Limit theorems for the super-hedging prices in general models with transaction costs

Emmanuel Lépinette, Duc Thinh Vu

Limit theorems for the super-hedging prices in general models with transaction costs

E. Lépinette [1,2] Duc Thinh Vu[1]

[1] *Ceremade, UMR CNRS 7534, Paris Dauphine University, PSL National Research, Place du Maréchal De Lattre De Tassigny, 75775 Paris cedex 16, France. Email: vu@ceremade.dauphine.fr*

[2] *Gosaef, Faculté des sciences de Tunis, Tunis*

Abstract: We propose numerical methods that provide estimations of super-hedging prices of European claims in financial market models with transaction costs. The transaction costs we consider are functions of the traded volumes and prices. Contrarily to the usual models of the literature, the transaction costs are not necessary proportional to the traded volumes, neither convex. The particular case of fixed cost is also considered. Limit theorem are established and allow to numerically compute the infimum super-hedging prices.

Keywords and phrases: Numerical methods in finance, Super-hedging prices, Transaction costs, No-arbitrage condition

Mathematics Subject Classification (2010): 49J53, 60D05, 91G20, 91G80.
JEL Classification: C02, C61, G13.

1. Introduction

Computing the super-hedging prices of a European option in presence of transaction costs is a difficult task. Indeed, the classical results of the literature focus on linear transaction costs and only dual characterizations of the super-hedging prices are formulated, see the FTAP theorems (Fundamental Theorem of Asset Pricing) by [14], [13], [20] among others. These results are formulated under rather strong no-arbitrage conditions (see [15], [18]) and the super-hedging prices are estimated through dual characterizations based on the so-called consistent price systems, see [4], [10].

The interesting question is how to implement the FTAP theorem and deduce numerical estimation of the prices. Few attempts have been achieved in that direction, e.g. [27] in the case of a finite probability space. The general case is difficult as we have first to identify the dual elements, i.e. the consistent price systems, which are martingales evolving in the positive duals of the solvency cones. The second step is to propose a numerical procedure to evaluate the possible super-hedging prices. There is no such a numerical method in the literature. Moreover, if the transaction costs are non linear, there is a priori no dual elements characterizing the no-arbitrage condition.

The methods we develop in this paper are based on the recent paper [22] where the super-hedging prices are characterized for a large class of transaction cost models which are not necessary linear. Indeed, using a new approach based on a weak no-arbitrage condition, mainly the prices of the non negative claims are supposed to be non negative, we prove that the infimum prices of a European claim are solutions to a dynamic programming problem it is possible to solve backwardly, at least in discrete-time. However, in [22], the results are merely theoretical. The authors do not provide algorithms to compute the super-hedging costs in practice. In this paper, we address this problem. To be precise, we consider financial markets with transaction costs defined by a cost process $(\mathrm{C}_t)_{0\le t\le T}$ depending on traded volumes and a process $(S_t)_{0\le t\le T}$ that includes the asset prices. We shall consider the case of countably infinite t-conditional supports for S_{t+1} where an exact characterization of the super-hedging costs is given. The randomized procedure we propose is based on the simulation of conditionally identically distributed random variables which share the same conditional support as the price process $(S_t)_{0\le t\le T}$. We formulate a limit theorem, see Theorem 3.15, that proves the efficiency of our method.

The paper is organized as follows. In Section 2, we recall the market model studied in [22] and the dynamic programming theorem. In Section 3, we describe the numerical scheme and the main convergence theorems. We present in Section 4 the special case of a model with one risky asset and a piecewise cost process $(\mathrm{C}_t)_{0\le t\le T}$. In Section 5, we also give the exact solution of the super-hedging cost in the models with proportional costs and with and without fixed cost. Finally, in Section 6, we prove a limit theorem for a sequence of financial markets defined by convex cost processes.

2. The model

Let $\xi \in L^0(\mathbf{R}^d, \mathcal{F}_T)$ be a contingent claim. Our goal is to characterize the set of all self-financing portfolio processes $(V_t)_{t=-1}^T$ such that $V_T = \xi$. Recall that a portfolio process is by definition (see [22]) a stochastic process $(V_t)_{t=-1}^T$ where $V_{-1} \in \mathbf{R}e_1$ is the initial endowment expressed in cash that we may convert immediately into $V_0 \in \mathbf{R}^d$ at time $t = 0$. By definition, we suppose that

$$\Delta V_t = V_t - V_{t-1} \in -\mathbf{G}_t, \ a.s., \quad t = 0, \cdots, T,$$

where, for every $t \leq T$, $\mathbf{G}_t$ is the random set of all solvent positions.

Our general model is defined by a set-valued process $(\mathbf{G}_t)_{t=0}^T$ adapted to the filtration $(\mathcal{F}_t)_{t=0}^T$. Precisely, we suppose that for all $t \leq T$, $\mathbf{G}_t$ is $\mathcal{F}_t$-measurable in the sense of the graph $\mathrm{Graph}(\mathbf{G}_t) = \{(\omega, x) : \ x \in G_t(\omega)\}$ that belongs to $\mathcal{F}_t \times \mathcal{B}(\mathbf{R}^d)$, where $\mathcal{B}(\mathbf{R}^d)$ is the Borel σ-algebra on $\mathbf{R}^d$ and $d \geq 1$ is the number of assets.

We suppose that $\mathbf{G}_t(\omega)$ is closed for every $\omega \in \Omega$ and $\mathbf{G}_t(\omega) + \mathbf{R}^d_+ \subseteq \mathbf{G}_t(\omega)$, for all $t \leq T$. The cost value process $\mathrm{C} = (\mathrm{C}_t)_{t=0}^T$ associated to $\mathbf{G}$ is defined as:

$$\mathrm{C}_t(z) = \inf\{\alpha \in \mathbf{R} : \ \alpha e_1 - z \in \mathbf{G}_t\} = \min\{\alpha \in \mathbf{R} : \ \alpha e_1 - z \in \mathbf{G}_t\}, \quad z \in \mathbf{R}^d.$$

We suppose that the right hand side in the definition above is non empty a.s. and $-e_1$ does not belong to $\mathbf{G}_t$ a.s. where $e_1 = (1, 0, \cdots, 0) \in \mathbf{R}^d$. Moreover, by assumption, $\mathrm{C}_t(z)e_1 - z \in \mathbf{G}_t$ a.s. for all $z \in \mathbf{R}^d$. Note that $\mathrm{C}_t(z)$ is the minimal amount of cash one needs to get the financial position $z \in \mathbf{R}^d$ at time t. In particular, we suppose that $\mathrm{C}_t(0) = 0$.

If we define the liquidation value process $\mathrm{Ł}_t(z) = -\mathrm{C}_t(-z)$, we get that $\mathbf{G}_t = \{z \in \mathbf{R}^d : \ \mathrm{Ł}_t(z) \geq 0\}$ and, as $\mathbf{G}_t$ is supposed to be closed a.s., $\mathrm{Ł}_t(z)$ is upper semicontinuous (u.s.c.) in z, see [24], or equivalently $\mathrm{C}_t(z)$ is lower semicontinuous (l.s.c.) in z. Naturally, $\mathrm{C}_t(z) = \mathrm{C}_t(S_t, z)$ depends on the available quantities and prices for the risky assets, described by an exogenous vector-valued $\mathcal{F}_t$-measurable random variable S_t of $\mathbf{R}^m_+$, $m \geq d$, and on the quantities $z \in \mathbf{R}^d$ to be traded. Here, we suppose that $m \geq d$ as an asset may be described by several prices and quantities offered by the market, e.g. bid and ask prices, or several pair of bid and ask prices of an order book and the associated quantities offered by the market.

Some examples of models are given in [22]. In the following, we are interested in the infimum cost one needs to super-hedge ξ, i.e. the infimum value of the initial capitals $V_{-1} \in \mathbf{R}$ among the portfolios $(V_t)_{t=-1}^T$ super-replicating ξ.

In the following, we use the notation $z = (z^1, z^2, ..., z^d) \in \mathbf{R}^d$ and we denote $z^{(2)} = (z^2, ..., z^d) \in \mathbf{R}^{d-1}$. Recall that the $\mathcal{F}_t$-measurable conditional essential supremum of a family of random variables is the smallest $\mathcal{F}_t$-measurable random variable that dominates the family with respect to the natural order between $[-\infty, \infty]$-valued random variables, i.e. $X \leq Y$ if $P(X \leq Y) = 1$, see [20, Section 5.3.1].

2.1. The one step hedging problem between time $t-1$ and t

Recall that $V_{t-1} \geq_{\mathbf{G}_t} V_t$ by the definition of portfolio process. Then, the hedging problem $V_t = \xi$ is equivalent at time $t-1$ to:

$$\begin{aligned}
\text{Ł}_t(V_{t-1} - \xi) \geq 0 \quad & \Longleftrightarrow V_{t-1}^1 \geq \xi^1 - \text{Ł}_t((0, V_{t-1}^{(2)})), \\
& \Longleftrightarrow V_{t-1}^1 \geq \text{ess sup}_{\mathcal{F}_{t-1}} \left(\xi^1 - \text{Ł}_t((0, V_{t-1}^{(2)} - \xi^{(2)}))\right), \\
& \Longleftrightarrow V_{t-1}^1 \geq \text{ess sup}_{\mathcal{F}_{t-1}} \left(\xi^1 + \text{C}_t((0, \xi^{(2)} - V_{t-1}^{(2)}))\right), \\
& \Longleftrightarrow V_{t-1}^1 \geq F_{t-1}^{\xi}(V_{t-1}^{(2)}),
\end{aligned}$$

where

$$F_{t-1}^{\xi}(y) \quad = \quad \text{ess sup}_{\mathcal{F}_{t-1}} \left(\xi^1 + \text{C}_t((0, \xi^{(2)} - y))\right). \tag{2.1}$$

By virtue of [22, Proposition 5.7], we may suppose that $F_{t-1}^{\xi}(\omega, y)$ is jointly $\mathcal{F}_{t-1} \times \mathcal{B}(\mathbf{R}^{d-1})$-measurable, a.s. l.s.c. (lower semi-continuous) as a function of y and convex in y if $\text{C}_t(s, y)$ is convex in y. As $\mathcal{F}_{t-1}$ is supposed to be complete, we conclude that $\mathcal{F}_{t-1}^{\xi}$ is an $\mathcal{F}_{t-1}$-normal integrand, see [26][Theorem Corollary 14.34].

2.2. The multi-step hedging problem

We denote by $\mathcal{P}_t(\xi)$ the set of all portfolio processes starting at time $t \leq T$ that replicates ξ at the terminal date T:

$$\mathcal{R}_t(\xi) := \left\{(V_s)_{s=t}^T : \; V_s \in L^0(\mathbf{R}^d, \mathcal{F}_s), \Delta V_s \in -\mathbf{G}_s, \, \forall s \geq t+1, V_T = \xi\right\}.$$

The set of replicating prices of ξ at time t is

$$\mathcal{P}_t(\xi) := \left\{ V_t = (V_t^1, V_t^{(2)}) : (V_s)_{s=t}^T \in \mathcal{R}_t(\xi) \right\}.$$

We define the the infimum replicating cost by:

$$c_t(\xi) := \text{ess inf}_{\mathcal{F}_t} \left\{ \mathrm{C}_t(V_t),\, V_t \in \mathcal{P}_t(\xi) \right\}.$$

By Section 2.1, we know that $V_{T-1} \in \mathcal{P}_{T-1}(\xi)$ if and only if

$$V_{T-1}^1 \geq \text{ess sup}_{\mathcal{F}_{T-1}} \left(\xi^1 + \mathrm{C}_T(0, \xi^{(2)} - V_{T-1}^{(2)}) \right) \text{ a.s.}$$

Similarly, $V_{T-2} \in \mathcal{R}_{T-2}(\xi)$ if and only if there exists $V_{T-1}^{(2)} \in L^0(\mathbf{R}^{d-1}, \mathcal{F}_{T-1})$ such that

$$V_{T-2}^1 \geq \text{ess sup}_{\mathcal{F}_{T-2}} \left(\text{ess sup}_{\mathcal{F}_{T-1}} \left(\xi^1 + \mathrm{C}_T(0, \xi^{(2)} - V_{T-1}^{(2)}) \right) + \mathrm{C}_{T-1}(0, V_{T-1}^{(2)} - V_{T-2}^{(2)}) \right).$$

By the tower property satisfied by the conditional essential supremum, we deduce that $V_{T-2} \in \mathcal{R}_{T-2}(\xi)$ if and only if there is $V_{T-1}^{(2)} \in L^0(\mathbf{R}^{d-1}, \mathcal{F}_{T-1})$ such that

$$V_{T-2}^1 \geq \text{ess sup}_{\mathcal{F}_{T-2}} \left(\xi^1 + \mathrm{C}_T(0, \xi^{(2)} - V_{T-1}^{(2)}) + \mathrm{C}_{T-1}(0, V_{T-1}^{(2)} - V_{T-2}^{(2)}) \right).$$

Recursively, we get that $V_t \in \mathcal{P}_t(\xi)$ if and only if, for some $V_s^{(2)} \in L^0(\mathbf{R}^{d-1}, \mathcal{F}_s)$, $s = t+1, \cdots, T-1$, and with $V_T^{(2)} = \xi^{(2)}$, we have

$$V_t^1 \geq \text{ess sup}_{\mathcal{F}_t} \left(\xi^1 + \sum_{s=t+1}^{T} \mathrm{C}_s(0, V_s^{(2)} - V_{s-1}^{(2)}) \right).$$

In the following, for $u \leq T-1$, $\xi_{u-1} \in L^0(\mathbf{R}^d, \mathcal{F}_{u-1})$, and $\xi \in L^0(\mathbf{R}^d, \mathcal{F}_T)$, we define the sets:

$$\Pi_u^T(\xi_{u-1}, \xi) := \{\xi_{u-1}^{(2)}\} \times \Pi_{s=u}^{T-1} L^0(\mathbf{R}^{d-1}, \mathcal{F}_s) \times \{\xi^{(2)}\}$$

of all families $(V_s^{(2)})_{s=u-1}^{t+1}$ such that $V_{u-1}^{(2)} = \xi_{u-1}$, $V_s^{(2)} \in L^0(\mathbf{R}^{d-1}, \mathcal{F}_s)$ for all $s = u, \cdots, T-1$ and $V_T^{(2)} = \xi^{(2)}$. We set $\Pi_u^T(\xi) := \Pi_u^T(0, \xi) = \Pi_u^T(\xi_{u-1}, \xi)$

when $\xi^{(2)}_{u-1} = 0$. When $u = T$, we set $\Pi^T_T(\xi_{T-1}, \xi) := \{\xi^{(2)}_{T-1}\} \times \{\xi^{(2)}\}$. Therefore, the infimum replicating cost at time 0 is given by

$$c_0(\xi) = \underset{V^2 \in \Pi^T_0(\xi)}{\text{ess inf}_{\mathcal{F}_0}} \text{ ess sup}_{\mathcal{F}_0} \left(\xi^1 + \sum_{s=0}^{T} \mathrm{C}_s(0, V^2_s - V^2_{s-1}) \right).$$

For $0 \leq t \leq T$ and $V_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_t)$, we define $\gamma^\xi_t(V_{t-1})$ as:

$$\gamma^\xi_t(V_{t-1}) := \underset{V^{(2)} \in \Pi^T_t(V_{t-1}, \xi)}{\text{ess inf}_{\mathcal{F}_t}} \text{ ess sup}_{\mathcal{F}_t} \left(\xi^1 + \sum_{s=t}^{T} \mathrm{C}_s(0, V^{(2)}_s - V^{(2)}_{s-1}) \right). \quad (2.2)$$

Note that $\gamma^\xi_t(V_{t-1})$ is the infimum cost to replicate the payoff ξ when starting from the initial position $(0, V^{(2)}_{t-1})$ at time t. Observe that $\gamma^\xi_t(V_{t-1})$ does not depend on the first component V^1_{t-1} of V_t. Moreover,

$$\gamma^\xi_T(V_{T-1}) = \xi^1 + \mathrm{C}_T(0, \xi^{(2)} - V^{(2)}_{T-1}).$$

As $\mathbf{G}_T + \mathbf{R}^d_+ \subseteq \mathbf{G}_T$, we also observe that $\gamma^\xi_T(V_{T-1}) \geq \gamma^0_T(V_{T-1})$ if $\xi \in L^0(\mathbf{R}^d_+, \mathcal{F}_T)$. At last, observe that $c_0(\xi) = \gamma^\xi_0(0)$.

