



HAL
open science

DISCRETE AND CONTINUOUS CURVED SURFACE ANTIPODAL PATHS. EXTENSIONS OF THE BORSUK-ULAM THEOREM AND APPLICATION OF THE FEYNMAN PATH INTEGRAL AND WODEHOUSE CONTOUR INTEGRAL

James F Peters, Roberto Alfano, Peter Smith, Arturo Tozzi

► **To cite this version:**

James F Peters, Roberto Alfano, Peter Smith, Arturo Tozzi. DISCRETE AND CONTINUOUS CURVED SURFACE ANTIPODAL PATHS. EXTENSIONS OF THE BORSUK-ULAM THEOREM AND APPLICATION OF THE FEYNMAN PATH INTEGRAL AND WODEHOUSE CONTOUR INTEGRAL. 2022. hal-03698584

HAL Id: hal-03698584

<https://hal.science/hal-03698584>

Preprint submitted on 18 Jun 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

**DISCRETE AND CONTINUOUS CURVED SURFACE
ANTIPODAL PATHS. EXTENSIONS OF THE BORSUK-ULAM
THEOREM AND APPLICATION OF THE FEYNMAN PATH
INTEGRAL AND WODEHOUSE CONTOUR INTEGRAL**

JAMES F. PETERS, ROBERTO ALFANO, PETER SMITH, AND ARTURO TOZZI

Dedicated to J.H.C. Whitehead & S. Lefschetz

ABSTRACT. Purpose: This paper introduces discrete and continuous paths over simply-connected surfaces with non-zero curvature as means of comparing and measuring paths between antipodes with either a Feynman path integral or Wodehouse contour integral, resulting in a number of extensions of the Borsuk Ulam Theorem.

Methods: All paths originate on a Riemannian surface S , which is simply-connected and has non-zero curvature at every point in S . A surface is a **simply connected**, provided every cross-cut divides the surface into disjoint regions. A **cross cut** is an arc that runs through the interior of the surface S without self-intersections and joins one boundary to another.

Results: A fundamental result in this paper is that for any pair of antipodal surface points, a path that begins and ends at the antipodal points can be found. This result is extended to Feynman path integrals on trajectory-of-particle curves and to N.M.J. Wodehouse contour integrals for antipodal vectors on twistor curves. Another fundamental result in this paper is that the Feynman trajectory of a particle is realizable as a Lefschetz arc.

1. INTRODUCTION

This paper introduces a path-Borsuk-Ulam Theorem, stemming from three main forms of paths over curved surfaces that have been identified, namely,

- 1^o **Poincaré Contour paths** were introduced by Poincaré in 1892 in his analysis situs paper [14]. In a **contour path**, each subpath is an infinitely small contour on a manifold [14, p. 240]. Recently, N.M.J. Woodhouse [19] introduced contour integrals defined on twistor curves on a complex manifold.
- 2^o **Whitehead Homotopic paths** were introduced during the late 1940s by J.H.C. Whitehead [17],[18] and S. Lefschetz [6]. For Lefschetz, a **homotopic path** h in an arcwise connected space S is simply a mapping of a directed (= oriented) closed arc $\widehat{v_0, v_1}$ into R [6, p. 158]. A space is **arcwise connected**, provided every vector in the space S is on a path containing an initial vector and a terminal vector.
- 3^o **Feynman paths** were introduced by R.P. Feynman in his thesis completed in 1942 [3, p. xiv]. A **Feynman path** is a trace of the trajectory of a particle

2010 *Mathematics Subject Classification.* 05C38 (paths and cycles), 14H55 (Riemann surfaces).
Key words and phrases. arc, cell, curvature, path, surface, vector.

between fixed endpoints [3, p. xiv], providing a framework for a path integral, also introduced by Feynman[3] and elaborated by R.P. Feynman and A.R. Hibbs in [4]. A **Penrose path** over a twistor curve (from R. Penrose’s 1968 paper [13]) and its refinement by R.S. Ward in his 1977 thesis [16] supervised by Penrose, is a form of Feynman path in which the trajectory of a particle is over a twistor curve.

The original BUT [2] from K. Borsuk in 1933 is given in terms of antipodal vectors $\vec{p}, -\vec{p}$ on the surface of an n -dimensional Euclidean sphere S^n , defined by

$$S^n = \{\vec{p}(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}, n \geq 2 | x_1^2 + \dots + x_{n+1}^2 = 1\}.$$

Points on the surface of a sphere are **antipodal**, provided the points are diametrically opposite each other. Examples of antipodal vectors are the poles on the surface of a planet.

In 1933, K. Borsuk introduced his Theorem 1.

BUT.

Theorem 1. (Borsuk-Ulam Theorem) [2, p. 178].

Let vector $\vec{p} \in S^n$. For a continuous map $f : S^n \rightarrow \mathbb{R}^n$, there is an antipodal point $-\vec{p}$ such that $f(\vec{p}) = f(-\vec{p})$.

