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Robust input redundancy for uncertain systems

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Abstract: For input redundant linear systems, it is possible to construct distinct inputs producing identical outputs, from the same initial state. Thus selecting an output does not determine uniquely the input, i.e., there exist degrees of freedom related to the choice of inputs to generate a given output. These degrees of freedom can be used to enhance the system performances without altering the output. However their determination might depend on system parameters. When some of them are not known or uncertain, it may be difficult or impossible to design distinct inputs leading to the same output. Considering uncertainties in linear systems, this paper determines if such degrees of freedom exist independently from the uncertain parameters and, in this case, provides a methodology to compute these inputs. This corresponds to the new concept of robust input redundancy, which is defined and characterized in this paper.

Keywords: Uncertainty, input redundancy, geometric control theory, robust control.

1. INTRODUCTION

Connecting more actuators than necessary to control a system has many advantages: examples include resilience to failure, yield improvement or state of health management (Johansen and Fossen, 2013; Huang and Tse, 2007; Kreiss et al., 2021). In the framework of disturbance decoupling problem, the disturbance can be viewed as a specific input which should be without effect on the output (Stikkel et al., 2003). This strategy leads to a so-called *input redundant (IR) system*. Such a system has the ability to have multiple input trajectories producing exactly the same output trajectory, starting from the same initial state. It thus contains degrees of freedom in the sense that selecting the output trajectory does not determine the input one. The methods using these degrees of freedom in order to improve the system performances are referred as control allocation (Bodson, 2002; Harkegard and Glad, 2005). They require the knowledge of the degrees of freedom. That is why their characterization is crucial. We mention that the input redundancy is in fact the lack of left invertibility (Kreiss and Tréguët, 2021; Ntogramatzidis and Prattichizzo, 2007).

Recently, the property of IR has been characterized (Zaccarian, 2009; Serrani, 2012; Kreiss and Tréguët, 2021) by the means of geometric control theory (Wonham, 2012; Trentelman et al., 2012; Basile and Marro, 1992), and in particular the notion of controlled invariant subspaces. At the same time a parametrization of the degrees of freedom, roughly speaking the input directions having no influence on the output, is provided.

In practice, due to modelling errors, linearization or unknown parameters, it is usual to deal with uncertainties. When the determination of the degrees of freedom depends on uncertain parameters, they become unusable and the

control allocation methods fail. Therefore, it is of utmost importance to determine when input redundancy is not affected by uncertainties, that is to say when the input redundancy is *robust*. This notion will be defined properly in this paper. Roughly speaking, the idea is to know if there exist degrees of freedom which hold for all values of the uncertainties, even if they are not known. In this case, the system will be called *robustly input redundant*.

Because of the recent characterization of input redundancy for Linear Time Invariant (LTI) systems, the presence of uncertainties leads to an open problem. Two main contributions, that are preliminaries for our work, should be nevertheless pointed in the literature: (Bhattacharyya, 1983) and (Basile and Marro, 1987). We will position our work with respect to these papers in a discussion presented after detailing our contribution.

The aim of this paper is to introduce the notion of *robust input redundancy* and to extend the geometric tools to the case of uncertain systems, in order to characterize the robust input redundancy. In particular, we will present extensions of the following notions in this context: the controlled invariant subspace, the weakly unobservable subspace and the friends that are related.

The paper is organized as follows. Section 2 establishes the context of the study and proposes the definition of robust input redundancy. In Section 3, useful tools of the geometric control theory for this paper are recalled. They are extended to the framework of uncertain systems in Section 4. Based on this extension, characterization of robust input redundancy is provided in Section 5. Then, Section 6 offers a comparison with existing results. Finally examples are given in Section 7 in order to illustrate our methodological contribution.

Notations Let \mathbb{R} and \mathbb{N} denote the set of real numbers and natural numbers respectively. I_n is the Identity matrix of dimension n and $0_{n_1 \times n_2}$ is the matrix composed of zeros of dimension $n_1 \times n_2$. For a rectangular matrix M , its transpose is denoted by M^\top . $\text{Im}\{V\}$ is the vector space that is spanned by the columns of the matrix V and $\text{Ker}\{V\}$ is the kernel of the related application to the matrix V . For a set \mathcal{V} , $\dim(\mathcal{V})$ denotes its dimension and \mathcal{V}^N stands for the Cartesian product $\mathcal{V} \times \dots \times \mathcal{V}$, N times. For a set $\mathcal{V} \subseteq \mathbb{R}^n$ and a rectangular matrix $B \in \mathbb{R}^{n \times m}$, we use the notation $B^{-1}\mathcal{V} = \{u \in \mathbb{R}^m, Bu \in \mathcal{V}\}$ for the inverse image of \mathcal{V} by the application B .

