq 2k$.

For general N, we have

$$(nN-1)! = \prod_{j=1}^{n-1} Nj \prod_{k=1}^{N-1} \prod_{j=0}^{n-1} (Nj+k).$$

We deduce that

$$\frac{(nN-1)!}{(n-1)!} = N^{n-1} \prod_{k=1}^{N-1} \prod_{j=0}^{n-1} (Nj+k).$$

We also have

$$(n(N-1))! = n(N-1) \prod_{j=1}^{n-1} (N-1)j \prod_{k=1}^{N-2} \prod_{j=0}^{n-1} ((N-1)j+k).$$

We deduce

$$\#\mathcal{A}_n \leq N^{n-1} \Big(\prod_{k=1}^{N-2} \prod_{j=0}^{n-1} \frac{Nj+k}{(N-1)j+k} \Big) \Big(\prod_{j=1}^{n-1} \frac{Nj+N-1}{(N-1)j} \Big) \frac{N-1}{n(N-1)}.$$

Let

$$I = \prod_{j=1}^{n-1} \frac{Nj+N-1}{(N-1)j}.$$

We have for all $j \in [1, n-1]$,

$$\frac{Nj+N-1}{(N-1)j} = \frac{N}{N-1} + \frac{1}{j} \leq \frac{2N-1}{N-1}.$$

We deduce

$$I \leq \Big(\frac{2N-1}{N-1}\Big)^{n-1}.$$

We have of course

$$\frac{N-1}{n(N-1)} = \frac{1}{n}.$$

We set

$$II = \prod_{k=1}^{N-2}\prod_{j=0}^{n-1}\frac{Nj+k}{(N-1)j+k}.$$

We have

$$\ln II = \sum_{k=1}^{N-2}\sum_{j=0}^{n-1}\ln\Big(1+\frac{j}{(N-1)j+k}\Big).$$

Because $\ln(1+x) \leq x$ for all $x \geq 0$, we have

$$\ln II \leq \sum_{j=1}^{n-1} j \sum_{k=1}^{N-2}\frac{1}{(N-1)j+k}.$$

We have

$$\sum_{k=1}^{N-2}\frac{1}{(N-1)j+k} \leq \int_{(N-1)j}^{(N-1)j+N-2}\frac{dx}{x} = \ln\Big(\frac{(N-1)j+N-2}{(N-1)j}\Big) = \ln(1+\frac{N-2}{(N-1)j}) \leq \frac{1}{j}.$$

We deduce

$$\ln II \leq (n-1)$$

and thus

$$II \leq e^{n-1}.$$

Summing up we get

$$\#\mathcal{A}_n \leq \Big(eN\frac{2N-1}{N-1}\Big)^{n-1}.$$

Roughly, we get

$$\#\mathcal{A}_n \leq (3eN)^{n-1}.$$

□

prop:estimateG

Proposition 2.11. *There exists $C = C(N,\Psi)$ such that for all $n_i n\mathbb{N}$, for all $\vec{k} \in (\mathbb{Z}^d_*)^{(N-1)n+1}$ and for all $A \in \mathcal{A}_n$, for all $t \in \mathbb{R}_+$, we have*

$$|G_{L,A,\vec{k}}(t)| \leq C^n t^n \max_{l=1}^{(N-1)n+1}\langle\frac{k_l}{L}\rangle^{rn}\prod_{j=1}^{(N-1)n+1}|a_{L,k_j}|.$$

Proof. We prove this by induction on the trees. If $A = ()$ then

$$|G_{L,(),(k_1)}(t)| = |a_{L,k_1}|.$$

If $A \in \mathcal{A}_{n+1}$ with $A = (A_1,\dots,A_N)$ and $A_j \in \mathcal{A}_{n_j}$, we have

$$|G_{L,A,\vec{k}}(t)| = \Big|\int_0^t e^{i(t-\tau)\omega(R(\vec{k})}\Psi(R(\vec{k}_1),\dots,R(\vec{k}_N))(G_{L,A_1,\vec{k}_1}(\tau),\dots,G_{L,A_N,\vec{k}_N}(\tau))\Big|.$$

We use the induction hypothesis to get that

$$\Big|\prod_{j=1}^{N} G_{L,A_j,\vec{k}_j}(\tau)\Big| \leq C^n \tau^n \max_{l=1}^{(N-1)n+1} \langle \frac{k_l}{L}\rangle^{rn} \prod_{l=\tilde{n}_j+1}^{\tilde{n}_j+(N-1)n_j+1} |a_{L,k_l}|.$$

We deduce

$$|G_{L,A,\vec{k}}(t)| \leq C^n \max_{l=1}^{(N-1)n+1} \langle \frac{k_l}{L}\rangle^{rn} \frac{t^{n+1}}{n+1} |\Psi(R(\vec{k}_1),\dots,R(\vec{k}_N))| \prod_l |a_{L,k_l}|.$$

We have that

$$|\Psi(R(\vec{k}_1),\dots,R(\vec{k}_N))| \leq C' \max_{j=1}^{N} \langle R(\vec{k}_j)\rangle^r.$$

Since $r \leq 1$, we have

$$\langle R(\vec{k}_j)\rangle^r \leq \sum_l \langle \frac{k_l}{L}\rangle^r \leq (n_j(N-1)+1) \max_l \langle \frac{k_l}{L}\rangle^r.$$

Since $n \geq n_j$ for all j, we have

$$|G_{L,A,\vec{k}}(t)| \leq C^n N C' \max_{l=1}^{(N-1)n+1} \langle \frac{k_l}{L}\rangle^{r(n+1)} t^{n+1} \prod_l |a_{L,k_l}|.$$

Taking $C = C'N$ we get the result. □

We now estimate

$$F_{L,\sigma}(t) = \frac{1}{(2\pi L)^{d(N-1)(\sum n_l)/2}} \sum_{A_l \in \mathcal{A}_{n_l}} \sum_{\Sigma_\sigma} \prod_{l=1}^{R} G_{L,A_l,\vec{k}_l}$$

Combining all we have done so far, we get the following proposition.

Proposition 2.12. *If for some $o \in O_\sigma$, we have*

$$\sum_{l\in o} \xi_l \neq 0$$

then

$$F_{L,\sigma}(t) = 0$$

otherwise, we have the estimate, with $\bar{C} = C3eN$ if $N > 2$ and $\bar{C} = 4C$ if $N = 2$,

$$|F_{L,\sigma}(t)| \leq \frac{(\bar{C} A_L^r t)^{\sum n_l}}{(2\pi L)^{d(R/2-\#O_\sigma)}} \|a_{L,k}\|_{\ell^\infty}^{\#S-2s_\sigma} \|a_{L,k}\|_{\ell^2}^{2s_\sigma}.$$

where we recall that A_L is defined as

$$A_L := \sup\{\langle \frac{k}{L}\rangle \mid a_{L,k} \neq 0\}.$$

prop:estExp **Proposition 2.13.** *If it exists, set O be a maximal partition of $[|1,R|]\cap\mathbb{N}$ such that for all $o\in O$,*

$$\sum_{l\in o}\xi_l = 0.$$

Then,

$$\mathbb{E}\Big(\prod_{l=1}^{R}\hat{u}_{n_l}^{(i_l)}(t)(\xi_l)\Big) \leq \#\mathfrak{S}\|a_{L,k}\|_{\ell^\infty\cap\ell^2}^{\#S}\frac{(\bar{C}A_L^r t)^{\sum n_l}}{(2\pi L)^{d(R/2-\#O)}}.$$

If such a partition does not exist then

$$\mathbb{E}\Big(\prod_{l=1}^{R}\hat{u}_{n_l}^{(i_l)}(t)(\xi_l)\Big) = 0$$

Proof. If such a partition does not exist then $F_{L,\sigma}=0$ for all the σ.

