# Analysing the scattering of electromagnetic ultra wide band pulses from large scale objects by the use of wavelets

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June 21, 2022

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**Abstract**

The aim of this paper is the study of the signal received by an antenna (RX) when a transmit antenna (TX) sends a short pulse in a large scale space, for instance, in an urban environment. Integral equations are established which link the densities of charges and currents inside the environment objects with the incident field created by the TX antenna. From this equations we define an integral operator $K$. The densities can be obtained inverting $1 - K$ . The introduction of Daubechies wavelets allows us to obtain sparse matrices for $KK^*$ which is computationally convenient to get $(1 - K)^{-1}$.

Keywords:
Electromagnetic scattering, UWB pulses, large scale, Daubechies wavelets, wireless telephony

Alex Grossmann started his career with two papers with Tai Tsun Wu [10] on "The Schrôdinger scattering amplitude for a fixed potential". The forecoming paper is an homage to his deepness and his clarity in a field which is still alive.

## 1 Introduction

With the development of wireless telephony there is a need to better understand the propagation of signals in an urban environment with its buildings, its inhabitants, its cars, its furniture... It becomes important to characterize the electrical field received by an user walking in the street when the antenna of the base station, situated on a roof or on a house front, transmits signals.

Previously the efforts were concentrated in the propagation of harmonic signals or narrow band signals. In this case, the signal at the user antenna, which, in many cases, is considered as pointwise, is determined once we get the complex amplitude of the electrical field at this point. To calculate the electric field scattered by the urban environment there are mainly two ways . The first one consists in using the geometrical optics approximation. Researchers and engineers introduce the reflections, transmission and diffraction of rays. Today, in the range of 0.3 GHz to 6 GHz, despite the great number of models and softwares implementing this technique, the calculations suffer many discrepancies from measurements. The problem often comes from the difficulty to take into account the irregularities of the house front, windows, balconies... The second way is to use Lippmann Schwinger type equations [3], [8]. As the wavelength is much smaller than the geometrical dimensions of the buildings, the matrix which has to be inverted numerically is very large. Even if new techniques are available for the inversion of large matrices [11], [12], they have

not been applied to the urban problem. The numerical results concern only smaller bodies and simple geometries.

Even if the harmonic problem was already difficult, the interest of the wireless community has been moving towards the propagation of ultra wide band (UWB) signals i.e short pulses in the nanosecond range, for which the study, of course, is more challenging [16], [17]. Apple launched the first three phones with ultra-wideband capabilities in september 2019. Once again, the technique which is commonly used is the geometrical optics approximation which obviously infers an imprecise description of the received signal at the user antenna, giving only some indications on the time arrival and the amplitude of the signal, showing some picks corresponding to the main reflections. Comparison with measured signals is not good, the latter are much more complex and they present an important tail, very badly described by the models deduced from ray tracing.

At our knowledge, in the UWB situation, analytic techniques starting from the Maxwell equations have not yet been considered as probably the idea is that they should lead to the inversion of such a large matrix that it is infeasible at present time.

The aim of this paper is to show that it exists a way which could help to solve the problem partly analytically and partly numerically with a control of the different approximations.

Section 2 is devoted to establish the definition of the $K$ operator (the equivalent of the Lippmann Schwinger operator in scattering theory ) acting on the Hilbert space $L^2(\mathbb{R}^4)\otimes\mathbb{C}^4$ of four functions $d_i(\mathbf{x},t), i=0,1,2,3$. $d_0(\mathbf{x},t)$ represents the density of charges at different times, inside the materials and $d_i(\mathbf{x},t), i=1,2,3$ represent the three components of the density of currents. These densities satisfy a Lippmann Schwinger type equation, $(1-K)d=d_{in}$. This equation links the densities $d$ to some functions $d_{in}$, called "incident densities", which depend on the incident electrical field alone i.e the field generated in free space by the TX antenna. The way to establish this equation seems to be new and relatively simple compared with the way Lippmann Schwinger type equation was obtained in electromagnetism for the harmonic signals, [8]. Notice that the usual Lippmann Schwinger (LS) type equations include the electric and magnetic field in full space, instead our equations involve functions which are defined only inside the materials.

Let us notice that once charge density and current density inside the materials are obtained, one can easily get the scattered electric field and adding to it the incident field one gets the total electric field anywhere in space and time.

To solve numerically the Lippmann Schwinger type equation, one could choose a subspace of $L^2(\mathbb{R}^4)\otimes\mathbb{C}^4$ of finite dimension, generated by a finite number of well chosen wavelets and write the LS type equation in this space only. This is problematic because the corresponding matrix, even if it is sparse, is not hermitian and contains large elements. The control of the stability of the solution as we enlarge the subspace is difficult. Does the densities converge when new basis vectors are added ?

The aim of this paper is to show that it is possible to tackle the problem differently. Multiplying the two terms of the LS type equation by $1-K^*$ we get $(1-K^*)(1-K)d=(1-K^*)d_{in}$. So to get $d$, we have to solve this new system or to invert $(1-K^*)(1-K)$. Notice that $(1-K^*)(1-K)$ is an hermitian operator which makes easier the problem of the stability of solutions. To invert $(1-K^*)(1-K)$ we write it as the sum of a diagonal matrix, $D$ and an off diagonal matrix, $A$, $(1-K^*)(1-K)=D+A=D^{1/2}(1+D^{-1/2}AD^{-1/2})D^{1/2}$. So $[(1-K^*)(1-K)]^{-1}=D^{-1/2}(1+D^{-1/2}AD^{-1/2})^{-1}D^{-1/2}$. We are going to prove that $D^{-1/2}AD^{-1/2}$ is sparse and its elements are small.

In section 3 we build a subspace of $L^2(\mathbb{R}^4)\otimes\mathbb{C}^4$ using the orthonormal basis of Daubechies$-p$ wavelets in the Hilbert space $L^2(\mathbb{R})$. The first $p$ moments of the Daubechies$-p$ wavelet are zero and their Fourier transform is essentially concentrated in two intervals in which its absolute value part looks like a smoothed characteristic function, (see Daubechies wavelets in Wikipedia [18] and Wolfram Mathworld [20]). From the mother wavelet $w$ whose support is compact, equal to $[-p+1,p]$, is built the spatial wavelet $w_{\mathbf{mn}}(\mathbf{x})=2^{m_1/2}2^{m_2/2}2^{m_3/2}w(2^{m_1}(x_1-2^{-m_1}n_1))w(2^{m_2}(x_2-2^{-m_2}n_2))w(2^{m_3}(x_3-2^{-m_3}n_3))$ where $\mathbf{m}$ denotes $(m_1,m_2,m_3)\in\mathbb{Z}^3$ and $\mathbf{n}$ denotes $(n_1,n_2,n_3)\in\mathbb{Z}^3$. Also is built the temporal wavelet $w_{jk}(t)=2^{j/2}w(2^j(t-2^{-j}k))$. Their product constitute a basis for $L^2(\mathbb{R}^4)$. We denote $\mathbf{x_{mn}}:=(2^{-m_1}n_1,2^{-m_2}n_2,2^{-m_3}n_3)$ , $\mathbf{x_{m'n'}}:=(2^{-m'_1}n'_1,2^{-m'_2}n'_2,2^{-m'_3}n'_3)$,

$t_{jk} := 2^{-j}k$ and $t_{j'k'} := 2^{-j'}k'$. One defines a subspace of $L^2(\mathbb{R}^4) \otimes \mathbb{C}^4$ as the linear span of a set of wavelets corresponding to a finite set of $\mathbf{m}$, $\mathbf{n}$, $j$ and $k$.

In sections 3 and 4 we will show that the matrix corresponding to $(1 - K^*)(1 - K)$ in this subspace is sparse, contrarily to what could be expected since multiplying two sparse matrices give us generally a non sparse matrix. This result is obtained through the use of spheroidal coordinates. The study of the regularities of the kernel of $K^*K$ allow us to prove the smallness of many matrix elements of $K^*K$.

In the figure below, using the results established in Theorems 3.1 and 3.4 for the matrix elements $\left(w_{\mathbf{m'n'}}w_{j'k'}, K^*Kw_{\mathbf{mn}}w_{jk}\right)$, we present, in the case the material volume is a parallelepiped,$\mathcal{P}$, three artist views of the amplitude of some matrix elements as $\mathbf{m}$, $\mathbf{n}$, $j$ and $k$ are fixed (so we are looking only to the elements of the column, $K^*Kw_{\mathbf{mn}}w_{jk}$ as functions of $\mathbf{x_{m'n'}} \in \mathcal{P}$ for three different $t_{j'k'}$. The coordinates of the center of the yellow spot in the upper figure are $\mathbf{x_{mn}}$. Notice that they are also the center coordinates of the spherical crowns in the middle and lower figures . The upper figure corresponds to the values for $\left(w_{\mathbf{m'n'}}w_{j'k'}, K^*Kw_{\mathbf{mn}}w_{jk}\right)$ as a function of $\mathbf{x_{m'n'}}$ when $t_{j'k'} - t_{jk}$ is fixed and close to zero. The figure in the middle corresponds to the values for $\left(w_{\mathbf{m'n'}}w_{j'k'}, K^*Kw_{\mathbf{mn}}w_{jk}\right)$ for a larger value of $t_{j'k'} - t_{jk}$ positive, the mean radius of the dark red crown is $c(t_{j'k'} - t_{jk})$ ( $c$ is the light velocity). The lower figure corresponds to the values for $\left(w_{\mathbf{m'n'}}w_{j'k'}, K^*Kw_{\mathbf{mn}}w_{jk}\right)$ foe an even greater value of $t_{j'k'} - t_{jk}$ , again the mean radius of the dark red crown is $c(t_{j'k'} - t_{jk})$.

Yellow color represents strongest amplitude, black color represents zero amplitude, while a gradation of black and red is associated with values, more or less large depending on the proportion of red.

Notice that if the time difference $t_{j'k'} - t_{jk}$ becomes sufficiently large the black region covers all the parallelepiped. So the time difference for which all the matrix elements become equal to zero can be easily estimated in terms of the initial position $\mathbf{x_{mn}}$ and the geometry. The presence of red scars results from theorem 4.3. They disappear in the lower figure.

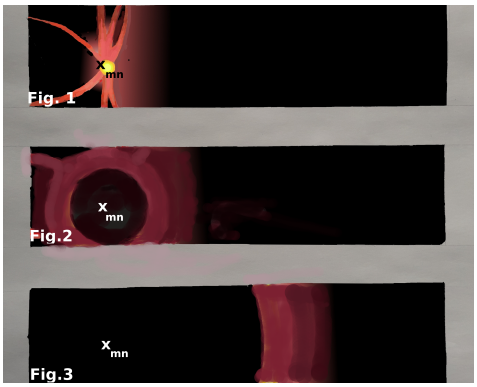
130622.pdf 


Figure 1: A representation of the modulus of the matrix element for $\left(w_{\mathbf{m'n'}}w_{j'k'}, K^*Kw_{\mathbf{mn}}w_{jk}\right)$ for three different couples $(j', k')$ corresponding to increasing times $t_{j'k'}$.

In section 5 we evaluate the proportion of matrix elements of $K^*K$ which are non negligible.

In section 6 we discuss the choice of the Daubechies wavelets order, in conjunction with the irregularities of the permittivity of the materials and their geometry. We will also discuss briefly the stability of the obtained densities as the number of elements of wavelets is increased and give some perspectives.

*Remark.* The reason why we will call $K^*K$ and $KK^*$ time reversal operators is because their sparsity has something to do with the time reversal technique . In the time reversal technique [6]

a pulse $s_i(t)$ is sent by the $i^{th}$ TX antenna. This signal propagates in the environment and it is registered by some RX antennas disposed around the materials. At the $j^{th}$ RX antenna, the received signal $r_{ji}(t) = (G_{ji}s_i)(t)$ is memorized and time reversed, that is transformed in $f_{ji}(t) = r_{ji}(2\tau - t)$. After, each $j^{th}$ RX antenna acting now as an emitter sends its reversed signal. Then the $i'^{th}$ antenna, which was previously a TX antenna, receives now a signal $r_{i'i}(t) = \sum_j (G_{i'j}f_{ji})(t)$.

What can be observed is that the $i^{th}$ TX antenna receives now a pulse $r_{ii}(t)$ which is concentrated in time and more important than the signals $r_{i'i}(t)$ received by the others TX antennas.

If one chooses $\tau$ as the time origin, the time reversal transform can be express in terms of the operator $G^*$ , and the signal received at the TX antennas calculated from $GG^*$ . The experiment indicates that in $G^*G$ the diagonal elements are greater than the non diagonal ones .

The analogy between $K$ and $G$ gave us the idea of the sparsity of $K^*K$.

In Appendix 7 in order to prefigure the calculation of the densities in the buildings of a street, we calculate the kernel in the case the volume is a set of parallelepipeds into each of which the permittivity is constant.

In appendix 8 we study the matrix elements of $K + K^*$ and finally prove that the non negligible matrix elements for $(1 + K^*)(1 + K)$ are the ones on the diagonal and its vicinity.

Finally in Appendix 9 we indicate a way to calculate the diagonal elements without the use of the spheroidal coordinates , using the Fourier transform of the wavelets.

# 2 Direct and scattered fields

Once the coordinate system has been chosen conveniently with respect to the city, ( for instance the 0z axis can be chosen vertical , the 0y axis parallel to the street axis), the TX antenna is at point A, whose coordinates are given by $\mathbf{a} \in R^3$. We choose some axis $Ax', Ay', Az'$ linked with the antenna geometry and denote by $r', \theta', \phi'$ the spherical coordinates of some point $\mathbf{u}$ with respect to the $Ax', Ay', Az'$ axis. We suppose that in free space, the electrical field at point $\mathbf{u}$, generated by a given voltage excitation, $v(t)$, is given in the far field approximation by

$$\mathbf{E}_{in}(\mathbf{u}, t) = \frac{1}{|\mathbf{u} - \mathbf{a}|}\mathbf{f}(\phi', \theta')\frac{dv}{dt}(t - |\mathbf{u} - \mathbf{a}|/c) \tag{2.1}$$

$\mathbf{f}(\phi', \theta')$ is a vector whose components $f_{r'}(\theta', \phi')$, $f_{\theta'}(\theta', \phi')$, $f_{\phi'}(\theta', \phi')$ have the dimension of a time. In the far field approximation (see [1]) $\mathbf{f}(\phi', \theta')$ is transversal i.e $f_{r'}(\theta', \phi') = 0$

We are going to choose an ultra wide band input voltage $v(t)$ which means that its Fourier transform has a support whose width is large, to fix the ideas, if the central frequency is $f_0 = 2\pi\omega_0$, the bandwidth is $0.1f_0$. Today in wireless telephony $0.8\ GHz < f_0 < 6\ GHz$ but higher frequencies will appear soon.

The environment materials respond to the incident field $\mathbf{E}_{in}(\mathbf{u}, t)$, charges and currents are created inside them. In turn, these charges and currents create everywhere an electrical field ( called the scattered field ) and summing it with the incident field, total field is obtained inside or outside the materials.

We are going to show that charge and current densities satisfy integral equations of Lippmann-Schwinger type.

Let us come back to the electromagnetic theory. The total field polarizes the materials. Polarization means that under the effect of the total electric field, the bound electrons of atoms move, the charge density inside the materials is modified (some "dipoles" are created) and currents are generated. Introducing the polarisation vector $\mathbf{P} = \epsilon_0\chi(\mathbf{E})$, the charge density inside the material becomes $\rho_p(\mathbf{u}, t) = -\nabla.\mathbf{P}(\mathbf{u}, t)$ and the current density is given by $\mathbf{J}_p(\mathbf{u}, t) = \frac{\partial}{\partial t}\mathbf{P}(\mathbf{u}, t)$. In the literature [9], [19], the relation between the total electric field and the polarisation vector has been discussed and many simplifications have been introduced.

To simplify our forthcoming analysis we will admit that in our situation, one can write that, approximatively, $\mathbf{P}(\mathbf{u}, t) = \epsilon_0\chi(\mathbf{u})\mathbf{E}(\mathbf{u}, t)$, then

$$\rho_p(\mathbf{u},t) = -\epsilon_0 \nabla.(\chi(\mathbf{u})\mathbf{E}(\mathbf{u},t)) = -\epsilon_0 \nabla\chi(\mathbf{u}).\mathbf{E}(\mathbf{u},t) - \epsilon_0\chi(\mathbf{u})\nabla.\mathbf{E}(\mathbf{u},t) \tag{2.2}$$

As $\nabla.\mathbf{E}(\mathbf{u},t) = \rho_p(\mathbf{u},t)/\epsilon_0$

$$\rho_p(\mathbf{u},t) = -\frac{\epsilon_0}{1+\chi(\mathbf{u})}\nabla\chi(\mathbf{u}).\mathbf{E}(\mathbf{u},t) \tag{2.3}$$

and

$$\mathbf{J}_p(\mathbf{u},t) = \epsilon_0\chi(\mathbf{u})\frac{\partial}{\partial t}\mathbf{E}(\mathbf{u},t). \tag{2.4}$$

If there are free electrons inside the materials (which is the case in metals) they can move under the effect of the electrical field and some current density is created. We will admit the linearity between the current density and the field i.e the Ohm law,

$$\mathbf{J}_f(\mathbf{u},t) = \sigma^e(\mathbf{u})\mathbf{E}(\mathbf{u},t) \tag{2.5}$$

,

where $\sigma^e(\mathbf{u})$ is the conductivity at point $\mathbf{u}$.