We recall the following result from [22]:

Proposition 2.1 (Dynamic Programming Principle)**.** *For any* $0 \leq t \leq T-1$ *and* $V_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_{t-1})$*, we have*

$$\gamma^\xi_t(V_{t-1}) = \underset{V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)}{\text{ess inf}_{\mathcal{F}_t}} \text{ ess sup}_{\mathcal{F}_t} \left(\mathrm{C}_t(0, V^{(2)}_t - V^{(2)}_{t-1}) + \gamma^\xi_{t+1}(V_t) \right). \quad (2.3)$$

Assumption 1. *The payoff* ξ *is hedgeable, i.e. there exists a portfolio process* $(V^\xi_u)^T_{u=0}$ *such that* $\xi = V^\xi_T$*.*

The dynamic programming principle (2.3) allows to get $\gamma^\xi_t(V_{t-1})$ from the cost function C_t and from γ^ξ_{t+1}. In the paper [22], we have shown that γ^ξ_t is l.s.c. for any t and convex, or piecewise linear, if γ^ξ_{t+1} satisfies the same properties.

As the term $\mathrm{C}_t(0, V^{(2)}_t - V^{(2)}_{t-1})$ in (2.3) is $\mathcal{F}_{t-1}$-measurable, we consider the conditional supremum

$$\theta^\xi_t(V_t) := \text{ess sup}_{\mathcal{F}_t} \gamma^\xi_{t+1}(V_t)$$

to compute the essential supremum of (2.3). In the following, we shall use the following notations:

$$D_t^{\xi}(V_{t-1}, V_t) = \mathrm{C}_t((0, V_t^{(2)} - V_{t-1}^{(2)})) + \theta_t^{\xi}(V_t), \tag{2.4}$$

$$D_t^{\xi}(S_t, V_{t-1}, V_t) = \mathrm{C}_t(S_t, (0, V_t^{(2)} - V_{t-1}^{(2)})) + \theta_t^{\xi}(S_t, V_t). \tag{2.5}$$

The second notation is used when we stress the dependence on S_t. Observe that $\gamma_t^{\xi}(V_{t-1}) = \underset{V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)}{\mathrm{ess\,inf}_{\mathcal{F}_t}} D_t^{\xi}(V_{t-1}, V_t)$.

In order to numerically compute the minimal costs, we need to impose the finiteness of $\gamma_t^{\xi}(V_{t-1})$, i.e. $\gamma_t^{\xi}(V_{t-1}) > -\infty$ a.s., at any time t and for all $V_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_{t-1})$. This is why, we consider the following condition, see [22]:

Definition 2.2. *We say that the financial market satisfies the Absence of Early Profit condition (AEP) if, at any time $t \leq T$, and for all $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$, $\gamma_t^0(V_t) > -\infty$ a.s..*

3. Numerical schemes

In the following, we suppose the following assumptions on the cost process C. For any $t \leq T$, the cost function C_t is a lower-semi continuous Borel function defined on $\mathbf{R}^k \times \mathbf{R}^d$ such that

$$\mathrm{C}_t(s, 0) = 0, \ \forall s \in \mathbf{R}_+^k,$$

$$\mathrm{C}_t(s, x + \lambda e_1) = \mathrm{C}_t(s, x) + \lambda, \ \lambda \in \mathbf{R}, \ x \in \mathbf{R}^d, \ s \in \mathbf{R}_+^k \ \text{(cash invariance)},$$

$$\mathrm{C}_T(s, x_2) \geq \mathrm{C}_T(s, x_1), \ \forall x_1, x_2 \text{ s.t. } x_2 - x_1 \in \mathbf{R}_+^d \ (\mathrm{C}_T \text{ is increasing w.r.t. } \mathbf{R}_+^d).$$

Note that C_T is increasing w.r.t. $\mathbf{R}_+^d$ is equivalent to $\mathbf{G}_T + \mathbf{R}_+^d \subseteq \mathbf{G}_T$. Moreover, for some $a \geq 0$, we say that C_t is a- super homogeneous if the following property holds:

$$\mathrm{C}_t(s, \lambda x) \geq \lambda \mathrm{C}_t(s, x), \forall \lambda \geq a, \ s \in \mathbf{R}_+^k, \ x \in \mathbf{R}^d.$$

3.1. The one period model

In this section, we consider two complete sub σ-algebras $\mathcal{F}_t$ and $\mathcal{F}_{t+1}$ such that $\mathcal{F}_t \subset \mathcal{F}_{t+1} \subset \mathcal{F}$ and an adapted price process $(S_s)_{s=t,t+1}$ satisfying the following assumption.

Assumption 2. *Suppose that there is a family of* $\mathcal{F}_t$*-measurable random variables* $(\alpha_t^m)_{m\geq 1}$ *such that* $S_{t+1} \in \{\alpha_t^m : \ m \geq 1\}$ *a.s. and suppose that* $P(S_{t+1} = \alpha_t^m|\mathcal{F}_t) > 0$ *a.s. for all* $m \geq 1$*. Moreover, we suppose that there exists continuous functions on* $\mathbf{R}^m$*, that we still denote by* α_t^m *with an abuse of notation, such that* $\alpha_t^m = \alpha_t^m(S_t)$*.*

In [22], we have shown the following:

Lemma 3.1. *Suppose that Assumption 2 holds. Then, for any Borel function* $f : \mathbf{R}^d \to \mathbf{R}$*, we have*

$$\text{ess sup}_{\mathcal{F}_t} f(S_{t+1}) = \sup_{m\geq 1} f(\alpha_t^m), \text{ a.s..}$$

Definition 3.2. *The random variables* $\{b_{t+1}^i, i \geq 1\}$*,* $b_{t+1}^i \in L^0(\mathbf{R}^k, \mathcal{F}_{t+1})$*, are said independent and identically distributed conditionally to* $\mathcal{F}_t$ *(for short* $\mathcal{F}_t$*-i.i.d.) if, for all finite set* $J \subset \mathbb{N}$*, and Borel sets* $B, B_j, j \in J$*:*

$$P\left[b_{t+1}^i \in B|\mathcal{F}_t\right] = P\left[b_{t+1}^j \in B|\mathcal{F}_t\right], \text{ a.s. } \forall i, j \geq 1,$$

$$P\left[\bigcap_{j\in J}\{b_{t+1}^j \in B_j\}\,|\mathcal{F}_t\right] = \prod_{j\in J} P\left[b_{t+1}^j \in B_j|\mathcal{F}_t\right], \text{ a.s..}$$

Lemma 3.3. *Consider a family of* $\mathcal{F}_t$*-i.i.d. random variables* b_{t+1}^i*,* $i \geq 1$ *and* $\theta_t \in L^0(\mathbf{R}^m, \mathcal{F}_t)$*. Let* $f^j : \mathbf{R}^k \times \mathbf{R}^m \to \mathbf{R}$*,* $j = 1, \cdots, n$ *be* $n \geq 1$ *measurable functions such that* $E\left[|f^j(b_{t+1}^1, \theta_t)||\mathcal{F}_t\right] < \infty$ *a.s. (resp.* f^j *is non negative), for all* $j \leq n$*. Then, for any finite set* $J \subset \mathbb{N}$ *of cardinality* n*, we have:*

$$E\left[f^k(b_{t+1}^i, \theta_t)|\mathcal{F}_t\right] = E\left[f^k(b_{t+1}^j, \theta_t)|\mathcal{F}_t\right], \text{ a.s., } i, j, k \geq 1,$$

$$E\left[\prod_{j\in J} f^j(b_{t+1}^j, \theta_t)|\mathcal{F}_t\right] = \prod_{j\in J} E\left[f^j(b_{t+1}^j, \theta_t)|\mathcal{F}_t\right], \text{ a.s..}$$

Proof. We prove the result by induction on n. Suppose that $f^j = 1_{D_j}$ where $D_j = B_j \times A_j$ and $B_j \in \mathcal{B}(\mathbf{R}^k)$, $A_j \in \mathcal{B}(\mathbf{R}^m)$. Then, the claim holds by definition of the $\mathcal{F}_t$-i.i.d. random variables for all $n \geq 1$ and the $\mathcal{F}_t$-measurability of θ_t. By the monotone class argument, this holds for any $D_1 \in \mathcal{B}(\mathbf{R}^k)\otimes\mathcal{B}(\mathbf{R}^m)$ if $n = 1$. If $n > 1$, we expand the product in the second claim and we use the induction hypothesis. Then, we repeat the arguments for $D_2 \in \mathcal{B}(\mathbf{R}^k)\otimes\mathcal{B}(\mathbf{R}^m)$ and so on. By linearity, and the induction argument after having expanding the product, we also deduce that the claim holds when $f^j = \sum_{h=1}^n c_h^j 1_{C_h^j}$

and for any $c_h^j \in \mathbf{R}, C_h^j \in \mathcal{B}(\mathbf{R}^k) \otimes \mathcal{B}(\mathbf{R}^m), h \geq 1$. By standard increasing approximations, we conclude in the case where $f^j \geq 0$. Otherwise, we write $f^j = (f^j)^+ - (f^j)^-$. In particular, we get that

$$E\left[|f^j(b_{t+1}^i, \theta_t)||\mathcal{F}_t\right] = E\left[|f^j(b_{t+1}^1, \theta_t)||\mathcal{F}_t\right] < \infty, \text{ a.s.}$$

in the case where $E\left[|f^j(b_{t+1}^1, \theta_t)||\mathcal{F}_t\right] < \infty$. □

Lemma 3.4. *Consider a Borel function $f : \mathbf{R}^k \to \mathbf{R}$ and a family of $\mathcal{F}_t$-i.i.d. random variables $(b_{t+1}^m)_{m\geq 1}$ with values in $\mathbf{R}^k$ and $\mathcal{F}_{t+1}$-measurable. Suppose that there exists $\mathcal{F}_t$-measurable random variables $(\alpha_t^n)_{n\geq 1}$ such that $b_{t+1}^m \in \{\alpha_t^n, n \geq 1\}$ a.s. and $P(b_{t+1}^m = \alpha_t^n|\mathcal{F}_t) > 0$ a.s. for all $n, m \geq 1$.*

Let us define $\theta_t := \sup_{m\geq 1} f(\alpha_t^m) = \text{ess sup}_{\mathcal{F}_t} f(S_{t+1})$ (by Lemma 3.1) and $\theta_t^m := \max_{i\leq m} f(b_{t+1}^i)$. The following holds:

$$\theta_t^m \to \theta_t, \text{ a.s. as } m \to \infty.$$

In particular, $\sup_m \theta_t^m = \theta_t$ a.s.

Proof. We may suppose w.l.o.g. that $\theta_t < \infty$. Indeed, we may consider $g(\theta_t)$ and the sequence $(g(\theta_t^m))_{m\geq 1}$ where g is a bounded strictly increasing continuous function in the contrary case. By Lemma 3.1, we get that $\text{ess sup}_{\mathcal{F}_t} f(b_{t+1}^1) = \sup_{m\geq 1} f(\alpha_t^m) = \theta_t$ a.s. For any $\epsilon > 0$, we deduce by assumption that

$$\begin{aligned} P[\theta_t - \theta_t^m > \epsilon|\mathcal{F}_t] &= P[\theta_t - \max_{i\leq m} f(b_{t+1}^i) > \epsilon|\mathcal{F}_t] \\ &= P[\theta_t - f(b_{t+1}^i) > \epsilon, \forall i \leq m|\mathcal{F}_t] \\ &= E\left[\prod_{i=1}^m 1_{\{\theta_t - f(b_{t+1}^i) > \epsilon\}}\big|\mathcal{F}_t\right], \text{ a.s..} \end{aligned}$$

By Lemma 3.3, we deduce that

$$\begin{aligned} P[\theta_t - \theta_t^m > \epsilon|\mathcal{F}_t] &= P[\theta_t - f(b_{t+1}^1) > \epsilon|\mathcal{F}_t]^m \\ &= P[\text{ess sup}_{\mathcal{F}_t} f(b_{t+1}^1) - f(b_{t+1}^1) > \epsilon|\mathcal{F}_t]^m, \text{ a.s.} \end{aligned}$$

We claim that $P[\text{ess sup}_{\mathcal{F}_t} f(b_{t+1}^1) - f(b_{t+1}^1) > \epsilon|\mathcal{F}_t] < 1$ a.s. Indeed, assume on the contrary that $P[\text{ess sup}_{\mathcal{F}_t} f(b_{t+1}^1) - f(b_{t+1}^1) > \epsilon|\mathcal{F}_t] = 1$ on some non null set $\Lambda_t \in \mathcal{F}_t$. In other words, we have

$$E\left[1_{\{\text{ess sup}_{\mathcal{F}_t} f(b_{t+1}^1) > f(b_{t+1}^1) + \epsilon\}}\big|\mathcal{F}_t\right] 1_{\Lambda_t} = 1_{\Lambda_t}.$$

Taking the expectation, we deduce that:

$$E\left[1_{\{\text{ess sup}_{\mathcal{F}_t} f(b^1_{t+1})>f(b^1_{t+1})+\epsilon\}}1_{\Lambda_t}\right] = E\left[1_{\Lambda_t}\right]$$

We then deduce that $1_{\{\text{ess sup}_{\mathcal{F}_t} f(b^1_{t+1})>f(b^1_{t+1})+\epsilon\}}1_{\Lambda_t} = 1_{\Lambda_t}$ a.s. We now define $\hat{\theta}_t := \text{ess sup}_{\mathcal{F}_t} f(b^1_{t+1})1_{\Omega\setminus\Lambda_t} + (\text{ess sup}_{\mathcal{F}_t} f(b^1_{t+1}) - \epsilon)1_{\Lambda_t}$. Observe that $\hat{\theta}_t$ is $\mathcal{F}_t$-measurable and $\hat{\theta}_t \geq f(b^1_{t+1})$ a.s. However, $\hat{\theta}_t < \text{ess sup}_{\mathcal{F}_t} f(b^1_{t+1})$ on the non null set Λ_t, in contradiction with the definition of the conditional essential supremum. Therefore,

$$\lim_{m\to\infty} P[\theta_t - \theta^m_t > \epsilon|\mathcal{F}_t] = 0, \text{ a.s.}$$

Finally, by the dominated convergence theorem, we have

$$\begin{aligned}\lim_{m\to\infty} P[\theta_t - \theta^m_t > \epsilon] &= \lim_{m\to\infty} E\left[E[1_{\{\theta_t-\theta^m_t>\epsilon\}}|\mathcal{F}_t]\right] \\ &= E\left[\lim_{m\to\infty} E[1_{\{\theta_t-\theta^m_t>\epsilon\}}|\mathcal{F}_t]\right] \\ &= 0.\end{aligned}$$

Hence θ^m_t increasingly tends to θ_t in probability, i.e. $\sup_m \theta^m_t = \theta_t$ a.s.. □

Assumption 3. *The payoff function ξ is of the form $\xi = g(S_T)$, where $g \in \mathbf{R}^k_+$ is continuous. Moreover, ξ is hedgeable, i.e. there exists a portfolio process $(V^\xi_u)^T_{u=-1}$ such that $\xi = V^\xi_T$.*

We recall here two weak no-arbitrage conditions introduced in [22]:

Definition 3.5. *We say that the condition AIP holds at time t if the minimal cost $c_t(0) = \gamma^0_t(0)$ of the European zero claim $\xi = 0$ is 0 at time $t \leq T$. We say that AIP holds if AIP holds at any time.*

The following condition is more technical.

Definition 3.6. *We say that the condition SAIP (Strong AIP condition) holds at time t if AIP holds at time t and, for any $Z_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$, we have $D^0_t(S_t, 0, Z_t) = 0$ if and only if $Z^{(2)}_t = 0$ a.s.. We say that SAIP holds if SAIP holds at any time.*

The condition SAIP states that the minimal cost of the zero payoff is 0 at time t and this minimal cost is only attained by the zero strategy $V_t = 0$, see [22]. This is intuitively clear as soon as any non null transaction implies positive costs.

We now introduce the sequence of functions which is defined recursively as follows:

$$\begin{aligned}
&\tilde{\gamma}_T^{\xi}(s, v_{T-1}) := \xi^1 + \mathrm{C}_T(s, (0, \xi^{(2)} - v_{T-1}^{(2)})), \quad v_{t-1}, \xi \in \mathbf{R}^d,\ s \in \mathbf{R}^k,\\
&\tilde{\theta}_t^{\xi}(s, v_t) := \sup_m \tilde{\gamma}_{t+1}^{\xi}(\alpha_t^m(s), v_t),\ t \le T-1,\ v_t \in \mathbf{R}^d,\\
&\tilde{D}_t^{\xi}(s, v_{t-1}, v_t) := \tilde{\theta}_t^{\xi}(s, v_t) + \mathrm{C}_t(s, (v_t^{(2)} - v_{t-1}^{(2)})),\\
&\tilde{\gamma}_t^{\xi}(s, v_{t-1}) := \mathrm{cl}\left(\inf_{v_t \in \mathbf{R}^d} \tilde{D}_t^{\xi}(s, v_{t-1}, v_t)\right). \qquad (3.6)
\end{aligned}$$

Here, the notation $\mathrm{cl}(f)$ designates the l.s.c. regularization of f. In this paper, we will impose later in the sequel a condition under which we have $\tilde{\gamma}_t^{\xi}(s, v_{t-1}) := \inf_{v_t \in \mathbf{R}^d} \tilde{D}_t^{\xi}(s, v_{t-1}, v_t)$.

