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ABSTRACT. In this paper, we focus on Carleson embeddings from Bergman spaces $\mathcal{A}^p$ into $L^p(\mu)$, where μ is a positive measure on the unit disk. We describe when this injection is r-summing on $\mathcal{A}^p$. We complete the full characterization of such operators when $p > 1$, and $r \geq 1$. As an immediate application, we get the characterization of absolutely summing weighted composition operators on Bergman spaces. In passing we also prove a new connection between the boundedness of the Berezin transform and the Carleson embedding on Bergman spaces.

Keywords: r-summing operators; Bergman spaces; Carleson embeddings; Berezin transform; Composition operators.

1. INTRODUCTION

In this paper, we investigate particular properties of Carleson embeddings on the classical Bergman spaces $\mathcal{A}^p$ when $p > 1$. In the following, the unit disk of the complex plane is denoted $\mathbb{D} = \{z \in \mathbb{C} \big| |z| < 1\}$. We denote by $\mathscr{H}(\mathbb{D})$ the class of holomorphic functions on the unit disk, and we recall that Bergman spaces are defined by

$$\mathcal{A}^p = \left\{ f \in \mathscr{H}(\mathbb{D}) \Big| \int_{\mathbb{D}} \big|f(z)\big|^p \, dA < \infty \right\}$$

with

$$\|f\|_{\mathcal{A}^p} = \Big(\int_{\mathbb{D}} \big|f(z)\big|^p \, dA \Big)^{1/p},$$

where A stands for the normalized area measure on $\mathbb{D}$. When there is no ambiguity, we write simply $\|\cdot\|_p$ instead of $\|\cdot\|_{\mathcal{A}^p}$. This definition makes sense for every $p > 0$. See for instance [8] and [10] to know more on Bergman spaces.

Now, let us turn to our main subject. Given a positive Borel measure μ on the open unit disk $\mathbb{D}$, we consider the formal identity J_μ from the Bergman space $\mathcal{A}^p$ into $L^p(\mathbb{D}, \mu)$ (we keep the notation J_μ instead of $J_{p,\mu}$ in the sequel for sake of lightness):

$$\begin{array}{rccc} J_\mu\colon & \mathcal{A}^p & \longrightarrow & L^p(\mathbb{D}, \mu) \\ & f & \longmapsto & f \end{array}$$

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∗ Corresponding author.

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A famous result of Carleson (see [3]), introduced and characterized measures such that the corresponding embedding on Hardy spaces are defined and bounded. On Bergman spaces, the boundedness of J_μ was characterized by Hastings [9]:

J_μ is well defined and bounded if and only if μ is a 2-Carleson measure, i.e.

$$\sup_{\xi\in\mathbb{T}} \mu\Big(\mathcal{W}(\xi,h)\Big) = O(h^2), \qquad \text{when } h \to 0,$$

where $\mathcal{W}(\xi,h)$ is the Carleson window

$$\mathcal{W}(\xi,h) = \{z \in \mathbb{D} \mid 1-h \le |z| \quad \text{and} \quad |\arg(z\bar{\xi})| \le h\}.$$

Once we know that J_μ is bounded, it is natural to wonder which specific operator properties are satisfied. The most natural is probably compactness and J_μ is compact if and only if μ is a vanishing Carleson measure:

$$\sup_{\xi\in\mathbb{T}} \mu\Big(\mathcal{W}(\xi,h)\Big) = o(h^2), \qquad \text{when } h \to 0.$$

Actually, our approach in this paper involves hyperbolic disks rather than Carleson windows (see (2.1) and Section 2 for definitions and the corresponding results about boundedness and compactness).

One standard application of Carleson embeddings is that it allows to recover some operator properties of weighted composition operators on $\mathcal{A}^p$. Let us recall that, given a symbol, *i.e.* an analytic function $\varphi : \mathbb{D} \to \mathbb{D}$, the composition operator $C_\varphi : \mathcal{A}^p \to \mathcal{A}^p$ is well defined and automatically bounded (see the monographs [4] or [22] for example). When we are given a weight u, *i.e.* an analytic function u on $\mathbb{D}$, then the weighted composition operator is (now formally) defined by $uC_\varphi(f)(z) = u(z)\cdot f(\varphi(z))$. Many operator properties of C_φ (and of $u\cdot C_\varphi$), can be expressed in terms of Carleson measures thanks to the transfer formula. We explain here how, for composition operators, the pullback measure of A associated to φ plays a crucial role:

$$A_\varphi(E) = A(\varphi^{-1}(E)) = A\Big(\{z\in\mathbb{D}|\ \varphi(z)\in E\}\Big) \quad \text{for every Borel subsets } E \text{ of } \mathbb{D}.$$

The transfer formula gives

$$\|f\circ\varphi\|_{\mathcal{A}^p} = \|f\|_{L^p(\mathbb{D},A_\varphi)} \qquad \text{for every } f\in\mathcal{A}^p\,.$$

Hence many properties of the operator C_φ are shared with the ones of the operator J_{A_φ}, in particular compactness, r-summingness.

The case of weighted composition operators can also be treated in the same manner.

The purpose of this paper is the characterization of (finite positive) measures μ on $\mathbb{D}$ such that J_μ is r-summing for some $r \ge 1$, and as a natural immediate application, we characterize r-absolutely summing weighted composition operators on Bergman spaces.

Let us recall that (1-)absolutely summing operators are the operators $T : X \to Y$ which do take unconditionally summable sequences $\{x_n\}$ in X to absolutely summable sequences $\{Tx_n\}$ in Y. More generally, let us recall

Definition 1.1. *Let $r \geq 1$ and $T : X \to Y$ be a linear operator between Banach spaces. We say that T is r-summing if there is a constant $C \geq 0$ such that regardless of the natural number n and regardless of the choice of $x_1, x_2, \ldots, x_n$ in X, we have*

$$\Big(\sum_{k=1}^{n} \|Tx_k\|_Y^r\Big)^{1/r} \leq C \sup_{x^* \in B_{X^*}} \Big(\sum_{k=1}^{n} |x^*(x_k)|^r\Big)^{1/r} = C \sup_{a \in B_{l^{r'}}} \Big\| \sum_{k=1}^{n} a_k x_k \Big\|_X$$

where X^* denotes the dual space of X, and B_{X^*} denotes the unit ball of the Banach space X^*. The best constant C for which the inequality always holds is denoted by $\pi_r(T)$.

The most natural example is the operator $f \in C(K) \longmapsto f \in L^r(K, \nu)$ where ν is a finite positive measure on the compact K. Pietsch's theorem expresses that it is the generic example of r-summing operators.

We shall write $\Pi_r(X, Y)$ for the set of all r-summing operators from X into Y. If $r = 1$, we say that $\Pi_1(X, Y)$ is the class of absolutely summing operators. The r-summing norm of T, denoted by $\pi_r(T)$, is the least suitable constant $C \geq 0$. The class of r-summing operators forms an operator ideal (for instance see [5] for more details).

This notion was studied for the first time by Grothendieck in the 50's. One of his famous result around this topic is that every continuous linear operator from ℓ^1 to ℓ^2 is absolutely summing. However, it was only in late 60's that this property received a new attention with many fruitful applications in functional analysis (operator theory and geometry of Banach spaces): for instance, Pietsch established the basic properties of this class of operators (in particular the famous factorization theorem which plays a key role in this paper) and studied these operators on Lebesgue type spaces, whereas Lindenstrauss and Pełczyński [15] enlighted how these properties are helpful in Banach space geometry. There are many other deep works in the 70's and after (see the monographs [5], [14] or [25] for some aspects). The monograph [21] enlights more particularly this topic on Banach spaces of analytic functions. Let us mention that some other ideals of operators have close connections with absolutely summing operators such as order bounded operators, r-integral operators, r-nuclear operators (see [5] or [25] for examples).

Recently, in [13], Lefèvre and Rodríguez-Piazza gave a complete characterization of r-summing Carleson embeddings on the Hardy spaces H^p when $p > 1$ (for every $r \geq 1$). We consider here the absolutely summing Carleson embedding viewed on Bergman spaces. In other words, we aim to characterize the measures μ such that the natural embedding map J_μ from the Bergman spaces $\mathcal{A}^p$ to $L^p(\mu)$ is r-summing:

Problem ($\mathcal{P}$): let $p > 1, r \geq 1$ and μ be a positive finite measure on $\mathbb{D}$. Which condition on μ ensures that J_μ is an r-summing operator?

We solve this problem for every $p > 1$ and every $r \geq 1$.

Let us mention that Domenig [6] (see too his PhD thesis [7]) characterized r-summing composition operators. Our approach is different from his point of view, which seems to rely rather strongly on the fact that he works with a pullback measure and the underlying analytic properties of the symbol. Nevertheless, a common feature of the two methods is that (in some cases only in our paper) a link is build with some specific diagonal operators on sequence spaces ℓ^p. This idea is not surprising, having in mind that the Bergman space $\mathcal{A}^p$ is

isomorphic to ℓ^p, but we do not use explicitly this isomorphism. Still, our approach is highly inspired by [13], though some specific difficulties appear, so the proofs require also some novelties. For instance, in passing, we prove a new connection between the boundedness of the Berezin transform viewed from $L^p(\mathbb{D}, A)$ to $L^q(\mathbb{D}, \mu)$, and the boundedness of the Carleson embedding from $\mathcal{A}^p$ to $L^q(\mathbb{D}, \mu)$. This result is an ingredient required for the success of our approach. Strangely we did not find this result in the literature. On the other hand, we make a heavy use of hyperbolic disks and tools whereas they do not appear in [13].

Concretely, let us describe the organization of the paper. The first section is an introduction to the framework of this paper. In the second section, we give basic properties of r-summing operators and remind some facts on hyperbolic geometry. We state the main results of the paper in the third section and give some direct applications to weighted composition operators. In the fourth section, we get some preliminary general results, mainly constructing some links between the Carleson embedding and some multipliers on classical sequence spaces. In Section 5, we focus on the Berezin transform and show a new link with the Carleson embedding. The case $1 < p \leq 2$ is solved in Section 6. In Sections 7, 8 and 9, we solve the case $p \geq 2$: actually we split this case into three cases $r \leq p'$, $p' \leq r \leq p$ and $r \geq p$, where as usual, p' is the conjugate exponent of p: $1/p + 1/p' = 1$. In Section 10, we state the corresponding results in the more general framework of weighted Bergman spaces $\mathcal{A}^p_\alpha$. In the last section, we give several applications, in particular an application to some composition operators between weak (resp. strong) vector valued Bergman spaces.

2. Definitions, notations and reminders.

2.1. **Absolutely summing operators.** We already gave the definition of absolutely summing operators in Definition 1.1. It is simple to check that $\Pi_r(X, Y)$ is included in $B(X, Y)$, the space of all continuous linear operators from X into Y, and that π_r defines a norm on $\Pi_r(X, Y)$ with

$$\|u\| \leq \pi_r(u) \qquad \text{for all } u \in \Pi_r(X, Y).$$

It is well known that $\Pi_r(X, Y)$ is a Banach space under the norm π_r .

The following results are well known properties about r-summing operators and will be used in this paper, see for example [5]. We give them here for sake of completeness

1) *For any Banach spaces X and Y, $T : X \to Y$ is r-summing ($r \geq 1$) if and only if there is a constant $C > 0$ such that for any measurable space (Ω, Σ, ν) and any continuous function $F : \Omega \to X$, we have*

$$\left(\int_\Omega \|T \circ F\|^r d\nu \right)^{1/r} \leq C \sup_{s \in B_{X^*}} \left(\int_\Omega |s(F)|^r d\nu \right)^{1/r}.$$

Moreover, the best C is $\pi_r(T)$.

2) *$\Pi_r(X, Y)$ is an operator ideal between Banach spaces: for any $T \in \Pi_r(X, Y)$, and for any two Banach spaces X_0, Y_0 such that $S \in B(X_0, X)$ and $U \in B(Y, Y_0)$, we have $UTS \in$*

$\Pi_r(X_0, Y_0)$ *with*

$$\pi_r(UTS) \le \|U\|\, \pi_r(T)\, \|S\|.$$

3) *The class of spaces* $\Pi_r(X, Y)$ *where* $r \ge 1$ *is monotone. That is, for any* $1 \le r \le s < \infty$, *we have* $\Pi_r(X, Y) \subset \Pi_s(X, Y)$ *and the relationship* $\pi_s(T) \le \pi_r(T)$, *for any* $T \in \Pi_r(X, Y)$.

4) *Cotype property: we say that a Banach space X has cotype* $q \ge 2$, *if there is a constant* $C \ge 0$ *such that no matter how we select finitely many vectors* $x_1, x_2, \ldots, x_n$ *from X, we have*

$$\Big(\sum_{k=1}^{n} \|x_k\|^q\Big)^{1/q} \le C \left(\int_0^1 \Big\|\sum_{k=1}^{n} r_k(t) x_k\Big\|^2 dt\right)^{1/2},$$

where (r_k) *is a Rademacher sequence.*

For any measure space (Ω, Σ, μ), the Lebesgue type space $L^q(\Omega, \mu)$ has cotype $\max\{q, 2\}$. Then for any Banach spaces X and Y, we have the following properties:

1) If X has cotype 2, then $\Pi_2(X, Y) = \Pi_1(X, Y)$.
2) If X has cotype $2 < q < \infty$, then $\Pi_1(X, Y) = \Pi_r(X, Y)$ for all $1 < r < q'$.
3) If X and Y both have cotype 2, then $\Pi_r(X, Y) = \Pi_1(X, Y)$ for every $1 < r < \infty$.

The following lemmas about absolutely summing operators are reformulation or direct consequences of Propositions 5.5 and 5.18, and Theorem 2.21 in [5] respectively.

Lemma 2.1. *Let X be a Banach space and* μ *be any Borel measure on* $\mathbb{D}$.
Let $T : X \to L^p(\mu)$, $1 \le p < \infty$, *be an order bounded operator. Then T is p-summing with*

$$\pi_p(T) \le \Big\| \sup_{f \in B_X} |T(f)| \Big\|.$$

Lemma 2.2. *Let X be a Banach space and H be a Hilbert space. We assume that the operator* $T : X \to H$ *is such that its adjoint* $T^* : H \to X^*$ *is r-summing for some* $r \ge 1$. *Then T is 1-summing with*

$$\pi_1(T) \lesssim \pi_r(T^*).$$

2.2. **Carleson embedding, Bergman metric and Luecking rectangles.**

Definition 2.3. *Suppose that* μ *is a finite positive Borel measure on the unit disk* $\mathbb{D}$. J_μ *is the identity operator from* $\mathcal{A}^p$ *into* $L^p(\mu)$

$$\begin{aligned} J_\mu : \mathcal{A}^p &\longrightarrow L^p(\mu) \\ f &\longmapsto f \end{aligned}$$

which is formally defined until now, but we recall below that it is defined precisely when μ *is a* 2*-Carleson measure.*

For simplicity, we shall use the terminology Carleson measure instead of "2-Carleson" or "$\mathcal{A}^p$-Carleson" measure. It may happen that we consider J_μ acting from $\mathcal{A}^{p_1}$ to $L^{p_2}(\mu)$ with

p_1, p_2 different from p, but keeping the same notation J_μ for simplicity. Nevertheless, we shall be careful to avoid ambiguity.

It is well known that (cf. [26], Theorem 7.4), J_μ is bounded if and only if

$$\sup\left\{\frac{\mu(D(a,r))}{(1-|a|^2)^2} : a \in \mathbb{D}\right\} < +\infty\,, \tag{2.1}$$

where $D(a,r)$ $(r>0)$ denotes the Bergman disk defined by

$$D(a,r) = \{z \in \mathbb{D} :\ \beta(z,a) < r\}.$$

Let us recall that the hyperbolic metric (Bergman metric) on the unit disk $\mathbb{D}$ is defined by

$$\beta(z,w) = \frac{1}{2}\log\frac{1+\rho(z,w)}{1-\rho(z,w)}$$

and $\rho(z,w)$ denotes the pseudo-hyperbolic, defined by

$$\rho(z,w) = \frac{|z-w|}{|1-z\overline{w}|}\,.$$

Recall that J_μ is compact if and only if μ is a vanishing Carleson measure (in the hyperbolic sense), that is,

$$\lim_{|a|\to 1^-}\frac{\mu(D(a,r))}{(1-|a|^2)^2} = 0\,.$$

In view of these results on Carleson embeddings, it is clear that the hyperbolic disks play a crucial role in this framework. Let us go further in this direction.

Definition 2.4. *A sequence $\{a_k\}$ in $\mathbb{D}$ is called a t-lattice ($t>0$) in the Bergman metric if the following conditions are satisfied:*
1) The unit disk is covered by the Bergman metric disks $D(a_k,t)$.
2) $\beta(a_i,a_j) \geq t/2$ for all i and j with $i \neq j$.

There exists a positive constant N with the following property: If $0<t<1$ and $\{a_k\}$ is a t-lattice in the Bergman metric, then every point $z \in \mathbb{D}$ belongs to at most N disks $D(a_k,t)$.

The existence of t-lattices in the Bergman metric is ensured by Lemma 4.8 in [26] :

Lemma 2.5. *For any $t>0$ there exists a t-lattice $\{a_k\}$ in the Bergman metric.*

Given a t-lattice, the sequence $\mathcal{D}_k$ given by the following Lemma 2.6 plays a crucial role in the sequel.

Lemma 2.6. [26, Lemma 4.10] *Suppose $0<t<1$ and $\{a_k\}$ is a t-lattice in the Bergman metric. For each k, there exists a measurable set $\mathcal{D}_k$ with the following properties:*
1) $D(a_k,t/4) \subset \mathcal{D}_k \subset D(a_k,t)$ for all $k \geq 1$.
2) $\mathcal{D}_j \bigcap \mathcal{D}_i = \emptyset$ for $i \neq j$.
3) $\mathcal{D}_1 \bigcup \mathcal{D}_2 \bigcup \cdots = \mathbb{D}$.

Observe that, for any $z \in \mathcal{D}_k$ there are several equivalence relations:

$$|1 - \bar{z}a_k| \approx 1 - |z|^2 \approx 1 - |a_k|^2$$

and

$$|\mathcal{D}_k| \approx (1 - |a_k|^2)^2$$

where the underlying constants depend only on t (see consequences of Proposition 4.5 in [26]).
These observations will be used repeatedly throughout this paper .

We also recall the definition of Luecking rectangles: let

$$\Gamma_n = \Big\{z \in \mathbb{D} \mid 1 - \frac{1}{2^n} \leq |z| < 1 - \frac{1}{2^{n+1}}\Big\},$$

where $n = 0, 1, 2, \cdots$. Then each Γ_n is divided into 2^n similar pieces $R_{n,j}$, $0 \leq j < 2^n$, that we call Luecking rectangles (or sometimes in the literature Luecking windows):

$$R_{n,j} = \Big\{z \in \mathbb{D} \mid 1 - \frac{1}{2^n} \leq |z| < 1 - \frac{1}{2^{n+1}}, \frac{2\pi j}{2^n} < arg(z) \leq \frac{2\pi(j+1)}{2^n}\Big\}.$$

So clearly the family of the $R_{n,j}$ with $n \geq 0$ and $0 \leq j < 2^n$, forms a disjoint partition of $\mathbb{D}$.

We shall use the following notations:

- D_z instead of $D(z, r_0)$ for some fixed $0 < r_0 < 1$ and L^p instead of $L^p(A)$.
- B_X denotes the unit ball of a Banach space X and B_X^+ the positive element in the unit ball of X, when X is a Banach lattice.
- C_p will denote a positive constant depending only on the parameter p, but it may change along the paper.
- The notation $A \lesssim B$ (and $A \gtrsim B$) for non negative quantities A and B means that there is a constant C such that $A \leq CB$ (and $A \geq CB$, respectively). Finally, $A \approx B$ means that both $A \lesssim B$ and $B \lesssim A$ hold.
- We are going to often use $|E|$ to denote the area of any measurable set E in $\mathbb{D}$ with respect to the normalized area measure on $\mathbb{D}$.
- We denote the cardinal of finite E by $\#(E)$.

3. The main results

In this section, we state our main results, first of all the characterization of absolutely summing Carleson embeddings on classical Bergman spaces.