2. CONTEXT OF THE STUDY AND DEFINITION

Consider the following uncertain system Σ

$$\dot{x}(t) = A(\theta)x(t) + B(\theta)u(t), \quad x(0) =: x_0, \quad (1a)$$

$$y(t) = C(\theta)x(t) + D(\theta)u(t), \quad (1b)$$

where $x(t), x_0 \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^p$ is the output and $\theta \in \mathbb{R}^\rho$, with $\rho \in \mathbb{N}$, is the constant uncertainty.

Suppose that a parametric representation of $A(\theta)$, $B(\theta)$, $C(\theta)$, $D(\theta)$ is given, i.e.,

$$\begin{bmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{bmatrix} = \begin{bmatrix} \tilde{A}_0 & \tilde{B}_0 \\ \tilde{C}_0 & \tilde{D}_0 \end{bmatrix} + \sum_{i=1}^{\rho} \theta_i \begin{bmatrix} \tilde{A}_i & \tilde{B}_i \\ \tilde{C}_i & \tilde{D}_i \end{bmatrix} \quad (2)$$

where $\forall i \in \{1, \dots, \rho\}$, θ_i is the i -th component of θ .

For the sake of simplicity let us rewrite (2) in a linear representation with respect to θ . From (2), we can easily derive the following form

$$\begin{aligned} \begin{bmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{bmatrix} &= \left(1 - \sum_{i=1}^{\rho} \theta_i\right) \begin{bmatrix} \tilde{A}_0 & \tilde{B}_0 \\ \tilde{C}_0 & \tilde{D}_0 \end{bmatrix} \\ &+ \sum_{i=1}^{\rho} \theta_i \begin{bmatrix} \tilde{A}_0 + \tilde{A}_i & \tilde{B}_0 + \tilde{B}_i \\ \tilde{C}_0 + \tilde{C}_i & \tilde{D}_0 + \tilde{D}_i \end{bmatrix} = \sum_{i=1}^N \alpha_i(\theta) \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \quad (3) \end{aligned}$$

with $N = \rho + 1$, $\alpha_1(\theta) = (1 - \sum_{i=1}^{\rho} \theta_i)$,

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} \tilde{A}_0 & \tilde{B}_0 \\ \tilde{C}_0 & \tilde{D}_0 \end{bmatrix}, \quad (4a)$$

and for all $i \in \{2, \dots, N\}$, $\alpha_i(\theta) = \theta_{i-1}$ and

$$\begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} = \begin{bmatrix} \tilde{A}_0 + \tilde{A}_{i-1} & \tilde{B}_0 + \tilde{B}_{i-1} \\ \tilde{C}_0 + \tilde{C}_{i-1} & \tilde{D}_0 + \tilde{D}_{i-1} \end{bmatrix}. \quad (4b)$$

Define also $\mathbf{A} = [A_1^\top, \dots, A_N^\top]^\top$, $\mathbf{B} = [B_1^\top, \dots, B_N^\top]^\top$, $\mathbf{C} = [C_1^\top, \dots, C_N^\top]^\top$ and $\mathbf{D} = [D_1^\top, \dots, D_N^\top]^\top$.

Our goal is to determine when input redundancy can be considered as *robust* to the vector of unknown parameters θ , i.e., is independent from them. To this end and based on the definitions given in (Kreiss and Tréguët, 2021), let us introduce the *robust input redundancy*.

Definition 1. System Σ is *robustly input redundant w.r.t.* θ if there exists an output y which can be produced by (at least) two distinct inputs, for some $x_0 \in \mathcal{X}$ and for all $\theta \in \mathbb{R}^\rho$, i.e., there exists $x_0 \in \mathcal{X}$ and two distinct input functions $u_a \neq u_b$ such that $\forall \theta \in \mathbb{R}^\rho$, $y_{u_a}(t, \theta, x_0) = y_{u_b}(t, \theta, x_0)$ for all $t \geq 0$.

3. BACKGROUND ON GEOMETRIC CONTROL THEORY

Complete characterization of input redundancy is usually based on geometric control theory (see (Kreiss and Tréguët, 2021; Serrani, 2012) for instance). Before proposing a characterization of robust input redundancy in Section 5, we recall now useful tools of this theory (see (Anderson, 1975), (Trentelman et al., 2012) or (Wonham, 2012)).

