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► **To cite this version:**

Honglu Sun, Maxime Folschette, Morgan Magnin. Limit cycle analysis of a class of hybrid gene regulatory networks. Journées Nationales de l'Informatique Mathématique (JNIM 2022), Mar 2022, Lille, France. hal-03700402

HAL Id: hal-03700402

<https://hal.science/hal-03700402>

Submitted on 29 Jun 2022

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Limit cycle analysis of a class of hybrid gene regulatory networks

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Abstract

- Our objective: Find **limit cycles** and analyze their **stability** in a **hybrid modeling framework** of gene regulatory networks [Cornillon et al. 2016] [Behaegel, Comet, Bernot, et al. 2016] [Behaegel, Comet, and Folschette 2017] which extends René Thomas' discrete modeling.
- Our idea: Extend **Poincaré map** to this hybrid framework.
- Our contribution: We search for potential cycles by **abstracting** the continuous trajectory into a succession of visited **discrete domains**. Then we introduce **transition matrices** and **stable zones** to analyze the concrete continuous dynamics and find the actual cycles that are stable.

Hybrid gene regulatory networks (HGRN)

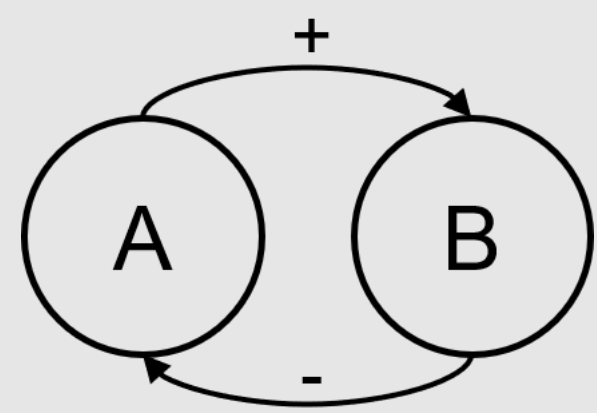


Figure 1. Influence graph of a simple gene regulatory networks in 2 dimensions

- A discrete state d_s is an integer vector which describes the discrete levels of all genes.
- A celerity c_s is a real vector assigned to d_s which describes the temporal derivative of the system in d_s . It is constant in all d_s .
- A (hybrid) state of a HGRN, noted $h = (\pi, d_s)$, contains a fractional part π and a discrete state d_s . The fractional part π is a real vector bounded in $[0, 1]^N$ with N the number of genes in the system.