Remark 1. *Theorem 1 is a translation from German, which is given by J. Matousšek [9, p. 21].* ■

Remark 2. *The basis for Theorem 1 came from K. Borsuk’s thesis completed in 1930 [1]. Ulam is credited by Borsuk (in a footnote [2, p. 178]) with the idea codified in Theorem 1, which Ulam stated as a conjecture. In effect, Borsuk proved Ulam’s conjecture in 1933. In 1930, L. Lusternik and S. Shnirel’man introduced the nonvoid intersection of sets of closed surface curves that have antipodal vectors in common.*

LST.

Theorem 2. (Lusternik- Shnirel’man Theorem) [7].

For any cover F_1, \dots, F_{n+1} of the sphere S^n by $n + 1$ closed sets, there is at least one set containing a pair of antipodal points common to $F_i, -F_i$ (i.e., $F_i \cap -F_i \neq \emptyset$).

Remark 3. *Theorem 2 is a translation from Russian, which is given by J. Matousšek [9, p. 21].* ■

Theorem 2 contrasts with Theorem 1. In the Lusternik-Shnirel’man Theorem 2, there is a closed set F_i that is a cover of a sphere S^n and that has an opposite set $-F_i$, in which the sets $F_i, -F_i$ contain antipodal points such that $F_i \cap -F_i \neq \emptyset$. This sharply contrasts with the Borsuk-Ulam Theorem, which asserts there is a continuous map f from S^n into \mathbb{R}^n over a surface containing antipodal surface vectors $\vec{p}, -\vec{p}$ such that $f(\vec{p}) = f(-\vec{p})$. Also, Theorem 2 concludes with the observation that the

intersection of $F_i, -F_i$ is nonvoid but the values of the shared antipodal points are not given. In the LS theorem formulation, it is possible that the antipodal points in $F_i \cap -F_i$ have different values. By contrast, in the Theorem 1 formulation, it is asserted that the antipodal points map to the same value. ■

Let Id be a discrete form of a unit interval $[0, 1]$ defined by

$$i \in \mathbb{Z}^+ \text{ natural numbers } 1, 2, 3, \dots$$

i.e., there is a gap between x_i and x_{i+1}

$$X = \overbrace{\{X_i \in S^n \mid |x_{i+1} - x_i| > 0\}}^{\text{gap}}$$

$$Id = \{0, 1\} \cup \{X : X_i \in S^n\}.$$

Example 1. Let $X_{0.00001}$ be a set of surface points on an n -dimensional Riemannian sphere S^n , defined by

i.e., there is a 0.00001 gap between x_i and x_{i+1} .

$$X_{0.00001} = \overbrace{\{X_i \in S^n \mid |x_{i+1} - x_i| = 0.00001\}}^{\text{gap}}$$

$$Id = \{0, 1\} \cup \{X_{0.1} : X_i \in S^n\}. \quad \blacksquare$$

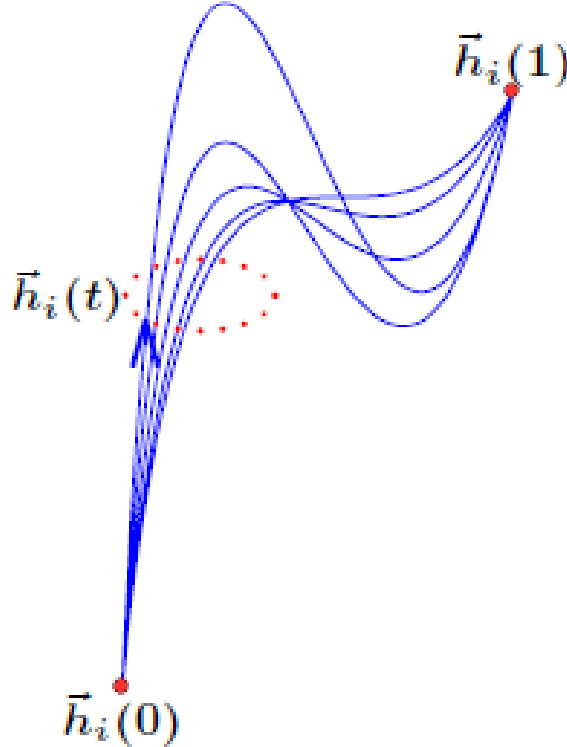


FIGURE 1. Discrete paths $h_i : Id \rightarrow S$ with all $t \in Id$.

2. PRELIMINARIES

More recent versions of BUT [11, §68,p.405] require the map $f : S^n \rightarrow R^n$ to be continuous, *i.e.*, the map f is continuous, provided, for each subset $E \subset S^n$, if a point \vec{p} is arbitrarily close to E , then $f(\vec{p})$ is arbitrarily close to E . However, in keeping with an interest in Riemannian surface surface points with gaps between them, we consider discrete maps.

Definition 1. Discrete map.

Discretely close surface points \vec{p}, \vec{q} such as water molecules always have a minute gap between them. A map $f : S^n \rightarrow R^n$ is discrete, provided, for each subset $E \subset S^n$, if a point \vec{p} is discretely close to E , then $f(\vec{p}) = \vec{q}$ is close to E .