Main Theorem:

Let $0 < t < 1/4$ and μ be a Carleson measure on $\mathbb{D}$. For any t-lattice $\{a_k\}$ and the relevant sets $\mathcal{D}_k$ as defined in Lemma 2.6, we have the following statements:

(1) Let $1 < p \leq 2$ and $r \geq 1$. Then $J_\mu : \mathcal{A}^p \to L^p(\mu)$ is r-summing if and only if

$$\int_{\mathbb{D}} \frac{(\mu(D_z))^{\frac{2}{p}}}{(1 - |z|^2)^{\frac{4}{p}+2}} dA(z) < \infty\,.$$

Moreover, we have

$$\begin{aligned}\pi_2(J_\mu) &\approx \left(\sum_{k=1}^{\infty}\left(\frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|}\right)^{2/p}\right)^{1/2} \approx \left(\sum_{n,j}(4^n\mu(R_{n,j}))^{2/p}\right)^{1/2}\\ &\approx \left(\int_{\mathbb{D}}\frac{(\mu(D_z))^{\frac{2}{p}}}{(1-|z|^2)^{\frac{4}{p}+2}}dA(z)\right)^{\frac{1}{2}} \approx \left(\int_{\mathbb{D}}\left(\int_{\mathbb{D}}\frac{1}{|1-\bar{w}z|^{p+2}}\,d\mu(w)\right)^{\frac{2}{p}}dA(z)\right)^{\frac{1}{2}}.\end{aligned} \tag{3.1}$$

(2) Let $p \geq 2$ and $1 \leq r \leq p'$. Then $J_\mu : \mathcal{A}^p \to L^p(\mu)$ is r-summing if and only if

$$\int_{\mathbb{D}}\frac{(\mu(D_z))^{\frac{p'}{p}}}{(1-|z|^2)^{2p'}}\,dA(z) < \infty.$$

Moreover, we have

$$\begin{aligned}\pi_1(J_\mu) \approx \pi_r(J_\mu) &\approx \left(\sum_{k=1}^{\infty}\left(\frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|}\right)^{p'/p}\right)^{1/p'} \approx \left(\sum_{n,j}(4^n\mu(R_{n,j}))^{p'/p}\right)^{1/p'}\\ &\approx \left(\int_{\mathbb{D}}\frac{(\mu(D_z))^{\frac{p'}{p}}}{(1-|z|^2)^{2p'}}dA(z)\right)^{1/p'} \approx \left(\int_{\mathbb{D}}\left(\int_{\mathbb{D}}\frac{1}{|1-\bar{w}z|^{2p}}\,d\mu(w)\right)^{\frac{p'}{p}}dA(z)\right)^{\frac{1}{p'}}.\end{aligned} \tag{3.2}$$

(3) Let $p \geq 2$ and $p' \leq r \leq p$. Then $J_\mu : \mathcal{A}^p \to L^p(\mu)$ is r-summing if and only if

$$\int_{\mathbb{D}}\frac{(\mu(D_z))^{\frac{r}{p}}}{(1-|z|^2)^{\frac{2r}{p}+2}}dA(z) < \infty.$$

Moreover, we have

$$\begin{aligned}\pi_r(J_\mu) &\approx \left(\sum_{k=1}^{\infty}\left(\frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|}\right)^{r/p}\right)^{1/r} \approx \left(\sum_{n,j}(4^n\mu(R_{n,j}))^{r/p}\right)^{1/r}\\ &\approx \left(\int_{\mathbb{D}}\frac{(\mu(D_z))^{\frac{r}{p}}}{(1-|z|^2)^{\frac{2r}{p}+2}}dA(z)\right)^{1/r} \approx \left(\int_{\mathbb{D}}\left(\int_{\mathbb{D}}\frac{1}{|1-\bar{w}z|^{2+\frac{2p}{r}}}\,d\mu(w)\right)^{\frac{r}{p}}dA(z)\right)^{\frac{1}{r}}.\end{aligned} \tag{3.3}$$

(4) Let $p \geq 2$ and $r \geq p$. Then $J_\mu : \mathcal{A}^p \to L^p(\mu)$ is r-summing if and only if

$$\int_{\mathbb{D}}\frac{1}{(1-|z|^2)^2}d\mu(z) < \infty\,.$$

Moreover, we have

$$\begin{aligned}\pi_r(J_\mu) \approx \pi_p(J_\mu) &\approx \left(\sum_{k=1}^{\infty}\frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|}\right)^{1/p} \approx \left(\sum_{n}4^n\mu(\Gamma_n)\right)^{1/p}\\ &\approx \left(\int_{\mathbb{D}}\frac{1}{(1-|z|^2)^2}d\mu(z)\right)^{1/p}\end{aligned} \tag{3.4}$$

Let us mention several remarks.

(R1) We shall prove below (see Prop.4.1) that, for any $\alpha > 0$,

$$\sum_{k=1}^{\infty}\Big(\frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|}\Big)^{\alpha} \approx \sum_{n\geq 0}\sum_{0\leq j<2^n}\Big(\frac{\mu(R_{n,j})}{|R_{n,j}|}\Big)^{\alpha}$$

and in Prop.4.2 the equivalence of the estimates with integrals, explaining why we obtain equivalent estimates of the π_r-norm in the preceding theorem.

(R2) The case $p = 2$ reduces to the characterization of Hilbert-Schmidt operator which is easy. More precisely, J_μ is r-summing on the Hilbert space $\mathcal{A}^2$ for some $r \geq 1$ if and only if J_μ is a Hilbert-Schmidt operator. Indeed, let $\{e_n = (\sqrt{n+1})z^n,\ n \in \mathbb{N}\}$ an orthonormal basis of $\mathcal{A}^2$, then we have

$$\begin{aligned}\|J_\mu\|_{HS} &= \Big(\sum_{n\geq 1}\|J_\mu(e_n)\|^2_{L^2(\mu)}\Big)^{1/2}\\ &= \Big(\sum_{n\geq 1}\int_{\mathbb{D}}(n+1)|z|^{2n}d\mu(z)\Big)^{1/2}\\ &= \left(\int_{\mathbb{D}}\frac{d\mu(z)}{(1-|z|^2)^2}\right)^{1/2}.\end{aligned}$$

(R3) The characterizations in the main theorem actually do not depend on the choice of t and r_0 (as soon as $0 < t < 1/4$ and $0 < r_0 < 1$).

Characterization of absolutely summing weighted composition operators:

As an application of our main results, we can characterize r-summing weighted composition operators for any $r \geq 1$ and $p > 1$. Let uC_φ be the weighted composition operator (formally) defined by

$$(uC_\varphi)(f) = u \cdot (f \circ \varphi),$$

where $u : \mathbb{D} \to \mathbb{C}$ is an analytic function and $\varphi : \mathbb{D} \to \mathbb{D}$ is an analytic map. When $u = 1$, we simply write C_φ, which is the composition operator with symbol φ.

We consider the measure $d\mu = (|u|^p dA)_\varphi$ and we have:

$$\big\|(uC_\varphi)f\big\|_{\mathcal{A}^p} = \left(\int_{\mathbb{D}}|f\circ\varphi|^p|u|^p\,dA\right)^{1/p} = \|f\|_{L^p(\mu)}.$$

Therefore, uC_φ is r-summing if and only if J_μ is r-summing, hence it suffices to apply the main theorem to the measure μ.

Theorem 3.1. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map and $u : \mathbb{D} \to \mathbb{C}$ be an analytic function. Let uC_φ be the weighted composition operator viewed from $\mathcal{A}^p$ to $\mathcal{A}^p$ and $d\mu = (|u|^p dA)_\varphi$, then uC_φ is r-summing if and only if the corresponding $\pi_r(uC_\varphi)$ norm is finite, where*

(1) When $1 < p \leq 2$, for any $r \geq 1$,

$$\pi_r(uC_\varphi) \approx \pi_2(uC_\varphi) \approx \left(\sum_{n,j}\big(4^n\mu(R_{n,j})\big)^{2/p}\right)^{1/2} \approx \left(\int_{\mathbb{D}}\left(\int_{\mathbb{D}}\frac{|u(w)|^p}{|1-\bar{z}\varphi(w)|^{p+2}}\,dA(w)\right)^{\frac{2}{p}}dA(z)\right)^{\frac{1}{2}}.$$

(2) When $p \geq 2$, for $1 \leq r \leq p'$,

$$\pi_r(uC_\varphi) \approx \pi_1(uC_\varphi) \approx \left(\sum_{n,j} (4^n \mu(R_{n,j}))^{p'/p} \right)^{1/p'} \approx \left(\int_{\mathbb{D}} \left(\int_{\mathbb{D}} \frac{|u(w)|^p}{|1-\bar{z}\varphi(w)|^{2p}} \, dA(w) \right)^{\frac{p'}{p}} dA(z) \right)^{\frac{1}{p'}}.$$

(3) When $p \geq 2$, for $p' \leq r \leq p$,

$$\pi_r(uC_\varphi) \approx \left(\sum_{n,j} (4^n \mu(R_{n,j}))^{r/p} \right)^{1/r} \approx \left(\int_{\mathbb{D}} \left(\int_{\mathbb{D}} \frac{|u(w)|^p}{|1-\bar{z}\varphi(w)|^{2+\frac{2p}{r}}} \, dA(w) \right)^{\frac{r}{p}} dA(z) \right)^{\frac{1}{r}}.$$

(4) When $p \geq 2$, for $p \leq r$,

$$\pi_r(uC_\varphi) \approx \left(\sum_{n,j} 4^n \mu(R_{n,j}) \right)^{1/p} \approx \left(\int_{\mathbb{D}} \frac{|u|^p}{(1-|\varphi|^2)^2} \, dA \right)^{1/p}.$$

As an immediate corollary we get a characterization for composition operators, recovering the results of Domenig (which were expressed in terms of Luecking windows in [6],[7]).

Corollary 3.2. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map. On $\mathcal{A}^p$, the composition operator C_φ with symbol φ is r-summing if and only if the corresponding $\pi_r(C_\varphi)$ norm is finite, where*

(1) When $1 < p \leq 2$, for any $r \geq 1$,

$$\pi_r(C_\varphi) \approx \pi_2(C_\varphi) \approx \left(\sum_{n,j} (4^n A_\varphi(R_{n,j}))^{2/p} \right)^{1/2} \approx \left(\int_{\mathbb{D}} \left(\int_{\mathbb{D}} \frac{1}{|1-\bar{z}\varphi(w)|^{p+2}} \, dA(w) \right)^{\frac{2}{p}} dA(z) \right)^{\frac{1}{2}}.$$

(2) When $p \geq 2$, for $1 \leq r \leq p'$,

$$\pi_r(C_\varphi) \approx \pi_1(C_\varphi) \approx \left(\sum_{n,j} (4^n A_\varphi(R_{n,j}))^{p'/p} \right)^{1/p'} \approx \left(\int_{\mathbb{D}} \left(\int_{\mathbb{D}} \frac{1}{|1-\bar{z}\varphi(w)|^{p+2}} \, dA(w) \right)^{\frac{2}{p}} dA(z) \right)^{\frac{1}{2}}$$

(3) When $p \geq 2$, for $p' \leq r \leq p$,

$$\pi_r(C_\varphi) \approx \left(\sum_{n,j} (4^n A_\varphi(R_{n,j}))^{r/p} \right)^{1/r} \left(\int_{\mathbb{D}} \left(\int_{\mathbb{D}} \frac{1}{|1-\bar{z}\varphi(w)|^{2+\frac{2p}{r}}} \, dA(w) \right)^{\frac{r}{p}} dA(z) \right)^{\frac{1}{r}}.$$

(4) When $p \geq 2$, for $p \leq r$,

$$\pi_r(C_\varphi) \approx \left(\int_{\mathbb{D}} \frac{dA}{(1-|\varphi|^2)^2} \right)^{1/p}.$$

Another consequence of the main theorem is the following corollary:

Corollary 3.3. *Let $p > 1$, for any two Carleson measures μ and ν, if $\mu(R_{n,j}) \approx \nu(R_{n,j})$ for all (n, j), then we have $\pi_r(J_\mu) \approx \pi_r(J_\nu)$.*

Of course, there is a similar statement in terms of the family $(\mathcal{D}_k)$.

4. Tools and preliminary results

In this section, we present several ingredients employed in the solution of problem $(\mathcal{P})$.

We begin with some estimates useful to prove the equivalence of the different viewpoints (hyperbolic disks, Luecking windows, or integrals) in our characterizations in the main theorem.

First of all we prove the following proposition proving the equivalent formulation either with help of Luecking rectangles or the family $(\mathcal{D}_k)_k$.

Proposition 4.1. *Let $a, \alpha > 0$. Then, we have*

$$\sum_{k=1}^{\infty} \frac{(\mu(\mathcal{D}_k))^{\alpha}}{|\mathcal{D}_k|^a} \approx \sum_{n\geq 0} \sum_{0\leq j<2^n} \frac{(\mu(R_{n,j}))^{\alpha}}{|R_{n,j}|^a}.$$

Proof. Let us mention that $R_{n,j}$ can be covered by a uniform finite number of $\mathcal{D}_k$ (actually, this is stated without proof in pages 339 and 344 in [18] and we shall give here an explanation but we postpone it until the end of our proof): there exists $N > 0$ (not depending on n and j) such that

$$R_{n,j} \subset \bigcup_{k\in J_{n,j}} \mathcal{D}_k \tag{4.1}$$

where $J_{n,j} = \{k \geq 1;\ \mathcal{D}_k \cap R_{n,j} \neq \emptyset\}$ and $\#(J_{n,j}) \leq N$.

In the same way, $\mathcal{D}_k$ can be covered by a (uniform) finite number of $R_{n,j}$: there exists $N' > 0$ (not depending on k) such that

$$\mathcal{D}_k \subset \bigcup_{(n,j)\in E_k} R_{n,j} \tag{4.2}$$

where $E_k = \{(n, j)\,;\ \mathcal{D}_k \cap R_{n,j} \neq \emptyset\}$ and $\#(E_k) \leq N'$.

We claim that when $k \in J_{n,j}$, we have $|R_{n,j}| \approx |D_k|$. Indeed, by definition, for any $k \in J_{n,j}$, there exists some $z \in R_{n,j} \cap \mathcal{D}_k \neq \emptyset$. But since $z \in R_{n,j}$, we have $1 - |z| \approx 2^{-n}$. Therefore, since $z \in \mathcal{D}_k$ as well, we obtain that $2^{-n} \approx 1 - |z| \approx 1 - |a_k|$. It is well known that $|\mathcal{D}_k| \approx (1 - |a_k|)^2$, hence we get that

$$|R_{n,j}| \approx 4^{-n} \approx |\mathcal{D}_k|\,. \tag{4.3}$$

Since $\#(J_{n,j}) \leq N$ for any $n \geq 0$ and for any $j \in \{0, 1, ..., 2^n - 1\}$, we have from (4.1) the following estimate:

$$(\mu(R_{n,j}))^{\alpha} = \left(\sum_{k\in J_{n,j}} \mu(\mathcal{D}_k \cap R_{n,j})\right)^{\alpha} \leq \left(\sum_{k\in J_{n,j}} \mu(\mathcal{D}_k)\right)^{\alpha} \lesssim \sum_{k\in J_{n,j}} (\mu(\mathcal{D}_k))^{\alpha}.$$

We get, using also (4.3) and (4.2),

$$\begin{aligned}
\sum_{n\geq 0} \sum_{0\leq j<2^n} \frac{(\mu(R_{n,j}))^{\alpha}}{|R_{n,j}|^a} &\lesssim \sum_{n\geq 0} \sum_{0\leq j<2^n} \sum_{k\in J_{n,j}} \frac{(\mu(\mathcal{D}_k))^{\alpha}}{|\mathcal{D}_k|^a} \\
&= \sum_{k=1}^{\infty} \sum_{(n,j)\in E_k} \frac{(\mu(\mathcal{D}_k))^{\alpha}}{|\mathcal{D}_k|^a} = \sum_{k=1}^{\infty} \#(E_k) \frac{(\mu(\mathcal{D}_k))^{\alpha}}{|\mathcal{D}_k|^a} \\
&\lesssim \sum_{k=1}^{\infty} \frac{(\mu(\mathcal{D}_k))^{\alpha}}{|\mathcal{D}_k|^a}\,.
\end{aligned}$$

By symmetry of the properties if the two families of sets (the $R_{n,j}$'s and the $\mathcal{D}_k$'s), the reverse inequality is proved in the same way.

Now let us justify (4.2) and (4.1). Of course, the relation $R_{n,j} \subset \bigcup_{k\in J_{n,j}} \mathcal{D}_k$ is obvious and the real interesting fact is that $J_{n,j}$ is uniformly finite. Let us fix an integer (n, j) with $0 \leq j < 2^n$ and $k \in J_{n,j}$. there exists $w \in \mathcal{D}_k \cap R_{n,j} \neq \emptyset$, so

$$1 - \left|a_k\right| \approx 1 - \left|w\right| \approx 2^{-n} .$$

On the other hand, since the $\mathcal{D}_k$'s are disjoint, we have $\sum_{k\in J_{n,j}} \left|\mathcal{D}_k\right| = \Big|\bigcup_{k\in J_{n,j}} \mathcal{D}_k\Big|$.

But $\bigcup_{k\in J_{n,j}} \mathcal{D}_k \subset D(z_{n,j}, \lambda 2^{-n})$ where $z_{n,j}$ is the "center" of $R_{n,j}$ and $\lambda > 0$ does not depend on n. Indeed every z belonging to some $\mathcal{D}_l$ with $l \in J_{n,j}$ satisfies

$$d(z_{n,j}, z) \leq d(z_{n,j}, w) + d(w, z) \lesssim 2^{-n} + 1 - \left|a_l\right| \lesssim 2^{-n}$$

where $w \in \mathcal{D}_l \cap R_{n,j}$.

We obtain

$$\sum_{k\in J_{n,j}} 4^{-n} \approx \sum_{k\in J_{n,j}} \left|\mathcal{D}_k\right| = \Big|\bigcup_{k\in J_{n,j}} \mathcal{D}_k\Big| \lesssim \Big|D(z_{n,j}, \lambda 2^{-n})\Big| \approx 4^{-n} .$$

Therefore $J_{n,j}$ is finite and there exists N (not depending on (n, j)) such that

$$4^{-n} \#(J_{n,j}) \leq N 4^{-n}$$

which proves (4.1).

Clearly (4.2) can be proved in the same way. □

We continue with some estimates comparing integrals.

Proposition 4.2. *Let $0 < t < 1/4$, and the set $\{D_j\}$ as defined in Lemma 2.6. Let μ be a positive finite measure on the unit disk $\mathbb{D}$, η, $\gamma > 0$. Then we have*

$$\int_{\mathbb{D}} \left(\int_{D(z,2t)} \frac{1}{(1-|w|^2)^{\eta/\gamma}} \, d\mu(w) \right)^{\gamma} dA(z) \quad \approx \quad \int_{\mathbb{D}} \frac{\mu(D(z,2t))^{\gamma}}{(1-|z|^2)^{\eta}} \, dA(z) \approx \sum_{j=1}^{\infty} \frac{\mu(\mathcal{D}_j)^{\gamma}}{|\mathcal{D}_j|^{\eta/2-1}} \tag{4.4}$$

Moreover when $\eta > \max(\gamma, 2)$ then we have

$$\int_{\mathbb{D}} \left(\int_{D(z,2t)} \frac{1}{(1-|w|^2)^{\eta/\gamma}} \, d\mu(w) \right)^{\gamma} dA(z) \quad \approx \quad \int_{\mathbb{D}} \left(\int_{\mathbb{D}} \frac{1}{|1-\bar{w}z|^{\eta/\gamma}} \, d\mu(w) \right)^{\gamma} dA(z) \tag{4.5}$$

with constants depending only on t, η and γ.