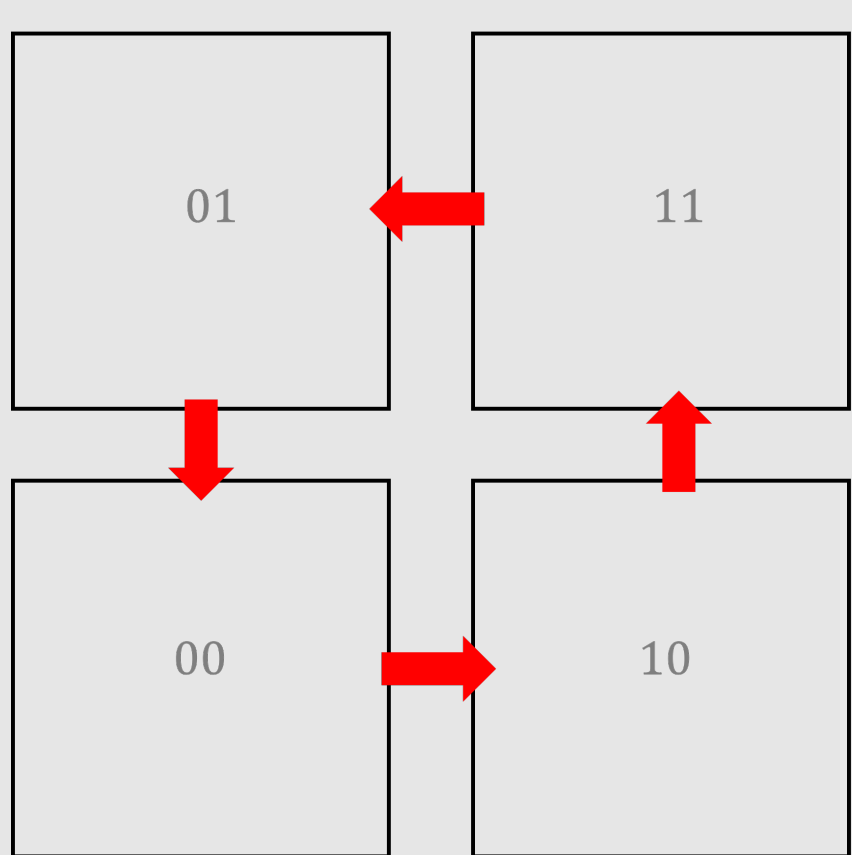


Figure 2. Example of dynamics of a discrete model (depending on discrete parameters)

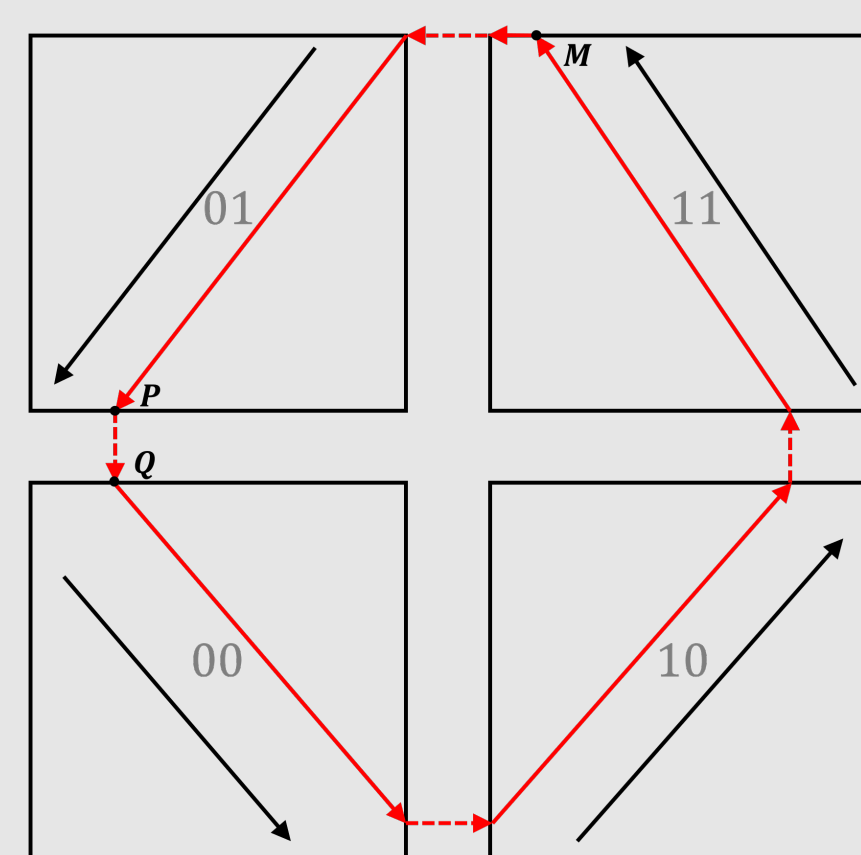


Figure 3. Example of dynamics of a HGRN (depending on real parameters); the black arrows represent the celerities

Compared with discrete models (Figure 2), HGRNs model a continuous dynamics inside each discrete state (Figure 3). [Cornillon et al. 2016]

1. Abstraction of a HGRN into discrete domains

A discrete domain is a set of hybrid states that touch the same boundaries. For example, discrete state $(1, 1)$ has 8 discrete domains (see Figure 4), among which:

- $(1^+, 1^+) = \{(\pi, (1, 1)) \mid \pi^1 = 1, \pi^2 = 1\}$,
- $(1, 1^+) = \{(\pi, (1, 1)) \mid \pi^1 \in]0, 1[, \pi^2 = 1\}$, ...