Suppose that inside the materials, the volume of which is denoted $V$, charge density is given by $\rho^{(1)}(\mathbf{u_1},t)$ and the current density by $\mathbf{J}^{(1)}(\mathbf{u}_1,t)$. In the Lorenz gauge the scalar potential and the vector potential created at some point $\mathbf{u}_2$, at time $t$, respectively by the charge density and the current density in the materials, are given by the so called retarded potential formulas :

$$\begin{aligned}\phi^{(1)}(\mathbf{u}_2,t) &= \frac{1}{4\pi\epsilon_0}\int_V d\mathbf{u}_1 \frac{\rho^{(1)}(\mathbf{u}_1, t-|\mathbf{u}_2-\mathbf{u}_1|/c)}{|\mathbf{u}_2-\mathbf{u}_1|}\\ \mathbf{A}^{(1)}(\mathbf{u}_2,t) &= \frac{\mu_0}{4\pi}\int_V d\mathbf{u}_1 \frac{\mathbf{J}^{(1)}(\mathbf{u}_1, t-|\mathbf{u}_2-\mathbf{u}_1|/c)}{|\mathbf{u}_2-\mathbf{u}_1|}\end{aligned} \tag{2.6}$$

Now let us recall that the scattered electric field is related to the potentials by: $\mathbf{E}_s = -\nabla\phi - \frac{\partial}{\partial t}\mathbf{A}$, then

$$\mathbf{E}_s^{(1)}(\mathbf{u}_2,t) = -\frac{1}{4\pi\epsilon_0}\nabla\int_V d\mathbf{u}_1 \frac{\rho^{(1)}(\mathbf{u}_1, t-|\mathbf{u}_2-\mathbf{u}_1|/c)}{|\mathbf{u}_2-\mathbf{u}_1|} - \frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\int_V d\mathbf{u}_1 \frac{\mathbf{J}^{(1)}(\mathbf{u}_1, t-|\mathbf{u}_2-\mathbf{u}_1|/c)}{|\mathbf{u}_2-\mathbf{u}_1|} \tag{2.7}$$

The charges density induced by the total field, $\mathbf{E}_t(\mathbf{u}_2,t) = \mathbf{E}_{in}(\mathbf{u}_2,t) + \mathbf{E}_s^{(1)}(\mathbf{u}_2,t)$, using (2.3) is

$$\begin{aligned}\rho^{(2)}(\mathbf{u}_2,t) &= -\frac{\epsilon_0}{1+\hat{\chi}(\mathbf{u}_2)}\nabla\chi(\mathbf{u}_2).\mathbf{E}_t(\mathbf{u}_2,t)\\ &= -\frac{\epsilon_0}{1+\hat{\chi}(\mathbf{u}_2)}\nabla\chi(\mathbf{u}_2).\mathbf{E}_{in}(\mathbf{u}_2,t)\\ &+\frac{1}{4\pi(1+\chi(\mathbf{u}_2))}\nabla\chi(\mathbf{u}_2).\nabla\int_V d\mathbf{u}_1\frac{1}{|\mathbf{u}_2-\mathbf{u}_1|}\rho^{(1)}(\mathbf{u}_1,t-|\mathbf{u}_2-\mathbf{u}_1|/c)\\ &+\frac{\epsilon_0\mu_0}{4\pi(1+\chi(\mathbf{u}_2))}\nabla\chi(\mathbf{u}_2).\frac{\partial}{\partial t}\int_V d\mathbf{u}_1\frac{1}{|\mathbf{u}_2-\mathbf{u}_1|}\mathbf{J}^{(1)}(\mathbf{u}_1,t-|\mathbf{u}_2-\mathbf{u}_1|/c)\end{aligned} \tag{2.8}$$

while using (2.4) and (2.5)

$$
\begin{aligned}
\mathbf{J}^{(2)}(\mathbf{u}_2,t) &= \epsilon_0\chi(\mathbf{u}_2)\frac{\partial \mathbf{E}_t}{\partial t}(\mathbf{u}_2,t) + \sigma^e(\mathbf{u}_2)\mathbf{E}_t(\mathbf{u}_2,t)\\
&= \epsilon_0\chi(\mathbf{u}_2)\frac{\partial \mathbf{E}_{in}}{\partial t}(\mathbf{u}_2,t) + \sigma^e(\mathbf{u}_2)\mathbf{E}_{in}(\mathbf{u}_2,t)\\
-\chi(\mathbf{u}_2)\frac{1}{4\pi}\nabla\int_V d\mathbf{u}_1\frac{1}{|\mathbf{u}_2-\mathbf{u}_1|}\frac{\partial\rho^{(1)}}{\partial t}(\mathbf{u}_1,t-|\mathbf{u}_2-\mathbf{u}_1|/c)\\
-\chi(\mathbf{u}_2)\frac{\epsilon_0\mu_0}{4\pi}\int_V d\mathbf{u}_1\frac{1}{|\mathbf{u}_2-\mathbf{u}_1|}\frac{\partial^2\mathbf{J}^{(1)}}{\partial t^2}(\mathbf{u}_1,t-|\mathbf{u}_2-\mathbf{u}_1|/c)\\
-\frac{\sigma^e(\mathbf{u}_2)}{4\pi\epsilon_0}\nabla\int_V d\mathbf{u}_1\frac{1}{|\mathbf{u}_2-\mathbf{u}_1|}\rho^{(1)}(\mathbf{u}_1,t-|\mathbf{u}_2-\mathbf{u}_1|/c)\\
-\frac{\mu_0\sigma^e(\mathbf{u}_2)}{4\pi}\int_V d\mathbf{u}_1\frac{1}{|\mathbf{u}_2-\mathbf{u}_1|}\frac{\partial\mathbf{J}^{(1)}}{\partial t}(\mathbf{u}_1,t-|\mathbf{u}_2-\mathbf{u}_1|/c) \qquad (2.9)
\end{aligned}
$$

.

Denoting by $\rho_{in}(\mathbf{u},t) = -\frac{\epsilon_0}{1+\chi(\mathbf{u})}\nabla\chi(\mathbf{u}).\mathbf{E}_{in}(\mathbf{u},t)$ and by $\mathbf{J}_{in}(\mathbf{u},t) = \epsilon_0\chi(\mathbf{u})\frac{\partial}{\partial t}\mathbf{E}_{in}(\mathbf{u},t) + \sigma(\mathbf{u})\mathbf{E}_{in}(\mathbf{u},t)$, we write (2.8) and (2.9) as $(\rho^{(2)},\mathbf{J}^{(2)}) = (\rho_{in},\mathbf{J}_{in}) + K(\rho^{(1)},\mathbf{J}^{(1)})$ where the matrix elements of $K$

$$
K = \begin{pmatrix} K_{00} & K_{01} & K_{02} & K_{03}\\ K_{10} & K_{11} & K_{12} & K_{13}\\ K_{20} & K_{21} & K_{22} & K_{23}\\ K_{30} & K_{31} & K_{32} & K_{33}\end{pmatrix}
$$

are integral operators which, in the case the conductivity is neglected, are given by,

$$
\begin{aligned}
(K_{00}\rho)(\mathbf{u}_2,t) &= \frac{1}{4\pi(1+\chi(\mathbf{u}_2))}\nabla\chi(\mathbf{u}_2).\nabla\int_V d\mathbf{u}_1\frac{1}{|\mathbf{u}_2-\mathbf{u}_1|}\rho(\mathbf{u}_1,t-|\mathbf{u}_2-\mathbf{u}_1|/c)\\
(K_{0i}J_i)(\mathbf{u}_2,t) &= \frac{\epsilon_0\mu_0}{4\pi(1+\chi(\mathbf{u}_2))}\frac{\partial\chi(u_{21},u_{22},u_{23})}{\partial u_{2i}}\frac{\partial}{\partial t}\int_V d\mathbf{u}_1\frac{1}{|\mathbf{u}_2-\mathbf{u}_1|}J_i(\mathbf{u}_1,t-|\mathbf{u}_2-\mathbf{u}_1|/c)\\
(K_{i0}\rho)(\mathbf{u}_2,t) &= -\chi(\mathbf{u}_2)\frac{1}{4\pi}\frac{\partial}{\partial u_{2i}}\int_V d\mathbf{u}_1\frac{1}{|\mathbf{u}_2-\mathbf{u}_1|}\frac{\partial\rho}{\partial t}(\mathbf{u}_1,t-|\mathbf{u}_2-\mathbf{u}_1|/c)\\
(K_{ii}J_i)(\mathbf{u}_2,t) &= -\chi(\mathbf{u}_2)\frac{\epsilon_0\mu_0}{4\pi}\int_V d\mathbf{u}_1\frac{1}{|\mathbf{u}_2-\mathbf{u}_1|}\frac{\partial^2 J_i}{\partial t^2}(\mathbf{u}_1,t-|\mathbf{u}_2-\mathbf{u}_1|/c)\\
K_{12} &= K_{13} = K_{21} = K_{23} = K_{31} = K_{32} = 0\\
i &= 1,2,3 \qquad (2.10)
\end{aligned}
$$

The true charges and currents densities have to be chosen consistently, so they have to satisfy $(\rho,\mathbf{J}) = (\rho_{in},\mathbf{J}_{in}) + K(\rho,\mathbf{J})$. Then we get, $(\rho,\mathbf{J}) = (1-K)^{-1}(\rho_{in},\mathbf{J}_{in})$

One uses (2.1) and (2.3) to get,

$$
\rho_{in}(\mathbf{u},t) = -\frac{\epsilon_0}{1+\chi(\mathbf{u})}\frac{1}{|\mathbf{u}-\mathbf{a}|}\nabla\chi(\mathbf{u}).\mathbf{f}_{in}(\hat{\mathbf{u}})\frac{dv}{dt}(t-|\mathbf{u}-\mathbf{a}|/c) \qquad (2.11)
$$

and (2.1) and (2.4) to get,

$$
\mathbf{J}_{in}(\mathbf{u},t) = \frac{1}{|\mathbf{u}-\mathbf{a}|}\mathbf{f}_{in}(\hat{\mathbf{u}})\Big(\sigma^e(\mathbf{u})\frac{dv}{dt}(t-|\mathbf{u}-\mathbf{a}|/c) + \epsilon_0\chi(\mathbf{u})\frac{d^2v}{dt^2}(t-|\mathbf{u}-\mathbf{a}|/c)\Big) \qquad (2.12)
$$

.

$K$ is an unbounded operator due to the presence of the spatial and the temporal derivatives, so the Neumann series cannot be used to calculate $(1-K)^{-1}$ . It is already clear from the numerical results obtained in small bodies that inside the materials the scattered field presents a shape very different from the shape of the incident field, the wavelength is changed. Instead if we use the Born approximation i.e $(1-K)^{-1} \approx (1+K)$ this would lead to densities whose shape would be similar to the incident density ones.

For the reasons already mentioned in the introduction, we are going to study the self-adjoint operator $(1-K)^*(1-K) = 1 - K^* - K + K^*K$. In particular we are going to study the sparsity of $K^*K$ in the Daubechies wavelet basis.

# 3 Time reversal operator

$K^*K =$

$$\begin{bmatrix} K_{00}^*K_{00} + K_{10}^*K_{10} + K_{20}^*K_{20} + K_{30}^*K_{30} & K_{00}^*K_{01} + K_{10}^*K_{11} & K_{00}^*K_{02} + K_{20}^*K_{22} & K_{00}^*K_{03} + K_{30}^*K_{33} \\ K_{01}^*K_{00} + K_{11}^*K_{10} & K_{01}^*K_{01} + K_{11}^*K_{11} & K_{01}^*K_{02} & K_{01}^*K_{03} \\ K_{02}^*K_{00} + K_{22}^*K_{20} & K_{02}^*K_{01} & K_{02}^*K_{02} + K_{22}^*K_{22} & K_{02}^*K_{03} \\ K_{03}^*K_{00} + K_{33}^*K_{30} & K_{03}^*K_{01} & K_{03}^*K_{02} & K_{03}^*K_{03} + K_{33}^*K_{33} \end{bmatrix}$$

$K^*K$ acts on a quadruplet constituted by the charge density function and the three components of the current density field, which will be expressed on the basis formed by the products of the spatial wavelets and the temporal wavelets. We want to study the operator $K^*K$ in the quadruplet basis

$\Big(w_{\mathbf{m}_0\mathbf{n}_0}(\mathbf{x})w_{j_0k_0}(t), w_{\mathbf{m}_1\mathbf{n}_1}(\mathbf{x})w_{jk}(t), w_{\mathbf{m}_2\mathbf{n}_2}(\mathbf{x})w_{j_2k_2}(t), w_{\mathbf{m}_3\mathbf{n}_3}(\mathbf{x})w_{j_3k_3}(t)\Big)$.

We are not going to study all the elements of $K^*K$ as this would be pretty tedious. To simplify the exposure we limit ourselves to the study of $K_{01}^*K_{01} + K_{11}^*K_{11}$ which acts only on the first component of the current density . To simplify we will also suppose that the conductivity is zero.

We are going to show that many matrix elements $(w_{\mathbf{m}_1'\mathbf{n}_1'}w_{j_1'k_1'}, (K_{01}^*K_{01}+K_{11}^*K_{11})w_{\mathbf{m}_1\mathbf{n}_1}w_{j_1k_1})$ are equal to zero or very small. To simplify the writing we suppress the index 1 in the wavelet notation.

The support of the Daubechies-p wavelet $w_{\mathbf{mn}}(\mathbf{x})$ is

$$\begin{aligned} &(2^{-m_1}n_1 - 2^{-m_1}(p-1), 2^{-m_1}n_1 + 2^{-m_1}p) \\ \times&(2^{-m_2}n_2 - 2^{-m_2}(p-1), 2^{-m_2}n_2 + 2^{-m_2}p) \\ \times&(2^{-m_3}n_3 - 2^{-m_3}(p-1), 2^{-m_3}n_3 + 2^{-m_3}p) \end{aligned} \tag{3.1}$$

.

$d_{\mathbf{m}} := p\sqrt{2^{-2m_1} + 2^{-2m_2} + 2^{-2m_3}}$ is such that for any point $\mathbf{x}$ of the $w_{\mathbf{mn}}(\mathbf{x})$ support , $|\mathbf{x} - \mathbf{x}_{\mathbf{mn}}| \leq d_{\mathbf{m}}$.

Similarly for the temporal wavelet $w_{jk}(t)$ we define $d_j := 2^{-j}p$ is such that for any point $t$ of the $w_{jk}(t)$ support , $|t - t_{jk}| \leq d_j$.

In the following theorem we want to show that some matrix elements of $K_{01}^*K_{01} + K_{11}^*K_{11}$ corresponding to the column $(j, k, \mathbf{m}, \mathbf{n})$ are equal to zero. .

**Theorem 3.1.** *For fixed* $(j, k, \mathbf{m}, \mathbf{n})$,
*if* $|\mathbf{x}_{\mathbf{m}'\mathbf{n}'} - \mathbf{x}_{\mathbf{mn}}| \leq c\big(t_{j'k'} - t_{jk} - d_j - d_{j'}\big) - d_{\mathbf{m}} - d_{\mathbf{m}'}$
*or if* $|\mathbf{x}_{\mathbf{m}'\mathbf{n}'} - \mathbf{x}_{\mathbf{mn}}| \leq c\big(t_{jk} - t_{j'k'} - d_j - d_{j'}\big) - d_{\mathbf{m}} - d_{\mathbf{m}'}$ .
*the matrix element* $\Big(w_{\mathbf{m}'\mathbf{n}'}w_{j'k'}, (K_{01}^*K_{01} + K_{11}^*K_{11})w_{\mathbf{mn}}w_{jk}\Big) = 0$

*Proof.* Let us introduce the Green function for the d'Alembert equation

$$G(\mathbf{x}_1, t_1, \mathbf{x}_2, t_2) = \frac{\delta(t_1 - t_2 - \frac{|\mathbf{x}_1 - \mathbf{x}_1|}{c})}{4\pi|\mathbf{x}_1 - \mathbf{x}_2|} \tag{3.2}$$

.

From (2.10) we get

$$(K_{11}w_{\mathbf{mn}}w_{jk})(\mathbf{u},t") = -\chi(\mathbf{u})\frac{1}{4\pi c^2}\int d\mathbf{x}\frac{1}{|\mathbf{u}-\mathbf{x}|}w_{\mathbf{mn}}(\mathbf{x})\frac{d^2w_{jk}}{dt^2}(t" - |\mathbf{u}-\mathbf{x}|/c)$$
$$= -\chi(\mathbf{u})\frac{1}{c^2}\int d\mathbf{x}w_{\mathbf{mn}}(\mathbf{x})\int_{-\infty}^{\infty}dt\frac{d^2w_{jk}}{dt^2}(t)\ G(\mathbf{u},t",\mathbf{x},t) \tag{3.3}$$

.