In particular, when $\gamma = 1$ and $\eta = 4$ we have:

$$\int_{\mathbb{D}} \frac{1}{(1-|z|^2)^2} \, d\mu(z) \approx \sum_{j=1}^{\infty} \frac{\mu(\mathcal{D}_j)}{|\mathcal{D}_j|} .$$

Proof. Let us begin with the particular case. Since the family $(\mathcal{D}_j)_{j\geq 1}$ forms a partition of $\mathbb{D}$ and $(1-|z|^2)^2 \approx |\mathcal{D}_j|$ whenever $z \in \mathcal{D}_j$, we get

$$\int_{\mathbb{D}} \frac{1}{(1-|z|^2)^2}\, d\mu(z) = \sum_{j\geq 1} \int_{\mathcal{D}_j} \frac{d\mu(z)}{(1-|z|^2)^2} \approx \sum_{j\geq 1} \frac{\mu(\mathcal{D}_j)}{|\mathcal{D}_j|}\,.$$

Obviously the first quantity is equivalent to the second one because

$$1-|z|^2 \approx 1-|w|^2, \qquad \text{for } w \in D(z,2t)\,.$$

Now, again since $(\mathcal{D}_j)_{j\geq 1}$ forms a partition of $\mathbb{D}$, $\mathcal{D}_j \subset D(z,2t)$ and $(1-|z|^2)^2 \approx |\mathcal{D}_j|$ whenever $z \in \mathcal{D}_j$, we get

$$\begin{aligned}
\int_{\mathbb{D}} \frac{\mu(D(z,2t))^\gamma}{(1-|z|^2)^\eta}\, dA(z) &= \sum_{j\geq 1} \int_{\mathcal{D}_j} \frac{\mu(D(z,2t))^\gamma}{(1-|z|^2)^\eta}\, dA(z) \\
&\gtrsim \sum_{j\geq 1} \int_{\mathcal{D}_j} \frac{\mu(\mathcal{D}_j)^\gamma}{|\mathcal{D}_j|^{\eta/2}}\, dA(z) \\
&= \sum_{j\geq 1} \frac{\mu(\mathcal{D}_j)^\gamma}{|\mathcal{D}_j|^{\eta/2}}\, |\mathcal{D}_j| \\
&= \sum_{j=1}^{\infty} \frac{\mu(\mathcal{D}_j)^\gamma}{|\mathcal{D}_j|^{\eta/2-1}}\,.
\end{aligned}$$

Hence, it remains to prove the reverse inequality: for every $z \in \mathbb{D}$, we consider the set $I_z = \{j;\ D(z,2t) \cap \mathcal{D}_j \neq \emptyset\}$. We claim that the cardinal

$$|I_z| \leq C\,, \tag{4.6}$$

where the constant C depends only on t. Indeed, let $j_1, ..., j_n$ distincts such that $D(z,2t) \cap \mathcal{D}_{j_i} \neq \emptyset$. Then there exists $w_i \in D(z,2t)$ such that $w_i \in \mathcal{D}_{j_i} \subset D(a_{j_i}, t)$, with $1 \leq i \leq n$. Take $w \in \bigcup_{1\leq i\leq n} D(a_{j_i}, t/4)$, then w belongs to at least one $D(a_{j_i}, t/4)$; and we have

$$\beta(w,z) \leq \beta(w,a_{j_i}) + \beta(a_{j_i}, w_i) + \beta(w_i, z) < t/4 + t + 2t < 4t.$$

Hence, $w \in D(z,4t)$ and we get $\bigcup_{i=1}^{n} D(a_{j_i}, t/4) \subset D(z,4t)$. Therefore, since the disks are disjoint, we have

$$\sum_{i=1}^{n} |D(a_{j_i}, t/4)| \leq |D(z,4t)|\,.$$

In other words, we have $\sum_{i=1}^{n} (1-|a_{j_i}|^2)^2 \leq (1-|z|^2)^2$ and since $1-|a_{j_i}|^2 \approx 1-|z|^2$ when $z \in \mathbb{D}$ satisfies $D(z,2t) \cap \mathcal{D}_{j_i} \neq \emptyset$, we obtain that $n \leq C$, where the constant C depends only on t, which is our claim (4.6).

Since the family $(\mathcal{D}_j)_{j\geq 1}$ forms a partition of $\mathbb{D}$, we have

$$\mu(D(z,2t)) = \sum_{j\in I_z} \mu(\mathcal{D}_j \cap D(z,2t)). \tag{4.7}$$

Hence, from (4.6) and (4.7), we get

$$\mu(D(z,2t))^\gamma \lesssim \sum_{j\in I_z} \mu(\mathcal{D}_j)^\gamma,$$

where the underlying constant depends only on t and γ. For every $z \in \mathbb{D}$ such that $j \in I_z$, we have $(1-|z|^2) \approx (1-|a_j|^2) \approx |\mathcal{D}_j|^{1/2}$. We obtain

$$\begin{aligned}\int_{\mathbb{D}} \frac{\mu(D(z,2t))^\gamma}{(1-|z|^2)^\eta}\, dA(z) &\lesssim \sum_j \int_{\mathbb{D}} \frac{\mu(\mathcal{D}_j)^\gamma}{(1-|z|^2)^\eta}\, \mathbb{1}_{\{j\in I_z\}}\, dA(z)\\ &\lesssim \sum_j \frac{\mu(\mathcal{D}_j)^\gamma}{|\mathcal{D}_j|^{\eta/2}}\, |\{z\in\mathbb{D};\ D(z,2t)\cap \mathcal{D}_j \neq \emptyset\}|\end{aligned}$$

where $\mathbb{1}_{\{j\in I_z\}} = 1$ when $j \in I_z$ and 0 else.

Observe that

$$|\{z\in\mathbb{D};\ D(z,2t)\cap \mathcal{D}_j \neq \emptyset\}| \le |D(a_j,3t)| \approx (1-|a_j|^2)^2 \approx |\mathcal{D}_j|\,.$$

Finally,

$$\int_{\mathbb{D}} \frac{\mu(D(z,2t))^\gamma}{(1-|z|^2)^\eta}\, dA(z) \lesssim \sum_{j=1}^{\infty} \frac{\mu(\mathcal{D}_j)^\gamma}{|\mathcal{D}_j|^{\eta/2-1}}\,.$$

This ends the proof of (4.4).

Now let us focus on (4.5).

First assume that $\gamma > 1$. If μ is the Dirac at point 0 then it is easily checked, so that we may assume that $\mu(\{0\}) = 0$. The argument follows almost directly from a result stated in [19, Th.3(iii)] (see especially (c),(d),(e); see [20, Th.2] too): for any ν positive measure on $\mathbb{D}$, with $\nu(\{0\}) = 0$, and $q > p > 1$, we have for $\lambda > 1$ (we must mention that actually the hypothesis in [19] is "for λ large enough" but following carefully the proof, it appears that $\lambda > 1$ is sufficient when we restrict to the case $q > p > 1$, and not every $q > p > 0$).

$$\|\Psi\|_{L^{p/(p-q)}} \approx \|\Phi\|_{L^{p/(p-q)}} \approx \|\Upsilon\|_{L^{p/(p-q)}}. \tag{4.8}$$

where

$$\Psi(z) := \int_{\mathbb{D}} \frac{(1-|w|)^{\lambda-2}}{|1-\bar{w}z|^\lambda} d\nu(w) \qquad \Phi(z) := \frac{\nu(D_z)}{(1-|z|^2)^2} \quad \text{and} \quad \Upsilon(z) := \int_{\Sigma_z} \frac{d\nu(w)}{(1-|w|^2)^2}.$$

and $\Sigma_z = \{\xi\in\mathbb{D} |\, \arg(z) - \arg(\xi)| < \frac{1}{2}(1-\frac{|\xi|}{|z|})\}$ is the non-tangential approach region.

In other words, we have

$$\begin{aligned}\int_{\mathbb{D}} \left(\int_{\mathbb{D}} \frac{(1-|w|)^{\lambda-2}}{|1-\bar{w}z|^\lambda} d\nu(w)\right)^{p/(p-q)} dA(z) &\approx \int_{\mathbb{D}} \left(\int_{D_z} \frac{1}{(1-|w|)^2} d\nu(w)\right)^{p/(p-q)} dA(z)\\ &\approx \int_{\mathbb{D}} \left(\int_{\Sigma_z} \frac{1}{(1-|w|)^2} d\nu(w)\right)^{p/(p-q)} dA(z)\end{aligned}$$

Now we apply this with the measure $d\nu = (1-|w|^2)^{2-\frac{\eta}{\gamma}} d\mu$, $q = 2$ and $p = \dfrac{2\gamma}{\gamma-1} > 2$. We also choose $\lambda = \dfrac{\eta}{\gamma} > 1$ and we get the conclusion once we point out again that $1-|w| \approx 1-|z|$ when $w \in D_z$.

Finally we assume that $\gamma \leq 1$ and that $\displaystyle\sum_{(n,j)} 2^{(\eta-2)n} (\mu(R_{n,j}))^\gamma$ is finite. With the preceding proposition and the first part of our proof, it is equivalent to

$$\int_{\mathbb{D}} \left(\int_{D(z,2t)} \frac{1}{(1-|w|^2)^{\eta/\gamma}} \, d\mu(w) \right)^\gamma dA(z) < +\infty$$

Of course there an obvious inequality since $|1-\bar{w}z| \lesssim 1-|z| \approx 1-|w|^2$ when $w \in D(z,2t)$. On the other hand, we have

$$\int_{\mathbb{D}} \left(\int_{\mathbb{D}} \frac{1}{|1-\bar{w}z|^{\eta/\gamma}} \, d\mu(w) \right)^\gamma dA(z) = \sum_{(m,k)} \int_{R_{m,k}} \left(\sum_{(n,j)} \int_{R_{n,j}} \frac{1}{|1-\bar{w}z|^{\eta/\gamma}} \, d\mu(w) \right)^\gamma dA(z)$$

which is lower than

$$\sum_{(m,k)} \sum_{(n,j)} 4^{-m} (\mu(R_{n,j}))^\gamma \sup \left\{ \frac{1}{|1-\bar{w}z|^\eta} \, \Big| \, z \in R_{m,k}, w \in R_{n,j} \right\}. \tag{4.9}$$

since $\|\cdot\|_{\ell^1} \leq \|\cdot\|_{\ell^\gamma}$ (recall that $\gamma \leq 1$). On the other hand, we observe that, for $w \neq z$,

$$\frac{1}{|1-\bar{w}z|} \leq \frac{1}{|w-z|}$$

hence when there is no contact between $R_{m,k}$ and $R_{n,j}$, then $z \in R_{m,k}$ and $w \in R_{n,j}$ are different and $\dfrac{1}{|1-\bar{w}z|} \leq d(R_{m,k}, R_{n,j})^{-1}$.

This explains why we are going to split our argument into several cases. More precisely, we consider the set $V_{(n,j)}$ of indexes (m,k) such that $\overline{R_{m,k}} \cap \overline{R_{n,j}} \neq \emptyset$. It is a (uniformly) finite set (cardinality less than 9). Moreover (n,j) belongs to $V_{(m,k)}$ if and only if (m,k) belongs to $V_{(n,j)}$. We get an upper estimate for these terms:

$$\sum_{(n,j)} \sum_{(m,k) \in V_{(n,j)}} 4^{-m} (\mu(R_{n,j}))^\gamma 2^{n\eta} \lesssim \sum_{(n,j)} 4^{-n} (\mu(R_{n,j}))^\gamma 2^{n\eta} < \infty \tag{4.10}$$

using the fact that $m \approx n$ when $(m,k) \in V_{(n,j)}$ and that $\#V_{(n,j)} \lesssim 1$.

When (n,j) does not belong to $V_{(m,k)}$ then $d(R_{m,k}, R_{n,j})$ can be under estimated by 2^{-n} when $m > n$ (looking the radius), but also by $|k2^{-m} - (j+1)2^{-n}|$ or $|(k+1)2^{-m} - j2^{-n}|$ (up to a constant) according on which side the rectangles are from each other, when we focus on the argument. Indeed, we have $2\sin(\theta/2) \approx \theta$, moreover the radius $1 - 2^{-\max(n,m)}$ may be considered as close to 1 since we focus on what happens near the boundary.

Now, we focus on an upper estimate for the quantity

$$\sum_{(m,k)} \sum_{\substack{(n,j) \notin V_{(m,k)} \\ n \geq m}} 4^{-m} (\mu(R_{n,j}))^\gamma \sup \left\{ \frac{1}{|1-\bar{w}z|^\eta} \, \Big| \, z \in R_{m,k}, w \in R_{n,j} \right\} \lesssim \sum_{(n,j)} \sum_{\substack{l \geq 1 \\ n \geq m}} 4^{-m} (\mu(R_{n,j}))^\gamma \Big(\frac{l}{2^m} \Big)^{-\eta}.$$

Indeed $d(R_{m,k}, R_{n,j})$ is larger (up to a constant) than the difference of angles between the borders of the corresponding Luecking rectangles and by symmetry, the value of these differences runs over $\{2\pi(d\,2^{-m} - 2^{-n}) \mid 2 \le d \le 2^{m-1}, n \ge m\}$ (the values are taken at most two times by symmetry)). We get

$$\sum_{(m,k)} \sum_{\substack{(n,j)\notin V_{(m,k)} \\ n\ge m}} 4^{-m}(\mu(R_{n,j}))^{\gamma} \sup\left\{\frac{1}{|1-\bar{w}z|^{\eta}} \,\Big|\, z \in R_{m,k}, w \in R_{n,j}\right\} \le \sum_{(n,j)} 4^{-n}(\mu(R_{n,j}))^{\gamma} 2^{n\eta} \quad (4.11)$$

since the series $\displaystyle\sum_{l\ge 2} \frac{1}{l^{\eta}}$ converges and $\displaystyle\sum_{m\le n} 4^{-m}2^{m\eta} \lesssim 4^{-n}2^{n\eta}$ (keep in mind $\eta > 2$).

Finally we focus on

$$\sum_{(m,k)} \sum_{\substack{(n,j)\notin V_{(m,k)} \\ n< m}} 4^{-m}(\mu(R_{n,j}))^{\gamma} \sup\left\{\frac{1}{|1-\bar{w}z|^{\eta}} \,\Big|\, z \in R_{m,k}, w \in R_{n,j}\right\}$$

which is the same than

$$\sum_{(n,j)} (\mu(R_{n,j}))^{\gamma} \sum_{\substack{(m,k)\notin V_{(n,j)} \\ m> n}} 4^{-m} \sup\left\{\frac{1}{|1-\bar{w}z|^{\eta}} \,\Big|\, z \in R_{m,k}, w \in R_{n,j}\right\}.$$

We split the set of the values of "k" such that $(m,k) \notin V_{(n,j)}$ into two subsets according that $R_{m,k}$ touches the sector $\{z \in \mathbb{D} |\ arg(z) \in [2\pi j 2^{-n}, 2\pi(j+1)2^{-n}]\}$ or not. We denote $S_{m,n,j}$ the set of values of k such that $R_{m,k}$ touches this sector: it has a cardinality $2^{m-n}+2$ so, taking into account the radius, the sum

$$4^{-m} \sum_{k\in S_{m,n,j}} \sup\left\{\frac{1}{|1-\bar{w}z|^{\eta}} \,\Big|\, z \in R_{m,k}, w \in R_{n,j}\right\}$$

is lower (up to a constant) than

$$4^{-m} \sum_{k\in S_{m,n,j}} \Big(\frac{1}{2^{-n}-2^{-m}}\Big)^{-\eta} \lesssim 4^{-m}2^{n\eta}2^{m-n}.$$

Summing over $m > n$, we get

$$\sum_{\substack{m>n \\ k\in S_{m,n,j}}} 4^{-m} \sup\left\{\frac{1}{|1-\bar{w}z|^{\eta}} \,\Big|\, z \in R_{m,k}, w \in R_{n,j}\right\} \lesssim 4^{-n}2^{n\eta}.$$

Taking into account the argument (like in (4.11)) and the symmetry if the situation, the sum

$$4^{-m} \sum_{k\notin S_{m,n,j}} \sup\left\{\frac{1}{|1-\bar{w}z|^{\eta}} \,\Big|\, z \in R_{m,k}, w \in R_{n,j}\right\}$$

is lower (up to a constant) than

$$4^{-m} \sum_{d\ge 2^{m-n}} \Big(\frac{1}{d2^{-m}}\Big)^{\eta} \lesssim 4^{-m}\frac{2^{m\eta}}{2^{(m-n)(\eta-1)}} = 2^{-m}2^{n(\eta-1)}.$$

Summing over $m > n$, we get

$$\sum_{\substack{m>n\\ k\notin S_{m,n,j}}} 4^{-m} \sup\left\{\frac{1}{|1-\bar{w}z|^{\eta}} \,\Big|\, z\in R_{m,k}, w\in R_{n,j}\right\} \lesssim 4^{-n}2^{n\eta}\,.$$

Gathering the previous estimates, we obtain

$$\sum_{(m,k)} \sum_{\substack{(n,j)\notin V_{(m,k)}\\ n<m}} 4^{-m}(\mu(R_{n,j}))^{\gamma} \sup\left\{\frac{1}{|1-\bar{w}z|^{\eta}} \,\Big|\, z\in R_{m,k}, w\in R_{n,j}\right\} \lesssim \sum_{(n,j)} (\mu(R_{n,j}))^{\gamma} 4^{-n}2^{n\eta} \quad (4.12)$$

which is the expected result in this case.

The inequalities (4.10), (4.11) and (4.12) together gives the final conclusion. □

Now, we focus on preliminary results about the membership of various ideals of operators.

The following proposition belongs to the folklore. We give a proof for sake of completeness.

Proposition 4.3. *Let μ be a Carleson measure on $\mathbb{D}$ and $p \geq 1$. Then $J_\mu : \mathcal{A}^p \to L^p(\mu)$ is order bounded if and only if*

$$\int_{\mathbb{D}} \frac{1}{(1-|z|^2)^2}\, d\mu(z) < \infty\,.$$

Proof. J_μ is order bounded if and only if there exists some $h \in L^p(\mu)$ such that for every $f \in B_{\mathcal{A}^p}$, we have $|f| \leq h$ a.e on $\mathbb{D}$. Since $\mathcal{A}^p$ is separable, it follows that J_μ is order bounded if and only if

$$\int_{\mathbb{D}} \sup_{\substack{f\in\mathcal{A}^p\\ \|f\|\leq 1}} |f(z)|^p\, d\mu(z) < \infty\,,$$

which is equivalent to

$$\int_{\mathbb{D}} \|\delta_z\|^p\, d\mu(z) < \infty\,,$$

where δ_z is the point evaluation at $z \in \mathbb{D}$, viewed as a functional on $\mathcal{A}^p$. It is well known that $\|\delta_z\| = 1/(1-|z|^2)^{2/p}$ and this completes the proof. □

For $p \geq 1$, the multiplier operator $\mathscr{M}_\beta$ on ℓ^p is defined by

$$\mathscr{M}_\beta(e_n) = \beta_n e_n\,, \quad n = 1, 2, \cdots, \quad (4.13)$$

where $\{e_n\}_{n\geq 1}$ denotes the canonical basis of ℓ^p and $\beta = (\beta_1, \beta_2, \ldots)$ is a sequence of complex numbers.

Lemma 4.4. [13, Proposition 4.1] *Define $\mathscr{M}_\beta$ as in* (4.13)*. Then with constants depending only on p and r, we have:*

1) For $1 \leq p \leq 2$ and every $r \geq 1$,

$$\pi_r(\mathscr{M}_\beta) \approx \|\beta\|_{\ell^2} \quad (4.14)$$

2) For $p \geq 2$ and $r \leq p'$,

$$\pi_r(\mathscr{M}_\beta) \approx \|\beta\|_{\ell^{p'}} \quad (4.15)$$

3) For $p \geq 2$ and $p' \leq r \leq p$,

$$\pi_r(\mathscr{M}_\beta) = \|\beta\|_{\ell^r} \tag{4.16}$$

4) For $p \geq 2$ and $r \geq p$,

$$\pi_r(\mathscr{M}_\beta) \approx \|\beta\|_{\ell^p} \,. \tag{4.17}$$

The following result is usually referred to as the atomic decomposition for Bergman spaces, and a proof can be found in [26].