In the graph of discrete domains (see Figure 5), a discrete domain \mathcal{D}_j is a successor of \mathcal{D}_i if, considering only the signs of celerities, a simple trajectory (straight line) could exist from \mathcal{D}_i to \mathcal{D}_j . We ignore the case of trajectories that reach more than one boundary at the same time.

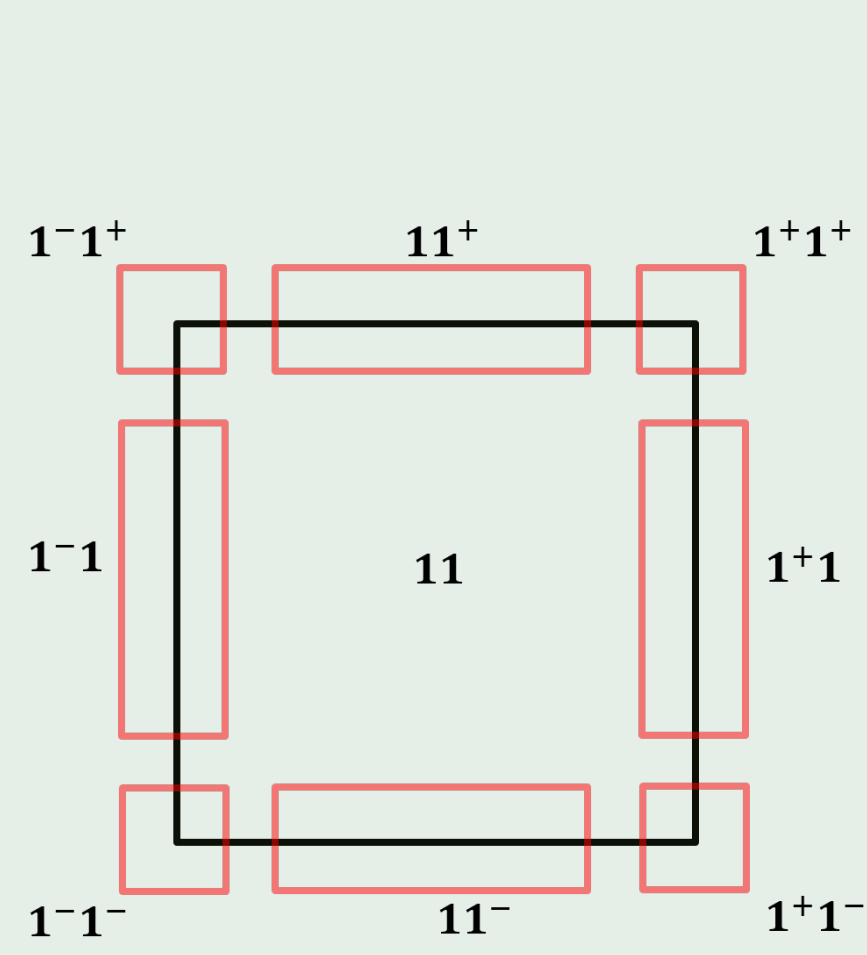


Figure 4. Discrete domains of the discrete state $(1, 1)^T$

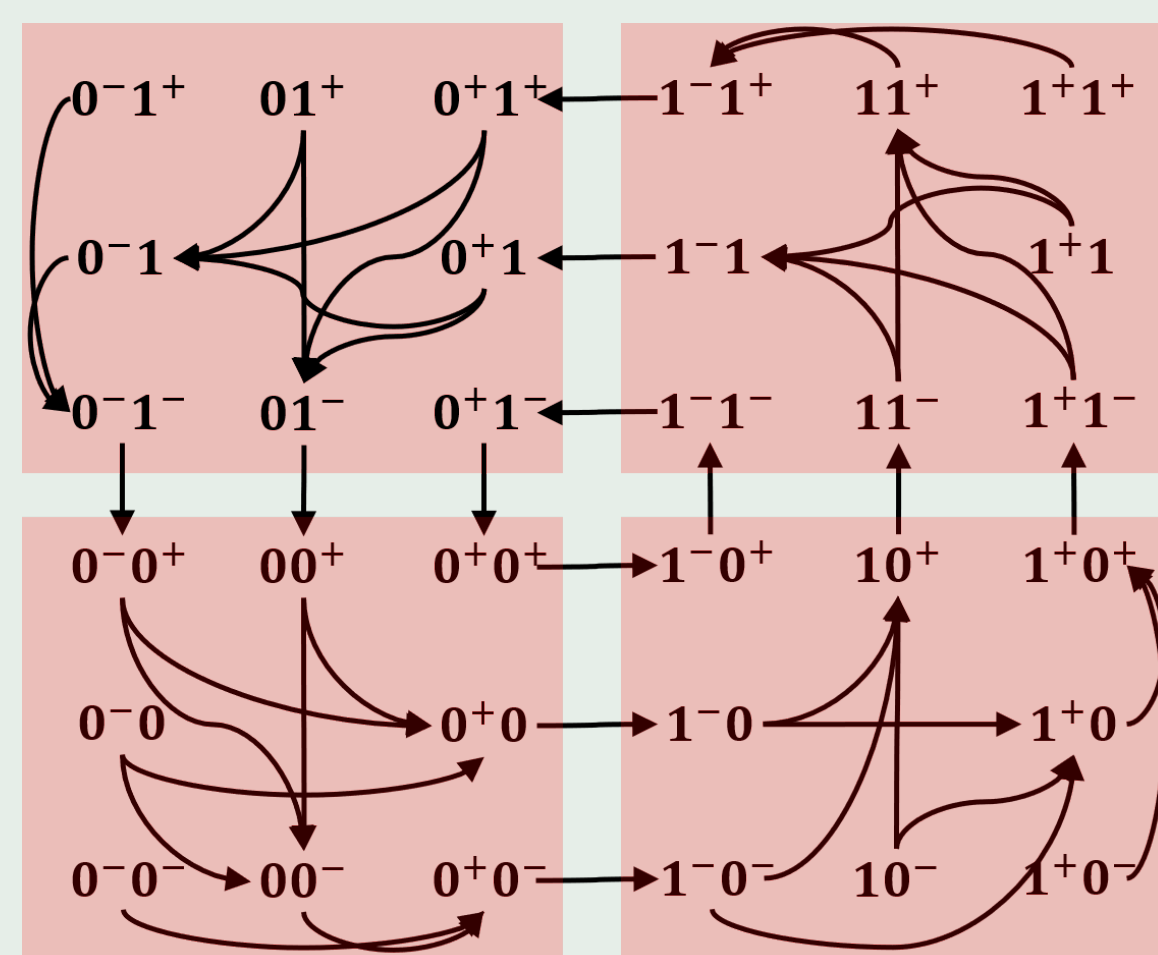


Figure 5. Graph of discrete domains of the HGRN

2. Find closed discrete trajectories

- The stable zone \mathcal{S} of a sequence of discrete domains $\mathcal{T} = (\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_m)$ is the maximal subset of \mathcal{D}_0 such that any trajectory starting from \mathcal{S} stays inside \mathcal{T} (begins from \mathcal{D}_0 and reaches by order all discrete domains of \mathcal{T}).
- If the stable zone \mathcal{S} of \mathcal{T} is not empty, then \mathcal{T} is called a discrete trajectory.
- For a discrete trajectory $\mathcal{T} = (\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_m)$, if $\mathcal{D}_0 = \mathcal{D}_m$, then \mathcal{T} is a closed discrete trajectory.

