



HAL
open science

A One-Pass Tree-Shaped Tableau for Defeasible LTL

Anasse Chafik, Fahima Cheikh-Alili, Jean-François Condotta, Ivan Varzinczak

► **To cite this version:**

Anasse Chafik, Fahima Cheikh-Alili, Jean-François Condotta, Ivan Varzinczak. A One-Pass Tree-Shaped Tableau for Defeasible LTL. 28th International Symposium on Temporal Representation and Reasoning (TIME 2021), Alpen-Adria-Universität Klagenfurt, Sep 2021, Klagenfurt, Austria. pp.16:1-16:18, 10.4230/LIPIcs.TIME.2021.16 . hal-03699248

HAL Id: hal-03699248

<https://hal.science/hal-03699248v1>

Submitted on 4 Apr 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A one-pass tree-shaped tableau for defeasible *LTL*

Anasse Chafik ✉

CRIL, Univ Artois & CNRS, France

Fahima Cheikh-Alili ✉

CRIL, Univ Artois & CNRS, France

Jean-François Condotta ✉

CRIL, Univ Artois & CNRS, France

Ivan Varzinczak ✉

CRIL, Univ Artois & CNRS, France

Abstract

Defeasible Linear Temporal Logic is a defeasible temporal formalism for representing and verifying exception-tolerant systems. It is based on Linear Temporal Logic (*LTL*) and builds on the preferential approach of Kraus et al. for non-monotonic reasoning, which allows us to formalize and reason with exceptions. In this paper, we tackle the satisfiability checking problem for defeasible *LTL*. One of the methods for satisfiability checking in *LTL* is the one-pass tree shaped analytic tableau proposed by Reynolds. We adapt his tableau to defeasible *LTL* by integrating the preferential semantics to the method. The novelty of this work is in showing how the preferential semantics works in a tableau method for defeasible linear temporal logic. We introduce a sound and complete tableau method for a fragment that can serve as the basis for further exploring tableau methods for this logic.

2012 ACM Subject Classification Theory of computation → Modal and temporal logics

Keywords and phrases Temporal logic, Non-monotonic reasoning, Tableau Calculi

Digital Object Identifier 10.4230/LIPIcs.CVIT.2016.23

Acknowledgements The open access publication of this article was supported by the Alpen-Adria-Universität Klagenfurt, Austria.

1 Introduction

Linear temporal logic (*LTL*) was introduced by Pnueli [13] as a formal tool for reasoning about programs execution. Many properties that an execution can be expressed elegantly using this formalism. The logic *LTL* is used for systems verification [16]. With advances in technologies, systems became more and more complex, displaying new features and behaviours. One of these behaviours is tolerating exceptions. In more general terms, if an error occurs, within an execution of a program, at certain points of time where it is tolerated, the program can still function properly.

Let us say, for the sake of argument, that there is an execution of a program in which a parameter cannot have a certain value. We notice that, at some given points of time, the execution produces the invalid value in the aforementioned parameter. Nevertheless, we do not mind that the program produces the error at these time points deemed to be harmless. The crucial point is that this behaviour is not present in other, more important, points of time. We want to be sure that the execution still continues and the program functions properly even in the presence of such benign time points.

We want a formalism for verifying properties of executions that can, on one hand, be strictly required at some points of time, and on the other hand, be missing in other points of time. That is why we introduced an extended formalism of *LTL*, called defeasible linear temporal logic (*LTL*[~]) [6]. It uses the preferential approach of Kraus et al. to non-monotonic reasoning [11] (a.k.a. the KLM approach). The defeasible aspect of *LTL*[~]



© Anasse Chafik, Fahima Cheikh Alili, Jean-François Condotta, Ivan Varzinczak; licensed under Creative Commons License CC-BY 4.0

42nd Conference on Very Important Topics (CVIT 2016).

Editors: John Q. Open and Joan R. Access; Article No. 23; pp. 23:1–23:18

Leibniz International Proceedings in Informatics



LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

45 adds a new dimension to the verification of a program's execution. We can order time points
46 from the important ones, which we call *normal*, to the lesser and lesser ones. *Normality* in
47 *LTL* indicates the importance of a time point within an execution compared to others.

48 We also introduced defeasible versions of the modalities *always* and *eventually*. With
49 these defeasible modalities, we can express properties similar to their classical counterparts,
50 targeting the most normal time points within the execution.

51 The main goal of this paper is to establish a *satisfiability checking* method for our logic, in
52 particular, for a fragment thereof. In the case of *LTL*, many tableau methods were proposed
53 in the literature. There are two types of tableau methods: *multi-pass* and *one-pass* tableaux.
54 Multi-pass tableau methods [22, 12, 10] go through an initial phase of building a tree-shaped
55 structure by putting the sentence in the root node and expanding the tableau via a systematic
56 application of a set of rules. The second phase is a *culling* phase, which uses an auxiliary
57 structure built from the tableau, and checks for the satisfiability of the input sentence in
58 this structure. Whereas in *one-pass* tableau methods [17, 14], the construction and the
59 verification are done simultaneously. Reynolds' tableau for *LTL* [15, 14] is a tree-shaped
60 one-pass tableau where each branch is independent from the others. Moreover, each successful
61 branch by itself is a representation of an interpretation that satisfies the sentence.

62 As for the KLM approach, tableau methods were developed for the preferential approach
63 of Kraus et al. logic [11] and formalisms extending the preferential approach [9, 4, 5]. In the
64 case of preferential modal logic, Britz and Varzinczak [4] proposed a tree-shaped tableau
65 that builds the ordering relation on worlds at the same time as the tableau is expanded. The
66 tableau method in this paper is based on both the one-pass tableau of Reynolds [14] and the
67 tableau for preferential modal logic by Britz and Varzinczak [4]. The novelty of this paper is
68 in showing how preferential semantics works in a tableau for a fragment of LTL^{\sim} .

69 The plan of this paper goes the following way: We talk briefly about *LTL* and LTL^{\sim} in
70 Section 2. We then describe a tableau method for a fragment of LTL^{\sim} in Section 3. We show
71 soundness, and completeness of our method in Section 4. Section 5 concludes the paper.

72 2 Preliminaries

73 Linear Temporal Logic [1] is a modal logic in which modalities are considered to be temporal
74 operators that describe events happening in different time points over a linearly ordered time-
75 line. Let \mathcal{P} be a finite set of *propositional atoms*. The set of operators in *LTL* can be split into
76 two parts: the set of *Boolean connectives* (\neg, \wedge, \vee), and that of *temporal operators* (\square, \diamond, \circ),
77 where \square reads as *always*, \diamond as *eventually*, and \circ as *next*. Let $p \in \mathcal{P}$, sentences in *LTL* are
78 built up according to the following grammar: $\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \square\alpha \mid \diamond\alpha \mid \circ\alpha$.

79 Standard abbreviations are included in *LTL*, such as: $\top \stackrel{\text{def}}{=} p \vee \neg p$, $\perp \stackrel{\text{def}}{=} p \wedge \neg p$, $\alpha \rightarrow \beta \stackrel{\text{def}}{=} \neg\alpha \vee \beta$
80 and $\alpha \leftrightarrow \beta \stackrel{\text{def}}{=} (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$. There are other temporal operators such as \mathcal{U} (until
81 operator) and \mathcal{R} (release operator) in *LTL*, but we chose to omit them in this paper.

82 The temporal semantics structure is a chronological linear succession of time points.
83 We use the set of natural numbers in order to label each of these time points i.e., $(\mathbb{N}, <)$.
84 Hence, a temporal interpretation associates each time point t with a truth assignment of all
85 propositional atoms. A temporal interpretation is defined as follows:

86 ► **Definition 1** (Temporal interpretation). *A temporal interpretation I is a mapping function*
87 $V : \mathbb{N} \rightarrow 2^{\mathcal{P}}$ *which associates each time point $t \in \mathbb{N}$ with a set of propositional atoms $V(t)$*
88 *corresponding to the set of propositions that are true in t . (Propositions not belonging to $V(t)$*
89 *are assumed to be false at the given time point.)*

90 The truth value of a sentence in an interpretation I at a time point $t \in \mathbb{N}$, denoted by
91 $I, t \models \alpha$, is recursively defined as follows:

- 92 ■ $I, t \models p$ if $p \in V(t)$; $I, t \models \neg\alpha$ if $I, t \not\models \alpha$;
- 93 ■ $I, t \models \alpha \wedge \alpha'$ if $I, t \models \alpha$ and $I, t \models \alpha'$; $I, t \models \alpha \vee \alpha'$ if $I, t \models \alpha$ or $I, t \models \alpha'$;
- 94 ■ $I, t \models \Box\alpha$ if $I, t' \models \alpha$ for all $t' \in \mathbb{N}$ s.t. $t' \geq t$; $I, t \models \Diamond\alpha$ if $I, t' \models \alpha$ for some $t' \in \mathbb{N}$ s.t.
95 $t' \geq t$;
- 96 ■ $I, t \models \bigcirc\alpha$ if $I, t + 1 \models \alpha$.

97 In previous work [6], we introduced a new formalism called preferential linear temporal
98 logic. The motivation is to provide a formalism for the specification and verification of
99 systems where exceptions can be tolerated.

100 Let $p \in \mathcal{P}$, sentences of the logic LTL^\sim are built up according to the following grammar:

$$101 \quad \alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \Box\alpha \mid \Diamond\alpha \mid \bigcirc\alpha \mid \boxminus\alpha \mid \diamond\alpha$$

102 The intuition behind the new temporal operators is the following: \boxminus reads as *non-*
103 *monotonic always* and \diamond reads as *non-monotonic eventually*. The set of all well-formed
104 LTL^\sim sentences is denoted by \mathcal{L}^\sim . It is worth to mention that any well-formed sentence α in
105 LTL is a sentence of \mathcal{L}^\sim .

106 A sentence such as $\boxminus\alpha$ reads as: in all normal future time points, α is true. A sentence
107 of the form $\diamond\alpha$ reads as: in some normal future time point, α is true. We can even express
108 properties using a mix of classical and non-monotonic operators. A sentence $\Box\Diamond\alpha$ reads as:
109 always, there is a normal future time point where α is true.

110 The preferential component of the interpretation of our language is directly inspired by
111 the preferential semantics proposed by Shoham [19] and used in the KLM approach [11].
112 The ordering relation, denoted by \prec , is a strict partial order on points of time. Following
113 Kraus et al. [11], $t \prec t'$ means that t is more preferred than t' . We use the pair notation
114 $(t, t') \in \prec$ to indicate that t is more normal than t' w.r.t. \prec .

115 ► **Definition 2 (Minimality w.r.t. \prec).** Let \prec be a strict partial order on a set \mathbb{N} and
116 $N \subseteq \mathbb{N}$. The set of the minimal elements of N w.r.t. \prec , denoted by $\min_{\prec}(N)$, is defined by
117 $\min_{\prec}(N) \stackrel{\text{def}}{=} \{t \in N \mid \text{there is no } t' \in N \text{ such that } (t', t) \in \prec\}$.

118 ► **Definition 3 (Well-founded set).** Let \prec be a strict partial order on a set \mathbb{N} . We say \mathbb{N}
119 is well-founded w.r.t. \prec iff $\min_{\prec}(N) \neq \emptyset$ for every $\emptyset \neq N \subseteq \mathbb{N}$.

120 In what follows, given a relation \prec and a time point $t \in \mathbb{N}$, the set of *preferred time points*
121 *relative to t* is the set $\min_{\prec}([t, \infty])$ which is denoted in short by $\min_{\prec}(t)$.

122 ► **Definition 4 (Preferential temporal interpretation).** An LTL^\sim interpretation on a
123 set of propositional atoms \mathcal{P} , also called *preferential temporal interpretation* on \mathcal{P} , is a pair
124 $I \stackrel{\text{def}}{=} (V, \prec)$ where V is a mapping function $V : \mathbb{N} \rightarrow 2^{\mathcal{P}}$, and $\prec \subseteq \mathbb{N} \times \mathbb{N}$ is a strict partial
125 order on \mathbb{N} such that \mathbb{N} is well-founded w.r.t. \prec . We denote the set of preferential temporal
126 interpretations by \mathfrak{I} .