Lemma 4.5. [26, Theorem 4.33] *Suppose $p > 0$ and*

$$b > \max\left\{1, \frac{1}{p}\right\} + \frac{1}{p} \cdot \tag{4.18}$$

Then there exists a constant $\delta > 0$ such that for any t-lattice $\{a_k\}$ in the Bergman metric, where $0 < t < \delta$, the space $\mathcal{A}^p$ consists exactly of functions of the form

$$f(z) = \sum_{k=1}^{\infty} c_k f_k(z)\,, \tag{4.19}$$

where

$$f_k(z) = \frac{(1-|a_k|^2)^{(pb-2)/p}}{(1-\bar{a}_k z)^b}$$

and $\{c_k\} \in \ell^p$. The series in (4.19) *converges in $\mathcal{A}^p$-norm and there exist constants C_1 and C_2 (depending only on p and δ) such that*

$$C_1\|f\|_p \leq \inf\left\{\sum_{k=1}^{\infty} |c_k|^p : \{c_k\} \textit{ satisfies } (4.19)\right\} \leq C_2\|f\|_p\,.$$

We can see from this lemma that for any $z \in \mathcal{D}_k$,

$$|f_k(z)|^p = \frac{(1-|a_k|^2)^{pb-2}}{|1-\bar{a}_k z|^{pb}} \approx \frac{1}{(1-|a_k|^2)^2} \approx |\mathcal{D}_k|^{-1}.$$

We will need the following important lemma.

Lemma 4.6. [26, Proposition 4.13] *Suppose $p > 0$ and $t > 0$. Then there exists a positive constant C such that*

$$|f(z)|^p \leq \frac{C}{(1-|z|^2)^2} \int_{D(z,t)} |f(w)|^p dA(w)$$

for all $f \in H(\mathbb{D})$ and every $z \in \mathbb{D}$.

We can get the following proposition, where the $\{f_k\}$ are defined as in Lemma 4.5 :

Proposition 4.7. *Let $p \geq 1$ and μ be a Carleson measure on $\mathcal{A}^p$. We fix a t-lattice $\{a_k\}$. Define the operator Δ from ℓ^p to $L^p(\mu)$ by*

$$\Delta : c = \{c_k\} \in \ell^p \longmapsto f = \sum_{k=1}^{\infty} c_k f_k \mathbb{1}_{\mathcal{D}_k} \in L^p(\mu), \tag{4.20}$$

and let β be the sequence defined by:

$$\beta = \left(\left(\frac{\mu(\mathcal{D}_1)}{|\mathcal{D}_1|} \right)^{1/p}, \left(\frac{\mu(\mathcal{D}_2)}{|\mathcal{D}_2|} \right)^{1/p}, \dots, \left(\frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|} \right)^{1/p}, \dots \right).$$

Then Δ is an r-summing operator if and only if $\mathscr{M}_\beta$ is r-summing on ℓ^p.

Moreover,

$$\pi_r(\Delta) \approx \pi_r(\mathscr{M}_\beta). \tag{4.21}$$

Proof. Without loss of generality, we may (and do) assume that $\mu(\mathcal{D}_k) \neq 0$ for every k. First, we consider the following composition:

$$\ell^p \xrightarrow{\mathscr{M}_\beta} \ell^p \xrightarrow{\Psi} L^p(\mu)\,,$$

where Ψ is defined by: for any $c = \{c_k\} \in \ell^p$

$$\Psi(c)(z) = \sum_{k=1}^{\infty} c_k \left(\frac{|\mathcal{D}_k|}{\mu(\mathcal{D}_k)} \right)^{1/p} f_k(z)\, \mathbb{1}_{\mathcal{D}_k}(z)\,.$$

Now we prove that Ψ is bounded from ℓ^p to $L^p(\mu)$:

$$\begin{aligned} \|\Psi(c)\|_{L^p(\mu)} &= \left(\int_{\mathbb{D}} \left| \sum_{k=1}^{\infty} c_k \left(\frac{|\mathcal{D}_k|}{\mu(\mathcal{D}_k)} \right)^{1/p} f_k(z)\, \mathbb{1}_{\mathcal{D}_k}(z) \right|^p d\mu(z) \right)^{1/p} \\ &= \left(\sum_{k=1}^{\infty} \int_{\mathcal{D}_k} |c_k|^p \, \frac{|\mathcal{D}_k|}{\mu(\mathcal{D}_k)} \, |f_k(z)|^p \, d\mu(z) \right)^{1/p} \\ &\approx \left(\sum_{k=1}^{\infty} |c_k|^p \right)^{1/p} \\ &= \|c\|_{\ell_p}\,, \end{aligned} \tag{4.22}$$

where we used in (4.22) the fact that $|f_k(z)|^p \approx |\mathcal{D}_k|^{-1}$ on $\mathcal{D}_k$. This implies that Ψ is a bounded operator with norm ≈ 1. It turns out that,

$$\Delta = \Psi \circ \mathscr{M}_\beta\,.$$

Furthermore, we have

$$\pi_r(\Delta) \lesssim \|\Psi\|\, \pi_r(\mathscr{M}_\beta) \approx \pi_r(\mathscr{M}_\beta)\,. \tag{4.23}$$

Now, let $Z = \overline{\operatorname{span}\{f_k \mathbb{1}_{\mathcal{D}_k}\}}^{L^p(\mu)}$ which is a closed subspace of $L^p(\mu)$. Let us consider :

$$\begin{aligned} \widetilde{\Psi} : \ell^p &\longrightarrow Z \\ c &\longmapsto \sum_{k=1}^{\infty} c_k \left(\frac{|\mathcal{D}_k|}{\mu(\mathcal{D}_k)} \right)^{1/p} f_k \mathbb{1}_{\mathcal{D}_k}. \end{aligned}$$

Then $\widetilde{\Psi}$ satisfies $\|\widetilde{\Psi}(c)\|_Z \approx \|c\|_{\ell_p}$. Hence the range of $\widetilde{\Psi}$ is closed, and contains the $\widetilde{\Psi}(e_k)$'s, a fortiori the vector space spanned by the $f_k \mathbb{1}_{\mathcal{D}_k}$. We obtain that $\operatorname{Im} \widetilde{\Psi}$ is closed and contains Z, therefore $\widetilde{\Psi}$ is onto. It follows that $\widetilde{\Psi}$ is an isomorphism.

When $f = \sum_{k=1}^{\infty} c_k f_k \mathbb{1}_{\mathcal{D}_k}$, then

$$\widetilde{\Psi}^{-1}(f) = (c_k \beta_k)_{k\geq 1}$$

Hence,

$$\mathscr{M}_\beta = \widetilde{\Psi}^{-1} \circ \tilde{\Delta}\,,$$

where $\tilde{\Delta}$ is the map Δ with range Z. It turns out that

$$\pi_r(\mathscr{M}_\beta) \lesssim \pi_r(\tilde{\Delta}) = \pi_r(\Delta). \tag{4.24}$$

With (4.23) and (4.24), we get the conclusion. □

Proposition 4.8. *Let $p \geq 1$ and μ be a Carleson measure on $\mathcal{A}^p$. If $J_\mu : \mathcal{A}^p \to L^p(\mu)$ is an r-summing operator for some $r \geq 1$, then $\mathscr{M}_\beta$ is an r-summing operator on ℓ^p.*

Proof. We assume that J_μ is an r-summing operator for some $r \geq 1$. Let $\delta = (\delta_1, \delta_2, \ldots, \delta_k, \ldots)$ be a sequence of Rademacher variables on a probability measurable space $(\Omega, \Sigma, \mathbb{P})$ taking their values in $\{\pm 1\}$.

We define an operator Φ_ω as follows: for every $\omega \in \Omega$,

$$\begin{aligned} \Phi_\omega : \ell^p &\longrightarrow \mathcal{A}^p \\ c &\longmapsto \sum_k c_k \delta_k(\omega) f_k, \end{aligned}$$

where $\{f_k\}$ are defined as in Lemma 4.5. Using Lemma 4.5 again, Φ_ω is bounded from ℓ^p to $\mathcal{A}^p$ with norm less than 1.

Define for every $\omega \in \Omega$

$$\mathcal{M}_\omega : f \in L^p(\mu) \longmapsto \sum_{k=1}^{\infty} \delta_k(\omega) \mathbb{1}_{\mathcal{D}_k} f \in L^p(\mu).$$

Then it is easy to check that $\mathcal{M}_\omega$ is an isometry, since it is actually a multiplier by the function M_ω where $M_\omega(z) = \delta_k(\omega)$ when $z \in \mathcal{D}_k$. We clearly have $|M_\omega| = 1$ everywhere since the family $(\mathcal{D}_k)_k$ forms a partition.

Now we can take the expectation $\mathbb{E}$ with respect to the measure $\mathbb{P}$ which satisfies $\mathbb{E}(\delta_j \delta_k) = \delta_{jk}$, with δ_{jk} is the Kronecker symbol. Then for each k, we have

$$\mathbb{E}(\mathcal{M}_\omega \circ J_\mu \circ \Phi_\omega) e_k = \int_\Omega \Big(\sum_j \delta_j(\omega) \mathbb{1}_{\mathcal{D}_j} \Big) f_k \, \delta_k(\omega) \, d\mathbb{P} = \mathbb{1}_{\mathcal{D}_k} f_k = \Delta e_k,$$

where Δ is defined as in Proposition 4.7. It follows that

$$\Delta = \mathbb{E}(\mathcal{M}_\omega \circ J_\mu \circ \Phi_\omega).$$

Then the convexity and the properties of ideal norm imply that

$$\pi_r(\mathscr{M}_\beta) \approx \pi_r(\Delta) \leq \pi_r(J_\mu).$$

This proves the proposition. □

5. Berezin Transform

The purpose of this section is to study the boundedness of the Berezin transform viewed from L^p to $L^q(\mu)$ for $p, q > 0$ and μ a positive Borel measure. Let us start with the definition of Berezin transform which is defined on $\mathbb{D}$ by

$$Bf(z) = \int_{\mathbb{D}} \frac{(1-|z|^2)^2 f(w)}{|1-\overline{w}z|^4}\, dA(w), \qquad \text{for } z \in \mathbb{D},$$

where $f \in L^1$. We refer the reader to [26] for more details about the Berezin transform properties.

In order to prove the main result in this section, we need various tools. We know that Bf is not subharmonic (in general) but the following proposition provides us with a very useful estimate for our purpose.

Proposition 5.1. *Let $0 < q < \infty$ and $r > 0$. Then we have*

$$(Bf)^q(z) \lesssim \frac{1}{|D(z,r)|} \int_{D(z,r)} (Bf)^q(w)\, dA(w)$$

for every measurable $f \geq 0$ and $z \in \mathbb{D}$, where the underlying constant depends only on r.

Proof. We begin with the case $q = 1$. Using Fubini's theorem and the fact that

$$|D(z,r)| \approx (1-|z|^2)^2 \approx (1-|w|^2)^2,$$

whenever $w \in D(z,r)$, we get:

$$\frac{1}{|D(z,r)|} \int_{D(z,r)} \int_{\mathbb{D}} \frac{(1-|w|^2)^2}{|1-\bar{u}w|^4} f(u)\, dA(u)\, dA(w) \approx \int_{\mathbb{D}} f(u) \frac{1}{(1-|z|^2)^2} \int_{D(z,r)} \frac{(1-|w|^2)^2}{|1-\bar{u}w|^4}\, dA(w)\, dA(u).$$

We can see from Lemma 4.30 in [26] that $|1-\bar{u}z| \approx |1-\bar{u}w|$ for every z, w, u in $\mathbb{D}$ with $\beta(z,w) < r$, hence

$$\begin{aligned}
\frac{1}{|D(z,r)|} \int_{D(z,r)} (Bf)(w)\, dA(w) &\approx \int_{\mathbb{D}} f(u) \frac{1}{(1-|z|^2)^2} \int_{D(z,r)} \frac{(1-|z|^2)^2}{|1-\bar{u}z|^4}\, dA(w)\, dA(u) \\
&\approx \int_{\mathbb{D}} f(u) \frac{(1-|z|^2)^2}{|1-\bar{u}z|^4}\, dA(u) \\
&= B(f)(z).
\end{aligned}$$

This special case together with Jensen's inequality (point out that $dA/|D(z,r)|$ is a probability measure on $D(z,r)$) implies that the desired inequality holds for every $q \geq 1$:

$$\begin{aligned}
(Bf)^q(z) &\approx \left(\int_{D(z,r)} Bf(w) \frac{dA(w)}{|D(z,r)|} \right)^q \\
&\lesssim \int_{D(z,r)} (Bf)^q(w) \frac{dA(w)}{|D(z,r)|} \\
&= \frac{1}{|D(z,r)|} \int_{D(z,r)} (Bf)^q(w)\, dA(w).
\end{aligned}$$

To deal with the case $q \in (0,1)$: we fix $z_0 \in \mathbb{D}$, then we have

$$P_w(u) = \frac{1-|w|^2}{|1-\bar{u}w|^2} \approx \frac{1-|z|^2}{|1-\bar{u}z|^2} = P_z(u),$$

for $w, z \in D(z_0, r)$ and $u \in \mathbb{D}$, where the constants depend only on r. Indeed, from Lemma 4.30 in [26], we have $|1-\bar{u}z| \approx |1-\bar{u}z_0|$ for every z, z_0, u in $\mathbb{D}$ with $\beta(z, z_0) < r$ and $|1-\bar{u}w| \approx |1-\bar{u}z_0|$ for every z_0, w, u in $\mathbb{D}$ with $\beta(w, z_0) < r$ where the constants depend only on r. Moreover, it is well known that $1-|w|^2 \approx 1-|z_0|^2$ whenever $\beta(z_0, w) < r$ and $1-|z|^2 \approx 1-|z_0|^2$ whenever $\beta(z_0, z) < r$ (cf. [26, p.69)].
Hence, we can see that there exist positive constants C_1, C_2 which depend only on r such that

$$C_1 P_z(u) \le P_{z_0}(u) \le C_2 P_z(u)$$

and

$$C_1 P_w(u) \le P_{z_0}(u) \le C_2 P_w(u)$$

for all $u \in \mathbb{D}$ and $w, z \in D(z_0, r)$. Therefore, there exist C and C' which depend only on r such that

$$C P_z(u) \le P_w(u) \le C' P_z(u)$$

for all $u \in \mathbb{D}$ and $w, z \in D(z_0, r)$.

Now we can focus on our estimate.
Since $|D(z_0, r)| \approx (1-|z_0|)^2$, we get for every $z \in D(z_0, r)$:

$$\begin{aligned}
&\frac{1}{|D(z_0,r)|}\int_{D(z_0,r)}\int_{\mathbb{D}} \frac{(1-|w|^2)^2}{|1-\bar{u}w|^4} f(u)\, dA(u)\, dA(w) \\
&\qquad \approx \frac{1}{(1-|z_0|)^2}\int_{\mathbb{D}} f(u) \int_{D(z_0,r)} \frac{(1-|w|^2)^2}{|1-\bar{u}w|^4}\, dA(w)\, dA(u) \\
&\qquad \approx \frac{1}{(1-|z_0|)^2}\int_{\mathbb{D}} f(u) \int_{D(z_0,r)} \frac{(1-|z|^2)^2}{|1-\bar{u}z|^4}\, dA(w)\, dA(u) \\
&\qquad \approx \frac{|D(z_0,r)|}{(1-|z_0|^2)^2}\int_{\mathbb{D}} f(u) \frac{(1-|z|^2)^2}{|1-\bar{u}z|^4}\, dA(u) \\
&\qquad \approx B(f)(z)\ .
\end{aligned}$$

It follows that

$$\begin{aligned}
\sup_{z\in D(z_0,r)} Bf(z) &\lesssim \frac{1}{(1-|z_0|^2)^2}\int_{D(z_0,r)} Bf(w)\, dA(w) \\
&\lesssim \frac{1}{(1-|z_0|^2)^2}\left(\int_{D(z_0,r)} (Bf)^q(w)\, dA(w)\right) \sup_{w\in D(z_0,r)} B(f)^{1-q}(w)\,.
\end{aligned}$$

Hence,

$$B(f)^q(z_0) \le \sup_{z\in D(z_0,r)} (Bf)^q(z) \lesssim \frac{1}{(1-|z_0|^2)^2}\int_{D(z_0,r)} (Bf)^q(w)\, dA(w)\,.$$

The proposition is proved for any $q > 0$. □

We state as a lemma the following result on Bergman Carleson embeddings:

Lemma 5.2. [17, Theorem 1] *Let $0 < q < p < \infty$, μ be a positive measure on $\mathbb{D}$ and $K(z) = \mu(D_z)/|D_z|$, $z \in \mathbb{D}$. Then*

$$\left(\int_{\mathbb{D}} |f(z)|^q \, d\mu(z)\right)^{1/q} \lesssim \left(\int_{\mathbb{D}} |f(z)|^p \, dA(z)\right)^{1/p}$$

for every $f \in \mathcal{A}^p$ if and only if $K \in L^{p/(p-q)}$. Note that this condition is independent of the choice of r_0 (cf. Lemma 2 in [17]*).*

Moreover,

$$\|J_\mu\|_{\mathcal{A}^p \to L^q(\mu)} \approx \left(\int_{\mathbb{D}} \Big(\int_{D_z} \frac{1}{(1-|w|^2)^2} \, d\mu(w)\Big)^{p/(p-q)} dA(z)\right)^{(p-q)/pq}.$$

The following crucial lemma is certainly well known from the specialists .

Lemma 5.3. *Let $0 < p \le q < \infty$ and $0 < r < 1$. Let μ be a positive Borel measure on $\mathbb{D}$. Then the following assertions are equivalent :*

1) The Bergman space $\mathcal{A}^p$ is boundedly contained in $L^q(\mu)$.

2) $\mu(D(z,r))/|D(z,r)|^{q/p}$ is bounded on $\mathbb{D}$.

3) $\mu(D(a_k,r))/|D(a_k,r)|^{q/p}$ is bounded on $\mathbb{D}$ for the sequence r-lattice $\{a_k\}$ as described in Section 2.

Proof. The equivalence of (1) and (2) is proved in [16]. (2) implies (3) is obvious. For the proof of (3) implies (1) see Theorem 3.1 in [24]. □

Now we can state the main result of this section.

Theorem 5.4. *Let $p > 1$ and $q > 0$. Let μ be a positive Borel measure on $\mathbb{D}$. Then the following assertions are equivalent:*

1) The Berezin transform B viewed from L^p to $L^q(\mu)$ is bounded;

2) The embedding $J_\mu : \mathcal{A}^p \to L^q(\mu)$ is bounded.