Example 2. Discretely close surface points \vec{p}, \vec{q} such as close water molecules always have a minute gap between them. ■

Example 3. The discrete unit interval Id is a collection **Discretely close surface points** $\vec{p}, \vec{q} \in Id$ such that $|\vec{p} - \vec{q}| > 0$. ■

Definition 2. Discrete Path.

Let S be a Riemannian surface. A **discrete path** h is a discrete mapping, defined by

$$h : Id \rightarrow S, \text{ such that}$$

$h(0) \in S$ is the initial point in path h , $h(1) \in S$ the endpoint of h and $h(t) \in S$ for all $t \in \{x : x \in X \subset Id\}$.

Example 4. A sample discrete path $h : Id \rightarrow S$ on a circular slide of a Riemannian sphere is shown in Fig. ???. This path begins at vector $\vec{h}(0) \in \mathbb{R}^n$ at \vec{v}_1 on the surface of S and ends at vector $\vec{h}(1) \in \mathbb{R}^n$, which is the value of antipode of \vec{v}_1 . The assumption made here is that $\vec{h}(0)$ and $\vec{h}(1)$ have the same value such as identical temperature. ■

That is, a **discrete path** $h : Id \rightarrow S$ is a mapping from the discrete unit interval $Id \subset I$ (for $I = [0, 1]$) to a bounded, simply connected surface S with non-zero curvature. Path h is discrete, since there gaps between all points $\vec{h}(t) \in S$ between 0 and 1 in $Id \subset [0, 1]$. The surface S is **simply connected**, provided every path h has end points $h(0), h(1) \in S$ and h has no self-loops.

Paths either lie entirely on a surface in the planar case or lie on a surface and, possibly, puncture a surface in the non-planar case. Paths that puncture a surface are called cross-cuts. A **cross cut path** P (also called an **ideal arc** [10, §3, p.11]) has both ends in P and path interior in the interior of S .

Remark 4. Homotopic paths were introduced by J.H.C. Whitehead[17]. For Whitehead, a path $h : [0, 1] \rightarrow X$ is a continuous mapping from the unit interval to a cell complex X . In the pursuit of discrete paths in a curved space, the focus is on 0-cells (single points) and 1-cells (arcs) in an n -dimensional Riemannian space S . A single surface vector is a 0-cell. A **Lefschetz arc** is a curvilinear line segment attached to a pair of 0-cells [5]. ■

Definition 3. (Path-Connected) A pair of vectors v_0, v_1 are **path-Connected**, provided there is a sequences of arcs between the vectors.

Example 5. All vectors on the circle in Fig. ?? are path-connected, since there is a sequence of arcs between each pair of vectors. ■

3. ANTIPODAL AND NON-ANTIPODAL PATH BORSUK-ULAM THEOREM

This section introduces results for paths on a Riemann surface.

Theorem 3. *The endpoints of a discrete Lefschetz arc can be the same.*

Proof. Let $\widehat{v_0, v_1}$ be a discrete Lefschetz arc in which the endpoints $v_0 = v_1$ are the same. There is an arc between v_0, v_1 , namely, itself. Hence, from Def. 3, the endpoints are path-connected. \square

Lemma 1. *Every discrete path constructs a discrete arc.*

Proof. Every $h(t)$ is a point on an arc $\widehat{h(0), h(1)}$ between $h(0)$ and $h(1)$ is a discrete path $h : Id \rightarrow S$. Id is discrete. Hence, $\widehat{h(0), h(1)}$ is also discrete. \square

Theorem 4. *Every discrete path constructs a vector field.*

Proof. Let $h : Id \rightarrow S$ be a discrete path. From Lemma 1, h constructs a discrete arc $\widehat{h(0), h(1)}$ on a surface S . Consequently, each $h(t) \in \widehat{h(0), h(1)}$ has a location $(x_1, \dots) \in S$ with its own magnitude and direction S , i.e., every $h(t)$ is a vector in S . Hence, h constructs a vector field. \square

Lemma 2. *Let \vec{v}_1, \vec{v}_2 be antipodal vectors on the surface of a sphere S . There exists a discrete path h with endpoints are antipodal on a surface S .*

Proof. Let \vec{v}_1, \vec{v}_2 be antipodal vectors on the surface of a sphere S . Then define a discrete path $h : Id \rightarrow S$ with $\vec{h}(0) = \vec{v}_1$ and $\vec{h}(1) = \vec{v}_2$. Hence, a discrete path can be defined for every pair of antipodal points on S . \square

From what we have observed about discrete paths on the surface of a sphere, we obtain

path-BUT.

Theorem 5. (Path-Borsuk-Ulam Theorem).

Let the discrete unit interval Id be an index set for vectors $v_0, \dots, v_t, \dots, v_1, t \in Id$ in S^n in a path $h : S^n \rightarrow \mathbb{R}^n$, i.e., a path discretely maps $v_0, \dots, v_t, \dots, v_1$ to a set of values $\{h(v_0), \dots, h(v_t), \dots, h(v_1)\}$ in \mathbb{R}^n . For a discrete path $h : S^n \rightarrow \mathbb{R}^n$ and a vector $\vec{p} \in S^n$, there is an antipodal vector $-\vec{p} \in S^n$ such that $h(\vec{p}) = h(-\vec{p})$.