127 Preferential temporal interpretations provide us with an intuitive way of interpreting
128 sentences of \mathcal{L}^\sim . Let $\alpha \in \mathcal{L}^\sim$, let $I = (V, \prec)$ be a preferential temporal interpretation, and let t
129 be a time point in I in \mathbb{N} . Satisfaction of α at t in I , denoted $I, t \models \alpha$, is defined as follows:

- 130 ■ The truth values of Boolean connectives and classical modalities are defined as in LTL .
- 131 ■ $I, t \models \boxminus\alpha$ if $I, t' \models \alpha$ for all $t' \in \min_{\prec}(t)$;
- 132 ■ $I, t \models \diamond\alpha$ if $I, t' \models \alpha$ for some $t' \in \min_{\prec}(t)$.

133 We say $\alpha \in \mathcal{L}^\sim$ is *satisfiable* if there is a preferential temporal interpretation I and a
 134 time point t in \mathbb{N} such that $I, t \models \alpha$. We can show that $\alpha \in \mathcal{L}^\sim$ is *satisfiable* iff there is a
 135 preferential temporal interpretation I s.t. $I, 0 \models \alpha$.

136 **3 A one-pass tableau for LTL^\sim**

137 In this paper, we address the computational task of *satisfiability checking* in LTL^\sim . That
 138 is, given a sentence α in LTL^\sim , decide whether or not there is an interpretation I that
 139 satisfies the sentence α . As mentioned in the Introduction, we propose a one-pass tree-shaped
 140 tableau for a fragment of LTL^\sim based on Reynolds' tableau [15] and inspired by the semantic
 141 rules for defeasible modalities in modal logic proposed by Britz and Varzinczak [4]. This
 142 fragment, denoted by \mathcal{L}_1 , serves as a starting point for showing how the ordering \prec is built
 143 for preferential interpretations in LTL^\sim .

144 **3.1 The fragment \mathcal{L}_1**

145 The fragment \mathcal{L}_1 considers that sentences are in NNF (negation is only allowed on the level
 146 of atomic propositions). On the other hand, the non-monotonic operator \boxminus is omitted from
 147 \mathcal{L}_1 . Furthermore, only Boolean sentences are permitted within the scope of \square sentences.
 148 In what follows, we define formally well formed sentences of \mathcal{L}_1 . In order to do that, we
 149 introduce first the set of Boolean sentences \mathcal{L}_{bool} . Let $p \in \mathcal{P}$, sentences $\alpha_{bool} \in \mathcal{L}_{bool}$ are
 150 defined recursively as such:

$$151 \quad \alpha_{bool} ::= p \mid \neg p \mid \alpha_{bool} \wedge \alpha_{bool} \mid \alpha_{bool} \vee \alpha_{bool}$$

152 Next, let $\alpha_{bool} \in \mathcal{L}_{bool}$, sentences in \mathcal{L}_1 are recursively defined as such:

$$153 \quad \alpha ::= \alpha_{bool} \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \diamond \alpha \mid \square \alpha_{bool} \mid \circ \alpha \mid \diamond \alpha$$

154 Sentences of the form $\diamond \alpha$ are called *eventualities*, because its truth depends on α being
 155 true in the future. Similarly, sentences of the form $\diamond \alpha$ are called *non-monotonic eventualities*.
 156 Their truth depends not only on α being true in some future, but it depends also on this
 157 future being preferred to the other future time points. Sentences of the form $\circ \diamond \alpha$ are called
 158 \circ -eventuality.

159 **3.2 Tableau method for \mathcal{L}_1**

160 A tableau for $\alpha \in \mathcal{L}_1$ is a tree of nodes. Each node has a positive integer n as a *label*. It
 161 has also two sets of sentences: one we denote as Γ and the other as *une* (which stands for
 162 unfulfilled non-monotonic eventualities, a notion to be detailed below). The set Γ is a subset
 163 of \mathcal{L}_1 which contains the sentences in the node. The set *une* is a set of pairs $(n_k, \diamond \alpha_k)$, where
 164 n_k is a label and $\diamond \alpha_k$ is a non-monotonic eventuality.

165 **► Definition 5 (Labelled node).** *A labelled node is a triple of the form $n : (\Gamma, \text{une})$ where*
 166 $n \in \mathbb{N}$, $\Gamma \subseteq \mathcal{L}_1$ and $\text{une} \subseteq [0, n] \times \mathcal{L}_1$.

167 It is worth to mention that different nodes can have the same label. Intuitively, the nodes
 168 labelled by a same integer n represent the set of sentences that are satisfied at the time point
 169 associated with n . Hence, these nodes correspond with a given temporal state.

170 A branch B is a sequence of nodes, we introduce also a strict partial ordering relation
 171 \prec_B on the labels of the nodes within the branch. The branch B has also a set of pairs of

172 labels denoted by min_B . The relation \prec_B represents a preference relation on the temporal
 173 states of the branch B . On the other hand, the set min_B represents some constraints that
 174 the final preference relation issued from B must satisfy. More precisely, each pair (n, n') in
 175 min_B indicates that n' represents a preferred temporal state compared to all $n'' \geq n$.

176 ► **Definition 6 (Branch).** A branch is a tuple $B \stackrel{\text{def}}{=} (\langle x_0, x_1, x_2, \dots \rangle, \prec_B, \text{min}_B)$ where the
 177 first element is a sequence of labelled nodes $x_i := n_i : (\Gamma_i, \text{une}_i)$, \prec_B is a strict partial order
 178 ($\prec_B \subseteq \mathbb{N} \times \mathbb{N}$) on labels within the branch, and min_B is a set of pairs of labels ($\text{min}_B \subseteq \mathbb{N} \times \mathbb{N}$).

179 Let $B := (\langle x_0, x_1, x_2, \dots \rangle, \prec_B, \text{min}_B)$ be a branch, x_n, x_m be two labelled nodes in B . If
 180 x_m comes after x_n in the sequence, then x_m is a *successor* of x_n , and x_n is a *predecessor* of
 181 x_m . We denote it by $x_n \leq x_m$. Moreover, if x_m is not the same labelled node as x_n , we say
 182 that x_m is a proper successor of x_n (same goes for a proper predecessor). We denote it by
 183 $x_n < x_m$. The last node of a branch is called a *leaf node*. When a leaf node is ticked with
 184 ✓, we say that the branch is a successful branch. On the other hand, when a leaf node is
 185 crossed with ✗, we say that the branch is a failed branch.

186 A tree is a set of branches $\mathcal{T} \stackrel{\text{def}}{=} \{B_0, B_1, B_2, B_3, \dots, B_k\}$ where $k \geq 0$. A tableau \mathcal{T} for
 187 α is the limit of a sequence of trees $\langle \mathcal{T}^0, \mathcal{T}^1, \mathcal{T}^2, \dots \rangle$ where the initial tree is $\mathcal{T}^0 := \{(\langle 0 : (\alpha, \emptyset), \emptyset, \emptyset \rangle)\}$
 188 and every \mathcal{T}^{i+1} is obtained from \mathcal{T}^i by applying a rule on one of its branches.
 189 We say that a tableau \mathcal{T} for α is *saturated* if no more rules can be applied after a tree \mathcal{T} .

190 We have two types of rules, static and dynamic rules. We introduce static rules first. Let
 191 \mathcal{T} be a tree, and let B be a branch of \mathcal{T} that has a leaf $n : (\Gamma, \text{une})$. We say that a static
 192 rule (ρ) is applicable at the leaf $n : (\Gamma, \text{une})$ if a sentence in Γ or a pair in une instantiates
 193 the pattern ρ . A static rule is a rule of the form:

$$194 \quad (\rho) \frac{n : (\Gamma, \text{une}), \prec_B, \text{min}_B}{n : (\Gamma_1, \text{une}_1), \prec_{B_1}, \text{min}_{B_1} \mid \dots \mid n : (\Gamma_k, \text{une}_k), \prec_{B_k}, \text{min}_{B_k}}$$

195 In a tree \mathcal{T}^i , after applying the static rule (ρ) , we obtain the tree \mathcal{T}^{i+1} by repla-
 196 cing the branch $B := (\langle x_0, x_1, x_2, \dots, n : (\Gamma, \text{une}) \rangle, \prec_B, \text{min}_B)$ by the branches $B_1 :=$
 197 $(\langle x_0, x_1, x_2, \dots, n : (\Gamma, \text{une}), n : (\Gamma_1, \text{une}_1) \rangle, \prec_{B_1}, \text{min}_{B_1})$, $B_2 := (\langle x_0, x_1, x_2, \dots, n : (\Gamma, \text{une}), n :$
 198 $(\Gamma_2, \text{une}_2) \rangle, \prec_{B_2}, \text{min}_{B_2})$, and so on. The symbol ‘|’ indicates the occurrence of a split in
 199 the branch, i.e., a non-deterministic choice of possible outcomes, each of which needs to be
 200 explored. It is worth to mention that after applying a static rule on $n : (\Gamma, \text{une})$, the leaf
 201 nodes of all the new branches keep the same label n .

202 In what follows, we show the rules for Boolean and the operators (\Box, \Diamond) . We also show
 203 two stopping conditions, namely, **Empty** and **Contradiction**. We chose to omit \prec_B and
 204 min_B to lighten these rules. The crucial detail to remember is that they do not change after
 205 applying the rules below, i.e., $\prec_{B_i} = \prec_B$ and $\text{min}_{B_i} = \text{min}_B$ for all resulting branches. The
 206 symbol \cup is the union of two sets. The symbol \uplus represents the union between disjoint sets.

$$\begin{array}{l} \text{(Contradiction)} \quad \frac{n : (\{\alpha, \neg\alpha\} \uplus \Sigma, \text{une})}{(\text{✗})} \qquad \qquad \qquad \text{(Empty)} \quad \frac{n : (\emptyset, \emptyset)}{(\text{✓})} \\ \\ \text{(}\wedge\text{)} \quad \frac{n : (\{\alpha_1 \wedge \alpha_2\} \uplus \Sigma, \text{une})}{n : (\{\alpha_1, \alpha_2\} \cup \Sigma, \text{une})} \qquad \qquad \qquad \text{(}\vee\text{)} \quad \frac{n : (\{\alpha_1 \vee \alpha_2\} \uplus \Sigma, \text{une})}{n : (\{\alpha_1\} \cup \Sigma, \text{une}) \mid n : (\{\alpha_2\} \cup \Sigma, \text{une})} \\ \\ \text{(}\Box\text{)} \quad \frac{n : (\{\Box\alpha_1\} \uplus \Sigma, \text{une})}{n : (\{\alpha_1, \Box\alpha_1\} \cup \Sigma, \text{une})} \qquad \qquad \qquad \text{(}\Diamond\text{)} \quad \frac{n : (\{\Diamond\alpha_1\} \uplus \Sigma, \text{une})}{n : (\{\alpha_1\} \cup \Sigma, \text{une}) \mid n : (\{\Box\Diamond\alpha_1\} \cup \Sigma, \text{une})} \end{array}$$

208 Before introducing the rule for the non-monotonic operator \Diamond , we discuss firsthand the
 209 notion of *fulfillment* for classical and non-monotonic eventualities. Following Reynolds’
 210 tableau, let an eventuality $\Diamond\alpha$ be in a node with a label n . If the sentence α appears in a

211 proper successor node x with the label $m \geq n$, we say that $\diamond\beta$ at the position n is *fulfilled* in
 212 m . In a similar fashion, we define the fulfillment for non-monotonic eventualities as follows:

213 ► **Definition 7** (Fulfillment of non-monotonic eventualities). *Let a non-monotonic eventuality*
 214 *$\diamond\alpha$ be in a node with a label n in a branch B . If α appears in a proper successor node x with*
 215 *a label $m \geq n$, and $(n, m) \in \text{min}_B$, we say $\diamond\alpha$ at the position n is fulfilled in m .*