For LTI systems captured by the quadruple (A, B, C, D) , consider a subspace $\mathcal{V} \subseteq \mathbb{R}^n$ of the state space. \mathcal{V} is called (A, B) -invariant if

$$A\mathcal{V} \subseteq \mathcal{V} + \text{Im}\{B\}$$

or equivalently there exists $F \in \mathbb{R}^{m \times n}$, called a friend, such that

$$(A + BF)\mathcal{V} \subseteq \mathcal{V}.$$

In addition, \mathcal{V} is called output invisible if it also satisfies

$$C\mathcal{V} \subseteq \text{Im}\{D\},$$

or equivalently if it reads

$$(C + DF)\mathcal{V} = 0.$$

It can be shown that there exists a largest control invariant and output invisible subspace of a system, denoted here by \mathcal{V}^* , namely the *weakly unobservable subspace*.

An algorithm to compute iteratively the set \mathcal{V}^* in a finite number of steps has been provided in (Wonham and Morse, 1970). It is noteworthy that this calculus avoids the explicit expression of the matrix F . There exist several techniques to design friends F , nevertheless, when the set \mathcal{V}^* is not restricted to the trivial singleton, we can always exhibit one friend (Anderson, 1975).

Finally, the IR property is closely related to the dimension of the set

$$B^{-1}\mathcal{V}^* \cap \text{Ker}\{D\},$$

as shown in (Kreiss and Tréguët, 2021).

4. EXTENSION OF GENERALIZED INVARIANT AND OUTPUT INVISIBLE SUBSPACES

Let us associate tractable conditions to definitions introduced in Sec. 2.

In the literature, the geometric objects we need for the characterization were sometimes approached (Bhattacharyya, 1983; Basile and Marro, 1987; Balas et al., 2003), such as *generalized controlled invariant subspace*. However, in (Basile and Marro, 1987; Balas et al., 2003), the friend of \mathcal{V} , F is possibly dependent on θ . For robust input redundancy, as we want to characterize inputs which are completely independent from θ , as detailed later, we also need an independent friend from θ . In (Bhattacharyya, 1983), matrices A , B or C cannot share the same unknown parameter, which is quite restrictive, and therefore (1) is not treated.

Let $x_u(t, \theta, x_0)$ and $y_u(t, x_0, \theta)$ be the state and output generated by system parametrized by θ when applying the input u , starting by the initial condition x_0 , respectively. As a consequence, let us extend the notion of generalized (A, B) -invariant and output invisible subspace of Σ .

Definition 2. A subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is a *generalized robust* $(A(\theta), B(\theta))$ -invariant and output invisible subspace if for any $x_0 \in \mathcal{V}$, there exists an input function u such that $\forall \theta \in \mathbb{R}^p, x_u(t, \theta, x_0) \in \mathcal{V}$ and $y_u(t, \theta, x_0) = 0$ for all $t \geq 0$.

A characterization of such a subspace is now provided:

Lemma 3. The following statements are equivalent

- (i) $\mathcal{V} \subset \mathbb{R}^n$ is a *robust generalized* $(A(\theta), B(\theta))$ -invariant and output invisible subspace;
- (ii) \mathcal{V} satisfies

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} \mathcal{V} \subseteq (\mathcal{V}^N \times \{0_{p \times 1}\}^N) + \text{Im} \left\{ \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} \right\}. \quad (5)$$

- (iii) There exists a matrix F , called a *robust friend* of \mathcal{V} , such that

$$\left(\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} F \right) \mathcal{V} \subseteq \mathcal{V}^N \times \{0_{p \times 1}\}^N. \quad (6)$$

Proof 1. (i) \Rightarrow (ii): Let us assume that \mathcal{V} is a generalized $(A(\theta), B(\theta))$ -invariant and output invisible subspace. For every $x_0 \in \mathcal{V}$, there exists an input function u such that $\forall \theta \in \mathbb{R}^p, x_u(t, \theta, x_0) \in \mathcal{V}$, then $\dot{x}(0^+) = \lim_{t \rightarrow 0^+} \frac{1}{t}(x_u(t, \theta, x_0) - x_0) \in \mathcal{V}$. It implies that there exists $u_0 \in \mathbb{R}^m$, such that $\forall \theta \in \mathbb{R}^p, A(\theta)x_0 + B(\theta)u_0 \in \mathcal{V}$ and $C(\theta)x_0 + D(\theta)u_0 = 0$, which is equivalent to

$$\forall i \in \{1, \dots, N\}, \begin{bmatrix} A_i \\ C_i \end{bmatrix} x_0 + \begin{bmatrix} B_i \\ D_i \end{bmatrix} u_0 \in \mathcal{V} \times 0_{p \times 1}, \quad (7)$$

by linearity of quadruple $(A(\theta), B(\theta), C(\theta), D(\theta))$ with respect to quadruples (A_i, B_i, C_i, D_i) (see (4)). Clearly, (7) can be written as

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} x_0 + \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} u_0 \in \mathcal{V}^N \times \{0_{p \times 1}\}^N, \quad (8)$$

leading to the inclusion in (5).