Proof. From Lemma 1, a path $h : S^n \rightarrow \mathbb{R}^n$ constructs of values from vectors on an arc with endpoints \vec{v}, \vec{v}' on S^n . Let the endpoints of this arc be antipodal. Hence, from Lemma 2, the desired result follows. \square

Remark 5. *An immediate consequence of Theorem 5 is that, for any pair of antipodal surface points, we can always introduce a discrete path h that begins and ends at the antipodal points such as places that have same latitude and longitude. For example, the antipode of Winnipeg, Manitoba, Canada with coordinates $49^\circ.53'N, 97^\circ.8'W$ is Port-aux-Français, Kerguelen, French Southern Territories. \blacksquare*

Example 6. An example of a discrete path that begins and ends at antipodal surface points is shown in Fig. ??.

Observe that a path can be constructed between any pair of surface vectors. This observation leads to more general form of Theorem 5.

Theorem 6. (Non-antipodal path-BUT).
 Let the discrete unit interval Id be an index set for vectors $v_0, \dots, v_t, \dots, v_1, t \in Id$ in S^n in a path $h : S^n \rightarrow \mathbb{R}^n$. There is a discrete path $h : S^n \rightarrow \mathbb{R}^n$ with endpoints $h(v_0), h(v_1)$ that are values in \mathbb{R}^n such that $h(v_0) = h(v_1)$.

Proof. In the proof of Theorem 5, replace antipodal points on a Leftschetz arc with endpoints that are any pair of surface points. In addition, notice that the initial point and the ending point of any discrete path can be the same. Hence. the desired result follows. \square

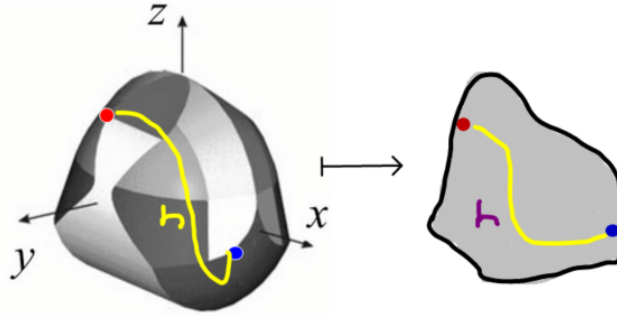


FIGURE 2. 2D and 3D views of discrete paths on a Gomboc Riemannian surface

Example 7. An example of a discrete path that begins and ends at antipodal surface vectors on a bumpy Riemannian sphere (aka Gomboc sphere) is shown in Fig. 2.

Example 8. An example of a discrete path $h : S^2 \rightarrow R^3$ on a 3D Gomboc Riemannian surface is shown in Fig. 2. The same path is also depicted on a 2D slice of the 3D surface. In keeping with Theorem 6, each vector $\vec{h}(v_t)$ is a signal value from the path h . For example, if we let the discrete path be an optical field flow containing a stream of photons reflected from a Riemannian surface, then there are number of possible signal values for $\vec{h}(v_t)$, e.g.,

- 1° wavelength of $\vec{h}(v_t)$.
- 2° frequency of $\vec{h}(v_t)$.
- 3° electron voltage of $\vec{h}(v_t)$.
- 4° lumens (luminosity) of $\vec{h}(v_t)$.

5^o gradient of $\vec{h}(v_t), t \in Id$, which would be perpendicular to the surface at (x, y, z) , defined by

$$\mathbf{grad}(\vec{h}(v_t)) = \frac{\partial \vec{h}}{\partial x} i + \frac{\partial \vec{h}}{\partial y} j + \frac{\partial \vec{h}}{\partial z} k.$$

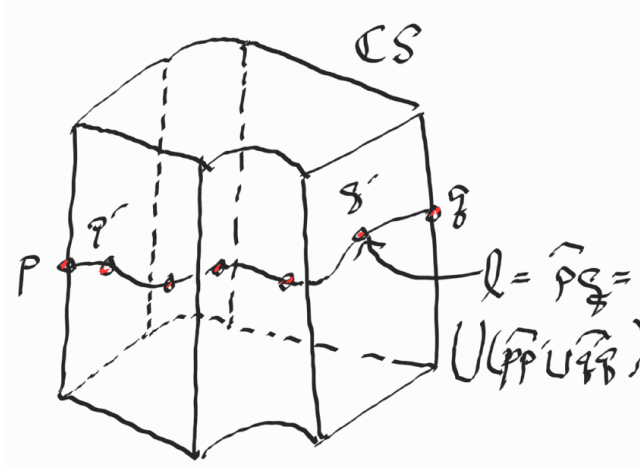


FIGURE 3. Trajectory of a particle over twistor curve realizable as the union of a sequence of sub-arcs on Lefschetz arc $\ell = \widehat{pq}$ on a R.S. Ward hypersurface \mathcal{CS} [16, p.62].