216 The truth value $\diamond\alpha$ in a temporal state n depends on α being true on a future temporal
 217 state m and m being minimal to all temporal states that come after n w.r.t. \prec_B . We say m
 218 is minimal to n as shorter way to say that m is minimal to all temporal states that come
 219 after n . Unfulfilled non-monotonic eventualities in a node x with the label n are represented
 220 by the set $\text{une} \stackrel{\text{def}}{=} \{(n_1, \diamond\alpha_1), (n_2, \diamond\alpha_2), \dots\}$, each pair $(n_k, \diamond\alpha_k)$ represents a non-monotonic
 221 eventuality $\diamond\alpha_k$ at a position n_k that needs to be fulfilled. Therefore each node x has three
 222 components: n is a label indicating the temporal state, Γ is the set of sentences within the
 223 node and une is the set of non-monotonic eventualities at x that need to be fulfilled. With
 224 all of our notions introduced, here is the rule for the \diamond operator:

$$225 \quad (\diamond) \frac{n : (\{\diamond\alpha_1\} \uplus \Sigma, \text{une}), \prec_B, \text{min}_B}{n : (\{\alpha_1\} \cup \Sigma, \text{une}), \prec_B, \text{min}_B \cup \{(n, n)\} \mid n : (\Sigma, \text{une} \cup \{(n, \diamond\alpha_1)\}), \prec_B, \text{min}_B}$$

226 For the rule (\diamond) , we explore two outcomes. The first outcome is when the non-monotonic
 227 eventuality $\diamond\alpha_1$ at n is fulfilled in n . We then add α_1 to the set of sentences Γ of the leaf
 228 node and add $(n, n) \in \text{min}$ of the branch. The second outcome is when $\diamond\alpha_1$ is not fulfilled in
 229 n , then we add the pair to $(n, \diamond\alpha_1)$ to une of the leaf node as a non-monotonic eventuality
 230 that needs to be fulfilled. Example 8 shows the application of $[\diamond]$ rule.

231 ► **Example 8.** Let a branch B have \prec_B, min_B and a leaf node $5 : (\{p, q, \Box(p \wedge q), \diamond r\}, \emptyset)$.
 232 After applying (\diamond) rule on $\diamond r$, we have two new branches B_1 and B_2 . The branch B_1 has a
 233 leaf node where the sentence r is in Γ of the leaf node and $(5, 5) \in \text{min}_{B_1}$. The branch B_2
 has $(5, \diamond r)$ in une of the leaf node.

$$\begin{array}{c} 5 : (\{p, q, \Box(p \wedge q), \diamond r\}, \emptyset), \prec_B, \text{min}_B \\ \swarrow \quad \searrow \\ 5 : (\{p, q, \Box(p \wedge q), r\}, \emptyset), \prec_B, \text{min}_B \cup \{(5, 5)\} \quad 5 : (\{p, q, \Box(p \wedge q)\}, \{(5, \diamond r)\}), \prec_B, \text{min}_B \end{array}$$

234

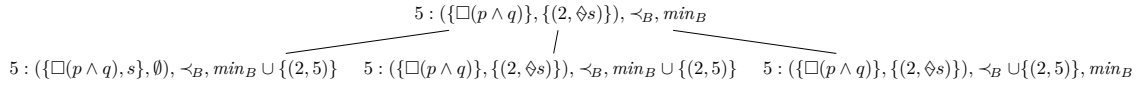
235 The next static rule we discuss is the rule (une) . Let n, n' be two labels such that $n' < n$,
 236 for each label n and a pair $(n', \diamond\alpha_1)$, the rule (une) is applied one and only one time. The
 237 rule goes as follows:

$$238 \quad (\text{une}) \frac{n : (\Gamma, \{(n', \diamond\alpha_1)\} \uplus U), \prec_B, \text{min}_B}{n : (\{\alpha_1\} \cup \Gamma, U), \prec_B, \text{min}_B \cup \{(n', n)\} \mid} \\ \frac{n : (\Gamma, \{(n', \diamond\alpha_1)\} \cup U), \prec_B, \text{min}_B \cup \{(n', n)\} \mid}{n : (\Gamma, \{(n', \diamond\alpha_1)\} \cup U), \prec_B \cup \{(n', n)\}, \text{min}_B}$$

239 For the rule (une) , we explore three outcomes. The first outcome is when $\diamond\alpha_1$ at the
 240 position n' is fulfilled at n . We remove $(n', \diamond\alpha_1)$ from une , then we add α in Γ of the leaf
 241 node and (n', n) in min of the branch. In the second and third branches, we explore the
 242 outcome of $\diamond\alpha_1$ not being fulfilled yet in n , we keep the pair $(n', \diamond\alpha_1)$ on the leaves of two
 243 branches. The second branch explore the outcome of n being minimal to n' w.r.t. to \prec of

244 the branch. We then add (n', n) to the min of the branch. In the third branch, we explore
 245 the outcome of n not being minimal to n' w.r.t. \prec of the branch. It means that there exists
 246 a temporal state m' that come after n' where m' is preferred to n w.r.t. to \prec of the branch,
 247 we add the pair (n', n) in \prec of the branch to represent this case. It is worth to mention that
 248 the rule (une) does not apply when the label of the node n is the same as $(n, \diamond\alpha_1)$. The
 249 reason behind this is that we have already explored the case when the eventuality is fulfilled
 250 in n thanks to (\diamond) rule. Example 9 shows the application of (une) rule.

251 ► **Example 9.** Let a branch B have \prec_B , min_B and a leaf node $5 : (\{\Box(p \wedge q)\}, \{(2, \diamond s)\})$.
 252 After the application of une on $(2, \diamond s)$, we have three branches B_1 , B_2 and B_3 . B_1 has the
 253 sentence s in Γ of its leaf node, it has also $(2, 5)$ in min_{B_1} . B_2 keeps $(2, \diamond s)$ in the une of its
 254 leaf node, with $(2, 5) \in min_{B_2}$. B_3 keeps also $(2, \diamond s)$ in une of its leaf node, with $(2, 5) \in \prec_{B_3}$.



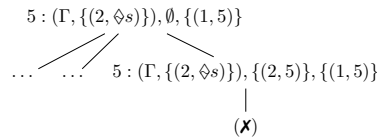
255

256 With the (une) and (\diamond) introduced, we need to check the consistency of \prec of all the
 257 new branches. We apply this check each time we apply (une) or (\diamond) rule. Let $B :=$
 258 $(\langle x_0, x_1, x_2, \dots \rangle, \prec_B, min_B)$ be a branch, the rule goes as follows:

259 ■ [**\prec -inconsistency**] If $(n, n') \in min_B$ and there exists $n'' \geq n$ s.t. $(n'', n') \in \prec_B$, then the
 260 branch is crossed (**X**).

261 In a branch B , if $(n, n') \in min_B$, then we are currently exploring a branch where n' is
 262 minimal to n w.r.t. \prec_B . Therefore there should be no $n'' \geq n$ where $(n'', n) \in \prec_B$. Each time
 263 we explore a branch where this inconsistency arises, we close the branch.

264 ► **Example 10.** Let B be a branch where \prec_B is empty, min_B has $(1, 5)$ in it, and a leaf node
 265 $5 : (\Gamma, \{(2, \diamond s)\})$. After applying une rule on $(2, \diamond s)$, we have three branches B_1 , B_2 and B_3 .
 266 The relation \prec_{B_1} is empty, and min_{B_1} has the pairs $(1, 5)$ and $(2, 5)$. In this case, there is no
 267 inconsistency w.r.t. \prec_{B_1} so far. The same goes for B_2 . However, we add $(2, 5)$ to \prec_{B_3} . Since
 we already have $(1, 5) \in min_{B_3}$, we then cannot have $(2, 5) \in \prec_{B_3}$. We close B_3 .



268

269 In a branch B of a tree \mathcal{T} with a leaf node x_i , after applying every static rule aforemen-
 270 tioned (the order of application these rules is non-deterministic) that can be applied, all leaf
 271 nodes of the generated branches contain only sentences of the form p , $\neg p$ or $\bigcirc\alpha$ in their Γ .
 272 When no more static rules can be applied in a node, this node is called a *state-labelled node*.
 273 State-labelled nodes mark the full expansion of all sentences that hold in a state n .

274 Once we are in a state-labelled node, in order to go from a temporal state to the next, we
 275 need a transition rule (a rule to go from a temporal state n to the next $n + 1$). In a branch
 276 B with a leaf state-labelled node, the rule **transition** goes the following way:

$$\text{(Transition)} \quad \frac{n : (\{\bigcirc\alpha_1, \bigcirc\alpha_2, \bigcirc\alpha_3, \dots, \bigcirc\alpha_k\} \uplus \Sigma, une), \prec_B, min_B}{n + 1 : (\{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_k\}, une), \prec_B, min_B}$$

277

278 After the **transition** rule is applied to a state-labelled node $n : (\Gamma, une)$, we add a node
 279 with the label $n + 1$. It marks the start of a new temporal state $n + 1$. We carry over to $n + 1$
 280 only sentences within the scope of $\bigcirc\alpha_i$ sentences. The set une gets transferred as well to the
 281 next temporal state. Any pair $(n', \diamond\alpha_1) \in une$ remaining in the state node with the label n
 282 indicates that the rule (une) was applied on the temporal state n and the current branch
 283 explores an outcome where $\diamond\alpha_1$ is not yet fulfilled in n . Therefore, these non-monotonic
 284 eventualities need to be fulfilled in $n'' \geq n + 1$.

285 Before applying the **transition** rule, we need to add a set of checks to prevent branches
 286 from expanding indefinitely. These checks are called **loop** and **prune** rules. These rules,
 287 together with the **transition** rule, are called *dynamic rules*.

288 Let $B := (\langle x_0, x_1, x_2, \dots, v \rangle, \prec_B, min_B)$ be a branch where v is a state-labelled node
 289 $n : (\Gamma_v, une_v)$. Let u be the last state-labelled node $n - 1 : (\Gamma_u, une_u)$ that comes before v in
 290 the branch B . Before applying the transition rule on v , we check for these rules:

291 ■ **[Loop]** Let v be a state-labelled node such that it has at least one sentence of the form
 292 $\bigcirc\Box\alpha_{bool}$ in Γ_v but has no $\bigcirc\alpha_{bool}, \bigcirc\diamond\beta, \bigcirc\diamond\beta$ in Γ_v and $une_v = \emptyset$. If for all $\bigcirc\Box\alpha_{bool}$ in Γ_v ,
 293 there exists $u < s \leq v$ such that $\Box\alpha_{bool} \in \Gamma_s$, then the branch B is ticked (\checkmark).

294 Notice that once an eventuality is fulfilled, it does not appear any longer in the successors
 295 of the node. In this case, we say that the sentence is *consumed*. On the other hand, sentences
 296 of the form $\Box\alpha_{bool}$ never get consumed and get replicated indefinitely. Once a branch has
 297 no eventuality left, $\Box\alpha_{bool}$ sentences give rise to an infinite tableau with repetitive nodes.
 298 Nevertheless, we can represent this by looping nodes of the last temporal state. We can, in
 299 this case, stop the branch from ever going infinite. The **loop** rule states that when the leaf
 300 state node v has no eventualities (classical or non-monotonic), has only $\bigcirc\Box\alpha_{bool}$ as sentences
 301 with the pattern \bigcirc , and each $\bigcirc\Box\alpha_{bool}$ is a result from applying the \Box rule to a node in B
 302 with label n , the branch is ticked and marked as a successful branch.

303 ■ **[Prune]** Let $u < v$ be two consecutive state-labelled nodes s.t. $\Gamma_v = \Gamma_u$ and $une_v = une_u$
 304 and that there is at least one eventuality in x_u (either $\bigcirc\diamond\beta \in \Gamma_u$ or $(n', \diamond\beta) \in une_u$),
 305 then the branch is crossed (\times).