Then the matrix element of $K_{11}^*K_{11}$ is

$$(w_{\mathbf{m'n'}}w_{j'k'}, K_{11}^*K_{11}w_{\mathbf{mn}}w_{jk}) = (K_{11}w_{\mathbf{m'n'}}w_{j'k'}, K_{11}w_{\mathbf{mn}}w_{jk})$$
$$= \frac{1}{c^4}\int d\mathbf{x}' \quad w_{\mathbf{m'n'}}(\mathbf{x}')\int d\mathbf{x} \quad w_{\mathbf{mn}}(\mathbf{x})\int_{-\infty}^{\infty}dt'\frac{d^2w_{j'k'}}{dt'^2}(t')\int_{-\infty}^{\infty}dt\frac{d^2w_{jk}}{dt^2}(t)$$
$$\int_{-\infty}^{\infty}dt"\int_V d\mathbf{u} \quad |\chi(\mathbf{u})|^2\ G(\mathbf{u},t",\mathbf{x}',t')G(\mathbf{u},t",\mathbf{x},t) \tag{3.4}$$

.

Introducing the expression for the Green function

$$\int_{-\infty}^{\infty}dt"\int_V d\mathbf{u} \quad |\chi(\mathbf{u})|^2G(\mathbf{u},t",\mathbf{x}',t')\,G(\mathbf{u},t",\mathbf{x}',t) =$$
$$\int_V d\mathbf{u} \quad |\chi(\mathbf{u})|^2\frac{1}{|\mathbf{u}-\mathbf{x}'||\mathbf{u}-\mathbf{x}|}\delta(t'-t+|\mathbf{u}-\mathbf{x}'|/c-|\mathbf{u}-\mathbf{x}|/c) \tag{3.5}$$

.

To calculate the integral over $\mathbf{u}$ in the material volume, we introduce the prolate spheroidal coordinates.

The Cartesian coordinates for $\mathbf{u} = (u_1', u_2', u_3')$ are given with respect to the following orthonormal vectors whose origins are the middle point in between $\mathbf{x}'$ and $\mathbf{x}$, $\mathbf{e}_3' = (\mathbf{x}'-\mathbf{x})/\|\mathbf{x}'-\mathbf{x}\|$, $\mathbf{e}_2'$ is parallel to the ground plane and perpendicular to $\mathbf{e}_3'$, while $\mathbf{e}_1'$ is perpendicular to the previous ones.

If one denotes by $2a$ the distance in between $\mathbf{x}'$ and $\mathbf{x}$, the spheroidal coordinates are given by

$$\tau = \frac{1}{2a}\sqrt{u_1'^2+u_2'^2+(u_3'+a)^2}+\sqrt{u_1'^2+u_2'^2+(u_3'-a)^2}$$
$$\sigma = \frac{1}{2a}\sqrt{u_1'^2+u_2'^2+(u_3'+a)^2}-\sqrt{u_1'^2+u_2'^2+(u_3'-a)^2}$$
$$\phi = \arctan\frac{u_2'}{u_1'} \tag{3.6}$$

Notice that $\tau \in [1,\infty)$ while $\sigma \in [-1,1)$. The cartesian coordinates , in term of the spheroidal coordinates are given by

$$u_1' = a\sqrt{(\tau^2-1)(1-\sigma^2)}\cos\phi$$
$$u_2' = a\sqrt{(\tau^2-1)(1-\sigma^2)}\sin\phi$$
$$u_3' = a\tau\sigma \tag{3.7}$$

The surfaces of constant $\tau$ are prolate spheroids while the surfaces of constant $\sigma$ are hyperboloids denoted $H_{\mathbf{x},\mathbf{x}'}(\sigma)$.

The volume element is $dV = a^3(\tau^2-\sigma^2)d\tau d\sigma d\phi$.

Notice that as $\tau = \frac{1}{2a}(|\mathbf{u}-\mathbf{x}'|+|\mathbf{u}-\mathbf{x}|)$ and $\sigma = \frac{1}{2a}(|\mathbf{u}-\mathbf{x}'|-|\mathbf{u}-\mathbf{x}|)$, then $|\mathbf{u}-\mathbf{x}'| = a(\tau+\sigma)$ and $|\mathbf{u}-\mathbf{x}| = a(\tau-\sigma)$, so $|\mathbf{u}-\mathbf{x}'||\mathbf{u}-\mathbf{x}| = a^2(\tau^2-\sigma^2)$ .

The integrand simplifies

$$\int_V d\mathbf{u}\,\delta(t'-t+|\mathbf{u}-\mathbf{x}'|/c-|\mathbf{u}-\mathbf{x}|/c)\,|\chi(\mathbf{u})|^2\frac{1}{|\mathbf{u}-\mathbf{x}||\mathbf{u}-\mathbf{x}'|}$$
$$= |\mathbf{x}'-\mathbf{x}|/2\int_{-1}^{1} d\sigma\,\delta(t'-t+\sigma|\mathbf{x}'-\mathbf{x}|/c)\int_1^\infty d\tau\int_0^{2\pi} d\varphi|\tilde{\chi}_{\mathbf{x}\mathbf{x}'}(\sigma,\tau,\varphi)|^2 \tag{3.8}$$

Using the fact that $w_{\mathbf{mn}}(\mathbf{x}) = w_{\mathbf{m0}}(\mathbf{x}-\mathbf{x}_{\mathbf{mn}})$ and $w_{jk}(t) = w_{j0}(t-t_{jk})$, performing the change of variables $\mathbf{x_0} = \mathbf{x}-\mathbf{x}_{\mathbf{mn}}$ , $\mathbf{x}'_{\mathbf{0}} = \mathbf{x}'-\mathbf{x}_{\mathbf{m'n'}}$, $t_0 = t-t_{jk}$ and $t'_0 = t'-t_{j'k'}$ in the integrals the matrix element of $K_{11}^*K_{11}$ is

$$(w_{\mathbf{m'n'}}w_{j'k'}, K_{11}^*K_{11}w_{\mathbf{mn}}w_{jk}) = (K_{11}w_{\mathbf{m'n'}}w_{j'k'}, K_{11}w_{\mathbf{mn}}w_{jk})$$
$$= \frac{1}{2c^4}\int d\mathbf{x}'_{\mathbf{0}}w_{\mathbf{m'0}}(\mathbf{x}'_{\mathbf{0}})\int d\mathbf{x_0}w_{\mathbf{m0}}(\mathbf{x_0})\int_{-\infty}^{\infty}dt'_0\frac{d^2w_{j'0}}{dt'^2}(t'_0)\int_{-\infty}^{\infty}dt_0\frac{d^2w_{j0}}{dt^2}(t_0)$$
$$\times|\mathbf{x}_{\mathbf{m'n'}}-\mathbf{x}_{\mathbf{mn}}+\mathbf{x}'_{\mathbf{0}}-\mathbf{x_0}|\int_{-1}^{1}d\sigma\,\delta(t_{j'k'}-t_{jk}+t'_0-t_0+\sigma|\mathbf{x}_{\mathbf{m'n'}}-\mathbf{x}_{\mathbf{mn}}+\mathbf{x}'_{\mathbf{0}}-\mathbf{x_0}|/c)$$
$$\times\int_1^\infty d\tau\int_0^{2\pi}d\varphi|\tilde{\chi}_{\mathbf{x}\mathbf{x}'}(\sigma,\tau,\varphi)|^2 \tag{3.9}$$

.

For fixed $(j,k,\mathbf{m},\mathbf{n})$, if $|\mathbf{x}_{\mathbf{m'n'}}-\mathbf{x}_{\mathbf{mn}}| \le c\big(t_{j'k'}-t_{jk}-d_j-d_{j'}\big)-d_{\mathbf{m}}-d_{\mathbf{m'}}$, then $\frac{c(t_{j'k'}-t_{jk}+t'_0-t_0)}{|\mathbf{x}_{\mathbf{m'n'}}-\mathbf{x}_{\mathbf{mn}}+\mathbf{x}'_{\mathbf{0}}-\mathbf{x_0}|} > 1$, whenever $\mathbf{x_0}$, $\mathbf{x}'_{\mathbf{0}}$, $t_0$, $t'_0$ are, so $\delta(t_{j'k'}-t_{jk}+t'_0-t_0+\sigma|\mathbf{x}_{\mathbf{m'n'}}-\mathbf{x}_{\mathbf{mn}}+\mathbf{x}'_{\mathbf{0}}-\mathbf{x_0}|/c)=0$ and $(w_{\mathbf{m'n'}}w_{j'k'}, K_{11}^*K_{11}w_{\mathbf{mn}}w_{jk})=0$

If $|\mathbf{x}_{\mathbf{m'n'}}-\mathbf{x}_{\mathbf{mn}}| \le c\big(t_{jk}-t_{j'k'}-d_j-d_{j'}\big)-d_{\mathbf{m}}-d_{\mathbf{m'}}$ , $\frac{c(t_{j'k'}-t_{jk}+t'_0-t_0)}{|\mathbf{x}_{\mathbf{m'n'}}-\mathbf{x}_{\mathbf{mn}}+\mathbf{x}'_{\mathbf{0}}-\mathbf{x_0}|} < -1$ whenever $\mathbf{x_0}$, $\mathbf{x}'_{\mathbf{0}}$, $t_0$, $t'_0$ are, so $(w_{\mathbf{m'n'}}w_{j'k'}, K_{11}^*K_{11}w_{\mathbf{mn}}w_{jk})=0$ QED

*Remark.* Theorem results from a property of functions supports. It could have been obtained more easily noticing that the operators $K_{ij}$ are the products of three operators, a time derivation operator, a free wave propagation operator and a space derivation operator. If $\mathbf{m}$ and $j$ are sufficiently large, it is easy to follow, as $t$ increases, the spatial support of $K_{ij}w_{\mathbf{mn}}w_{jk}$. At time $t=t_{jk}$ the support is in a small ball, while at time $t$ larger than $t=t_{jk}$, the support is a spherical crown centred at $\mathbf{x_{mn}}$ whose mean radius is $c(t-t_{jk})$ and width $d_{\mathbf{m}}$. Similarly the support $K_{ij}w_{\mathbf{m'n'}}w_{j'k'}$ at time $t$ is a spherical crown centred at $\mathbf{x_{m'n'}}$ whose mean radius is $c(t-t_{j'k'})$ and width $d_{\mathbf{m'}}$. It is easy to see that if $|\mathbf{x}_{\mathbf{m'n'}}-\mathbf{x}_{\mathbf{mn}}| \le c\big(t_{j'k'}-t_{jk}-(2^{-j}+2^{-j'})p+2^{-j'}\big)-d_{\mathbf{m}}-d_{\mathbf{m'}}$ or if $|\mathbf{x}_{\mathbf{m'n'}}-\mathbf{x}_{\mathbf{mn}}| \le c\big(t_{jk}-t_{j'k'}-(2^{-j}+2^{-j'})p+2^{-j}\big)-d_{\mathbf{m}}-d_{\mathbf{m'}}$ the supports never intersect as t evolves , so the scalar product of $K_{ij}w_{\mathbf{mn}}w_{jk}$ and $K_{ij}w_{\mathbf{m'n'}}w_{j'k'}$ is equal to zero. So Theomem 3.1 is almost trivial and its proof could be obtained without the use of spheroidal coordinates.

# 4 Kernels regularity and matrix elements smallness

We want now to examine the values for the matrix elements $(w_{\mathbf{m'n'}}w_{j'k'}, K_{11}^*K_{11}w_{\mathbf{mn}}w_{jk})$ when $\mathbf{x_{mn}}, \mathbf{x_{m'n'}}, t_{j'k'}, t_{j'k'}$ do not satisfy the conditions stated in the previous theorem.

In (3.9) we perform the integration w.r.to the $\sigma$ variable , obtaining,

$$(w_{\mathbf{m'n'}}w_{j'k'}, K_{11}^*K_{11}w_{\mathbf{mn}}w_{jk}) = (K_{11}w_{\mathbf{m'n'}}w_{j'k'}, K_{11}w_{\mathbf{mn}}w_{jk})$$
$$= \frac{1}{2c^3}\int d\mathbf{x'_0}w_{\mathbf{m'0}}(\mathbf{x'_0})\int d\mathbf{x_0}w_{\mathbf{m0}}(\mathbf{x_0})\int_{-\infty}^{\infty}dt'_0\frac{d^2w_{j'0}}{dt'^2}(t'_0)\int_{-\infty}^{\infty}dt_0\frac{d^2w_{j0}}{dt^2}(t_0)$$
$$\times\int_1^{\infty}d\tau\int_0^{2\pi}d\varphi|\tilde{\chi}_{\mathbf{x_0}+\mathbf{x_{mn}},\mathbf{x'_0}+\mathbf{x_{m'n'}}}\Big(\frac{c(t_{jk}-t_{j'k'}+t_0-t'_0)}{|\mathbf{x_{m'n'}}-\mathbf{x_{mn}}+\mathbf{x'_0}-\mathbf{x_0}|},\tau,\varphi\Big)|^2 \tag{4.1}$$

.

In the following, to simplify the writing we will use the notations

$$f(\sigma,\mathbf{x},\mathbf{x}') := \int_1^{\infty}d\tau\int_0^{2\pi}d\varphi\ |\tilde{\chi}_{\mathbf{x},\mathbf{x}'}(\sigma,\tau,\varphi)|^2 \tag{4.2}$$

and

$$f_{\mathbf{m},\mathbf{n},\mathbf{m'},\mathbf{n'},j,k,j',k'}(\mathbf{x_0},\mathbf{x'_0},t_0,t'_0) := f(\frac{c(t_{jk}-t_{j'k'}+t_0-t'_0)}{|\mathbf{x_{m'n'}}-\mathbf{x_{mn}}+\mathbf{x'_0}-\mathbf{x_0}|},\mathbf{x_0}+\mathbf{x_{mn}},\mathbf{x'_0}+\mathbf{x_{m'n'}})$$
$$= \int_1^{\infty}d\tau\int_0^{2\pi}d\varphi\Big|\tilde{\chi}_{\mathbf{x_0}+\mathbf{x_{mn}},\mathbf{x'_0}+\mathbf{x_{m'n'}}}\Big(\frac{c(t_{jk}-t_{j'k'}+t_0-t'_0)}{|\mathbf{x_{m'n'}}-\mathbf{x_{mn}}+\mathbf{x'_0}-\mathbf{x_0}|},\tau,\varphi\Big)\Big|^2 \tag{4.3}$$

So

$$(w_{\mathbf{m'n'}}w_{j'k'}, K_{11}^*K_{11}w_{\mathbf{mn}}w_{jk}) = (K_{11}w_{\mathbf{m'n'}}w_{j'k'}, K_{11}w_{\mathbf{mn}}w_{jk})$$
$$= \frac{1}{2c^3}\int d\mathbf{x'_0}w_{\mathbf{m'0}}(\mathbf{x'_0})\int d\mathbf{x_0}w_{\mathbf{m0}}(\mathbf{x_0})\int_{-\infty}^{\infty}dt'_0\,\frac{d^2w_{j'0}}{dt^2}(t'_0)\int_{-\infty}^{\infty}dt_0\,\frac{d^2w_{j0}}{dt^2}(t_0)f_{\mathbf{m},\mathbf{n},\mathbf{m'},\mathbf{n'},j,k,j',k'}(\mathbf{x_0},\mathbf{x'_0},t_0,t'_0) \tag{4.4}$$

.

Notice that $f_{\mathbf{m},\mathbf{n},\mathbf{m'},\mathbf{n'},j,k,j',k'}(\mathbf{x_0},\mathbf{x'_0},t,t') = 0$ if $\frac{c(t_{jk}-t_{j'k'}+t_0-t'_0)}{|\mathbf{x_{m'n'}}-\mathbf{x_{mn}}+\mathbf{x'_0}-\mathbf{x_0}|} > 1$ or $< -1$.

In order to study the smallness of the matrix elements we are going to examine the regularity of the function $f_{\mathbf{m},\mathbf{n},\mathbf{m'},\mathbf{n'},j,k,j',k'}(\mathbf{x_0},\mathbf{x'_0},t,t')$ inside $I_{\mathbf{mm'}jj'}$ which is defined as the set of points $(\mathbf{x_0},\mathbf{x'_0},t,t')$ such that $x_{0i}\in(-2^{-m_i}(p-1),2^{-m_i}p)$, $x'_{0i}\in(-2^{-m'_i}(p-1),2^{-m'_i}p)$, $t\in(-2^{-j}(p-1),2^{-j}p)$, $t'\in(-2^{-j'}(p-1),2^{-j'}p)$ i.e the product of the supports of the wavelets $w_{\mathbf{m0}}(\mathbf{x_0})$, $w_{\mathbf{m'0}}(\mathbf{x'_0})$, $w_{j0}(t_0)$, $w_{j'0}(t'_0)$.

To start, we consider simple geometries for the material volume and regularity of the permittivity inside it .