Otherwise, $F_{L,\sigma}\neq 0$ implies that for all $o\in O_\sigma$,

$$\sum_{l\in o}\xi_l = 0.$$

In particular,

$$\#O_\sigma \leq \#O,$$

and thus

$$\frac{1}{(2\pi L)^{d(R/2-\#O_\sigma)}} \leq \frac{1}{(2\pi L)^{d(R/2-\#O)}}.$$

□

Remark 2.5. *The cardinal of O is necessarily smaller than $\frac{R}{2}$ since ξ_l cannot be null.*

Proposition 2.14. *Assume that a partition $\sqcup_{o\in O} o = [|1,R|]$ such that for all $o\in O$,*

$$\sum_{l\in o}\xi_l = 0$$

exists. Then, let $O_1,\dots,O_F$ be the maximal partitions of this type, then

$$\Big|\mathbb{E}\Big(\prod_{l=1}^{R}\hat{u}_{n_l}^{(i_l)}(t)(\xi_l)\Big) - \sum_{J=1}^{F}\prod_{o\in O_J}\mathbb{E}\Big(\prod_{l\in o}\hat{u}_{n_l}^{(i_l)}(t)(\xi_l)\Big)\Big| \leq \#\mathfrak{S}\|a_{L,k}\|_{\ell^\infty\cap\ell^2}^{\#S}\frac{(\bar{C}tA_L^r)^{\sum n_l}}{(2\pi L)^{d(R/2-\#O+1)}}.$$

Proof. The σs that correspond to the leading order in L of

$$\mathbb{E}\Big(\prod_{l=1}^{R}\hat{u}_{n_l}(t)(\xi_l)\Big)$$

are the ones such that $O_\sigma = O_J$ for some J. And thus, this σs decompose into involutions σ_o without fixed points of

$$S_o = \bigcup_{l\in o}\{l\}\times([1,n_l(N-1)+1]\cap\mathbb{N})$$

with only one orbit.

Conversely, the σ_o that yield a non-zero contribution to

$$\mathbb{E}\Big(\prod_{l\in o}\hat{u}_{n_l}^{(i_l)}(t)(\xi_l)\Big)$$

have necessarily only one orbit due to the maximality of O_J.

Note that if $O_\sigma = O_J$ and $O_{\sigma'} = O_{J'}$ then $J\neq J'$ implies $\sigma\neq\sigma'$. □

Corollary 2.15. *We have*

$$\Big|\mathbb{E}\Big(\prod_{l=1}^{R}\hat{u}_{n_l}^{(i_l)}(t)(\xi_l)\Big)-\sum_{O\in\mathcal{P}_R}\prod_{\{l,l'\}\in O}\mathbb{E}(\hat{u}_{n_l}^{(i_l)}(t)(\xi_l)\hat{u}_{n_{l'}}^{(i_{l'})}(t)(\xi_{l'}))\Big|\leq \#\mathfrak{S}\|a_{L,k}\|_{\ell^\infty\cap\ell^2}^{\#S}\frac{(\bar{C}tA_L^r)^{\sum n_l}}{(2\pi L)^{d/2})}$$

where $\mathcal{P}_R$ is the set of partitions of $[|1,R|]$ whose elements are pairs of $[|1,R|]$.

Remark 2.6. *In other words, $\mathcal{P}_R$ is the set of involutions of $[|1,R|]$ without fixed points.*

3 Application to the Euler equation

We consider the Euler equation :

$$\begin{cases}\partial_t u_L+u_L\cdot\nabla u_L=-\nabla p \quad \text{on } L\mathbb{T}^d\\ \nabla\cdot u_L=0\end{cases}. \tag{9}$$

Euler1

Remark that the quantity

$$\int_{L\mathbb{T}^d}u_L(x)dx$$

is a priori conserved under the action of the flow of the equation, we chose it null.

Applying the Leray projection defined in Fourier mode as

$$\widehat{Pv}(\xi)=\hat{v}(\xi)-\sum_j\xi_j\hat{v}^{(j)}(\xi)\frac{\xi}{|\xi|^2}$$

we get that $u_L=Pu_L$ satisfies

$$\partial_t u_L+P(u_L\cdot\nabla u_L)=0.$$

Therefore, J writes

$$J(u,v)=P(u\cdot\nabla v)$$

and Ψ writes

$$\Psi(\xi-\eta,\eta)(X,Y)=\sum_{j=1}^{d}i\eta_jX^{(j)}Y-\sum_{j,k=1}^{d}\frac{\xi_k}{|\xi|}i\eta_jX^{(j)}Y^{(k)}\frac{\xi}{|\xi|}$$

for all $\eta=(\eta_1,\dots,\eta_d)\in\frac{1}{L}\mathbb{Z}_*^d$, $\xi=(\xi_1,\dots,\xi_d)\in\frac{1}{L}(\mathbb{Z}_*^d)$ such that $\xi-\eta\neq 0$ and all $X=(X^{(1)},\dots,X^{(d)}),Y=(Y^{(1)},\dots,Y^{(d)})\in\mathbb{C}^d$.

Therefore, we are in the framework afore-mentioned with $r=1$.

Note that the initial datum must satisfy $\nabla\cdot a_L=0$ which is implied by the condition $\xi\cdot a(\xi)=0$ for all ξ.

3.1 Well-posedness in the analytic framework

sec:wellposedness

Let $\psi:\mathbb{R}$ be a smooth increasing map with values in $[0,1]$ which is equal to 1 on $[1,\infty)$ and to 0 on $(-\infty,0]$. Set $\varphi(x)=\psi(x+1)$ on $[-1,0]$, $\varphi(x)=1-\psi(x)$ on $[0,1]$ and $\varphi(x)=0$ elsewhere. We set for $n\in\mathbb{Z}^d$, and $\xi\in\mathbb{R}^d$,

$$\varphi_n(\xi)=\prod_{j=1}^{d}\varphi(\xi_j-n_j).$$

We get that φ_n is smooth, supported in the rectangle

$$R_n = \prod_{j=1}^{d} [n_j - 1, n_j + 1].$$

We also have that

$$\sum_{n \in \mathbb{Z}^d} \varphi_n = Id_{\mathbb{R}^d}.$$

Definition 3.1. Let $\rho > 0$, we introduce the space E_ρ induced by the norm

$$\|f\|_\rho := \sum_n e^{\rho|n|} \|\phi_n * f\|_{L^\infty(\mathbb{R}^d)}$$

where ϕ_n is the inverse Fourier transform of φ_n.

prop:estimBilin1

Proposition 3.2. *There exists $C = C(d, \varphi)$ such that for all $\rho > 0$ and all $f, g \in E_\rho$, we have*

$$\|fg\|_\rho \leq e^{2\rho} \|f\|_\rho \|g\|_\rho.$$

Proof. Let $f_n = \phi_n * f$ and $g_n = \phi_n * g$. We have for all n,

$$\phi_n * (fg) = \phi_n * (\sum_k f_k \sum_l g_l) = \sum_{k,l} \phi_n * (f_k g_l).$$

We have that f_k is supported in Fourier mode in R_k and g_l is supported in Fourier in R_l hence $f_k g_l$ is supported in the rectangle

$$\prod_{j=1}^{d} [k_j + l_j - 2, k_j + l_j + 2].$$

Therefore,

$$\phi_n * (f_k g_l) \neq 0$$

implies that for all $j = 1, \ldots, d$,

$$[n_j - 1, n_j + 1] \cap [k_j + l_j - 2, k_j + l_j + 2]$$

is not of null Lebesgue measure. In other words, $n_j - 1$ has to be strictly smaller that $k_j + l_j + 2$ and $n_j + 1$ has to be strictly greater than $k_j + l_j - 2$, that is

$$n_j \in [|k_j + l_j - 2, k_j + l_j + 2|].$$

In particular, $|n_j| \leq |k_j| + |l_j| + 2$ and there are 5^d tuples n that satisfy this thus

$$\phi_n * (fg) = \sum_{|n_j| \leq |k_j| + |l_j| + 2} \phi_n * (f_k g_l).$$

Therefore, we have

$$\|\phi_n * (fg)\|_{L^\infty} \leq \sum_{|n| \leq |k| + |l| + 2} \|\phi_n * (f_k g_l)\|_{L^\infty}.$$

Since

$$\phi_n(x) = e^{in \cdot x} \phi_0(x),$$

we get that ϕ_n belongs to L^1 its norm is uniformly bounded in n, we get

$$\|\phi_n * (fg)\|_{L^\infty} \lesssim \sum_{|n|\leq|k|+|l|+2} 1_n(k,l)\|(f_k g_l)\|_{L^\infty}$$

where $1_n(k,l)$ equals 1 if $n_j \in [k_j + l_j - 2, k_j + l_j + 2]$ for all j and 0 otherwise.