306 The **prune** rule states that when the last two state nodes u and v have the same set of
 307 classical and non-monotonic eventualities that need to be fulfilled, and there is at least one
 308 eventuality in u , the branch is then crossed and marked as an unsuccessful branch. Any
 309 branch that does not fulfill at least one eventuality between the current and the last temporal
 310 state is closed, to prioritize the exploration of branches that fulfill one or more eventuality of
 311 the last temporal state. If neither **prune** or **loop** apply on v , we apply the **transition** rule
 312 on the node v . Note that the **loop** and **prune** rules are fundamentally different from the
 313 ones proposed in Reynolds' tableau [14]. These rules are tailored to the restrictions of the
 314 fragment \mathcal{L}_1 , in particular, the restriction of not allowing temporal sentences inside the \Box
 315 operator. We argue in this paper that when eventualities (either classical or non-monotonic)
 316 are not infinitely replicated inside *globally* operators, we only need to check the current state
 317 node with the last one that comes beforehand. It is the reason why we also omit also the
 318 operator \mathcal{U} , since the right part of a \mathcal{U} -sentence can also replicate eventualities.

319 Once we are in a state-labelled node, we check for the loop and prune within the branch
 320 before applying the transition rule. If the transition rule is applied on a state node with a
 321 label n , we obtain a new node with the label $n + 1$. We can then expand the tree from this
 322 node by applying static rules until we find ticked branches (thanks to the **empty** rule), closed
 323 branches (thanks to the **contradiction** or **\prec -inconsistency** rules), or branches with a state

node that has the label $n + 1$. We then repeat the cycle between static and dynamic rules. We can see that the tableau method does not go indefinitely. Thanks to prune rule, we close any branch (\times) that does not fulfill any eventuality in the current temporal state. Anytime we apply a transition rule (from n to $n + 1$), we need to fulfill at least one eventuality in n . Therefore, as long as a branch is not closed with prune rule, eventuality sentences (either classical or non-monotonic) get consumed one by one over the execution of the method. Thus any branch that is not closed with prune has no eventualities left to fulfill. Note that if a branch contains at least one sentence of the form $\Box\alpha_{bool}$, it is then ticked thanks to the loop rule ($\Box\alpha_{bool}$ sentences do not get consumed). Otherwise, it is ticked thanks to the empty rule. Therefore any tableau \mathcal{T} for a sentence in \mathcal{L}_1 is a *saturated* tableau.

4 Soundness and completeness

4.1 Soundness

Here we prove that the tableau method is sound, that is, when a tableau \mathcal{T} of a sentence $\alpha \in \mathcal{L}_1$ has a successful branch, then α is satisfiable. As a first step, we show that we can extract an interpretation $I \in \mathfrak{J}$ from the successful branch. Let $B := (\langle x_0, x_1, x_2, \dots, x_n, (\checkmark) \rangle, \prec_B, \text{min}_B)$ be a successful branch of a tableau \mathcal{T} for α , the sequence of nodes contains normal and state-labelled nodes. Each state-labelled node, denoted by x_{j_i} , within this sequence has a distinct label i . Figure 1 shows an example of the branch B .

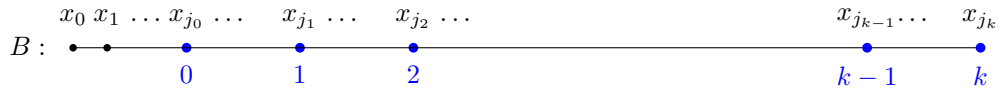


Figure 1 Illustration of the branch B .

From the aforementioned branch B , we can build an interpretation $I_B = (V, \prec)$. In this section, k denotes the label of the last state node. The function V is defined as follows:

$$V(i) := \begin{cases} \{p \in \mathcal{P} \mid p \in \Gamma_{x_{j_i}}\}, & \text{if } 0 \leq i \leq k; \\ V(k), & \text{otherwise.} \end{cases}$$

The ordering relation \prec is defined as follows $\prec := \{(n, n') \mid (n, n') \in \prec_B\}$. We can see that \prec is irreflexive, since there is no $(n, n) \in \prec_B$. The relation \prec is also transitive, since for all (n_1, n_2) and (n_2, n_3) in \prec_B , there is no $(n_3, n_1) \in \prec_B$. Finally, since \prec_B has no infinitely descending chains, then we can conclude that \prec preserves the well-foundedness condition over \mathbb{N} . Therefore the interpretation $I_B \in \mathfrak{J}$.

With the model construction introduced, we can move on to the second part of the proof of soundness. We need to show that the model I satisfies the sentence α . In order to do so, we introduce a mapping function, denoted by Δ_B , that links each time point $i \in \mathbb{N}$ to a set of sentences that are true in said i . These sentences come from the branch B . Depending on how the branch is ticked, the function Δ_B is defined in the following way.

If the branch was ticked with the empty rule:

$$\Delta_B(i) := \begin{cases} \bigcup_{x_0 \leq x \leq x_{j_0}} \Gamma_x, & \text{if } i = 0; \\ \bigcup_{x_{j_{i-1}} < x \leq x_{j_i}} \Gamma_x, & \text{if } 1 \leq i \leq k - 1; \\ \{\}, & \text{otherwise.} \end{cases}$$

357 If the branch was ticked with the loop rule:

$$358 \quad \Delta_B(i) := \begin{cases} \bigcup_{x_0 \leq x \leq x_{j_0}} \Gamma_x, & \text{if } i = 0; \\ \bigcup_{x_{j_{i-1}} < x \leq x_{j_i}} \Gamma_x, & \text{if } 1 \leq i \leq k; \\ \Delta_B(k), & \text{otherwise.} \end{cases}$$

359 For a time point $0 \leq i \leq k$, $\Delta_B(i)$ contains the set of all sentences in Γ of the node
360 between the two consecutive state nodes $x_{j_{i-1}}$ and x_{j_i} , $x_{j_{i-1}}$ not included. If B is ticked
361 thanks to the empty rule, then $\Delta_B(i)$ is empty for all $i \geq k$. If B is ticked thanks to the
362 loop rule, then $\Delta_B(i)$ has the same set of sentences as $\Delta_B(k)$ for all $i \geq k$. We can show
363 next that if a sentence α_1 is in $\Delta_B(i)$, then $I_B, i \models \alpha_1$. In what follows, let B be a successful
364 branch of a tableau \mathcal{T} , let k be the label of the last state node in B , and let I_B, Δ_B be the
365 interpretation and the mapping function of sentences extracted from B .

366 ► **Lemma 11.** *Let B be a successful branch, and $i \in \mathbb{N}$. If $\bigcirc \alpha_1 \in \Delta_B(i)$, then $\alpha_1 \in \Delta_B(i+1)$.*

367 ► **Lemma 12.** *Let B be a successful branch, and $i \in \mathbb{N}$. If $\square \alpha_1 \in \Delta_B(i)$, then for all $f \geq i$,
368 we have $\{\alpha_1, \square \alpha_1, \bigcirc \square \alpha_1\} \subseteq \Delta_B(f)$.*

369 ► **Lemma 13.** *Let B be a successful branch, and $i \in \mathbb{N}$. If $\diamond \alpha_1 \in \Delta_B(i)$, then there exists
370 $d \geq i$ s.t. $\alpha_1 \in \Delta_B(d)$ and for all $i \leq f < d$, we have $\{\diamond \alpha_1, \bigcirc \diamond \alpha_1\} \subseteq \Delta_B(f)$.*

371 Lemma 11 to 13 are analogous to Reynolds' method [14]. Their proof are in Appendix A.

372 ► **Proposition 14.** *Let B be a successful branch. If $(i, i') \in \min_B$, then there is no $i \leq i''$
373 where $(i'', i') \in \prec_B$.*

374 **Proof.** Let B be a successful branch s.t. $(i, i') \in \min_B$. Since the branch is successful, then
375 it is not closed with \prec -inconsistency and therefore there is no $i \leq i''$ where $(i'', i') \in \prec_B$. ◀

376 ► **Lemma 15.** *Let B be a successful branch and $0 \leq i \leq k$. If $\diamond \alpha_1 \in \Delta_B(i)$, then there exists
377 $d \geq i$ s.t. $(i, d) \in \min_B$ and $\alpha_1 \in \Delta_B(d)$.*

378 **Proof.** Let B be a ticked branch of the tableau, k be the label of the last state node and
379 $i \in \mathbb{N}$. We discuss two possibilities:

380 ■ When the branch B is ticked with empty rule, whenever $\diamond \alpha_1 \in \Delta_B(i)$, then we have
381 $0 \leq i \leq k - 1$. Since $\diamond \alpha_1 \in \Delta_B(i)$, then $\diamond \alpha_1 \in \Gamma_x$ where $x_{j_{i-1}} < x \leq x_{j_i}$. Let x be
382 the node where we apply the rule (\diamond) on $\diamond \alpha_1$, then we either have α_1 in Γ of the next
383 node with $(i, i) \in \min_B$ or we have $(i, \diamond \alpha_1) \in \text{une}$ of the next node. If α_1 is in Γ of
384 the next node, then the lemma holds. If $(i, \diamond \alpha_1) \in \text{une}$ of the next node, then we find
385 $(i, \diamond \alpha_1) \in \text{une}_{x_{j_i}}$. Thanks to the transition rule, we have $(i, \diamond \alpha_1) \in \text{une}_{x_{j_{i+1}}}$. By applying
386 the rule *une* on a node with the label $i + 1$, then we either have α_1 in Γ of the next node
387 with $(i, i + 1) \in \min_B$ or we have $(i, \diamond \alpha_1) \in \text{une}$ (the two remaining branches) of the
388 next node. In a similar way as in i , we can conclude that either $\alpha_1 \in \Delta_B(i + 1)$ with
389 $(i, i + 1) \in \min_B$ (the lemma holds) or $(i, \diamond \alpha_1) \in \text{une}_{x_{j_{i+1}}}$. Without loss of generality,
390 $(i, \diamond \alpha_1)$ is in $\text{une}_{x_{j_f}}$ for $i \leq f \leq k - 1$ unless we find $i \leq d \leq f$ with $\alpha_1 \in \Delta_B(d)$
391 and $(i, d) \in \min_B$. Since the branch is closed thanks to the empty rule, it means that
392 $(i, \diamond \alpha_1) \notin \text{une}_{x_{j_{k-1}}}$. Therefore, there is a state $i \leq d \leq k - 1$ where $\alpha_1 \in \Delta_B(d)$ with
393 $(i, d) \in \min_B$.

394 ■ When the branch B is ticked with loop rule, the proof is analogous to the case of the
395 empty rule (notice that we also have $(i, \diamond \alpha_1) \notin \text{une}_{x_{j_k}}$).

396

397 ► **Theorem 16.** *Let B be a ticked branch from a saturated tableau, and $I_B = (V, \prec)$ be the*
 398 *model built from the branch B . For all $\alpha \in \mathcal{L}_1$, for all $i \geq 0$, if $\alpha \in \Delta_B(i)$ then $I_B, i \models \alpha$.*

399 **Proof.** We prove this lemma using structural induction on the size of the sentence α . Let B
 400 be a successful branch for a tableau \mathcal{T} , and $I_B = (V, \prec)$ be the model built from B .

401 ■ $\alpha = p$. Let $p \in \Delta_B(i)$. By construction of the model I_B , we have $p \in V(i)$. Therefore,
 402 we have $I_B, i \models p$.

403 ■ $\alpha = \neg p$. Let $\neg p \in \Delta_B(i)$. Since B is a ticked branch, then it was not closed with the
 404 contradiction rule, therefore we have $p \notin V(i)$. Therefore, we have $I_B, i \models \neg p$.

405 ■ $\alpha = \alpha_1 \wedge \alpha_2$. Let $\alpha_1 \wedge \alpha_2 \in \Delta_B(i)$. By \wedge -rule, we have $\alpha_1, \alpha_2 \in \Delta_B(i)$. By induction
 406 hypothesis on α_1, α_2 , we have $I_B, i \models \alpha_1$ and $I_B, i \models \alpha_2$. Thus, we have $I_B, i \models \alpha_1 \wedge \alpha_2$.