The introduction of the functions above is motivated by the following result proved in [22].

Theorem 3.7. *Suppose that either AIP holds and $\mathrm{C}_t(s,.)$ is convex for fixed s or SAIP holds. Then, we have $\gamma_t^{\xi}(S_t, V_t) = \tilde{\gamma}_t^{\xi}(S_t, V_t)$ a.s. and, also, $\theta_t^{\xi}(S_t, V_t) = \tilde{\theta}_t^{\xi}(S_t, V_t)$ a.s. and $D_t^{\xi}(S_t, V_{t-1}, V_t) = \tilde{D}_t^{\xi}(S_t, V_{t-1}, V_t)$ for any $V_{t-1}, V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. Moreover, $\tilde{\gamma}_t^{\xi}(s, v)$ is l.s.c. on $\mathbf{R}^k \times \mathbf{R}^d$ and convex in v when $\mathrm{C}_t(s,.)$ is convex.*

Recall that the family of $\mathcal{F}_t$-measurable random variables $(\alpha_t^n(S_t))_{n \ge 1}$ is defined in Assumption 2. We now consider an $\mathcal{F}_t$-i.i.d. sample of random variables $\{b_{t+1}^i, i \ge 1\}$ that satisfies $P[b_{t+1}^1 = \alpha_t^n(S_t)|\mathcal{F}_t] > 0$ a.s. for all $n \ge 1$ and $b_{t+1}^1 \in \{\alpha_t^n(S_t), n \ge 1\}$ a.s. Now, let us define the (random) functions

$$\begin{aligned}
\bar{D}_T^{\xi}(s, x, y) &:= \tilde{\gamma}_T^{\xi}(s, y),\\
\bar{D}_t^{\xi}(s, x, y) &:= \mathrm{C}_t(s, (0, y^{(2)} - x^{(2)})) + \tilde{\gamma}_{t+1}^{\xi}(s, y),\\
\bar{D}_T^{n}(\omega, x, y) &:= \bar{D}_T^{\xi}(s, x, y)\\
\bar{D}_t^{n}(\omega, x, y) &:= \max_{i \le n} \bar{D}_t^{\xi}(b_{t+1}^i(\omega), x, y). \qquad (3.7)
\end{aligned}$$

Since $\tilde{\gamma}_{t+1}^{\xi}(s, x)$ is l.s.c. in s, it is Borel in s for fixed x. Then, by Lemma 3.4, we deduce that:

$$\lim_{n\to\infty} \max_{i \le n} \tilde{\gamma}_{t+1}^{\xi}(b_{t+1}^i(\omega), y) = \sup_n \tilde{\gamma}_{t+1}^{\xi}(\alpha_t^n(S_t(\omega)), y) = \tilde{\theta}_t^{\xi}(S_t(\omega), y), \text{a.s.}$$

In particular, $\lim_{n\to\infty} \bar{D}_t^n(\omega, x, y) = \tilde{D}_t^{\xi}(S_t(\omega), x, y)$. We now investigate the question whether $\inf_{y \in \mathbf{R}^d} \bar{D}_t^n(\omega, x, y)$ converge a.s.(ω) to $\inf_{y \in \mathbf{R}^d} \tilde{D}_t^{\xi}(\omega, x, y)$

as $n \to \infty$. To do so, we first recall the definition of epi-convergence, see [25, Chapter 3] or [26, Chapter 7]. In the following, the notation $B(x,r)$ designates the closed ball of $\mathbf{R}^d$, where $d \geq 1$ depends on the context, centered a point $x \in \mathbf{R}^d$ and of radius $r \geq 0$.

Definition 3.8. *Let $f_n : \mathbf{R}^k \to \overline{\mathbf{R}}$, $n \geq 1$, be a sequence of functions. The **epi-limit inferior** $\mathrm{li}_e f_n$ and **epi-limit superior** $\mathrm{ls}_e f_n$ of $(f_n)_{n\geq 1}$ are defined as:*

$$\mathrm{li}_e[(f_n)_{n\geq 1}](u) := \sup_{k\geq 1} \liminf_{n\to\infty} \inf_{v\in B(u,1/k)} f_n(v),$$
$$\mathrm{ls}_e[(f_n)_{n\geq 1}](u) := \sup_{k\geq 1} \limsup_{n\to\infty} \inf_{v\in B(u,1/k)} f_n(v).$$

The sequence $(f_n)_{n\geq 1}$ is said to be epi-convergent at point u if

$$\mathrm{li}_e[(f_n)_{n\geq 1}](u) = \mathrm{ls}_e[(f_n)_{n\geq 1}](u).$$

We also introduce the definition of almost sure epi-convergence for random functions.

Definition 3.9. *If $(f_n)_{n\geq 1}$, is a sequence of functions $f_n : \Omega \times \mathbf{R}^k \to \overline{\mathbf{R}}$ such that f_n is $\mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^d)$-measurable for each n, we say that f_n epi-converges to f almost surely (notation $f_n \xrightarrow{epi} f$ a.s.) if, for any ω outside a P-null set, and for all u: $\mathrm{li}_e[(f_n(\omega,\cdot))_{n\geq 1}](u) = \mathrm{ls}_e[(f_n(\omega,\cdot))_{n\geq 1}](u) = f(\omega,u)$.*

Theorem 3.10. *Suppose that AIP holds and $\mathrm{C}_t(s,y)$ is convex in y. We then have $\bar{D}_t^n(\omega,.,.) \vee \big(-\mathrm{C}_t(S_t(\omega),(0,x^{(2)}))\big) \xrightarrow{\mathrm{epi}} \tilde{D}_t^{\xi}(S_t(\omega),.,.)$ a.s.(ω), as $n \to \infty$.*

Suppose that SAIP holds and, for any t, $\mathrm{C}_t(s,v_t^1) \geq \mathrm{C}_t(s,v_t^2)$ if $v_t^1 \geq_{\mathbf{R}^d_+} v_t^2$. Then, $\bar{D}_t^n(\omega,.,.) \xrightarrow{\mathrm{epi}} \tilde{D}_t^{\xi}(S_t(\omega),.,.)$ a.s..

Proof. We first consider the case where AIP holds and $\mathrm{C}_t(s,y)$ is convex in y. Let us define $\bar{L}_t^{\xi}(\omega,x,y) := \bar{D}_t^n(\omega,x,y) \vee \big(-\mathrm{C}_t(S_t(\omega),(0,x^{(2)}))\big)$. Observe that $\bar{L}_t^n(\omega,x,y)$ is l.s.c. in (x,y) as a maximum of two l.s.c. functions. As the sequence $(\bar{L}_t^n)_{n\geq 1}$ is also non decreasing, we deduce by [26, Proposition 7.4], that for any ω:

$$\mathrm{li}_e[(\bar{L}_t^n(\omega,\cdot,\cdot)_{n\geq 1})](x,y) = \mathrm{ls}_e[(\bar{L}_t^n(\omega,\cdot,\cdot)_{n\geq 1})](x,y) = \sup_n \bar{L}_t^n(\omega,x,y).$$

We now prove that there exists a negligible set H such that for any $\omega \in \Omega \setminus H$ and $x, y \in \mathbf{R}^d \times \mathbf{R}^d$ the following holds:

$$\sup_n \bar{L}_t^n(\omega, x, y) = \tilde{D}_t^\xi(\omega, x, y). \tag{3.8}$$

By assumption on $(\mathrm{C}_t)_{t\geq 0}$, we get by induction that $\theta_t^\xi(V_t) \geq \theta_t^0(V_t)$ a.s. for any $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. We deduce that $D_t^\xi(V_{t-1}, V_t) \geq -\mathrm{C}_t(S_t(\omega), (0, V_{t-1}^{(2)}))$ for any for any $V_{t-1}, V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. Indeed, under AIP, $D_t^0(0, V_t) \geq 0$ a.s. hence

$$\begin{aligned}
D_t^\xi(V_{t-1}, V_t) &= \theta_t^\xi(V_t) + \mathrm{C}_t(S_t, (0, V_t^{(2)} - V_{t-1}^{(2)})) \\
&\geq \theta_t^\xi(V_t) + \mathrm{C}_t(S_t, (0, V_t^{(2)})) - \mathrm{C}_t(S_t, (0, V_{t-1}^{(2)})), \text{ (by subadditivity)} \\
&\geq \theta_t^0(V_t) + \mathrm{C}_t(S_t, (0, V_t^{(2)})) - \mathrm{C}_t(S_t, (0, V_{t-1}^{(2)})) \\
&\geq D_t^0(0, V_t) - \mathrm{C}_t(S_t, (0, V_{t-1}^{(2)})) \geq -\mathrm{C}_t(S_t, (0, V_{t-1}^{(2)})), \text{ a.s.}
\end{aligned}$$

for any $V_{t-1}, V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$.

We now deduce that $\tilde{D}_t^\xi(S_t(\omega), x, y) \geq -\mathrm{C}_t\left(S_t(\omega), (0, x^{(2)})\right)$ for every x, y a.s.(ω). Indeed, suppose on the contrary that the $\mathcal{F}_t$-measurable set

$$\Gamma_t(\omega) := \left\{(x, y) \in \mathbf{R}^d \times \mathbf{R}^d : \tilde{D}_t^\xi(S_t(\omega), x, y) < -\mathrm{C}_t\left(S_t(\omega), (0, x^{(2)})\right)\right\}$$

is non-empty on the non-null set $G_t := \{\omega : \Gamma_t(\omega) \neq \emptyset\}$. We then deduce a measurable selection $(\bar{V}_{t-1}, \bar{V}_t) \in L^0(\mathbf{R}^d, \mathcal{F}_t) \times L^0(\mathbf{R}^d, \mathcal{F}_t)$ such that we have $\tilde{D}_t^\xi(S_t, \bar{V}_t, \bar{V}_{t-1}) < -\mathrm{C}_t(S_t, (0, \bar{V}_{t-1}^{(2)}))$ on G_t and we extend to the whole space by putting $\bar{V}_{t-1} = 0 = \bar{V}_t$ on the complementary set $\Omega \setminus G_t$. Moreover, by Theorem 5.5, we then deduce that $D_t^\xi(\bar{V}_t, x) < -\mathrm{C}_t(S_t, (0, \bar{V}_{t-1}^{(2)})$ on the non-null set G_t, which is a contradiction.

Similarly, under AEP and Assumption 3, we have that $D_t^\xi(V_{t-1}, V_t) \in \mathbf{R}$ a.s. for any $V_{t-1}, V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$, see [22]. Then, by a measurable selection argument, using the fact that $D_t^\xi(V_{t-1}, V_t) = \tilde{D}_t^\xi(S_t, V_{t-1}, V_t)$ a.s., we deduce that $\tilde{D}_t^\xi(S_t(\omega), x, y) \in \mathbf{R}$ for any x, y, for any ω outside a negligible set.

By Lemma 3.4, $\bar{L}_t^n(\omega, x, y) \to \tilde{D}_t^\xi(S_t(\omega), x, y) \vee (-\mathrm{C}_t(S_t(\omega), (0, x^{(2)}))$ as $n \to \infty$ for any ω outside a negligible set $N(x, y)$. Moreover, by the discussion above, we deduce a negligible set M such that for any $\omega \in \Omega \setminus M$, we have $\tilde{D}_t^\xi(S_t(\omega), x, y) \geq -\mathrm{C}_t(S_t(\omega), (0, x^{(2)}))$ and $\tilde{D}_t^\xi(S_t(\omega), x, y) \in \mathbf{R}$ for any

x, y. We set $H := \cup_{y\in\mathbf{Q}^d} N(x,y) \cup M$, we claim that for any $\omega \in \Omega \setminus H$, $\sup_n \bar{L}_t^n(\omega, x, y) = \tilde{D}_t^\xi(S_t(\omega), x, y)$ for all $x, y \in \mathbf{R}^d$. Indeed, by the definition of H, we deduce that (3.8) holds for any $y \in \mathbf{Q}^d$. Now, since $\tilde{D}_t^\xi(S_t(\omega), ., .)$ is convex and takes values in $\mathbf{R}$, it is continuous for any $\omega \in \Omega \setminus H$. Moreover, we claim that $\sup_n \bar{L}_t^n(\omega, x, y) < \infty$ for any $x, y \in \mathbf{R}^d$ and $\omega \in \Omega \setminus H$. Indeed, by lower semicontinuity, we have:

$$\sup_n \bar{L}_t^n(\omega, x, y) \le \liminf_k \sup_n \bar{L}_t^n(\omega, x_k, y_k)$$

for any sequence $x_k, y_k \in \mathbf{Q}^d$ such that $x_k \to x$ and $y_k \to y$. Moreover, by the definition of H and the continuity of $\tilde{D}_t^\xi(S_t(\omega), ., .)$ for any $\omega \in \Omega \setminus H$, we have $\liminf_k \sup_n \bar{L}_t^n(\omega, x_k, y_k) = \liminf_k \tilde{D}_t^\xi(S_t(\omega), x_k, y_k) = D_t^\xi(S_t(\omega), x, y) \in \mathbf{R}$. We deduce that $\sup_n \bar{L}_t^n(\omega, x, y) \in \mathbf{R}$ for any $x, y \in \mathbf{R}^d$, and $\omega \in \Omega \setminus H$. Moreover, $\sup_n \bar{L}_t^n(\omega, ., .)$ also convex as a supremum of convex functions, it is also continuous. We then deduce by continuity that (3.8) holds for any $y \in \mathbf{R}^d$.

Now, we consider the second case where $\mathrm{C}_t(s, v_t^1) \ge \mathrm{C}_t(s, v_t^2)$ for any $v_t^1, v_t^2 \in \mathbf{R}^d$ such that $v_t^1 \ge_{\mathbf{R}_+^d} v_t^2$. Similarly to the first case, we only need to prove $\sup_n \bar{D}_t^n(\omega, x, y) = \tilde{D}_t^\xi(S_t(\omega), x, y)$ for all x, y and ω outside a negligible set. By the definition of $\tilde{\gamma}_t^\xi$ and $\tilde{\theta}_t^\xi$, we can show by induction an by Lemma 3.11 that the mappings $y \mapsto \tilde{\theta}_t^\xi(s, y)$ and $y \mapsto \tilde{\gamma}_t^\xi(s, y)$ are decreasing with respect to $\mathbf{R}_+^d$.

Recall the definition of $N(x, y)$, we also denote $H := \cup_{y\in\mathbf{Q}^d} N(x,y) \cup M$ and claim that for any $\omega \in \Omega \setminus H$, $\sup_n \tilde{\gamma}_{t+1}^\xi(b_{t+1}^n(\omega), y) = \tilde{\theta}_t^\xi(S_t(\omega), y)$, for all $y \in \mathbf{R}^d$. Indeed, fix some $y \in \mathbf{R}^d$ and a sequence $(y_k)_{k\ge1}$ in $\mathbf{Q}^d$ such that $y_k \to y$ and $y_k \ge_{\mathbf{R}_+^d} y$. By lower semicontinuity and the discussion above, we have for any $\omega \in \Omega \setminus H$:

$$\tilde{\theta}_t^\xi(S_t(\omega), y) \le \liminf_k \tilde{\theta}_t^\xi(S_t(\omega), y_k) \le \tilde{\theta}_t^\xi(S_t(\omega), y), \text{ and}$$

$$\sup_n \tilde{\gamma}_{t+1}^\xi(b_{t+1}^n(\omega), y) \le \liminf_k \sup_n \tilde{\gamma}_{t+1}^\xi(b_{t+1}^n(\omega), y_k) \le \sup_n \tilde{\gamma}_{t+1}^\xi(b_{t+1}^n(\omega), y).$$

Then, we have

$$\begin{aligned} \tilde{\theta}_t^\xi(S_t(\omega), y) &= \liminf_k \tilde{\theta}_t^\xi(S_t(\omega), y_k), \\ \sup_n \tilde{\gamma}_{t+1}^\xi(b_{t+1}^n(\omega), y) &= \liminf_k \sup_n \tilde{\gamma}_{t+1}^\xi(b_{t+1}^n(\omega), y_k). \end{aligned}$$

Moreover, by the definition of H, we have $\sup_n \tilde{\gamma}^{\xi}_{t+1}(b^n_{t+1}(\omega), y_k) = \tilde{\theta}^{\xi}_t(S_t(\omega), y_k)$ for any $\omega \in \Omega \setminus H$. We then deduce that $\sup_n \tilde{\gamma}^{\xi}_{t+1}(b^n_{t+1}(\omega), y) = \tilde{\theta}^{\xi}_t(S_t(\omega), y)$ for any $\omega \in \Omega \setminus H$. At last, by the definition of $\tilde{D}^{\xi}_t$ and $\bar{D}^n_t$, we conclude that $\sup_n \bar{D}^n_t(\omega, x, y) = \tilde{D}^{\xi}_t(S_t(\omega), x, y)$ for any x, y and $\omega \in \Omega \setminus H$. □

In the Proof of Theorem 3.10, we have used the following result:

Lemma 3.11. *Let $f : \mathbf{R}^k \to \mathbf{R}$ be a function such that f that is non increasing with respect to the partial order $\geq_{\mathbf{R}^k_+}$. Consider $\mathrm{cl}(f)$ the lower semicontinuous regularization of f. Then, $\mathrm{cl}(f)$ is non increasing w.r.t. the partial order $\geq_{\mathbf{R}^k_+}$.*

Proof. From [26, Lemma 1.7], we have the following representation of the l.s.c. closure:

$$\mathrm{cl}(f)(x) = \liminf_{y \to x} f(x) = \left\{ \alpha \in \overline{\mathbf{R}} :\ \exists (x_n)_{n \geq 1},\ x_n \to x, \lim_n f(x_n) = \alpha \right\}.$$

Consider $x^1, x^2 \in \mathbf{R}^d$ such that $x^1 \geq_{\mathbf{R}^d_+} x^2$ and a sequence $(x_n)_{n\geq 1}$ such that $x_n \to x^2$ and $f(x_n) \to \mathrm{cl}(f)(x^2)$ as $n \to \infty$. Observe that $x_n + x^1 - x^2 \to x^1$ as $n \to \infty$. We then have $f(x_n + x^1 - x^2) \leq f(x_n)$ by our hypothesis. We deduce that

$$\mathrm{cl}(f)(x^1) \leq \liminf_n f(x_n + x^1 - x^2) \leq \lim_n f(x_n) = \mathrm{cl}(f)(x^2).$$

□

Definition 3.12. *We say that a set-valued mapping $K_t : \mathbf{R}^k_+ \times \mathbf{R}^d \twoheadrightarrow \mathbf{R}^d$ is a reachability set at time $t \leq T$ for the super-hedging problem if K_t has compact set values and satisfies:*

$$\inf_{y \in \mathbf{R}^d} \tilde{D}^{\xi}_t(S_t(\omega), x, y) = \inf_{y \in K_t(S_t(\omega), x)} \tilde{D}^{\xi}_t(S_t(\omega), x, y), \text{ a.s..}$$

Moreover, we suppose that $K_t(s, x)$ is upper hemicontinuous in (s, x), see [1, Definition 17.2].