**Proposition 4.1.** *If the material volume is a polyhedron in which the permittivity is infinitely differentiable , if*

$|\mathbf{x_{m'n'}}-\mathbf{x_{mn}}| > c(t_{jk}-t_{j'k'})+c(d_j+d_{j'})+d_{\mathbf{m}}+d_{\mathbf{m'}}$ *(H1) or*

$|\mathbf{x_{m'n'}}-\mathbf{x_{mn}}| > c(t_{j'k'}-t_{jk})+c(d_j+d_{j'})+d_{\mathbf{m}}+d_{\mathbf{m'}}$ *(H2),*

*and if for fixed* $\mathbf{m},\mathbf{n},\mathbf{m'},\mathbf{n'},j,k,j',k'$ *the set of hyperboloids*

$H_{\mathbf{x_{mn}}+\mathbf{x_0},\mathbf{x_{m'n'}}+\mathbf{x'_0}}\Big(\frac{c(t_{jk}+t-t_{j'k'}-t')}{|\mathbf{x_{m'n'}}+\mathbf{x'_0}-\mathbf{x_{mn}}-\mathbf{x_0}|}\Big)$ *with* $(\mathbf{x_0},\mathbf{x'_0},t_0,t'_0)\in I_{\mathbf{mm'}jj'}$ *hits only one face* $F_i$ *(H3) or if it hits several faces,* $F_i$, $F_{i+1}$,...$F_{i+k}$ *but do not hit a corner point (H4),*

*then function* $f_{\mathbf{m},\mathbf{n},\mathbf{m'},\mathbf{n'},j,k,j',k'}(\mathbf{x_0},\mathbf{x'_0},t_0,t'_0)$ *is regular for any* $(\mathbf{x_0},\mathbf{x'_0},t_0,t'_0)$ *in* $I_{\mathbf{mm'}jj'}$ .

*Proof.* For fixed $\mathbf{m},\mathbf{n},\mathbf{m'},\mathbf{n'},j,k,j',k'$ one chooses $(\mathbf{x_0},\mathbf{x'_0},t_0,t'_0)\in I_{\mathbf{mm'}jj'}$ . From (H1) or (H2) $-1 < \frac{c(t_{jk}+t_0-t_{j'k'}-t'_0)}{|\mathbf{x_{m'n'}}+\mathbf{x'_0}-\mathbf{x_{mn}}-\mathbf{x_0}|} < 1$ .

Consider first , $(H3)$, the case the intersection of the hyperboloid $H_{\mathbf{x_{mn}}+\mathbf{x_0},\mathbf{x_{m'n'}}+\mathbf{x'_0}}\Big(\frac{c(t_{jk}+t_0-t_{j'k'}-t'_0)}{|\mathbf{x_{m'n'}}+\mathbf{x'_0}-\mathbf{x_{mn}}-\mathbf{x_0}|}\Big)$ and the polyhedron is totally inside the face $F_i$.

The spheroidal coordinates of the points of the intersection are denoted $\Big(\sigma=\frac{c(t_{jk}+t_0-t_{j'k'}-t'_0)}{|\mathbf{x_{m'n'}}+\mathbf{x'_0}-\mathbf{x_{mn}}-\mathbf{x_0}|},\varphi,\tau(\sigma,\varphi)\Big)$ We are going to calculate $\tau(\sigma,\varphi)$. We need to establish the equations for the plane $P_i$ in the spheroidal coordinates.

We are going first to establish the link between the cartesian coordinates and the spheroidal coordinates linked with $\mathbf{x}$ and $\mathbf{x}'$. Vectors of the new basis are $\mathbf{e}'_3 = \frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|}$ , $\mathbf{e}'_2$ is chosen parallel to the ground and perpendicular to $\mathbf{e}'_3$, so $\mathbf{e}'_2 = \frac{x'_2-x_2}{\sqrt{(x'_1-x_1)^2+(x'_2-x_2)^2}}\mathbf{e}_1 - \frac{x'_1-x_1}{\sqrt{(x'_1-x_1)^2+(x'_2-x_2)^2}}\mathbf{e}_2$, and

$$\begin{aligned}\mathbf{e}'_1 = \mathbf{e}'_2 \times \mathbf{e}'_3 = &-\frac{(x'_1-x_1)(x'_3-x_3)}{\sqrt{(x'_1-x_1)^2+(x'_2-x_2)^2+(x'_3-x_3)^2}\sqrt{(x'_1-x_1)^2+(x'_3-x_3)^2}}\mathbf{e}_1\\ &-\frac{(x'_3-x_3)(x'_2-x_2)}{\sqrt{(x'_1-x_1)^2+(x'_2-x_2)^2+(x'_3-x_3)^2}\sqrt{(x'_1-x_1)^2+(x'_3-x_3)^2}}\mathbf{e}_2\\ &+\frac{x_1^2+x_2^2-2x_1x'_1+x_1^{'2}-2x_2x'_2+x_2^{'2}}{\sqrt{(x'_1-x_1)^2+(x'_2-x_2)^2+(x'_3-x_3)^2}\sqrt{(x'_1-x_1)^2+(x'_3-x_3)^2}}\mathbf{e}_3. \qquad (4.5)\end{aligned}$$

The old vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ can be expressed in terms of the new ones $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3)$ by the formulas

$$\begin{aligned}\mathbf{e}_1 &= (\mathbf{e}_1.\mathbf{e}'_1)\mathbf{e}'_1 + (\mathbf{e}_1.\mathbf{e}'_2)\mathbf{e}'_2 + (\mathbf{e}_1.\mathbf{e}'_3)\mathbf{e}'_3\\ \mathbf{e}_2 &= (\mathbf{e}_2.\mathbf{e}'_1)\mathbf{e}'_1 + (\mathbf{e}_2.\mathbf{e}'_2)\mathbf{e}'_2 + (\mathbf{e}_2.\mathbf{e}'_3)\mathbf{e}'_3\\ \mathbf{e}_3 &= (\mathbf{e}_3.\mathbf{e}'_1)\mathbf{e}'_1 + (\mathbf{e}_3.\mathbf{e}'_2)\mathbf{e}'_2 + (\mathbf{e}_3.\mathbf{e}'_3)\mathbf{e}'_3 \qquad (4.6)\end{aligned}$$

where the scalar products can be extracted from the formulas (4.5)
The old coordinates $u_1, u_2, u_3$ can be expressed in terms of the new ones

$$\begin{aligned}u_1 &= \frac{x_1+x'_1}{2} + (\mathbf{e}_1.\mathbf{e}'_1)u'_1 + (\mathbf{e}_1.\mathbf{e}'_2)u'_2 + (\mathbf{e}_1.\mathbf{e}'_3)u'_3\\ u_2 &= \frac{x_1+x'_1}{2} + (\mathbf{e}_2.\mathbf{e}'_1)u'_1 + (\mathbf{e}_2.\mathbf{e}'_2)u'_2 + (\mathbf{e}_2.\mathbf{e}'_3)u'_3\\ u_3 &= \frac{x_1+x'_1}{2} + (\mathbf{e}_3.\mathbf{e}'_1)u'_1 + (\mathbf{e}_3.\mathbf{e}'_2)u'_2 + (\mathbf{e}_3.\mathbf{e}'_3)u'_3 \qquad (4.7)\end{aligned}$$

Now we are going to study the intersection of the hyperboloid $H_\sigma$ and the plane $P_i$ containing the polyhedron face $F_i$.

The normal $\mathbf{n}_i$ to the plane $P_i$ is written, in the original basis, as $\mathbf{n}_i = n_{i1}\mathbf{e}_1 + n_{i2}\mathbf{e}_2 + n_{i3}\mathbf{e}_3$, in the basis $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$, it becomes

$$\begin{aligned}\mathbf{n}_i &= n_{i1}\mathbf{e}_1 + n_{i2}\mathbf{e}_2 + n_{i3}\mathbf{e}_3\\ &= n_{i1}\big[(\mathbf{e}_1.\mathbf{e}'_1)\mathbf{e}'_1 + (\mathbf{e}_1.\mathbf{e}'_2)\mathbf{e}'_2 + (\mathbf{e}_1.\mathbf{e}'_3)\mathbf{e}'_3\big]\\ &+n_{i2}\big[(\mathbf{e}_2.\mathbf{e}'_1)\mathbf{e}'_1 + (\mathbf{e}_2.\mathbf{e}'_2)\mathbf{e}'_2 + (\mathbf{e}_2.\mathbf{e}'_3)\mathbf{e}'_3\big]\\ &+n_{i3}\big[(\mathbf{e}_3.\mathbf{e}'_1)\mathbf{e}'_1 + (\mathbf{e}_3.\mathbf{e}'_2)\mathbf{e}'_2 + (\mathbf{e}_3.\mathbf{e}'_3)\mathbf{e}'_3\big]\\ &= \big[n_{i1}(\mathbf{e}_1.\mathbf{e}'_1) + n_{i2}(\mathbf{e}_2.\mathbf{e}'_1) + n_{i3}(\mathbf{e}_3.\mathbf{e}'_1)\big]\mathbf{e}'_1\\ &+\big[n_{i1}(\mathbf{e}_1.\mathbf{e}'_2) + n_{i2}(\mathbf{e}_2.\mathbf{e}'_2) + n_{i3}(\mathbf{e}_3.\mathbf{e}'_2)\big]\mathbf{e}'_2\\ &+\big[n_{i1}(\mathbf{e}_1.\mathbf{e}'_3) + n_{i2}(\mathbf{e}_2.\mathbf{e}'_3) + n_{i3}(\mathbf{e}_3.\mathbf{e}'_3)\big]\mathbf{e}'_3 \qquad (4.8)\end{aligned}$$

.

All the scalar products can be extracted from (4.5).

The equation of the plane $P_i$ is obtained writing that if $M = (u'_1, u'_2, u'_3) \in P_i$, $O'M.\mathbf{n}_i = s_i\mathbf{e}'_3.\mathbf{n}_i$, where $O'$ is the middle point $\frac{\mathbf{x}+\mathbf{x}'}{2}$ , $(0, 0, s_i)$ are the coordinates, in the new basis, of the intersection point of the line $\mathbf{x}\mathbf{x}'$ and the plane $P_i$ . The previous equation becomes,

$$(u'_1\mathbf{e}'_1 + u'_2\mathbf{e}'_2 + u'_3\mathbf{e}'_3).\mathbf{n}_i = s_i\mathbf{e}'_3.\mathbf{n}_i \qquad (4.9)$$

.

The scalar products $\mathbf{e'}_1.\mathbf{n}_i$, $\mathbf{e'}_2.\mathbf{n}_i$, $\mathbf{e'}_3.\mathbf{n}_i$ will be denoted respectively $\alpha_{i1}$, $\beta_{i1}$, $\gamma_{i1}$, they can be calculated from equation (4.8).

In spheroidal coordinates the plane equation becomes

$$\alpha_{i,1}\Big(a\sqrt{(\tau^2-1)(1-\sigma^2)}\cos\varphi\Big)+\beta_{i,1}\Big(a\sqrt{(\tau^2-1)(1-\sigma^2)}\sin\varphi\Big)+\gamma_{i,1}\Big(a\tau\sigma\Big)=\gamma_{i,1}s_i \quad (4.10)$$

.

Introducing $\varphi_{i,0}$ such that $\cos\varphi_{i,0}=\frac{\alpha_{i,1}}{\alpha_{i,1}^2+\beta_{i,1}^2}$ and $\sin\varphi_{i,0}=\frac{\beta_{1,1}}{\alpha_{i,1}^2+\beta_{i,1}^2}$ we get from (4.10)

$$\cos(\varphi-\varphi_{i,0})=\frac{\gamma_{i,1}(s_i-a\tau\sigma)}{a\sqrt{(\tau^2-1)(1-\sigma^2)}} \quad (4.11)$$

.

To determine the limits of integration for the integrals which enter in $f(\sigma,\mathbf{x},\mathbf{x}')$ we have to find the relationship in between $\tau$ and $\varphi$. As for fixed $-1<\sigma<1$ the function $F_\sigma(\tau):=\frac{\gamma_{i1}(s_i-a\tau\sigma)}{a\sqrt{(\tau^2-1)(1-\sigma^2)}}$ is a decreasing function on $[1,\infty)$ its inverse $F_\sigma^{-1}$ exists , so $\tau(\sigma,\varphi)=F_\sigma^{-1}(\cos(\varphi-\varphi_{0,i}))$

So finally in the case $H_\sigma$ hits only the face $F_i$, we obtain

$$f(\sigma,\mathbf{x},\mathbf{x}')=\int_0^{2\pi}d\varphi\int_1^{\tau(\sigma,\varphi)}d\tau|\tilde{\chi}(\sigma,\tau,\varphi|^2, \quad (4.12)$$

In the case the permittivity is a constant, $\chi_0$

$$f(\sigma,\mathbf{x},\mathbf{x}')=\chi_0^2\Big(-2\pi+\int_0^{2\pi}d\varphi\ \tau(\sigma,\varphi)\Big), \quad (4.13)$$

It is easy to see that $\gamma_i$, $s_i$, $a$ are regular functions in $\mathbf{x}$ and $\mathbf{x}'$ , so $F$ and $F^{-1}$ are regular and finally $\tau(\sigma,\varphi)$ is also regular.

Using now the Leibniz integral function formula several times , the regularity of $|\chi_{\mathbf{xx}'}(\sigma,\tau,\varphi)|^2$ and $\tau(\sigma,\varphi)$ with respect to $\sigma$ one gets that $f_{\mathbf{mn},\mathbf{m'n'},jk,j'k'}(\mathbf{x_0},\mathbf{x'_0},t_0,t'_0)$ is regular for any $(\mathbf{x_0},\mathbf{x'_0},t_0,t'_0)$ in $I_{\mathbf{mm'}jj'}$

Let us now examine the case for fixed $\mathbf{m},\mathbf{n},\mathbf{m}',\mathbf{n}',j,k,j',k'$ the set of hyperboloids $H_{\mathbf{x_{mn}}+\mathbf{x_0},\mathbf{x_{m'n'}}+\mathbf{x'_0}}\Big(\frac{c(t_{jk}+t_0-t_{j'k'}-t'_0)}{|\mathbf{x_{m'n'}}+\mathbf{x'_0}-\mathbf{x_{mn}}-\mathbf{x_0}|}\Big)$ with $(\mathbf{x_0},\mathbf{x'_0},t_0,t'_0)\in I_{\mathbf{mm'}jj'}$ hits several faces, $F_i$, $F_{i+1}$,...$F_{i+k}$ but do not hit a corner point ( (H4) hypothesis).

The intersections of the hyperboloid with the polyhedron faces are pieces of ellipses. We have to determine the spheroidal coordinates of the points which are at the intersection of the hyperboloid with the edges of the polyhedron. Let us denote $E_{ii'}$ the edge corresponding to the faces $F_i$ and $F'_i$. The spheroidal coordinates of the intersection point of $H_\sigma$ and $E_{ii'}$ are denoted $(\sigma,\varphi_{ii'}(\sigma),\tau_{ii'}(\sigma))$.

We have to distinguish two cases . In the first case $H_\sigma$ hits only the edge $E_{i,i+1}$ and it does at two points. In the second case $H_\sigma$ hits successively edges $E_{i,i+1},...,E_{i+l,i+l+1},...,E_{i+k,i}$ and it does at an unique point .

In the first case integrating always w.r.to $\varphi$ in the counterclockwise

$$f(\sigma,\mathbf{x},\mathbf{x}')=\Big(\int_{\varphi_{i,i+1}(\sigma)}^{\varphi_{i+1,i}(\sigma)}d\varphi\int_1^{\tau(\sigma,\varphi)}d\tau|\tilde{\chi}_{\mathbf{xx}'}(\sigma,\tau,\varphi|^2+\int_{\varphi_{i+1,i}}^{\varphi_{i,i+1}}d\varphi\int_1^{\tau(\sigma,\varphi)}d\tau|\tilde{\chi}(\sigma,\tau,\varphi|^2\Big) \quad (4.14)$$

In the second case

$$f(\sigma,\mathbf{x},\mathbf{x}')=\sum_{l=0}^{k}\int_{\varphi_{i+l,i+l+1}(\sigma)}^{\varphi_{i+l+1,i+l+2}(\sigma)}d\varphi\int_1^{\tau(\sigma,\varphi)}d\tau|\tilde{\chi}(\sigma,\tau,\varphi|^2 \quad (4.15)$$

To calculate $\varphi_{i+l,i+l+1}(\sigma)$ we use the fact that, at an intersection point of the two ellipses, the value for the coordinate $\tau$ has to satisfy from (4.11) $\gamma_{i+l,1}(s_{i+l}-a\tau\sigma)=\gamma_{i+l+1,1}(s_{i+l+1}-$

$a\tau\sigma$), so it takes the value denoted, $\tau_{i+l} := \frac{\gamma_{i+l+1,1}s_{i+l+1}-\gamma_{i+l,1}s_{i+l}}{a\sigma(\gamma_{i+l+1,1}-\gamma_{i+l,1})}$ and $\varphi_{i+l,i+l+1}(\sigma) = \varphi_{i+l,0} + \arccos\frac{\gamma_{i+l,1}(s_{i+l}-a\tau_{i+l}\sigma)}{a\sqrt{(\tau_{i+l}^2-1)(1-\sigma^2)}}$

It may be another value for $\varphi$, at the intersection of the hyperboloid with the edge, $\pi - (\varphi_{i+l,i+l+1}(\sigma) - \varphi_{i+l,0})$. In the first case we have to take the two values for $\varphi$ while in the second case we have to choose the convenient one. From the regularity of the integral bounds we deduce the regularity of function $f_{\mathbf{m},\mathbf{n},\mathbf{m}',\mathbf{n}',j,k,j',k'}(\mathbf{x_0},\mathbf{x'_0},t_0,t'_0)$. QED

*Remark.* If $\frac{c(t_{jk}-t_{j'k'})}{|\mathbf{x_{m'n'}}-\mathbf{x_{mn}}|}$ is close to 1, when $(\mathbf{x_0},\mathbf{x'_0},t_0,t'_0)$ belong to the wavelets supports it may occur that $\frac{c(t_{jk}+t-t_{j'k'}-t')}{|\mathbf{x_{m'n'}}-\mathbf{x_{mn}}+\mathbf{x'}-\mathbf{x}|} < 1$ or it may occur that $\frac{c(t_{jk}+t-t_{j'k'}-t')}{|\mathbf{x_{m'n'}}-\mathbf{x_{mn}}+\mathbf{x'}-\mathbf{x}|} > 1$. Since $f(\sigma,\mathbf{x},\mathbf{x'})$ has a jump at $\sigma = 1$, $f_{\mathbf{m},\mathbf{n},\mathbf{m}',\mathbf{n}',j,k,j',k'}(\mathbf{x_0},\mathbf{x'_0},t_0,t'_0) = f\Big(\frac{c(t_{j'k'}-t_{jk}+t'_0-t_0)}{|\mathbf{x_{m'n'}}-\mathbf{x_{mn}}+\mathbf{x'_0}-\mathbf{x_0}|},\mathbf{x_{mn}}+\mathbf{x_0},\mathbf{x_{m'n'}}+\mathbf{x'_0}\Big)$ is not regular.