4. FEYNMAN TRAJECTORIES OF A PARTICLE

This section introduces particle trajectories as continuous paths over the curvature of space-time, which leads to the counterpart of the discrete path results already given. The transition from discrete paths results from the geometry of space-time generated by quantum processes [8], which is in keeping with the observation by R. Penrose [13] that the link between space-time curvature and quantum processes such as those found in Feynman trajectory of a particle is supplied by the use of twistors. A **twistor space** is a complex manifold CM . For example, a Lefschetz arc in curved space-time is a R.S. Ward hypersurface \mathcal{S} twistor [16, p.56], which is a complex curve ℓ in \mathcal{CS} .

Example 9. A sample twistor curve $\ell \in \mathcal{CS}$ is shown in Fig. 3, which is a geometric realization of a Feynman trajectory of a particle (see Def. 4), which leads to a space-time view of a Lefschetz arc (see Def. 4 and Lemma 3). ■

Definition 4. The trajectory of a particle in a 2-plane in curved space-time is a mapping

$$h : \{\mathbb{R}^2 \times S^2\} \rightarrow \{\mathbb{R}^2 \times S^2\}$$

defined by

$$h(\{t^{x_i}\}) = \widehat{t_{x_0}, t_{x_i}} \cup \bigcup \widehat{t_{x_i}, t'_{x_i}} = \ell, t, t' \in \mathbb{R}, x_i \in S^2, i \in I,$$

in which each $\widehat{t_{x_i}, t'_{x_i}}$ is a space-time line segment in a curve ℓ starting with subarc $\widehat{t_{x_0}, t_{x_i}}$ in a Lefschetz arc at times t_t (instant t in region time t) with index i in the

unit interval $I = [0, 1]$ is mapped to an arcwise-connected set, i.e., the line segments in the trajectory are attached to each other and starting with t_{x_0}, t'_{x_i} , there is a path from any subarc a sequence of subarcs can be traversed to reach an ending subarc $t_{x_n}, t'_{x_{n'}}$ in a N.M.J. Woodhouse [19] twistor space $\mathbb{R}^2 \times S^2$ with metric signature $++--$. ■

Remark 6. From Def. 4, the vectors in $h(\{t_{x_i}\})$ are J.H.C. Wodehouse zero cells in an arcwise-connected space $\mathbb{R} \times S^2 \times \mathbb{R}$. ■

Definition 5. (Lefschetz arc).

A **Lefschetz arc** E is a curve ℓ attached between a pair of 0-cells p, p' . The curve ℓ is dense and the points in ℓ are path-connected, i.e., between every pair of points q, q' in ℓ , there is a sequence of sub-arcs traversable between q and q' . ■

Lemma 3. (Lefschetz trajectory arc).

A trajectory of a particle is realizable as a Lefschetz arc.

Proof. From Def. 4, a trajectory h is a curve ℓ that starts and ends with a 0-cell and is the union of subarcs in an arcwise-connected space. Hence, from Def. 5, the trajectory h is realizable as a Lefschetz arc. □

Example 10. A sample trajectory of a particle as a Lefschetz arc over a twistor curve realized as a Lefschetz arc $\ell = \widehat{pc}$ with endpoints (0-cells) \vec{p}, \vec{q} and which is the union of sub-arcs is shown in Fig. 3. ■

Definition 6. (Unit interval).

The unit $I = [0, 1] \in \mathbb{R}$ is the set of all real values in the closed interval with initial value 0 and ending 1 and an unbounded number of consecutive everywhere dense subintervals of real values between 0 and 1. That is, every real number x in a subinterval of $A \subset I$ has another real number $x' \in A$ that is arbitrarily close to x . ■

Lemma 4. The trajectory of a particle is continuous.

Proof. From Def. 6, I is dense and is the index set for the points in the trajectory of a particle. From L. Susskind and A. Friedman [15, p. 238], a particle moving along the Lefschetz curve can be found at any real value in the unit interval $I = [0, 1]$. Let h be the trajectory of a particle. Then if $i, i' \in I$ are close, then $t_{x_i}, t_{x_{i'}}$ are close. Hence, the trajectory h is continuous. ■ □

Example 11. (Trajectory of a Particle).

Let $g : I \rightarrow \mathbb{R}$ be defined by $g(t_i) = t_i, i \in I, g(t_i) \in \mathbb{R}$ for each time t_i in the trajectory of a particle. That is, I is a index set for the instants of time of occurrence of the points in the trajectory of a particle over a vector field. The map f is continuous, since, for every pair $t_i, t_j, i \neq j$, if i, j are arbitrarily close, then $g(t_i), g(t_j)$ are arbitrarily close. Hence, the g is continuous. ■

5. FEYNMAN PATH INTEGRAL

In this section, it is observed that a Feynman path is continuous (Lemma 5), which leads to the results in Theorem 7 and Theorem 8 for Feynman paths, which are consequences of the Borsuk-Ulam Theorem.