Remark 3.13. *By [22, Theorem 4.14], under SAIP, the determining set $K_t(s, x)$ is constructed for $s = S_t(\omega)$ as a closed ball $\overline{B}(0, r_t(s, x) + 1)$, where $r_t(s, x)$ is an u.s.c. function. We shall see later in the model with one risky asset how to characterize $K_t(s, x)$ explicitely for* ***every*** *$(s, x) \in \mathbf{R} \times \mathbf{R}$ such that $K_t(s, x)$ is compact for all (s, x) and upper hemicontinuous. Moreover, By [1, Lemma 17.29], the upper hemicontinuity of K implies that*

$$\tilde{\gamma}^{\xi}_t(s, v_{t-1}) := \inf_{v_t \in \mathbf{R}^d} \tilde{D}^{\xi}_t(s, v_{t-1}, v_t). \tag{3.9}$$

Theorem 3.14. *Suppose that SAIP holds and* $\mathrm{C}_t(s, v_t^1) \geq \mathrm{C}_t(s, v_t^2)$ *for any* $v_t^1, v_t^2 \in \mathbf{R}^d$ *such that* $v^1 \geq_{\mathbf{R}_+^d} v^2$. *Then, we have:*

$$\lim_{n\to\infty} \inf_{y\in K_t(S_t(\omega),x)} \bar{D}_t^n(\omega, x, y) = \inf_{y\in K_t(S_t(\omega),x)} \tilde{D}_t^{\xi}(S_t(\omega), x, y), \quad \forall x, y, \text{ a.s.} \tag{3.10}$$

Moreover, for each fixed $x_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ *such that the random set* $K_t(S_t, x_t)$ *is* $\mathcal{F}_t$*-measurable, there exists a sequence* $(\hat{y}_{t+1}^n)_{n\geq 1}$ *of* $L^0(\mathbf{R}^d, \mathcal{F}_{t+1})$ *such that* $\hat{y}_{t+1}^n \in \arg\min_{K_t(S_t,x_t)}(\bar{D}_t^n(\omega, x_t, .))$ *a.s. and* $\hat{y}_{t+1}^n \to \hat{y}_{t+1}^0 \in L^0(\mathbf{R}^d, \mathcal{F}_{t+1})$ *along a random* $\mathcal{F}_{t+1}$*-measurable subsequence where* $\hat{y}_{t+1}^0 \in \arg\min(\tilde{D}_t^{\xi}(S_t, x_t, .))$.

In the case where $\mathrm{C}_t(s, y)$ *is convex in* y*, the same conclusion holds if we replace* $\bar{D}_t^n(\omega, x, y)$ *by* $\bar{D}_t^n(\omega, x, y) \vee \left(-\mathrm{C}_t(S_t(\omega), (0, x^{(2)}))\right)$. *Moreover, in that case, if* $K_t(S_t, x_t)$ *is also convex, for fixed* $x_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ *such that the random set* $K_t(S_t, x_t)$ *is* $\mathcal{F}_t$*-measurable,* $\hat{y}_t^n = E(\hat{y}_{t+1}^n|\mathcal{F}_t) \in K_t(S_t, x_t)$ *a.s. and converges a.s. to* $\hat{y}_t^0 = E(\hat{y}_{t+1}^0|\mathcal{F}_t) \in \arg\min(\tilde{D}_t^{\xi}(S_t, x_t, .))$.

Proof. We prove the claim in the first case, the second case is deduced similarly using Theorem 3.10.

Consider the negligible set H in the proof of Theorem 3.10 such that $\bar{D}_t^n(\omega, x, y) \leq \tilde{D}_t^{\xi}(\omega, x, y)$, for all x, y and for any $\omega \in \Omega \setminus H$ and $n \geq 1$. We then have:

$$\lim_{n\to\infty} \inf_{y\in K_t(S_t(\omega),x)} \bar{D}_t^n(\omega, x, y) \leq \inf_{y\in K_t(S_t(\omega),x)} \tilde{D}_t^{\xi}(S_t(\omega), x, y), \ \forall x, \tag{3.11}$$

for any $\omega \in \Omega \setminus H$. We now establish the reversed inequality. Since each $\bar{D}_t^n$ is an $\mathcal{F}$-normal integrand, then by [26, Theorem 13.37], we deduce that $\inf_{y\in K_t(S_t(\omega),x)} \bar{D}_t^n(\omega, x, y)$ is almost surely attained at some $\hat{y}_t^n(\omega, x)$. In other words, we have $\hat{y}_t^n(\omega, x) \in \arg\min_{K_t(S_t(\omega),x)}(\bar{D}_t^n(\omega, x, .))$ for any ω outside a negligible set N such that $H \subset N$.

Since $K_t(s, x)$ is compact, for any $\omega \in \Omega \setminus N$ and $x \in \mathbf{R}^d$, there is a random subsequence $\{\hat{y}_t^{n_k}(\omega, x), k \geq 1\}$ of $\{\hat{y}_t^n(\omega, x), n \geq 1\}$ converging to some $\hat{y}_t^0(\omega, x) \in K_t(S_t(\omega), x)$. Since $\bar{D}_t^n(\omega, ., .) \xrightarrow{\text{epi}} \tilde{D}_t^{\xi}(S_t(.), ., .)$ a.s.(ω) by Theorem 3.10, we deduce by [26, Proposition 7.2] that:

$$\liminf_{k\to\infty} \bar{D}_t^k(\omega, x, \hat{y}_t^k(\omega, x)) \geq \tilde{D}_t^{\xi}(S_t(\omega), x, \hat{y}_t^0(\omega, x)) \tag{3.12}$$

for any $\omega \in \Omega \setminus N$. As $\tilde{D}_t^\xi(S_t(\omega), x, \hat{y}_t^0(\omega, x)) \geq \inf_{y \in K_t(S_t(\omega), x)} \tilde{D}_t^\xi(S_t(\omega), x, y)$, we deduce that for any $\omega \in \Omega \setminus N$:

$$\liminf_{k \to \infty} \bar{D}_t^k(\omega, x, \hat{y}_t^k(\omega, x)) \geq \inf_{y \in K_t(S_t(\omega), x)} \tilde{D}_t^\xi(S_t(\omega), x, y). \tag{3.13}$$

We deduce from (3.11) and (3.13) and, finally (3.12), that

$$\liminf_{k \to \infty} \bar{D}_t^k(\omega, x, \hat{y}_t^k(\omega, x)) = \inf_{y \in K_t(S_t(\omega), x)} \tilde{D}_t^\xi(S_t(\omega), x, y) = \tilde{D}_t^\xi(S_t(\omega), x, \hat{y}_t^0(\omega, x))$$

We then deduce that $\hat{y}_t^0(\omega, x) \in \arg\min_{K_t(S_t(\omega), x)}(\tilde{D}_t^\xi(S_t(\omega), x, .))$ for any $\omega \in \Omega \setminus N$, i.e. (3.10) holds. Using the definition of the reachability set-valued mapping K_t, we conclude that $\hat{y}_t^0(\omega, x) \in \arg\min(\tilde{D}_t^\xi(S_t(\omega), x, .))$ outside a negligeable set.

Recall that $\inf_{y \in K_t(S_t(\omega), x_t)} \bar{D}_t^n(\omega, x_t, y)$ is $\mathcal{F}_{t+1}$-measurable, see [22]. Therefore, by a measurable selection argument, we may deduce the existence of $\hat{y}_{t+1}^n \in L^0(\mathbf{R}^d, \mathcal{F}_{t+1})$ such that $\bar{D}_t^n(\omega, x_t, \hat{y}_{t+1}^n) = \inf_{y \in K_t(S_t(\omega), x_t)} \bar{D}_t^n(\omega, x_t, y)$ and $\hat{y}_{t+1}^n \in K_t(S_t, x_t)$ a.s.. By [20, Lemma 2.1.2], we may suppose that $\hat{y}_{t+1}^n \in K_t(S_t, x_t)$ is convergent for some random subsequence towards a $\mathcal{F}_{t+1}$-measurable limit $\hat{y}_{t+1}^0 \in K_t(S_t, x_t)$. Moreover, by the first step, we have $\hat{y}_{t+1}^0 \in \arg\min_{K_t(S_t, x_t)}(\tilde{D}_t^\xi(S_t, x_t, .))$.

If $K_t(S_t, x_t)$ is $\mathcal{F}_t$-measurable, consider a Castaing representation $(z_t^m)_{m \geq 1}$ of $K_t(S_t, x_t)$. The generalized conditional expectation $E(\hat{y}_{t+1}^n | \mathcal{F}_t)$ exists as $\hat{y}_{t+1}^n \in K_t(S_t, x_t)$ is $\mathcal{F}_t$-bounded. Note that $\hat{y}_{t+1}^n$ may be approximated by a sequence of $\mathcal{F}_{t+1}$-measurable random variables in the set $\{z_t^m : \ m \geq 1\}$. We deduce that $E(\hat{y}_{t+1}^n | \mathcal{F}_t) \in K_t(S_t, x_t)$ if $K_t(S_t, x_t)$ is convex. It is clear that $E(\hat{y}_{t+1}^n | \mathcal{F}_t)$ converges to $E(\hat{y}_{t+1}^0 | \mathcal{F}_t) \in K_t(S_t, x_t)$.

When the cost function is convex, $\bar{D}_t^n(\omega, x_t, y)$ is convex. Using the Jensen inequality for conditional expectations, we get that

$$\begin{aligned}
\tilde{D}_t^\xi(S_t, x_t, E(\hat{y}_{t+1}^0 | \mathcal{F}_t)) &\leq E\left(\tilde{D}_t^\xi(S_t, x_t, \hat{y}_{t+1}^0) | \mathcal{F}_t\right), \\
&\leq E\left(\inf_{y \in \mathbf{R}^d} \tilde{D}_t^\xi(S_t, x_t, y) | \mathcal{F}_t\right), \\
&\leq \inf_{y \in \mathbf{R}^d} \tilde{D}_t^\xi(S_t, x_t, y).
\end{aligned}$$

The last inequality holds since $\inf_{y \in \mathbf{R}^d} \tilde{D}_t^\xi(S_t, x_t, y)$ is $\mathcal{F}_t$-measurable. This implies that $E(\hat{y}_{t+1}^0 | \mathcal{F}_t) \in \arg\min(\tilde{D}_t^\xi(S_t, x_t, .))$.

□

3.2. Multi-period framework

In this section, we consider the multi-period setting $t = 0, \cdots, T$. Our goal is to determine the infimum super-hedging cost of $\xi := g(S_T) = (g^1(S_T), g^{(2)}(S_T))$ at time 0, where $g : \mathbf{R}_+^k \to \mathbf{R}_+^d$ is a deterministic continuous function. To do so, we apply the dynamic programming principle of Proposition 2.1 to recursively compute $\gamma_t^\xi(V_{t-1})$ for $t = 0, \cdots, T$. Moreover, since $\gamma_0^\xi(0) = \tilde{\gamma}_0^\xi(S_0, 0)$ under the weak no-arbitrage condition we suppose, it is then sufficient to compute $\tilde{\gamma}_0(S_0, V_0)$ for $V_0 = 0$. We work under the following assumption:

Assumption 4. *For each t, suppose that there is a reachability set-valued mapping $K_t : \mathbf{R}_+^k \times \mathbf{R}^d \twoheadrightarrow \mathbf{R}^d$ such that $K_t(s, v_{t-1})$ is a compact upper hemicontinuous set-valued mapping, i.e.*

$$\inf_{y \in \mathbf{R}^d} \tilde{D}_t^\xi(s, x, y) = \inf_{y \in K_t(s,x)} \tilde{D}_t^\xi(s, x, y), \text{ a.s..}$$

For simplicity, we consider the model where the price process satisfies

$$\mathrm{supp}_{\mathcal{F}_t}(S_{t+1}) = \{a_t S_t : \ a_t \in \Theta\}, \, t \le T - 1,$$

such that $P[S_{t+1} = a_t S_t | \mathcal{F}_t] > 0$ a.s. for all $a_t \in \Theta$, where $\Theta = \{a_t^n, n \ge 1\}$ is a deterministic sequence of positive numbers. Consider a sequence of random variables $\{b_t^i, i \in J_t, \, t = 0, \cdots, T\}$ in $\mathbf{R}^{k \times T}$ generated by the following procedure:

1) $b_0^i = S_0$ for all $i \in J_0 = \mathbb{N} \setminus \{0\}$.

2) For given $t \ge 0$, we denote $\tilde{\mathcal{F}}_t = \sigma(b_u^k : \ k \in J_u, u \le t)$ where $(b_u^k)_{k \in J_u}$ are the random variables constructed at time t. Then, for time $t + 1$, and for each $i \in J_t$, we generate a sequence of i.i.d. random variables α_{t+1}^j, $j \ge 1$, independent of $\mathcal{F}_t$ such that $\alpha_{t+1}^j \in L^0(\Theta, \mathcal{F}_{t+1})$ for each j. Moreover, $\mathrm{supp}_{\mathcal{F}_t} \alpha_{t+1}^j = \Theta$. We then define for each $i \in J_t$ and $j \ge 1$, $b_{t+1}^{i,j} = \alpha_{t+1}^j b_t^i$. Then, $J_{t+1} = \{(i, j) : \ i \in J_t, \, j \ge 1\}$.

To compute $\tilde{\gamma}_0^\xi(S_0, 0)$, we approximate $\tilde{\gamma}_t^\xi(b_t^i, v_{t-1})$ by the randomization method considered in the last section that we extend to the multi-period setting.

We denote $\mathbf{n}^1 = (\mathbf{n}_u^1)_{u=1,\cdots,T}$ a generic element in $\mathbb{N}^T$ and, for $t = 1, \cdots, T$, we define $\mathbf{n}^t = (\mathbf{n}_u^t)_{u=t,\cdots,T} \in \mathbb{N}^{T-t+1}$. If $b_t^i \in \{\alpha_t^k b_{t-1}^j ; \, j \in J_{t-1}, k \ge 1\}$, $i \in J_t$, we set:

$$\hat{\theta}_{T-1}^{\mathbf{n}^T}(b_{T-1}^i, v_{T-1}) := \max_{m\le \mathbf{n}_T^T} \tilde{\gamma}_T^{\xi}(\alpha_T^m b_{T-1}^i, v_{T-1}),$$

$$\hat{\theta}_t^{\mathbf{n}^{t+1}}(b_t^i, v_t) := \max_{m\le \mathbf{n}_{t+1}^{t+1}} \hat{\gamma}_{t+1}^{\mathbf{n}^{t+2}}(\alpha_{t+1}^m b_t^i, v_t),\ \mathbf{n}^{t+2} = (\mathbf{n}_u^{t+1})_{u=t+2,\cdots,T},\ t\le T-1,$$

$$\hat{D}_t^{\mathbf{n}^{t+1}}(b_t^i, v_{t-1}, v_t) := \hat{\theta}_t^{\mathbf{n}^{t+1}}(b_t^i, v_t) + \mathrm{C}_t(b_t^i, (0, v_t^{(2)} - v_{t-1}^{(2)})),\ t\le T-1,$$

$$\hat{\gamma}_t^{\mathbf{n}^{t+1}}(b_t^i, v_{t-1}) := \inf_{v_t\in K_t(b_t^i, v_{t-1})} \hat{D}_t^{\mathbf{n}^{t+1}}(b_t^i, v_{t-1}, v_t),\ t\le T-1.$$

Note that by assumption

$$\tilde{\gamma}_T^{\xi}(s, v_{T-1}) := g^1(s) + \mathrm{C}_T(s, (0, g^2(s) - v_{T-1}^{(2)})).$$

Therefore, $\tilde{\gamma}_T^{\xi}$ is l.s.c. Since K_t is an upper hemicontinuous compact set-valued mapping by assumption, see [22, Corollary 5.14 and Proof of Theorem 4.15], and $\hat{D}_t^{\mathbf{n}_{t+1}}$ is l.s.c. by induction, $\hat{\gamma}_t^{\mathbf{n}_{t+1}}(b_t^i, v_{t-1})$ is l.s.c. in b_t^i and v_{t-1} by [1, Lemma 17.29].