(ii) \Rightarrow (iii): Assume that the inclusion (5) holds. Let us introduce the matrix $V \in \mathbb{R}^{n \times q}$ as a basis of the set $\mathcal{V} \subset \mathbb{R}^n$. For each column v of V , there exist N vectors $\tilde{v}_j \in \mathcal{V}, j = 1, \dots, N$ and a vector $\tilde{u} \in \mathbb{R}^m$ such that

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} v = \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} \tilde{u} + (\tilde{v}_1^\top \cdots \tilde{v}_N^\top \ 0_{1 \times p} \cdots 0_{1 \times p})^\top.$$

We can deduce that there exist N matrices $Y_i \in \mathbb{R}^{q \times q}$ and a matrix $U \in \mathbb{R}^{m \times q}$ such that

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} V = \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} U + \begin{bmatrix} \text{diag} \{VY_1; \dots; VY_N\} \\ 0_{pN \times qN} \end{bmatrix}.$$

The matrix V being a basis of \mathcal{V} , the matrix V is full column rank and $V^\top V$ is invertible. We introduce the gain

$$F = -U (V^\top V)^{-1} V^\top, \quad (9)$$

which is a friend and satisfied inclusion (6).

(iii) \Rightarrow (i): the implication is trivial by identifying a satisfactory control as the state feedback $u(t) = Fx(t)$ in Definition 2.

Remark 4. It is important to note that the vector u_0 , in the proof, is common for all the parameters $\theta \in \mathbb{R}^p$, or in other words to all the quadruples (A_i, B_i, C_i, D_i) in relation (8). The concatenation of the pairs (\mathbf{A}, \mathbf{C}) and (\mathbf{B}, \mathbf{D}) in the inclusion (5) is characteristic of the robustness in Definition 2.

Remark 5. The relation (9) highlights a possible robust friend F , but this is not necessarily the unique solution. We denote $\mathbf{F}(\mathcal{V})$ their set.

Let denote $\mathcal{V}(\Sigma)$ the set of subspaces \mathcal{V} that satisfy (5) for the system Σ .

Lemma 6. The set $\mathcal{V}(\Sigma)$ admits a unique maximal (in the sense of standard ordering of subspaces by inclusion) element, which is denoted by $\mathcal{V}^*(\Sigma)$. In other words, we have that if $\mathcal{V} \in \mathcal{V}(\Sigma)$, then $\mathcal{V} \subseteq \mathcal{V}^*(\Sigma)$.

Proof 2. The proof is inspired by the development in (Wonham and Morse, 1970). By definition, $\{0\} \in \mathcal{V}(\Sigma)$. Thus, the set $\mathcal{V}(\Sigma)$ is not empty. Furthermore, $\mathcal{V}(\Sigma)$ is closed by addition. Indeed, if \mathcal{V}^a and \mathcal{V}^b belong to $\mathcal{V}(\Sigma)$, then $\mathcal{V}^a + \mathcal{V}^b \in \mathcal{V}(\Sigma)$. Since the elements of $\mathcal{V}(\Sigma)$ have finite dimension, it induces that there exists a maximal element in $\mathcal{V}(\Sigma)$. In fact, using Zorn's Lemma (Zorn, 1935) or (Wonham, 2012, Lemma 4.4), there exists an element $\mathcal{V}^*(\Sigma)$ of greatest dimension; if $\mathcal{V} \in \mathcal{V}(\Sigma)$, $\dim(\mathcal{V} + \mathcal{V}^*(\Sigma)) \leq \dim(\mathcal{V}^*(\Sigma))$, which implies $\mathcal{V} \subset \mathcal{V}^*(\Sigma)$. Let us prove by contradiction that the maximal element in $\mathcal{V}(\Sigma)$ is unique. Assume that there exist two distinct maximal elements, then they are included in each others by definition of the maximality and finally they are identical.

$\mathcal{V}^*(\Sigma)$ can be computed by Algorithm 1, recalling that the operation $B^{-1}\mathcal{V}$ for some matrix B and subspace \mathcal{V} refers to the inverse image of \mathcal{V} by B where B is not necessarily invertible. Note that MATLAB tools to compute the geometric subspaces are developed in Marro (2010).