In order to find limit cycles, we find closed discrete trajectories at first, because a limit cycle must be inside a closed discrete trajectory.

A depth first algorithm is used to find closed discrete trajectories (see Figure 6). The HGRN of Figure 5 has 5 closed discrete trajectories (see Figure 7).

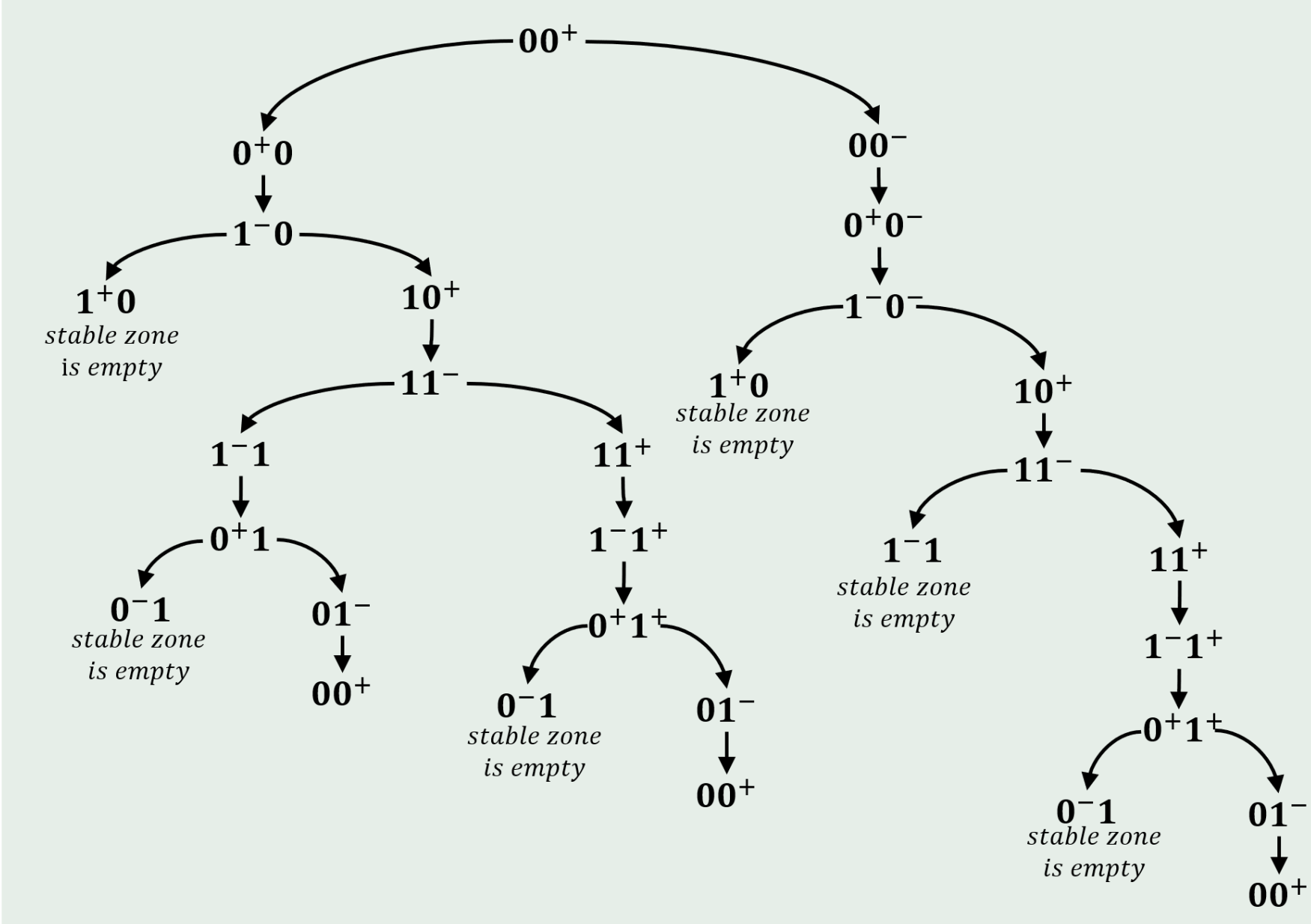


Figure 6. Illustration of the depth first algorithm starting from discrete domain $(0, 0^+)^T$