**Theorem 4.2.** *If the volume material volume is a polyhedron in which the permittivity is infinitely differentiable , if*

$|\mathbf{x_{m'n'}} - \mathbf{x_{mn}}| > c(t_{j'k'} - t_{jk}) + c(d_j + d_{j'}) + d_{\mathbf{m}} + d_{\mathbf{m}'}$ *, or if* $|\mathbf{x_{m'n'}} - \mathbf{x_{mn}}| > c(t_{jk} - t_{j'k'}) + c(d_j + d_{j'}) + d_{\mathbf{m}} + d_{\mathbf{m}'}$ *,*

*if for fixed* $\mathbf{m},\mathbf{n},\mathbf{m}',\mathbf{n}',j,k,j',k'$ *the set of hyperboloids* $H_{\mathbf{x_{mn}}+\mathbf{x_0},\mathbf{x_{m'n'}}+\mathbf{x'_0}}\Big(\frac{c(t_{jk}+t-t_{j'k'}-t')}{|\mathbf{x_{m'n'}}+\mathbf{x'_0}-\mathbf{x_{mn}}-\mathbf{x_0}|}\Big)$ *with* $(\mathbf{x_0},\mathbf{x'_0},t,t') \in I_{\mathbf{mm'}jj'}$ *do not hit a corner point*

*the matrix element satisfies, for some constant* $C$

$\Big(w_{\mathbf{m'n'}}w_{j'k'},(K_{01}^*K_{01}+K_{11}^*K_{11})w_{\mathbf{mn}}w_{jk}\Big) < \frac{C}{8((p+1))!}2^{-(p+1)\mathbf{m}/2}2^{-(p+1)\mathbf{m}'/2}2^{-(p+1)j/2}2^{-(p+1)j'/2}$

.

*Proof.* Recall that

$$(w_{\mathbf{m'n'}}w_{j'k'},K_{11}^*K_{11}w_{\mathbf{mn}}w_{jk})$$
$$= \frac{1}{2c^3}\int d\mathbf{x'_0}w_{\mathbf{m'0}}(\mathbf{x'_0})\int d\mathbf{x_0}w_{\mathbf{m0}}(\mathbf{x_0})\int_{-\infty}^{\infty}dt'_0\,\frac{d^2w_{j'0}}{dt'^2}(t')\int_{-\infty}^{\infty}dt_0\,\frac{d^2w_{j0}}{dt_0^2}(t_0)\quad f_{\mathbf{mn},\mathbf{m'n'},jk,j'k'}(\mathbf{x_0},\mathbf{x'_0},t_0,t'_0) \tag{4.16}$$

Integrating by parts

$$(w_{\mathbf{m'n'}}w_{j'k'},K_{11}^*K_{11}w_{\mathbf{mn}}w_{jk})$$
$$= \frac{1}{2c^3}\int d\mathbf{x'_0}w_{\mathbf{m'0}}(\mathbf{x'_0})\int d\mathbf{x_0}w_{\mathbf{m0}}(\mathbf{x_0})\int_{-\infty}^{\infty}dt'_0\,w_{j'0}(t'_0)\int_{-\infty}^{\infty}dt_0\,w_{j0}(t_0)\quad \frac{\partial^4 f_{\mathbf{mn},\mathbf{m'n'},jk,j'k'}}{\partial t^2\partial t'^2}(\mathbf{x_0},\mathbf{x'_0},t_0,t'_0) \tag{4.17}$$

From proposition 4.1, $f_{\mathbf{mn},\mathbf{m'n'},jk,j'k'}(\mathbf{x_0},\mathbf{x'_0},t_0,t'_0)$ is regular . To simplify the notations we replace $\mathbf{x_0},\mathbf{x'_0},t,t'$ by $\mathcal{X}$ such that $\mathcal{X}_1 = x_{01},\mathcal{X}_2 = x_{02},Y_3 = x_{03},\mathcal{X}_4 = x'_{01},\mathcal{X}_5 = x'_{02},\mathcal{X}_6 = x'_{03},\mathcal{X}_7 = t_0,\mathcal{X}_8 = t'_0$,

$$\hat{f}(\mathcal{X}) := \frac{\partial^4 f_{\mathbf{mn},\mathbf{m'n'},jk,j'k'}}{\partial t_0^2\partial t_0'^2}(\mathbf{x_0},\mathbf{x'_0},t_0,t'_0) \tag{4.18}$$

It has the following Taylor expansion at point $(\mathbf{0})$.

$$\hat{f}(\mathcal{X}) = \hat{f}(\mathbf{0}) + \sum_{i=1}^{8} y_i \frac{\partial \hat{f}}{\partial y_i}(\mathbf{0}) + \frac{1}{2!}\sum_{i=1}^{8}\sum_{j=1}^{8} y_i y_j \frac{\partial^2 \hat{f}}{\partial y_i \partial y_j}(\mathbf{0}) + ....$$
$$\sum_{\alpha_1+\alpha_2+...+\alpha_8 \leq k} \frac{1}{\alpha_1!\alpha_2!... + \alpha_8!} y_1^{\alpha_1} y_2^{\alpha_2} ... y_8^{\alpha_8} \frac{\partial^{\alpha_1+\alpha_2+...+\alpha_8}\hat{f}}{\partial y_1^{s_1}\partial y_2^{s_2}....\partial y_8^{s_8}}(\mathbf{0}) + \sum_{\beta_1+\beta_2+...+\beta_8=k+1} y_1^{\beta_1} y_2^{\beta_2} ... y_8^{\beta_8} R_\beta(\mathcal{X}_0) \tag{4.19}$$

, where $R_\beta(\mathcal{X}_0)$ is a bounded function which satisfies $|R_\beta(\mathcal{X}_0)| \leq \frac{1}{\beta!} max_{|\alpha|=|\beta|} max_{(\mathcal{X}_0)} |D^\alpha F(\mathcal{X}_0)|$.

Inserting the expression (4.19) in (4.17), the integration of the term $y_1^{\alpha_1} y_2^{\alpha_2} ... y_8^{\alpha_8}$ give us 0 if one of the exponents $\alpha_i$ is smaller or equal to $p$, as the first $p$ moments of the Daubechies -p wavelets are null.

$$\begin{aligned}(w_{\mathbf{m'n'}} w_{j'k'}, K_{11}^* K_{11} w_{\mathbf{mn}} w_{jk}) \\ = \frac{1}{2c^4} \int d\mathbf{x_0'} w_{\mathbf{m'0}}(\mathbf{x_0'}) \int d\mathbf{x_0} w_{\mathbf{m0}}(\mathbf{x_0}) \int_{-\infty}^{\infty} dt'\, w_{j0}(t') \int_{-\infty}^{\infty} dt\, w_{j0}(t) \\ \times \sum_{\beta_1+\beta_2+...+\beta_8=8(p+1)} x_1^{\beta_1} x_2^{\beta_2} ... t'^{\beta_8} R_{p+1}(\mathbf{x_0}, \mathbf{x_0'}, t, t')\end{aligned} \tag{4.20}$$

.

In this case the matrix element is of order $\frac{1}{(8(p+1))!} 2^{-(p+1)\mathbf{m}/2} 2^{-(p+1)\mathbf{m'}/2} 2^{-(p+1)j/2} 2^{-(p+1)j'/2}$.

**Theorem 4.3.** *If the volume material volume is a polyhedron in which the permittivity is infinitely differentiable and if $|\mathbf{x_{m'n'}} - \mathbf{x_{mn}}| > d_{\mathbf{m}} + d_{\mathbf{m'}}$ (H5), if the regularity of the wavelets is sufficiently high ,( for the Daubechies-p wavelets , p has to be sufficiently large) the matrix element satisfies, for some constant C , $\left(w_{\mathbf{m'n'}} w_{j'k'}, (K_{01}^* K_{01} + K_{11}^* K_{11}) w_{\mathbf{mn}} w_{jk}\right) < \frac{C}{(7(p+1))!} 2^{-(p+1)\mathbf{m}/2} 2^{-(p+1)\mathbf{m'}/2} 2^{-(p+1)j/2}$*

*Proof.* The kernel $f(\mathbf{x}, \mathbf{x'}, t, t')$ is a function of the variables $\mathbf{x}$, $\mathbf{x'}$ and $t - t'$. For fixed $\mathbf{x_{mn}}$ and $\mathbf{x_{m'n'}}$ satisfying (H5), $|\mathbf{x'} - \mathbf{x}|$ is strictly positive whatever $\mathbf{x}, \mathbf{x'}$ are resp. in the support of the wavelets $w_{\mathbf{mn}}$ and $w_{\mathbf{m'n'}}$. From the previous calculations it appears that if $|\mathbf{x'} - \mathbf{x}| > 0$, $f(\mathbf{x}, \mathbf{x'}, t - t')$ is regular except on surfaces corresponding to the values of $(\mathbf{x}, \mathbf{x'}, t - t')$ for which the hyperbolod $H_{\frac{c(t-t')}{|\mathbf{x}-\mathbf{x'}|}}$ hits a corner.

Recall that the matrix element is written as

$$\begin{aligned}(w_{\mathbf{m'n'}} w_{j'k'}, K_{11}^* K_{11} w_{\mathbf{mn}} w_{jk}) = \frac{1}{2c^3} \int d\mathbf{x_0} w_{\mathbf{m0}}(\mathbf{x_0}) \int_{-\infty}^{\infty} dt_0 \frac{d^2 w_{j0}}{dt_0^2}(t_0) \\ \times \int d\mathbf{x_0'} w_{\mathbf{m'0}}(\mathbf{x_0'}) \int_{-\infty}^{\infty} dt_0' \frac{d^2 w_{j'0}}{dt_0'^2}(t_0') \quad f_{\mathbf{m,n,m',n'},j,k,j',k'}(\mathbf{x_0}, \mathbf{x_0'}, t_0 - t_0')\end{aligned} \tag{4.21}$$

We denote $F_{\mathbf{m,n,m',n'},j,k,j',k'}(\mathbf{x_0}, \mathbf{x_0'}, t_0) := \int_{-\infty}^{\infty} dt_0' \frac{d^2 w_{j'0}}{dt_0'^2}(t_0') \quad f_{\mathbf{m,n,m',n'},j,k,j',k'}(\mathbf{x_0}, \mathbf{x_0'}, t_0 - t_0')$.

It appears as the convolution of a regular function and a function $f_{\mathbf{m,n,m',n'},j,k,j',k'}(\mathbf{x_0}, \mathbf{x_0'}, t"_0)$ which is regular except on surfaces corresponding to the values of $(\mathbf{x_0}, \mathbf{x_0'}, t"_0)$ for which the hyperboloid $H_{\frac{c(t_{jk}+t"_0-t_{j'k'})}{|\mathbf{x_{m'n'}}+\mathbf{x_0'}-\mathbf{x_{mn}}-\mathbf{x_0}|}}$ hits a corner. The partial derivatives of $f_{\mathbf{m,n,m',n'},j,k,j',k'}(\mathbf{x_0}, \mathbf{x_0'}, t"_0)$ exist also except on the same surfaces. Partial derivatives of first order contain jumps , partial derivatives of second order contain delta functions... We suppose that the derivatives of the wavelets exist up to order $N$, from Leibniz integral rule , the partial derivatives of $F_{\mathbf{m,n,m',n'},j,k,j',k'}(\mathbf{x_0}, \mathbf{x_0'}, t_0)$ with respect to $\mathbf{x_0}$,$\mathbf{x_0'}$,$t_0$ exist, because the regularisation effect introduced by the convolution with the regular wavelets.

From the existence of the partial derivatives of $F_{\mathbf{m},\mathbf{n},\mathbf{m}',\mathbf{n}',j,k,j',k'}(\mathbf{x_0},\mathbf{x'_0},t_0)$, on the support of the wavelets $w_{\mathbf{m0}}(\mathbf{x_0})$,$w_{\mathbf{m'0}}(\mathbf{x'_0})$, $w_{j0}(t_0)$ by the same arguments as in theorem 4.2, one can prove that $\left(w_{\mathbf{m'n'}}w_{j'k'},(K_{01}^*K_{01}+K_{11}^*K_{11})w_{\mathbf{mn}}w_{jk}\right) < \frac{C}{(7(p+1))!}2^{-(p+1)\mathbf{m}/2}2^{-(p+1)\mathbf{m}'/2}2^{-(p+1)j/2}$

# 5 Evaluating the sparsity of the time reversal operator

In the case the Hilbert space $L^2(\mathbb{R}^4)\otimes\mathbb{C}^4$ is replaced by a finite dimensional space generated by a finite set of wavelets we are going to estimate the proportion of matrix elements that are not negligible. There is no clear limit to the wave propagation of pulses in cities, nevertheless considering that TX antenna, situated at the origin, emits a short pulse at a time which can be considered as the time origin, considering that this pulse will create charge and currents densities in the materials which decrease rapidly in time and become negligible for $t$ larger than some $t_{max}$, we will impose a time limit and a space limit to the study, (measurements show that the signal fade at the receiver antenna approximatively 300 ns after the emission, this could be $t_{max}$). Then we will limit the study to a ball of radius $ct_{max}$. To simplify the calculations we suppose that the selected volume is a parallelepiped $(-L_1,L_1)\times(-L_2,L_2)\times(-L_3,L_3)$.

We limit the indices $j,j',m_i,m'_i$ to some set of values, $j\in(\underline{J},\overline{J})$, $j'\in(\underline{J},\overline{J})$, $m_i\in(\underline{M}_i,\overline{M}_i)$ , $m'_i\in(\underline{M}_i,\overline{M}_i)$. The values of $\underline{M}_i$ depend on the support of the Daubechies-p wavelet and the lengths $L_1,L_2,L_3$. $\underline{J}$ depend on the support of the Daubechies-p wavelet and $t_{max}$. The values of $\overline{M}_i$ and $\overline{J}$ will be chosen depending on the objective, in terms of precision for the densities to be calculated.

Once $m_i$ is given the $n_i$ is chosen such that the support of $w_{m_in_i}(x_i)$ i.e $(2^{-m_i}n_i-2^{-m_i}(p-1),2^{-m_i}n_i+2^{-m_i}p)$ intersects the interval $(-L_i,L_i)$. Once $j$ is given, $k$ is chosen so that the support of $w_{jk}(t)$ i.e $(t_{jk}-2^{-j}(p-1),t_{jk}+2^{-j}p)$ intersects the interval $(0,t_{max})$.

The total number of considered $w_{\mathbf{m}_i\mathbf{n}_i}$ spatial wavelets, is equal to

$(1+2+...+2^{\overline{M}_1-\underline{M}_1})\times(1+2+...+2^{\overline{M}_2-\underline{M}_2})\times(1+2+...2^{\overline{M}_3-\underline{M}_3})$

$=2^{\overline{M}_1+\overline{M}_2+\overline{M}_1-\underline{M}_1-\underline{M}_2-\underline{M}_3+3}$, while the total number of considered $w_{\mathbf{jk}}$ temporal wavelets is equal to $(1+2+...+2^{\overline{J}-\underline{J}})N=2^{\overline{J}-\underline{J}+1}$

So the truncated $K_{11}^*K_{11}$ operator acts on a space of dimension

$\mathcal{N}_0=2^{\overline{M}_1+\overline{M}_2+\overline{M}_3+\overline{J}-\underline{M}_1-\underline{M}_2-\underline{M}_3-\underline{J}+4}$

We are going to consider that the matrix elements of $(1+K^*)(1+K)$ which are of order $\frac{1}{(p+1)!}2^{-(p+1)\mathbf{m}/2}2^{-(p+1)\mathbf{m}'/2}$ or smaller are negligible. From theorems 3.1, 4.3 and proposition 9.1 those which are not negligible are those such that $|\mathbf{x_{m'n'}}-\mathbf{x_{mn}}|<d_{\mathbf{m}}+d_{\mathbf{m}'}$ and $c|t_{j'k'}-t_{jk}|<2(d_{\mathbf{m}}+d_{\mathbf{m}'})+(2^{-j}+2^{-j'})p$. In fact if $|\mathbf{x_{m'n'}}-\mathbf{x_{mn}}|<d_{\mathbf{m}}+d_{\mathbf{m}'}$ and $c(t_{j'k'}-t_{jk})\geq 2(d_{\mathbf{m}}+d_{\mathbf{m}'})+(2^{-j}+2^{-j'})p$, $|\mathbf{x_{m'n'}}-\mathbf{x_{mn}}|<c(t_{j'k'}-t_{jk})-d_{\mathbf{m}}-d_{\mathbf{m}'}$ which infers by theorem 3.1 that the corresponding matrix element is 0.