Algorithm 1: Computation of $\mathcal{V}^*(\Sigma)$

Data: Matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$

Result: $\mathcal{V}^*(\Sigma)$

Initialization;

$\mathcal{V}_0 \leftarrow \mathbb{R}^n; k = 0;$

repeat

$$\left| \begin{array}{l} \mathcal{V}_{k+1} \leftarrow \begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}^{-1} \left(\mathcal{V}_k^N \times \{0_{p \times 1}\}^N + \text{Im} \left\{ \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} \right\} \right); \\ \\ k \leftarrow k + 1; \end{array} \right. \quad (10)$$

until $\mathcal{V}_{k+1} = \mathcal{V}_k;$

return $\mathcal{V}^*(\Sigma) = \mathcal{V}_k$

This algorithm extends the one related to the LTI case (see (Wonham and Morse, 1970)). Let us prove by recurrence that the sequence $\{\mathcal{V}_k\}_{k \in \mathbb{N}}$ is not increasing in the sense of inclusion. We have $\mathcal{V}_0 = \mathbb{R}^n \supseteq \mathcal{V}_1$. Assume that $\mathcal{V}_{k-1} \supseteq \mathcal{V}_k$ and let us prove that $\mathcal{V}_k \supseteq \mathcal{V}_{k+1}$:

$$\begin{aligned} \mathcal{V}_{k-1}^N &\supseteq \mathcal{V}_k^N, \\ \mathcal{V}_{k-1}^N \times \{0_{p \times 1}\}^N &\supseteq \mathcal{V}_k^N \times \{0_{p \times 1}\}^N, \end{aligned}$$

$$\begin{aligned} \mathcal{V}_{k-1}^N \times \{0_{p \times 1}\}^N + \text{Im} \left\{ \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} \right\} \\ \supseteq \mathcal{V}_k^N \times \{0_{p \times 1}\}^N + \text{Im} \left\{ \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} \right\}, \end{aligned}$$

and finally

$$\mathcal{V}_k = \begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}^{-1} \left((\mathcal{V}_{k-1}^N \times \mathbf{0}^N) + \text{Im} \left\{ \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} \right\} \right) \supseteq \begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}^{-1} \left((\mathcal{V}_k^N \times \mathbf{0}^N) + \text{Im} \left\{ \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} \right\} \right) = \mathcal{V}_{k+1}. \quad (11)$$

We can prove by recurrence also that $\mathcal{V}_k \supseteq \mathcal{V}^*(\Sigma)$. $\mathcal{V}_0 = \mathbb{R}^n \supseteq \mathcal{V}^*(\Sigma)$. Assume that $\mathcal{V}_k \supseteq \mathcal{V}^*(\Sigma)$ and let us show that $\mathcal{V}_{k+1} \supseteq \mathcal{V}^*(\Sigma)$:

$$\mathcal{V}_{k+1} = \begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}^{-1} \left((\mathcal{V}_k^N \times \mathbf{0}^N) + \text{Im} \left\{ \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} \right\} \right) \supseteq \begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}^{-1} \left(((\mathcal{V}^*(\Sigma))^N \times \mathbf{0}^N) + \text{Im} \left\{ \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} \right\} \right) = \mathcal{V}^*(\Sigma). \quad (12)$$

Finally, the sequence $\{\mathcal{V}_k\}_{k \in \mathbb{N}}$ is a non-increasing sequence and there exists one integer $K \leq n-1$ such that Algorithm 1 converges in finite time and such that $\mathcal{V}_{K+1} = \mathcal{V}_K$, then \mathcal{V}_K belongs to $\mathcal{V}(\Sigma)$ and due to the maximality of $\mathcal{V}^*(\Sigma)$, we have $\mathcal{V}_K \supseteq \mathcal{V}^*(\Sigma) \supseteq \mathcal{V}_K$, that is the equality $\mathcal{V}_K = \mathcal{V}^*(\Sigma)$.

Let us define \mathcal{V}_i^* as the weakly unobservable subspace of LTI system $\Sigma_i(A_i, B_i, C_i, D_i)$.

Lemma 7. The set $\mathcal{V}^*(\Sigma)$ satisfies the following inclusion:

$$\mathcal{V}^*(\Sigma) \subset \bigcap_{i=1}^N \mathcal{V}_i^*. \quad (13)$$

Proof 3. Let us assume that the inclusion (5) holds. By extracting block rows, we obtain

$$\begin{bmatrix} A_i \\ C_i \end{bmatrix} \mathcal{V} \subset (\mathcal{V} \times \{0_{p \times 1}\}) + \text{Im} \left\{ \begin{bmatrix} B_i \\ D_i \end{bmatrix} \right\}. \quad (14)$$

We infer that the $\mathcal{V}^*(\Sigma)$ is then a weakly unobservable subspace of the LTI system $\Sigma_i(A_i, B_i, C_i, D_i)$, that is $\mathcal{V}^*(\Sigma) \subset \mathcal{V}_i^*$, $\forall i = 1, \dots, N$. We finally obtain the inclusion (13) and ends the proof.

Remark 8. Let us note that the inclusion is not an equality in general. Example 2 in Section 7 points out a case for which this inclusion is strict.