Definition 7. *A Feynman path is a function $h(t_{x_i}) = x_i$ for a particle at point x_i at time t [4, p.31].* ■

Lemma 5. (Feynman Path).

Every Feynman path is continuous.

Proof. Let $h : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow S^2$ be a Feynman path, defined by $h(t_{x_a}) = x_a$ which is the trajectory of a particle at point x_a at time t_t . Let ℓ represent that a particle travels over during its trajectory and let $h(t_{x_a}) = x_a$ be a point in ℓ . For simplicity, the curve ℓ is referred to as the trajectory of a particle and p is point in ℓ . During the passage of a particle over ℓ , ℓ has no gaps in it. That is, for each point $h(t_{x_a}) = x_a \in \ell$ at time t_t , if x_a is arbitrarily close to point $h(t_{x_{k+1}}) = x_{k+1} \in \ell$ at time t_t , then $h(t_{x_a})$ is arbitrarily close to $h(t_{x_{k+1}})$. Hence, a Feynman path h is continuous. □

Remark 7. *In Lemma 5, the continuity of a Feynman path h is explained in terms of the closeness (nearness) paradigm from [12, §1.5, p. 8], instead of the abstract (less intuitive) $\epsilon - \delta$ view of continuity. This approach befits the character of the trajectory of a particle over a curve ℓ , where both the trajectory of a particle and the curve ℓ (without gaps) traced by particle trajectory. Just as pairs of points in the curve ℓ can be arbitrarily close, so too, from Lemma 4, the vectors $h(t_{x_a}), h(t_{x_{k+1}})$ in the trajectory of a particle can be arbitrarily close.* ■

The value of a path between points a and b on a curve ℓ (the positions of a particle trajectory at times t_a, t_b , respectively), is $K(b, a)$, defined in a complex space $\mathbb{C}S$ with respect to Planck's constant \hbar by Feynman and Hibbs [4, p. 45] by

$V(x, t) = \mathbf{Potential\ energy\ of\ particle\ with\ mass\ } m.$

$$L = \frac{m}{2} \dot{x}^2 - V(x, y) (\mathbf{Lagrangian\ for\ the\ system}).$$

$$S = \int_{t_a}^b L(\dot{x}, x, t) dt.$$

$$K(b, a) = \int_a^b e^{(\frac{i}{\hbar}) S[b, a]} \mathcal{D}x(t).$$

Theorem 7. (Feynman Path Theorem).

Let $h : S^3 \times \mathbb{R}^3 \rightarrow S^3$ be a Feynman path over a curved surface S^3 and let $K(b_k, x_a), K(-x_a, b_k)$ be the value of path h containing points b_k, x_a in a segment $\overline{b_k, x_a}$ in a curve ℓ starting at point x_a and with endpoint b_k attached to a segment $\overline{-x_a, b_k}$ starting at point b_k and with endpoint $-x_a$. For each initial point x_a , there exists an antipodal point $-x_a$ in path h such that $K(b_k, x_a) = K(-x_a, b_k)$.

Proof. From Lemma 5, a Feynman path h is continuous. Hence, from Theorem 1, we obtain the desired result for antipodal points x_a, x_b in path h . □

Trajectory Results.

Theorem 8. (Feynman Trajectory-of-Particle Theorem).

The Feynman trajectory of a particle satisfies Borsuk-Ulam Theorem 1. Let $h : S^n \rightarrow \mathbb{R}^n$ be the trajectory of a particle on the surface of sphere. There is at least one pair vectors $\vec{p}, \vec{p}' \in S^n$ such that $h(\vec{p}) = h(\vec{p}')$.

Proof. From Lemma 5, a Feynman trajectory is continuous. Hence, from Theorem 1, we obtain the desired result for antipodal points $\vec{p}, \vec{p}' \in S^n$ in Feynman trajectory h . \square

Theorem 9. (Feynman Path Integral Theorem).

There exists a Feynman path with an initial path integral $K(b_k, x_a)$ for an initial vector \vec{x}_a that equals the path integral $K(b_k, -x_a)$ for a later vector $-\vec{x}_a$, which may or may not be the antipode of vector \vec{x}_a .

Proof. $K(b_k, x_a)$ are Feynman path integrals that resonate (have values) for a particle that has gradients on either two different surface curvatures along a surface curve ℓ or on the same surface curvature on a path ℓ' for a boomerang trajectory that follows a path that is a cycle. In either case, choose an intermediate point b_k in the path between \vec{x}_a and b_k so that the two segments on ℓ have the same length. In that case, $K(b_k, x_a) = K(b_k, -x_a)$ \square

Remark 8. The significance of Theorem 9 is that the endpoints on a particle trajectory curve ℓ need not be antipodal points. That is, Theorem 9 is more general than Theorem 7. \blacksquare

6. WOODHOUSE BORSUK-ULAM THEOREM

This section briefly introduces N.M.J. Woodhouse contour integrals [19, p. 198], defined with respect to real α -planes in a twistor space $\mathbb{R}^2 \times S^1$ in which S^1 represents the α -planes that lie in the null cone at ∞ , over which we have

$\xi = x_1 + ix_2$ and $\tau = t_1 + it_2$, representing α -planes as surfaces, with $w = \xi + \bar{z}r, \bar{w} = \bar{\xi} + zr$, constant for $z = e^{i\theta}$.

$$\phi(\xi, \tau, \bar{\xi}, \bar{\tau}) = \frac{1}{2\pi} \oint_{|z|=1} f(\xi + \bar{z}\tau, \bar{\xi} + z\bar{\tau}, z) \frac{dz}{z}.$$

Definition 8. (Contour Integral as a Smooth function).