Moreover,

$$\|B\| \approx \|J_\mu\|$$

Proof. For the proof we need to split cases:

(a) The case $0 < q < p < \infty$:

First, we assume that B is bounded from L^p to $L^q(\mu)$. Let us consider this composition

$$\mathcal{A}^p \xrightarrow{i_p} L^p \xrightarrow{B} L^q(\mu)$$

where i_p is the natural embedding map. From the fact that (cf. [26, Proposition 6.13])

$$Bg = g \quad \text{for every } g \in \mathcal{A}^p,$$

it follows that $J_\mu = B \circ i_p$. Since i_p is a contraction, we get

$$\|J_\mu\| \le \|B\|.$$

Now assume that J_μ is bounded from $\mathcal{A}^p$ to $L^q(\mu)$. From now on, we may assume that $f \geq 0$ since $|Bf| \leq B(|f|)$, and $\left\|f\right\|_{L^p} = \left\||f|\right\|_{L^p}$. Using Theorem 5.1 with $r > 0$, we have

$$(Bf)^q(w) \lesssim \frac{1}{(1-|w|^2)^2} \int_{D(w,r)} (Bf)^q(z)\, dA(z)\,.$$

The sufficiency will be clarified by the following computation:

$$\begin{aligned}\int_{\mathbb{D}} \Big(Bf(w)\Big)^q d\mu(w) &\lesssim \int_{\mathbb{D}} \frac{1}{(1-|w|^2)^2} \int_{D(w,r)} \Big(Bf(z)\Big)^q dA(z)\, d\mu(w)\\ &= \int_{\mathbb{D}} \frac{1}{(1-|w|^2)^2} \int_{\mathbb{D}} \mathbb{1}_{D(w,r)}(z) \Big(Bf(z)\Big)^q dA(z)\, d\mu(w)\\ &= \int_{\mathbb{D}} \Big(Bf(z)\Big)^q \int_{\mathbb{D}} \frac{\mathbb{1}_{D(w,r)}(z)}{(1-|w|^2)^2}\, d\mu(w)\, dA(z),\end{aligned}$$

thanks to Fubini's theorem. We get,

$$\int_{\mathbb{D}} \Big(Bf(w)\Big)^q d\mu(w) \lesssim \int_{\mathbb{D}} \Big(Bf(z)\Big)^q \int_{\mathbb{D}} \frac{\mathbb{1}_{D(z,r)}(w)}{(1-|z|^2)^2}\, d\mu(w)\, dA(z),$$

since $\mathbb{1}_{D(w,r)}(z) = \mathbb{1}_{D(z,r)}(w)$ and $1-|z|^2 \approx 1-|w|^2$ whenever $\beta(z,w) < r$. We obtain,

$$\begin{aligned}\int_{\mathbb{D}} \Big(Bf(w)\Big)^q d\mu(w) &\lesssim \int_{\mathbb{D}} \Big(Bf(z)\Big)^q \frac{\mu(D(z,r))}{(1-|z|^2)^2}\, dA(z)\\ &\approx \int_{\mathbb{D}} \Big(Bf(z)\Big)^q\, K(z)\, dA(z),\end{aligned}$$

where

$$K(z) = \frac{\mu(D(z,r))}{(1-|z|^2)^2}$$

is introduced in Lemma 5.2. Since J_μ is bounded from $\mathcal{A}^p$ to $L^p(\mu)$, Lemma 5.2 implies that $K \in L^{p/p-q}$ and is a multiplier from L^p to L^q, hence

$$\int_{\mathbb{D}} \Big(Bf(w)\Big)^q d\mu(w) \lesssim \left(\int_{\mathbb{D}} \Big(Bf(z)\Big)^p dA(z) \right)^{q/p} \|K\|_{L^{p/(p-q)}}.$$

Finally, we conclude that

$$\begin{aligned}\int_{\mathbb{D}} \Big(Bf(w)\Big)^q d\mu(w) &\lesssim \|f\|_{L^p}^q . \|K\|_{L^{p/(p-q)}}\\ &\approx \|f\|_{L^p}^q . \|J_\mu\|^q_{\mathcal{A}^p \to L^q(\mu)},\end{aligned}$$

since $\|K\|_{L^{p/(p-q)}} \approx \|J_\mu\|^q_{\mathcal{A}^p \to L^q(\mu)}$ (recall Lemma 5.2) and B is bounded from L^p to L^p.
We get,

$$\|B\| \lesssim \|J_\mu\|\,.$$

(b) The case $1 < p \leq q < \infty$:
Once again, we only need to prove that (2) implies (1). Using Theorem 5.1 with $r > 0$, the

fact that $D(w,r) \subset D(a_k, 2r)$ and $|D(w,r)| \approx |D(a_k,r)|$ whenever $w \in D(a_k,r)$, we have

$$\begin{aligned}\int_{\mathbb{D}} \Big(Bf(w)\Big)^q d\mu(w) &\lesssim \sum_{k\geq 1} \int_{D(a_k,r)} \left(\frac{1}{|D(w,r)|}\int_{D(w,r)} \Big(Bf(z)\Big)^q dA(z)\right) d\mu(w)\\ &\lesssim \sum_{k\geq 1} \int_{D(a_k,r)} \left(\frac{1}{|D(a_k,r)|}\int_{D(a_k,2r)} \Big(Bf(z)\Big)^q dA(z)\right) d\mu(w)\\ &= \sum_{k\geq 1} \frac{\mu(D(a_k,r))}{|D(a_k,r)|}\left(\int_{D(a_k,2r)} \Big(Bf(z)\Big)^q dA(z)\right).\end{aligned}$$

We claim that

$$Bf(z) \lesssim \frac{\|f\|_{\mathcal{A}^p}}{(1-|z|^2)^{2/p}}\ .$$

Indeed, it is clear for $p = 1$. When $p > 1$, using Hölder's inequality and Lemma 3.10 in [26], we have

$$\begin{aligned}Bf(z) &= \int_{\mathbb{D}} \frac{(1-|z|^2)^2}{|1-z\overline{w}|^4}\, f(w)\, dA(w)\\ &\leq \|f\|_{\mathcal{A}^p}\left(\int_{\mathbb{D}} \left(\frac{(1-|z|^2)^2}{|1-z\overline{w}|^4}\right)^{p'} dA(w)\right)^{1/p'}\\ &\lesssim \|f\|_{\mathcal{A}^p}\, \frac{1}{(1-|z|^2)^{2/p}}\,.\end{aligned}$$

Now, since (b) is satisfied, it follows from Lemma 5.3 and the fact that $(1-|z|^2)^2 \approx |D(a_k,r)|$ whenever $z \in D(a_k,r)$, that

$$\begin{aligned}\int_{\mathbb{D}} \Big(Bf(w)\Big)^q d\mu(w) &\lesssim \sum_{k\geq 1} \int_{D(a_k,2r)} |D(a_k,r)|^{(q-p)/p} \Big(Bf(z)\Big)^{q-p} \Big(Bf(z)\Big)^p dA(z)\\ &\lesssim \|f\|_{\mathcal{A}^p}^{q-p} \sum_{k\geq 1} \int_{D(a_k,2r)} \Big(Bf(z)\Big)^p dA(z)\\ &\lesssim \|f\|_{\mathcal{A}^p}^{q-p}\, \|Bf\|_{L^p}^p\\ &\lesssim \|f\|_{\mathcal{A}^p}^q\,,\end{aligned}$$

where we used the fact that B is bounded from L^p to L^p. □

6. The case $1 < p \leq 2$

Let us recall that, since $1 < p \leq 2$, $\mathcal{A}^p$ and $L^p(\mu)$ have both cotype 2, we know that $\pi_r(J_\mu) \approx \pi_2(J_\mu)$ for any $r \geq 1$. Therefore, in this section we focus on 2-summing operators.

Theorem 6.1. *Let $1 < p \leq 2$ and μ be a positive measure on $\mathbb{D}$. The following assertions are equivalent:*

1) $J_\mu : \mathcal{A}^p \to L^2(\mu)$ is 2-summing.

2) $\displaystyle\int_{\mathbb{D}}\Big(\int_{\mathbb{D}}\frac{d\mu(z)}{|1-z\overline{w}|^4}\Big)^{p'/2}dA(w)<\infty\,.$

3) The Bergman projection viewed from L^p to $L^2(\mu)$

$$T: f\in L^p\longmapsto Tf\in L^2(\mu)\quad \textit{with}\ \ T(f)(z)=\int_{\mathbb{D}}\frac{f(w)}{(1-z\overline{w})^2}\,dA(w)$$

is 1-summing.

Moreover, we have

$$\pi_2(J_\mu:\mathcal{A}^p\to L^2(\mu))\approx\left(\int_{\mathbb{D}}\Big(\int_{\mathbb{D}}\frac{d\mu(z)}{|1-z\overline{w}|^4}\Big)^{p'/2}dA(w)\right)^{1/p'}. \tag{6.1}$$

Proof. We assume that $p<2$ since the case $p=2$ is clear (cf. Remarks in Section 3).

1)⇒2). For $r\in(0,1)$ and for any $w\in\overline{\mathbb{D}}$, we consider the function

$$k_w(z)=\frac{1}{(1-rz\overline{w})^2}\,,\quad z\in\mathbb{D},$$

then $k_w\in\mathcal{A}^p$ for every $w\in\overline{\mathbb{D}}$.

For any ξ in the unit ball of the dual of $\mathcal{A}^p$, there exists a function g_1 in the unit ball of $L^{p'}$, such that

$$\xi(k_w)=\int_{\mathbb{D}}k_w(z)\overline{g_1(z)}dA(z)=\overline{g(rw)}\,,$$

where $Pg_1=g$ and P is the Bergman projection from $L^{p'}$ onto $\mathcal{A}^{p'}$ which is bounded. Then

$$\begin{aligned}\int_{\mathbb{D}}|\xi(k_w)|^{p'}dA(w)&=\int_{\mathbb{D}}|\overline{g(rw)}|^{p'}dA(w)\\&\le\|g\|^{p'}_{\mathcal{A}^{p'}}\\&\le\|P\|^{p'}\|g_1\|^{p'}_{L^{p'}}\,.\end{aligned}$$

So

$$\sup_{\xi\in B_{(\mathcal{A}^p)^*}}\int_{\mathbb{D}}|\xi(k_w)|^{p'}dA(w)\le\|P\|^{p'}.$$

Then, since $p'\ge2$, thanks to Fubini's Theorem and the definition of p'-summing, we have

$$\begin{aligned}\pi_2(J_\mu)&\ge\pi_{p'}(J_\mu)\\&\ge\frac{1}{\|P\|}\left(\int_{\mathbb{D}}\|k_w\|^{p'}_{L^2(\mu)}dA(w)\right)^{1/p'}\\&=\frac{1}{\|P\|}\left(\int_{\mathbb{D}}\Big(\int_{\mathbb{D}}\frac{d\mu(z)}{|1-rz\overline{w}|^4}\Big)^{p'/2}dA(w)\right)^{1/p'}.\end{aligned}$$

Let $r\to1^-$ and thanks to the Fatou's Lemma, we obtain

$$\left(\int_{\mathbb{D}}\Big(\int_{\mathbb{D}}\frac{d\mu(z)}{|1-z\overline{w}|^4}\Big)^{p'/2}dA(w)\right)^{1/p'}\le\|P\|\,\pi_2(J_\mu)\,.$$

2)⇒3). Consider,

$$R : f \in L^2(\mu) \longmapsto R(f) \in L^{p'}$$

formally defined by

$$R(f)(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - z\overline{w})^2}\, d\mu(w),$$

which is an analytic function on $\mathbb{D}$.

We are going to justify that R actually defines an order bounded operator. Indeed, for every $z \in \mathbb{D}$, we have

$$\sup_{\|f\|_{L^2(\mu)} \leq 1} \left| \int_{\mathbb{D}} \frac{f(w)}{(1 - z\overline{w})^2}\, d\mu(w) \right| = \left(\int_{\mathbb{D}} \frac{d\mu(w)}{|1 - z\overline{w}|^4} \right)^{1/2}.$$

Under the hypothesis conditions, we have

$$\int_{\mathbb{D}} \sup_{f \in B_{L^2(\mu)}} |Rf|^{p'}\, dA(z) = \int_{\mathbb{D}} \Big(\int_{\mathbb{D}} \frac{d\mu(w)}{|1 - z\overline{w}|^4} \Big)^{p'/2} dA(z) < \infty.$$

This shows that R is order bounded. From Lemma 2.1, we get that R is p'-summing. Actually its adjoint operator $R^* : L^p \to L^2(\mu)$ is the operator T:

$$\begin{aligned} L^p &\longrightarrow L^2(\mu) \\ f &\longmapsto Tf \quad \text{with } Tf(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - z\overline{w})^2}\, dA(w). \end{aligned}$$

Hence, R^* is the Bergman projection acting from L^p to $L^2(\mu)$. Moreover, we have $T^* = R^{**} = R$. Therefore, $T = R^*$ is 1-summing by Lemma 2.2 and we have

$$\pi_1(T) \lesssim \pi_{p'}(T^*) \lesssim \left(\int_{\mathbb{D}} \Big(\int_{\mathbb{D}} \frac{d\mu(w)}{|1 - z\overline{w}|^4} \Big)^{p'/2} dA(z) \right)^{1/p'}.$$

3)⇒1). For any $f \in \mathcal{A}^p$, we have $Tf = f$. Considering the maps

$$\mathcal{A}^p \xrightarrow{i_p} L^p \xrightarrow{T} L^2(\mu),$$

where $i_p : \mathcal{A}^p \to L^p$ is the natural embedding with a norm equal to 1. Then

$$J_\mu = T \circ i_p\,.$$

J_μ is a 2-summing operator since it is 1-summing with norm

$$\pi_2(J_\mu) \leq \pi_1(J_\mu) \leq \pi_1(T)\, \|i_p\| \leq \pi_1(T)\,.$$

□

Proposition 6.2. *Let $1 < p \leq 2$, $1 \leq q \leq 2$ and assume that*

$$J_\mu : f \in \mathcal{A}^p \mapsto f \in L^q(\mu)$$

is an r-summing operator for some $r \geq 1$, then

$$\int_{\mathbb{D}} \frac{1}{(1 - |z|^2)^q}\, d\mu(z) < \infty\,.$$

In other words, $d\nu = d\mu/(1-|z|^2)^q$ is a finite positive measure on $\mathbb{D}$.

Proof. Since $\mathcal{A}^p$ and $L^q(\mu)$ have both cotype 2 when $1 \le p, q \le 2$, then

$$\Pi_1(\mathcal{A}^p, L^q(\mu)) = \Pi_r(\mathcal{A}^p, L^q(\mu))$$

for any $r \ge 1$. In particular, J_μ is a q-summing operator. We shall work with a probability space $(\Omega, \Sigma, \mathbb{P})$. We consider a (finite) sequence of independent Bernoulli variables $(r_n)_{0\le n\le N}$ on this space taking values in $\{\pm 1\}$ (i.e. a Rademacher sequence), and the function F_ω defined by:

$$F_\omega(z) = \sum_{n=0}^{N} r_n(\omega)(n+1)^{1/2} z^n \ , \ \text{where } \omega \in \Omega.$$

Applying property 1 (cf. Section 2) to the $\mathcal{A}^p$-valued random variable $\omega \in \Omega \mapsto F_\omega$, we get

$$\int_\Omega \|J_\mu \circ F_\omega\|^q_{L^q(\mu)}\, d\mathbb{P}(\omega) \le \pi_q(J_\mu)^q \sup_{\alpha \in B_{(\mathcal{A}^p)^*}} \int_\Omega |\alpha \circ F_\omega|^q \, d\mathbb{P}(\omega). \tag{6.2}$$

On one hand, using Fubini's theorem and Khinchin's inequality, we have

$$\begin{aligned}\int_\Omega \int_{\mathbb{D}} \Big| \sum_{n=0}^{N} r_n(\omega)(n+1)^{1/2} z^n \Big|^q d\mu(z)\, d\mathbb{P}(\omega) &= \int_{\mathbb{D}} \int_\Omega \Big| \sum_{n=0}^{N} r_n(\omega)(n+1)^{1/2} z^n \Big|^q d\mathbb{P}(\omega)\, d\mu(z) \\ &\approx \int_{\mathbb{D}} \Big(\sum_{n=0}^{N} (n+1)|z|^{2n} \Big)^{q/2} d\mu(z) \ .\end{aligned} \tag{6.3}$$

On the other hand, for every α in $B_{(\mathcal{A}^p)^*}$, we can write

$$\alpha(F_\omega) = \langle g, F_\omega \rangle = \int_{\mathbb{D}} g(\bar{z}) F_\omega(z)\, dA(z) \, ,$$

where $g \in \mathcal{A}^{p'}$ with a norm bounded by some c_p (depending only on p). Then, by Khinchin's inequality

$$\begin{aligned}\int_\Omega \big|\alpha(F_\omega)\big|^q \, d\mathbb{P}(\omega) &= \int_\Omega \Big| \sum_{k=0}^{N} \frac{\hat{g}(k) r_k(\omega)}{(k+1)^{1/2}} \Big|^q d\mathbb{P}(\omega) \\ &\approx \Big(\sum_{k=0}^{N} \frac{|\hat{g}(k)|^2}{k+1} \Big)^{q/2} \\ &\le \|g\|^q_{\mathcal{A}^2} \le \|g\|^q_{\mathcal{A}^{p'}} \\ &\le c_p \lesssim 1 \, .\end{aligned} \tag{6.4}$$

Combining (6.3) with (6.4) and the definition of q-summing operator, we get for arbitrary large N

$$\int_{\mathbb{D}} \Big(\sum_{k=0}^{N} (k+1)|z|^{2k} \Big)^{q/2} d\mu(z) \lesssim \pi_q(J_\mu)^q \, .$$

Note that

$$\lim_{N\to\infty}\Big(\sum_{k=0}^{N}(k+1)|z|^{2k}\Big)^{q/2}=\frac{1}{(1-|z|^2)^q}\cdot$$

Finally, we obtain:

$$\int_{\mathbb{D}}\frac{1}{(1-|z|^2)^q}\,d\mu(z)<\infty\,,\qquad\text{for every } z\in\mathbb{D}.$$

□

We shall need the following tools for the proof of our main theorem in this section.

Lemma 6.3. [13, Lemma 7.5] *Let $\delta>0$, (Ω,Σ,μ) be a measure space and $h:\Omega\to[0,+\infty)$ be a measurable function. Then*

$$\inf\left\{\int_\Omega\frac{h}{F}\,d\mu : F\in L^\delta(\mu), F\geq 0, \int_\Omega F^\delta d\mu\leq 1\right\}=\left(\int_\Omega h^{\delta/(\delta+1)}d\mu\right)^{(\delta+1)/\delta}. \tag{6.5}$$

Lemma 6.4. [13, Lemma 7.6] *Let $1\leq q<2$, X be a Banach space and $T:X\to L^q(\mu)$ be a continuous linear operator. Then T is a 2-summing operator if and only if there exists $F\in L^{2q/(2-q)}(\mu)$ with $F\geq 0$ (μ-a.e.), such that*

$$T:X\to L^2(\mu/F^2)$$

is well defined and 2-summing operator.

Moreover

$$\pi_2(T:X\to L^q(\mu))\approx\inf\left\{\pi_2(T:X\to L^2(\mu/F^2)) : F\geq 0,\ \|F\|_{L^{2q/(2-q)}(\mu)}\leq 1\right\}. \tag{6.6}$$

Lemma 6.5. *(Ky Fan's Lemma) (See* [5]*, page 190)*
Let E be a Hausdorff topological vector space, and F be a compact convex subset of E. Let M be a set of functions on F with values in $(-\infty,+\infty]$ having the following properties:

1) Each $f\in M$ is convex and lower semi-continuous.

2) If $g\in conv(M)$, the convex closure of M, there is a $f\in M$ with $g(x)\leq f(x)$, for all $x\in F$.

3) There is an $r\in\mathbb{R}$ such that for each $f\in M$ has a value less and equal to r.

Then, there exists at least one $x_0\in F$, such that $f(x_0)\leq r$ for all $f\in M$.

The following lemma is a variant of the claim in the proof of Theorem 7.3 in [13].

Lemma 6.6. *Let $1<p,q<2$, set $t=q/(2-q)$, $k=p/(2-p)$. Define*

$$I_1=\inf_{f\in B^+_{L^t(\mu)}}\sup_{g\in B^+_{L^k}}\int_{\mathbb{D}}\int_{\mathbb{D}}\frac{g(w)}{|1-\overline{w}z|^4 f(z)}\,d\mu(z)\,dA(w)\,,$$

$$I_2=\sup_{g\in B^+_{L^k}}\inf_{f\in B^+_{L^t(\mu)}}\int_{\mathbb{D}}\int_{\mathbb{D}}\frac{g(w)}{|1-\overline{w}z|^4 f(z)}\,d\mu(z)\,dA(w)\,.$$

Then $I_1=I_2$.

Proof. The proof is nothing but a copy and paste of the one in [13] up to a small change of the kernel, but we include it for sake of completeness.
The fact that $I_1 \geq I_2$ is obvious by the definitions of I_1 and I_2. In the sequel, we assume that I_2 is finite else there is nothing to do and the inequality is trivial.

Let

$$\Phi_g(f) = \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{g(w)}{|1-\overline{w}z|^4 f(z)}\, d\mu(z)\, dA(w)\,, \quad \text{for } f \in B^+_{L^t(\mu)}$$

and

$$M = \{\Phi_g |\ g \in B^+_{L^k}\}\,.$$

We point out that $B^+_{L^t(\mu)}$ is a compact set under its weak topology by the Alaoglu's theorem once $t > 1$. Then M appears as a set of functions on $B^+_{L^t(\mu)}$ and since the maps $g \mapsto \Phi_g$ are linear and $B^+_{L^t(\mu)}$ is convex, M is also a convex set. We have:

(a) For any $\Phi_g \in M$, Φ_g is convex (thanks to the convexity of $1/x$) and lower semi-continuous. Indeed, fix $\delta > 0$, $g > 0$ and take

$$K_g = \{f \in B^+_{L^t(\mu)} |\, \Phi_g(f) \leq \delta\}\,.$$

Let $\{f_n\}$ in K_g, converging to some $f \in L^t(\mu)$. Or, we can assume that it is a pointwise convergence a.e up to an extraction, then by Fatou's lemma, we have

$$\Phi_g(f) \leq \liminf_{n\to\infty} \Phi_g(f_n) \leq \delta\,.$$

It implies that K_g is a closed set of $L^t(\mu)$ and so Φ_g is lower semi-continuous on $B^+_{L^t(\mu)}$.