5. CHARACTERIZATION OF ROBUST INPUT REDUNDANCY

Thanks to the technical material developed in the previous section, we can provide a technical theorem (Theorem 9, which is an extension of (Trentelman et al., 2012, Theorem 7.11)) and the main result of the paper for the characterization of robust input redundancy.

Theorem 9. Let $F \in \mathbf{F}(\mathcal{V}^*(\Sigma))$ and let L be a matrix of maximal rank such that

$$\text{Im}\{L\} = \bigcap_{i=1}^N B_i^{-1} \mathcal{V}^*(\Sigma) \cap \text{Ker}\{D_i\}. \quad (15)$$

Let $x_0 \in \mathcal{V}^*(\Sigma)$ and u be an input function, then the output resulting from u and x_0 is zero if and only if u has the form

$$u(t) = Fx(t) + Lw(t), \quad (16)$$

for some function w .

Proof 4. \Rightarrow : Let us assume that the output is zero: $y_u(t, \theta, x_0) = 0$, $\forall (t, \theta) \in \mathbb{R}^+ \times \mathbb{R}^\rho$. We can deduce that

the state trajectory $x_u(t, \theta, x_0)$ belongs to the set $\mathcal{V}^*(\Sigma)$, $\forall (t, \theta) \in \mathbb{R}^+ \times \mathbb{R}^\rho$. We now introduce an auxiliary input $v(t) = u(t) - Fx_u(t, \theta, x_0)$, that satisfies, $\forall (t, \theta) \in \mathbb{R}^+ \times \mathbb{R}^\rho$:

$$\begin{aligned} \dot{x}_u(t, \theta, x_0) &= (A(\theta) + B(\theta)F)x_u(t, \theta, x_0) + B(\theta)v(t), \\ 0 &= (C(\theta) + D(\theta)F)x_u(t, \theta, x_0) + D(\theta)v(t). \end{aligned}$$

It yields that $B(\theta)v(t) \in \mathcal{V}^*(\Sigma)$ and $D(\theta)v(t) = 0$, $\forall (t, \theta) \in \mathbb{R}^+ \times \mathbb{R}^\rho$. The dependency in θ of $B(\theta)$ and $D(\theta)$ being linear, that results in

$$v(t) \in \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix}^{-1} (V^*(\Sigma))^N \times \{0_{\mathbf{p} \times 1}\}^N.$$

The latter preimage set can be rewritten as follows:

$$\begin{aligned} &\begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix}^{-1} (V^*(\Sigma))^N \times \{0_{\mathbf{p} \times 1}\}^N \\ &= \bigcap_{i=1}^N B_i^{-1} \mathcal{V}^*(\Sigma) \cap D_i^{-1} \{0_{\mathbf{p} \times 1}\} \\ &= \bigcap_{i=1}^N B_i^{-1} \mathcal{V}^*(\Sigma) \cap \text{Ker}\{D_i\}. \quad (17) \end{aligned}$$

Using the definition of the matrix L (see (15)), there exists a function $w(t)$ such that $v(t) = Lw(t)$. By computing the original control $u(t)$, Equation (16) holds.

\Leftarrow : This implication follows from the computation of the state trajectory.

We are now ready to provide the main result of the paper: the characterization of robust input redundancy.

Theorem 10. The following statements are equivalent

- Σ is robustly input redundant;
- $\dim \bigcap_{i=1, \dots, N} B_i^{-1} \mathcal{V}^*(\Sigma) \cap \text{Ker}\{D_i\} > 0$.

Proof 5. Let us consider two trajectories x_a and x_b for the uncertain system (1a)–(1b) related to two inputs u_a and u_b respectively. Let us write the conditions to fit the situation described in Definition 1 for robust input redundancy. The two trajectories start from the same initial condition: $x_a(0) = x_b(0) = x_0 \in \mathbb{R}^n$. In addition, the error $x_a(t) - x_b(t)$ satisfies the following dynamics for any $\theta \in \mathbb{R}^\rho$:

$$\begin{aligned} \dot{x}_a(t) - \dot{x}_b(t) &= A(\theta)(x_a(t) - x_b(t)) + B(\theta)(u_a(t) - u_b(t)), \\ y_a(t) - y_b(t) &= C(\theta)(x_a(t) - x_b(t)) + D(\theta)(u_a(t) - u_b(t)), \end{aligned}$$

with the initial condition $x_a(0) - x_b(0) = 0$ being zero.

In order to impose the same outputs $y_a(t) = y_b(t)$, the error $y_a(t) - y_b(t) = 0$ is zero. The initial condition $x_a(0) - x_b(0) = 0$ belongs to \mathcal{V}^* . Applying Theorem 9, we infer that

$$u_a(t) - u_b(t) = F(x_a(t) - x_b(t)) + Lw(t),$$

for some vector valued function $w(t)$. The robust input redundancy is equivalent to the fact that L does not have a zero dimension. That ends the proof.