We look at the column of the matrix $(1+K^*)(1+K)$ corresponding to the basis vector $w_{\mathbf{mn}}w_{jk}$ . We order the basis vectors $w_{\mathbf{m'n'}}w_{j'k'}$ , the large scale wavelets with small $\mathbf{m'}$ and $j'$ come first , notice they are few , the small scale wavelets with large $\mathbf{m'}$ and $j'$ are many , their number is of the order $2^{m'_1}L_1\times 2^{m'_2}L_2\times 2^{m'_3}L_3$. For fixed $\mathbf{m}$, $\mathbf{n}$, $j$ , $k$, $\mathbf{m'}$ and $j'$ the number of matrix elements $(w_{\mathbf{m'n'}}w_{j'k'}(1+K^*)(1+K)w_{\mathbf{mn}}w_{jk})$ which are not negligible is the product of the number of lattice points $\mathbf{n'}$ which are inside the material volume , in the ball of radius $d_{\mathbf{m}}+d_{\mathbf{m}'}$ centred at $\mathbf{x_{mn}}$ by the number of $k'$ which satisfy $t_{j'k'}-t_{jk}<2(d_{\mathbf{m}}+d_{\mathbf{m}'})/c+(2^{-j}+2^{-j'})p/c$ . It is smaller than $2^{m'_1+m'_2+m'_3}2^{j'+1}(d_{m_1}+d_{m'_1})(d_{m_2}+d_{m'_2})(d_{m_3}+d_{m'_3})$. Then the total number of matrix elements $(w_{\mathbf{m'n'}}w_{j'k'}K_{11}^*K_{11}w_{\mathbf{mn}}w_{jk})$ which are not negligible is $\sum_{\mathbf{m'}=\underline{\mathbf{M}}}^{\overline{\mathbf{M}}}\sum_{j'=\underline{J}}^{\overline{J}}2^{m'_1+m'_2+m'_3}2^{j'+1}(d_{m_1}+d_{m'_1})(d_{m_2}+d_{m'_2})(d_{m_3}+d_{m'_3})$. Let us notice that for large $\mathbf{m}$ and $j$ the proportion of non negligible

elements with respect to the negligible ones becomes very small .

Let us estimate the sparsity in the case the volume is a parallelepipedic wall ($L_1 = 10m, L_2 = 3m, L_3 = 0.3m$), signals are in the GHz range, the wavelength outside is of the order of the centimetre. The unit of time is the nanosecond , then the light velocity is 0.3 m/ns. We choose the Daubechies 10 wavelet. Then, $p = 10$ , as $\log_2 L_1/p = 0$ , $-\log_2 L_2/p = 1.73$, $-\log_2 L_1/cp = 5.05$ , one chooses $\underline{M}_1 = 0$, $\underline{M}_2 = 1$, $\underline{M}_3 = 5$, $\underline{J} = 0$. The values for $\overline{M}_1$ $\overline{M}_2$ and $\overline{M}_3$ can be determined once the desired precision on the densities is chosen. We can for instance look for space details of the order $1cm$ and temporal details of $0.01ns$ in which case $\overline{M}_1 = \overline{M}_2 = \overline{M}_3 = -\log_2 10^{-3}/p = 9$ and $\overline{J} = 9$. Then the dimension of the column vector is

$\mathcal{N}_0 = 2^{\overline{M}_1+\overline{M}_2+\overline{M}_3+\overline{J}-\underline{M}_1-\underline{M}_2-\underline{M}_3-\underline{J}+4} = 2^{34} = 1.7 \times 10^{10}$

The total number of non negligible elements of the column $\overline{\mathbf{M}}\mathbf{n}\overline{J}k$ is

$\sum_{\mathbf{m}'=\underline{\mathbf{M}}}^{\overline{\mathbf{M}}} \sum_{j'=\underline{J}}^{\overline{J}} p^4(1+2^{m_1'-\overline{M}_1})(1+2^{m_2'-\overline{M}_2})(1+2^{m_3'-\overline{M}_3})(1+2^{j'-\overline{J}})$ .

So for this column the ratio in between the non negligible elements and the vector dimension is approximatively $3^4 10^4/1.7 \times 10^{10} \approx 5 \times 10^{-5}$.

# 6 Discussion and conclusion

Here we want to address some open problems and perspectives. The choice of $p$, which affects the support of the Daubechies-p wavelet and its regularity , has to be tuned. From theorem 3.1 hypothesis, the number of matrix elements

$\left(w_{\mathbf{m}'\mathbf{n}'}w_{j'k'}, K_{01}^*K_{01} + K_{11}^*K_{11}w_{\mathbf{mn}}w_{jk}\right)$ such that $|\mathbf{x}_{\mathbf{m}'\mathbf{n}'} - \mathbf{x}_{\mathbf{mn}}| < c(t_{j'k'} - t_{jk} - (2^{-j} + 2^{-j'})p + 2^{-j'}) - d_{\mathbf{m}} - d_{\mathbf{m}'}$, which are equal to zero, decreases with the increase of $p$ as can be checked from the inequality. Similarly from the hypothesis of theorem 4.2 , the number of matrix elements of $K_{01}^*K_{01} + K_{11}^*K_{11}$ such that $|\mathbf{x}_{\mathbf{m}'\mathbf{n}'} - \mathbf{x}_{\mathbf{mn}}| > c(t_{j'k'} - t_{jk} + (2^{-j} + 2^{-j'})p - 2^{-j}) + d_{\mathbf{m}} + d_{\mathbf{m}'}$ decreases with the increase of $p$ but some of them become much smaller since of order $\frac{1}{8((p+1))!}2^{-(p+1)\mathbf{m}/2}2^{-(p+1)\mathbf{m}'/2}2^{-(p+1)j/2}2^{-(p+1)j'/2}$. Notice also that the number of singularities of $f(\sigma, \mathbf{x}, \mathbf{x}')$, inside intervals whose length depends on $p$, changes. All that affects the number of matrix elements which will be neglected.

To solve numerically the $(1-K)^*(1-K)d = (1-K)^*d_{in}$ system, one has first to calculate the non negligible matrix elements. Notice that for the $K^*K$ matrix elements, using spheroidal coordinates the involved integrals do not have strong singularities. Instead in the appendix-Diagonal terms- we establish formulas, using Fourier transform which involve Cauchy principal values. In fact it is a major benefit since the time used calculating numerically the matrix elements becomes small .

Solving the system, even if the sparsity is important, will need clever algorithms and powerful computers.

Finally we conjecture that the irregularities of the house fronts, windows, balconies...greatly affect the signals, their presence modifies $f(\sigma, \mathbf{x}, \mathbf{x})$ and increases the number of non negligible elements of the matrix $(1-K)^*(1-K)$. To study better this point we could start with a simple rectangular wall $20m$ wide, $10m$ high, $0.4m$ thick and after we would introduce several windows of size $1m \times 2m$ distributed regularly. The aim would be to compare the resulting densities for the two cases. We could also determine the behaviour of the signals at the user antenna and compare them with the signals obtained by ray tracing.

An important question is the stability of the charge and current densities when we increase the dimension of the space generated by a finite set of basis functions. Using the fact that $(1-K)^*(1-K)$ is Hermitian one could control the stability in the following way. Denoting $\mathbf{A}$ the matrix $(1-K)^*(1-K)$ restricted to the finite space spanned by a finite number of basis functions, adding for instance the function $w_{\mathbf{mn}}w_{\overline{J}+1,0}$ to the previous basis, a column and a line are added to matrix $\mathbf{A}$. Then the new matrix is formed by 4 sub-blocks $\mathbf{A}$ for the original matrix, $\mathbf{B}$ for the added column without the last element, $\mathbf{C}$ for the added line without the last element, $\mathbf{D}$ for the added diagonal element. Using the following analytic inversion formula

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} (\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & -(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & \mathbf{D}^{-1}+\mathbf{D}^{-1}\mathbf{C}(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1} \end{bmatrix}.$$

one notices that as $\mathbf{D}$ is large and $\mathbf{B}$ and $\mathbf{C}$ have many zero elements, it appears that $\mathbf{B}\mathbf{D}^{-1}\mathbf{C}$ is small so the block $\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C}$ is close to $\mathbf{A}$. As the component of the "incident densities", $d_{in}$, on the added wavelet is small, then the modified densities are very close to the densities previously calculated since if we compare the first components of the new densities with the components of the old densities, they are close since $\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C}$ is close to $\mathbf{A}$ and the added component of the new densities is small.

# 7 Appendix - The K*K kernel for an union of parallelepipeds

We are going to calculate the $K^*K$ kernel, $f(\mathbf{x},\mathbf{x}',t,t')$ in the case the volume material is the union of parallelepipeds $P_l$ with constant permittivity, $\chi_l$. The $P_l$ parallelepiped center is the point $(c_{l1},c_{l2},c_{l3})$ The parallelepipeds summits coordinates are : $(c_{l1}\pm\frac{d_{l1}}{2},c_{l2}\pm\frac{d_{l2}}{2},c_{l3}\pm\frac{d_{l3}}{2})$. The faces perpendicular to $\mathbf{e}_i$ are denoted $F_{li}^{\pm}$ and the edge at the intersection of the faces $F_{li}^{s}$, $F_{lj}^{s'}$ are denoted $E_{lij}^{ss'}$.

We are going to evaluate $f(\sigma,\mathbf{x},\mathbf{x}')$ using formula (3.9) and the fact that $\chi(\mathbf{u})=\sum_l \chi_l \mathbb{I}_{P_l}(\mathbf{u})$ .

To evaluate the integrals we need to see how the hyperboloid $H_\sigma$ intersects the parallelepiped, $P_l$. The intersection may be simple if it intersects only one face in which case the curve is an ellipse or it may be a close curve made of several pieces of ellipses if the hyperboloid intersects the edges of the parallelepiped . To examine the different cases we need to establish the equations for the planes and the edges in the spheroidal coordinates.

Now we are going to study the intersection of $H_\sigma$ and the planes $P_1^-$, $P_1^+$, $P_2^-$, $P_2^+$, $P_3^-$, $P_3^+$ containing resp.the parallelepiped faces $F_1^-$, $F_1^+$, $F_2^-$, $F_2^+$, $F_3^-$, $F_3^+$. They are ellipses denoted resp.$E_1$, $E_1^+$, $E_2$, $E_2^+$, $E_3$, $E_3^+$. Each of these ellipses may be totally inside the face or partially inside or totally outside the face. First we have to write the equation of the planes in the new coordinates. The equations for the line which contains $\mathbf{x}$ and $\mathbf{x}'$ is

$$\frac{y_1-x_1}{x_1'-x_1}=\frac{y_2-x_2}{x_2'-x_2}=\frac{y_3-x_3}{x_3'-x_3}. \tag{7.1}$$

The intersection of this line (or $\mathbf{e}'_3$ axis ) with the plane $P_1^-$ is obtained in introducing $y_1=c_1-\frac{d_1}{2}$ in the previous equation, so the coordinates of the intersection point in the old basis are $y_1=c_1-\frac{d_1}{2}$, $y_2=x_2-\frac{x_2'-x_2}{x_1'-x_1}(c_1-\frac{d_1}{2}-x_1)$, $y_3=x_3-\frac{x_3'-x_3}{x_1'-x_1}(c_1-\frac{d_1}{2}-x_1)$. The distance from $(\mathbf{x}'+\mathbf{x})/2$ to this point , denoted $s_1^-$ is given by

$$s_1^-=\sqrt{\Big(c_1-\frac{d_1}{2}-\frac{x_1+x_1'}{2}\Big)^2+\Big(x_2-\frac{x_2'-x_2}{x_1'-x_1}(c_1-\frac{d_1}{2}-x_1)-\frac{x_2'+x_2}{2}\Big)^2+\Big(x_3-\frac{x_3'-x_3}{x_1'-x_1}(c_1-\frac{d_1}{2}-x_1)-\frac{x_3+x} \tag{7.2}$$

.

Notice that the intersection of the line with the plane $P_1^-$ exits if $x_1'\neq x_1$.

The normals to the planes $P_1^-$, $P_1^+$, $P_2^-$, $P_2^+$, $P_3^-$, $P_3^+$ are in the original basis resp . $-\mathbf{e}_1,\mathbf{e}_1,-\mathbf{e}_2,\mathbf{e}_2,-\mathbf{e}_3,\mathbf{e}_3$ in the $\mathbf{e}'_1,\mathbf{e}'_2,\mathbf{e}'_3$ basis .

The equation of the plane $P_1^-$ is obtained writing that if $M=(u_1',u_2',u_3')\in P_1^-$, $O'M.\mathbf{e}_1=s_1^-\mathbf{e}'_3.\mathbf{e}_1$ or

$$(u_1'\mathbf{e}'_1+u_2'\mathbf{e}'_2+u_3'\mathbf{e}'_3).\mathbf{e}_1=s_1^-\mathbf{e}'_3.\mathbf{e}_1 \tag{7.3}$$

.

The scalar product $\mathbf{e}'_1.\mathbf{e}_1$ will be denoted $\alpha_1$ , $\mathbf{e}'_2.\mathbf{e}_1$ will be denoted $\beta_1$ , $\mathbf{e}'_3.\mathbf{e}_1$ will be denoted $\gamma_1$ they are calculated from equation (4.6).

In spheroidal coordinates the plane equation becomes

$$\alpha_1\Big(a\sqrt{(\tau^2-1)(1-\sigma^2)}\cos\phi\Big)+\beta_1\Big(a\sqrt{(\tau^2-1)(1-\sigma^2)}\sin\phi\Big)+\gamma_1\Big(a\tau\sigma\Big)=\gamma s_1^- \tag{7.4}$$

.

The intersection of the hyperboloid $H_\sigma$ and the plane of constant $\phi$ is a parabola. This parabola intersects a face plane, for instance $P_1^-$, at a point $(\sigma,\phi,\tau_1^-(\sigma,\phi))$ . $\tau_1^-(\sigma,\phi)$ using (4.19) is given by

$$\tau_1^-(\sigma,\phi)=\frac{a\gamma_i^2 s_i^-\sigma+\sqrt{a^2\gamma_i^4 s_i^-\sigma^2+a^2(\sin(\phi-\phi_0)^2(1-\sigma^2)-\gamma_i^2\sigma^2)(a^2\sin(\phi-\phi_0)^2(1-\sigma^2)-\gamma_i^2 s_i^-)}}{a^2(\sin(\phi-\phi_0)^2(1-\sigma^2)-\gamma_i^2\sigma^2)} \tag{7.5}$$

In (4.12) to determine the bounds for the integral with respect to $\phi$ we need to calculate the coordinates of the intersection points of the parallelepiped edges with the hyperboloid of constant $\sigma$. Each edge intersects $H_\sigma$ at 0 or 1 or 2 points. The edge which is adjacent to the faces $F_1^-$ and $F_2^+$ is denoted $E_{12}^{-+}$. $E_{12}^{-+}$ goes from the point points $A=(c_1-d_1/2,c_2+d_2/2,c_3-d_3/2)$ to the point $B=(c_1-d_1/2,c_2+d_2/2,c_3+d_3/2)$ .In the new basis $(\mathbf{e}'_1,\mathbf{e}'_2,\mathbf{e}'_3)$ the coordinates of these points are

$$\begin{aligned}
u'_{A1}&=(c_1-d_1/2-\frac{x_1+x'_1}{2})(\mathbf{e}'_1.\mathbf{e}_1)+(c_2+d_2/2-\frac{x_2+x'_2}{2})(\mathbf{e}'_2.\mathbf{e}_1)+(c_3-d_3/2-\frac{x_3+x'_3}{2})(\mathbf{e}'_3.\mathbf{e}_1)\\
u'_{A2}&=(c_1-d_1/2-\frac{x_1+x'_1}{2})(\mathbf{e}'_1.\mathbf{e}_2)+(c_2+d_2/2-\frac{x_2+x'_2}{2})(\mathbf{e}'_2.\mathbf{e}_2)+(c_3-d_3/2-\frac{x_3+x'_3}{2})(\mathbf{e}'_3.\mathbf{e}_2)\\
u'_{A3}&=(c_1-d_1/2-\frac{x_1+x'_1}{2})(\mathbf{e}'_1.\mathbf{e}_3)+(c_2+d_2/2-\frac{x_2+x'_2}{2})(\mathbf{e}'_2.\mathbf{e}_3)+(c_3-d_3/2-\frac{x_3+x'_3}{2})(\mathbf{e}'_3.\mathbf{e}_3)\\
u'_{B1}&=(c_1-d_1/2-\frac{x_1+x'_1}{2})(\mathbf{e}'_1.\mathbf{e}_1)+(c_2+d_2/2-\frac{x_2+x'_2}{2})(\mathbf{e}'_2.\mathbf{e}_1)+(c_2+d_2/2-\frac{x_3+x'_3}{2})(\mathbf{e}'_3.\mathbf{e}_1)\\
u'_{B2}&=(c_1-d_1/2-\frac{x_1+x'_1}{2})(\mathbf{e}'_1.\mathbf{e}_2)+(c_2+d_2/2-\frac{x_2+x'_2}{2})(\mathbf{e}'_2.\mathbf{e}_2)+(c_2+d_2/2-\frac{x_3+x'_3}{2})(\mathbf{e}'_3.\mathbf{e}_2)\\
u'_{B3}&=(c_1-d_1/2-\frac{x_1+x'_1}{2})(\mathbf{e}'_1.\mathbf{e}_3)+(c_2+d_2/2-\frac{x_2+x'_2}{2})(\mathbf{e}'_2.\mathbf{e}_3)+(c_2+d_2/2-\frac{x_3+x'_3}{2})(\mathbf{e}'_3.\mathbf{e}_3)
\end{aligned} \tag{7.6}$$

The points of the edge line satisfy the equations

$$\frac{u'_1-u'_{B1}}{u'_{A1}-u'_{B1}}=\frac{u'_2-u'_{B2}}{u'_{A2}-u'_{B2}}=\frac{u'_3-u'_{B3}}{u'_{A3}-u'_{B3}}. \tag{7.7}$$

.

or the equations

$$\begin{aligned}
u'_1&=u'_{B1}+\frac{(u'_{A1}-u'_{B1})(u'_3-u'_{B3})}{u'_{A3}-u'_{B3}}\\
u'_2&=u'_{B2}+\frac{(u'_{A2}-u'_{B2})(u'_3-u'_{B3})}{u'_{A3}-u'_{B3}}
\end{aligned} \tag{7.8}$$

.