(b) For fixed $\varepsilon > 0$, by the definition of I_2, for any $g \in B^+_{L^k}$

$$\inf_{f \in B^+_{L^t(d\mu)}} \Phi_g(f) < I_2 + \varepsilon = r\,,$$

so, there is a $f_0 \in B^+_{L^t(\mu)}$ such that

$$\Phi_g(f_0) \leq r\,.$$

Now, Ky Fan's theorem shows that there exists at least one $f_1 \in B^+_{L^t(\mu)}$ such that

$$\Phi_g(f_1) \leq r = I_2 + \varepsilon\,,$$

for any $g \in B^+_{L^k}$. Note that the second condition in Ky Fan's theorem is obvious since M is convex. This implies that $I_1 \leq I_2 + \varepsilon$.

□

Now we can state and prove the main theorem of this section. Point out that the measure ν in the following statement is finite as soon as J_μ is r-summing for some $r \geq 1$, thanks to Proposition 6.2. On the other hand, as soon as J_ν is bounded, it is clear that ν is a finite measure (just testing the constant function $\mathbb{1}$).

Theorem 6.7. *Let $1 < p < 2$, $1 \le q \le 2$. Let μ be a positive Borel measure and $d\nu = d\mu/(1-|z|^2)^q$.*
Then the following statements are equivalent:

1) $J_\mu : \mathcal{A}^p \longrightarrow L^q(\mu)$ is 2-summing.

2) $J_\mu : \mathcal{A}^p \longrightarrow L^q(\mu)$ is r-summing for some $r \ge 1$.

3) $J_\nu : \mathcal{A}^{\frac{p}{2-p}} \longrightarrow L^{\frac{q}{2}}(\nu)$ is bounded.

4) $K(z) = \dfrac{\mu(D_z)}{|D_z|(1-|z|^2)^q} \in L^s$, where $s = 2p/(2p-2q+pq)$.

Moreover, we have

$$\pi_2(J_\mu) \approx \Big(\int_{\mathbb{D}} |K(z)|^s \, dA(z) \Big)^{1/sq}.$$

Therefore (3.1) follows from the following immediate corollary.

Corollary 6.8. *Let $r \ge 1$, $1 < p \le 2$ and μ be a Carleson measure on $\mathcal{A}^p$. Then $J_\mu : \mathcal{A}^p \to L^p(\mu)$ is an r-summing operator if and only if*

$$\int_{\mathbb{D}} \Big(\frac{\mu(D_z)}{(1-|z|^2)^{p+2}} \Big)^{2/p} dA(z) < \infty \,.$$

Moreover

$$\pi_r(J_\mu) \approx \pi_2(J_\mu) \approx \Big(\int_{\mathbb{D}} \Big(\frac{\mu(D_z)}{(1-|z|^2)^{p+2}} \Big)^{2/p} dA(z) \Big)^{1/2}.$$

Proof of Theorem 6.7. Since $\mathcal{A}^p$ and $L^q(\mu)$ have both cotype 2 when $1 \le p, q \le 2$, then $J_\mu : \mathcal{A}^p \longrightarrow L^q(\mu)$ is r-summing for some $r \ge 1$ if and only if J_μ is 2-summing.
Since $p/(2-p) > 1 \ge q/2 > 0$ then Lemma 5.2 shows that $K \in L^s$ if and only if

$$J_\nu : \mathcal{A}^{p/(2-p)} \longrightarrow L^{q/2}(\nu)$$

is bounded. So we only need to prove that 1) is equivalent to 3).

We already mentioned before the statement of the theorem why we know that ν is a finite positive Borel measure.

Set $k = p/(2-p)$ and $t = q/(2-q)$. From Lemma 6.4 and Theorem 6.1, we know that:

$$\begin{aligned} \Big(\pi_2\big(J_\mu : \mathcal{A}^p \to L^q(\mu)\big) \Big)^2 &\approx \inf_{F \in B^+_{L^{2t}(\mu)}} \Big\{ \Big(\pi_2\big(J_\mu : \mathcal{A}^p \to L^2(\mu/F^2)\big) \Big)^2 \Big\} \\ &\approx \inf_{f \in B^+_{L^t(\mu)}} \left(\int_{\mathbb{D}} \Big(\int_{\mathbb{D}} \frac{1}{|1-\overline{w}z|^4 f(z)} \, d\mu(z) \Big)^{p'/2} dA(w) \right)^{2/p'}. \end{aligned} \tag{6.7}$$

Hence, by duality between L^k and $L^{p'/2}$, we get

$$\Big(\pi_2\Big(J_\mu : \mathcal{A}^p \to L^q(\mu)\Big)\Big)^2 \approx \inf_{f\in B^+_{L^{t}(\mu)}} \sup_{g\in B^+_{L^k}} \int_{\mathbb{D}}\int_{\mathbb{D}} \frac{g(w)}{|1-\overline{w}z|^4 f(z)}\, d\mu(z)\, dA(w)\,.$$

Using Lemma 6.6, Lemma 6.3 and then Lemma 5.4, it follows

$$\begin{aligned}\Big(\pi_2\Big(J_\mu : \mathcal{A}^p \to L^q(\mu)\Big)\Big)^2 &\approx \sup_{g\in B^+_{L^k}} \inf_{f\in B^+_{L^{t}(\mu)}} \int_{\mathbb{D}} \frac{Bg(z)}{(1-|z|^2)^2 f(z)}\, d\mu(z)\\ &= \sup_{g\in B^+_{L^k}} \left\| \frac{Bg}{(1-|z|^2)^2}\right\|_{L^{q/2}(\mu)}\\ &= \sup_{g\in B^+_{L^k}} \big\|Bg\big\|_{L^{q/2}(\nu)}\\ &\approx \sup_{g\in B_{L^k}} \|Bg\|_{L^{q/2}(\nu)}\\ &\approx \|J_\nu\|_{\mathcal{A}^k\hookrightarrow L^{q/2}(\nu)}\end{aligned} \tag{6.8}$$

where the (positive) operator B is the Berezin transformation viewed from $L^k = L^k(A)$ to $L^{q/2}(\nu)$. In other words, we obtain that (1) is equivalent to (3). □

7. THE CASE $2 \le p \le r$

In this section, we focus on the case $2 \le p \le r$. Actually, we shall also obtain some results useful to treat the other cases. In particular, some general necessary conditions are obtained below.

Let us begin with the simple case $r = p$.

Proposition 7.1. *Let $p \ge 2$ and μ be a Carleson measure. Then $J_\mu : \mathcal{A}^p \to L^p(\mu)$ is p-summing if and only if $d\mu(z)/(1-|z|^2)^2$ is a finite (positive) measure, that is*

$$\int_{\mathbb{D}} \frac{1}{(1-|z|^2)^2}\, d\mu(z) < \infty\,. \tag{7.1}$$

Moreover

$$\pi_p(J_\mu) \approx \Big(\int_{\mathbb{D}} \frac{1}{(1-|z|^2)^2}\, d\mu(z)\Big)^{1/p}.$$

This result is the counterpart in the Bergman space $\mathcal{A}^p$ of the corresponding result in [23] which characterizes p-summing composition operators on H^p (their proof extends obviously to general Carleson measures on H^p).

Proof. First we assume that the measure $d\mu(z)/(1-|z|^2)^2$ is finite, then by Proposition 4.3, this is equivalent to the fact that J_μ is an order bounded operator, therefore J_μ is p-summing as well. Hence,

$$\pi_p(J_\mu) \lesssim \left(\int_{\mathbb{D}} \frac{d\mu(z)}{(1-|z|^2)^2}\right)^{1/p}.$$

Now we define the following (normalized) sequence in $\mathcal{A}^p$.

$$e_n(z) = \Big(\frac{pn+2}{2}\Big)^{1/p} z^n, \qquad n \geq 0.$$

Let $g \in \mathcal{A}^{p'}$ with a Taylor expansion $g(z) = \sum_{k=0}^{\infty} \hat{g}(k) z^k$. It is well known (cf. [11, Corollary 8.2.10]) that, since $p \geq 2$,

$$\Big(\sum_{k=0}^{\infty} \frac{|\hat{g}(k)|^p}{(k+1)^{p-1}} \Big)^{1/p} \lesssim \|g\|_{\mathcal{A}^{p'}} .$$

Then, we have for every g in the unit ball of $\mathcal{A}^{p'}$:

$$\sum_{n=0}^{\infty} |\langle e_n, g\rangle|^p \lesssim \sum_{n=0}^{\infty} \frac{|\hat{g}(n)|^p}{(n+1)^{p-1}} \lesssim 1 .$$

The definition of the p-summing operator yields

$$\begin{aligned} \sum_{n=0}^{\infty} \|e_n\|_{L^p(\mu)}^p &\leq \pi_p(J_\mu)^p \sup_{g \in B_{\mathcal{A}^{p'}}} \sum_{n=0}^{\infty} |\langle e_n, g\rangle|^p \\ &\lesssim \pi_p(J_\mu)^p . \end{aligned}$$

But

$$\begin{aligned} \sum_{n=0}^{\infty} \|e_n\|_{L^p(\mu)}^p &= \int_{\mathbb{D}} \sum_{n=0}^{\infty} \Big(\frac{pn}{2} + 1\Big) |z|^{pn} \, d\mu(z) \\ &\geq \int_{\mathbb{D}} \sum_{n=0}^{\infty} (n+1) |z|^{pn} \, d\mu(z) \\ &= \int_{\mathbb{D}} \frac{1}{(1-|z|^p)^2} \, d\mu(z) . \end{aligned}$$

We obtain

$$\Big(\int_{\mathbb{D}} \frac{d\mu(z)}{(1-|z|^2)^2} \Big)^{1/p} \lesssim \pi_p(J_\mu) .$$

□

We recall that the multiplier operator $\mathscr{M}_\beta$ on ℓ^p is defined by $\mathscr{M}_\beta(e_n) = \beta_n e_n$, where $\{e_n\}$ denotes the canonical basis of ℓ^p and $\beta = (\beta_1, \beta_2, \cdots)$ is a sequence.

Proposition 7.2. *Let $p \geq 2$ and μ be a Carleson measure on $\mathcal{A}^p$. Assume that J_μ is r-summing. Then*

1) For $1 \leq r \leq p'$,

$$\Big(\sum_{k=1}^{\infty} \Big(\frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|} \Big)^{p'/p} \Big)^{1/p'} \lesssim \pi_r(J_\mu) . \tag{7.2}$$

2) For $p' \leq r \leq p$,

$$\Big(\sum_{k=1}^{\infty} \Big(\frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|} \Big)^{r/p} \Big)^{1/r} \lesssim \pi_r(J_\mu) . \tag{7.3}$$

3) For $p \le r$,

$$\left(\int_{\mathbb{D}} \frac{1}{(1-|z|^2)^2}\, d\mu(z)\right)^{1/p} \approx \left(\sum_{k=1}^{\infty} \frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|}\right)^{1/p} \lesssim \pi_r(J_\mu)\,. \tag{7.4}$$

Proof. Assume that J_μ is r-summing. From Proposition 4.8 , we get that $\mathscr{M}_\beta$ is r-summing as well, where

$$\beta = \left(\Big(\frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|}\Big)^{1/p}\right)_{k\ge 1}.$$

Now the comparison (1); (2) and (3) of $\pi_r(J_\mu)$ with the norm of β follows directly from Lemma 4.4 : more precisely , (1) follows from (4.15); (2) follows from (4.16) and (3) follows from (4.17). □

Now we can prove the main result for $2 \le p \le r$.

Theorem 7.3. *Let $2 \le p \le r$ and μ be a Carleson measure on $\mathcal{A}^p$. Then $J_\mu : \mathcal{A}^p \to L^p(\mu)$ is an r-summing operator if and only if*

$$\int_{\mathbb{D}} \frac{1}{(1-|z|^2)^2}\, d\mu(z) < \infty\,. \tag{7.5}$$

Furthermore

$$\pi_r(J_\mu) \approx \Big(\int_{\mathbb{D}} \frac{1}{(1-|z|^2)^2}\, d\mu(z)\Big)^{1/p} \approx \Big(\sum_{k\ge 1} \frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|}\Big)^{1/p}. \tag{7.6}$$

Proof. Let us assume that (7.5) is satisfied. By Proposition 7.1, J_μ is a p-summing operator, hence it is r-summing since $p \le r$ and we get

$$\pi_r(J_\mu) \lesssim \pi_p(J_\mu) \lesssim \Big(\int_{\mathbb{D}} \frac{1}{(1-|z|^2)^2}\, d\mu(z)\Big)^{1/p}.$$

The reverse inequality follows immediately from Proposition 7.2 . □

8. The case $p' \le r \le p$

In this section, we focus on the case $p' \le r \le p$ with $p \ge 2$.
We shall denote $\mathcal{A}^p(\mathcal{D}_k, A) = \mathcal{H}(\mathbb{D}) \bigcap L^p(\mathcal{D}_k, A)$, with

$$\|f\|^p_{\mathcal{A}(\mathcal{D}_k,A)} = \left(\int_{\mathcal{D}_k} |f|^p dA\right)^{\frac{1}{p}}.$$

Given a sequence $\{X_k\}$ of Banach spaces, the space $\bigoplus_{\ell^p} X_k$ is equipped by the norm

$$\big\|\{x_k\}\big\|_{\bigoplus_{\ell^p} X_k} = \Big(\sum_{k\ge 1} \|x_k\|^p_{X_k}\Big)^{\frac{1}{p}},$$

where $x_k \in X_k$ for every $k \ge 1$.

For any $f \in L^p(\mu)$, since the family $(\mathcal{D}_k)_{k\geq 1}$ forms a partition of $\mathbb{D}$,

$$\begin{aligned}\int_{\mathbb{D}} |f|^p d\mu &= \sum_{k=1}^{\infty} \int_{\mathcal{D}_k} |f|^p d\mu \\ &= \sum_{k=1}^{\infty} \|f\|^p_{L^p(\mathcal{D}_k,\mu)} \,.\end{aligned}$$

In other words,

$$f \in L^p(\mu) \longmapsto (f_{|_{\mathcal{D}_k}})_k \in \bigoplus_{\ell^p} L^p(\mathcal{D}_k,\mu)\,, \tag{8.1}$$

is an isomorphic isometry. This can be rewritten as

$$L^p(\mu) = \bigoplus_{\ell^p} L^p(\mathcal{D}_k,\mu)\,.$$

Point out that $\mathbb{D} = \bigcup_k D(a_k, 4t)$, with $0 < t < 1/4$, and recall that there exists a positive integer N such that each point of $\mathbb{D}$ belongs to at most N of the sets $D(a_k, 4t)$ (cf. Section 2). For $p \geq 1$, the following map is continous:

$$f \in \mathcal{A}^p \longmapsto (f_{|_{D(a_k,4t)}})_k \in \bigoplus_{\ell^p} \mathcal{A}^p(D(a_k,4t),A)\,. \tag{8.2}$$

Indeed, for any $f \in \mathcal{A}^p$ we have

$$\begin{aligned}\left\|(f_{|_{D(a_k,4t)}})_k\right\|_{\bigoplus_{\ell^p} \mathcal{A}^p(D(a_k,4t),A)} &= \left(\sum_{k\geq 1} \int_{D(a_k,4t)} |f(z)|^p dA(z)\right)^{1/p} \\ &\leq N^{1/p}\left(\int_{\mathbb{D}} |f(z)|^p dA(z)\right)^{1/p} \\ &= N^{1/p}\|f\|_{\mathcal{A}^p}\,.\end{aligned}$$

For any t-lattice $\{a_k\}$ $(0 < t < 1/2)$, we shall use the notation D'_k instead of $D(a_k, 2t)$.

Lemma 8.1. *Let $1 \leq r \leq p < \infty$, $0 < t < 1/2$, $\{a_k\}$ be a t-lattice on $\mathbb{D}$ and $\{\mathcal{D}_k\}$ be the relevant subsets of $\mathbb{D}$ defined in Lemma 2.6. Then for a positive Borel measure μ on $\mathcal{D}_k$ $(k = 1, 2, \ldots)$, the natural embedding map*

$$\mathcal{T}_{r,p} : \mathcal{A}^r(D'_k, A) \hookrightarrow L^p(\mathcal{D}_k,\mu)$$

is bounded if and only if

$$\frac{\mu(\mathcal{D}_k)^{1/p}}{|\mathcal{D}_k|^{1/r}} < \infty\,.$$

Furthermore,

$$\|\mathcal{T}_{r,p}\| \approx \frac{\mu(\mathcal{D}_k)^{1/p}}{|\mathcal{D}_k|^{1/r}}\,.$$

Proof. Using Lemma 4.6, the fact that $(1 - |z|^2)^2 \approx (1 - |a_k|^2)^2 \approx |\mathcal{D}_k|$ whenever $z \in \mathcal{D}_k$ (cf. Section 2), and the fact that $D(z,t) \subset D(a_k, 2t) = D'_k$ for every $z \in \mathcal{D}_k$, we get, for any

$f \in \mathcal{A}^r(D'_k, A)$,

$$\begin{aligned}\int_{\mathcal{D}_k} |f(z)|^p d\mu(z) &\le \sup_{z\in\mathcal{D}_k} |f(z)|^p \mu(\mathcal{D}_k)\\ &\lesssim \sup_{z\in\mathcal{D}_k} \frac{1}{(1-|z|^2)^{2p/r}} \left(\int_{D(z,t)} |f(w)|^r dA(w)\right)^{p/r} \mu(\mathcal{D}_k)\\ &\lesssim \frac{1}{(1-|a_k|^2)^{2p/r}} \left(\int_{D'_k} |f(w)|^r dA(w)\right)^{p/r} \mu(\mathcal{D}_k)\\ &\lesssim \|f\|_r^p \, \frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|^{p/r}}\,.\end{aligned} \tag{8.3}$$

This gives that

$$\|\mathcal{T}_{r,p}\| \lesssim \frac{\mu(\mathcal{D}_k)^{1/p}}{|\mathcal{D}_k|^{1/r}}\,.$$

Now take

$$f(z) = \left(\frac{(1-|a_k|^2)^2}{(1-\bar{a}_k z)^4}\right)^{1/r}, \qquad \text{for } z \in \mathbb{D}\,.$$

Since $(1-|a_k|^2)^2 \approx |1-\bar{a}_k z|^2 \approx |D'_k| \approx |\mathcal{D}_k|$ whenever $z \in \mathcal{D}_k$ (so $z \in D'_k$) with constants independent of k, we get

$$\|f\|_{\mathcal{A}^r(D'_k,\, A)} \approx 1\,,$$

and

$$\int_{\mathcal{D}_k} |f(z)|^p \, d\mu(z) \approx \frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|^{p/r}}\,.$$

Therefore

$$\|\mathcal{T}_{r,p}\| \gtrsim \frac{\mu(\mathcal{D}_k)^{1/p}}{|\mathcal{D}_k|^{1/r}}\,.$$

□

As an immediate corollary, we have by renormalization:

Corollary 8.2. *Let $1 \le r \le p < \infty$, $0 < t < 1/2$, $\{a_k\}$ be a t-lattice on $\mathbb{D}$ and $\{\mathcal{D}_k\}$ be the relevant subsets of $\mathbb{D}$ defined in Lemma 2.6. Then the operator $T^{(k)}_{r,p}$ defined by*

$$T^{(k)}_{r,p} : f \in \mathcal{A}^r\left(D'_k, \frac{A}{|\mathcal{D}_k|}\right) \longmapsto \frac{1}{|\mathcal{D}_k|^{1/p}} f_{|_{\mathcal{D}_k}} \in L^p(\mathcal{D}_k, \mu)$$

is bounded if and only if

$$\frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|} < \infty\,.$$

Moreover its norm is

$$\left\|T^{(k)}_{r,p}\right\| \approx \left(\frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|}\right)^{1/p}.$$

Theorem 8.3. *Let $p \geq 1$, $0 < t < 1/4$, a t-lattice $\{a_k\}$ and $\{\mathcal{D}_k\}$ is defined as in Lemma* 2.6. *For any k, assume that*

$$\frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|} < \infty\,.$$

Then,

$$S_k : f \in \mathcal{A}^p(D(a_k, 4t), A) \longmapsto f_{|_{\mathcal{D}_k}} \in L^p(\mathcal{D}_k, \mu)$$

is 1*-summing with*

$$\pi_1(S_k) \lesssim \left(\frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|}\right)^{1/p}.$$

Proof. Let $H^\infty(D'_k)$ be the class of the bounded analytic functions on D'_k with the natural supremum norm. We consider the commutative diagram

$$\begin{array}{ccc} \mathcal{A}^p(D(a_k, 4t), A) & \xrightarrow{S_k} & L^p(\mathcal{D}_k, \mu) \\ \Big\downarrow{\scriptstyle P_1} & & \Big\uparrow{\scriptstyle P_2} \\ H^\infty(D'_k) & \xrightarrow[j_1]{} & \mathcal{A}^1\Big(D'_k, \nu_k\Big) \end{array}$$

where ν_k is the normalized area measure on $\mathcal{D}_k$: $\frac{1}{|\mathcal{D}_k|}\, dA$, and $P_2 := T^{(k)}_{1,p}$ and with

$$P_1 f(z) = |\mathcal{D}_k|^{1/p} f(z),$$

for $f \in \mathcal{A}^p(D(a_k, 4t), A)$, we know from Lemma 8.2 that P_2 is bounded with norm $(\mu(\mathcal{D}_k)/|\mathcal{D}_k|)^{1/p}$.