407 ■ $\alpha = \alpha_1 \vee \alpha_2$. Let $\alpha_1 \vee \alpha_2 \in \Delta_B(i)$. By \vee -rule, we either have α_1 or α_2 in $\Delta_B(i)$. Suppose
 408 that $\alpha_1 \in \Delta_B(i)$, by induction hypothesis on α_1 , we have $I_B, i \models \alpha_1$. Therefore, we have
 409 $I_B, i \models \alpha_1 \vee \alpha_2$. Same reasoning applies when $\alpha_2 \in \Delta_B(i)$.

410 ■ $\alpha = \bigcirc \alpha_1$. Let $\bigcirc \alpha_1 \in \Delta_B(i)$. Thanks to Lemma 11, we have $\alpha_1 \in \Delta_B(i+1)$. By induction
 411 hypothesis on α_1 , we have $I_B, i+1 \models \alpha_1$. Therefore, we have $I_B, i \models \bigcirc \alpha_1$.

412 ■ $\alpha = \square \alpha_1$. Let $\square \alpha_1 \in \Delta_B(i)$. Thanks to Lemma 12, we have $\alpha_1 \in \Delta_B(f)$ for all $f \geq i$.
 413 By induction hypothesis on α_1 , we have $I_B, f \models \alpha_1$ for all $f \geq i$. Therefore, we have
 414 $I_B, i \models \square \alpha_1$.

415 ■ $\alpha = \diamond \alpha_1$. Let $\diamond \alpha_1 \in \Delta_B(i)$. Thanks to Lemma 13, we have $\alpha_1 \in \Delta_B(d)$ for some $d \geq i$.
 416 By induction hypothesis on α_1 , we have $I_B, d \models \alpha_1$. Therefore, we have $I_B, i \models \diamond \alpha_1$.

417 ■ $\alpha = \diamond \alpha_1$. Let $\diamond \alpha_1 \in \Delta_B(i)$. Depending on where i is, we have two cases:

418 ■ In the case of $i > k$, since $\diamond \alpha_1 \in \Delta_B(i)$, then we have $\diamond \alpha_1 \in \Delta_B(k)$. Furthermore,
 419 since the branch is ticked with loop rule, we know that $(i, \diamond \alpha_1) \notin \text{une}_{x_{j_k}}$. Therefore
 420 $\alpha_1 \in \Delta_B(k)$, thus $\alpha_1 \in \Delta_B(i)$. Furthermore, since $\prec := \prec_B$, and there is no $f \geq i$ such
 421 $(f, i) \in \prec_B$, then $i \in \min_{\prec}(i)$, and therefore, $I_B, i \models \diamond \alpha_1$.

422 ■ $0 \leq i \leq k$. Thanks to Lemma 15, there exists $d \geq i$ s.t. $\alpha_1 \in \Delta_B(d)$ and $(i, d) \in \min_B$.
 423 By induction hypothesis on α_1 , we have $I_B, d \models \alpha_1$. Thanks to Proposition 14, there
 424 is no $i \leq f \leq k$ where $(f, d) \in \prec_B$ and therefore there is no $i \leq f \leq k$ where $(f, d) \in \prec$.
 425 Furthermore, by the construction of the model I_B , there is no $f \geq k$ where $(f, d) \in \prec$.
 426 Therefore, we have $d \in \min_{\prec}(i)$. Thus, we have $I_B, i \models \diamond \alpha_1$.

427

428 Let $\alpha \in \mathcal{L}_1$, B be a ticked branch from a saturated tableau for α , $I_B = (V, \prec)$ be a model
 429 built from B . Since we have $\alpha \in \Delta_B(0)$, then we have $I_B, 0 \models \alpha$.

430 4.2 Completeness

431 We conclude this paper by proving the completeness of the tableau method for sentences
 432 in \mathcal{L}_1 i.e., if a sentence α is satisfiable, then any tableau for α has a successful branch, no
 433 matter the order of applying the rules. We use a model I for α to find a ticked node.

434 ► **Theorem 17.** *Let $\alpha \in \mathcal{L}_1$ be a satisfiable sentence of LTL^{\sim} . Then any tableau for α has*
 435 *a successful branch.*

436 The idea behind this proof is to have an intermediate sequence s that serves as a link
 437 between an interpretation I that satisfies the sentence α and a tableau \mathcal{T} for α . The sequence
 438 s is a tuple $s := (\langle x_0, x_1, x_2, \dots \rangle, \prec_s, \min_s)$ where each x_i is a pair (Γ, une) , \prec_s, \min_s are
 439 the set of constraints that the sequence s must follow in order to be coherent with \prec of

440 the interpretation. The set \prec_s is not an ordering relation, it records instances of points of
 441 time not being minimal to other points of time w.r.t. the ordering relation \prec . Remember
 442 that each when we apply the *une* rule, we add a pair (n', n) to \prec in order to symbolize the
 443 outcome of n not being minimal to n' . The set of min_s records the instances of points of
 444 points of time being minimal to other points of time w.r.t. the ordering relation \prec .

445 We link each node of the sequence x_i to a time point $J(x_i)$ of the interpretation I and a
 446 labelled node $f(x_i)$ of the tableau \mathcal{T} . Depending on I , we can build the sequence s using the
 447 tableau, we then show the sequence s ends up with a tick (\checkmark). We make sure that for each
 448 node x_i with the index time point $J(x_i)$ of the sequence, we have the following invariant:

$$449 \quad Inv(x_i, J(x_i)) \left\{ \begin{array}{l} \text{For each } \alpha \in \Gamma_{x_i}, \text{ we have } I, J(x_i) \models \alpha; \\ \text{For each } (J_1, \diamond\alpha_1) \in une_{x_i}, \text{ there exists } J_2 \geq J(x_i) \text{ where} \\ \quad J_2 \in min_{\prec}(J_1) \text{ and } I, J_2 \models \alpha_1; \\ \text{For each } (J_1, J_2) \in min_s, \text{ we have } J_2 \in min_{\prec}(J_1); \\ \text{For each } (J_1, J_2) \in \prec_s, \text{ there exists } J_3 \geq J_1 \text{ s.t. } (J_3, J_2) \in \prec \\ \quad (\text{in other words } J_2 \notin min_{\prec}(J_1)). \end{array} \right.$$

450 We start by putting the root node $0 : (\{\alpha\}, \emptyset)$ with the index time point $J(x_0) := 0$ at the
 451 start of the sequence. For the first node x_0 with the index time point 0 (since there is no rule
 452 applied before the root node, the sets min_s and \prec_s are empty at the start), we have $I, 0 \models \alpha$.
 453 Therefore the invariant $Inv(x_0, 0)$ holds. Suppose that the invariant holds up to x_i , and a
 454 rule was applied to x_i , we then add a new node x_{i+1} to the sequence depending on which
 455 outcome of the rule represents the interpretation I . We then move to the outcome node in
 456 the tableau, and see which rule is applied to it, and so on and so forth. Each time we add a
 457 new node x_{i+1} to the sequence s , we need to make sure that the invariant $Inv(x_{i+1}, J(x_{i+1}))$
 458 holds. In general, the sequence will head from the parent node to a child node but it might
 459 occasionally jump backwards (only in the case of the parent being a prune node, more on
 460 that later). It is worth to point out that since we might be jumping back and forth between
 461 nodes of \mathcal{T} , each time we are add a new node x_{i+1} to the sequence s , we are going to rename
 462 labels within the sets une_x , \prec_B and min_B by their respective indexed time points J . The
 463 function f links each node x_i of the sequence s to a labelled node $f(x_i)$ of the tableau \mathcal{T} .
 464 It is worth to mention that, since we are only renaming labels of other sets, then we have
 465 $\Gamma_{x_i} = \Gamma_{f(x_i)}$. In Appendix B, we discuss the case of each rule that is applied to x_i .

466 5 Conclusion

467 We introduced the basis for a tableau method for LTL^\sim . We showed how preferential
 468 semantics work in a one-pass tree-shaped tableau. We also established semantic rules for the
 469 \diamond operator. We showed how to handle non-monotonic eventualities using *une*, \prec_B and min_B .
 470 In the end, we proved that our method is sound and complete. The loop/prune checkers
 471 proposed in this paper are specific to \mathcal{L}_1 , and work well under these restrictions.

472 With the foundation laid in this work, the next step is to establish semantic rules for
 473 the \boxtimes operator. The next fragment of LTL^\sim that we are investigating is the sub-language
 474 that allows only Boolean sentences within \square and \boxtimes . We conjecture that the satisfiability of
 475 this fragment is decidable and has an upper bound model property similar to one that we
 476 published in [6].

477 ——— **References** ———

- 478 1 M. Ben-Ari. *Mathematical Logic for Computer Science, third edition*. Springer, 2012.
- 479 2 K. Britz, T. Meyer, and I. Varzinczak. Preferential reasoning for modal logics. *Electronic*
480 *Notes in Theoretical Computer Science*, 278:55 – 69, 2011. doi:[https://doi.org/10.1016/j.](https://doi.org/10.1016/j.entcs.2011.10.006)
481 [entcs.2011.10.006](https://doi.org/10.1016/j.entcs.2011.10.006).
- 482 3 K. Britz, T. Meyer, and I. Varzinczak. Semantic foundation for preferential description logics.
483 In *AI 2011: Advances in Artificial Intelligence*, pages 491–500, Berlin, Heidelberg, 2011.
484 Springer Berlin Heidelberg.
- 485 4 K. Britz and I. Varzinczak. From KLM-style conditionals to defeasible modalities, and back.
486 *Journal of Applied Non-Classical Logics*, 28(1):92–121, 2018. doi:[10.1080/11663081.2017.](https://doi.org/10.1080/11663081.2017.1397325)
487 [1397325](https://doi.org/10.1080/11663081.2017.1397325).
- 488 5 K. Britz and I. Varzinczak. Preferential tableaux for contextual defeasible \mathcal{ALC} . In *Proceedings*
489 *of the 28th International Conference on Automated Reasoning with Analytic Tableaux and*
490 *Related Methods (TABLEAUX)*, number 11714 in LNCS, pages 39–57, 2019.
- 491 6 A. Chafik, F. Cheikh Alili, J.-F. Condotta, and I. Varzinczak. On the decidability of a
492 fragment of preferential LTL. In *27th International Symposium on Temporal Representation*
493 *and Reasoning, TIME 2020*, volume 178 of *LIPICs*, pages 19:1–19:19, 2020. doi:[10.4230/](https://doi.org/10.4230/LIPICs.TIME.2020.19)
494 [LIPICs.TIME.2020.19](https://doi.org/10.4230/LIPICs.TIME.2020.19).
- 495 7 D. M. Gabbay. The declarative past and imperative future: Executable temporal logic for
496 interactive systems. In *Temporal Logic in Specification, Altrincham, UK, April 8-10, 1987,*
497 *Proceedings*, volume 398 of *Lecture Notes in Computer Science*, pages 409–448. Springer, 1987.
498 doi:[10.1007/3-540-51803-7_36](https://doi.org/10.1007/3-540-51803-7_36).
- 499 8 L. Giordano, V. Gliozzi, N. Olivetti, and G.L. Pozzato. Preferential description logics. In
500 *Logic for Programming, Artificial Intelligence, and Reasoning (LPAR)*, number 4790 in LNAI,
501 pages 257–272. Springer, 2007.
- 502 9 L. Giordano, V. Gliozzi, N. Olivetti, and G.L. Pozzato. Analytic tableaux calculi for KLM logics
503 of nonmonotonic reasoning. *ACM Transactions on Computational Logic*, 10(3):18:1–18:47,
504 2009.
- 505 10 Y. Kesten, Z. Manna, H. Mcguire, and A. Pnueli. A decision algorithm for full propositional
506 temporal logic. *LNCS*, 697, 04 1999.
- 507 11 S. Kraus, D. Lehmann, and M. Magidor. Nonmonotonic reasoning, preferential models and
508 cumulative logics. *Artificial Intelligence*, 44:167–207, 1990.
- 509 12 Z. Manna and A. Pnueli. *Temporal verification of reactive systems - safety*. Springer, 1995.
- 510 13 A. Pnueli. The temporal logic of programs. In *18th Annual Symposium on Foundations of*
511 *Computer Science (SFCS 1977)*, pages 46–57, Oct 1977. doi:[10.1109/SFCS.1977.32](https://doi.org/10.1109/SFCS.1977.32).
- 512 14 M. Reynolds. A new rule for LTL tableaux. *Electronic Proceedings in Theoretical Computer*
513 *Science*, 226:287–301, Sep 2016. URL: <http://dx.doi.org/10.4204/EPTCS.226.20>, doi:
514 [10.4204/eptcs.226.20](https://doi.org/10.4204/eptcs.226.20).
- 515 15 M. Reynolds. A traditional tree-style tableau for LTL, 2016. arXiv:[1604.03962](https://arxiv.org/abs/1604.03962).
- 516 16 K. Y. Rozier and M. Y. Vardi. LTL satisfiability checking. In *Model Checking Software*, pages
517 149–167, Berlin, Heidelberg, 2007. Springer Berlin Heidelberg.
- 518 17 S. Schwendimann. A new one-pass tableau calculus for PLTL. In *Automated Reasoning with*
519 *Analytic Tableaux and Related Methods*, pages 277–291. Springer Berlin Heidelberg, 1998.
- 520 18 Y. Shoham. *A Semantical Approach to Nonmonotonic Logics*, page 227–250. Morgan Kaufmann
521 Publishers Inc., San Francisco, CA, USA, 1987.
- 522 19 Y. Shoham. *Reasoning about Change: Time and Causation from the Standpoint of Artificial*
523 *Intelligence*. MIT Press, 1988.
- 524 20 A. P. Sistla and E. M. Clarke. The complexity of propositional linear temporal logics. *J. ACM*,
525 32(3):733–749, July 1985. doi:[10.1145/3828.3837](https://doi.org/10.1145/3828.3837).
- 526 21 P. Wolper. Temporal logic can be more expressive. *Information and Control*, 56(1):72 – 99,
527 1983. doi:[https://doi.org/10.1016/S0019-9958\(83\)80051-5](https://doi.org/10.1016/S0019-9958(83)80051-5).
- 528 22 P. Wolper. The tableau method for temporal logic. *Logique Et Analyse*, 28:110–111, 01 1985.