The following theorem is our main contribution of this section. We use the convention that $\mathbf{n}^1\to\infty$, $\mathbf{n}^1\in\mathbb{N}^T$, if and only if $\mathbf{n}_i^1\to\infty, \forall i=1,\cdots,T$.

Theorem 3.15. *[Limit theorem to approximate the infimum super-hedging price] Suppose that Assumption 4 holds and suppose that* C_t *satisfies* $\mathrm{C}_t(s, v_t^1)\ge \mathrm{C}_t(s, v_t^2)$ *whenever* $v^1 \ge_{\mathbf{R}_+^d} v_t^2$. *Then:*

$$\lim_{\mathbf{n}^1\to\infty} \hat{\gamma}_0^{\mathbf{n}^1}(S_0, 0) = \tilde{\gamma}_0^{\xi}(S_0, 0),\ \text{a.s..}$$

Proof. By Remark 3.13, Assumption 4 implies that

$$\tilde{\gamma}_0^{\xi}(S_0, 0) = \inf_{v_1\in K_0(S_0,0)} \tilde{D}_0^{\xi}(S_0, 0, v_1)$$

where $K_0(S_0,0)$ is a compact set-valued mapping. Moreover, since $\tilde{\gamma}_{t+1}^{\xi}(., v_t)$ is l.s.c. hence Borel, Theorem 3.14 applies when we replace S_t by each random variable $b_t^i\in\left\{\alpha_t^k b_{t-1}^j;\ j\in J_{t-1}, k\ge 1\right\}$. Precisely, in accordance with (3.7), we shall consider:

$$\bar{D}_t^{\mathbf{n}_{t+1}^{t+1}}(b_t^i, v_{t-1}, v_t) = \sup_{n\le n_{t+1}^s} \tilde{\gamma}_{t+1}^{\xi}(\alpha_t^n b_t^i, v_t) + \mathrm{C}_t(\alpha_t^n b_t^i, (0, v_t^{(2)} - v_{t-1}^{(2)})), t\le T-1,$$

$$\bar{\gamma}_t^{\mathbf{n}_{t+1}^{t+1}}(b_t^i, v_{t-1}) := \inf_{v_t\in K_t(b_t^i, v_{t-1})} \bar{D}_t^{\mathbf{n}_{t+1}^{t+1}}(b_t^i, v_{t-1}, v_t),\ t\le T-1,$$

$$\sup_{\mathbf{n}_{t+1}^{t+1}} \bar{\gamma}_t^{\mathbf{n}_{t+1}^{t+1}}(b_t^i, v_{t-1}) = \tilde{\gamma}_t^{\xi}(b_t^i, v_{t-1}),\ t\le T-1,\ \text{by Theorem 3.14.} \tag{3.14}$$

We now prove by induction that $\lim_{\mathbf{n}^1\to\infty}\hat{\gamma}_0^{\mathbf{n}^1}(S_0,0)=\tilde{\gamma}_0^{\xi}(S_0,0)$ a.s. Observe that, at time $T-1$, $\mathbf{n}^T=: n^T\in N$ and $\hat{\gamma}_{T-1}^{\mathbf{n}^T}(b^i_{T-1},v_{T-1})$ and $\bar{\gamma}_{T-1}^{n^T}(b^i_{T-1},v_{T-1})$ coincide. So, by Theorem 3.14, we have

$$\lim_{\mathbf{n}^T\to\infty}\hat{\gamma}_{T-1}^{\mathbf{n}^T}(b^i_{T-1},v_{T-2})=\lim_{n^T\to\infty}\bar{\gamma}_{T-1}^{n^T}(b^i_{T-1},v_{T-2})=\tilde{\gamma}_{T-1}^{\xi}(b^i_{T-1},v_{T-2})$$

Now, we suppose that $\sup_{\mathbf{n}^{t+2}\in\mathbb{N}^{T-t-1}}\hat{\gamma}_{t+1}^{\mathbf{n}^{t+2}}(b^i_{t+1},v_t)=\tilde{\gamma}_{t+1}^{\xi}(b^i_{t+1},v_t)$ for any $b^i_{t+1}\in\{\alpha^k_{t+1}b^j_t;\, j\in J_t, k\geq 1\}$. We have by definition:

$$\begin{aligned}\hat{D}_t^{\mathbf{n}^{t+1}}(b^i_t,(0,v^{(2)}_t-v^{(2)}_{t-1})) &= \hat{\theta}_t^{\mathbf{n}^{t+1}}(b^i_t,v_t)+\mathrm{C}_t(b^i_t,(0,v^{(2)}_t-v^{(2)}_{t-1}))\\ &= \max_{m\leq n^{t+1}_{t+1}}\hat{\gamma}_{t+1}^{\mathbf{n}^{t+2}}(\alpha^m_t b^i_t,v_t)+\mathrm{C}_t(b^i_t,(0,v^{(2)}_t-v^{(2)}_{t-1})),\\ &\quad \mathbf{n}^{t+2}=(n^{t+1}_u)_{u=t+2,\cdots,T}.\end{aligned}$$

Consider the directed set of all $\mathbf{n}^{t+1}\in\mathbb{N}^{T-t}$ endowed with the partial order $\mathbf{n}^{t+1}\geq\mathbf{m}^{t+1}$ if and only if $\mathbf{n}^{t+1}_i\geq\mathbf{m}^{t+1}_i$ for all $t+1\leq i\leq T$. By construction and by induction, it is easy to check that $(\hat{D}^{\mathbf{n}}_t)_{\mathbf{n}\in N^{[t+1,T]}}$ is increasing, i.e. $\bar{D}^{\mathbf{n}}_t\geq\bar{D}^{\mathbf{m}}_t$ whenever $\mathbf{n}\geq\mathbf{m}$. Also, we may show by induction that $\hat{D}^{\mathbf{n}}_t(b^i_t,.)$ is l.s.c. for all $\mathbf{n}$. By Lemma 3.16 that allows us to exchange the supremum and infimum in the following first equality, plus the induction hypothesis, we deduce that

$$\begin{aligned}
\sup_{\mathbf{n}^{t+1}} \hat{\gamma}_t^{\mathbf{n}_{t+1}}(b_t^i, v_{t-1}) &= \sup_{\mathbf{n}^{t+1}} \inf_{v_t \in K_t(b_t^i, v_{t-1})} \hat{D}_t^{\mathbf{n}^{t+1}}(b_t^i, v_{t-1}, v_t) \\
&= \inf_{v_t \in K_t(b_t^i, v_{t-1})} \sup_{\mathbf{n}^{t+1}} \hat{D}_t^{\mathbf{n}^{t+1}}(b_t^i, v_{t-1}, v_t) \\
&= \inf_{v_t \in K_t(b_t^i, v_{t-1})} \sup_{\mathbf{n}_{t+1}^{t+1} \in \mathbb{N}} \sup_{\mathbf{n}^{t+2}} \hat{D}_t^{\mathbf{n}^{t+1}}(b_t^i, v_{t-1}, v_t) \\
&= \inf_{v_t \in K_t(b_t^i, v_{t-1})} \sup_{\mathbf{n}_{t+1}^{t+1} \in \mathbb{N}} \sup_{\mathbf{n}^{t+2}} \Bigg(\max_{m \leq \mathbf{n}_{t+1}^{t+1}} \hat{\gamma}_{t+1}^{\mathbf{n}^{t+2}}(\alpha_{t+1}^m b_t^i, v_t) \\
&\qquad + \mathrm{C}_t(b_t^i, (0, v_t^{(2)} - v_{t-1}^{(2)})) \Bigg) \\
&= \inf_{v_t \in K_t(b_t^i, v_{t-1})} \sup_{\mathbf{n}_{t+1}^{t+1} \in \mathbb{N}} \max_{m \leq \mathbf{n}_{t+1}^{t+1}} \Bigg(\sup_{\mathbf{n}^{t+2}} \hat{\gamma}_{t+1}^{\mathbf{n}^{t+2}}(\alpha_{t+1}^m b_t^i, v_t) \\
&\qquad + \mathrm{C}_t(b_t^i, (0, v_t^{(2)} - v_{t-1}^{(2)})) \Bigg) \\
&= \inf_{v_t \in K_t(b_t^i, v_{t-1})} \sup_{\mathbf{n}_{t+1}^{t+1} \in \mathbb{N}} \max_{m \leq \mathbf{n}_{t+1}^{t+1}} \Bigg(\tilde{\gamma}_{t+1}^{\xi}(\alpha_{t+1}^m b_t^i, v_t) \\
&\qquad + \mathrm{C}_t(b_t^i, (0, v_t^{(2)} - v_{t-1}^{(2)})) \Bigg) \\
&= \inf_{v_t \in K_t(b_t^i, v_{t-1})} \tilde{D}_t^{\xi}(b_t^i, , v_{t-1}, v_t) = \tilde{\gamma}_t^{\xi}(b_t^i, v_{t-1}).
\end{aligned}$$

To deduce the last two equalities, we use the definition of $\tilde{\theta}_t^{\xi}(b_t^i, , v_{t-1}, v_t)$ and $\tilde{D}_t^{\xi}(b_t^i, , v_{t-1}, v_t)$, see (3.6) but also (3.9) in Remark 3.13. The conclusion follows by induction. □

In the proof above, we have used the following lemma:

Lemma 3.16 (Dini-Cartan)**.** *Consider a family of l.s.c. functions* $(f_n)_{n \in I}$*,* $f_n : \mathbf{R}^d \to \overline{\mathbf{R}}$ *such that for every finite set* $J \subset I$*, there is* $n_0 \in I$ *with* $\sup_{j \in J} f_j \leq f_{n_0}$*. Consider a compact set* G*, then the following holds:*

$$\sup_n \inf_{x \in G} f_n(x) = \inf_{x \in G} \sup_n f_n(x).$$

Proof. By considering an increasing homeomorphism from $[-\infty, +\infty]$ onto $[0, 1]$, we then restrict ourselves to the case $\sup_n f_n$ is bounded. It is clear that

$\sup_n \inf_{x\in G} f_n(x) \le \inf_{x\in G} \sup_n f_n(x)$ so that the inequality holds if the second term is $-\infty$. For the reverse inequality, consider any $a < \inf_{x\in G} \sup_n f_n(x)$. For all $x \in G$, we have $a < \sup_n f_n(x)$. Then, there exists some $k = k_x$ such that $a < f_k(x)$. Note that the set $O_k := \{x : a < f_k(x)\}$ is open since f_k is l.s.c. By compactness argument, we deduce a finite covering of G by some O_{k_i}, $j = 1, \cdots, N$. By our hypothesis, there exists n_0 such that $a \le f_{k_i}(x) \le f_{n_0}(x)$, for all $x \in G$ and $i = 1, \cdots, N$ hence we have $a \le \inf_{x\in G} f_{n_0}(x) \le \sup_n \inf_{x\in G} f_n(x)$. □

Lemma 3.17. *For all t , for all $j \in J_{t+1}$, consider $b^j_{t+1} = \alpha^k_{t+1} b^i_t$ where $i \in J_t$ and $k \ge 1$. Then, $b^j_{t+1} \in \{a^n_t b^i_t, n \ge 1\}$ a.s. and $P[b^j_{t+1} = a^n_t b^i_t|\mathcal{F}_t] > 0$ a.s. Moreover, $\{b^j_{t+1}, j \in J_{t+1}\}$ are $\mathcal{F}_t$-i.i.d.*

Proof. For all $n \ge 1$, we have almost surely :

$$P\left[b^j_{t+1} = a^n_t b^i_t|\mathcal{F}_t\right] = P\left[\alpha^k_{t+1} b^i_t = a^n_t b^i_t|\mathcal{F}_t\right] \ge P\left[\alpha^k_{t+1} = a^n_t|\mathcal{F}_t\right] > 0.$$

The last statement follows directly from Lemma 3.3 as $(\alpha^j_{t+1})_{j\ge 1}$ are $\mathcal{F}_t$-i.i.d. by assumption.

□

4. Model with one risky asset and piecewise linear costs

As we may observe in the previous section, the reachability set-valued mapping plays an important role in propagating the lower semicontinuity which, in turn, propagates the convergence property. We consider in this section a special case of convex cost functions and provide explicit expressions for the minimal super-hedging costs. In particular, under SAIP condition, we obtain an explicit expression of the reachability set $K_t(s, v_{t-1})$ when the payoff is of linear growth, i.e. $\xi = (\xi^1, \xi^2) \le_{\mathbf{R}^2_+} (aS_T + b, c)$ for some $a, b, c \in \mathbf{R}_+$.

We suppose the the market consists of one risk-free asset and one risky asset denoted by $(S_t)_{0\le t\le T}$. We impose the following assumption for the conditional support of the price and cost processes.

Assumption 5. *The price process satisfies $S_{t+1} \in \{a^n_t S_t, n \ge 1\}$ where the sequence $(a^n_t)_{n\ge 1}$ is deterministic and satisfies $a^1_t = \min_n a^n_t = k^d_t \ge 0$, $a^2_t = \max_n a^n_t = k^u_t \in \mathbf{R}_+$, where k^d_t, k^u_t are deterministic. The cost process C_t is given by $\mathrm{C}_t(S_t, (x, v_t)) = x + S_t \tilde{\mathrm{C}}_t(v^2_{t-1})$ for some deterministic piecewise linear function $\tilde{\mathrm{C}}_t : \mathbf{R} \to \mathbf{R}$.*

We recall the AEP condition in [22]

Definition 4.1. *We say that the financial market satisfies the Absence of Early Profit condition (AEP) if, at any time* $t \le T$*, and for all* $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$*,* $\gamma_t^0(V_t) > -\infty$ *a.s..*

By Lemma 4.11 in [22], AIP implies AEP if the cost function C_t is either sub-additive or super-additive. Moreover, by Theorem 4.5 in [22], AEP implies that $\tilde{\gamma}_t^\xi(S_t, .) > -\infty$ a.s. This property will be used in the proof of the following result.

Proposition 4.2. *Suppose that Condition AEP and Assumption 5 hold. Then the minimal hedging cost of the payoff* $\xi = (mS_T + G, K)$*,* $m, G, K \in \mathbf{R}$*, is given by* $\tilde{\gamma}_t^\xi(S_t, v_{t-1}) = G + S_t h_t(v_{t-1}^2)$*, where* $h_t : \mathbf{R} \to \mathbf{R}$ *is a deterministic piecewise linear function.*

Moreover, $\tilde{D}_t(S_t, v_t, v_{t-1}) = S_t \tilde{h}_t(v_t, v_{t-1})$ *for some deterministic piecewise linear function* $\tilde{h}_t : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$*.*

Proof. We first show by induction that, if $\tilde{\gamma}_{t+1}^\xi(S_{t+1}, v_t) = S_{t+1}\tilde{f}_{t+1}(v_t^2)$ where $\tilde{f}_{t+1} : \mathbf{R} \to \mathbf{R}$ is a piecewise linear function, then $\tilde{\gamma}_t^\xi(S_t, v_{t-1}) = S_t \tilde{f}_t(v_{t-1}^2)$ for some piecewise linear function $\tilde{f}_t : \mathbf{R} \to \mathbf{R}$. To do so, observe that:

$$\begin{aligned}\tilde{\theta}_t^\xi(S_t, v_t) = \sup_{s \in \{a_t^n S_t, n \ge 1\}} \left(s\tilde{f}_{t+1}(v_t^2)\right) &= \max\left\{k_t^d S_t \tilde{f}_{t+1}(v_t^2), k_t^u S_t \tilde{f}_{t+1}(v_t^2)\right\} \\ &= S_t \max\left\{k_t^d \tilde{f}_{t+1}(v_t^2), k_t^u \tilde{f}_{t+1}(v_t^2)\right\}.\end{aligned}$$

Since $\tilde{f}_{t+1}$ is piecewise linear function by the hypothesis, we deduce that $\tilde{g}_t(v_t^2) := \max\{k_t^d \tilde{f}_{t+1}(v_t^2), k_t^u \tilde{f}_{t+1}(v_t^2)\}$ is also piecewise linear by [26, Proposition 3.55]. Therefore,

$$\begin{aligned}\tilde{\gamma}_t^\xi(S_t, v_{t-1}) = \inf_{v^2 \in \mathbf{R}} \tilde{D}_t^\xi(S_t, v_{t-1}, v_t) &= \inf_{v^2 \in \mathbf{R}} \left(\tilde{\theta}_t^\xi(S_t, v_t) + \mathrm{C}_t(S_t, v_t^2 - v_{t-1}^2)\right) \\ &= S_t \inf_{v_t^2 \in \mathbf{R}} \left(\tilde{g}_t(v_t^2) + \tilde{\mathrm{C}}_t(v_t^2 - v_{t-1}^2)\right).\end{aligned}$$

By [26, Proposition 3.55], we also deduce that $\tilde{g}_t(v_t^2) + \tilde{\mathrm{C}}_t(v_t^2 - v_{t-1}^2)$ is a piecewise linear function in (v_t^2, v_{t-1}^2). Moreover, under AEP, we know that $\tilde{\gamma}_t^\xi(S_t, v_{t-1}) > -\infty$ a.s.. Therefore, by [26, Proposition 3.55],

$$\tilde{f}_t(v_{t-1}^2) := \inf_{v^2 \in \mathbf{R}} \left(\tilde{g}_t(v_t^2) + \tilde{\mathrm{C}}_t(v_t^2 - v_{t-1}^2)\right)$$

is a piecewise linear function in v_{t-1}^2.