We have that L^∞ is an algebra and thus

$$\|\phi_n * (fg)\|_{L^\infty} \lesssim \sum_{|n|\leq|k|+|l|+2} 1_n(k,l)\|f_k\|_{L^\infty}\|g_l\|_{L^\infty}. \tag{10}$$ StartAgain

We sum over n and get

$$\|fg\|_\rho \lesssim \sum_{k,l} \|f_k\|_{L^\infty}\|g_l\|_{L^\infty} \sum_{|n|\leq|k|+|l|+2} 1_n(k,l)e^{\rho'|n|}.$$

We deduce

$$\|fg\|_\rho \lesssim e^{2\rho} \sum_{k,l} e^{\rho|k|}\|f_k\|_{L^\infty} e^{\rho|l|}\|g_l\|_{L^\infty}.$$

Hence the result. □

Proposition 3.3. *Set χ_1 a smooth map that is equal to 1 on $\{|\xi| \geq 1\}$ and null on $\{|\xi| \leq 1/2\}$. We identify χ_1 and the Fourier multiplier by χ_1. There exists $C = C(d,\varphi)$ such that for all $\rho \geq 0$ and all $f \in E_\rho$,*

$$\|P\chi_1 f\|_\rho \leq C\|f\|_\rho.$$

Proof. Because P and χ_1 commute with Fourier multipliers, we have that

$$\phi_n * (P\chi_1 f) = P\chi_1(\phi_n * f).$$

We have that $P\chi_1$ acts as a smooth Fourier multiplier on $\phi_n * f$. We set χ a $\mathbb{C}^\infty$ map that is non-negative, equal to 1 on $[-1,1]^d$ and null outside $[-3/2, 3/2]^d$. We write also P the kernel of the Leray projection and

$$P_n(\xi) = \chi(\xi - n)\chi_1(\xi)P(\xi).$$

We have that the inverse Fourier transform of P_n is in L^1 and that its norm is less than

$$\|P_n\|_{H^s}$$

for $s > d/2$. We get that

$$\|P_n\|_{H^s} \leq \|\chi(\cdot - n)\|_{H^s}\|\chi_1 P\|_{W^{s,\infty}}$$

we deduce that the L^1 norm of the inverse Fourier transform of P_n is uniformly bounded on n and thus

$$\|\phi_n * (Pf)\|_{L^\infty} = \|P_n\phi_n * f\|_{L^\infty} \leq \|f_n\|_{L^\infty}.$$

□

Proposition 3.4. *There exists $C = C(d,\varphi)$ such that for all $\rho > \rho' \geq 0$, and for all $f \in E_\rho$, we have*

$$\|\nabla f\|_{\rho'} \leq Ce^{\rho'}(\rho - \rho')^{-1}\|f\|_{\rho'}.$$

Proof. We have

$$(\nabla f)_n = (\nabla \phi_n) * f.$$

For the usual support considerations, we have

$$(\nabla f)_n = \sum_{|n'-n|\leq 1} (\nabla \phi_n) * f_{n'}.$$

Indeed, we have

$$(\nabla f)_n = \sum_{n'} (\nabla \phi_n) * f_{n'}.$$

What is more, $\nabla \phi_n$ is supported in Fourier modes in R_n and $f_{n'}$ is supported in Fourier modes in $R_{n'}$. If $R_n \cap R_{n'}$ is not negligible, then for all j, we have

$$n_j \in (n'_j - 1, n'_j + 1)$$

that is for all j, $|n_j - n'_j| \leq 1$.

We have that

$$\nabla \phi_n = ine^{ix\cdot n}\phi_0 + e^{ix\cdot n}\nabla \phi_0$$

and thus

$$|\nabla \phi_n||_{L^1} \lesssim (|n| + 1).$$

We deduce

$$||(\nabla f)_n||_{L^\infty} \leq \sum_{|n'-n|\leq 1} (|n| + 1)||f_{n'}||_{L^\infty}.$$

We sum on n and get the result using that

$$e^{\rho'|n|}|n| \leq \frac{1}{\rho - \rho'} e^{\rho|n|}.$$

□

Proposition 3.5. *There exists $C = C(d, \varphi)$ such that for all $u = Pu \in E_\rho$ and all $v \in E_\rho$ such that $\nabla v \in E_\rho$, we have*

$$||P(u \cdot \nabla v)||_\rho \leq Ce^{2\rho}||u||_\rho(||v||_\rho + ||\nabla v||_\rho)$$

and

$$||\nabla P(u \cdot \nabla v)||_\rho \leq Ce^{2\rho}(||u||_\rho||\nabla \otimes \nabla v||_\rho + ||\nabla u||_\rho||\nabla v||_\rho + ||u||_\rho||\nabla v||_\rho).$$

Proof. We write $P = P\chi_1 + P(1 - \chi_1)$. As we have already seen, $P\chi_1$ is smooth in Fourier modes which, combined with the estimates on the product of two maps is sufficient to conclude that

$$||P\chi_1(u \cdot \nabla v)||_\rho \lesssim e^{2\rho}||u||_\rho||\nabla v||_\rho, \quad ||\nabla P\chi_1(u \cdot \nabla v)||_\rho \leq Ce^{2\rho}(||u||_\rho||\nabla \otimes \nabla v||_\rho + ||\nabla u||_\rho||\nabla v||_\rho).$$

Now $\nabla P(1 - \chi_1)$ is not a C^∞ Fourier multiplier but it is compactly supported and its behaviour at 0 is sufficiently smooth. Indeed, we have, for v in the Schwartz class,

$$\nabla P(1 - \chi_1)v(x) = \int d\xi (1 - \chi_1(\xi))\Big(\xi \otimes \hat{v}(\xi) - \sum_j \xi_j \hat{v}^{(j)}(\xi)\frac{\xi \otimes \xi}{|\xi|^2}\Big)e^{i\xi(x-y)}.$$

We use the inverse Fourier transform and get

$$\nabla P(1-\chi_1)v(x) = \int dy v(y) \otimes F(x-y) - \sum_j \int dy v^{(j)}(y) F_j(x-y)$$

where

$$F(z) = \int d\xi (1-\chi_1(\xi))\xi e^{iz}, \quad F_j(z) = \int d\xi \frac{\xi \otimes \xi}{|\xi|^2} \xi_j (1-\chi_1(\xi)) e^{i\xi z}.$$

These two functions are well-defined because

$$\xi \mapsto (1-\chi_1(\xi))\xi, \quad \xi \mapsto \frac{\xi \otimes \xi}{|\xi|^2}\xi_j(1-\chi_1(\xi))$$

are continuous and compactly supported.