A point of $H_\sigma$ has coordinates

$$\begin{aligned}
u'_1&=a\sqrt{(\tau^2-1)(1-\sigma^2)}\cos\phi\\
u'_2&=a\sqrt{(\tau^2-1)(1-\sigma^2)}\sin\phi\\
u'_3&=a\tau\sigma
\end{aligned} \tag{7.9}$$

Introducing these values for $u'_1, u'_2, u'_3$ in the line equations we get

$$u_1^{'2} + u_2^{'2} = a^2(\tau^2-1)(1-\sigma^2) = \Big(u'_{B1} + \frac{(u'_{A1}-u'_{B1})(a\tau\sigma - u'_{B3})}{u'_{A3}-u'_{B3}}\Big)^2 + \Big(u'_{B2} + \frac{(u'_{A2}-u'_{B2})(a\tau\sigma - u'_{B3})}{u'_{A3}-u'_{B3}}\Big)^2 \tag{7.10}$$

For fixed $\sigma$ from this equation of degree 2 has two solutions denoted $\tau_{12}^{-+}(\sigma)$ and $\tau_{12}^{'-+}(\sigma)$. For each of these values for $\tau$ one can extract the value for $\phi$,

$$\phi_{12}^{-+}(\sigma) = \arccos\Big(\frac{u'_{B1} + \frac{(u'_{A1}-u'_{B1})(a\tau_{12}^{-+}(\sigma)\sigma - u'_{B3})}{u'_{A3}-u'_{B3}}}{a\sqrt{(\tau_{12}^{-+}(\sigma)^2-1)(1-\sigma^2)}}\Big)$$
$$\phi_{12}^{'-+}(\sigma) = \arccos\Big(\frac{u'_{B1} + \frac{(u'_{A1}-u'_{B1})(a\tau_{12}^{'-+}(\sigma)\sigma - u'_{B3})}{u'_{A3}-u'_{B3}}}{a\sqrt{(\tau_{12}^{'-+}(\sigma)^2-1)(1-\sigma^2)}}\Big) \tag{7.11}$$

It remains to verify if the obtained points are inside or outside the edge. Doing the same calculus for all the edges, finally we obtain a set of cardinality smaller than 24 ( at most 2 values for $\phi$ by edge),$(\phi_{12}^{-+}(\sigma), \phi_{12}^{'-+}(\sigma), ...)$ that we are going to order . For instance one can obtain the following ordered set $(\phi_{31}^{-+} < \phi_{12}^{++} < \phi_{13}^{--} < \phi_{21}^{+-})$

Then the contribution of the parallelepiped $\mathcal{P}_l$ to $f(\sigma, \mathbf{x}, \mathbf{x}')$ in the case of the ordered set $(\phi_{31}^{-+} < \phi_{12}^{++} < \phi_{13}^{--} < \phi_{21}^{+-})$, by inspection of the successive involved faces, can be written as

$$\begin{aligned} f_l(\sigma, \mathbf{x}, \mathbf{x}') &= \chi_l^2\Big(\int_{\phi_{31}^{-+}(\sigma)}^{\phi_{12}^{++}(\sigma)} d\phi \int_{\tau_3^-(\sigma,\phi)}^{\tau_1^+(\sigma,\phi)} d\tau + \int_{\phi_{12}^{++}(\sigma)}^{\phi_{12}^{--}(\sigma)} d\phi \int_{\tau_3^-(\sigma,\phi)}^{\tau_2^+(\sigma,\phi)} d\tau + \int_{\phi_{12}^{--}(\sigma)}^{\phi_{21}^{+-}(\sigma)} d\phi \int_{\tau_1^-(\sigma,\phi)}^{\tau_2^+(\sigma,\phi)} d\tau\Big) \\ &= \chi_l^2\Big(\int_{\phi_{31}^{-+}(\sigma)}^{\phi_{12}^{++}(\sigma)} d\phi(\tau_2^+(\sigma,\phi) - \tau_3^-(\sigma,\phi)) + \int_{\phi_{12}^{++}(\sigma)}^{\phi_{12}^{--}(\sigma)} d\phi(\tau_2^+(\sigma,\phi) - \tau_3^-(\sigma,\phi)) \\ &\quad + \int_{\phi_{12}^{--}(\sigma)}^{\phi_{21}^{+-}(\sigma)} d\phi(\tau_2^+(\sigma,\phi) - \tau_3^-(\sigma,\phi))\Big) \end{aligned} \tag{7.12}$$

.

Using Mathematica it appears that the integrals have analytic expressions which include the variables coming from the parallelepiped $\mathcal{P}_l$. So we can obtain an analytic expression for $f(\sigma, \mathbf{x}, \mathbf{x}') = \sum_l f_l(\sigma, \mathbf{x}, \mathbf{x}')$.

It is clear that the kernel is very regular as long as the hyperboloid $H_\sigma$ do not hit the parallelepiped corners. Denoting $\mathbf{c}_{li}$ the $i^{th}$ corner of parallelepiped $\mathcal{P}_l$ , once $\mathbf{x_{mn}}$, $t_{jk}$, $t_{j'k'}$ are fixed , the singularity appears when $|\mathbf{c}_{li} - \mathbf{x_{m'n'}}| - |\mathbf{c}_{li} - \mathbf{x_{mn}}| = c(t_{j'k'} - t_{jk})$ i.e for the points $\mathbf{x_{m'n'}}$ which are on the circle of radius $|\mathbf{c}_{li} - \mathbf{x_{mn}}| + c(t_{j'k'} - t_{jk})$ centred at $\mathbf{c}_{li}$. Then the corresponding matrix elements are of order $(2^{-j} + 2^{-j'})$.

In Figure 1 the red scars correspond to the singularities of $f$. . Notice that they disappeared in the lower figure as for large time the intersection in the outer region of the crown with the spheres centred at the corners is void.

# 8 Appendix - Study for $K + K^*$ -

**Proposition 8.1.** *If the volume material volume is a polyhedron in which the permittivity is infinitely differentiable and if $|\mathbf{x_{m'n'}} - \mathbf{x_{mn}}| > d_{\mathbf{m}} + d_{\mathbf{m'}}$ (H5), if the regularity of the wavelets is sufficiently high ,( for the Daubechies-p wavelets , p has to be sufficiently large) the matrix element satisfies, for some constant C , $\Big(w_{\mathbf{m'n'}} w_{j'k'}, K^*_{ii} + K_{ii} w_{\mathbf{mn}} w_{jk}\Big) < \frac{C}{(p+1)!} 2^{-(p+1)\mathbf{m}/2} 2^{-(p+1)\mathbf{m'}/2}$*

*Proof.* From (3.3) the matrix element of $K_{ii}$ is

$$
\begin{aligned}
&(w_{\mathbf{m'n'}} w_{j'k'}, K_{ii} w_{\mathbf{mn}} w_{jk}) \\
&= \frac{1}{c^2} \int d\mathbf{x'} \quad w_{\mathbf{m'n'}}(\mathbf{x'}) \chi(\mathbf{x'}) \int d\mathbf{x} \quad w_{\mathbf{mn}}(\mathbf{x}) \int_{-\infty}^{\infty} dt' w_{j'k'}(t') \int_{-\infty}^{\infty} dt \frac{d^2 w_{jk}}{dt^2}(t) G(\mathbf{x'}, t', \mathbf{x}, t) \\
&= \frac{1}{4\pi c^2} \int d\mathbf{x'} \quad w_{\mathbf{m'n'}}(\mathbf{x'}) \chi(\mathbf{x'}) \int d\mathbf{x} \quad w_{\mathbf{mn}}(\mathbf{x}) \frac{1}{|\mathbf{x'} - \mathbf{x}|} \int_{-\infty}^{\infty} dt' w_{j'k'}(t') \frac{d^2 w_{jk}}{dt^2}(t' - \frac{|\mathbf{x'} - \mathbf{x}|}{c}) \\
&= \frac{1}{4\pi c^2} \int d\mathbf{x'_0} \quad w_{\mathbf{m'0}}(\mathbf{x'_0}) \chi(\mathbf{x_{m'n'}} + \mathbf{x'_0}) \int d\mathbf{x_0} \quad w_{\mathbf{m0}}(\mathbf{x_0}) \frac{1}{|\mathbf{x_{m'n'}} - \mathbf{x_{mn}} + \mathbf{x'_0} - \mathbf{x_0}|} \\
&\qquad \times \int_{-\infty}^{\infty} dt' w_{j'0}(t') \frac{d^2 w_{j0}}{dt^2} \Big(t' + t_{j'k'} - t_{jk} - \frac{|\mathbf{x_{m'n'}} - \mathbf{x_{mn}} + \mathbf{x'_0} - \mathbf{x_0}|}{c}\Big)
\end{aligned}
\tag{8.1}
$$

Denoting $W^{(2)}_{jj'}(\tau) = \int_{-\infty}^{\infty} dt' w_{j'0}(t') \frac{d^2 w_{j0}}{dt^2}(t' + \tau)$

$$
\begin{aligned}
&(w_{\mathbf{m'n'}} w_{j'k'}, K_{ii} w_{\mathbf{mn}} w_{jk}) \\
&= \frac{1}{4\pi c^2} \int d\mathbf{x'_0} w_{\mathbf{m'0}}(\mathbf{x'_0}) \chi(\mathbf{x_{m'n'}} + \mathbf{x'_0}) \int d\mathbf{x_0} w_{\mathbf{m0}}(\mathbf{x_0}) \frac{1}{|\mathbf{x_{m'n'}} - \mathbf{x_{mn}} + \mathbf{x'_0} - \mathbf{x_0}|} \\
&\qquad \times W^{(2)}_{jj'} \Big(t_{j'k'} - t_{jk} - \frac{|\mathbf{x_{m'n'}} - \mathbf{x_{mn}} + \mathbf{x'_0} - \mathbf{x_0}|}{c}\Big)
\end{aligned}
\tag{8.2}
$$

Noticing that $W^{(2)}_{jj'}(\tau)$ has its support in the interval $\Big(-(2^{-j} + 2^{-j'})(p-1), (2^{-j} + 2^{-j'})p\Big)$ the matrix element is equal to zero if $|\mathbf{x_{m'n'}} - \mathbf{x_{mn}}| \geq c|t_{j'k'} - t_{jk}| + d_{\mathbf{m}} + d_{\mathbf{m'}} + (2^{-j} + 2^{-j'})(p-1)$ or if $|\mathbf{x_{m'n'}} - \mathbf{x_{mn}}| \leq c|t_{j'k'} - t_{jk}| - d_{\mathbf{m}} - d_{\mathbf{m'}} - (2^{-j} + 2^{-j'})p$.

If $|\mathbf{x_{m'n'}} - \mathbf{x_{mn}}| > d_{\mathbf{m}} + d_{\mathbf{m'}}$, $\frac{1}{|\mathbf{x_{m'n'}} - \mathbf{x_{mn}} + \mathbf{x'_0} - \mathbf{x_0}|}$

$W^{(2)}_{jj'} \Big(t_{j'k'} - t_{jk} - \frac{|\mathbf{x_{m'n'}} - \mathbf{x_{mn}} + \mathbf{x'_0} - \mathbf{x_0}|}{c}\Big)$ are regular when $\mathbf{x_0}$, $\mathbf{x'_0}$, belong resp.to the supports of $w_{\mathbf{m0}}$ and $w_{\mathbf{m'0}}$. In this case the matrix element $(w_{\mathbf{m'n'}} w_{j'k'}, K_{ii} w_{\mathbf{mn}} w_{jk})$ is of order $\frac{1}{((p+1))!} 2^{-(p+1)\mathbf{m}/2} 2^{-(p+1)\mathbf{m'}/2}$. QED

In conclusion, the non zero matrix elements are negligible except those for which $|\mathbf{x_{m'n'}} - \mathbf{x_{mn}}| < d_{\mathbf{m}} + d_{\mathbf{m'}}$ and $c|t_{j'k'} - t_{jk}| < 2(d_{\mathbf{m}} + d_{\mathbf{m'}}) + (2^{-j} + 2^{-j'})p$.

# 9 Appendix -Diagonal matrix elements using Fourier transform-

The diagonal elements of the matrix $K^*_{0i} K_{0i} + K^*_{ii} K_{ii}$ for i=1,2,3, can be evaluated using the spheroidal coordinates and formula (4.4). In this appendix we will use another way , introducing the Fourier transform of the wavelets and known expressions for the d'Alembert kernel in the wave vector space . The Fourier transform of the wavelets look like smoothed indicator functions (see Wikipedia ) and they can be obtained in Python, Julia or Mathematica libraries.

$\hat{w}_{\mathbf{mn}} := \mathfrak{F} w_{\mathbf{mn}}$, $\hat{w}_{jk} := \mathfrak{F} w_{jk}$ and the permittivity $\hat{\chi} := \mathfrak{F}\chi$. Let us start with

$\Big(w_{\mathbf{mn}} w_{jk}, K^*_{ii} K_{ii} w_{\mathbf{mn}} w_{jk}\Big) = \Big(\mathfrak{F} K_{ii} \mathfrak{F}^{-1} \hat{w}_{\mathbf{mn}} \hat{w}_{jk}, \mathfrak{F} K_{ii} \mathfrak{F}^{-1} \hat{w}_{\mathbf{mn}} \hat{w}_{jk}\Big)$ .