If the payoff is $\xi = (mS_T + G, K)$, then $\tilde{\gamma}_T^\xi(S_T, v_{T-1}) = G + S_T \tilde{f}_T(v_{T-1}^2)$ where $\tilde{f}_T(v_{T_1}^2) := m + \tilde{\mathrm{C}}_T(K - v_{T-1}^2)$ is a piecewise linear function by assumption on C_T. We then argue by induction as previously done to deduce that $\gamma_{t-1}^\xi(S_{t-1}, v_{t-2}) = G + S_{t-1}\tilde{f}_{t-1}(v_{t-2}^2)$ for some piecewise linear function $\tilde{f}_{t-1}$.

At last, since $D_t^\xi(S_t, v_t, v_{t-1}) = \tilde{\theta}_t(S_t, v_t) + \mathrm{C}_t(S_t, (0, v_t^{(2)} - v_{t-1}^{(2)}))$, the conclusion on $\tilde{D}_t$ follows. □

The following is our main result of this section. It states the existence of the reachability set under SAIP.

Proposition 4.3. *Suppose that the payoff $\xi = (g^1(S_T), g^2(S_T))$ satisfies $g^1(S_T) \leq aS_T + b$ and $g^2(S_T) \leq c$ for some $a, b, c \in \mathbf{R}_+$. We also suppose that $\mathrm{C}_t(s, v^1) \geq \mathrm{C}_t(s, v^2)$ whenever $v^1 \geq_{\mathbf{R}_+^2} v^2$ and suppose that $\mathrm{C}_t(s, .)$ is subadditive and 1-homogeneous.*

*Under the no-arbitrage condition SAIP, the reachability set $K_t(s, v_{t-1})$ is defined for **every** $(s, v_{t-1}) \in \mathbf{R} \times \mathbf{R}$ and is explicitly given by:*

$$K_t(s, v_{t-1}) = \bar{B}_t(0, r_t(s, v_{t-1}) + 1)$$

where $r_t(s, v_{t-1}) = sf_t(v_{t-1})/g_t(s)$ and f_t, g_t are deterministic piecewise linear functions such that $g_t(s) > 0$ for all $s > 0$.

Proof. We define $\tilde{\xi} := (aS_T + b, c)$ so that $\xi \leq_{\mathbf{R}_+^2} \tilde{\xi}$. We show by induction that $\tilde{D}_t^0(s, v_{t-1}, v_t) \leq \tilde{D}_t^\xi(s, v_{t-1}, v_t) \leq \tilde{D}_t^{\tilde{\xi}}(s, v_{t-1}, v_t)$. By the proof of [22, Theorem 4.15], we get that

$$K_t(s, v_{t-1}) \subseteq \left\{ v_t : \tilde{D}_t^\xi(s, v_{t-1}, v_t) \leq \tilde{D}_t^{\tilde{\xi}}(s, v_{t-1}, 0) \right\}$$

Moreover, by sub-additivity and 1-homogeneity.

$$\tilde{D}_t^0(s, v_{t-1}, v_t) = \mathrm{C}_t(s, (0, v_t^2 - v_{t-1}^{(2)})) + \tilde{\theta}_t^0(s, v_t) \geq -\mathrm{C}_t(s, (0, v_{t-1}^2)) + \tilde{D}_t^0(s, 0, v_t)$$
$$\tilde{D}_t^0(s, 0, v_t) \geq |v_t| \tilde{D}_t^0(s, 0, v_t/|v_t|) \geq |v_t| \min_{z \in \{-1,1\}} \tilde{D}_t^0(s, 0, z), \ \forall\ |v_t| \geq 1.$$

We deduce that $K_t(S_t, v_{t-1}) \subseteq \bar{B}(0, r_t(S_t, v_{t-1}) + 1)$, where the radius $r_t(S_t, v_{t-1})$ is given by

$$r_t(S_t, v_{t-1}) := \frac{\tilde{D}_t^{\tilde{\xi}}(S_t, v_{t-1}, 0) + \mathrm{C}_t(S_t, (0, v_{t-1}^2))}{\min_{z \in \{-1,1\}} \tilde{D}_t^0(S_t, 0, z)} =: \frac{S_t f_t(v_{t-1}^2)}{g_t(S_t)}.$$

Note that by Proposition 4.2, $f_t : \mathbf{R} \to \mathbf{R}$ and $g_t : \mathbf{R} \to \mathbf{R}$ are deterministic piecewise linear functions. Moreover, we have $g_t(S_t) = S_t \inf_{z \in \{-1,1\}} a_t(z)$ for some deterministic piecewise linear function a_t. Since SAIP holds, we deduce that $\inf_{z \in \{-1,1\}} a_t(z) > 0$. We then define $g_t(s) := s \inf_{z \in \{-1,1\}} a_t(z) > 0$ for all $s > 0$. The conclusion follows. □

5. Examples

In this section, we consider two classical examples. The first one corresponds to the market with proportional transaction cost and the second one is with fixed cost. We provide the explicit expression of the reachability set-valued mapping K_t for the Put option. Then, as a by-product, the minimal super-hedging cost for Put option is computed.

For a sake of simplicity, we consider the binomial market model, i.e. the price process satisfies $\mathrm{supp}_{\mathcal{F}_t} S_{t+1} = \left\{ k_t^d S_t, k_t^u S_t \right\}$, where $k_t^d, k_t^u \in \mathbf{R}_+$.

5.1. Market model with proportional transaction costs

We consider a particular case of section 4 where

$$\mathrm{C}_t(S_t, v) = v^1 + (1+\epsilon_t) S_t v^2 1_{v^2 \geq 0} + (1-\epsilon_t) S_t v^2 1_{v^2 \leq 0}. \tag{5.15}$$

for some deterministic coefficient $\epsilon_t \in \mathbf{R}_+$. By a direct computation, see Appendix, we obtain the following

Proposition 5.1. *If $v_{t-1} \in \mathbf{R}^2$, the following holds:*

$$\begin{aligned} \tilde{\theta}_{t-1}^0(S_{t-1}, v) &= -(1-\epsilon_t) k_{t-1}^d S_{t-1} v^2 1_{v^2 \geq 0} - (1+\epsilon_t) k_{t-1}^u S_{t-1} v^2 1_{v^2 \leq 0} \\ \tilde{D}_{t-1}^0(S_{t-1}, 0, v) &= ((1+\epsilon_{t-1}) S_{t-1} - (1-\epsilon_t) k_{t-1}^d S_{t-1}) v^2 1_{v^2 \geq 0} \\ &\quad + ((1-\epsilon_{t-1}) S_{t-1} - (1+\epsilon_t) k_{t-1}^u S_{t-1}) v^2 1_{v^2 \leq 0} \end{aligned}$$

Moreover, AIP_{t-1} *holds if and only if:*

$$k_{t-1}^d \leq \frac{1+\epsilon_{t-1}}{1-\epsilon_t} \text{ and } k_{t-1}^u \geq \frac{1-\epsilon_{t-1}}{1+\epsilon_t}. \tag{5.16}$$

Moreover, SAIP_{t-1} *holds if and only if the above inequalities are strict. If* AIP_{t-1} *holds, we then deduce that:*

$$\begin{aligned} \inf_{v^2 \in \{-1,1\}} \tilde{D}_{t-1}^0(S_{t-1}, 0, v) = S_{t-1} \min \big\{ &(1+\epsilon_{t-1}) - (1-\epsilon_t) k_{t-1}^d, \\ &(1+\epsilon_t) k_{t-1}^u - (1-\epsilon_{t-1}) \big\}. \end{aligned}$$

Proof. Recall that AIP_{t-1} holds if and only if $\tilde{D}^0_{t-1}(S_{t-1},0,v) \geq 0$ for any $v \in \mathbf{R}^d$ which is equivalent to (5.16). Moreover, suppose that SAIP_{t-1} holds. If $k^d_{t-1} = \dfrac{1+\epsilon_{t-1}}{1-\epsilon_t}$, $D^0_{t-1}(S_{t-1},0,v) = 0$ for any $v^2 > 0$, i.e. SAIP_{t-1} fails. Similarly, we get that $k^u_{t-1} > (1-\epsilon_{t-1})/(1+\epsilon_t)$. At last, suppose that the inequalities in (5.16) are strict. Since $S_{t-1} > 0$ a.s.,

$$\inf_{v^2\in\{-1,1\}} \tilde{D}^0_{t-1}(S_{t-1},0,v) > 0, \text{ a.s.}$$

so that SAIP_{t-1} holds by [22, Theorem 4.15]. □

We apply the result above at time T and we proceed by induction, see Appendix, to deduce the following result at time $T-2$.

Proposition 5.2. *Assume that* $1+\epsilon_{T-1} \leq (1+\epsilon_T)k^u_{T-1}$ *and* $1-\epsilon_{T-1} \geq (1-\epsilon_T)k^d_{T-1}$*, we have:*

$$\begin{aligned}
\tilde{\theta}^0_{T-2}(S_{T-2},z) &= -(1+\epsilon_{T-1})k^d_{T-2}S_{T-2}z^2 1_{z^2\geq 0} - (1-\epsilon_T)k^d_{T-1}k^u_{T-2}S_{T-2}z^2 1_{z^2\leq 0},\\
\tilde{D}^0_{T-2}(S_{T-2},0,z) &= \big((1+\epsilon_{T-2})S_{T-2} - (1+\epsilon_{T-1})k^d_{T-2}S_{T-2})\big)\, z^2 1_{z^2\geq 0}\\
&\quad + \big((1-\epsilon_{T-2})S_{T-2} - (1-\epsilon_T)k^d_{T-1}k^u_{T-2}S_{T-2}\big)\, z^2 1_{z^2\leq 0}.
\end{aligned}$$

and AIP_{T-2} *holds if and only if:*

$$k^d_{T-2} \leq \frac{1+\epsilon_{T-2}}{1+\epsilon_{T-1}} \text{ and } k^u_{T-2} \geq \frac{1-\epsilon_{T-2}}{(1-\epsilon_T)k^d_{T-1}}.$$

Moreover, SAIP_{T-2} *holds if and only if the above inequalities are strict. Moreover, under* SAIP_{T-2}*, we have:*

$$\begin{aligned}
\inf_{v^2\in\{-1,1\}} \tilde{D}^0_{T-2}(S_{T-2},0,v) = S_{T-2}\min\big\{&((1+\epsilon_{T-2}) - (1-\epsilon_{T-1})k^d_{T-2}),\\
&-((1-\epsilon_{T-2}) - (1+\epsilon_T)k^d_{T-1}k^u_{T-2})\big\}.
\end{aligned}$$

The assumptions of Proposition 5.2 are chosen for a sake of simplification. The computations for $t < T-2$ are similar. In particular, for a Put option with payoff $(K-S_T)^+$, $K>0$, we obtain a simple formula for the reachability set.

Lemma 5.3. *Suppose that SAIP holds and* $\xi = (g(S_T),0)$ *where* g *is a continuous function bounded from above by a constant* $M \in \mathbf{R}_+$*. Then, there* *exists a reachability set* $K_t(s, v_{t-1}) = \bar{B}_t(0, r_t(s, v_{t-1}) + 1)$, $t \leq T - 1$, *closed ball of radius* $r_t(s, v_{t-1}) := \lambda_t(s, v_{t-1})/i_t(s)$ *where the functions*

$$i_t(s) := \inf_{v^2 \in \{-1,1\}} \tilde{D}_t^0(s, 0, v),$$
$$\lambda_t(s, v_{t-1}) := \mathrm{C}_t(s, (0, v_{t-1}^2)) + M + \mathrm{C}_t(s, (0, -v_{t-1}^2)),$$

are explicitely given by Proposition 5.1 and Proposition 5.2. In particular, we have $i_t(s) > 0$ *for all* $s > 0$.

We illustrate the results above by a numerical example. We consider the put option payoff $g(S_T) := (K - S_T)^+$ at time $T = 2$. We suppose that the proportional cost coefficients $\epsilon_1 = \epsilon_2 = 0.02$. We assume that SAIP condition holds and choose $k_2^d = 0.9, k_2^u = 1.1$, $k_1^d = 0.9, k_1^u = 1.2$. The price function at time $t = 0$ is presented in Figure 1.

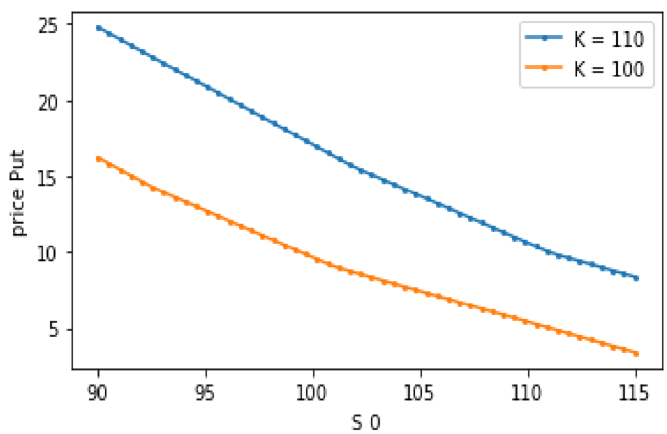

Fig 1: Price of Put option

We also visualize the ratio of put option to asset price S_0

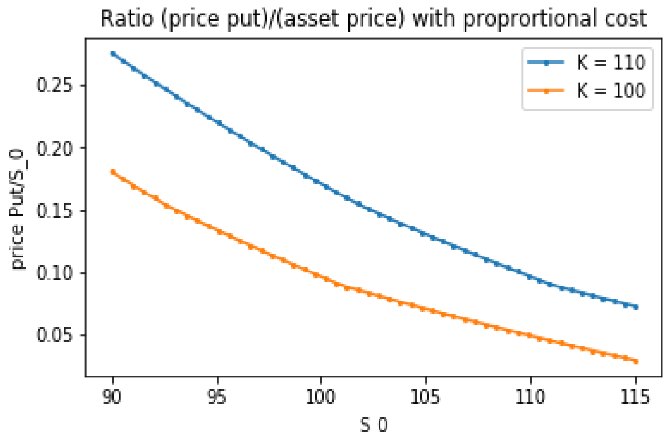

Fig 2: Ratio Put option/Asset Price

5.2. Market model with fixed cost

In this section, we consider a financial market model in presence of both proportional and fixed costs modeled by the following liquidation and cost functions:

$$\text{Ł}_t(S_t, v_t) := v_t^1 + (v_t^2(1-\epsilon_t)S_t - c_t)^+ 1_{v_t>0} + (v_t^2(1+\epsilon_t)S_t - c_t)1_{v_t<0}$$
$$\mathrm{C}_t(S_t, v_t) := -\text{Ł}_t(S_t, -v_t).$$

for some deterministic constant $c_t > 0$ representing the fixed cost we need to pay to obtain a non-null position.