The operator j_1 is obviously bounded and $\|j_1\| \approx 1$.

Using Lemma 4.6 and the fact that $D(z, 2t) \subset D(a_k, 4t)$ for every $z \in D'_k$, we get for any $f \in \mathcal{A}^p(D(a_k, 4t), A)$

$$\begin{aligned} \sup_{z \in D'_k} |f(z)| &\leq \sup_{z \in D'_k} \frac{1}{(1 - |z|^2)^{2/p}} \left(\int_{D(z,2t)} |f(w)|^p \, dA(w)\right)^{1/p} \\ &\lesssim \sup_{z \in D(a_k, 4t)} \frac{1}{(1 - |z|^2)^{2/p}} \left\|f\right\|_{\mathcal{A}^p(D(a_k, 4t), A)} \\ &\lesssim \frac{1}{(1 - |a_k|^2)^{2/p}} \left\|f\right\|_{\mathcal{A}^p(D(a_k, 4t), A)} \\ &\approx \frac{1}{|\mathcal{D}_k|^{1/p}} \left\|f\right\|_{\mathcal{A}^p(D(a_k, 4t), A)}. \end{aligned}$$

So P_1 is bounded with norm $\lesssim 1$. Obviously, $S_k = T^{(k)}_{1,p} \circ j_1 \circ P_1$.

Now let us observe that j_1 is 1-summing: indeed, it suffices to see that the restriction to $H^\infty(D'_k)$ of the map $id : L^\infty(D'_k) \to L^1(D'_k, \nu_k)$ is absolutely summing. Thus,

$$\pi_1(j_1) \leq \frac{|D'_k|}{|\mathcal{D}_k|} \approx 1\,.$$

Therefore, the ideal property for 1-summing operators leads to

$$\pi_1(S_k) \lesssim \|P_1\|\,\pi_1(j_1)\,\|T_{1,p}^{(k)}\| \lesssim \Big(\frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|}\Big)^{1/p}.$$

□

We shall need the following lemma.

Lemma 8.4. [13, Corollary 4.8] *Let $p \geq 2$, $p' \leq r \leq p$. Assume that for every integer $n \geq 1$, we have bounded operators $T_n : X_n \to Y_n$ for Banach spaces X_n and Y_n. Consider the operator*

$$T : \bigoplus_{\ell^p} X_n \longrightarrow \bigoplus_{\ell^p} Y_n$$
$$(x_n)_n \longmapsto (T_n(x_n))_{n\geq 1}.$$

Then T is r-summing if and only if each T_n is r-summing and

$$\sum_{n\geq 1} \pi_r(T_n)^r < +\infty.$$

Moreover, we have

$$\pi_r(T) \approx \Big(\sum_{n\geq 1} \pi_r(T_n)^r\Big)^{1/r}.$$

Now we can get easily the main theorem of this section .

Theorem 8.5. *Let $p \geq 2$, $p' \leq r \leq p$ and $0 < t < 1/4$. Let μ be a Carleson measure on $\mathcal{A}^p$. For any t-lattice $\{a_k\}$ and relevant set $\{\mathcal{D}_k\}$ as defined in Lemma 2.6, $J_\mu : \mathcal{A}^p \to L^p(\mu)$ is r-summing if and only if*

$$\sum_k \Big(\frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|}\Big)^{r/p} < \infty\,.$$

Moreover

$$\pi_r(J_\mu) \approx \Big(\sum_{k=1}^{\infty} \Big(\frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|}\Big)^{r/p}\Big)^{1/r}. \tag{8.4}$$

Proof. We consider the commutative diagram:

$$\begin{array}{ccc} \mathcal{A}^p & \xrightarrow{\quad J_\mu \quad} & L^p(\mu) \\ \downarrow M & & \uparrow M' \\ \bigoplus_{\ell^p} \mathcal{A}^p(D(a_k,4t),A) & \xrightarrow[S]{} & \bigoplus_{\ell^p} L^p(\mathcal{D}_k,\mu) \end{array}$$

where M is the map defined as in (8.2), which is continuous, and M' is bounded since the map as defined in (8.1) is an isomorphic isometry. Finally, S in the natural block diagonal map whose entries are the operator S_k. Theorem 8.3 implies that

$$S_k : f \in \mathcal{A}^p(D(a_k,4t),A) \longmapsto f \in L^p(\mathcal{D}_k,\mu)$$

is 1-summing with

$$\pi_1(S_k) \lesssim \left(\frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|}\right)^{1/p}.$$

It is obvious that $J_\mu = M' \circ S \circ M$ and it follows from Lemma 8.4 that

$$\pi_r(J_\mu) \lesssim \pi_r(S) \approx \Big(\sum_{k\geq 1} \pi_r(S_k)^r\Big)^{1/r} \leq \Big(\sum_{k\geq 1} \pi_1(S_k)^r\Big)^{1/r} \lesssim \Big(\sum_{k=1}^{\infty} \Big(\frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|}\Big)^{r/p}\Big)^{1/r}.$$

We get the reverse inequality from Proposition 7.2 :

$$\Big(\sum_{k=1}^{\infty} \Big(\frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|}\Big)^{r/p}\Big)^{1/r} \lesssim \pi_r(J_\mu).$$

□

9. The case $1 \leq r \leq p'$

We focus now on the last remaining case $1 \leq r \leq p'$.

Proposition 9.1. *Let X_n and Y_n be two Banach spaces. Let $T_n : X_n \to Y_n$ be a 1-summing operator with $\pi_1(T_n) \lesssim 1$. Then the operator*

$$T : x = (x_n)_n \in \bigoplus_{\ell^1} X_n \longmapsto \big(T_n(x_n)\big)_{n\geq 1} \in \bigoplus_{\ell^2} Y_n$$

is 1-summing and $\pi_1(T) \lesssim 1$.

We are indebted to L. Rodríguez-Piazza for the following argument which seems to us clearer than an alternative previous one.

Proof. Let $n \geq 1$. Since T_n is 1-summing with $\pi_1(T_n) \lesssim 1$, by the Pietsch Theorem (see for instance [5], page 44), there exist a constant $C > 0$ depending only on n and a probability measure μ_n on the unit ball K_n of X_n^* (K_n is a compact set endowed with the weak-star topology) such that

$$\forall u \in X_n\,, \qquad \|T_n(u)\| \leq C \int_{K_n} |\,\chi(u)|\, d\mu_n(\,\chi) \tag{9.1}$$

for each n.

Let $(\Omega, \mathbb{P})$ be a probability space and $(r_n)_n$ be a Rademacher sequence (i.e. a sequence of independent random equidistributed variables taking their values in $\{\pm 1\}$; i.e. random choice of signs). Actually we can assume that Ω is a compact set (take $\{\pm 1\}^{\mathbb{N}}$, the Cantor group and r_n is the n^{th} coordinate map). Then, writing $K = \prod_n K_n$, we define $\hat{K} = \Omega \times K$ which is a compact set and the measure $\nu = \mathbb{P} \otimes \mu$ is a probability measure on $\hat{K}$, where $\mu = \otimes_n \mu_n$.

Let $x_n^*(\xi_n) = \xi_n(x_n)$ where $\xi = (\xi_n) \in K$. The map:

$$E : x = (x_n)_n \in \bigoplus_{\ell^1} X_n \longmapsto \sum_n r_n \otimes x_n^* \in C(\hat{K})$$

is an isometry. Indeed,

$$\|E(x)\| = \sup_{\substack{t\in\Omega \\ \xi_n\in K_n}} \Big|\sum_n r_n(t) x_n^*(\xi_n)\Big| = \sup_{\xi_n\in K_n} \sum_n |\xi_n(x_n)| = \sum_n \|x_n\|\,.$$

On the other hand, let Z_1 be the closure in $L^1(\hat{K}, \nu)$ of the vector space V_1 spanned by $\{r_n \otimes x_n^*;\ x_n \in X_n\}$. The formal identity i from $C(\hat{K})$ to $Z_1 \subset L^1(\hat{K}, \nu)$ is 1-summing with $\pi_1(i) = \nu(\hat{K}) = 1$ (ν is a probability measure).

Finally the map

$$F : f = \sum_n r_n \otimes x_n^* \in V_1 \longmapsto (T_n(x_n))_n \in \bigoplus_{\ell^2} Y_n$$

is well defined, bounded and extends to a bounded operator on Z_1. Indeed, since (9.1) holds then

$$\Big(\sum_n \|T_n(x_n)\|^2 \Big)^{1/2} \leq C \Big(\sum_n \|x_n^*\|^2_{L^1(K)} \Big)^{1/2}$$

which is nothing but

$$\Big\| \Big(\int_K |x_n^*| d\mu \Big)_n \Big\|_{\ell^2}$$

and is then lower than

$$\int_K \Big(\sum_n |x_n^*|^2 \Big)^{1/2} d\mu.$$

Then the Khinchin's inequality implies that

$$\Big(\sum_n \|T_n(x_n)\|^2 \Big)^{1/2} \lesssim \int_K \int_\Omega \Big| \sum_n r_n(t) x_n^* \Big| \, d\,\mathbb{P}\, d\mu = \Big\| \sum_n r_n \otimes x_n^* \Big\| .$$

Hence,

$$\pi_1(T) \leq \|F\| \, \pi_1(i) \, \|E\| \lesssim 1.$$

The conclusion follows. □

Corollary 9.2. *Let X_n and Y_n be Banach spaces. Let $T_n : X_n \longrightarrow Y_n$ be a 1-summing operator. We assume that*

$$\sum_{n \geq 1} \pi_1(T_n)^{p'} < \infty.$$

Then the operator

$$T : x = (x_n)_n \in \bigoplus_{\ell^p} X_n \longmapsto \Big(T_n(x_n) \Big)_{n \geq 1} \in \bigoplus_{\ell^p} Y_n$$

is 1-summing with

$$\pi_1(T) \lesssim \|\pi_1(T_n)\|_{\ell^{p'}}.$$

Proof. Write $\lambda_n = \pi_1(T_n)$. We have $(\lambda_n)_n \in \ell^{p'}$ by hypothesis. The operator $S_n = \lambda_n^{-1} T_n$ satisfies $\pi_1(S_n) = 1$ by definition. From the previous proposition, we know that the operator

$$S : x = (x_n)_n \in \bigoplus_{\ell^1} X_n \longmapsto (S_n(x_n))_n \in \bigoplus_{\ell^2} Y_n$$

is 1-summing with $\pi_1(S) \lesssim 1$.

On the other hand, we have the factorization: $T = J \circ S \circ D$ where

$$D : x = (x_n)_n \in \bigoplus_{\ell^p} X_n \longmapsto (\lambda_n x_n)_n \in \bigoplus_{\ell^1} X_n$$

is bounded since $(\lambda_n)_n \in \ell^{p'}$ with $\|D\| \lesssim \|\lambda_n\|_{\ell^{p'}}$, and

$$J : y = (y_n)_n \in \bigoplus_{\ell^2} Y_n \longmapsto (y_n)_n \in \bigoplus_{\ell^p} Y_n$$

is bounded since $p \geq 2$ with norm $\|J\| \lesssim 1$. Therefore,

$$\pi_1(T) \leq \|J\|\, \pi_1(S)\, \|D\| \lesssim \|\lambda_n\|_{\ell^{p'}} = \|\pi_1(T_n)\|_{\ell^{p'}}.$$

□

Now we can state the main theorem of this section.

Theorem 9.3. *Let $p \geq 2$ and $1 \leq r \leq p'$. Let μ be a Carleson measure on $\mathcal{A}^p$, a t-lattice $\{a_k\}$ and $\{\mathcal{D}_k\}$ as defined as in Lemma 2.6. Then the following statements are equivalent:*

1) $J_\mu : \mathcal{A}^p \to L^p(\mu)$ is 1-summing.

2) $J_\mu : \mathcal{A}^p \to L^p(\mu)$ is r-summing.

3) $\displaystyle\sum_{k\geq 1} \Big(\frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|}\Big)^{p'/p} < \infty.$

Moreover,

$$\pi_1(J_\mu) \approx \pi_r(J_\mu) \approx \Big(\sum_{k\geq 1} \Big(\frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|}\Big)^{p'/p}\Big)^{1/p'} \approx \left(\int_{\mathbb{D}} \Big(\int_{D_z} \frac{1}{(1-|w|^2)^{2(1+p/p')}} d\mu(w)\Big)^{p'/p} dA(z)\right)^{1/p'}.$$

Proof. 1) $\Rightarrow$ 2). The proof is obvious.

2) $\Rightarrow$ 3). It follows from Theorem 8.5 and the fact that $r \leq p'$,

$$\Big(\sum_{k\geq 1} \Big(\frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|}\Big)^{p'/p}\Big)^{1/p'} \lesssim \pi_{p'}(J_\mu) \leq \pi_r(J_\mu).$$

3) $\Rightarrow$ 1). By hypothesis, we have in particular:

$$\sup_k \frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|} < \infty.$$

Theorem 8.3 implies that

$$S_k : f \in \mathcal{A}^p(D(a_k, 4t), A) \longmapsto f \in L^p(\mathcal{D}_k, \mu)$$

is 1-summing with

$$\pi_1(S_k) \lesssim \Big(\frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|}\Big)^{1/p}.$$

Following the same proof of Theorem 8.5 and using Corollary 9.2, we get J_μ is 1-summing with

$$\pi_1(J_\mu) \lesssim \Big(\sum_{k\geq 1} \Big(\frac{\mu(\mathcal{D}_k)}{|\mathcal{D}_k|}\Big)^{p'/p}\Big)^{1/p'}.$$

This completes the proof. □

10. The results on weighted Bergman spaces

In this section, we give similar characterizations on weighted Bergman spaces $\mathcal{A}^p_\alpha$. For any $\alpha > -1$, let $dA_\alpha = (\alpha+1)(1-|z|^2)^\alpha dA(z)$ and

$$\mathcal{A}^p_\alpha = \mathcal{H}(\mathbb{D}) \bigcap L^p(\mathbb{D}, A_\alpha).$$

The following theorem extends the main theorem to the framework of weighted Bergman spaces. We can prove it in a similar way, and so we omit the details here.

Theorem 10.1. *Let $0 < t < 1/4$ and $\alpha > -1$. Let μ be a Carleson measure on $\mathcal{A}^p_\alpha$. Then*

(1) Let $1 < p \le 2$ and $r \ge 1$. Then $J_\mu : \mathcal{A}^p_\alpha \to L^p(\mu)$ is r-summing if and only if

$$\int_{\mathbb{D}} \frac{(\mu(D_z))^{\frac{2}{p}}}{(1-|z|^2)^{(\alpha+2)(\frac{2}{p}+1)}}\, dA_\alpha(z) < \infty\,.$$

Moreover, we have

$$\pi_2(J_\mu) \approx \left(\sum_{n,j}\left(4^{n(1+\alpha/2)}\mu(R_{n,j})\right)^{2/p}\right)^{1/2} \approx \left(\int_{\mathbb{D}} \frac{(\mu(D_z))^{\frac{2}{p}}}{(1-|z|^2)^{(\alpha+2)(\frac{2}{p}+1)}}\, dA_\alpha(z)\right)^{1/2}. \tag{10.1}$$

(2) Let $p \ge 2$ and $1 \le r \le p'$. Then $J_\mu : \mathcal{A}^p_\alpha \to L^p(\mu)$ is r-summing if and only if

$$\sum_{n,j}\left(4^{n(1+\alpha/2)}\mu(R_{n,j})\right)^{p'/p} < \infty.$$

Moreover, we have

$$\pi_1(J_\mu) \approx \pi_r(J_\mu) \approx \left(\sum_{n,j}\left(4^{n(1+\alpha/2)}\mu(R_{n,j})\right)^{p'/p}\right)^{1/p'} \approx \left(\int_{\mathbb{D}} \frac{(\mu(D_z))^{\frac{p'}{p}}}{(1-|z|^2)^{(\alpha+2)(\frac{p'}{p}+1)}}\, dA_\alpha(z)\right)^{1/p'}. \tag{10.2}$$

(3) Let $p \ge 2$ and $p' \le r \le p$. Then $J_\mu : \mathcal{A}^p_\alpha \to L^p(\mu)$ is r-summing if and only if

$$\sum_{n,j}\left(4^{n(1+\alpha/2)}\mu(R_{n,j})\right)^{r/p} < \infty\,.$$

Moreover, we have

$$\pi_r(J_\mu) \approx \left(\sum_{n,j}\left(4^{n(1+\alpha/2)}\mu(R_{n,j})\right)^{r/p}\right)^{1/r} \approx \left(\int_{\mathbb{D}} \frac{(\mu(D_z))^{\frac{r}{p}}}{(1-|z|^2)^{(\alpha+2)(\frac{r}{p}+1)}}\, dA_\alpha(z)\right)^{1/r}. \tag{10.3}$$

(4) Let $p \geq 2$ and $r \geq p$. Then $J_\mu : \mathcal{A}^p_\alpha \to L^p(\mu)$ is r-summing if and only if

$$\int_{\mathbb{D}} \frac{1}{(1-|z|^2)^{2+\alpha}} d\mu(z) < \infty .$$

Moreover, we have

$$\pi_r(J_\mu) \approx \left(\sum_{n,j}^{\infty} \left(4^{n(1+\alpha/2)} \mu(R_{n,j}) \right) \right)^{1/p} \approx \left(\int_{\mathbb{D}} \frac{1}{(1-|z|^2)^{2+\alpha}} d\mu(z) \right)^{1/p}. \tag{10.4}$$

In the same way, we get for free the characterizations for weighted composition operators.

Theorem 10.2. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map and $u : \mathbb{D} \to \mathbb{C}$ be an analytic function. Let uC_φ be the composition operator viewed from $\mathcal{A}^p_\alpha$ to $\mathcal{A}^p_{\alpha'}$ where $\alpha, \alpha' > -1$, and $d\mu = (|u|^p dA_{\alpha'})_\varphi$. The operator uC_φ is r-summing if and only if the following π_r norm is finite.*

(1) When $1 < p \leq 2$, for any $r \geq 1$,

$$\pi_r(uC_\varphi) \approx \pi_2(uC_\varphi) \approx \left(\int_{\mathbb{D}} \left(\int_{\varphi^{-1}(D_z)} \frac{|u|^p}{(1-|\varphi|^2)^{(2+\alpha)(1+p/2)}} dA_{\alpha'}(w) \right)^{2/p} dA_\alpha(z) \right)^{1/2}.$$

(2) When $p \geq 2$, for $1 \leq r \leq p'$,

$$\pi_r(uC_\varphi) \approx \pi_1(uC_\varphi) \approx \left(\int_{\mathbb{D}} \left(\int_{\varphi^{-1}(D_z)} \frac{|u|^p}{(1-|\varphi|^2)^{(2+\alpha)(1+p/p')}} \, dA_{\alpha'}(w) \right)^{p'/p} dA_\alpha(z) \right)^{1/p'}.$$

(3) When $p \geq 2$, for $p' \leq r \leq p$,

$$\pi_r(uC_\varphi) \approx \left(\int_{\mathbb{D}} \left(\int_{\varphi^{-1}(D_z)} \frac{|u|^p}{(1-|\varphi|^2)^{(2+\alpha)(1+p/r)}} \, dA_{\alpha'}(w) \right)^{r/p} dA_\alpha(z) \right)^{1/r}.$$

(4) When $p \geq 2$, for $p \leq r$,

$$\pi_r(uC_\varphi) \approx \left(\int_{\mathbb{D}} \frac{|u|^p}{(1-|\varphi|^2)^{2+\alpha}} \, dA_{\alpha'} \right)^{1/p}.$$

Remark: Of course, we could add some estimates involving the family $(\mathcal{D}_k)_k$ or the Luecking rectangles, using the fact that, for every $s > 0$,

$$\sum_{k=1}^{\infty} \left(\mu(\mathcal{D}_k)/|\mathcal{D}_k|^{1+\alpha/2} \right)^s \approx \sum_{n \geq 0} \sum_{0 \leq j < 2^n} \left(\mu(R_{n,j})/|R_{n,j}|^{1+\alpha/2} \right)^s.$$

11. Some other applications

We mention in this section some direct applications of our results.