6. COMPARISON WITH EXISTING RESULTS

First of all, our results in the framework of uncertain systems are an extension of the Linear Time Invariant

framework. When there is no uncertain parameter, that is $\rho = 0$ and $N = 1$, we recover the results of the literature for LTI framework, that are shortly summarized in Section 3.

Paper (Bhattacharyya, 1983) introduces the notion of generalized invariant subspaces and characterizes it when each matrix depends on independent parameters, i.e., if θ is an unknown parameter of A , it cannot be an unknown parameter of B , C or D . Obviously, for this case, it is possible to retrieve the results in (Bhattacharyya, 1983) by applying our methodology. However, in Example 1 of Section 7, we show that $\dim(\mathcal{V}^*(\Sigma)) > 0$ whereas the strategy of (Bhattacharyya, 1983) does not apply.

7. NUMERICAL EXAMPLES AND DISCUSSIONS

In this section, several numerical examples are presented to emphasize the derivation of the main results of this paper.

Example 1: Let us consider the following example given by $\theta \in \mathbb{R}$ and

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ \theta & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & \theta \end{bmatrix} u, \quad y = [0 \ 1] x$$

The matrices $A(\theta)$ and $B(\theta)$ share a common uncertain parameter and the example does not fit to the framework considered in (Bhattacharyya, 1983). However, by applying Algorithm 1, we obtain that $\mathcal{V}^*(\Sigma) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbb{R}$, and by

applying the relation (9), we can exhibit $F = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}$ as an associated friend to $\mathcal{V}^*(\Sigma)$. We can check that

$$(A(\theta) + B(\theta)F)\mathcal{V}^*(\Sigma) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathcal{V}^*(\Sigma) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \times \mathcal{V}^*(\Sigma).$$

To answer the question of robust input redundancy, we compute the set defined by (15):

$$B_i^{-1}\mathcal{V}^*(\Sigma) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbb{R}, \quad \text{Ker}\{D_i\} = \mathbb{R}^2, \quad i = 1, 2,$$

and finally it allows us to identify a possible $L = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ using (15).

We can conclude that this system is robust input redundant. In order to verify Definition 1, let us set $\theta = 2/3$, $x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, that is $x_0 \notin \mathcal{V}^*(\Sigma)$ and exhibit two distinct inputs:

$$u_a(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} x_a(t);$$

$$u_b(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} x_b(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} 2 \sin(t),$$

which are depicted on Fig. 1 as well as the resulting state trajectories $x_a(t)$ and $x_b(t)$. We clearly see that u_a and u_b are different whereas $y_a(t) = y_b(t)$, $\forall t \in \mathbb{R}_{\geq 0}$.

We also apply the same inputs for a different $\theta = -1/2$, starting from the same initial state. We observe on Fig. 2 that with a different θ these inputs also lead to the same output.