529 **A** Soundness proof

530 ► **Lemma 11.** *Let B be a successful branch, and $i \in \mathbb{N}$. If $\circ\alpha_1 \in \Delta_B(i)$, then $\alpha_1 \in \Delta_B(i+1)$.*

531 **Proof.** Let B be a ticked branch of the tableau, k be the label of the last node and $i \in \mathbb{N}$.
532 We discuss two possibilities:

533 ■ When the branch B is ticked with empty rule. We can see that when $i \geq k$, $\Delta_B(i) = \{\}$
534 and therefore $\circ\alpha_1 \notin \Delta_B(i)$. We also know that since $\Delta_B(k) = \{\}$, then there is no
535 $\circ\alpha_1 \in \Gamma_{x_{j_{k-1}}}$. Furthermore, there is no static rule that removes $\circ\alpha_1$, we can conclude
536 that there is no $\circ\alpha_1 \in \Delta_B(k-1)$.

537 Otherwise, in the case of $0 \leq i < k-1$, if $\circ\alpha_1 \in \Delta_B(i)$, then $\circ\alpha_1 \in \Gamma_x$ where
538 $x_{j_{i-1}} < x \leq x_{j_i}$. Since there is no static rule that removes $\circ\alpha_1$, we have $\circ\alpha_1 \in \Gamma_{x_{j_i}}$.
539 Furthermore, after applying the transition rule on the node x_{j_i} , we have $\alpha_1 \in \Gamma_{x_{j_{i+1}}}$.
540 Thus, we have $\alpha_1 \in \Delta_B(i+1)$.

541 ■ When the branch B is ticked with loop rule. In the case of $0 \leq i < k$, the proof is analogous
542 to the case of empty rule. When $i = k$, if $\circ\alpha_1 \in \Delta_B(k)$, then $\circ\alpha_1$ is subsequently in $\Gamma_{x_{j_k}}$.
543 Since B is ticked with loop, then α_1 is a sentence of the form $\Box\alpha_{bool}$ and $\Box\alpha_{bool} \in \Gamma_x$
544 ($x_{j_{k-1}} < x \leq x_{j_k}$) and therefore $\Box\alpha_{bool} \in \Delta_B(k)$. Moreover, we have $\Delta_B(k) = \Delta_B(k+1)$.
545 Therefore, we have $\Box\alpha_{bool} \in \Delta_B(k+1)$ and thus $\alpha_1 \in \Delta_B(k+1)$.

546 In the case where $i \geq k$. If $\circ\alpha_1 \in \Delta_B(i)$, then $\circ\alpha_1 \in \Delta_B(k-1)$. As mentioned before,
547 since $\circ\alpha_1 \in \Delta_B(k-1)$, then α_1 is $\Box\alpha_2$ and $\Box\alpha_2 \in \Delta_B(k-1)$. Since $\Box\alpha_2 \in \Delta_B(k-1)$,
548 then $\Box\alpha_2 \in \Delta_B(i+1)$ and therefore $\alpha_1 \in \Delta_B(i+1)$.

549 ◀

550 ► **Lemma 12.** *Let B be a successful branch, and $i \in \mathbb{N}$. If $\Box\alpha_1 \in \Delta_B(i)$, then for all $f \geq i$,
551 we have $\{\alpha_1, \Box\alpha_1, \circ\Box\alpha_1\} \subseteq \Delta_B(f)$.*

552 **Proof.** Let B be a ticked branch of the tableau, k be the label of the last node and $i \in \mathbb{N}$.
553 For all $0 \leq i \leq k$, whenever $\Box\alpha_1 \in \Delta_B(i)$, then both α_1 and $\circ\Box\alpha_1$ is in $\Delta_B(i)$. By
554 Lemma 11, since $\circ\Box\alpha_1 \in \Delta_B(i)$, then we have $\Box\alpha_1 \in \Delta_B(i+1)$. By successive applications
555 of Lemma 11, we have $\{\alpha_1, \Box\alpha_1, \circ\Box\alpha_1\} \subseteq \Delta_B(f)$ for all $i \leq f \leq k$. Note that in the case
556 of a branch ticked with empty rule, since $\Delta_B(k) = \{\}$, $\Box\alpha_1$ cannot be in any $\Delta_B(i)$ where
557 $0 \leq i \leq k$. In other words, if a branch contains $\Box\alpha_1$, it can only be ticked with loop rule.

558 Since $\{\alpha_1, \Box\alpha_1, \circ\Box\alpha_1\} \subseteq \Delta_B(k)$, and for all $f \geq k$, we have $\Delta_B(f) = \Delta_B(k)$, then
559 $\{\alpha_1, \Box\alpha_1, \circ\Box\alpha_1\} \subseteq \Delta_B(f)$. Thus, the lemma holds when $0 \leq i \leq k$.

560 In the case of $i > k$, since $\Box\alpha_1 \in \Delta_B(i)$ and $\Delta_B(i) = \Delta_B(k-1)$. Thanks to \Box -rule,
561 $\{\alpha_1, \Box\alpha_1, \circ\Box\alpha_1\} \subseteq \Delta_B(k-1)$. Thus, we have $\{\alpha_1, \Box\alpha_1, \circ\Box\alpha_1\} \subseteq \Delta_B(f)$ for all $f \geq k$ and
562 subsequently $\{\alpha_1, \Box\alpha_1, \circ\Box\alpha_1\} \subseteq \Delta_B(f)$ for all $f \geq i$. ◀

563 ► **Lemma 13.** *Let B be a successful branch, and $i \in \mathbb{N}$. If $\Diamond\alpha_1 \in \Delta_B(i)$, then there exists
564 $d \geq i$ s.t. $\alpha_1 \in \Delta_B(d)$ and for all $i \leq f < d$, we have $\{\Diamond\alpha_1, \circ\Diamond\alpha_1\} \subseteq \Delta_B(f)$.*

565 **Proof.** Let B be a ticked branch of the tableau, k be the label of the last node and $i \in \mathbb{N}$.
566 We discuss two possibilities:

567 ■ When the branch B is ticked with empty rule. In the case of $0 \leq i \leq k-1$, whenever
568 $\Diamond\alpha_1 \in \Delta_B(i)$, then either α_1 is in $\Delta_B(i)$ or $\circ\Diamond\alpha_1$ is in $\Delta_B(i)$. If $\alpha_1 \in \Delta_B(i)$, the lemma
569 holds. Otherwise, by Lemma 11, if $\circ\Diamond\alpha_1 \in \Delta_B(i)$ then $\Diamond\alpha_1 \in \Delta_B(i+1)$. By successive
570 applications of Lemma 11, $\{\Diamond\alpha_1, \circ\Diamond\alpha_1\}$ is in $\Delta_B(f)$ for $i \leq f \leq k-1$, unless we find
571 $i \leq d \leq f$ with $\alpha_1 \in \Delta_B(d)$.

572 It remains to show that there is a time point d where $\alpha_1 \in \Delta_B(d)$. Since the branch is
 573 closed thanks to the empty rule, it means that $\circ\Diamond\alpha_1 \notin \Delta_B(k-1)$. Therefore, there is a
 574 state $i \leq d \leq k-1$ where $\alpha_1 \in \Delta_B(d)$.

575 ■ When the branch B is ticked with loop rule and in the case of $0 \leq i \leq k$, the proof
 576 is analogous to the case of empty rule (notice that $\circ\Diamond\alpha_1 \notin \Delta_B(k)$ also in the case of
 577 branches ticked with loop).

578 In the case of $i > k$, since $\Diamond\alpha_1 \in \Delta_B(i)$, then we have $\Diamond\alpha_1 \in \Delta_B(k-1)$. Furthermore,
 579 since the branch is ticked with loop rule, we know that $\circ\Diamond\alpha_1 \notin \Delta_B(k)$. Therefore
 580 $\alpha_1 \in \Delta_B(k)$, thus $\alpha_1 \in \Delta_B(i)$.

581

B Completeness proof

583 **Proof.** In this section, suppose that we build the sequence s up to x_i and the invariant holds
 584 for all the nodes in the sequence.

585 **[Empty, Loop]:** If we end up with a ticked node in the sequence s , the theorem holds.

586 **[Contradiction]:** If the sequence s is closed, then we have p and $\neg p$ in Γ_{x_i} . Since we
 587 have $Inv(x_i, J(x_i))$, then we $I, J(x_i) \models p$ and $I, J(x_i) \models \neg p$. This cannot happen in a
 588 interpretation $I \in \mathcal{I}$.

589 **[\wedge]:** Suppose that the rule \wedge is applied to the sentence $\alpha_1 \wedge \alpha_2$ on the node $f(x_i)$
 590 of the tableau \mathcal{T} . Let y be the child node of the node $f(x_i)$ in the branch. We have
 591 $\Gamma_y = (\Gamma_{f(x_i)} \setminus \{\alpha_1 \wedge \alpha_2\}) \cup \{\alpha_1, \alpha_2\}$. We define the next node in the sequence x_{i+1} with
 592 $\Gamma_{x_{i+1}} = \Gamma_y$, $une_{x_{i+1}} = une_{x_i}$, and the sets min_s, \prec_s remain unchanged. Since we have
 593 $Inv(x_i, J(x_i))$ and $\alpha_1 \wedge \alpha_2 \in \Gamma_{x_i}$, then $I, J(x_i) \models \alpha_1$ and $I, J(x_i) \models \alpha_2$. For the node x_{i+1} ,
 594 we have $\Gamma_{x_{i+1}} = (\Gamma_{x_i} \setminus \{\alpha_1 \wedge \alpha_2\}) \cup \{\alpha_1, \alpha_2\}$ and $une_{x_{i+1}} = une_{x_i}$. Therefore the first and
 595 second conditions of $Inv(x_{i+1}, J(x_i))$ are met. Moreover, since min_s, \prec_s remain unchanged
 596 and we have $Inv(x_i, J(x_i))$, then the third and fourth conditions of $Inv(x_{i+1}, J(x_i))$ are met.
 597 Consider that $J(x_{i+1}) = J(x_i)$, the invariant $Inv(x_{i+1}, J(x_i))$ holds.