The contour integral ϕ is a smooth function on a twistor space [19]. That is, ϕ is continuous. \blacksquare

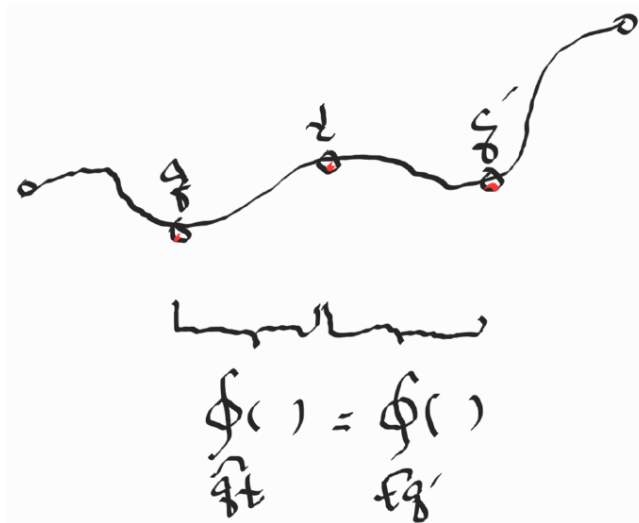


FIGURE 4. Wodehouse contour integrals on sub-twistor curve antipodes

Antipodal Contour Integrals Result.

Theorem 10. *The contour integral ϕ satisfies the Borsuk-Ulam Theorem.*

Proof. From Def. 8, the contour integral ϕ is a continuous function. Let $p, -p$ be antipodes on a twistor curve in $\mathbb{R}^2 \times S^1$. Replace the Feynman path integral with the Wodehouse contour integral in the proof of Theorem 9, and the desired result follows. \square

Example 12. *Sample contour integrals on sub-twistor vectors that are antipodal are shown in Fig. 4.* ■

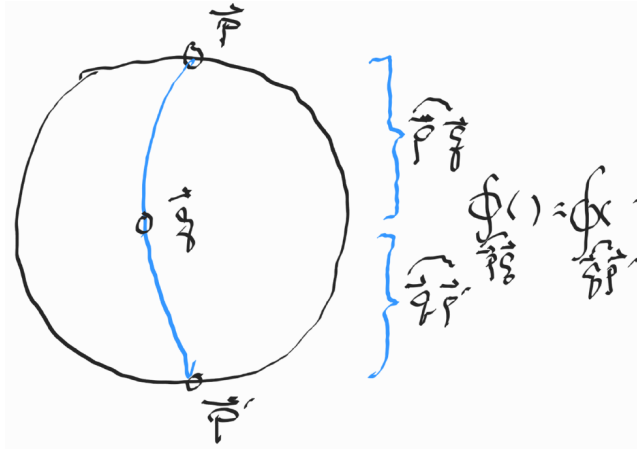


FIGURE 5. Wodehouse contour integrals on sub-twistor curves

Non-Antipodal Contour Integrals Result.

Theorem 11. *Let $\phi_p, \phi_{p'}$ be the Wodehouse contour integral over a twistor curve ℓ and let p, p' be any two distinct points on ℓ . Then there are ϕ, ϕ' such that $\phi_p = \phi'_{p'}$.*

Proof. Again, replace the Feynman path integral with the Wodehouse contour integral in the proof of Theorem 9, and the desired result follows. That is, we can always find a point q between p, p' on the twistor ℓ such that $\phi_p = \phi'_{p'}$. \square

Remark 9. *Theorem 11 covers a broader spectrum of twistor length measurements than Theorem 10. That is, for any pair of distinct vectors on a twistor curve, we can always find an intermediate vector so that the contour integrals over the resulting twistor sub-arcs have equal value. \blacksquare*

Example 13. *Sample contour integrals on sub-twistor curves $\widehat{p, q}, \widehat{p', q'}$ with endpoints \vec{p}, \vec{p}' that may or may not be non-antipodal are shown in Fig. 5. \blacksquare*

7. CONCLUDING REMARKS

The Borsuk-Ulam Theorem is topological with an implicit surface geometry. That is, the focus in the Borsuk-Ulam Theorem is on a continuous map from the surface of a sphere S^n to real values of antipodes in \mathbb{R}^n . The geometry underlying the Borsuk-Ulam Theorem looms up, when we consider where to look for antipodes such as the endpoints on an arc stretching over the curved surface of the Earth between a place such as the University of the Manitoba and another place at varying space-times with the same latitude and longitude.