Example 2: Consider the following system

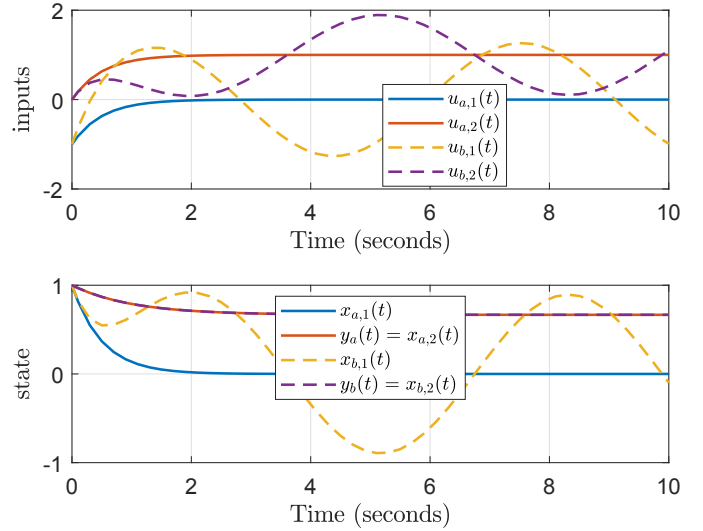


Fig. 1. Input and state trajectories with $\theta = 2/3$

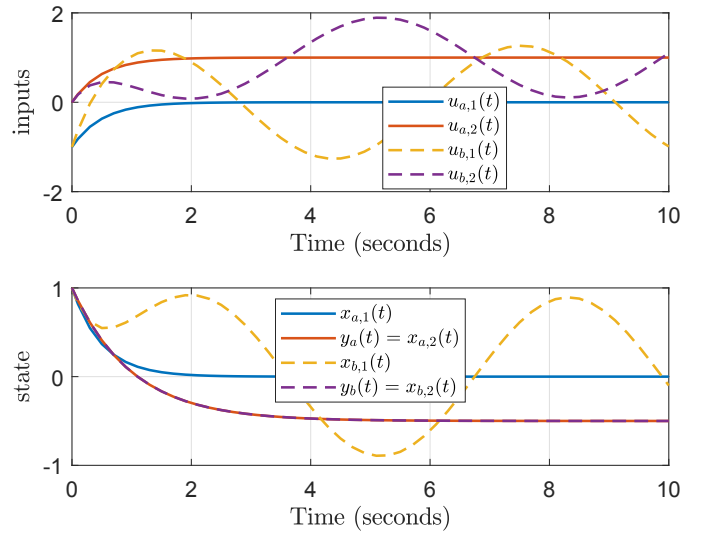


Fig. 2. Input and state trajectories with $\theta = -1/2$

$$\dot{x} = \begin{bmatrix} 1 & \theta \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u, \quad y = [1 \ 0] x,$$

with $\theta \in \mathbb{R}$.

By applying Algorithm 1, we obtain $\mathcal{V}^*(\Sigma) = \{0_{2 \times 1}\}$. That induces that there does not exist a robust friend F , as commented in (Bokor et al., 2002). Nevertheless, for the two sets (A_1, B_1, C_1) and (A_2, B_2, C_2) given by (4), we have $\mathcal{V}_1^* = \mathcal{V}_2^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbb{R}$. It results that the inclusion (13) is strict. The matrix B has a full column rank, that implies that $B^{-1}\mathcal{V}^*(\Sigma) = \{0_{2 \times 1}\}$ and consequently the set defined by Equation (15) is reduced to the trivial singleton. Thanks to Theorem 10, this uncertain system is not robust input redundant, even if $B_1^{-1}\mathcal{V}_1^* = B_2^{-1}\mathcal{V}_2^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbb{R}$.

Example 3: this third example, considering two parameters, is also taken from (Bokor et al., 2002).

$$A(\theta_1, \theta_2) = \begin{bmatrix} -1.05 + \theta_2 & -2.55 + \theta_1 & 0 & 0 & 0 & 0 \\ 2.55 + \theta_1 & -1.05 + \theta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -77.53 & 0 & -8.8 & 0 \\ 0 & 0 & 39.57 & -20.20 & 0 & 0 \\ -169.66 + \theta_1 & 57.09 + \theta_1 & 0 & 0 & -20.20 & 0 \\ -0.01 & 0.01 & 0 & 0 & 0 & -0.1 \end{bmatrix}$$

$$B(\theta_1, \theta_2) = [0 \ 0 \ 0 \ 0 \ 1 \ 0]^\top, \quad D(\theta_1, \theta_2) = 0_{2 \times 1},$$

$$C(\theta_1, \theta_2) = \begin{bmatrix} -0.01 & 0.09 & 0.07 & 0 & 0 & 0 \\ -0.48 & -0.59 & 0 & 0 & -49.51 & -0.02 \end{bmatrix}^\top$$

and adding the constraints $(\theta_1, \theta_2) \in \mathbb{R}^2$. We obtain the subspace $\mathcal{V}^*(\Sigma) = [0 \ 0 \ 0 \ 1 \ 0 \ 0]^\top \mathbb{R}$, as expected. Thanks to relation (9), one possible robust friend is obtained as $F = 0_{1 \times 6}$, which is an other possibility than the one in (Bokor et al., 2002), that is $F_{\text{Bokor}} = [0 \ 0 \ 0 \ 1 \ 0 \ 0]$. To answer to the robust input redundancy, we can compute the set defined in Equation (15). The matrix B being independent of the parameter θ , we can check that

$$B^{-1}\mathcal{V}^*(\Sigma) = \{0\},$$

which ensures that the set (15) is also zero. This system is not robust input dependent according to Theorem 10. Note that the answer is negative even if the set $\mathcal{V}^*(\Sigma)$ is not trivial.

8. CONCLUSIONS

The notion of input redundancy has been extended to uncertain linear systems in order to determine when it can be independent from uncertainties: this is called robust input redundancy. A complete characterization of the notion has been provided in this paper. This work opens the door to various relevant extensions. Classical input redundancy originates from several reasons. A proper taxonomy is proposed in the LTI context. As the control problem has to be dealt appropriately to the origin of redundancy, generalizing the taxonomy for uncertain systems is a relevant extension. Furthermore, the parameters are considered constant in the paper. Looking at time varying ones is also an interesting problem. Finally, we also plan in future work to consider known and possibly time varying parameters, and to characterize an adaptative input redundancy in this context.

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