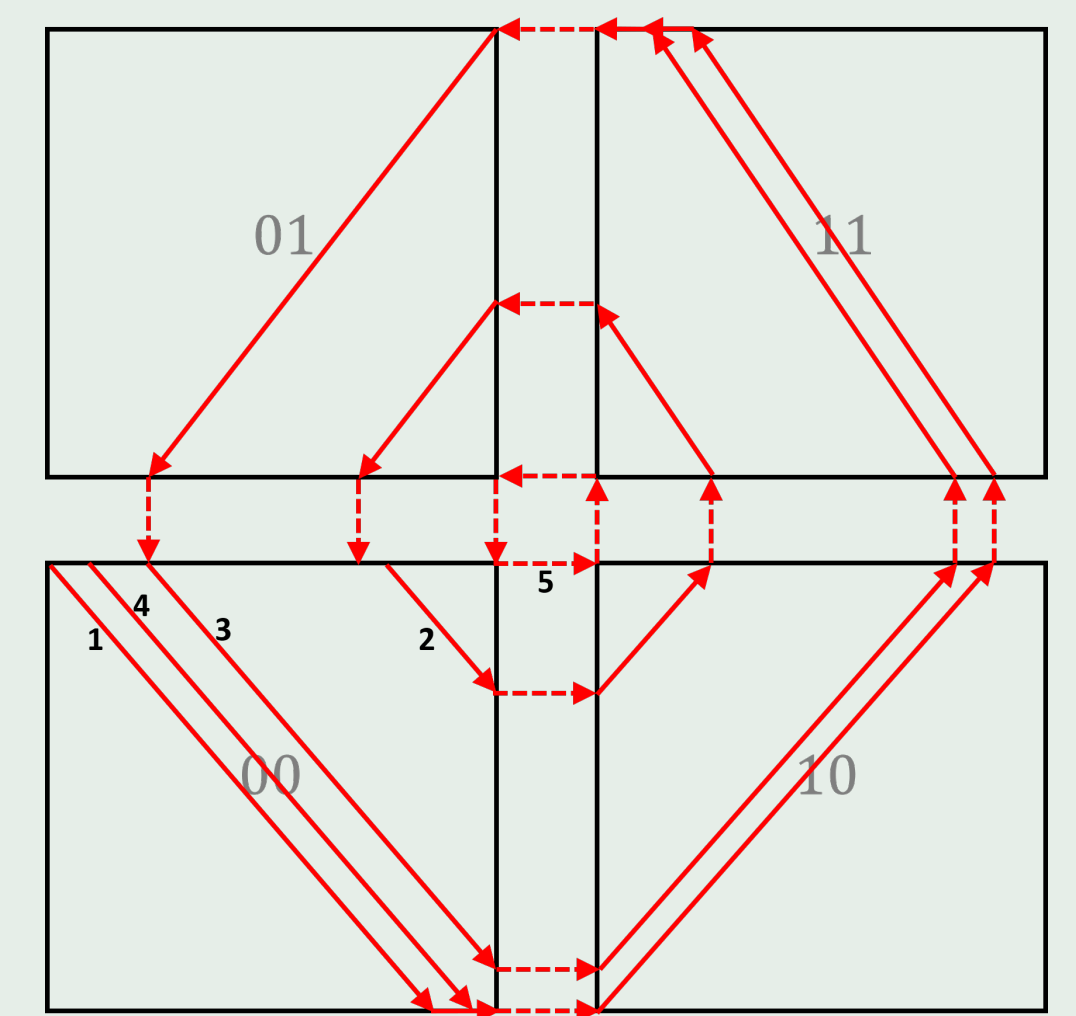


Figure 7. Examples of hybrid trajectories contained inside 5 different discrete trajectories