The first one concerns a monotony property relatively to the index of the Bergman space.

Proposition 11.1. *Let μ be a positive measure on $\mathbb{D}$ and $1 < p_2 < p_1$.*

If J_μ is r-summing on $\mathcal{A}^{p_1}$, then J_μ is r-summing on $\mathcal{A}^{p_2}$.

Proof. Fix $r \geq 1$, $p_1 > p_2 > 1$ and let $\xi := \left(\frac{\mu(D_k)}{|D_k|}\right)_{k\geq 1}$.

We have to separate the cases since our characterizations depend on the values of (p_1, p_2, r).

- If $1 < p_2 < p_1 \leq 2$.
  Assume that $J_\mu : \mathcal{A}^{p_1} \to L^{p_1}(\mu)$ is absolutely summing, then we have $\xi \in \ell^{2/p_1}$ by our characterization in Section 3. It follows that $\xi \in \ell^{2/p_2}$, hence $J_\mu : \mathcal{A}^{p_2} \to L^{p_2}(\mu)$ is absolutely summing thanks to our characterization (3.1).

- If $2 \leq p_2 < p_1$. Let $r \geq 1$. we split this case in sub-cases:
  - If $1 \leq r \leq p_1'$, we have that $\xi \in \ell^{p_1'/p_1}$. This implies that $\xi \in \ell^{p_2'/p_2}$ since $p_1'/p_1 < p_2'/p_2$. Then, again from our main theorem in Section 3, it follows that J_μ is r-summing on $\mathcal{A}^{p_2}$ thanks to (3.2).
  - If $p_1' \leq r \leq p_2'$, we get that $\xi \in \ell^{p_2'/p_2}$, since $\xi \in \ell^{r/p_1}$ and $r/p_1 < p_2'/p_2$, hence J_μ is r-summing on $\mathcal{A}^{p_2}$ thanks to (3.2).
  - If $p_2' \leq r \leq p_2$, we get that $\xi \in \ell^{r/p_2}$, since $\xi \in \ell^{r/p_1}$ and $r/p_1 < r/p_2$, hence J_μ is r-summing on $\mathcal{A}^{p_2}$ thanks to (3.3).
  - If $p_2 \leq r \leq p_1$, we get that $\xi \in \ell^1$, since $\xi \in \ell^{r/p_1}$ and $r < p_1$, hence J_μ is r-summing on $\mathcal{A}^{p_2}$ thanks to (3.4).
  - If $p_1 \leq r$, then we have that $\xi \in \ell^1$. It turns out that J_μ is r-summing on $\mathcal{A}^{p_2}$ thanks to (3.4).
- The case $p_2 \leq 2 < p_1$ follows immediately from the previous cases: if J_μ is r-summing on $\mathcal{A}^{p_1}$ then it is r-summing on $\mathcal{A}^2$ by the previous case, hence it is r-summing on $\mathcal{A}^{p_2}$ by the first case.

□

We give another direct application of our main theorem: we have a sufficient condition to ensure that J_μ is absolutely summing in terms of Carleson's condition.

Proposition 11.2. *Let μ be a positive measure on $\mathbb{D}$. Let $\gamma > 5/2$.*

Assume that μ is a positive measure on $\mathbb{D}$ which is γ-Carleson: $\mu(D_z) = O((1-|z|)^\gamma)$, when $|z| \to 1^-$, equivalently:

$$\sup_{\xi\in\mathbb{T}} \mu\Big(\mathcal{W}(\xi, h)\Big) = O(h^\gamma), \qquad \text{when } h \to 0.$$

Then

- *Let $p \in (1, 2)$. If $\gamma > \frac{p}{2} + 2$, then J_μ is absolutely summing on $\mathcal{A}^p$.*
- *Let $p \geq 2$.*
  - *(i) If $\gamma > 3$, then J_μ is p-summing on $\mathcal{A}^p$.*
  - *(ii) If $\gamma > \frac{p}{r} + 2$ (for some $r \in [p', p]$), then J_μ is r-summing on $\mathcal{A}^p$.*
  - *(iii) If $\gamma > p + 1$, then J_μ is absolutely summing on $\mathcal{A}^p$.*

Recall that the boundedness of J_μ on $\mathcal{A}^p$ is ensured exactly as soon as μ is 2-Carleson.

We can read the previous result in another way. Assume that μ is γ-Carleson. If $\gamma \in (5/2, 3]$ then J_μ is absolutely summing on $\mathcal{A}^p$ for any $p \in (1, 2(\gamma-2)) \subset (1,2)$.

If $\gamma > 3$, then J_μ is r-summing on $\mathcal{A}^p$ for some suitable (r,p) with $p > 2$. In particular, it is absolutely summing on $\mathcal{A}^p$ for any $p \in (1, \gamma - 1)$.

Proof. It suffices to check (easily) that the conditions in the main theorem ar satisfied.

For instance, for $p \in (1,2)$, since

$$\frac{(\mu(D_z))^{\frac{2}{p}}}{(1-|z|^2)^{\frac{4}{p}+2}} \lesssim (1-|z|^2)^{\frac{2(\gamma-2)}{p}-2}$$

a sufficient condition to ensure the convergence of the integral is

$$\frac{2(\gamma-2)}{p} - 2 > -1 \ .$$

The other cases are treated in the same way. □

Another application concerns composition operators on vector valued (weak and strong) Bergman spaces. A partial result is given by Laitila-Tylli-Wang in [12]. They also treat partially the case of vector valued Hardy spaces and the result is completed in [1] by Blasco for L^q valued functions. We give below the corresponding result for Bergman spaces.

Before stating precisely the results, let us recall the framework. For any Banach space X, we use $\ell^p_{weak}(X)$ to show the weak p-summable sequence (x_n) in X. It is equipped with the norm defined by

$$\|(x_n)\|_{\ell^p_{weak}(X)} := \sup_{x^* \in B_{X^*}} \left(\sum_k |\langle x^*, x_k \rangle|^p \right)^{\frac{1}{p}}$$

where $(x_n) \in \ell^p_{weak}(X)$.

For any Banach space X, let $p \geq 1$, the norm on $\mathcal{A}^p(\mathbb{D}, X)$ (resp. $\mathcal{A}^p_{weak}(\mathbb{D}, X)$) is defined by

$$\|f\|_{\mathcal{A}^p(\mathbb{D},X)} := \left(\int_{\mathbb{D}} \|f(z)\|_X^p dA(z) \right)^{\frac{1}{p}}$$

and

$$\|f\|_{\mathcal{A}^p_{weak}(\mathbb{D},X)} := \sup_{x^* \in B_{X^*}} \left(\int_{\mathbb{D}} |\langle x^*, f(z) \rangle|^p dA(z) \right)^{\frac{1}{p}}$$

respectively.

For any symbol $\varphi : \mathbb{D} \to \mathbb{D}$ (analytic), the composition operator C_φ is well defined (and bounded) from $\mathcal{A}^p$ into $\mathcal{A}^p$. For any Banach space X, we are interested in the operator C^X_φ (until now formally) defined by

$$C^X_\varphi : f \in \mathcal{A}^p_{weak}(\mathbb{D}, X) \to f \circ \varphi \in \mathcal{A}^p(\mathbb{D}, X).$$

That is, to any $f(z) = \sum_{n=0}^{\infty} x_n z^n \in \mathcal{A}^p_{weak}(\mathbb{D}, X)$ where $x_n \in X$ for $n \geq 0$, we associate the X-valued analytic function

$$C^X_\varphi(f)(z) = \sum_{n=0}^{\infty} x_n (\varphi(z))^n.$$

In [12, Th 3.2], the authors prove the following theorem:

Theorem A. *Let $2 \leq p < \infty$, φ be a symbol and X be any complex infinite-dimensional Banach space. Then*

$$\|C_\varphi\|_{\mathcal{A}^p_{weak}(\mathbb{D},X)\to\mathcal{A}^p(\mathbb{D},X)} \approx \left(\int_{\mathbb{D}} \frac{1}{(1-|\varphi(z)|^2)^2} dA(z)\right)^{1/p}. \tag{11.1}$$

They also prove a Hardy space version of the previous theorem, which is completed for other values of p and when $X = L^q(\mu)$ by Blasco in [1, Th 1.1]:

Theorem B. *Let φ be a symbol and $1 < p \leq 2$. Let $X = L^q(\mu)$ for $1 \leq q \leq p$ and infinite dimensional. Then*

$$\|C^X_\varphi\|_{H^p_{weak}(\mathbb{D},X)\to H^p(\mathbb{D},X)} \approx \left(\int_0^{2\pi}\left(\int_0^{2\pi} \frac{1}{|1-e^{-is}\varphi(e^{it})|^{1+p/2}} \frac{dt}{2\pi}\right)^{2/p} \frac{ds}{2\pi}\right)^{1/2}. \tag{11.2}$$

The following result is the analogue of Theorem B for vector valued Bergman spaces, and completes Theorem A in this framework.

Theorem 11.3. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic function, $1 < p \leq 2$ and $1 \leq q \leq p$. Let (Σ, Ω, μ) be a probability space and $X := L^q(\mu)$. Then*

$$C^X_\varphi : \mathcal{A}^p_{weak}(\mathbb{D}, L^q(\mu)) \to \mathcal{A}^p(\mathbb{D}, L^q(\mu))$$

is bounded if and only if

$$\int_{\mathbb{D}}\left(\int_{\mathbb{D}} \frac{1}{|1-\bar{z}\varphi(w)|^{p+2}}\, dA(w)\right)^{\frac{2}{p}} dA(z) < \infty, \tag{11.3}$$

and moreover

$$\|C^X_\varphi\|_{\mathcal{A}^p_{weak}\to\mathcal{A}^p} \approx \left(\int_{\mathbb{D}}\left(\int_{\mathbb{D}} \frac{1}{|1-\bar{z}\varphi(w)|^{p+2}}\, dA(w)\right)^{\frac{2}{p}} dA(z)\right)^{\frac{1}{2}}.$$

Proof. Suppose (11.3) is true. Then by Corollary 3.2, we get $C_\varphi \in \Pi_q(\mathcal{A}^p, \mathcal{A}^p)$.
Let $F(z) = \sum_{n=0}^{\infty} f_n z^n \in \mathcal{A}^p_{weak}(\mathbb{D}, L^q(\mu))$, where $f_n \in L^q(\mu)$ for every $n \geq 0$. Hence, for each $h \in (L^q(\mu))^*$, we have that

$$z \in \mathbb{D} \longmapsto \langle h, F(z)\rangle = \sum_{n=0}^{\infty}\left(\int_\Omega h(\omega) f_n(\omega) d\mu(\omega)\right) z^n.$$

belongs to $\mathcal{A}^p$.

So we have

$$\begin{aligned}\|F\|_{\mathcal{A}^p_{weak}(\mathbb{D},L^q(\mu))} &= \sup_{h\in B_{L^{q'}(\mu)}}\left(\int_{\mathbb{D}}|\langle h,F(z)\rangle|^p dA(z)\right)^{1/p}\\ &= \sup_{h\in B_{L^{q'}(\mu)}}\sup\left\{\Big|\sum_{n=0}^{\infty}\int_{\Omega}h(\omega)f_n(\omega)d\mu(\omega)\bar{a_n}\Big|\ ;\ g\in B_{\mathcal{A}^{p'}}\text{ with } g=\sum_{n=0}^{\infty}a_nz^n\right\}\\ &= \sup_{g\in B_{\mathcal{A}^{p'}}}\left(\int_{\Omega}|\langle G(\omega),g(.)\rangle|^q d\mu(\omega)\right)^{1/q},\end{aligned}$$

where $G(\omega)=\sum_{n=0}^{\infty}f_n(\omega)z^n\in\mathcal{A}^p$.

The fact that $C_\varphi\in\Pi_q(\mathcal{A}^p,\mathcal{A}^p)$ shows that

$$\left(\int_{\Omega}\left\|C_\varphi(G)\right\|^q_{\mathcal{A}^p}d\mu(\omega)\right)^{1/q}\leq\pi_q(C_\varphi)\sup_{g\in B_{\mathcal{A}^{p'}}}\left(\int_{\Omega}|\langle G(\omega),g(.)\rangle|^q d\mu(\omega)\right)^{1/q}.$$

So the Minkowski's inequality gives that

$$\begin{aligned}\left(\int_{\mathbb{D}}\left|\int_{\Omega}|C_\varphi(F)|^q d\mu(\omega)\right|^{p/q}dA(z)\right)^{1/p} &\leq \left(\int_{\Omega}\left|\int_{\mathbb{D}}|C_\varphi(F)|^p dA(z)\right|^{q/p}d\mu(\omega)\right)^{1/q}\\ &\leq \pi_q(C_\varphi)\|F\|_{\mathcal{A}^p_{weak}\left(\mathbb{D},L^q(\mu)\right)},\end{aligned}$$

and moreover

$$\|C^X_\varphi\|_{\mathcal{A}^p_{weak}\to\mathcal{A}^p}\lesssim\left(\int_{\mathbb{D}}\left(\int_{\mathbb{D}}\frac{1}{|1-\bar{z}\varphi(w)|^{p+2}}\,dA(w)\right)^{\frac{2}{p}}dA(z)\right)^{\frac{1}{2}}.$$

For the converse, assume that C^X_φ is bounded and we are going to show that

$$\pi_2(C_\varphi)\leq\left\|C^X_\varphi\right\|_{\mathcal{A}^p_{weak}\to\mathcal{A}^p}.$$

Let $(f_k)\in\ell^w_2(\mathcal{A}^p)$ with

$$\sup_{g\in B_{\mathcal{A}^{p'}}}\left(\sum_k|\langle g,f_k\rangle|^2\right)^{1/2}\leq 1.$$

The proof will be finished as soon as we will have proved that

$$\left(\sum_k\|C_\varphi(f_k)\|^2_{\mathcal{A}^p}\right)^{1/2}\leq\|C^X_\varphi\|_{\mathcal{A}^p_{weak}\to\mathcal{A}^p}.$$

For that, we make use of the Dvoretzky theorem (see [5, p.397]): for every $\varepsilon>0$, and $n\in N$, there exists a linear map $T_n:\ell^n_2\to L^q(\mu)$ such that for every $(a_j)\in\ell^n_2$

$$(1+\varepsilon)^{-1}\left(\sum_{j=1}^n|a_j|^2\right)^{1/2}\leq\Big\|\sum_{j=1}^n a_jT_n(e_j)\Big\|_{L^q(\mu)}\leq\left(\sum_{j=1}^n|a_j|^2\right)^{1/2}.$$

Now define $F_n(z)=\sum_{k=1}^n f_k(z)T_n(e_k)$.

For each $x^* \in L^{q'}(\mu)$

$$\begin{aligned}\langle x^*, F_n(z)\rangle &= \sum_{k=1}^{n} \langle x^*, T_n(e_k)\rangle f_k(z) \\ &= \left(\sum_{k=1}^{n} \langle T_n^*(x^*), e_k\rangle f_k\right)(z).\end{aligned}$$

Therefore, for every x^* in the unit ball of $L^{q'}(\mu)$, we have

$$\begin{aligned}\|\langle x^*, F_n\rangle\|_{\mathcal{A}^p} &= \Big\| \sum_{k=1}^{n} \langle T_n^*(x^*), e_k\rangle f_k \Big\|_{\mathcal{A}^p} \\ &= \sup_{g\in B_{\mathcal{A}^{p'}}} \left| \sum_k \langle T_n^*(x^*), e_k\rangle . \langle f_k, g\rangle \right| \\ &\le \left(\sum_k |\langle T_n^*(x^*), e_k\rangle|^2\right)^{1/2} . \sup_{g\in B_{\mathcal{A}^{p'}}} \left(\sum_k |\langle f_k, g\rangle|^2\right)^{1/2} \\ &\le \|T_n^*\| . \|x^*\| \le 1\end{aligned}$$

since $\|T_n\| = \|x^*\| = 1$. This implies that

$$\|F_n\|_{\mathcal{A}^p_{weak}(\mathbb{D}, L^q(\mu))} \le 1.$$

Since C_φ^X is bounded from $\mathcal{A}^p_{weak}(\mathbb{D}, L^q(\mu))$ into $\mathcal{A}^p(\mathbb{D}, L^q(\mu))$, using Dvoretzky theorem, we have

$$\begin{aligned}\left\|C_\varphi^X\right\|^p_{\mathcal{A}^p_{weak}\to\mathcal{A}^p} &\ge \left\|C_\varphi^X(F_n)\right\|^p_{\mathcal{A}^p} \\ &= \int_{\mathbb{D}} \left\| \sum_{k=1}^{n} T_n(e_k) f_k(\varphi(z)) \right\|^p_{L^q(\mu)} dA(z) \\ &\ge (1+\varepsilon)^{-p} \int_{\mathbb{D}} \left(\sum_{k=1}^{n} |f_k(\varphi(z))|^2\right)^{p/2} dA(z) \\ &\ge (1+\varepsilon)^{-p} \left(\sum_{k=1}^{n} \left(\int_{\mathbb{D}} |f_k(\varphi(z))|^p dA(z)\right)^{2/p}\right)^{p/2},\end{aligned}$$

where the last inequality holds since $2/p \ge 1$ and using the Minkowski's inequality.

Since ε is arbitrary, we get

$$\left(\sum_{k=1}^{n} \|C_\varphi(f_k)\|_p^2\right)^{1/2} \le \left\|C_\varphi^X\right\|^p_{\mathcal{A}^p_{weak}\to\mathcal{A}^p}.$$

By the definition of 2-summing, we have

$$\pi_2(C_\varphi) \le \|C_\varphi^X\|_{\mathcal{A}^p_{weak}\to\mathcal{A}^p}.$$

At last, by Corollary 3.2, we have

$$\left(\int_{\mathbb{D}}\left(\int_{\mathbb{D}}\frac{1}{|1-\bar{z}\varphi(w)|^{p+2}}\,dA(w)\right)^{\frac{2}{p}}dA(z)\right)^{\frac{1}{2}}\lesssim\left\|C_{\varphi}^{X}\right\|_{\mathcal{A}^{p}_{weak}\to\mathcal{A}^{p}}.$$

□

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BO HE, DEPARTMENT OF MATHEMATICS, SHANTOU UNIVERSITY, SHANTOU GUANGDONG 515063, PEOPLE'S REPUBLIC OF CHINA.
Email address: 16bhe@stu.edu.cn

JOELLE JREIS, UNIV. ARTOIS, UR 2462, LABORATOIRE DE MATHÉMATIQUES DE LENS (LML), F-62300 LENS, FRANCE
Email address: joelle.jreis@univ-artois.fr

PASCAL LEFÈVRE, UNIV. ARTOIS, UR 2462, LABORATOIRE DE MATHÉMATIQUES DE LENS (LML), F-62300 LENS, FRANCE
Email address: pascal.lefevre@univ-artois.fr

ZENGJIAN LOU, DEPARTMENT OF MATHEMATICS, SHANTOU UNIVERSITY, SHANTOU GUANGDONG 515063, PEOPLE'S REPUBLIC OF CHINA.
Email address: zjlou@stu.edu.cn