598 We can see that by applying a static rule of the from $(\wedge, \vee, \square, \diamond)$ on the node $f(x_i)$, we do
 599 not add in either une , \prec_B or min_B while applying these rules nor add a new non-monotonic
 600 eventuality to be fulfilled in the outcome nodes. In order to lighten the proof, we skip the
 601 check for the second, third and fourth conditions of Inv up until \diamond and une rules.

602 **[\vee]:** Suppose that the rule \vee is applied to the sentence $\alpha_1 \vee \alpha_2$ on the node $f(x_i)$ of
 603 the tableau \mathcal{T} . We obtain two children nodes y and z of $f(x_i)$. We have $\Gamma_y = (\Gamma_{f(x_i)} \setminus$
 604 $\{\alpha_1 \vee \alpha_2\}) \cup \{\alpha_1\}$ and $\Gamma_z = (\Gamma_{f(x_i)} \setminus \{\alpha_1 \vee \alpha_2\}) \cup \{\alpha_2\}$. Since we have $Inv(x_i, J(x_i))$,
 605 and $\alpha_1 \vee \alpha_2 \in \Gamma_{x_i}$, then we either have $I, J(x_i) \models \alpha_1$ or $I, J(x_i) \models \alpha_2$. Consider that
 606 $J(x_{i+1}) = J(x_i)$, we discuss two cases:

607 ■ Case 1: If $I, J(x_i) \models \alpha_1$, then we define the next node x_{i+1} with $\Gamma_{x_{i+1}} = \Gamma_y$ and
 608 $une_{x_{i+1}} = une_{x_i}$. We know that $\Gamma_{x_{i+1}} = (\Gamma_{x_i} \setminus \{\alpha_1 \vee \alpha_2\}) \cup \{\alpha_1\}$. Therefore for all
 609 $\gamma \in \Gamma_{x_{i+1}}$, we have $I, J(x_i) \models \gamma$. Thus, the invariant $Inv(x_{i+1}, J(x_i))$ holds.

610 ■ Case 2: Otherwise, when $I, J(x_i) \models \alpha_2$, then we define the node x_{i+1} with $\Gamma_{x_{i+1}} = \Gamma_z$
 611 and $une_{x_{i+1}} = une_{x_i}$. We know that $\Gamma_{x_{i+1}} = (\Gamma_{x_i} \setminus \{\alpha_1 \vee \alpha_2\}) \cup \{\alpha_2\}$. Therefore for all
 612 $\gamma \in \Gamma_{x_{i+1}}$, we have $I, J(x_i) \models \gamma$. Thus, the invariant $Inv(x_{i+1}, J(x_i))$ holds.

613 **[\diamond]:** Suppose that the rule \diamond is applied to the sentence $\Diamond\alpha_1$ on the node $f(x_i)$ of the tableau
 614 \mathcal{T} . We obtain two children nodes y and z of $f(x_i)$. We have $\Gamma_y = (\Gamma_{f(x_i)} \setminus \{\Diamond\alpha_1\}) \cup \{\alpha_1\}$
 615 and $\Gamma_z = (\Gamma_{f(x_i)} \setminus \{\Diamond\alpha_1\}) \cup \{\circ\Diamond\alpha_1\}$. Since we have $Inv(x_i, J(x_i))$, and $I, J(x_i) \models \Diamond\alpha_1$, then

616 we have $I, J(x_i) \models \alpha_1 \vee \circ\Diamond\alpha_1$. Therefore, we either have $I, J(x_i) \models \alpha_1$ or $I, J(x_i) \models \circ\Diamond\alpha_1$.
 617 Consider that $J(x_{i+1}) = J(x_i)$, we discuss two cases:

- 618 ■ Case 1: If $I, J(x_i) \models \alpha_1$, then we define the next node x_{i+1} with $\Gamma_{x_{i+1}} = \Gamma_y$ and
 619 $une_{x_{i+1}} = une_{x_i}$. We know that $\Gamma_{x_{i+1}} = (\Gamma_{x_i} \setminus \{\Diamond\alpha_1\}) \cup \{\alpha_1\}$. Therefore for all $\gamma \in \Gamma_{x_{i+1}}$,
 620 we have $I, J(x_i) \models \gamma$. Thus, the invariant $Inv(x_{i+1}, J(x_i))$ holds.
- 621 ■ Case 2: When $I, J(x_i) \models \circ\Diamond\alpha_1$, then we define the next node x_{i+1} with $\Gamma_{x_{i+1}} = \Gamma_z$
 622 and $une_{x_{i+1}} = une_{x_i}$. We know that $\Gamma_{x_{i+1}} = (\Gamma_{x_i} \setminus \{\Diamond\alpha_1\}) \cup \{\circ\Diamond\alpha_1\}$. Therefore for all
 623 $\gamma \in \Gamma_{x_{i+1}}$, we have $I, J(x_i) \models \gamma$. Thus, the invariant $Inv(x_{i+1}, J(x_i))$ holds.

624 $[\Box]$: Suppose that the rule \Box is applied to the sentence $\Box\alpha_1$ on the node $f(x_i)$ of the
 625 tableau \mathcal{T} . Let y be the child node of the node $f(x_i)$ in the branch. We have $\Gamma_y = (\Gamma_{f(x_i)} \setminus$
 626 $\{\Box\alpha_1\}) \cup \{\alpha_1, \circ\Box\alpha_1\}$. We define the next node x_{i+1} with $\Gamma_{x_{i+1}} = \Gamma_y$ and $une_{x_{i+1}} = une_{x_i}$
 627 and $I, J(x_i) \models \Box\alpha_1$, then we have $I, J(x_i) \models \alpha_1 \wedge \circ\Box\alpha_1$. Therefore, we have $I, J(x_i) \models \alpha_1$
 628 and $I, J(x_i) \models \circ\Box\alpha_1$. We know that $\Gamma_{x_{i+1}} = (\Gamma_{x_i} \setminus \{\Box\alpha_1\}) \cup \{\alpha_1, \circ\Box\alpha_1\}$. Therefore
 629 for all $\gamma \in \Gamma_{x_{i+1}}$, we have $I, J(x_i) \models \gamma$. Consider that $J(x_{i+1}) = J(x_i)$, the invariant
 630 $Inv(x_{i+1}, J(x_i))$ holds.

631 $[\Diamond]$: When the rule $[\Diamond]$ is applied to $\Diamond\alpha_1$ on the node $f(x_i)$ of \mathcal{T} , we explore two outcomes.
 632 Let n be the label of the node $f(x_i)$ in the branch. In the first outcome, we have a child y
 633 with $\Gamma_y = (\Gamma_{f(x_i)} \setminus \{\Diamond\alpha_1\}) \cup \{\alpha_1\}$ and (n, n) in min of the branch. In the second outcome,
 634 we have a child node z with $\Gamma_z = (\Gamma_{f(x_i)} \setminus \{\Diamond\alpha_1\})$ and $une_z = une_{f(x_i)} \cup (n, \Diamond\alpha_1)$. Since
 635 we have $Inv(x_i, J(x_i))$, and $\Diamond\alpha_1 \in \Gamma_{x_i}$, then we have $I, J(x_i) \models \Diamond\alpha_1$. It means that there
 636 exists $J_1 \geq J(x_i)$ s.t. $J_1 \in min_{\prec}(J(x_i))$ and $I, J_1 \models \alpha_1$. Consider that $J(x_{i+1}) = J(x_i)$, we
 637 discuss two cases:

- 638 ■ Case 1: If $J_1 = J(x_i)$, then we have $I, J(x_i) \models \alpha_1$ and $J(x_i) \in min_{\prec}(J(x_i))$. We then
 639 define the next node x_{i+1} of the sequence with $\Gamma_{x_{i+1}} = \Gamma_y$, $une_{x_{i+1}} = une_{x_i}$ and add the
 640 pair $(J(x_i), J(x_i))$ to min_s . Notice that we swap the labels of nodes with the position of
 641 their indexed time point $J(x_i)$, we will be using indexed time point J instead of labels
 642 throughout this proof. We know that $\Gamma_{x_{i+1}} = (\Gamma_{x_i} \setminus \{\Diamond\alpha_1\}) \cup \{\alpha_1\}$ with $I, J(x_i) \models \alpha_1$.
 643 Additionally, we have $min_s := min_s \cup \{(J(x_i), J(x_i))\}$ with $J(x_i) \in min_{\prec}(J(x_i))$. The
 644 sets $une_{x_{i+1}}, \prec_s$ remains unchanged. Therefore, the invariant $Inv(x_{i+1}, J(x_i))$ holds.
- 645 ■ Case 2: when $J_1 > J(x_i)$, then we define the next node x_{i+1} of the sequence with
 646 $\Gamma_{x_{i+1}} = \Gamma_z$, $une_{x_{i+1}} = une_{x_i} \cup \{(J(x_i), \Diamond\alpha_1)\}$. We also know that $J_1 > J(x_i)$ and
 647 $J_1 \in min_{\prec}(J(x_i))$ and $I, J_1 \models \alpha_1$. Therefore, the second condition of $Inv(x_{i+1}, J(x_i))$
 648 holds on the pair $(J(x_i), \Diamond\alpha_1)$. The sets min_s and \prec_s remain unchanged. The invariant
 649 $Inv(x_{i+1}, J(x_i))$ holds.

650 $[une]$: When the rule $[une]$ is applied on a pair $(n_1, \Diamond\alpha_1)$ in une of $f(x_i)$. Let n be the
 651 label of the node $f(x_i)$. Let x be the predecessor of x_i in s where the rule $[\Diamond]$ was applied on
 652 $\Diamond\alpha_1$, let $J(x)$ be the indexed time point of x . Note that the label of $f(x)$ is n_1 . In the first
 653 outcome, we have a child y where $\Gamma_y = \Gamma_{f(x_i)} \cup \{\alpha_1\}$, $une_y = une_{f(x_i)} \setminus \{(n_1, \Diamond\alpha_1)\}$ and
 654 (n_1, n) in min of the branch. In the second outcome, we have a child z where $\Gamma_z = \Gamma_{f(x_i)}$,
 655 $une_z = une_{f(x_i)}$ and (n_1, n) in min of the branch. In the third outcome, we have a child v
 656 where $\Gamma_v = \Gamma_{f(x_i)}$, $une_v = une_{f(x_i)}$ and (n_1, n) in \prec of the branch.