For example, in Figure 7, trajectory number 3 is contained in sequence: $(0, 0^+) \rightarrow (0^+, 0) \rightarrow (1^-, 0) \rightarrow (1, 0^+) \rightarrow (1, 1^-) \rightarrow (1, 1^+) \rightarrow (1^-, 1^+) \rightarrow (0^+, 1^+) \rightarrow (0, 1^-) \rightarrow (0, 0^+)$, which is also a closed discrete trajectory.

3. Find a limit cycle and analyze its stability

We note $G(\cdot)$ the Poincaré map on the stable zone of \mathcal{T} which is calculated based on the transition matrix. The Poincaré map, which is widely used to study periodic trajectories in nonlinear dynamical systems, is the intersection of periodic trajectories with a certain lower-dimensional subspace, called the Poincaré section.

To verify if there is a **limit cycle** inside a closed discrete trajectory $\mathcal{T} = (\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_m, \mathcal{D}_0)$, we only need to verify:

- $\exists (\pi_0, d_0) \in \mathcal{D}_0$ such that $G(\pi_0) = \pi_0$, and
- (π_0, d_0) belongs to the stable zone of \mathcal{T} .

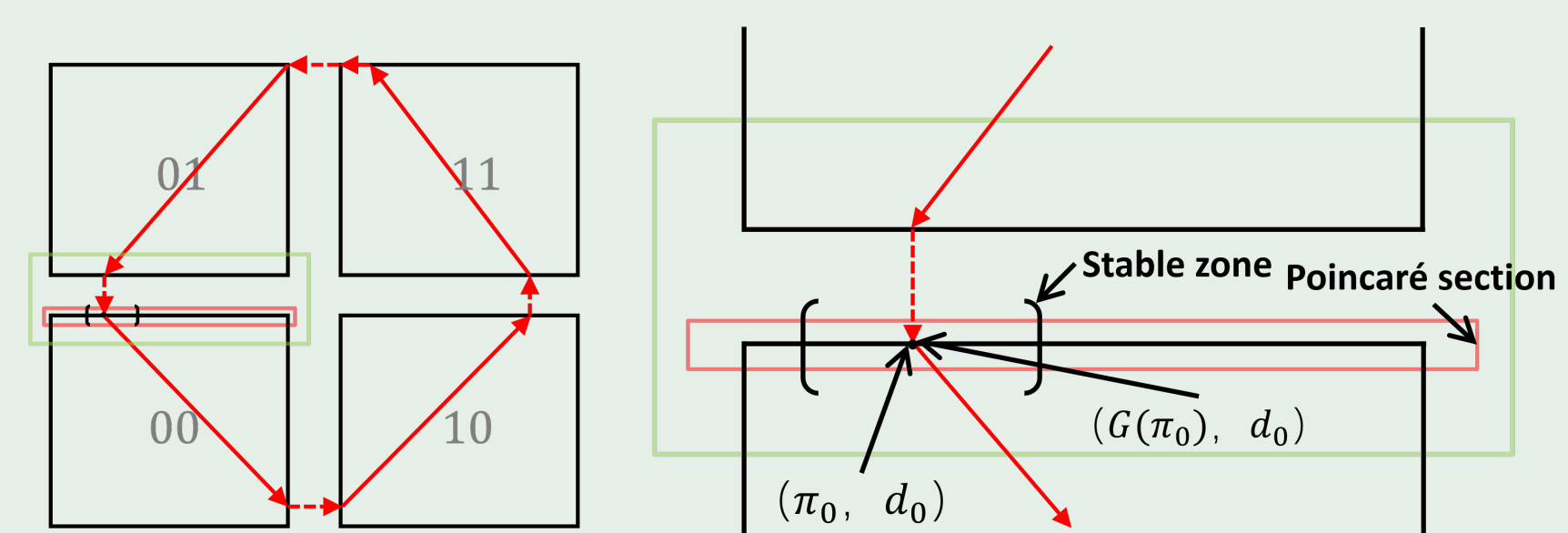


Figure 8. Illustration of the method to find limit cycle

To verify if the limit cycle inside $\mathcal{T} = (\mathcal{D}_0, \dots, \mathcal{D}_m, \mathcal{D}_0)$ that crosses (π_0, d_0) is **stable**, we only need to verify whether $\max_{i \in \{1, 2, \dots, p\}} |\lambda_i| < 1$, where:

- $\lambda_1, \lambda_2, \dots, \lambda_p$ are the eigenvalues of A , and
- $x = Ax + b$ is equivalent to $G(\pi_0) = \pi_0$,

where x is the reduction of π_0 which only contains the dimensions in which the boundaries are not reached and A is the same reduction on the transition matrix.

References

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Application

We applied our method on a HGRN of the cell cycle in 5 dimensions [Behaegel, Comet, Bernot, et al. 2016] and successfully found two limit cycles.

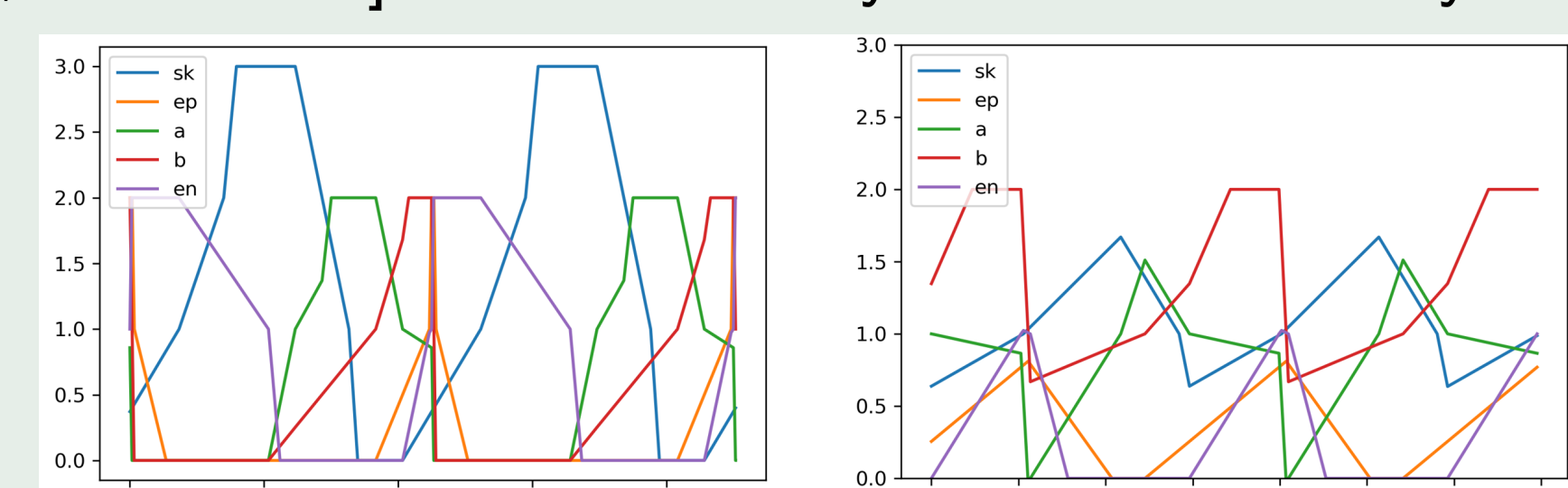


Figure 9. Simulation of limit cycles. Left: stable limit cycle. Right: unstable limit cycle.