We prove that F and F_j are in L^1. For F, this is the case because $\xi \mapsto (1-\chi_1(\xi))\xi$ is C^∞ and compactly supported. For F_j, we have that

$$\|F_j\|_{L^1} \lesssim \|\hat{F}_j\|_{H^{d/2+\eta}}$$

for any $\eta > 0$. We have that $\hat{F}_j$ is smooth outside of 0 and compactly supported. At 0, it behaves like

$$\frac{(\xi \times \xi)\xi_j}{|\xi|^2}$$

and thus

$$|\langle \nabla_\xi \rangle^\alpha \hat{F}_j(\xi)| \sim c|\xi|^{1-\alpha}.$$

We have that $\xi \mapsto |\xi|^{1-\alpha}$ belongs to L^2 if $1-\alpha > -\frac{d}{2}$. In particular, we have that $\hat{F}_j$ belongs to H^α for $\alpha \in (\frac{d}{2}, \frac{d}{2}+1)$. Therefore, F_j belongs to L^1. By duality, we get that for $v \in L^\infty$, we have

$$\nabla P(1-\chi_1)v(x) = \int dy v(y) \otimes F(x-y) - \sum_j \int dy v^{(j)}(y) F_j(x-y)$$

and that

$$\|\nabla P(1-\chi_1)v(x)\| \leq (\|F\|_{L^1} + \|F_j\|_{L^1})\|v\|_{L^\infty}.$$

For $P(1-\chi_1)(u \cdot \nabla v)$, we use that $u = Pu$ and thus $u \cdot \nabla v = \partial_j(u^{(j)}v)$ and then we use the same arguments as for $\nabla P(1-\chi_1)$ to get

$$\|P(1-\chi_1)(u \cdot \nabla v)\|_\rho \lesssim \|u\|_\rho \|v\|_\rho.$$

□

We now prove bilinear estimates such that we can make the Picard expansion converge. The idea is to render explicit the Cauchy-Kowalevskaia abstract theorem.

Definition 3.6. Let $\rho_0 > 0, \beta \in (0,1)$ and $\theta > 0$. We set for all $\rho \in (0, \rho_0)$, $\theta(\rho) = \theta(\rho_0 - \rho)$. We define $M(\rho_0, \beta, \theta)$ the space induced by the norm

$$\|u\|_{\rho_0,\beta,\theta} = \sup_{0<\rho<\rho_0} \sup_{0 \leq t < \theta(\rho)} \Big(\|u(t)\|_\rho + \|\nabla u(t)\|(\theta(\rho) - t)^\beta \Big).$$

Proposition 3.7. *Let $\rho_0 > 0$, and $\beta \in (0,1)$. There exists $C = C(d, \varphi, \rho_0, \beta)$ such that for all $u, v \in M(\rho_0, \beta, \theta)$ such that $u = Pu$, we have*

$$\Big\| \int_0^t P(u(\tau) \cdot \nabla v(\tau)) d\tau \Big\|_{\rho_0,\beta,\theta} \leq C\theta^{1+\beta} \|u\|_{\rho_0,\beta,\theta} \|v\|_{\rho_0,\beta,\theta}.$$

Besides

$$P \int_0^t P(u(\tau) \cdot \nabla v(\tau)) d\tau = \int_0^t P(u(\tau) \cdot \nabla v(\tau)) d\tau.$$

Proof. For the sake of this proof, we set

$$A(t) \int_0^t P(u(\tau) \cdot \nabla v(\tau)) d\tau, \quad \|\cdot\| = \|\cdot\|_{\rho_0,\beta,\theta}.$$

Let $\rho \in (0, \rho_0)$ and $t \in (0, \theta(\rho))$. We have

$$\|A(t)\|_\rho \leq \int_0^t \|P(u \cdot \nabla v)(\tau)\|_\rho d\tau.$$

We have

$$\|A(t)\|_\rho \leq \int_0^t e^{2\rho} \|u(\tau)\|_\rho (\|v(\tau)\|_\rho + \|\nabla v(\tau)\|_\rho) d\tau.$$

We use that $\|\nabla v(\tau)\|_\rho \leq \|v\|(\theta(\rho) - \tau)^{-\beta}$ and that

$$\int_0^t (\theta(\rho) - \tau)^{-\beta} d\tau \leq \frac{1}{1-\beta} (\theta(\rho))^{1-\beta}$$

to get

$$\|A(t)\|_\rho \leq Ce^{2\rho} \Big(\frac{(\theta\rho_0)^{1-\beta}}{1-\beta} + \theta\Big) \|u\| \, \|v\|.$$

We have that

$$\|\nabla A(t)\| \leq Ce^{2\rho} \int_0^t (\|\nabla u(\tau)\|_\rho \|\nabla v(\tau)\|_\rho + \|u(\tau)\|_\rho \|\nabla \otimes \nabla v(\tau)\|_\rho + \|u(\tau)\|_\rho \|v(\tau)\|_\rho) d\tau.$$

We estimate

$$I(t) = e^{2\rho} \int_0^t \|\nabla u(\tau)\|_\rho \|\nabla v(\tau)\|_\rho d\tau.$$

We use that u and v belong to $M(\rho_0, \beta, \theta)$ to get

$$I(t) \leq e^{3\rho} \|u\| \, \|v\| \int_{0^t} (\theta(\rho) - \tau)^{-2\beta} d\tau.$$

We use that $1 - 2\beta \geq -\beta$ to get

$$I(t) \leq \begin{cases} \frac{e^{3\rho}}{1-2\beta} (\theta\rho_0)^{1-2\beta} \|u\| \, \|v\| (\theta(\rho) - t)^{-\beta} & \text{if } \beta \neq \frac{1}{2} \\ \frac{2}{e} \sqrt{\theta\rho_0} \frac{1}{\sqrt{\theta(\rho) - t}} \|u\| \, \|v\| \text{ otherwise.} \end{cases}$$

We estimate

$$II(t) = e^{2\rho} \int_0^t \|u(\tau)\|_\rho \|\nabla \otimes \nabla v(\tau)\|_\rho d\tau.$$

We set

$$\rho(\tau) = \rho_0 - \frac{\theta(\rho) + \tau}{2\theta}.$$

By definition $\rho(\tau) < \rho_0$.

Because $\tau < \theta(\rho)$, we have

$$\rho(\tau) > \rho_0 - \frac{\theta(\rho)}{\theta} = \rho.$$

We deduce

$$II(t) \leq e^{3\rho}\|u\| \int_0^t \frac{1}{\rho(\tau) - \rho} \|\nabla v(\tau)\|_{\rho(\tau)} d\tau.$$

We also have that

$$\theta(\rho(\tau)) = \frac{\theta(\rho) + \tau}{2}$$

and thus

$$\theta(\rho(\tau)) - \tau = \frac{\theta(\rho) - \tau}{2} > 0.$$

We deduce that since $u, v \in M(\rho_0, \beta, \theta)$,

$$II(t) \leq e^{3\rho}\|u\| \, \|v\| \int_0^t (\theta(\rho(\tau)) - \tau)^{-\beta} (\rho(\tau) - \rho)^{-1} d\tau$$

and thus

$$II(t) \leq 2e^{3\rho}\|u\| \, \|v\| \int_0^t (\theta(\rho) - \tau)^{-\beta} (\rho(\tau) - \rho)^{-1} d\tau.$$

By definition, we have

$$\rho(\tau) - \rho = \rho_0 - \rho - \frac{\theta(\rho) + \tau}{2\theta} = \frac{\theta(\rho) - \tau}{2\theta},$$

We get

$$II(t) \leq 4\theta e^{3\rho}\|u\| \, \|v\| \int_0^t (\theta(\rho) - \tau)^{-\beta - 1} d\tau.$$

We deduce

$$II(t) \leq 4\theta^{1-\beta} \rho_0^{-\beta} e^{3\rho}\|u\| \, \|v\| (\theta(\rho) - t)^{-\beta}.$$

Finally,

$$III(t) := \int_0^t \|u(\tau)\|_\rho \|v(\tau)\|_\rho d\tau \leq \|u\| \, \|v\| t \lesssim \|u\| \, \|v\| (\theta(\rho) - t)^{-\beta} \theta^{1+\beta}.$$

□

rop:wellPosedness

Proposition 3.8. *Let $\rho_0 > 0$ and $\theta \geq 1$. Let $u_0 \in E_{\rho_0}$ and define by induction on n,*

$$u_{n+1} = \sum_{n_1 + n_2 = n} \int_0^t P(u_{n_1}(\tau) \cdot u_{n_2}(\tau)) d\tau.$$

There exists $C = C(d, \varphi, \beta, \rho_0)$ such that for all n, we have

$$\|u_n\|_{\rho_0, \beta, \theta} \leq \theta^{(1+\beta)n} C^n \|u_0\|_{\rho_0}^{n+1}$$

such that the series u_n converge in $M(\rho_0, \beta, \theta)$ if $\|u_0\|_{\rho_0} < \theta^{\beta - 1} C^{-1}$ towards the unique solution to the Euler equation with initial datum u_0.