In this model, the cost function does not satisfy the condition property that $\mathrm{C}_t(S_t, \lambda z) \geq \lambda \mathrm{C}_t(S_t, z)$ for any $\lambda \geq 1$. Then, the propagation of lower semicontinuity is not guaranteed if we only assume the SAIP condition on the market defined by this cost function. In [22], we have introduced the *horizon* cost function defined as follows:

$$\mathrm{C}_t^{\infty}(s, y) := \liminf_{\alpha \to \infty} \frac{\mathrm{C}_t(s, \alpha y)}{\alpha}. \tag{5.17}$$

Definition 5.4. *We say that the robust no-arbitrage condition RSAIP holds at time t if the SAIP condition holds at time t for the enlarged model defined by C_t^{∞}. We say that RSAIP holds if it holds at any time.*

In [22], we proved the following theorem analogous to Theorem 5.5:

Theorem 5.5. *Suppose that the condition RSAIP holds. Then, we have* $\gamma_t^\xi(S_t, V_t) = \tilde{\gamma}_t^\xi(S_t, V_t)$ *a.s.,* $\theta_t^\xi(S_t, V_t) = \tilde{\theta}_t^\xi(S_t, V_t)$ *a.s. and, also, we have* $D_t^\xi(S_t, V_{t-1}, V_t) = \tilde{D}_t^\xi(S_t, V_{t-1}, V_t)$ *a.s. for any* $V_{t-1}, V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$*, where* $\tilde{\theta}_t^\xi, \tilde{D}_t^\xi$ *are given by (3.6).*

As the horizon cost function coincides with the cost function (5.15) without fixed costs, the results stated in Propositions 5.16 and 5.2 allows us to characterize the reachability set-valued mapping K_t for this market. In particular, since $\mathrm{C}_t \leq \mathrm{C}_t^\infty + c_t$, by a straightforward computation, we deduce a simple formula of K_t for the Put option:

Lemma 5.6. *Suppose that* $\xi = (g(S_T), 0)$ *where* g *is a continuous function bounded from above by* $M \in \mathbf{R}_+$*. Then, a reachability set* $K_t(s, v_{t-1})$ *is explicitly given at any time* $t \leq T-1$ *by* $K_t(s, v_{t-1}) = \bar{B}_t(0, r_t(s, v_{t-1}) + 1)$*, closed ball of radius* $r_t(s, v_{t-1}) := \lambda_t(s, v_{t-1})/i_t(s)$ *where*

$$i_t(s) := \inf_{v^2 \in \{-1,1\}} D_t^{0,\infty}(s, 0, v),$$

$$\lambda_t(s, v_{t-1}) := \mathrm{C}_t^\infty(s, (0, v_{t-1}^2)) + M + \mathrm{C}_t^\infty(s, (0, -v_{t-1}^2)) + \sum_{s=t}^{T} c_s,$$

and $D_t^{0,\infty}$ *is given in the model without fixed cost given by Proposition 5.1 or Proposition 5.2. In particular, we have* $i_t(s) > 0$ *for all* $s > 0$.

As a numerical example, we also consider the put option payoff $(K - S_T)^+$ at time $T = 2$. We consider the binomial tree model as previously. In the case where the conditional support $\mathrm{supp}_{\mathcal{F}_t} S_t$ is countable, we can use the randomized method established in section 3.

We use the same parameters as in Section 5.1 and we consider fixed costs $c_1 = c_2 = 0.8$. The price function is illustrated in Figure 3.

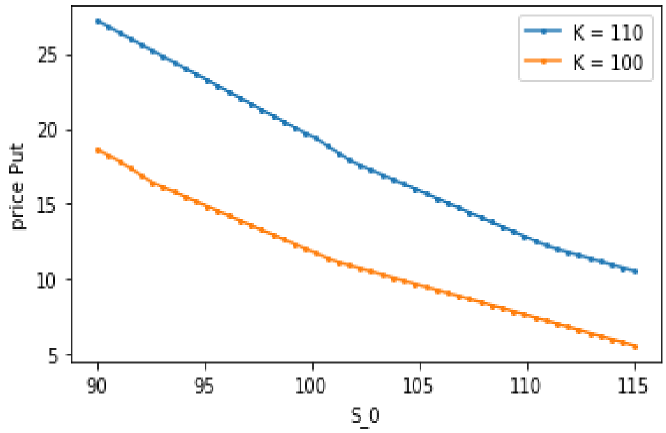

Fig 3: Price of put option with fixed costs.

We also visualize the ratio of put price to asset price S_0

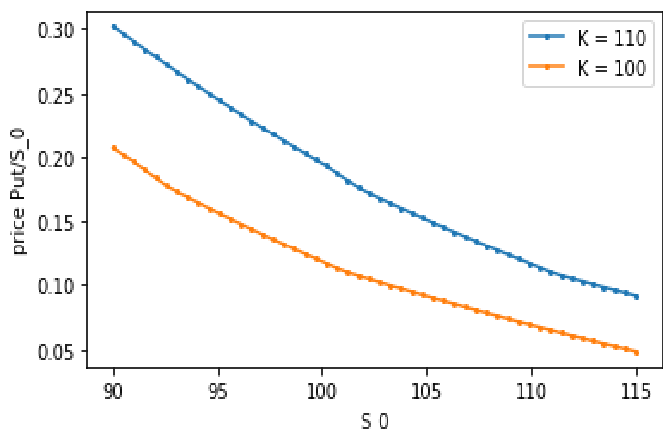

Fig 4: Ratio price of put to asset price with fixed costs.

We also compare the price of put option with and without fixed costs.

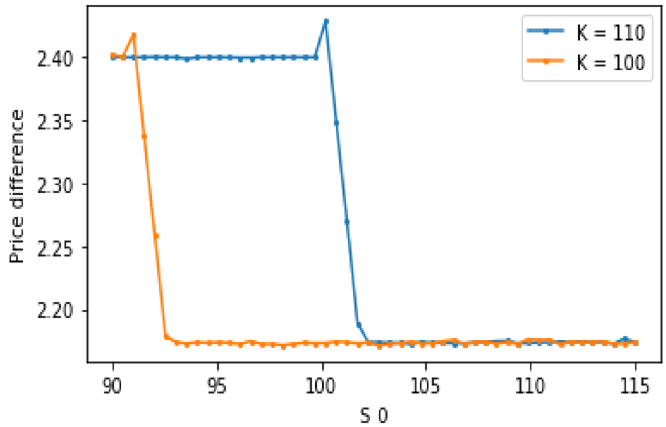

Fig 5: Price difference between two cases

6. Limit theorem for convex markets

In the literature, there is few results providing limit theorems for financial market models with transaction costs, see [12] and [2], but also [21] and [17] without transaction costs. In this section, we consider a sequence of markets defined by convex cost functions $\{\mathrm{C}_t^n(S_t,x), n\geq 1\}$ such that $\mathrm{C}_t^n(S_t,x)\downarrow \mathrm{C}_t(S_t,x)$ as $n\to\infty$ for some convex function C_t. We associate to each C_t^n a dynamic programming scheme deduced by our general analysis, see [22]:

$$\begin{aligned}
\gamma_T^{\xi,n}(S_T,V_{T-1}) &:= g^1(S_T)+\mathrm{C}_T^n(S_T,(0,g^{(2)}(S_T)-V_{T-1})),\\
\theta_t^{\xi,n}(S_t,v_t) &:= \operatorname{ess\,sup}_{\mathcal{F}_t}\gamma_{t+1}^{\xi,n}(S_{t+1},V_t),\\
D_t^{\xi,n}(S_t,V_{t-1},V_t) &:= \theta_t^{\xi,n}(S_t,V_t)+\mathrm{C}_t^n(S_t,(0,V_t^{(2)}-V_{t-1}^{(2)})),\\
\gamma_t^{\xi,n}(S_t,V_{t-1}) &:= \operatorname*{ess\,inf_{\mathcal{F}_t}}_{V_t\in L^0(\mathbf{R}^d,\mathcal{F}_t)} D_t^{\xi,n}(S_t,V_{t-1},V_t).
\end{aligned}$$

Assumption 6. *We suppose that* $\mathrm{supp}_{\mathcal{F}_t}S_{t+1}=\phi_t(S_t)=\mathrm{conv}\{\phi_t^1(S_t),...\phi_t^J(S_t)\}$ *where* $\phi_t^j:\mathbf{R}^d\to\mathbf{R}^d$*,* $j\leq J$*, are piecewise linear mappings in the sense of Definition 7.3.*

We define $\tilde{\gamma}_t^{\xi,n} : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$ recursively as follows:

$$
\begin{aligned}
&\tilde{\gamma}_T^{\xi,n}(s, v_{T-1}) := \gamma_T^{\xi,n}(s, v_{T-1}),\\
&\tilde{\theta}_{T-1}^{\xi,n}(s, v_{T-1}) := \max_{j\leq J} \tilde{\gamma}_T^{\xi,n}(\phi_{T-1}^j(s), v_{t-1}),\\
&\tilde{D}_t^{\xi,n}(s, v_{t-1}, v_t) := \tilde{\theta}_t^{\xi}(s, v_t) + \mathrm{C}_t^n(s, v_t^{(2)} - v_{t-1}^{(2)}),\\
&\tilde{\gamma}_t^{\xi,n}(s, v_{t-1}) := \mathrm{cl}\left(\inf_{v_t\in\mathbf{R}^d} \tilde{D}_t^{\xi,n}(s, v_{t-1}, v_t)\right).
\end{aligned}
$$

Assumption 7. *Suppose that for any $t \leq T-1$, $\inf_{v_t\in S^{d-1}(0,1)} \tilde{D}_t^0(s, 0, v_t) > 0$ for all $s \in \mathbf{R}_+^k$, so that there is a upper hemicontinuous reachability set-valued mapping $K_t(s, v_{t-1})$ for the super-hedging problem in the market defined by C_t. Moreover, we suppose that K_t is an* ***universal reachability*** *set in the sense that it satisfies for all $n \geq 1$ and (s, v_{t-1}):*

$$
\tilde{\gamma}_t^{\xi,n}(s, v_{t-1}) = \inf_{v_t\in K_t(s,v_{t-1})} \tilde{D}_t^{\xi,n}(s, v_{t-1}, v_t).
$$

Remark 6.1. *Consider the case where C, C^n and S_t satisfy the assumptions specified in section 4. Since $\mathrm{C} \leq \mathrm{C}^n$ for all $n \geq 1$ by assumption, we deduce that $\inf_{v_t\in S^{d-1}(0,1)} \tilde{D}_t^0(s, 0, v_t) > 0$ implies $\inf_{v_t\in S^{d-1}(0,1)} \tilde{D}_t^{0,n}(s, 0, v_t) > 0$ for all n. By the proof of Proposition 4.3, it is sufficient to suppose that SAIP holds for the market defined by C. If we suppose that $\mathrm{C}_t(s, v_t), \mathrm{C}_t^n(s, v_t)$ are bounded above by $|h_t(s, v_t)|$ for some continuous function h_t, by the same argument as in Lemma 5.21 in [22], we deduce that the quantities $\tilde{D}_t^0(s, v_{t-1}, 0)$ and $\tilde{D}_t^{0,n}(s, v_{t-1}, 0)$ are bounded above by a continuous function $\hat{h}_t(s, v_{t-1})$. Hence, an universal reachability set exits as $K_t(s, v_{t-1}) = \bar{B}(0, r_t(s, v_{t-1}))$ where*

$$
r_t(s, v_{t-1}) = \frac{\hat{h}_t(s, v_{t-1}) + |h_t(s, v_{t-1})|}{\inf_{v_t\in S^{d-1}(0,1)} \tilde{D}_t^0(s, 0, v_t)}.
$$

Since r_t is u.s.c., we deduce by Lemma 5.12 *in [22] that K_t is upper hemicontinuous.*

Theorem 6.2. *Suppose that the functions $\phi_t^j : \mathbf{R}_+^k \to \mathbf{R}_+^k$, $j \leq J$ satisfy Assumption 6. Suppose that Assumption 7 holds. Then, for any $t \leq T-1$ and for any $v_{t-1} \in \mathbf{R}^d$, $\lim_{n\to\infty} \tilde{\gamma}_t^{\xi,n}(s, v_{t-1}) = \tilde{\gamma}_t^{\xi}(s, v_{t-1})$. Moreover, SAIP condition holds for the markets defined by C^n and $\lim_{n\to\infty} \gamma_t^{\xi,n}(S_t, V_t) = \gamma_t^{\xi}(S_t, V_t)$ a.s. as $n \to \infty$ for any $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ and $t \leq T$.*

Proof. We first observe that $\tilde{\gamma}_t^{\xi,n}$ is convex in (s, v_{t-1}) for any n. We now prove that $\tilde{D}_t^{\xi,n}(s, v_{t-1}, .) \xrightarrow{epi} \tilde{D}_t^{\xi}(s, v_{t-1}, .)$. Indeed, by the definition of $\tilde{\gamma}_T^{\xi,n}$ we have that $\tilde{\gamma}_T^{\xi,n}(s, .) \downarrow \tilde{\gamma}_T^{\xi}(s, .)$. Since $\tilde{\gamma}_T^{\xi}(s, .)$ is convex and takes values in $\mathbf{R}$, it is continuous. We deduce by [26, Proposition 7.4(c)] that $\tilde{\gamma}_T^{\xi,n}(s, .) \xrightarrow{epi} \tilde{\gamma}_T^{\xi}(s, .)$. Moreover, by convexity and by assumption, we get that

$$\begin{aligned}\tilde{\theta}_{T-1}^{\xi,n}(s, v_{T-1}) &= \max_{j\leq J} \tilde{\gamma}_T^{\xi,n}(\phi_{T-1}^j(s), v_{t-1}),\\ \tilde{\theta}_{T-1}^{\xi}(s, v_{T-1}) &= \max_{j\leq J} \tilde{\gamma}_T^{\xi}(\phi_j(s), v_{t-1}).\end{aligned}$$

Under Assumption 6 holds, the mapping $(s, v_{t-1}) \mapsto (\phi_j(s), v_{t-1})$ is piecewise linear in the sense of Definition 7.3. Since, $\tilde{\gamma}_T^{\xi,n}$ is convex, we deduce by [26, Exercies 2.20] that $\tilde{\gamma}_T^{\xi,n}(\phi_j(.), .)$ is jointly convex. Moreover, since we have $\lim_{n\to\infty} \tilde{\gamma}_T^{\xi,n}(\phi_j(s), .) \overset{epi}{=} \tilde{\gamma}_T^{\xi}(\phi_j(s), .)$, for any $j \leq J$, we deduce by [26, Proposition 7.48] that:

$$\tilde{\theta}_{T-1}^{\xi,n}(s, .) = \max_{j\leq J} \tilde{\gamma}_T^{\xi,n}(\phi_j(s), .) \xrightarrow{epi} \max_{j\in J} \tilde{\gamma}_T^{\xi}(\phi_j(s), .) = \tilde{\theta}_{T-1}^{\xi}(s, .),\ n \to \infty.$$

Since $\mathrm{C}_{T-1}^n(s, .) \downarrow \mathrm{C}_{T-1}(s, .)$ and $\mathrm{C}_{T-1}(s, .)$ is continuous, we deduce by the Dini theorem that the convergence is uniform on any compact subset K of $\mathbf{R}^d$. By [26, Theomrem 7.14], we deduce that $\mathrm{C}_{T-1}^n(s, .)$ converges continuously to $\mathrm{C}_{T-1}(s, .)$ in the sense that $\mathrm{C}_{T-1}^n(s, x^n) \to \mathrm{C}_{T-1}(s, x)$ whenever $x^n \to x$. We then deduce by [26, Theorem 7.46] that

$$\tilde{D}_{T-1}^{\xi,n}(s, v_{T-2}, .) \xrightarrow{epi} \tilde{D}_{T-1}^{\xi,n}(s, v_{T-2}, .),\ n \to \infty.$$

Suppose that $\lim_{n\to\infty} \tilde{D}_{t+1}^{\xi,n}(s, v_t, .) \overset{epi}{=} \tilde{D}_{t+1}^{\xi,n}(s, v_t, .)$ and, by induction, let us show that $\lim_{n\to\infty} \tilde{D}_t^{\xi,n}(s, v_{t-1}, .) \overset{epi}{=} \tilde{D}_t^{\xi,n}(s, v_{t-1}, .)$. Since $K_{t+1}(s, .)$ is compact, we deduce that $\tilde{\gamma}_{t+1}^{\xi,n}(s, .) \downarrow \tilde{\gamma}_{t+1}^{\xi}(s, .)$. Since $\tilde{\gamma}_{t+1}^{\xi}(s, .)$ is convex and takes real values, it is also continuous. We deduce by [26, Proposition 7.4] that $\lim_{n\to\infty} \tilde{\gamma}_{t+1}^{\xi,n}(s, .) \overset{epi}{=} \tilde{\gamma}_{t+1}^{\xi}(s, .)$. As in the case $t = T - 1$, we deduce by induction that $\lim_{n\to\infty} \tilde{D}_t^{\xi,n}(s, v_{t-1}, .) \overset{epi}{=} \tilde{D}_t^{\xi}(s, v_{t-1}, .)$.