657 On the other hand, since x is a predecessor of x_i in s , then we have $Inv(x, J(x))$.
 658 Furthermore, since we have $(n_1, \Diamond\alpha_1) \in une_{f(x_i)}$, it means that when the rule $[\Diamond]$ is applied
 659 on the node $f(x)$, the branch where $(n_1, \Diamond\alpha_1) \in une_{f(x+1)}$ is the path that corresponds with
 660 the interpretation I . By $[\Diamond]$ rule, since we have $Inv(x+1, J(x+1))$, $(n_1, \Diamond\alpha_1) \in une_{f(x+1)}$
 661 and we know that $J(x+1) = J(x)$, then we have $(J(x), \Diamond\alpha_1) \in une_{x+1}$. Furthermore, since
 662 no rule application consumed $(n_1, \Diamond\alpha_1)$ up to $f(x_i)$, then the pair $(J(x), \Diamond\alpha_1)$ remains also

663 in une_{x_i} . Also, we have $Inv(x_i, J(x_i))$, then there is $J' \geq J(x_i)$ where $J' \in \min_{\prec}(J(x))$ and
 664 $I, J' \models \alpha_1$. Consider that $J(x_{i+1}) = J(x_i)$, we discuss all possibilities below:

- 665 ■ Case 1: If $J' = J(x_i)$, then we have $J(x_i) \in \min_{\prec}(J(x))$ and $I, J(x_i) \models \alpha_1$. We
 666 define the next node x_{i+1} with $\Gamma_{x_{i+1}} = \Gamma_y$, $une_{x_{i+1}} = une_{x_i} \setminus \{(J(x), \diamond\alpha_1)\}$ and add
 667 $(J(x), J(x_i))$ to \min_s . We have $\Gamma_{x_{i+1}} = \Gamma_{x_i} \cup \{\alpha_1\}$ with $I, J(x_i) \models \alpha_1$. Additionally, we
 668 have $(J(x), J(x_i)) \in \min_s$ with $J(x_i) \in \min_{\prec}(J(x))$. The set \prec_s remains unchanged.
 669 Thus, the invariant $Inv(x_{i+1}, J(x_i))$ holds.
- 670 ■ Case 2: when $J' > J(x_i)$, we have two possibilities:
 - 671 ■ Case 2.1: If $J(x_i) \in \min_{\prec}(J(x))$, then we define the next node x_{i+1} with $\Gamma_{x_{i+1}} = \Gamma_z$,
 672 $une_{x_{i+1}} = une_{x_i}$ and add $(J(x), J(x_i))$ to \min_s . We have $(J(x), J(x_i)) \in \min_s$ with
 673 $J(x_i) \in \min_{\prec}(J(x))$. The sets $\Gamma_{x_{i+1}}$, $une_{x_{i+1}}$ and \prec_s remain unchanged. Thus, the
 674 invariant $Inv(x_{i+1}, J(x_i))$ holds.
 - 675 ■ Case 2.2: If $J(x_i) \notin \min_{\prec}(n_1)$, then there exists $J'' \geq J(x)$ s.t. $(J'', J(x_i)) \in \prec$. We
 676 define the next node x_{i+1} with $\Gamma_{x_{i+1}} = \Gamma_v$, $une_{x_{i+1}} = une_{x_i}$ and add $(J(x), J(x_i))$ to
 677 \prec_s . We have $(J(x), J(x_i)) \in \prec_s$ with $J(x_i) \notin \min_{\prec}(n_1)$. The sets $\Gamma_{x_{i+1}}$, $une_{x_{i+1}}$ and
 678 \min_s remain unchanged. Thus, the invariant $Inv(x_{i+1}, J(x_i))$ holds.

679 **[Transition]:** Suppose that the transition rule is applied on the state node $f(x_i)$. Let
 680 y be the child node of the node x_i in the branch. We have $\Gamma_y = \{\alpha_1 \mid \bigcirc\alpha_1 \in \Gamma_{f(x_i)}\}$ and
 681 $une_y = une_{f(x_i)}$. We define the next node x_{i+1} in s with $\Gamma_{x_{i+1}} = \Gamma_y$ and $une_{x_{i+1}} = une_{x_i}$.
 682 We consider that $J(x_{i+1}) = J(x_i) + 1$.

683 Since we have $Inv(x_i, J(x_i))$, then for all $\bigcirc\alpha_1 \in \Gamma_{x_i}$, we have $I, J(x_i) \models \bigcirc\alpha_1$ and
 684 therefore $I, J(x_i) + 1 \models \alpha_1$. The first condition of the invariant $Inv(x_{i+1}, J(x_i) + 1)$ is met.

685 Secondly, since x_i is a state node, then for each remaining $(n_1, \diamond\alpha_1) \in une_{f(x_i)}$, either the
 686 rule $[\diamond]$ or $[une]$ was applied to a node $f(x'_i)$ with the index $J(x'_i) = J(x_i)$ and $(n_1, \diamond\alpha_1)$ was
 687 carried over to $f(x_i)$. In both rules, for each $(n_1, \diamond\alpha_1) \in une_{f(x_i)}$, we have $(J(x_1), \diamond\alpha_1) \in$
 688 une_{x_i} s.t. $f(x_1)$ is the node where the rule $[\diamond]$ was applied to $\diamond\alpha_1$ (see Case 2 for $[\diamond]$ and
 689 $[une]$ rules). Furthermore, since we have $Inv(x_i, J(x_i))$ and $f(x_i)$ is a state node, then for
 690 each $(J(x_1), \diamond\alpha_1) \in une_{x_i}$, there exists $J_2 > J(x_i)$ where $J_2 \in \min_{\prec}(J(x_1))$ and $I, J_2 \models \alpha_1$.
 691 Without loss of generality, there exists $J_2 \geq J(x_i) + 1$ where $J_2 \in \min_{\prec}(J(x_1))$ and $I, J_2 \models \alpha_1$.
 692 The second condition of the invariant $Inv(x_{i+1}, J(x_i) + 1)$ is met. Since \min_s and \prec_s remain
 693 unchanged, the invariant $Inv(x_{i+1}, J(x_i) + 1)$ holds.

694 **[\prec -inconsistency]:** Suppose that the \prec -inconsistency rise on the node $f(x_i)$, and let n
 695 be the label of the $f(x_i)$ on the branch B . If this inconsistency rises, we have (n_1, n) in \min_B
 696 and (n_2, n) in \prec_B where $n_1 \leq n_2 \leq n$. These two pairs come from applying $[\diamond]$ or $[une]$ rule
 697 on two predecessors $f(x), f(x')$ of $f(x_i)$ with the same label n and the same indexed time
 698 point $J(x) = J(x') = J(x_i)$.

699 Let J_1 be the time point corresponding to the node $f(x_1)$ with the label n_1 , and let
 700 J_2 be the time point corresponding to the node $f(x_2)$ with the label n_2 . It is worth to
 701 mention that $J_1 \leq J_2 \leq J(x_i)$. Since x, x' are predecessors of x , we have $Inv(x, J(x))$,
 702 $Inv(x', J(x'))$ and $Inv(x_i, J(x_i))$. Therefore, we the rules are applied on x and x' , we end
 703 up with $(J_1, J(x_i)) \in \min_s$ and $(J_2, J(x_i)) \in \prec_s$. Since $(J_1, J(x_i)) \in \min_s$, then we have
 704 $J(x_i) \in \min_{\prec}(J_1)$. On the other hand, since $(J_2, J(x_i)) \in \prec_s$, then there exists $J_3 \geq J_2$
 705 s.t. $(J_3, J(x_i)) \in \prec$. Moreover, we have $J_1 \leq J_2$, this entails that there exists $J_3 \geq J_1$ s.t.
 706 $(J_3, J(x_i)) \in \prec$. This contradicts Definition 4 of minimality w.r.t. to the relation \prec . Therefore
 707 this cannot happen in a interpretation $I \in \mathcal{I}$.

708 **[Prune]:** Let $f(x_i)$ be a state node where the prune condition is met. There is a sequence
 709 within s that goes the following way, $x_h = u, x_{h+1}, x_{h+2}, \dots, v = x_i$. The node u or x_h is

710 the state node that comes before x_i and the node v is the current state node. Since v is a
 711 prune node, we have $\Gamma_v = \Gamma_u$ and $une_u = une_v$. We can see that if we apply the transition
 712 rule to the node x_i , we will have $\Gamma_{x_{i+1}} = \Gamma_{x_{h+1}}$ and $une_{x_{i+1}} = une_{x_{h+1}}$. Therefore, we can
 713 proceed with the construction of s as if x_i was linked to $f(u)$ instead of $f(v)$. Thanks to the
 714 transition, since we have $Inv(x_u, J(x_u))$, then we have $Inv(x_{i+1}, J(x_i) + 1)$.

715 Each time we find a pair (u, v) in the sequence s , we call it a *jump*. These jumps may
 716 occur once or many times (and it may go infinite) in s . In a sequence s , if a pair (u, v) jumps
 717 repeatedly in succession, we call the pair a *recurring jump*. It is worth to point out that,
 718 each time we jump backwards because of a node closed with prune, we return to the state
 719 labelled node that comes before. In general, the sequence s explores one branch B of \mathcal{T} , and
 720 it deviates sometime to a prune node and goes back to B . Furthermore, since no eventuality
 721 is fulfilled within a prune loop, eventualities and their fulfillment are in the same branch B .

722 What we showed so far is that for an interpretation I and its corresponding sequence s ,
 723 we have $Inv(x_i, J(x_i))$ for each $i \geq 0$. Going back to the start of the proof, we need to prove
 724 that the sequence finishes with a ticked node (such is the case when we end up in [loop] or
 725 [empty] node). We can see that if the sequence s is on a [prune] node, we jump back to the
 726 state node that comes before it. Theoretically, this jump can recur infinitely many times.
 727 This means that sequence goes infinite on this case (and never find a ticked node). We need
 728 to prove that this case cannot happen in the sequence s of I . Suppose that is the case, that
 729 means the last jump (u_k, v_k) in the sequence s is a recurring jump that goes infinitely many
 730 times. The jumps (u_j, v_j) that come before may recur many times but not infinitely many
 731 times (otherwise, (u_k, v_k) would not be the last jump). In the recurring jump (u_k, v_k) , no
 732 eventuality is fulfilled (whether it is classical or non-monotonic). This entails that when we
 733 are in a parent node $u_k < x_l < v_k$ that applies either $[\diamond]$ or $[une]$ rule, we move to the child
 734 node that delays the propagation of the eventuality (we are in Case 2 for both rules).

735 It is worth to point out that we have at least one eventuality in u_k . Let us take $\circ\Diamond\alpha_1 \in \Gamma_{u_k}$
 736 for example, since we have $Inv(u_k, J(u_k))$, that means that $I, J(u_k) \models \circ\Diamond\alpha_1$. Thus, we take
 737 the *first* time point $J_{\alpha_1} > J(u_k)$ s.t. $I, J_{\alpha_1} \models \alpha_1$. We also have $I, J_{\alpha_1} \models \Diamond\alpha_1$. On the other
 738 hand, for all $J(u_k) < J < J_{\alpha_1}$, we have $I, J \models \Diamond\alpha_1$ $I, J \models \circ\Diamond\alpha_1$. In other words, each time
 739 we encounter $\Diamond\alpha_1 \in \Gamma_{x_{l-1}}$ within our jumps (keep in mind we have $Inv(x_{l-1}, J)$), we pick
 740 the node in Case 2 of the $[\diamond]$ rule i.e., $\circ\Diamond\alpha_1 \in \Gamma_{x_l}$. However, in the node indexed with J_{α_1} ,
 741 when we encounter $\Diamond\alpha_1 \in \Gamma_{x_{l'-1}}$ (keep in mind we have $Inv(x_{l'-1}, J_{\alpha_1})$), we pick the node
 742 in Case 1 of the $[\diamond]$ rule i.e., $\alpha_1 \in \Gamma_{x'_l}$. This raises a contradiction, because the node $x_{l'}$ is
 743 not present within the jump (u_k, v_k) . Thus breaking the infinite recurring jump (u_k, v_k) .

744 If the eventuality is a non-monotonic one, namely $(J_1, \Diamond\alpha_1) \in une_{u_k}$. Since we have
 745 $Inv(u_k, J(u_k))$ with u_k being a state node, there exists $J' > J(u_k)$ s.t. $J' \in \min_{\prec}(J_1)$
 746 and $I, J' \models \alpha_1$. Let J_{α_1} be the first time point that met these criteria. For all $J(u_k) <$
 747 $J < J_{\alpha_1}$, each time we encounter $(J_1, \Diamond\alpha_1) \in une_{x_{l-1}}$ with the index J , we have $J_{\alpha_1} > J$,
 748 $J_{\alpha_1} \in \min_{\prec}(J_1)$ and $I, J_{\alpha_1} \models \alpha_1$. Therefore, we pick Case 2 of $[une]$ rule i.e., $(J_1, \Diamond\alpha_1) \in$
 749 une_{x_l} . However, when we encounter $(J_1, \Diamond\alpha_1) \in une_{x_{l'-1}}$ with the index J_{α_1} , we have
 750 $J_{\alpha_1} \in \min_{\prec}(J_1)$ and $I, J_{\alpha_1} \models \alpha_1$, then we pick the node in Case 1 of $[une]$ rule i.e., $\alpha_1 \in x_{l'}$.
 751 This raises a contradiction, because the node $x_{l'}$ is not present within the jump (u_k, v_k) .

752 We proved that since $I, 0 \models \alpha$, then the corresponding sequence s cannot finish on a
 753 contradiction, \prec -inconsistency or a prune jump. Therefore it must finish with a ticked node.
 754 Hence, the tableau \mathcal{T} of α has a ticked node and therefore a successful branch.