The beauty of the Borsuk-Ulam Theorem is that it serves as a roadmap, telling us how to look for surface antipodes (with continuous maps), where to look (*e.g.*, endpoints of twistor curves that are antipodal) and how to measure antipodal distances. In this paper, the Borsuk-Ulam Theorem is an emperor with new clothes to wear, namely,

1^o **How to look** (with either a discrete or continuous map).

- 2° **Where to look** (endpoints of twistor curves that are either antipodal or non-antipodal).
- 3° **Length-of-arc measure** (with either Feynman path integrals or Wodehouse contour integrals over arcs having antipodal endpoints).

REFERENCES

1. K. Borsuk, *On retractions and related sets*, Ph.D. thesis, University of Warsaw, Warsaw, Poland, 1930, supervisor: Stefan Mazurkiewicz.
2. ———, *Drei sätze über die n -dimensionale euklidische sphäre*, *Fundamenta Mathematicae* **XX** (1933), 177–190.
3. R.P. Feynman, *The principle of least action in quantum mechanics*, Ph.D. thesis, Princeton University, Princeton, N.J., 1942, supervisor: John Wheeler.
4. R.P. Feynman and A.R. Hibbs, *Quantum mechanics and path integrals, extended edition, preface by d.f. styer*, Dover Publications, Inc., Mineola, NY, 2010, xii+371 pp., MR2797644.
5. S. Lefschetz, *Applications of algebraic topology. Graphs and networks, the picard-lefschetz theory and feynman integrals, applied mathematical sciences, vol. 16*, Springer-Verlag, New York-Heidelberg, 1976, viii+189 pp., MR0494126, reviewed by P. Orlik.
6. ———, *Introduction to topology*, Princeton University Press, Princeton, N.U., 1976, v+218 pp.
7. L. Lusternik and S. Shnirel'man, *Topological problems in variational problems (russisa)*, Issledowatel'skii Institut Matematiki i Mehaniki pri O.M.G.U., Moscow **17** (1930), no. 1, 166–217.
8. L.J. Mason, *Twistors in curved space-time*, Ph.D. thesis, Wadham College, Oxford University, Oxford, U.K., 1985, supervisor: N.M.J. Woodhouse.
9. J. Matoušek, *Using the borsuk-ulam theorem. lectures on topological methods in combinatorics and geometry. written in cooperation with anders bj rner and g nter m. ziegler*, Springer-Verlag, Berlin, 2003, xii+196 pp., MR1988723.
10. L. Mosher, *Tiling the projective foliation space of a punctured surface*, *Trans. Amer. Math. Soc.* **306** (1988), no. 1, 1–70, MR0927683, reviewer N.V. Ivanov.
11. J.R. Munkres, *Elements of algebraic topology, 2nd ed.*, Perseus Publishing, Cambridge, MA, 1984, ix + 484 pp., ISBN: 0-201-04586-9, MR0755006.
12. S.A. Naimpally and J.F. Peters, *Topology with applications. Topological spaces via near and far*, World Scientific, Singapore, 2013, xv + 277 pp, Amer. Math. Soc. MR3075111.
13. R. Penrose, *Twistor quantization and curved space-time*, *Int. J. of Theor. Phys.* **1** (1968), no. 1, 61–99.
14. J.H. Poincaré, *Sur l'analyse situs*, *Comptes Rendus Acad. Sci. Paris* **115** (1892), 633–636.
15. L. Susskind and A. Friedman, *Quantum mechanics. the theoretical minimum*, Penguin Books, UK, 2014, xx+364 pp., ISBN: 978-0-141-977812.
16. R.S. Ward, *Curved twistors spaces*, Ph.D. thesis, St. John's College, Oxford University, Oxford, U.K., 1977, supervisor: R. Penrose.
17. J.H.C. Whitehead, *Combinatorial homotopy. I*, *Bulletin of the American Mathematical Society* **55** (1949), no. 3, 213–245, Part 1.
18. ———, *Combinatorial homotopy. II*, *Bulletin of the American Mathematical Society* **55** (1949), no. 5, 453–496, Part 2.
19. N.M.J. Woodhouse, *Contour integrals for the ultrahyperbolic wave equation*, *Proc. Roy. Soc. London Ser. A* **438** (1992), no. 1902, 197–2006, MR1174728, reviewed by S. Ofman.

DEPARTMENT OF ELECTRICAL & COMPUTER ENGINEERING, UNIVERSITY OF MANITOBA, WINNIPEG, R3T 5V6, MANITOBA, CANADA

Email address: `james.peters3@umanitoba.ca`

THE INSTITUTE OF PHYSICS, LONDON N1 9BU, UK

Email address: `roberto.alfano@physics.org`

FACULTY OF APPLIED SCIENCES, UNIVERSITY OF SUNDERLAND, SUNDERLAND, TYNE AND WEAR, UK

Email address: `peter.smith@sunderland.ac.uk`

CENTER FOR NONLINEAR SCIENCE, UNIVERSITY OF NORTH TEXAS 1155 UNION CIRCLE, 311427, DENTON, TX 76203-5017, USA

Email address: `tozziarturo@libero.it`