At last, since $\inf_{v_t\in S(0,1)} \tilde{D}_t^0(s, 0, v_t) > 0$, SAIP holds for the market defined by C_t, see [22, Theorem 4.16]. By Theorem 5.5, we have $\tilde{\gamma}_t^{\xi}(S_t, V_t) = \gamma_t^{\xi}(S_t, V_t)$ a.s. for any $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. Moreover, since $\tilde{D}_t^{0,n}(s, 0, v_t) \geq \tilde{D}_t^0(s, 0, v_t)$, we deduce that SAIP also holds for market defined by C_t^n and, similarly, we have $\tilde{\gamma}_t^{\xi,n}(S_t, V_t) = \gamma_t^{\xi,n}(S_t, V_t)$ a.s. for any $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. The conclusion follows. □

7. Appendix

By [26, Theorem 14.37], we have:

Proposition 7.1. *If f is an $\mathcal{F}_t$-normal integrand,* $\inf_{y\in\mathbf{R}^d} f(\omega,y)$ *is $\mathcal{F}_t$-measurable and* $\{(\omega,x)\in\Omega\times\mathbf{R}^d : f(\omega,x)=\inf_{y\in\mathbf{R}^d} f(\omega,y)\}\in\mathcal{F}_t\otimes\mathcal{B}(\mathbf{R}^d)$ *is a measurable closed set.*

We now recall a result from [3] which characterizes a conditional essential supremum as a pointwise supremum on a random set. Let $\mathcal{H}$ and $\mathcal{F}$ be two complete sub-σ-algebras of $\mathcal{F}_T$ such that $\mathcal{H}\subseteq\mathcal{F}$. The conditional support of $X\in L^0(\mathbf{R}^d,\mathcal{F})$ with respect to $\mathcal{H}$ is the smallest $\mathcal{H}$-graph measurable random set $\mathrm{supp}_{\mathcal{H}}X$ containing the singleton $\{X\}$ a.s., see [3].

Proposition 7.2. *Let $h:\Omega\times\mathbf{R}^k\to\mathbf{R}$ be a $\mathcal{H}\otimes\mathcal{B}(\mathbf{R}^k)$-measurable function which is l.s.c. in x. Then, for all $X\in L^0(\mathbf{R}^k,\mathcal{F})$,*

$$\text{ess sup}_{\mathcal{H}}\, h(X)=\sup_{x\in supp_{\mathcal{H}}X} h(x)\quad a.s.$$

7.1. Piecewise linear cost function

We recall from [26] the definiton of piecewise linear function:

Definition 7.3. *A mapping $F:D\to\mathbf{R}^m$ defined on a set $D\in\mathbf{R}^n$ is piecewise linear on D if D is the union of finitely many polyhedral sets $(P_i)_{i\in J}$ such that, for all $x\in P_i$, $F(x)=A_ix+B_i$, for some matrix $A_i\in\mathbf{R}^{m\times n}$ and $B_i\in\mathbf{R}^m$.*

A function $f:\mathbf{R}^n\to\bar{\mathbf{R}}$ is piecewise linear if it is a real-valued piecewise linear function on its domain $\mathrm{dom}f=\{x:\ f(x)\in\mathbf{R}\}$.

7.2. Complement to Section 4

Recall that the model is defined by one risk-free asset and one risky asset denoted by S. The cost function is given by

$$\mathrm{C}_t(S_t,v)=v^1+S_t\tilde{\mathrm{C}}_t(v^2), \tag{7.18}$$

where $\tilde{\mathrm{C}}_t:\mathbf{R}\to\mathbf{R}$ is a piecewise linear function.

By Proposition 7.2, we have:

$$\begin{aligned}
\theta^0_{T-1}(S_{T-1},v) &:= \text{ess sup}_{\mathcal{F}_{T-1}} \mathrm{C}_T(S_T,(0,-v^2)) = \sup_{s\in \text{supp}_{\mathcal{F}_{T-1}} S_T} \mathrm{C}_T(s,(0,-v^2))\\
&= \sup_{s\in \text{supp}_{\mathcal{F}_{T-1}} S_T} \big(-(1+\epsilon_T)sv^2 1_{v^2\le 0} - (1-\epsilon_T)sv^2 1_{v^2\ge 0}\big)\\
&= \sup_{s\in [k^d_{T-1}S_{T-1},k^u_{T-1}S_{T-1}]} \big(-(1+\epsilon_T)sv^2 1_{v^2\le 0} - (1-\epsilon_T)sv^2 1_{v^2\ge 0}\big)\\
&= \max\big\{-(1+\epsilon_T)k^d_{T-1}S_{T-1}v^2 1_{v^2\le 0} - (1-\epsilon_T)k^d_{T-1}S_{T-1}v^2 1_{v^2\ge 0},\\
&\quad -(1+\epsilon_T)k^u_{T-1}S_{T-1}v^2 1_{v^2\le 0} - (1-\epsilon_T)k^u_{T-1}S_{T-1}v^2 1_{v^2\ge 0}\big\}\\
&= -(1-\epsilon_T)k^d_{T-1}S_{T-1}v^2 1_{v^2\ge 0} - (1+\epsilon_T)k^u_{T-1}S_{T-1}v^2 1_{v^2\le 0}.
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{C}_{T-1}(S_{T-1},(0,v^2-z^2)) &= (1+\epsilon_{T-1})S_{T-1}v^2 1_{v^2-z^2\ge 0} + (1-\epsilon_{T-1})S_{T-1}v^2 1_{v^2-z^2\le 0}\\
&\quad -(1+\epsilon_{T-1})S_{T-1}z^2 1_{v^2-z^2\ge 0} + (1-\epsilon_{T-1})S_{T-1}z^2 1_{v^2-z^2\le 0}.
\end{aligned}$$

We then have:

$$\begin{aligned}
D^0_{T-1}(S_{T-1},0,v) &= \theta^0_{T-1}(S_{T-1},v) + \mathrm{C}_{T-1}(S_{T-1},(0,v^2))\\
&= ((1+\epsilon_{T-1})S_{T-1} - (1-\epsilon_T)k^d_{T-1}S_{T-1})v^2 1_{v^2\ge 0}\\
&\quad + ((1-\epsilon_{T-1})S_{T-1} - (1+\epsilon_T)k^u_{T-1}S_{T-1})v^2 1_{v^2\le 0}
\end{aligned}$$

More generally:

$$\begin{aligned}
D^0_{T-1}(S_{T-1},z,v) &= \theta^0_{T-1}(S_{T-1},v) + \mathrm{C}_{T-1}(S_{T-1},(0,v-z))\\
&= (1+\epsilon_{T-1})S_{T-1}v^2 1_{v^2-z^2\ge 0} + (1-\epsilon_{T-1})S_{T-1}v^2 1_{v^2-z^2\le 0}\\
&\quad -(1+\epsilon_{T-1})S_{T-1}z^2 1_{v^2-z^2\ge 0} + (1-\epsilon_{T-1})S_{T-1}z^2 1_{v^2-z^2\le 0}\\
&\quad -(1-\epsilon_T)k^d_{T-1}S_{T-1}v^2 1_{v^2\ge 0} - (1+\epsilon_T)k^u_{T-1}S_{T-1}v^2 1_{v^2\le 0}.
\end{aligned}$$

In the following, we assume that $1+\epsilon_{T-1} \le (1+\epsilon_T)k^u_{T-1}$ and, also, that $1-\epsilon_{T-1} \ge (1-\epsilon_T)k^d_{T-1}$. We shall use the usual convention that $\inf\emptyset = \infty$. We get that:

$$\gamma^0_{T-1}(z) = \inf_{v\in\mathbf{R}^2} D^0_{T-1}(S_{T-1},z,v) = \min_{i=1,\cdots,4} D^{0,i}_{T-1}(S_{T-1},z,v),$$

where:

$$\begin{aligned}
D^{0,1}_{T-1} &= \inf_{v^2:v^2\ge z^2, v^2\ge 0} \big((1+\epsilon_{T-1})S_{T-1}(v^2-z^2) - (1-\epsilon_T)k^d_{T-1}S_{T-1}v^2\big)\\
&= -(1-\epsilon_T)k^d_{T-1}S_{T-1}z^2 1_{z^2\le 0} - (1+\epsilon_{T-1})S_{T-1}z^2 1_{z^2\ge 0}.
\end{aligned}$$

$$
\begin{aligned}
D^{0,2}_{T-1} &= \inf_{v^2 : v^2 \geq z^2, v^2 \leq 0} \left((1+\epsilon_{T-1}) S_{T-1}(v^2 - z^2) - (1+\epsilon_T) k^u_{T-1} S_{T-1} v^2 \right) \\
&= \infty 1_{z^2>0} - (1+\epsilon_{T-1}) S_{T-1} z^2 1_{z^2 \leq 0}.
\end{aligned}
$$

$$
\begin{aligned}
D^{0,3}_{T-1} &= \inf_{v^2 : v^2 \leq z^2, v^2 \geq 0} \left((1-\epsilon_{T-1}) S_{T-1}(v^2 - z^2) - (1-\epsilon_T) k^d_{T-1} S_{T-1} v^2 \right) \\
&= \infty 1_{z^2<0} - (1-\epsilon_{T-1}) S_{T-1} z^2 1_{z^2 \geq 0}.
\end{aligned}
$$

$$
\begin{aligned}
D^{0,4}_{T-1} &= \inf_{v^2 : v^2 \leq z^2, v^2 \leq 0} \left((1-\epsilon_{T-1}) S_{T-1}(v^2 - z^2) - (1+\epsilon_T) k^u_{T-1} S_{T-1} v^2 \right) \\
&= -(1-\epsilon_{T-1}) S_{T-1} z^2 1_{z \geq 0} - (1+\epsilon_T) k^u_{T-1} S_{T-1} z^2 1_{z^2 \leq 0}.
\end{aligned}
$$

We deduce that

$$
\begin{aligned}
\gamma^0_{T-1}(S_{T-1}, z) &= \inf_{v \in \mathbf{R}^2} D^0_{T-1}(S_{T-1}, z, v) \\
&= -(1+\epsilon_{T-1}) S_{T-1} z^2 1_{z^2 \geq 0} - (1-\epsilon_T) k^d_{T-1} S_{T-1} z^2 1_{z^2 \leq 0}.
\end{aligned}
$$

We now compute $D^0_{T-2}(S_{T-1}, 0, z)$. We have:

$$
\begin{aligned}
\theta^0_{T-2}(S_{T-2}, z) &= \text{ess sup}_{\mathcal{F}_{T-2}} \gamma^0_{T-1}(S_{T-1}, z) \\
&= \sup_{s \in [k^d_{T-2} S_{T-2}, k^u_{T-2} S_{T-2}]} \gamma^0_{T-1}(s, z) \\
&= \sup_{s \in [k^d_{T-2} S_{T-2}, k^u_{T-2} S_{T-2}]} \left(-(1+\epsilon_{T-1}) s z^2 1_{z^2 \geq 0} - (1-\epsilon_T) k^d_{T-1} s z^2 1_{z^2 \leq 0} \right) \\
&= -(1+\epsilon_{T-1}) k^d_{T-2} S_{T-2} z^2 1_{z^2 \geq 0} - (1-\epsilon_T) k^d_{T-1} k^u_{T-2} S_{T-2} z^2 1_{z^2 \leq 0}.
\end{aligned}
$$

$$
\begin{aligned}
D^0_{T-2}(S_{T-2}, 0, z) &= \theta^0_{T-2}(S_{T-2}, z) + \mathrm{C}_{T-2}(S_{T-2}, (0, z^2)) \\
&= -(1+\epsilon_{T-1}) k^d_{T-2} S_{T-2} z^2 1_{z^2 \geq 0} - (1-\epsilon_T) k^d_{T-1} k^u_{T-2} S_{T-2} z^2 1_{z^2 \leq 0} \\
&\quad + (1+\epsilon_{T-2}) S_{T-2} z^2 1_{z^2 \geq 0} + (1-\epsilon_{T-2}) S_{T-2} z^2 1_{z^2 \leq 0} \\
&= \left((1+\epsilon_{T-2}) S_{T-2} - (1+\epsilon_{T-1}) k^d_{T-2} S_{T-2}) \right) z^2 1_{z^2 \geq 0} \\
&\quad + \left((1-\epsilon_{T-2}) S_{T-2} - (1-\epsilon_T) k^d_{T-1} k^u_{T-2} S_{T-2} \right) z^2 1_{z^2 \leq 0}.
\end{aligned}
$$

We then get the following:

Proposition 7.4. *AIP holds at time $T-2$ if and only if the following holds:*

$$
k^d_{T-2} \leq \frac{1+\epsilon_{T-2}}{1+\epsilon_{T-1}} \text{ and } k^u_{T-2} \geq \frac{1-\epsilon_{T-2}}{(1-\epsilon_T) k^d_{T-1}}.
$$

References

[1] Aliprantis, C. D. and K. C. Border. Infinite Dimensional Analysis : A Hitchhicker's Guide, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 3rd edition, 2006.

[2] Bank P., Dolinsky Y. and Perkkio A-P. The scaling limit of super-replication prices with small transaction costs in the multivariate case. Finance and Stochastics, 21, 487-508, 2017.

[3] Baptiste, J., Carassus, L. and Lépinette E., Pricing without martingale measure, 2018. https://hal.archives-ouvertes.fr/hal-01774150.

[4] Campi L. and Schachermayer W. A super-replication theorem in Kabanov's model for transaction costs. Finance and Stochastics, 10, 4, 2006.

[5] Cherny A. Pricing with coherent risk. Theory of Probability and Its Applications, 2007, 52(3), 389-415.

[6] Carassus, Laurence and Gobet, Emmanuel and Temam, Emmanuel. A class of financial products and models where super-replication prices are explicit. Stochastic Processes and Applications to Mathematical Finance. World Scientific 2007, 67-84.

[7] Dalang E.C., Morton A. and Willinger W. Equivalent martingale measures and no-arbitrage in stochastic securities market models. Stochastics and Stochastic Reports, 1990, 29, 185-201.

[8] Delbaen F. and Schachermayer W. A general version of the fundamental theorem of asset pricing. Mathematische Annalen, 1994, 300, 463-520.

[9] Delbaen F. and Schachermayer W. The fundamental theorem of asset pricing for unbounded stochastic processes. Mathematische Annalen, 1996, 312, 215-250.

[10] De Vallière D., Kabanov Y. and Lépinette E. Hedging of American options under transaction costs. Finance and Stochastics 13, 1, 105-119, 2009.

[11] EL Mansour M and Lépinette E. Conditional interior and conditional closure of a random sets. Journal of Optimization Theory and Applications, 5051 187, 356-369, 2020.

[12] Grépat J. and Kabanov Y. On a multi-asset version of the Kusuoka limit theorem: convergence of hedging sets. Finance and Stochastics, 25, 167-187, 2021.

[13] Guasoni P., Lépinette E. and Rásonyi M. The fundamental theorem of asset pricing under transaction costs. Finance and Stochastics, 16, 4, 741-777, 2012.

[14] Guasoni P., Rásonyi M. and Schachermayer W. The fundamental theorem of asset pricing for continuous processes under small transaction costs. Annals of Finance, 6, 157-191, 2010.

[15] P. Guasoni, M. Rásonyi and Schachermayer W. Consistent Price Systems and Face-Lifting Pricing under Transaction Costs. Annals of Applied Probability, 18, 2, 491-520, 2008.

[16] Hess C. Epi-convergence of sequences of normal integrands and strong consistency of the maximum likelihood estimator. The Annals of Statistics, 1996.

[17] Hubalek F. and Schachermayer W. When does convergence of asset price process imply convergence of option prices ? Math. Finance 8, 215-233, 1998.

[18] Consistent price systems and arbitrage opportunities of the second kind in models with transaction costs. Finance and Stochastics. 16, 1, 135-154, 2011.

[19] Kabanov Y. and Stricker C. A Teachers' note on no-arbitrage criteria. In Séminaire de Probabilités, XXXV, volume 1755 of Lecture Notes in Math., Springer Berlin, 2001,149-152.

[20] Kabanov Y. and Safarian, M. Markets with transaction costs. Mathematical Theory. Springer-Verlag, 2009.

[21] Kusuoka S. Limit theorem on option replication cost with transaction costs. Annals of Applied Probability, 5, 198-221, 1995.

[22] Lépinette E. and Vu D.T. Dynamic programming principle and computable prices in financial market models with transaction costs. Preprint. https://hal.archives-ouvertes.fr/hal-03284655/

[23] Lépinette E. and Molchanov I. Conditional cores and conditional convex hulls of random sets. Journal of Mathematical Analysis and Applications, 478, 2, 368-392, 2019.

[24] Lépinette E. and Tran T.Q. Arbitrage theory for non convex financial market models. Stochastic Processes and Applications, 127, 10, 3331-3353, 2017.

[25] Molchanov I. Theory of Random Sets. 2nd edition. Springer, London, 2017.

[26] Rockafellar, R Tyrrell and Wets, Roger J-B. Variational analysis. Springer Science & Business Media, 2009.

[27] Löhne A. and Rudloff B. An algorithm for calculating the set of superhedging portfolios in markets with transaction costs. IJTAF, 17, 02, 1-33, 2014.

[28] Lépinette E. and Zhao J. Super-hedging a European option with a coherent risk-measure and without no-arbitrage condition. Stochastics, 2022, online.