Proof. We can check by induction on n that

$$\|u_n\|_{\rho_0,\beta,\theta} \leq \theta^{(1+\beta)n} C^n c_n \|u_0\|_{\rho_0}^{n+1}$$

where C is the constant of the previous proposition and where $c_n = \#\mathcal{A}_n s$ are the Catalan numbers. We then use that $c_n \leq 4^n$. □

3.2 Estimations on the norm of the initial datum

Here, we set $a_{L,k} = \varepsilon(L) a(\frac{k}{L})$ where a is a bounded, compactly supported function and where $\varepsilon(L) = O((\ln L)^{-1/2})$ for some $\varepsilon > 0$.

prop:estDI

Proposition 3.9. *There exists $C = C(d, a, \rho_0, \varphi)$ and $c = c(d, a, \rho_0, \varphi) > 0$ such that for all $L \geq e^2$, and all $R \geq \sqrt{\ln L}\varepsilon(L)C$,*

$$\mathbb{P}(\|a_L\|_{\rho_0} \geq R) \leq e^{-cR^2/\varepsilon(L)^2}.$$

Proof. Let $p \geq 2$, we estimate

$$\mathbb{E}_p^p := \mathbb{E}(\|a_L\|_{\rho_0}^p).$$

We have

$$\mathbb{E}_p \leq \sum_n e^{\rho|n|} \|(a_L)_n\|_{L^p(\Omega,L^\infty)}.$$

Since a_L is L periodic, so is $(a_L)_n = \phi_n * a_L$.

We deduce that

$$\|(a_L)_n\|_{L^\infty} \leq \sup_X \|\chi_X (a_L)_n\|_{L^\infty}$$

where the the supremum is taken over the $X \in \mathbb{Z}^d$ such that $|X| \leq L$ and such that $\chi_X = \chi_0(\cdot - X) = \prod_j \chi(\cdot_j - X_j)$ where χ is a smooth function supported on $[-2, 2]$ and equal to 1 on $[-1, 1]$. By the Sobolev injection, for $s \in (\frac{d}{2}, \infty) \cap \mathbb{N}$, we have

$$\|\chi_X (a_L)_n\|_{L^\infty} \lesssim \|\chi_X (a_L)_n\|_{H^s} \lesssim \sup_{Y, s' \leq s} \|\chi_Y (a_L^{s'})_n\|_{L^2}$$

where $a_L^{s'} = \nabla^{\otimes s'} a_L$. We deduce

$$\|(a_L)_n\|_{L^p(\Omega,L^\infty)} \leq \| \sup_{Y,s'} \|\chi_Y (a_L^{s'})_n\|_{L^2} \|_{L^p} \leq \sup_{s' \leq s} \Big(\sum_Y \|\chi_Y (a_L^{s'})_n\|_{L^p(\Omega,L^2)}^p \Big)^{1/p}.$$

The sum on Y is for $Y \in [|-L, L|]^d$ hence we sum on $(2L+1)^d$ factors.

By Minskowski's inequality, since $p \geq 2$,

$$\|\chi_Y (a_L^{s'})_n\|_{L^p(\Omega,L^2)} \leq \|\chi_Y (a_L^{s'})_n\|_{L^2(\mathbb{R}^d,L^p(\Omega))}.$$

The law of a_L is invariant under the action of space translations, hence so is the law of $(a_L^{s'})_n$ and thus

$$\|\chi_Y (a_L^{s'})_n\|_{L^2(\mathbb{R}^d,L^p(\Omega))} = \|\chi_0 (a_L^{s'})_n\|_{L^2(\mathbb{R}^d,L^p(\Omega))}.$$

We get

$$\|(a_L)_n\|_{L^p(\Omega,L^\infty)} \lesssim L^{d/p} \sup_{s' \leq s} \|\chi_0 (a_L^{s'})_n\|_{L^2(\mathbb{R}^d,L^p(\Omega))}.$$

Still using the invariance under space translations, we get

$$\|(a_L)_n\|_{L^p(\Omega,L^\infty)} \lesssim L^{d/p}\|\chi_0\|_{L^2(\mathbb{R}^d)} \sup_{s'\le s} \|(a_L^{s'})_n(0)\|_{L^p(\Omega)}.$$

We use that $(a_L^{s'})_n(0)$ is a Gaussian and that χ_0 does not depend on L to get

$$\|(a_L)_n\|_{L^p(\Omega,L^\infty)} \lesssim L^{d/p}\sqrt{p} \sup_{s'\le s} \|(a_L^{s'})_n(0)\|_{L^2(\Omega)}.$$

We note that

$$\|(a_L^{s'})_n(0)\|^2_{L^2(\Omega)} = \frac{1}{2\pi L}\sum_k \Big|\frac{k}{L}\Big|^{2s'} |a_{L,k}|^2 \varphi_n(k/L)^2 \lesssim \int \langle\xi\rangle^{2s}\varphi_n^2(\xi)\varepsilon(L)^2|a(\xi)|^2.$$

Summing over n and using Cauchy-Schwarz inequality, we get

$$\mathbb{E}_p \lesssim \sum_n e^{\rho_0|n|}\|(a_L)_n\|_{L^p(\Omega,L^\infty)} \lesssim L^{d/p}\sqrt{p}\Big(\sum_n e^{2\rho_0|n|}\langle n\rangle^{4s}\|(a_L^s)_n(0)\|^2_{L^2(\Omega)}\Big)^{1/2}.$$

We get

$$\mathbb{E}_p \lesssim L^{d/p}\varepsilon(L)\sqrt{p}\Big(\int d\xi\langle\xi\rangle^{4s}e^{2\rho_0|\xi|}|a(\xi)|^2\Big)^{1/2}$$

In other words, there exists $C(a,\rho_0,d)$ such that for all $p\ge 2$, and all L

$$\mathbb{E}_p \le CL^{d/p}\varepsilon(L)\sqrt{p}.$$

By Markov's inequality, we deduce that for all p, R, L, we have

$$\mathbb{P}(\|a_L\|_{\rho_0}\ge R) \le R^{-p}C^pL^d\varepsilon(L)^p p^{p/2},$$

that is

$$\mathbb{P}(\|a_L\|_{\rho_0}\ge R) \le \Big(\frac{CL^{d/p}\varepsilon(L)\sqrt{p}}{R}\Big)^p.$$

We set $p = \frac{R^2}{C^2e^{d+1}\varepsilon(L)^2}$ taking R as in the hypothesis with a big enough constant, we get

$$p \ge \ln L \ge 2.$$

We get

$$\mathbb{P}(\|a_L\|_{\rho_0}\ge R) \le (L^{d/p}e^{-(d+1)})^p.$$

We have $L^{d/p} = e^{d\ln L/p} \le e^d$, hence

$$\mathbb{P}(\|a_L\|_{\rho_0}\ge R) \le e^{-p} = e^{-R^2/(\varepsilon(L)^2e^{d+1}C^2)}$$

hence the result. □

3.3 Conclusion

Let $\rho_0 > 0, \beta \in (0,1)$ and $\theta \geq 1$ and set $A = A(\theta) = \frac{\theta^{-\beta-1}}{2C}$ where C is the constant mentioned in Proposition 3.8. Now set

$$\mathcal{E}_L = \mathcal{E}_L(\rho_0, \beta, \theta) = \{\|a_L\|_{\rho_0} \leq A\}.$$

If $\varepsilon(L) = o((\ln L)^{-1/2})$ then for L big enough, we have that A is big enough to get

$$\mathbb{P}(\mathcal{E}_L) \geq 1 - e^{-cA^2\varepsilon(L)^{-2}}.$$

If $\varepsilon(L) = O((\ln L)^{-1/2})$, for A to be big enough to get the above inequality, one needs θ to be small enough. We assume then that θ is small enough to get the estimate on the measure of $\mathcal{E}_L$.