As $\mathfrak{F}K_{ii}\mathfrak{F}^{-1}\hat{w}_{\mathbf{mn}}(\mathbf{p})\hat{w}_{jk}(\omega) = \mathfrak{F}\chi\mathfrak{F}^{-1}\frac{\omega^2}{|\mathbf{p}|^2-\omega^2}\hat{w}_{\mathbf{mn}}(\mathbf{p})\hat{w}_{jk}(\omega)$, then

$$\begin{aligned}
&\Big(w_{\mathbf{mn}}w_{jk}, K_{ii}^*K_{ii}w_{\mathbf{mn}}w_{jk}\Big)\\
&= \int d\mathbf{p}\int d\mathbf{p}'\overline{\hat{\chi}(\mathbf{p}-\mathbf{p}')\hat{w}_{\mathbf{mn}}(\mathbf{p}')}\int d\mathbf{p}"\hat{\chi}(\mathbf{p}-\mathbf{p}")\ \hat{w}_{\mathbf{mn}}(\mathbf{p}")\\
&\quad\times\int d\omega\frac{\omega^2}{|\mathbf{p}'|^2c^2-\omega^2}\frac{\omega^2}{|\mathbf{p}"|^2c^2-\omega^2}\overline{\hat{w}_{jk}(\omega)}\hat{w}_{jk}(\omega)
\end{aligned}\tag{9.1}$$

Writing the decomposition of the rational expression

$$\begin{aligned}
&\frac{\omega^2}{|\mathbf{p}'|^2c^2-(\omega+i0)^2}\frac{\omega^2}{|\mathbf{p}"|^2c^2-(\omega+i0)^2}\\
&1+\frac{|\mathbf{p}'|^3c^3}{2(\omega+i0-|\mathbf{p}'|c)(|\mathbf{p}'|^2c^2-(|\mathbf{p}"|^2c^2)}\\
&-\frac{|\mathbf{p}'|^3c^3}{2(\omega+i0+|\mathbf{p}'|c)(|\mathbf{p}'|^2c^2-|\mathbf{p}"|^2c^2)}\\
&-\frac{|\mathbf{p}"|^3c^3}{2(\omega+i0-|\mathbf{p}"|c)(|\mathbf{p}'|^2c^2-|\mathbf{p}"|^2c^2)}\\
&+\frac{|\mathbf{p}"|^3c^3}{2(\omega+i0+|\mathbf{p}"|c)(|\mathbf{p}'|^2c^2-|\mathbf{p}"|^2c^2)}
\end{aligned}\tag{9.2}$$

Using the Plemelj formula $\lim_{\epsilon\to0^+}\frac{1}{x\pm i\epsilon} = \mp i\pi f(0) + pv\Big(\frac{1}{x}\Big)$ in the integration w.r.to $\omega$ and performing the integration w.r.to $\mathbf{p}$, we obtain

$$\begin{aligned}
&\Big(w_{\mathbf{mn}}w_{jk}, K_{ii}^*K_{ii}w_{\mathbf{mn}}w_{jk}\Big)\\
&= \int d\mathbf{p}'\overline{\hat{w}_{\mathbf{mn}}(\mathbf{p}')}\int d\mathbf{p}"\hat{\chi}^{(2)}(\mathbf{p}'-\mathbf{p}")\ \hat{w}_{\mathbf{mn}}(\mathbf{p}")\\
&+\int d\mathbf{p}'\overline{\hat{w}_{\mathbf{mn}}(\mathbf{p}')}\int d\mathbf{p}"\hat{\chi}^{(2)}(\mathbf{p}'-\mathbf{p}")\ \hat{w}_{\mathbf{mn}}(\mathbf{p}")\frac{|\mathbf{p}'|^3c^3}{2(|\mathbf{p}'|^2c^2-|\mathbf{p}"|^2c^2)}\\
&\quad\times\Big(-i\pi\overline{\hat{w}_{jk}(|\mathbf{p}'|c)}\hat{w}_{jk}(|\mathbf{p}'|c)+pv\int d\omega\frac{1}{\omega-|\mathbf{p}'|c}\overline{\hat{w}_{jk}(\omega)}\hat{w}_{jk}(\omega)\Big)\\
&-\int d\mathbf{p}'\overline{\hat{w}_{\mathbf{mn}}(\mathbf{p}')}\int d\mathbf{p}"\hat{\chi}^{(2)}(\mathbf{p}'-\mathbf{p}")\ \hat{w}_{\mathbf{mn}}(\mathbf{p}")\frac{|\mathbf{p}'|^3c^3}{2(|\mathbf{p}'|^2c^2-|\mathbf{p}"|^2c^2)}\\
&\times\Big(-i\pi\overline{\hat{w}_{jk}(-|\mathbf{p}'|c)}\hat{w}_{jk}(-|\mathbf{p}'|c)+pv\int d\omega\frac{1}{\omega+|\mathbf{p}'|c}\overline{\hat{w}_{jk}(\omega)}\hat{w}_{jk}(\omega)\Big)\\
&-\int d\mathbf{p}'\overline{\hat{w}_{\mathbf{mn}}(\mathbf{p}')}\int d\mathbf{p}"\hat{\chi}^{(2)}(\mathbf{p}'-\mathbf{p}")\ \hat{w}_{\mathbf{mn}}(\mathbf{p}")\frac{|\mathbf{p}"|^3c^3}{2(|\mathbf{p}'|^2c^2-|\mathbf{p}"|^2c^2)}\\
&\quad\times\Big(-i\pi\overline{\hat{w}_{jk}(|\mathbf{p}"|c)}\hat{w}_{jk}(|\mathbf{p}"|c)+pv\int d\omega\frac{1}{\omega-|\mathbf{p}"|c}\overline{\hat{w}_{jk}(\omega)}\hat{w}_{jk}(\omega)\Big)\\
&+\int d\mathbf{p}'\overline{\hat{w}_{\mathbf{mn}}(\mathbf{p}')}\int d\mathbf{p}"\hat{\chi}^{(2)}(\mathbf{p}'-\mathbf{p}")\ \hat{w}_{\mathbf{mn}}(\mathbf{p}")\frac{|\mathbf{p}"|^3c^3}{2(|\mathbf{p}'|^2c^2-|\mathbf{p}"|^2c^2)}\\
&\times\Big(-i\pi\overline{\hat{w}_{jk}(-|\mathbf{p}"|c)}\hat{w}_{jk}(-|\mathbf{p}"|c)+pv\int d\omega\frac{1}{\omega+|\mathbf{p}"|c}\overline{\hat{w}_{jk}(\omega)}\hat{w}_{jk}(\omega)\Big)
\end{aligned}\tag{9.3}$$

.

where $\hat{\chi}^{(2)}(\mathbf{p}'-\mathbf{p}") := \int d\mathbf{p}\overline{\hat{\chi}(\mathbf{p}-\mathbf{p}')}\hat{\chi}(\mathbf{p}-\mathbf{p}")$
Noticing that $|\hat{w}_{jk}(-|\mathbf{p}'|c)| = |\hat{w}_{jk}(|\mathbf{p}'|c)|$ and that

$pv\int d\omega \frac{1}{\omega+|\mathbf{p}'|c}|\hat{w}_{jk}(\omega)|^2 = -pv\int d\omega \frac{1}{\omega-|\mathbf{p}'|c}|\hat{w}_{jk}(\omega)|^2$, one obtains

$$
\begin{aligned}
&\Big(w_{\mathbf{mn}}w_{jk}, K^*_{ii}K_{ii}w_{\mathbf{mn}}w_{jk}\Big)\\
&= \int d\mathbf{p}'\overline{\hat{w}_{\mathbf{mn}}(\mathbf{p}')}\int d\mathbf{p}"\,\hat{\chi}^{(2)}(\mathbf{p}'-\mathbf{p}")\;\hat{w}_{\mathbf{mn}}(\mathbf{p}")\\
&+2\int d\mathbf{p}'\overline{\hat{w}_{\mathbf{mn}}(\mathbf{p}')}\int d\mathbf{p}"\,\hat{\chi}^{(2)}(\mathbf{p}'-\mathbf{p}")\;\hat{w}_{\mathbf{mn}}(\mathbf{p}")\frac{|\mathbf{p}'|^3c^3}{2(|\mathbf{p}'|^2c^2-|\mathbf{p}"|^2c^2)}\\
&\times pv\int d\omega\frac{1}{\omega-|\mathbf{p}'|c}\overline{\hat{w}_{jk}(\omega)}\hat{w}_{jk}(\omega)\Big)\\
&-2\int d\mathbf{p}'\overline{\hat{w}_{\mathbf{mn}}(\mathbf{p}')}\int d\mathbf{p}"\,\hat{\chi}^{(2)}(\mathbf{p}'-\mathbf{p}")\;\hat{w}_{\mathbf{mn}}(\mathbf{p}")\frac{|\mathbf{p}"|^3c^3}{2(|\mathbf{p}'|^2c^2-|\mathbf{p}"|^2c^2)}\\
&\times pv\int d\omega\frac{1}{\omega-|\mathbf{p}"|c}\overline{\hat{w}_{jk}(\omega)}\hat{w}_{jk}(\omega)\Big)\\
&= \int d\mathbf{p}'\overline{\hat{w}_{\mathbf{mn}}(\mathbf{p}')}\int d\mathbf{p}"\,\hat{\chi}^{(2)}(\mathbf{p}'-\mathbf{p}")\;\hat{w}_{\mathbf{mn}}(\mathbf{p}")\\
&+2\int d\mathbf{p}'\overline{\hat{w}_{\mathbf{mn}}(\mathbf{p}')}\int d\mathbf{p}"\,\hat{\chi}^{(2)}(\mathbf{p}'-\mathbf{p}")\;\hat{w}_{\mathbf{mn}}(\mathbf{p}")\frac{|\mathbf{p}'|^3c^3}{2(|\mathbf{p}'|^2c^2-|\mathbf{p}"|^2c^2)}\\
&\times pv\int d\omega\frac{1}{\omega-|\mathbf{p}'|c}|\hat{w}_{jk}(\omega)|^2\Big)\\
&-2\int d\mathbf{p}"\overline{\hat{w}_{\mathbf{mn}}(\mathbf{p}")}\int d\mathbf{p}'\hat{\chi}^{(2)}(\mathbf{p}"-\mathbf{p}')\;\hat{w}_{\mathbf{mn}}(\mathbf{p}')\frac{|\mathbf{p}'|^3c^3}{2(|\mathbf{p}"|^2c^2-|\mathbf{p}'|^2c^2)}\\
&\times pv\int d\omega\frac{1}{\omega-|\mathbf{p}'|c}|\hat{w}_{jk}(\omega)|^2
\end{aligned}
\tag{9.4}
$$

.

Then

$$
\begin{aligned}
&\Big(w_{\mathbf{mn}}w_{jk}, K^*_{ii}K_{ii}w_{\mathbf{mn}}w_{jk}\Big)\\
&= \int d\mathbf{p}'\overline{\hat{w}_{\mathbf{mn}}(\mathbf{p}')}\int d\mathbf{p}"\,\hat{\chi}^{(2)}(\mathbf{p}'-\mathbf{p}")\;\hat{w}_{\mathbf{mn}}(\mathbf{p}")\\
&+c\int\int d\mathbf{p}'d\mathbf{p}"\Big(\overline{\hat{w}_{\mathbf{mn}}(\mathbf{p}')}\hat{\chi}^{(2)}(\mathbf{p}'-\mathbf{p}")\;\hat{w}_{\mathbf{mn}}(\mathbf{p}")-\overline{\hat{w}_{\mathbf{mn}}(\mathbf{p}")}\hat{\chi}^{(2)}(\mathbf{p}"-\mathbf{p}')\;\hat{w}_{\mathbf{mn}}(\mathbf{p}')\Big)\\
&\times\frac{|\mathbf{p}'|^3}{|\mathbf{p}'|^2-|\mathbf{p}"|^2}\;pv\int d\omega\frac{1}{\omega-|\mathbf{p}'|c}|\hat{w}_{jk}(\omega)|^2
\end{aligned}
\tag{9.5}
$$

.

The integrand presents a singularity when $|\mathbf{p}'|=|\mathbf{p}"|$, the integral has to be considered as

$$
\begin{aligned}
&\lim_{\eta\to 0}c\int\int d\mathbf{p}'d\mathbf{p}"\Big(\overline{\hat{w}_{\mathbf{mn}}(\mathbf{p}')}\hat{\chi}^{(2)}(\mathbf{p}'-\mathbf{p}")\;\hat{w}_{\mathbf{mn}}(\mathbf{p}")-\overline{\hat{w}_{\mathbf{mn}}(\mathbf{p}")}\hat{\chi}^{(2)}(\mathbf{p}"-\mathbf{p}')\;\hat{w}_{\mathbf{mn}}(\mathbf{p}')\Big)\\
&\times\frac{|\mathbf{p}'|^3}{|\mathbf{p}'|^2-(|\mathbf{p}"|+i\eta)^2}\;pv\int d\omega\frac{1}{\omega-|\mathbf{p}'|c}|\hat{w}_{jk}(\omega)|^2\\
&= \lim_{\eta\to 0}2ic\int d\mathbf{p}'pv\int d\omega\frac{1}{\omega-|\mathbf{p}'|c}|\hat{w}_{jk}(\omega)|^2\\
&\times\int d\mathbf{p}"\,Im\Big(\overline{\hat{w}_{\mathbf{mn}}(\mathbf{p}')}\hat{\chi}^{(2)}(\mathbf{p}'-\mathbf{p}")\;\hat{w}_{\mathbf{mn}}(\mathbf{p}")\Big)\frac{|\mathbf{p}'|^3}{|\mathbf{p}'|^2-(|\mathbf{p}"|+i\eta)^2}
\end{aligned}
\tag{9.6}
$$

, to which once more the Plemelj formula can be applied .

One notices that the first term is equal to $2^{\mathbf{m}} \int d\mathbf{x}\; w(2^{\mathbf{m}}(\mathbf{x}-\mathbf{x_{mn}})\; \chi^2(\mathbf{x})$

Let us look now at the diagonal term of $K_{0i}^* K_{0i}$ i=1,2,3
$\Big(w_{\mathbf{mn}} w_{jk}, K_{0i}^* K_{0i} w_{\mathbf{mn}} w_{jk}\Big) = \Big(\mathfrak{F} K_{0i} \mathfrak{F}^{-1} \hat{w}_{\mathbf{mn}} \hat{w}_{jk}, \mathfrak{F} K_{0i} \mathfrak{F}^{-1} \hat{w}_{\mathbf{mn}} \hat{w}_{jk}\Big)$ .

As $\mathfrak{F} K_{0i} \mathfrak{F}^{-1} \hat{w}_{\mathbf{mn}}(\mathbf{p}) \hat{w}_{jk}(\omega) = \mathfrak{F} \frac{1}{4\pi c^2} \frac{\frac{\partial \chi}{\partial u_i}}{1+\chi} \mathfrak{F}^{-1} \frac{\omega c^2}{(|\mathbf{p}|+i0)^2 c^2 - \omega^2} \hat{w}_{\mathbf{mn}}(\mathbf{p}) \hat{w}_{jk}(\omega)$, denoting $f_i(\mathbf{u}) := \frac{1}{4\pi} \frac{\frac{\partial \chi}{\partial u_i}}{1+\chi(\mathbf{u})}$ then

$$\begin{aligned}
&\Big(w_{\mathbf{mn}} w_{jk}, K_{0i}^* K_{0i} w_{\mathbf{mn}} w_{jk}\Big) = \\
&\int d\mathbf{p} \int d\omega \int d\mathbf{p}' \overline{\hat{f}_i(\mathbf{p}-\mathbf{p}')} \frac{\omega}{(|\mathbf{p}'|+i0)^2 c^2 - \omega^2} \overline{\hat{w}_{\mathbf{mn}}(\mathbf{p}')}\; \overline{\hat{w}_{jk}(\omega)} \\
&\times \int d\mathbf{p}" \hat{f}_i(\mathbf{p}-\mathbf{p}") \frac{\omega}{(|\mathbf{p}"|+i0)^2 c^2 - \omega^2} \hat{w}_{\mathbf{mn}}(\mathbf{p}") \hat{w}_{jk}(\omega)
\end{aligned} \tag{9.7}$$

Writing the decomposition of the rational expression

$$\begin{aligned}
&\frac{\omega}{|\mathbf{p}'|^2 c^2 - (\omega+i0)^2}\; \frac{\omega}{|\mathbf{p}"|^2 c^2 - (\omega+i0)^2} \\
&= \frac{|\mathbf{p}'| c}{2(\omega+i0-|\mathbf{p}'|c)(|\mathbf{p}'|^2 c^2 - |\mathbf{p}"|^2 c^2)} \\
&- \frac{|\mathbf{p}'| c}{2(\omega+i0+|\mathbf{p}'|c)(|\mathbf{p}'|^2 c^2 - |\mathbf{p}"|^2 c^2)} \\
&- \frac{|\mathbf{p}"| c}{2(\omega+i0-|\mathbf{p}"|c)(|\mathbf{p}'|^2 c^2 - |\mathbf{p}"|^2 c^2)} \\
&+ \frac{|\mathbf{p}"| c}{2(\omega+i0+|\mathbf{p}"|c)(|\mathbf{p}'|^2 c^2 - |\mathbf{p}"|^2 c^2)}
\end{aligned} \tag{9.8}$$

we can as before obtain the matrix elements in terms of principal value integrals in the same manner as in (**??**)

$$\begin{aligned}
&\Big(w_{\mathbf{mn}} w_{jk}, K_{0i}^* K_{0i} w_{\mathbf{m}}\Big) \\
&= \lim_{\eta \to 0} \frac{1}{c} \int \int d\mathbf{p}' d\mathbf{p}" \Big(\overline{\hat{w}_{\mathbf{mn}}(\mathbf{p}')} \hat{f}_i^{(2)}(\mathbf{p}'-\mathbf{p}")\; \hat{w}_{\mathbf{mn}}(\mathbf{p}") - \overline{\hat{w}_{\mathbf{mn}}(\mathbf{p}")} \hat{f}_i^{(2)}(\mathbf{p}"-\mathbf{p}')\; \hat{w}_{\mathbf{mn}}(\mathbf{p}')\Big) \\
&\times \frac{|\mathbf{p}'|}{|\mathbf{p}'|^2 - (|\mathbf{p}"|+i\eta)^2}\; pv \int d\omega \frac{1}{\omega - |\mathbf{p}'|c} |\hat{w}_{jk}(\omega)|^2 \\
&= \lim_{\eta \to 0} \frac{2i}{c} \int d\mathbf{p}' pv \int d\omega \frac{1}{\omega - |\mathbf{p}'|c} |\hat{w}_{jk}(\omega)|^2 \\
&\times \int d\mathbf{p}" Im\Big(\overline{\hat{w}_{\mathbf{mn}}(\mathbf{p}')} \hat{f}_i^{(2)}(\mathbf{p}'-\mathbf{p}")\; \hat{w}_{\mathbf{mn}}(\mathbf{p}")\Big) \frac{|\mathbf{p}'|}{|\mathbf{p}'|^2 - (|\mathbf{p}"|+i\eta)^2}
\end{aligned} \tag{9.9}$$

where $\hat{f}_i^{(2)}(\mathbf{p}'-\mathbf{p}") := \int d\mathbf{p} \overline{\hat{f}_i(\mathbf{p}-\mathbf{p}')} \hat{f}_i(\mathbf{p}-\mathbf{p}")$

The same technic can be used to evaluate the diagonal term of $K_{00}^* K_{00}$.

These evaluations have the interest to include the wavelets Fourier transform which contrarily to the wavelets do not present oscillations but they have the drawback to need Cauchy principal values estimation because of the singularities of the kernel in the Fourier representation.

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