We also have that for all $u_0 \in \mathcal{E}_L$. The solution u to the Euler equation exists and is unique in $M(\rho_0, \beta, \theta)$ and satisfies that for all $n \in \mathbb{N}$, $t < \theta$,

$$\|u_n(t)\|_0 \leq 2^{-n}A.$$

Therefore, for the rest of this subsection, we fix $R \in \mathbb{N}^*$, $(\xi_1, \dots, \xi_R) \in \frac{1}{L}(\mathbb{Z}^d_*)^R$, $(i_1, \dots, i_R) \in ([1,d] \cap \mathbb{N})^R$ and finally

$$I = \mathbb{E}\Big(1_{\mathcal{E}_L} \prod_{l=1}^{R} \hat{u}^{i_l}(\xi_l)(t)\Big).$$

We assume that $\varepsilon^{-1}(L) \lesssim L^\alpha$ for some $\alpha \geq 0$ and we set $M = M(L)$ such that for L big enough,

$$2 \ln 2 \frac{M(L)}{\ln L} > \frac{(R+1)d}{2} + R\alpha$$

and

$$C(a,\theta)\sqrt{2R}\sqrt{M(L)+1}\varepsilon(L) \leq \frac{1}{2}$$

for $C(a,\theta)$ a constant that appears in the proof of Lemma 3.11 and depends only on a and θ. For such a $M(L)$ to exists, this requires that

$$\varepsilon(L)\sqrt{\ln L} \leq c(a,\theta,R)$$

for a constant $c(a,\theta,R) > 0$ that depends only on a, θ and R, which is small enough.

We now write

$$u(t) = \mathcal{U}_M(t) + \mathcal{R}_M(t)$$

with

$$\mathcal{U}_M(t) = \sum_{n=0}^{M} u_n(t), \quad \mathcal{R}_M(t) = \sum_{n>M} u_n(t).$$

We get

$$\|\mathcal{U}_M(t)\|_0 \leq 2A, \quad \|\mathcal{R}_M(t)\|_0 \leq 2^{-M}A.$$

Lemma 3.10. *We have*

$$I = \mathbb{E}\Big(1_{\mathcal{E}_L} \prod_{l=1}^{R} \hat{\mathcal{U}}_M{}^{(i_l)}(t)(\xi_l)\Big) + O_{d,\varphi,\rho_0,\beta,\theta,R}(\varepsilon(L)^R L^{-d/2}).$$

Proof. Take $v \in E_0$ and $2\pi L$ periodic. We have that

$$v = \sum_n \phi_n * v$$

and that for all n, $\phi_n * v$ is $2\pi L$ periodic. Therefore

$$\hat{v} = \sum_n \widehat{\phi_n * v}.$$

We deduce that

$$\|\hat{v}\|_{L^\infty} \leq \sum_n \|\widehat{\phi_n * v}\|_{L^\infty}.$$

We recall that in the torus $L\mathbb{T}^d$, we define

$$\hat{w}(\xi) := \int_{L\mathbb{T}^d} w(x) \frac{e^{-ix\xi}}{(2\pi L)^{d/2}} dx$$

and thus

$$\|\hat{w}\|_{L^\infty} \leq (2\pi L)^{d/2} \|w\|_{L^\infty}.$$

Therefore,

$$\|\hat{v}\|_{L^\infty} \leq \sum_n (2\pi L)^{d/2} \|\phi_n * v\|_{L^\infty} = (2\pi L)^{d/2} \|v\|_0.$$

We deduce that on $\mathcal{E}_L$,

$$\|\hat{u}(t)\|_{L^\infty} \leq (2\pi L)^{d/2} 2A, \quad \|\hat{\mathcal{U}}_M(t)\|_{L^\infty} \leq (2\pi L)^{d/2} 2A, \quad \|\hat{\mathcal{R}}_M(t)\|_{L^\infty} \leq (2\pi L)^{d/2} 2^{-M} A.$$

We deduce that

$$I = \mathbb{E}\Big(1_{\mathcal{E}_L} \prod_{l=1}^{R} \hat{\mathcal{U}_M}^{(i_l)}(t)(\xi_l)\Big) + O_d(2^{-M} R (2A)^R L^{(dR)/2}).$$

We get the result since

$$2^{-M} L^{(dR)/2} = O(\varepsilon(L)^R L^{-d/2}).$$

Indeed, we have

$$2^{-M} L^{d/2(R+1)} \varepsilon(L)^{-R} \leq e^{-(\ln L)(\ln 2M(L)/\ln L - (R+1)d/2 - R\alpha)}.$$

For $L \gg 1$, we have

$$2 \ln 2M(L)/\ln L - (R+1)d/2 - R\alpha > 0$$

which ensures the result. □

lem:genEst **Lemma 3.11.** *We have*

$$\mathbb{E}\Big(\prod_{l=1}^{R} \hat{\mathcal{U}}_M^{(i_l)}(\xi_l)(t)\Big) = O_{d,\varphi,\rho_0,\beta,\theta,R,a,\varepsilon}(\varepsilon(L)^R).$$

Proof. Set

$$II := \mathbb{E}\Big(\prod_{l=1}^{R} \hat{\mathcal{U}}_M^{(i_l)}(\xi_l)(t)\Big).$$

We have

$$II = \sum_{n_l \le M} \mathbb{E}\Big(\prod_{l=1}^{R} \hat{u}_{n_l}^{(i_l)}(\xi_l)(t) \Big).$$

By Proposition 2.13, we have

$$\Big| \mathbb{E}\Big(\prod_{l=1}^{R} \hat{u}_{n_l}^{(i_l)}(\xi_l)(t) \Big) \Big| \le \#\mathfrak{S} \|a_{L,k}\|_{\ell^2 \cap \ell^\infty}^{\#S} (t\bar{C}A_L)^{\sum n_l} (2\pi L)^{d(R/2 - \#O)}$$

where

$$S = \{(l,k) \mid l \in [|1,R|],\ k \in [|1, n_j + 1|]\}$$

where $\mathfrak{S}$ is the set of involutions of S without fixed points, where

$$A_L = \sup\{\langle k/L \rangle \mid a_{k,L} \neq 0\},$$

and where O is a maximal partition of $[1,R] \cap \mathbb{N}$ such that for all $o \in O$,

$$\sum_{l \in o} \xi_l = 0.$$

We have that $\#S = \sum n_l + R$ and

$$a_{L,k} = \varepsilon(L) a(k/L)$$

and thus

$$A_L \le A_\infty := sup\{\langle \xi \rangle \mid a(\xi) \neq 0\}$$

and

$$\|a_{L,k}\|_{\ell^2 \cap \ell^\infty}^{\#S} (2t\bar{C}A_L)^{\sum n_l} \le (C(a,\theta)\varepsilon(L))^{\sum n_l} (C'(a)\varepsilon(L))^R$$

where $C(a,\theta) = 2\bar{C}A_\infty^{1+d/2}\theta \|a\|_{L^\infty}^2$ and $C'(a) = A_\infty^{1+d/2}\|a\|_{L^\infty}^2$.

Since O cannot contain singletons, we have

$$\#O \le R/2, \quad d(R/2 - \#O) \ge 0.$$

We deduce

$$\Big| \mathbb{E}\Big(\prod_{l=1}^{R} \hat{u}_{n_l}^{i_l}(\xi_l)(t) \Big) \Big| \le \#\mathfrak{S} (C(a,\theta)\varepsilon(L))^{\sum n_l} (C'(a)\varepsilon(L))^R.$$

We have that

$$\#\mathfrak{S} = \binom{\#S}{\#S/2} (\#S/2)! \le 2^{\#S} (\#S/2)! \le 2^R R! (2\#S)^{\frac{1}{2}\sum n_l}.$$

Since $\#S \le (M+1)R$, we get

$$\Big| \mathbb{E}\Big(\prod_{l=1}^{R} \hat{u}_{n_l}^{i_l}(\xi_l)(t) \Big) \Big| \lesssim_R (C(a,\theta)\sqrt{2R(M+1)}\varepsilon(L))^{\sum n_l} (C'a)\varepsilon(L))^R.$$

For L big enough, $C(a,\theta)\alpha R(M(L)+1)\varepsilon(L) \le \frac{1}{2}$. We deduce that for L big enough

$$II \lesssim_{a,\theta,R,\varepsilon} \varepsilon(L)^R \sum_{n_l} 2^{-\sum n_l - R} = 1.$$

Hence the result. □

lem:removeEL **Lemma 3.12.** *We have*

$$\mathbb{E}\Big((1-1_{\mathcal{E}_L})\prod_{l=1}^{R}\hat{\mathcal{U}_M}^{(i_l)}(\xi_l)\Big)=O_{d,\varphi,\rho_0,\beta,\theta,R,a,\varepsilon}(\varepsilon(L)^R L^{-d/2}).$$

Proof. We simply use Cauchy-Schwarz inequality to get

$$\mathbb{E}\Big((1-1_{\mathcal{E}_L})\prod_{l=1}^{R}\hat{\mathcal{U}}_M^{(i_l)}(\xi_l)\Big)^2\leq\mathbb{E}(1-1_{\mathcal{E}_L})\mathbb{E}\Big(\prod_{l=1}^{2R}\hat{\mathcal{U}}_M^{(i_l)}(\xi_l)\Big)$$

with $\xi_{R+l}=\xi_l$. We use Lemma 3.11 with R replaced by $2R$ to get

$$\mathbb{E}\Big(\prod_{l=1}^{2R}\hat{\mathcal{U}}_M(\xi_l)\Big)=O_{d,\varphi,\rho_0,\beta,\theta,R,a,\delta,\varepsilon}(1).$$

We also have that

$$\mathbb{E}(1-1_{\mathcal{E}_L})=\mathbb{P}(\mathcal{E}_L^c)\leq e^{-cA^2/\varepsilon(L)^2}.$$

Since

$$\varepsilon(L)^{-2}\geq\frac{\ln L}{c(R)^2},$$

we get that

$$\sqrt{\mathbb{E}(1-1_{\mathcal{E}_L})}\leq L^{-cA^2/2c(a,\theta,R)^2}$$

which is a $O(\varepsilon(L)^R L^{-d/2})$ for $c(a,\theta,R)$ small enough and concludes the proof. □

lem:explicitComp **Lemma 3.13.** *We have*

$$\mathbb{E}\Big(\prod_{l=1}^{R}\hat{\mathcal{U}}_M^{(i_l)}(\xi_l)(t)\Big)=\sum_{O\in\mathcal{P}_R}\prod_{\{l,l'\}\in O}\mathbb{E}(\hat{\mathcal{U}}_M(t)(\xi_l)^{(i_l)}\hat{\mathcal{U}}_M^{(i_{l'})}(t)(\xi_{l'}))+O_{d,\varphi,\rho_0,\beta,\theta,R,a,\varepsilon}(\varepsilon(L)^R L^{-d/2})$$

where $\mathcal{P}_R$ is the set of partitions of $[|1,R|]$ whose elements are pairs of $[|1,R|]$.

Proof. We write that

$$\mathbb{E}\Big(\prod_{l=1}^{R}\hat{\mathcal{U}}_M^{(i_l)}(\xi_l)(t)\Big)=\sum_{n_l\leq M}\mathbb{E}\Big(\prod_{l=1}^{R}\hat{u}_{n_l}^{(i_l)}(\xi_l)(t)\Big).$$

We use the result (2.15) to get that for L big enough

$$\Big|\mathbb{E}\Big(\prod_{l=1}^{R}\hat{u}_{n_l}^{(i_l)}(\xi_l)(t)\Big)-\sum_{O\in\mathcal{P}_R}\prod_{\{l,l'\}\in O}\mathbb{E}(\hat{u}_{n_l}^{(i_l)}(t)(\xi_l)\hat{u}_{n_{l'}}^{(i_{l'})}(t)(\xi_{l'}))\Big|\lesssim_R L^{-d/2}2^{-\sum n_l}\varepsilon(L)^R.$$

We sum back on n_l to conclude. □

We now use Lemmas 3.10, 3.11, and 3.12 with $R=2$ to get that

$$\mathbb{E}(\hat{\mathcal{U}}_M^{(i_l)}(t)(\xi_l)\hat{\mathcal{U}}_M^{(i_{l'})}(t)(\xi_{l'}))=\mathbb{E}(\mathbf{1}_{\mathcal{E}_L}\hat{u}(t)^{(i_l)}(\xi_l)\hat{u}^{(i_{l'})}(t)(\xi_{l'}))+O_{d,\varphi,\rho_0,\beta,\theta,R,a,\varepsilon}(\varepsilon(L)^2L^{-d/2})$$

and conclude.

A Estimates on the norm of the initial datum

app:sizeofaL

For this appendix, we set

$$a_L(x) = \sum_{k\in\mathbb{Z}^d_*} \frac{e^{ikx/L}}{(2\pi L)^{d/2}} a(k/L) g_k$$

where a is even. For $\xi \in ([-1,1]\setminus\{0\})^d$, we assume that $a(\xi)$ lies in the orthogonal of $\{\xi\}$ and is of norm 1. Outside, we assume that $a = 0$.

prop:estDI

We have seen in Proposition 3.9 that for R big enough

$$\mathbb{P}(\|a_L\|_{\rho_0} \geq R\sqrt{\ln L}) \leq e^{-cR^2 \ln L}.$$

We now want to check that for δ small enough if $a \neq 0$,

$$\mathbb{P}(\|a_L\|_{\rho_0} > \delta\sqrt{\ln L}) \geq \frac{1}{2}$$

which would tell us that $\sqrt{\ln L}$ is the typical size of $\|a_L\|_{\rho_0}$. We note that this typical size is more due to the number of independent Gaussian variables we sum that to the size of the box.

We have that

$$\|a_L\|_{L^\infty} = \|\sum_n \phi_n * a_L\|_{L^\infty} \leq \sum_n \|\phi_n * a_L\|_{L^\infty} \leq \|a_L\|_{\rho_0}$$

and thus

$$\mathbb{P}(\|a_L\|_{\rho_0} \geq \delta\sqrt{\ln L}) \geq \mathbb{P}(\|a_L\|_{L^\infty} \geq \delta\sqrt{\ln L}).$$

We fix for $n \in [|1, L|]^d$, $x_n = 2n\pi$. We have

$$\mathbb{P}(\|a_L\|_{\rho_0} \geq \delta\sqrt{\ln L}) \geq \mathbb{P}(\exists n,\ |a_L(x_n)| \geq \delta\sqrt{\ln L}).$$

We have that

$$\mathbb{P}(\exists n,\ |a_L(x_n)| \geq \delta\sqrt{\ln L}) = 1 - \mathbb{P}(\forall n,\ |a_L(x_n)| < \delta\sqrt{\ln L}).$$

We have

$$\mathbb{E}(\langle a_L(x_n), a_L(x_m)\rangle_{\mathbb{C}^d}) = \frac{1}{(2\pi L)^d} \sum_{k\in([|-L,L|]\setminus\{0\})^d} e^{ik(x_n - x_m)/L} = \frac{1}{\pi^d}\delta_{n,m}.$$

Therefore,

$$\mathbb{P}(\forall n,\ |a_L(x_n)| \leq \delta\sqrt{\ln L}) = \mathbb{P}(|a_L(x_0)| < \delta\sqrt{\ln L})^{L^d}.$$

We have that

$$\mathbb{P}(|a_L(x_0)| < \delta\sqrt{\ln L}) = 1 - \mathbb{P}(|a_L(x_0)| \geq \delta\sqrt{\ln L}).$$

The random variable $a_L(x_0)$ is a real Gaussian variable with variance π^{-d} we deduce that for L big enough,

$$\mathbb{P}(|a_L(x_0)| \geq \delta\sqrt{\ln L}) \geq e^{-2\pi^{-d}\delta^2 \ln L} = L^{-\tilde{\delta}}$$

with $\tilde{\delta} = 2\pi^{-d}\delta^d$.

We deduce that

$$\mathbb{P}(|a_L(x_0)| < \delta\sqrt{\ln L}) \leq (1 - L^{-\tilde{\delta}})^{L^d}.$$

If $\tilde{\delta} < d$, we have that

$$(1 - L^{-\tilde{\delta}})^{L^d} \to 0$$

as $L \to \infty$, and thus for L big enough

$$\mathbb{P}(\|a_L\|_{\rho_0} \geq \delta\sqrt{\ln L}) \geq \frac{1}{2}.$$

app:Polish

B Polish notations

Definition B.1. Let $n \in \mathbb{N}$. Let $\bar{\mathcal{A}}_n$ be the subset of $\{0,1\}^{nN+1}$ such that if $(a_1, \dots, a_{nN+1}) \in \bar{\mathcal{A}}_n$ we have that for all $k < nN+1$,

$$k < (\sum_{j \leq k} a_j)N + 1$$

and such that $\sum_{j=1}^{nN+1} a_j = n$.

Remark B.1. *The second condition means that there are exactly n ones in the sequence. It can be rewritten as*

$$nN + 1 = (\sum_{j \leq nN+1} a_j)N + 1.$$

Remark B.2. *For $n = 0$, we have*

$$\bar{\mathcal{A}}_0 = \{0\}.$$

For $n = 1$, we have that

$$1 < a_1 N + 1,$$

hence $a_1 = 1$. We get

$$\bar{\mathcal{A}}_1 = \{(1, 0, \dots, 0)\}.$$

Notation B.2. If $M \in \mathbb{N}$, $n_j \in \mathbb{N}$) and $(A_1, \dots, A_M) \in \prod_{j \leq M} \{0,1\}^{n_j}$, we write

$$A_1 \dots A_M = (a_{1,1}, \dots, a_{1,n_1}, a_{1,1}, \dots, a_{2,n_2}, \dots, a_{M,1}, \dots, a_{M,n_M})$$

the contatenation of A_1 up to A_M with

$$A_j = (a_{j,1}, \dots, a_{j,n_j})$$

for all j.

Proposition B.3. *Let $n \in \mathbb{N}$. Let $A \in \bar{\mathcal{A}}_{n+1}$. There exists $(n_1, \dots, n_N) \in \mathbb{N}^N$ and $(A_1, \dots, A_N) \in \prod_j \bar{\mathcal{A}}_j$ such that*

$$n = \sum_{j=1}^{N} n_j, \quad A = 1A_1 \dots A_N;$$

this decomposition is unique.

Conversely, if $A \in \{0,1\}^{nN+1}$ is equal to

$$1A_1 \dots A_N$$

with $A_j \in \bar{\mathcal{A}}_{n_j}$ for all j and such that $\sum n_j = n$ then

$$A \in \bar{\mathcal{A}}_{n+1}.$$

Proof. Let $n \in \mathbb{N}$ and $A \in \bar{\mathcal{A}}_{n+1}$. We write

$$A = (a_1, \dots, a_{(n+1)N+1}), \quad \forall k \leq nN+1,\ b_k = \sum_{j \leq k} a_j.$$

Since

$$1 < a_1 N + 1,$$

we deduce that $a_1 = 1$. We set for $m = 1, \dots, N$,

$$k_m = \min(k | k \geq (b_k - 1)N + m + 1).$$

We prove that $(k_m)_m$ is well-defined, strictly increasing and that for all m, $k_m = (b_{k_m} - 1)N + m + 1$.

First of all, we have

$$(n+1)N + 1 = b_{nN+1}N + 1 = (b_{nN+1} - 1)N + N + 1 \geq (b_{nN+1} - 1)N + m + 1.$$

Hence k_m is well-defined.

Set $c_{k,m} = (b_k - 1)N + m + 1 - k$. We have that

$$c_{k+1,m} - c_{k,m} = (b_{k+1} - b_k)N - 1$$

and because (b_k) is increasing, this ensures that

$$c_{k+1,m} - c_{k,m} \leq -1.$$

We deduce that we cannot pass from a (strictly) positive $c_{k,m}$ to a (strictly) negative $c_{k+1,m}$.

We have $c_{1,1} = 1 > 0$ thus $k_1 > 1$, $c_{1,k_1} = 0$ and thus $c_{2,k_1} > 0$. By induction, we get that (k_m) is strictly increasing and that for all m, $c_{k_m,m} = 0$.

What is more,

$$k_N = \min(k | k \geq b_k N + 1) = (n+1)N + 1$$

by definition of $\bar{\mathcal{A}}_{n+1}$.

We set $\tilde{n}_m = k_m - k_{m-1}$ with the convention $k_0 = 1$. We write

$$A = (1, a_{1,1}, \dots, a_{1,\tilde{n}_1}, \dots, a_{N,1}, \dots, a_{N,\tilde{n}_N})$$

and

$$A_m = (a_{m,1}, \dots, a_{m,\tilde{n}_j}).$$

We have $A = 1A_1 \dots A_N$.

We also have that $\tilde{n}_m = (b_{k_m} - b_{k_{m-1}})N + 1$ hence $\tilde{n}_m = n_m N + 1$ with $n_m = b_{k_m} - b_{k_{m-1}}$. We prove that

$$A_m \in \bar{\mathcal{A}}_{n_m}.$$

We have $a_{m,k} = a_{k+k_{m-1}}$ and $b_{m,k} = \sum_{j \leq k} a_{m,j} = b_k - b_{k_{m-1}}$.

We use the definition of the sequence $(k_m)_m$ to that for $k < n_m N + 1$,

$$k + k_{m-1} < (b_{k+k_{m-1}} - 1)N + m + 1.$$

Using that $k_{m-1} = (b_{k_{m-1}} - 1)N + m$, we get

$$k < (b_{k+k_{m-1}} - b_{k_{m-1}})N + 1 = b_{m,k}N + 1$$

and since $b_{m,n_m N+1} = n_m$ by definition, we have $A_m \in \bar{\mathcal{A}}_m$.

The construction of the decomposition ensures its uniqueness.

Conversely, if $A = 1A_1 \dots A_N$ with $A_m \in \bar{\mathcal{A}}_{n_m}$ and $\sum n_m = n$ then $A \in \bar{\mathcal{A}}_{n+1}$. Indeed, for $k = 1$, we have

$$1 = k < b_1 N + 1 = N + 1.$$

and for $k \in [k_{m-1}+1, k_m]$ with $k_0 = 1$, $k_m = \sum_{l\le m} n_l N + m + 1$, we have

$$b_k = \sum_{l<m} n_l + 1 + b_{m,k-k_{m-1}}.$$

We deduce

$$b_k N + 1 = \sum_{l<m} n_l N + N b_{m,k-k_{m-1}} + N + 1 = k_{m-1} - (m-1) + N b_{m,k-k_{m-1}} + N.$$

If $m < N$, then we use that $N b_{m,k-k_{m-1}} \ge k - k_{m-1} - 1$, and get

$$b_k N + 1 \ge N - m + k > k.$$

If $m = N$ then we have

$$b_k N + 1 = k_{N-1} + N b_{N,k-k_{N-1}} + 1.$$

If $k < (n+1)N + 1$ then $k - k_{N-1} < (n+1)N + 1 - N - \sum_{l<N} n_l N = n_N N + 1$, and thus

$$b_k N + 1 > k_{N-1} + k - k_{N-1} = k.$$

If $k = (n+1)N + 1$ then $k - k_{N-1} = n_N N + 1$ and thus

$$b_k N + 1 = k.$$

which concludes the proof. □

We deduce that $\bar{\mathcal{A}}_n$ is isomorphic to $\mathcal{A}_n$ and in particular they have the same cardinal.

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